

# A Quick Guide for the QZ Package

Wei-Chen Chen

Computer Science and Mathematics Division,  
Oak Ridge National Laboratory,  
Oak Ridge, TN, USA

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**Warning:** This document is written to explain the main functions of **QZ** package (Chen 2013), version 0.1-3. Every effort will be made to ensure future versions are consistent with these instructions, but features in later versions may not be explained in this document.

## 1. Introduction

This article is to explain the **QZ** package (Chen 2013), and is organized as the following. Section 2 introduces briefly background of generalized eigenvalues problem and QZ decomposition. Section 3 lists the main functions and detail Fortran functions of LAPACK library (Anderson *et al.* 1999).

## 2. Methods

Some details can be found on wikipedia website at

[http://en.wikipedia.org/wiki/Eigendecomposition\\_of\\_a\\_matrix](http://en.wikipedia.org/wiki/Eigendecomposition_of_a_matrix)

for generalized eigenvalues, and at

[http://en.wikipedia.org/wiki/Schur\\_decomposition](http://en.wikipedia.org/wiki/Schur_decomposition)

about QZ decomposition or generalized Schur form. The LAPACK (Anderson *et al.* 1999) also provides functions to solve these problems.

### 2.1. Generalized Eigenvalues for Pair Matrices

Suppose  $\mathbf{A}$  and  $\mathbf{B}$  are two  $N \times N$  non-symmetric matrices which can be both in real or in complex. The goal is to find right generalized eigen vectors  $\mathbf{v}$  such that  $\mathbf{A}\mathbf{v} = \lambda\mathbf{B}\mathbf{v}$ , or left generalized eigen vectors  $\mathbf{u}$  such that  $\mathbf{u}^H\mathbf{A} = \lambda\mathbf{u}^H\mathbf{B}$  where  $\mathbf{u}^H$  is the conjugate-transpose of  $\mathbf{u}$ . Also,  $\lambda$  is called generalized eigenvalues of  $\mathbf{A}$  and  $\mathbf{B}$  which obeys  $\det(\mathbf{A} - \lambda\mathbf{B}) = 0$ . Note that  $\lambda$ ,  $\mathbf{u}$ , and  $\mathbf{v}$  may be complex even  $\mathbf{A}$  and  $\mathbf{B}$  are in real.

Suppose  $\mathbf{B}$  is an identity matrix  $\mathbf{I}$ , then the problem reduces to traditional eigenvalue problem. i.e. This is a special case.

### 2.2. QZ Decomposition for Pair Matrices

Suppose  $\mathbf{A}$  and  $\mathbf{B}$  are two  $N \times N$  non-symmetric matrices which can be both in real or in complex. The QZ decomposition factorizes both matrices as

- $\mathbf{A} = \mathbf{Q}\mathbf{S}\mathbf{Z}^\top$  and  $\mathbf{B} = \mathbf{Q}\mathbf{T}\mathbf{Z}^\top$  if  $\mathbf{A}$  and  $\mathbf{B}$  are real, or
- $\mathbf{A} = \mathbf{Q}\mathbf{S}\mathbf{Z}^H$  and  $\mathbf{B} = \mathbf{Q}\mathbf{T}\mathbf{Z}^H$  if  $\mathbf{A}$  and  $\mathbf{B}$  are complex

where  $\mathbf{Q}$  and  $\mathbf{Z}$  are unitary and  $\mathbf{S}$  and  $\mathbf{T}$  are upper triangular. The unitary means  $\mathbf{X}\mathbf{X}^H = \mathbf{I}$  if  $\mathbf{X}$  is complex or  $\mathbf{X}\mathbf{X}^\top = \mathbf{I}$  if  $\mathbf{X}$  is real where  $\mathbf{I}$  is the identity matrix.

The QZ decomposition is also called generalized Schur decomposition where  $\mathbf{S}$  and  $\mathbf{T}$  are the Schur form of  $\mathbf{A}$  and  $\mathbf{B}$ . The generalized eigenvalues  $\lambda$  that solve the generalized eigenvalue problem  $\mathbf{A}\mathbf{x} = \lambda\mathbf{B}\mathbf{x}$  where  $\mathbf{x}$  is an unknown nonzero vector and  $\lambda_i = \mathbf{S}_{ii}/\mathbf{T}_{ii}$ .

Suppose  $B$  is an identity matrix  $I$ , then the problem reduces to find  $Q$  such that  $A = QSQ^{-1}$  for real  $A$  or  $A = QSQ^H$  for complex  $A$ . i.e. This is a special case.

### 3. Implementation

Two main functions are `geigen()` for generalized eigenvalues, and `qz()` for QZ decomposition with reordering capability. Both functions are able to deal a single matrix  $A$  or a paired matrices  $(A, B)$  in both complex and real systems. Both functions are wrapper functions for several lower level R functions `qz.*()` which are also wrapper functions via `.Call()` for C and Fortran functions to LAPACK library version 3.4.2.

LAPACK library is incorporated in **QZ** including complex\*16 and double precision for complex and real systems respectively. **QZ** has functions of LAPACK and BLAS (Blackford *et al.* 2002) independently to the R's LAPACK and BLAS libraries since some functions are not available. Table 1 provides a detail lists for the `qz.*()` functions.

Table 1: **QZ** functions

Function	Wrapper	Main Input	System	Purpose
<code>geigen()</code>	<code>qz.zgeev</code>	$A$	Complex	Generalized eigenvalues
	<code>qz.dgeev</code>	$A$	Real	
<code>qz()</code>	<code>qz.zgees</code>	$A$	Complex	QZ decomposition
	<code>qz.dgees</code>	$A$	Real	
	<code>qz.ztrsen</code>	$T, Q$	Complex	Reordering
	<code>qz.dtrsen</code>	$T, Q$	Real	
<code>geigen()</code>	<code>qz.zggev</code>	$(A, B)$	Complex	Generalized eigenvalues
	<code>qz.dggev</code>	$(A, B)$	Real	
<code>qz()</code>	<code>qz.zgges</code>	$(A, B)$	Complex	QZ decomposition
	<code>qz.dgges</code>	$(A, B)$	Real	
	<code>qz.ztgssn</code>	$(S, T), Q, Z$	Complex	Reordering
	<code>qz.dtgssn</code>	$(S, T), Q, Z$	Real	

An extral MATLAB-like function `ordqz()` is also available to reordering generalized eigenvalues and QZ decomposition results. The function which is the combinations of `qz()` and `qz.ztrsen()/qz.dtrsen()` for specified ordering keywords in Table 2. Note that `select`

Table 2: The `ordez()` keyword for reording.

keyword	Purpose
<code>lhp</code>	Left-half ( $\text{real}(E) < 0$ )
<code>rhp</code>	Right-half ( $\text{real}(E) > 0$ )
<code>udi</code>	Interior of unit disk ( $\text{abs}(E) < 1$ )
<code>udo</code>	Exterior of unit disk ( $\text{abs}(E) > 1$ )
<code>ref</code>	Real eigenvalues first (top-left conner)
<code>cef</code>	Complex eigenvalues first (top-left conner)

argument of `qz()` allows users to specify any order to group and reorder the decompositions.

## 4. Data Example

There are four demos for the **QZ** package which are listed in Table 3

Table 3: The demos of **QZ** package.

demo	Purpose
ex1_geigen	<code>geigen()</code> for double/complex single/paired matrices
ex2_qz	<code>qz()</code> for double/complex single/paired matrices
ex3_ordqz	<code>ordqz()</code> and arbitrary reordering
ex4_fda_geigen	generalized eigen analysis of <b>fda</b> package (Ramsay <i>et al.</i> 2013)

There are also several datasets for **QZ** package to verify results which are listed in Table 4.

Table 4: The datasets of **QZ** package.

data	Source
exAB1	<a href="http://www.nag.com/lapack-ex/node124.html">http://www.nag.com/lapack-ex/node124.html</a>
exAB2	<a href="http://www.nag.com/lapack-ex/node119.html">http://www.nag.com/lapack-ex/node119.html</a>
exAB3	<a href="http://www.nag.com/numeric/fl/nagdoc_fl123/xhtml/F08/f08yuf.xml">http://www.nag.com/numeric/fl/nagdoc_fl123/xhtml/F08/f08yuf.xml</a>
exAB4	<a href="http://www.nag.com/numeric/fl/nagdoc_fl123/xhtml/F08/f08yuf.xml">http://www.nag.com/numeric/fl/nagdoc_fl123/xhtml/F08/f08yuf.xml</a>
exA1	<a href="http://www.nag.com/lapack-ex/node94.html">http://www.nag.com/lapack-ex/node94.html</a>
exA2	<a href="http://www.nag.com/lapack-ex/node89.html">http://www.nag.com/lapack-ex/node89.html</a>
exA3	<a href="http://www.nag.com/numeric/fl/nagdoc_fl123/xhtml/F08/f08quf.xml">http://www.nag.com/numeric/fl/nagdoc_fl123/xhtml/F08/f08quf.xml</a>
exA4	<a href="http://www.nag.com/numeric/fl/nagdoc_fl122/xhtml/F08/f08quf.xml">http://www.nag.com/numeric/fl/nagdoc_fl122/xhtml/F08/f08quf.xml</a>

These demos can be obtained in R by the following.

### QZ demo ex1\_geigen

```
> demo(ex1_geigen, 'QZ')

      demo(ex1_geigen)
      ---- ~~~~~

Type  <Return>   to start :

> library(QZ, quiet = TRUE)

> ### http://www.nag.com/lapack-ex/node122.html
> (ret <- geigen(exAB1$A, exAB1$B))
ALPHA:
[1] 19.03-57.10i 11.88-29.70i 10.96- 3.65i 21.87-27.34i
BETA:
```

```

[1] 6.344+0i 5.941+0i 3.654+0i 5.468+0i

U:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.0358-0.1155i 0.0725-0.3001i 0.1650-0.0068i 0.01727-0.02542i
[2,] 0.2152+0.2357i -0.2139+0.7641i 0.0999-0.8330i -0.01045-0.09180i
[3,] -0.2425+0.4271i 0.7520-0.2317i -0.9374-0.0626i -0.17518-0.82482i
[4,] 0.5658-0.4342i -0.1782-0.8218i -0.0529+0.1385i -0.84361-0.01589i

V:
      [,1]      [,2]      [,3]
      [,4]
[1,] -0.8238-0.1762i 0.63974+0.360259i 0.9775+0.0225i
      -0.90623+0.093766i
[2,] -0.1530+0.0707i 0.00416-0.000547i 0.1591-0.1137i
      -0.00743+0.006875i
[3,] -0.0707-0.1530i 0.04021+0.022645i 0.1209-0.1537i
      0.03021-0.003126i
[4,] 0.1530-0.0707i -0.02264+0.040212i 0.1537+0.1209i
      -0.01459-0.140970i

> ### http://www.nag.com/lapack-ex/node117.html
> (ret <- geigen(exAB2$A, exAB2$B))
ALPHA:
[1] 3.801+0.000i 3.030+4.040i 1.563-2.084i 4.000+0.000i

BETA:
[1] 1.900 1.010 0.521 1.000

U:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.53333+0i 0.2171-0.1284i 0.2171+0.1284i -7.276e-17+0i
[2,] -0.06667+0i 0.1744-0.1851i 0.1744+0.1851i -1.000e+00+0i
[3,] -1.00000+0i -0.7928+0.2072i -0.7928-0.2072i 1.000e+00+0i
[4,] 0.60000+0i 0.3912+0.0911i 0.3912-0.0911i -3.695e-16+0i

V:
      [,1]      [,2]      [,3]      [,4]
[1,] 1.000000+0i -0.4398-0.5602i -0.4398+0.5602i -1.000000+0i
[2,] 0.005714+0i -0.0880-0.1120i -0.0880+0.1120i -0.011111+0i
[3,] 0.062857+0i -0.1424+0.0031i -0.1424-0.0031i 0.033333+0i
[4,] 0.062857+0i -0.1424+0.0031i -0.1424-0.0031i -0.15556+0i

> ### http://www.nag.com/lapack-ex/node92.html
> (ret <- geigen(exA1$A))
W:
[1] -6.000-7.000i -5.000+2.006i 7.998-0.996i 3.002-4.000i

U:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.8357+0.0000i -0.3510+0.1013i -0.1689+0.2595i 0.1099-0.2007i
[2,] -0.0794+0.3372i -0.4035+0.4540i 0.6762+0.0000i 0.0336+0.2312i
[3,] 0.0917+0.3097i 0.6239+0.0000i 0.3032+0.5642i 0.0944-0.3947i

```

```

[4,] 0.0456-0.2741i -0.0816-0.3190i 0.1328+0.1376i 0.8534+0.0000i
V:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.8457+0.0000i -0.3865+0.1732i -0.1730+0.2669i -0.0356-0.1782i
[2,] -0.0177+0.3036i -0.3539+0.4529i 0.6924+0.0000i 0.1264+0.2666i
[3,] 0.0875+0.3115i 0.6124+0.0000i 0.3324+0.4960i 0.0129-0.2966i
[4,] -0.0561-0.2906i -0.0859-0.3284i 0.2504-0.0147i 0.8898+0.0000i

> ### http://www.nag.com/lapack-ex/node87.html
> (ret <- geigen(exA2$A))
W:
[1] 0.7995+0.0000i -0.0994+0.4008i -0.0994-0.4008i -0.1007+0.0000i
U:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.62447+0i 0.5330+0.0000i 0.5330+0.0000i 0.6641+0i
[2,] -0.59949+0i -0.2666+0.4041i -0.2666-0.4041i -0.1068+0i
[3,] 0.49992+0i 0.3455+0.3153i 0.3455-0.3153i 0.7293+0i
[4,] -0.02709+0i -0.2541-0.4451i -0.2541+0.4451i 0.1249+0i
V:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.65509+0i -0.1933+0.2546i -0.1933-0.2546i 0.1253+0i
[2,] -0.52363+0i 0.2519-0.5224i 0.2519+0.5224i 0.3320+0i
[3,] 0.53622+0i 0.0972-0.3084i 0.0972+0.3084i 0.5938+0i
[4,] -0.09561+0i 0.6760+0.0000i 0.6760+0.0000i 0.7221+0i

```

## QZ demo ex2\_qz

```

> demo(ex2_qz, 'QZ')

      demo(ex2_qz)
      ---- ~~~~~~

Type <Return> to start :

> library(QZ, quiet = TRUE)

> ### http://www.nag.com/lapack-ex/node124.html
> (ret <- qz(exAB1$A, exAB1$B))
ALPHA:
[1] 19.03-57.10i 11.88-29.70i 10.96- 3.65i 21.87-27.34i

BETA:
[1] 6.344+0i 5.941+0i 3.654+0i 5.468+0i
S:
      [,1]      [,2]      [,3]      [,4]
[1,] 19.03-57.1i 53.59-89.82i -81.31-63.23i 106.66-44.79i
[2,] 0.00+ 0.0i 11.88-29.70i 3.56+27.63i -0.67-16.42i
[3,] 0.00+ 0.0i 0.00+ 0.00i 10.96- 3.65i -25.02- 8.20i
[4,] 0.00+ 0.0i 0.00+ 0.00i 0.00+ 0.00i 21.87-27.34i

```

```

T:
      [,1]      [,2]      [,3]      [,4]
[1,] 6.344+0i 3.399+0.712i -0.515-2.382i 6.582+2.430i
[2,] 0.000+0i 5.941+0.000i -2.448-0.343i 5.739-0.702i
[3,] 0.000+0i 0.000+0.000i 3.654+0.000i -1.410-3.933i
[4,] 0.000+0i 0.000+0.000i 0.000+0.000i 5.468+0.000i

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.3347+0.7387i 0.2872-0.4789i 0.1725+0.0093i 0.01443-0.02124i
[2,] -0.1277+0.2493i -0.0282+0.4999i 0.1541-0.8008i -0.00873-0.07670i
[3,] -0.3557+0.0396i -0.4615-0.0822i -0.3939+0.0258i -0.14637-0.68917i
[4,] -0.0126-0.3682i 0.1508-0.4417i 0.1517-0.3555i -0.70486-0.01328i

Z:
      [,1]      [,2]      [,3]
[1,] -0.9240-0.1977i 0.2460+0.2090i -0.00543+0.05421i
      0.000e+00+0.000e+00i
[2,] -0.1716+0.0793i -0.5943+0.0905i 0.74673-0.21271i
      -1.092e-16-3.690e-16i
[3,] -0.0793-0.1716i 0.0943-0.5082i 0.01020-0.44383i
      7.034e-01-7.277e-02i
[4,] 0.1716-0.0793i 0.5082+0.0943i 0.44383+0.01020i
      -7.277e-02-7.034e-01i

> ### http://www.nag.com/lapack-ex/node119.html
> (ret <- qz(exAB2$A, exAB2$B))
ALPHA:
[1] 3.801+0.000i 3.030+4.040i 1.563-2.084i 4.000+0.000i

BETA:
[1] 1.900 1.010 0.521 1.000

S:
      [,1]      [,2]      [,3]      [,4]
[1,] 3.801 31.326 -61.485 -66.836
[2,] 0.000 3.351 7.074 6.692
[3,] 0.000 -1.192 1.410 4.379
[4,] 0.000 0.000 0.000 4.000

T:
      [,1]      [,2]      [,3]      [,4]
[1,] 1.9 -1.078 -5.6252 -9.987
[2,] 0.0 1.176 0.0000 1.751
[3,] 0.0 0.000 0.4474 1.090
[4,] 0.0 0.000 0.0000 1.000

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.4642 0.81159 0.3547 -5.145e-17
[2,] 0.5002 -0.06975 -0.4950 -7.071e-01
[3,] 0.5002 -0.06975 -0.4950 7.071e-01

```



```

[4,] 0.5331 -0.57585 0.6198 -2.613e-16

Z:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.996056 0.08183 -0.03428 0.000e+00
[2,] 0.005692 -0.44454 -0.89574 5.145e-17
[3,] 0.062609 -0.63075 0.31343 7.071e-01
[4,] 0.062609 -0.63075 0.31343 -7.071e-01

> ### http://www.nag.com/lapack-ex/node94.html
> (ret <- qz(exA1$A))
W:
[1] -6.000-7.000i -5.000+2.006i 7.998-0.996i 3.002-4.000i

T:
      [,1]      [,2]      [,3]      [,4]
[1,] -6-7i 0.1618+0.4896i 0.4761-0.1946i 0.8633-0.3014i
[2,] 0+0i -5.0000+2.0060i 0.6907+0.2115i 0.2281+0.1328i
[3,] 0+0i 0.0000+0.0000i 7.9982-0.9964i -1.0155+0.3626i
[4,] 0+0i 0.0000+0.0000i 0.0000+0.0000i 3.0023-3.9998i

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.5312-0.6581i -0.0799-0.3774i -0.0935-0.2736i 0.1869-0.1321i
[2,] 0.2474-0.1769i -0.4108-0.4021i -0.4015+0.6010i -0.0713+0.2225i
[3,] 0.1874-0.2637i -0.0937+0.6241i -0.5752-0.0389i 0.2581-0.3132i
[4,] -0.1909+0.2262i 0.3457-0.0537i -0.1537+0.1951i 0.7668+0.3747i

> ### http://www.nag.com/lapack-ex/node89.html
> (ret <- qz(exA2$A))
W:
[1] 0.7995+0.0000i -0.0994+0.4008i -0.0994-0.4008i -0.1007+0.0000i

T:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.7995 0.006037 -0.11445 -0.03357
[2,] 0.0000 -0.099412 -0.64834 -0.20258
[3,] 0.0000 0.247764 -0.09941 -0.34742
[4,] 0.0000 0.000000 0.00000 -0.10066

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.65509 -0.3450 -0.1037 0.6641
[2,] -0.52363 0.6141 0.5807 -0.1068
[3,] 0.53622 0.2935 0.3073 0.7293
[4,] -0.09561 0.6463 -0.7467 0.1249

```

## QZ demo ex3\_ordqz

```
> demo(ex3_ordqz, 'QZ')
```

```
demo(ex3_ordqz)
```

```

----- ~~~~~~

Type <Return> to start :

> # Reordering eigenvalues
> library(QZ, quiet = TRUE)

> select <- c(TRUE, FALSE, FALSE, TRUE)

> (ret <- qz(exAB1$A, exAB1$B, select = select))
ALPHA:
[1] 19.033-57.099i 17.897-22.371i 18.175-45.437i 8.757- 2.919i

BETA:
[1] 6.344+0i 4.474+0i 9.087+0i 2.919+0i

S:
      [,1]      [,2]      [,3]      [,4]
[1,] 19.03-57.1i 0.07-93.12i -128.250- 6.366i -98.392+ 9.509i
[2,] 0.00+ 0.0i 17.90-22.37i 0.581- 4.575i 6.972+17.755i
[3,] 0.00+ 0.0i 0.00+ 0.00i 18.175-45.437i -19.992- 6.063i
[4,] 0.00+ 0.0i 0.00+ 0.00i 0.000+ 0.000i 8.757- 2.919i

T:
      [,1]      [,2]      [,3]      [,4]
[1,] 6.344+0i 1.427-1.821i -4.137-6.323i -1.783-1.262i
[2,] 0.000+0i 4.474+0.000i -0.003-3.720i -2.992-0.076i
[3,] 0.000+0i 0.000+0.000i 9.087+0.000i -0.777-1.003i
[4,] 0.000+0i 0.000+0.000i 0.000+0.000i 2.919+0.000i

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.3347+0.7387i 0.0511-0.3524i -0.2997+0.3302i 0.08899-0.09359i
[2,] -0.1277+0.2493i 0.3749+0.5365i 0.2504+0.0137i -0.39709-0.52213i
[3,] -0.3557+0.0396i -0.4717+0.2407i 0.0591+0.2199i -0.56045+0.47485i
[4,] -0.0126-0.3682i 0.4020-0.0522i 0.1201+0.8198i 0.04567+0.10657i

Z:
      [,1]      [,2]      [,3]
      [,4]
[1,] -0.9240-0.1977i 0.2234+0.1906i -0.08922-0.09991i
      0.0338+0.04268i
[2,] -0.1716+0.0793i -0.5288+0.0772i 0.27684-0.00803i
      0.3880-0.67191i
[3,] -0.0793-0.1716i -0.1722-0.6151i -0.57435-0.20679i
      -0.2753-0.32832i
[4,] 0.1716-0.0793i 0.3215+0.3418i -0.58658+0.43433i
      0.3229-0.32726i

> ### http://www.nag.com/lapack-ex/node119.html
> select <- c(TRUE, FALSE, FALSE, TRUE)

> (ret <- qz(exAB2$A, exAB2$B, select = select))

```

```

ALPHA:
[1] 3.801+0.000i 9.203+0.000i 0.857+1.143i 0.857-1.143i

BETA:
[1] 1.9005 2.3008 0.2857 0.2857

S:
      [,1]      [,2]      [,3]      [,4]
[1,] 3.801 -69.451 50.3135 -43.288
[2,] 0.000  9.203 -0.2001  5.988
[3,] 0.000  0.000  1.4279  4.445
[4,] 0.000  0.000  0.9019 -1.196

T:
      [,1]      [,2]      [,3]      [,4]
[1,] 1.9 -10.228 0.8658 -5.2134
[2,] 0.0  2.301 0.7915  0.4262
[3,] 0.0  0.000 0.8101  0.0000
[4,] 0.0  0.000 0.0000 -0.2823

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.4642  0.78862  0.29148 -0.2786
[2,] 0.5002 -0.59864  0.56379 -0.2713
[3,] 0.5002  0.01541 -0.01074  0.8657
[4,] 0.5331 -0.13952 -0.77270 -0.3151

Z:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.996056 -0.00140  0.08868 -0.002602
[2,] 0.005692 -0.04037 -0.09376 -0.994760
[3,] 0.062609  0.71938 -0.69084  0.036273
[4,] 0.062609 -0.69344 -0.71140  0.095554

> (ret <- ordqz(exAB2$A, exAB2$B, keyword = "ref"))
ALPHA:
[1] 3.801+0.000i 9.203+0.000i 0.857+1.143i 0.857-1.143i

BETA:
[1] 1.9005 2.3008 0.2857 0.2857

S:
      [,1]      [,2]      [,3]      [,4]
[1,] 3.801 -69.451 50.3135 -43.288
[2,] 0.000  9.203 -0.2001  5.988
[3,] 0.000  0.000  1.4279  4.445
[4,] 0.000  0.000  0.9019 -1.196

T:
      [,1]      [,2]      [,3]      [,4]
[1,] 1.9 -10.228 0.8658 -5.2134
[2,] 0.0  2.301 0.7915  0.4262
[3,] 0.0  0.000 0.8101  0.0000

```

```

[4,] 0.0 0.000 0.0000 -0.2823

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.4642 0.78862 0.29148 -0.2786
[2,] 0.5002 -0.59864 0.56379 -0.2713
[3,] 0.5002 0.01541 -0.01074 0.8657
[4,] 0.5331 -0.13952 -0.77270 -0.3151

Z:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.996056 -0.00140 0.08868 -0.002602
[2,] 0.005692 -0.04037 -0.09376 -0.994760
[3,] 0.062609 0.71938 -0.69084 0.036273
[4,] 0.062609 -0.69344 -0.71140 0.095554

> (ret <- ordqz(exAB2$A, exAB2$B, keyword = "cef"))
ALPHA:
[1] 0.8571+1.143i 0.8571-1.143i 0.6172+0.000i 4.0000+0.000i

BETA:
[1] 0.2857 0.2857 0.3086 1.0000

S:
      [,1]      [,2]      [,3]      [,4]
[1,] -38.566 41.488 37.2809 65.427
[2,] 6.827 -5.244 -12.9545 -15.482
[3,] 0.000 0.000 0.6172 3.252
[4,] 0.000 0.000 0.0000 4.000

T:
      [,1]      [,2]      [,3]      [,4]
[1,] -3.368 0.0000 4.9228 9.696
[2,] 0.000 0.9621 -1.1839 -2.988
[3,] 0.000 0.0000 0.3086 1.027
[4,] 0.000 0.0000 0.0000 1.000

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.5521 -0.67876 0.4842 -5.145e-17
[2,] -0.5106 0.06994 -0.4842 -7.071e-01
[3,] -0.5106 0.06994 -0.4842 7.071e-01
[4,] -0.4169 0.72767 0.5447 -2.613e-16

Z:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.8775 0.43756 1.961e-01 0.000e+00
[2,] 0.1755 0.08751 -9.806e-01 5.145e-17
[3,] -0.3155 0.63281 -2.387e-15 7.071e-01
[4,] -0.3155 0.63281 -2.498e-15 -7.071e-01

> select <- c(TRUE, FALSE, FALSE, TRUE)

```

```

> (ret <- qz(exA1$A, select = select))
W:
[1] -6.000-7.000i  3.002-4.000i -5.000+2.006i  7.998-0.996i

T:
      [,1]      [,2]      [,3]      [,4]
[1,] -6-7i  0.3254-0.8854i  0.5349-0.0829i  0.0083+0.4285i
[2,]  0+0i  3.0023-3.9998i  0.1669+0.2948i -0.2477-1.0389i
[3,]  0+0i  0.0000+0.0000i -5.0000+2.0060i -0.5188-0.4792i
[4,]  0+0i  0.0000+0.0000i  0.0000+0.0000i  7.9982-0.9964i

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.5312-0.6581i -0.0184-0.2122i -0.3775+0.0311i  0.3003+0.0754i
[2,]  0.2474-0.1769i  0.2150+0.2457i -0.4610+0.3622i -0.2198-0.6395i
[3,]  0.1874-0.2637i -0.0469-0.2699i  0.6166+0.1728i  0.4350-0.4701i
[4,] -0.1909+0.2262i  0.8352-0.2747i -0.0033-0.3207i  0.0869-0.1703i

> ### http://www.nag.com/lapack-ex/node89.html
> select <- c(TRUE, FALSE, FALSE, TRUE)

> (ret <- qz(exA2$A, select = select))
W:
[1]  0.7995+0.0000i -0.1007+0.0000i -0.0994+0.4008i -0.0994-0.4008i

T:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.7995 -0.005914 -0.07508 -0.09268
[2,] 0.0000 -0.100657  0.39367  0.35692
[3,] 0.0000  0.000000 -0.09941 -0.51282
[4,] 0.0000  0.000000  0.31324 -0.09941

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.65509 -0.1210 -0.50323  0.55043
[2,] -0.52363 -0.3286  0.78570  0.02287
[3,]  0.53622 -0.5974  0.09038  0.58945
[4,] -0.09561 -0.7215 -0.34825 -0.59081

> (ret <- ordqz(exA2$A, keyword = "lhp"))
W:
[1] -0.0994+0.4008i -0.0994-0.4008i -0.1007+0.0000i  0.7995+0.0000i

T:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.09941  0.24919  0.3491  0.089393
[2,] -0.64462 -0.09941  0.2049  0.090443
[3,]  0.00000  0.00000 -0.1007  0.009467
[4,]  0.00000  0.00000  0.0000  0.799482

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.1733 -0.3607 -0.6707 -0.62447

```

```

[2,] 0.5173 0.6024 0.1005 -0.59949
[3,] 0.3629 0.3067 -0.7241 0.49992
[4,] -0.7554 0.6426 -0.1252 -0.02709

> (ret <- ordqz(exA2$A, keyword = "rhp"))
W:
[1] 0.7995+0.0000i -0.0994+0.4008i -0.0994-0.4008i -0.1007+0.0000i

T:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.7995 0.006037 -0.11445 -0.03357
[2,] 0.0000 -0.099412 -0.64834 -0.20258
[3,] 0.0000 0.247764 -0.09941 -0.34742
[4,] 0.0000 0.000000 0.00000 -0.10066

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.65509 -0.3450 -0.1037 0.6641
[2,] -0.52363 0.6141 0.5807 -0.1068
[3,] 0.53622 0.2935 0.3073 0.7293
[4,] -0.09561 0.6463 -0.7467 0.1249

> (ret <- ordqz(exA2$A, keyword = "ref"))
W:
[1] 0.7995+0.0000i -0.1007+0.0000i -0.0994+0.4008i -0.0994-0.4008i

T:
      [,1]      [,2]      [,3]      [,4]
[1,] 0.7995 -0.005914 -0.07508 -0.09268
[2,] 0.0000 -0.100657 0.39367 0.35692
[3,] 0.0000 0.000000 -0.09941 -0.51282
[4,] 0.0000 0.000000 0.31324 -0.09941

Q:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.65509 -0.1210 -0.50323 0.55043
[2,] -0.52363 -0.3286 0.78570 0.02287
[3,] 0.53622 -0.5974 0.09038 0.58945
[4,] -0.09561 -0.7215 -0.34825 -0.59081

> (ret <- ordqz(exA2$A, keyword = "cef"))
W:
[1] -0.0994+0.4008i -0.0994-0.4008i 0.7995+0.0000i -0.1007+0.0000i

T:
      [,1]      [,2]      [,3]      [,4]
[1,] -0.09941 0.24919 0.09306 -0.348147
[2,] -0.64462 -0.09941 0.09259 -0.203889
[3,] 0.00000 0.00000 0.79948 0.009467
[4,] 0.00000 0.00000 0.00000 -0.100657

Q:
      [,1]      [,2]      [,3]      [,4]

```

```
[1,] -0.1733 -0.3607 -0.6315  0.6641
[2,]  0.5173  0.6024 -0.5984 -0.1068
[3,]  0.3629  0.3067  0.4923  0.7293
[4,] -0.7554  0.6426 -0.0284  0.1249
```

## QZ demo ex4\_fda\_geigen

```
> demo(ex4_fda_geigen, 'QZ')

      demo(ex4_fda_geigen)
      ---- ~~~~~

Type  <Return>   to start :

> library(QZ, quiet = TRUE)

> ### Generate Data
> set.seed(123)

> X <- matrix(rnorm(500), nrow = 25)

> X <- t(X) %*% X

> A <- X[1:8, 9:20]

> B <- X[1:8, 1:8]

> C <- X[9:20, 9:20]

> ### Perform generalized eigenanalysis
> ret.qz <- fda.geigen(A, B, C)

> ret.fda <- fda::geigen(A, B, C)

> ### Verify
> round(abs(ret.qz$values - ret.fda$values))
[1] 0 0 0 0 0 0 0 0

> round(abs(ret.qz$Lmat - ret.fda$Lmat))
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    0    0    0    0    0    0    0    0
[2,]    0    0    0    0    0    0    0    0
[3,]    0    0    0    0    0    0    0    0
[4,]    0    0    0    0    0    0    0    0
[5,]    0    0    0    0    0    0    0    0
[6,]    0    0    0    0    0    0    0    0
[7,]    0    0    0    0    0    0    0    0
[8,]    0    0    0    0    0    0    0    0

> round(abs(ret.qz$Mmat - ret.fda$Mmat))
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    0    0    0    0    0    0    0    0
```

[2,]	0	0	0	0	0	0	0	0
[3,]	0	0	0	0	0	0	0	0
[4,]	0	0	0	0	0	0	0	0
[5,]	0	0	0	0	0	0	0	0
[6,]	0	0	0	0	0	0	0	0
[7,]	0	0	0	0	0	0	0	0
[8,]	0	0	0	0	0	0	0	0
[9,]	0	0	0	0	0	0	0	0
[10,]	0	0	0	0	0	0	0	0
[11,]	0	0	0	0	0	0	0	0
[12,]	0	0	0	0	0	0	0	0



## References

- Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J, Du Croz J, Greenbaum A, Hammarling S, McKenney A, Sorensen D (1999). *LAPACK Users' Guide*. Third edition. Society for Industrial and Applied Mathematics, Philadelphia, PA. ISBN 0-89871-447-8 (paperback).
- Blackford L, Demmel J, Dongarra J, Duff I, Hammarling S, Henry G, Heroux M, Kaufman L, Lumsdaine A, Petitet A, Pozo R, Remington K, Whaley R (2002). "An Updated Set of Basic Linear Algebra Subprograms (BLAS)." *ACM Trans. Math. Soft.*, **28**, 135–151. URL <http://www.netlib.org/blas/>.
- Chen WC (2013). "QZ: Generalized Eigenvalues and QZ Decomposition." R Package, URL <http://cran.r-project.org/package=QZ>.
- Ramsay J, Wickham H, Graves S, Hooker G (2013). "fda: Functional Data Analysis." R Package, URL <http://cran.r-project.org/package=fda>.