Lecture 6: The Kalman filter

STAT 545: Introduction to computational statistics

Vinayak Rao

Department of Statistics, Purdue University

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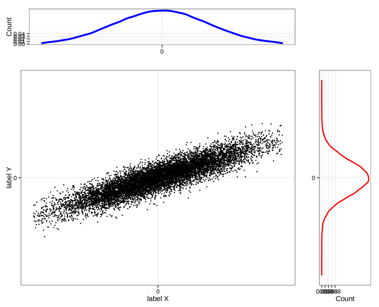
Marginalization:

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix} \right), \quad X \sim ?$$



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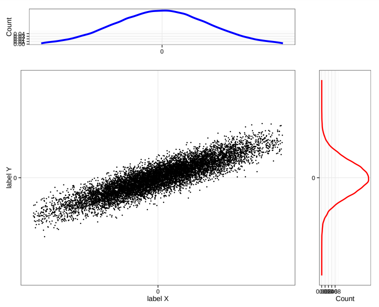

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$$X \sim \mathcal{N}(\mu_X, \Sigma_{XX})$$

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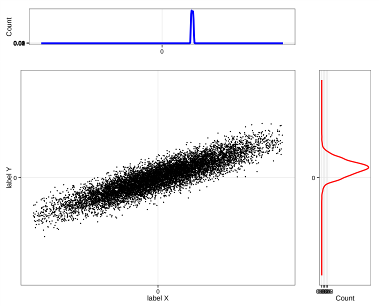

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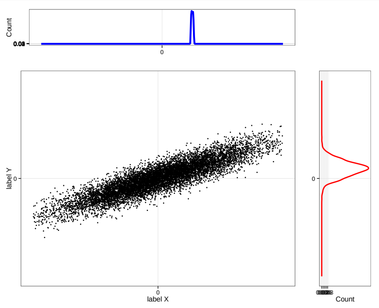

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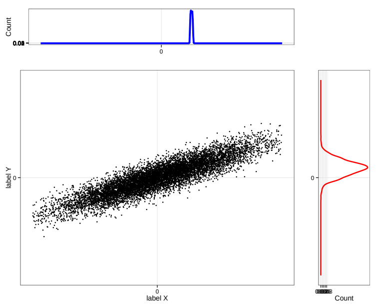

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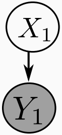
Cost?

The Gaussian distribution, conjugacy and Bayes' rule

$$X_1 \sim \mathcal{N}(\mu, \Sigma)$$



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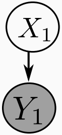

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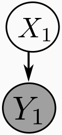

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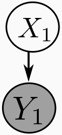

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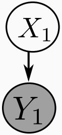

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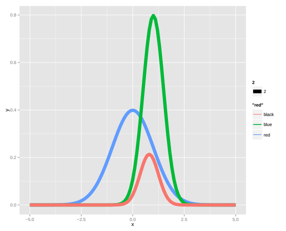
$X|Y$ is Gaussian: what is its mean and covariance?

Product of Gaussian densities:

Product of Gaussian densities is Gaussian a Gaussian density (upto a multiplication constant)



Intuition: sum of two quadratic functions is a quadratic

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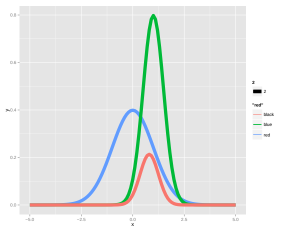


Intuition: sum of two quadratic functions is a quadratic

Aside: need only specify prob. distrib. to a constant

- $p(x)$ and $C \cdot p(x)$ represents the same, if C is independent of x
- Probabilities must integrate to 1

Markov chains

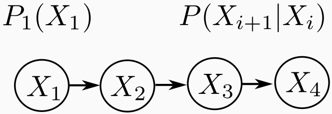
$$P_1(X_1) \qquad\qquad P_i(X_{i+1}|X_i)$$

$$X_1 \to X_2 \to X_3 \to X_4$$

A sequence of random variables such that

$$P(X_{i+1}|X_i, X_{i-1}, \cdots, X_1) = P(X_{i+1}|X_i)$$

We'll stick to homogeneous chains:



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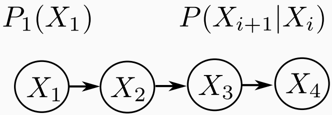

In fact, with $X_i \in \Re^D$, we will consider:

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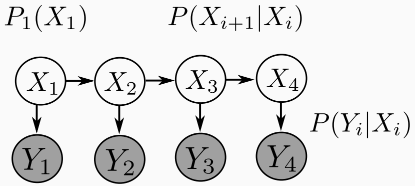
If our chain has T steps, a TD-dimensional Gaussian!
In the figure, $T = 4$. In practice: thousands to millions.

We don't observe the chain directly:



$$Y_i = BX_i + \zeta_i, \quad \zeta \sim \mathcal{N}(0, \Sigma_z), \quad Y_i \in \Re^d$$

A hidden Markov model

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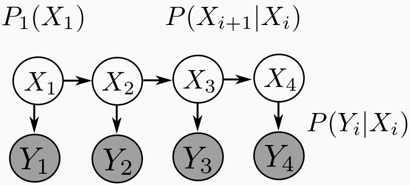

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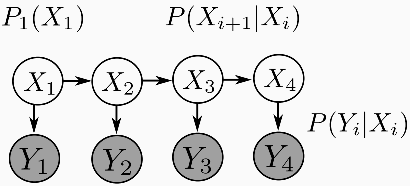

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We 'just' have to look at conditionals?

[board]