

HW # 9

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1 Solutions

1. First, I show that the spectral norm is equivalent to finding $\sup_{\|x\|=1} \|Ax\|$. This is fairly obvious since

$$\sup_{x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|} = \sup_{x \in \mathbb{R}^n} \frac{\left\| \frac{Ax}{\|x\|} \|x\| \right\|}{\|x\|} = \sup_{x \in \mathbb{R}^n} \frac{\|x\| \left\| \frac{Ax}{\|x\|} \right\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|.$$

From now on, any vector x will be assumed to have norm one. Next, I maximize $\|Ax\|^2$ by the Lagrangian method. $\mathcal{L} = x' A' A x + \lambda(1 - x' x)$. Using basic matrix calculus, I differentiate with respect to x . Because $A' A$ is symmetric, the derivative of $x' A' A x$ with respect to x will be $2A' A x$. Then we obtain $2A' A x = 2\lambda x$. It is clear the x is an eigenvector of $A' A$ and λ is the corresponding eigenvalue. But then $\|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, A' A x \rangle = \langle x, \lambda x \rangle = |\lambda|$. The global maximum is achieved with the eigenvector corresponding to the eigenvalue whose absolute value is the spectral radius. Thus $\|Ax\| = \sqrt{\rho(A' A)}$ \square

2. Consider the power series $\sum_{k=0}^{\infty} \|A\| x^k$. By Gelfand's formula, $\lim_{k \rightarrow \infty} \|A^k\|^{\frac{1}{k}} = \limsup_{k \rightarrow \infty} \|A^k\|^{\frac{1}{k}} = \rho(A) < 1$. From the convergence theorems for power series, the power series is absolutely convergence if $|x| < \frac{1}{\rho(A)}$. Since $\frac{1}{\rho(A)} > 1$, the power series converges when $x = 1$, which is $\sum_{k=0}^{\infty} \|A\|$. Since the series is summable, it is also Abel summable, so $\lim_{x \rightarrow 1^-} \sum_{k=0}^{\infty} \|A\| x^k = \sum_{k=0}^{\infty} \|A\|$. We could use the relationship to find C and $r \in (0, 1)$ such that $\|A^k\| \leq C r^k \forall k$, but the details are fuzzy \square .

3. Since A is a diagonalizable square matrix, $A = P D P^{-1}$. Where D is a diagonal matrix of the eigenvalues of A . Then $|\lambda_i| < |\rho(A)| < 1$. We use the famous result that $A^k = P D^k P^{-1}$. But then D^k is a diagonal matrix of all the eigenvalues raised to the power k . Then since $\lim_{k \rightarrow \infty} |\lambda_i^k| = \lim_{k \rightarrow \infty} |\rho(A)^k| = 0$, we conclude that $\lim_{k \rightarrow \infty} A^k = \lim_{k \rightarrow \infty} P D P^{-1} = \lim_{k \rightarrow \infty} P D^k P^{-1} = 0 \square$.

4. Suppose $\{M_n\}_{n \in \mathbb{N}}$ is a sequence of square non-negative definite matrices that converges to some matrix M which is not. Then there exists some non-zero vector x such that $x'Mx < 0$. Consider the sequence $\{x'M_nx\}_{n \in \mathbb{N}}$. Then it is clear that this sequence converges to $x'Mx$, which means that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $n \geq N$ implies $|x'M_nx - x'Mx| < \epsilon$. Suppose $x'Mx = -c$, where $c \in \mathbb{R}^+$. I claim there must be some N such that for $n \geq N$, $x'M_nx < 0$. Suppose this were not true. Then $\forall n$ we would have $|x'M_nx - x'Mx| = |x'M_nx + c| \geq |0 + c| = c$. This not possible because then the limit definition would not hold if we pick $\epsilon = c/2$. On the other hand, since M_n is positive definite for each n , $x'M_nx \geq 0 \forall n$. Thus we have obtained a contradiction and so M must be non-negative definite also, allowing us to conclude that the set of non-negative definite matrices in $\mathcal{M}(n \times n)$ is closed \square .

5. a) We know that $X^* = AX^*A' + M$ and $*$ is unique. Suppose M is symmetric. Then $X^{*'} = (AX^*A' + M)' = A(X^*)'A' + M' = A(X^*)'A' + M$. Then $(X^*)'$ is a solution to the Lyapunov equation, but since the solution is unique $X^* = (X^*)'$ and so X^* is symmetric \square

b) Suppose M is non-negative definite. Take non-zero vector z . Then $z'X^*z = z'(AX^*A' + M)z = z'AX^*A'z + z'Mz$. Suppose we repeatedly substitute the expression for X^* . We then obtain $z'X^*z = \lim_{k \rightarrow \infty} z'A^kX^*A^{k'} + \sum_{k=1}^{\infty} z'A^kMA^{k'}z + z'Mz$.

Since the absolute value of the maximum eigenvalue of A is strictly less than 1, A^k goes to zero in the power. Let $y_k = A^{k'}z$. Each y_k is a nonzero vector and M is non-negative definite so, $\sum_{k=1}^{\infty} y_k'My_k + z'Mz \geq 0$. Thus X^* is non-negative definite. \square

c) The argument is the same as part (b) except the last step gives strict inequality since M is positive definite.