Review of Linear Algebra

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STAT 4690–Applied Multivariate Analysis

Eigenvalues and Eigenvectors

Eigenvalues

- Let $\mathbf{A}$ be a square $n \times n$ matrix.
- The equation

  $$\det(\mathbf{A} - \lambda I_n) = 0$$

  is called the *characteristic equation* of $\mathbf{A}$.
- This is a polynomial equation of degree n, and its roots are called the *eigenvalues* of $\mathbf{A}$.

Example

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \det(\mathbf{A} - \lambda I_n) &= (1-\lambda)^2 - 0.25 \\ &= (\lambda - 1.5)(\lambda - 0.5) \end{aligned}$$

Therefore, $\mathbf{A}$ has two (real) eigenvalues, namely

$$\lambda_1 = 1.5, \lambda_2 = 0.5.$$

A few properties

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$ (with multiplicities).

1. $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$;
2. $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$;
3. The eigenvalues of $\mathbf{A}^k$ are $\lambda_1^k, \ldots, \lambda_n^k$, for k a nonnegative integer;
4. If $\mathbf{A}$ is invertible, then the eigenvalues of $\mathbf{A}^{-1}$ are $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$.

Eigenvectors

- If λ is an eigenvalues of $\mathbf{A}$, then (by definition) we have $\det(\mathbf{A} - \lambda I_n) = 0$.
- In other words, the following equivalent statements hold:
  - The matrix $\mathbf{A} - \lambda I_n$ is singular;
  - The kernel space of $\mathbf{A} - \lambda I_n$ is nontrivial (i.e. not equal to the zero vector);
  - The system of equations $(\mathbf{A} - \lambda I_n)v = 0$ has a nontrivial solution;
  - There exists a nonzero vector v such that

$$\mathbf{A}v = \lambda v.$$

- Such a vector is called an *eigenvector* of $\mathbf{A}$.