Comp Econ HW9

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Excercise 1 :

First, we can show that

$$\|\mathbf{A}\| = sup\{\frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|} \ : \ \mathbf{x} \in R^k, \ \mathbf{x} \neq \mathbf{x}\} = sup\{\|\mathbf{Ax}\| \ : \ \mathbf{x} \in R^k, \ \|\mathbf{x}\| = 1\}$$

Observing that $\|\mathbf{x}\|$ is a real number and that (by definition) $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ is a vector of length 1, we obtain

$$\frac{\|\mathbf{Ax} \cdot \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|}\|}{\|\mathbf{x}\|} = \|\mathbf{x}\| \cdot \frac{\|\mathbf{A}\frac{\mathbf{x}}{\|\mathbf{x}\|}\|}{\|\mathbf{x}\|} = \left\|\mathbf{A}\frac{\mathbf{x}}{\|\mathbf{x}\|}\right\|$$

Hence, the restriction over vectors of length 1 doesn't change the value of $\|\mathbf{A}\|$. Now, we want to consider the problem

$$\|\mathbf{A}\| = \max_{\mathbf{x}} \|\mathbf{Ax}\|$$

subject to $\|\mathbf{x}\| = 1$. This can be easily solved using the Lagrangian Method for constrained optimization. In other words, we can write

$$\max_{\|\mathbf{x}\|=1} [\ \|\mathbf{Ax}\| - \lambda\|\mathbf{x}\|\ ] = \max_{\|\mathbf{x}\|=1} \left[\ (\mathbf{x}^T\mathbf{A}^T\mathbf{Ax}) - \lambda\mathbf{x}\mathbf{x}^T\ \right]$$

which can be differentiated using matrix calculus. Observing that $\mathbf{A}^T\mathbf{A}$ is a symmetric matrix, we have F.O.C

$$2\mathbf{A}^T\mathbf{Ax} - 2\lambda\mathbf{x} = 0$$

and therefore

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Here, we notice that λ must be an eigenvalue of the symmetric matrix $\mathbf{A}^T\mathbf{A}$. Further, we proceed by left multipling the last equation by $\mathbf{x}^T$, obtaining

$$\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = \lambda\ \mathbf{x}^T\mathbf{x}$$

which implies

$$\|\mathbf{A}\mathbf{x}\|^2 = |\lambda|\ \|\mathbf{x}\|^2 = |\lambda|$$

Now, taking the max over the set of all eigenvectors $\mathbf{x}$ and corresponding eigenvalues λ of $\mathbf{A}^T\mathbf{A}$, we finally have

$$\|\mathbf{A}\|_{Spectral} = \sqrt{\max_i \|\mathbf{A}\mathbf{x_i}\|^2} = \sqrt{\max_i |\lambda_i|} = \sqrt{\rho(\mathbf{A}^T, \mathbf{A})}$$

Excercise 2 :

From Gelfand's Formula, we know that $\lim_{k\to\infty} \|\mathbf{A}^k\|^{1/k} = \rho(\mathbf{A})$. Since this limit exists, we also have

$$\lim_{k\to\infty} \left[\|\mathbf{A}^k\|^{1/k}\right]^k = \lim_{k\to\infty} \|\mathbf{A}^k\| = \lim_{k\to\infty} \rho(\mathbf{A})^k$$

If we assume that $\rho(\mathbf{A}) < 1$ then $\lim_{k\to\infty} \rho(\mathbf{A})^k = \lim_{k\to\infty} \|\mathbf{A}^k\| = 0$.

Now we want to show that $\sum_{k=0}^{\infty} \|\mathbf{A}^k\| < \infty$. Thus, consider the power series $\sum_{k=0}^{\infty} \|\mathbf{A}^k\| x^k$, and let $L = \limsup_{k\to\infty} \|\mathbf{A}^k\|^{1/k} = \rho(\mathbf{A}) < 1$, by assumption. From the theorem for convergence of power series, we know that $\sum_{k=0}^{\infty} \|\mathbf{A}^k\| x^k$ converges for $|x| < \frac{1}{L} = \frac{1}{\rho(\mathbf{A})}$. Since $\frac{1}{\rho(\mathbf{A})} > 1$; thus, if we set $x = 1$, we have that $\sum_{k=0}^{\infty} \|\mathbf{A}^k\| x^k = \sum_{k=0}^{\infty} \|\mathbf{A}^k\|$ converges absolutely.

We could have also shown convergence using Gelfand's Formula more explicitly. In fact, from that one we know that $\forall\varepsilon > 0$, $\exists$ N such that if

$k > N$, we have $\|\mathbf{A}^k\|^{1/k} - \rho(\mathbf{A}) < \varepsilon$. This implies that $\|\mathbf{A}^k\|^{1/k} < \varepsilon + \rho(\mathbf{A})$, and hence $\|\mathbf{A}^k\| < (\varepsilon + \rho(\mathbf{A}))^k$. Therefore, if we let $r = \varepsilon + \rho(\mathbf{A})$ and pick an ε sufficiently small, we can obtain that for $r < 1$, and $\forall\ k > N$

$$\|\mathbf{A}^k\| < r^k$$

.

Since $\sum_{k=0}^{\infty} \|\mathbf{A}^k\|$ is increasing and bounded for $k > N$, it must converge.

Excercise 3 :

Suppose $A \in \mathcal{M}(n \times n)$ and A is diagonalizable. Then we can write $A = PDP^{-1}$, where P is the usual eigenvector matrix and D is the diagonal matrix with the corresponding eigenvalues as entries. If $\rho(A) < 1$, then all the eigenvalues of the matrix A are less than 1 in absolute value. Therefore

$$\lim_{k\to\infty} D^k = 0$$

This implies that $\| \lim_{k\to\infty} A^k\| = \| \lim_{k\to\infty} PD^kP^{-1}\| = \|\mathbf{0}\| = 0 = \lim_{k\to\infty} \|A^k\|$.

Excercise 4 :

Consider the sequence $\{M_n\}$ of $(n \times n)$ matrices such that M_n is nonnegative definite for all $n \in N$ and such that $M_n \to M$. Now consider the mapping $T : (\mathcal{M}(n \times n), \|.\|) \to \mathbf{R}$, defined as $T(Q) = x^TQx$ - which is continuous on its domain.

Now assume that M is not nonnegative definite, i.e., there exist a vector x such that $T(M) = x^TMx < 0$. By definition of continuity of T, we must have that as $M_n \to M$, $\lim_{n\to\infty} T(M_n) = T(M)$, which implies that $\forall \varepsilon > 0$, $\exists\ \delta > 0$ such that when $\|M_n - M\| < \delta$, then $|T(M_n) - T(M)| =$ $= |x^TM_nx - x^TMx| < \varepsilon$. Moreover, since $T(M_n) \geq 0$ for all n, and since $T(M) < 0$, we have that $T(M_n) - T(M) > 0$, and hence $T(M_n) - T(M) < \varepsilon$

for all $\varepsilon > 0$. It follows that $T(M_n) < T(M)$, and that $T(M_n)$ is negative, which is the desired contradiction.

Therefore, M must also be nonnegative definite, and the set of nonnegative definite matrices is a closed subset of $(\mathcal{M}(n \times n), \|.\|)$.

Excercise 5 :

1. Let X^* be the unique solution to the Lyapunov equation $X = AXA^T + M$, and assume that M is symmetric. Then, if we transpose this solution we obtain $X^{*T} = AX^*A^T + M^T = AX^*A^T + \mathrm{M}^T = \left[AX^*A^T\right]^T + M^T$ $= \mathrm{AX}^{*T}A^T + M$. Since the solution is proved to be unique, we must have that $X^* = X^{*T}$.

2. Let X^* be the unique solution to the Lyapunov equation $X = AXA^T + M$, and assume that M is nonnegative definite. Then, X^* satisfies the equation

$$X^* = AXA^T + M$$

Therefore, if $z \in R^n$, we also have

$$z^T X^* z = z^T AX^*A^T z + z^T Mz$$

From the last expression, we see that X^* must be nonnegative definite, because AX^*A^T is nonnegative definite for all $X \in \mathcal{M}(n \times n)$, and M is nonnegative definite by assumption.

3. Same setting as part 2), but now assume that M is positive definite. Again, we have

$$z^T X^* z = z^T AX^*A^T z + z^T Mz$$

In this expression, we know that AX^*A^T is nonnegative definite for all $X \in \mathcal{M}(n \times n)$, and M is positive definite by assumption. Therefore we see that $z^T X^* z > 0$ and, thus X^* must be positive definite.