IB Extended Essay
Subject: Mathematics

Title:

An Exploration into Fourier Transform’s Effectiveness

Research Question:

To what extent is Fourier Transform efficient in filtering noise for sound processing purposes under real life conditions?

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Introduction

As a big fan for Soviet and Russian music, I always wanted to investigate into methods of noise filtration, as the recordings of myself playing the piano are often full of noise. I found that Fourier Transform is widely applied in signal denoising, so I want to investigate its effectiveness for that purpose. I created a Python program (Check Appendix I for the code) that converts any time-domain sound signal (.wav) to a frequency-domain plot for exploration purposes using Discrete Fourier Transform (DFT). I decided to use a pure tone to obtain preliminary insight. Below is the program-generated frequency-domain plot of the recording of a 338Hz pure tone in a noisy environment, spanning 6 seconds:

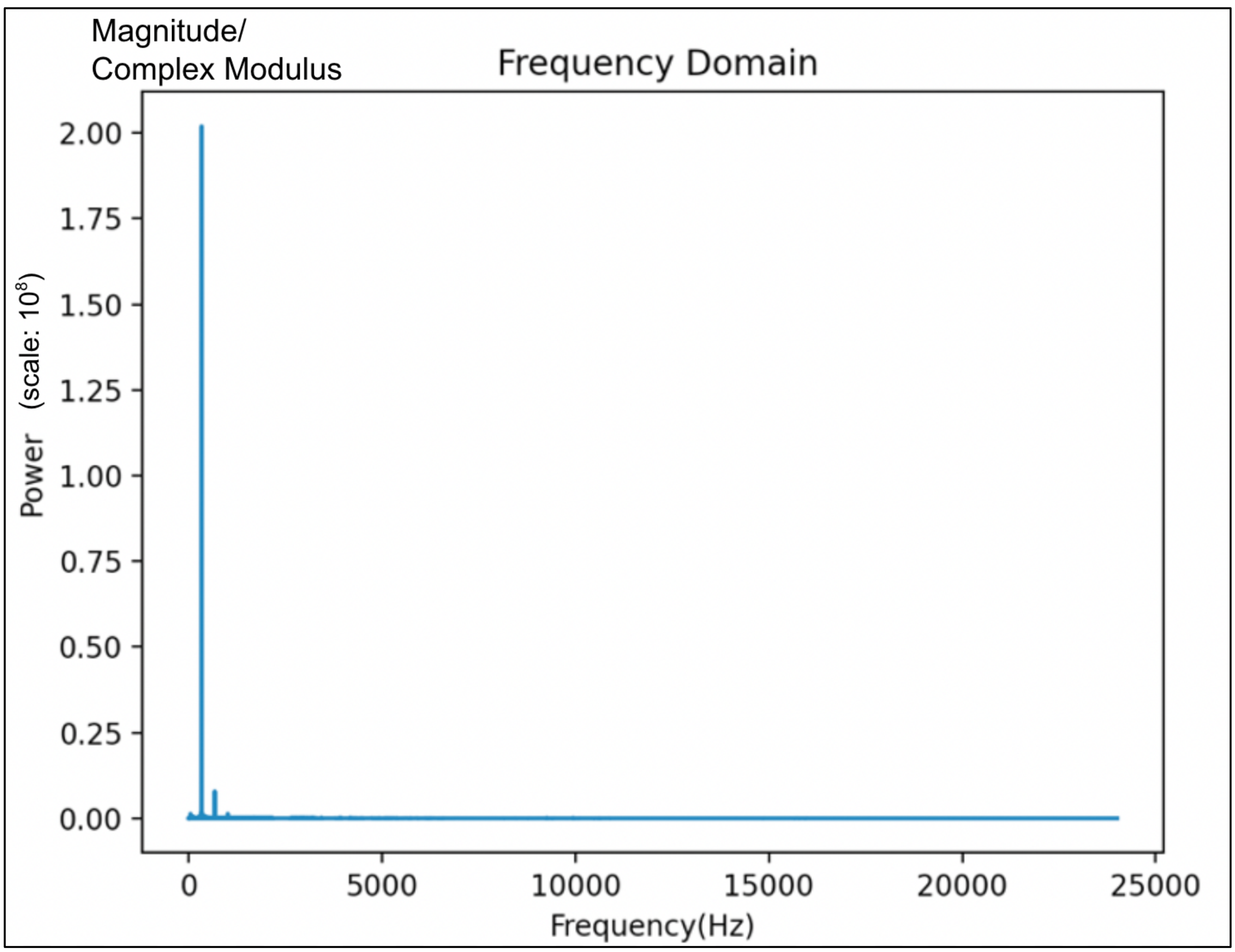


Fig. 1a

In Fig. 1a, I see a huge spike at 338Hz, along with a few smaller spikes. I became curious: Why and how are these small spikes induced? Is noise the only factor? In my Extended Essay, I will explore how real-life factors affect the efficiency of Fourier Transform-denoising, including the generation of frequency domain plots, mathematically and empirically with numerous examples. Originally, I aimed to further build on my Python Program, to allow directly denoising of a signal through filtering selected frequencies, but that was not successful as I lack programming knowledge. Nevertheless, I will strive to obtain insights from various perspectives, including with generated frequency-domain plots. I will investigate the effects of the following real-life factors on Fourier Transform's effectiveness:

1. Length of recorded sound signal (i.e. evaluated domain length)

2. Complexity of recorded sound signal

3. Sampling frequency of sound-recording device

4. Type and nature of noise

It is important to conceptually understand how Fourier Transform works, before analysing the different effects of real-life variables, hence that will be the focus of my next section.

Conceptual Understandings

In 1822, Joseph Fourier derived a method to decompose any function into a summation of different sinusoids **(Jha)**. Fourier Transform is often performed on sound signals, which have a time domain. For any time-domain function, h(t), Fourier Transform is applied onto it at a specific frequency f, $f \in \mathbb{R}$. The classical Fourier Transform is evaluated from $-\infty$ to ∞ in the evaluated domain. Since time-domain functions often start from t = 0 (no negative time), and have finite length, the Fourier Transform on h(t) will be evaluated from $a \leq t \leq b$ in my

Extended Essay, where a and b are adjustable values, and $a,$ b ∈ ℝ. "$a \leq t \leq b$" will also be referred to as the "evaluated domain" of Fourier Transform across my Extended Essay.

A crucial step of Fourier Transform is the winding of a segment of the to-be-evaluated function (as accordance to the evaluated domain) around the origin of a complex plane, at a specific evaluated frequency, before integration is performed on the wound-up curve. The winding is in **clockwise** direction due to convention. Below, I will extensively use a style of representation called "wound-up graphs" (produced with help of Tim Brzezinski's "Polar Function Graphing" applet on GeoGebra), which graphically demonstrates the above concept, and introduce the concepts of **evaluated frequency** and **evaluated domain** in the context of Fourier Transform.

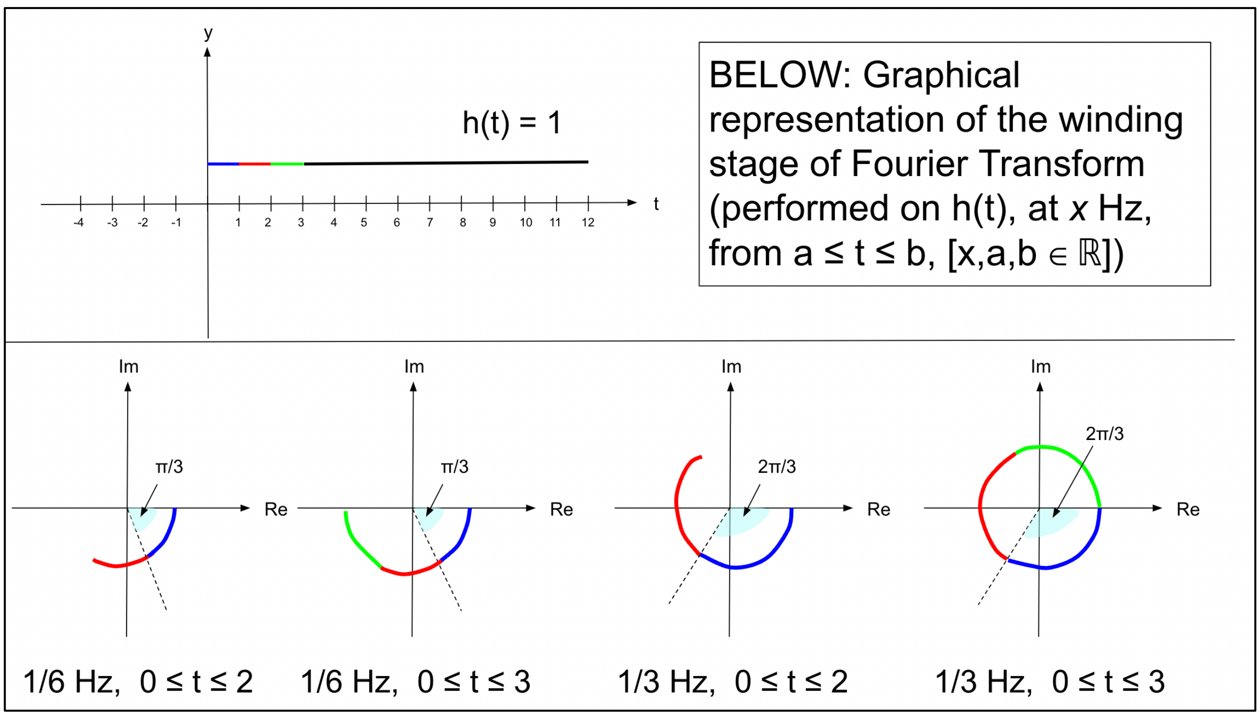


Fig. 1b

We start with the simplest example in Fig. 1b – a flat line with expression $h(t) = 1$. Note that each color-coded subsegment of h(t) spans 1 unit across time-domain. When the evaluated frequency is $\frac{1}{6}$ Hz, each colored subsegment spans through $\frac{\pi}{3}$ radians when being wound, as shown by the first two sub-diagrams on the bottom-left. And as shown by the other two sub-diagrams, each colored subsegment spans through $\frac{2\pi}{3}$ radians when being wound, when the evaluated frequency is $\frac{1}{3}$ Hz. Meanwhile, the length of evaluated domain impacts the length of segment of h(t) being wound.

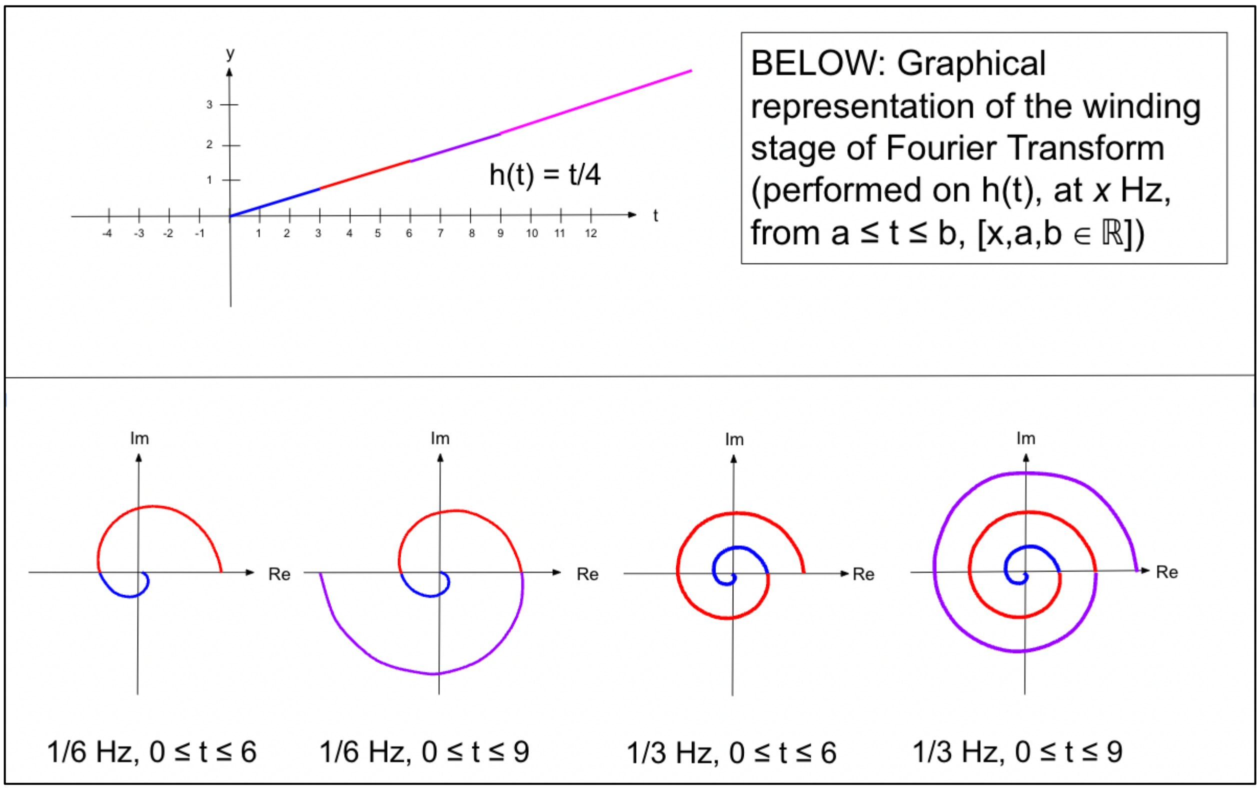


Fig. 1c

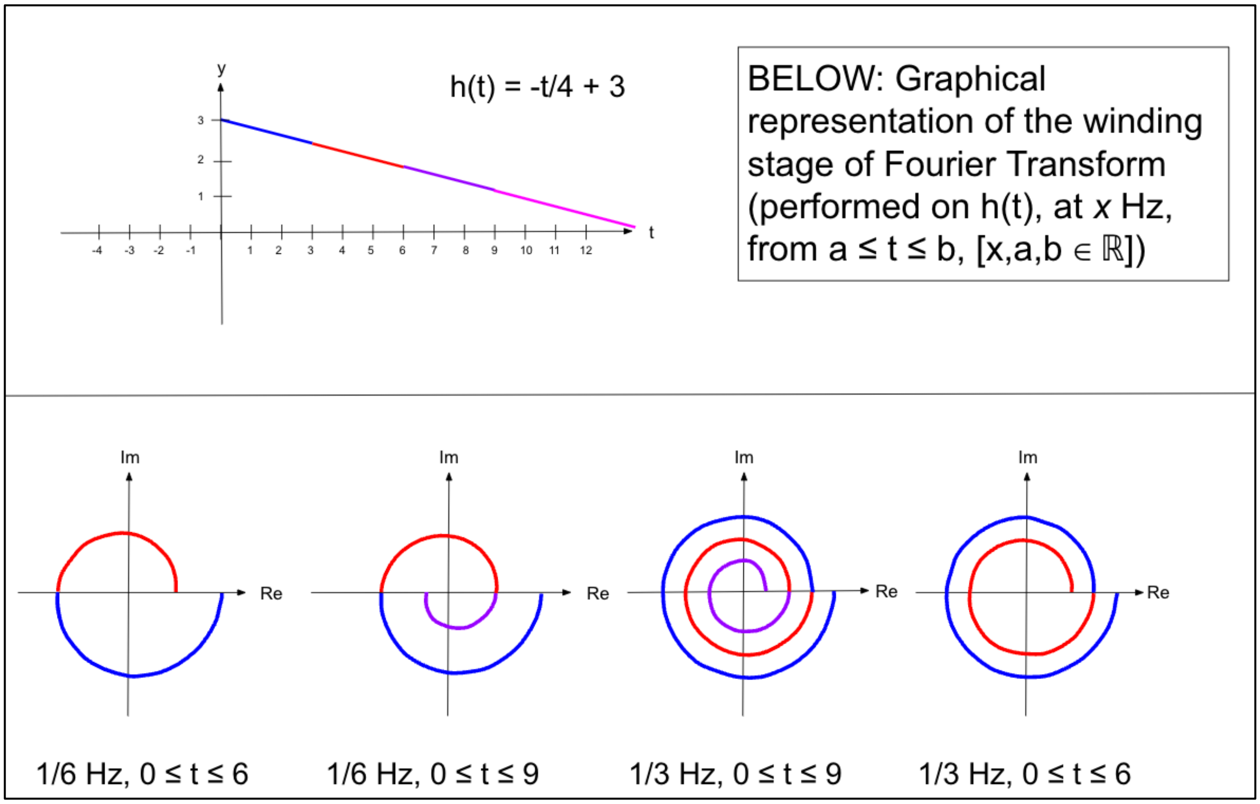


Fig. 1d

Figures 1c and 1d furthers the concepts of Fig. 1b, by showing the wound-up graphs of upward-sloping and downward-sloping linear functions, respectively. Spiral-looking shapes are produced, as the gradient is nonzero in both original functions.

With insights from the above examples, For the winding stage of Fourier Transform being performed on a time-domain signal, h(t), one can generalize that:

1. This process of "winding" continues until the entirety of h(t)'s designated segment has been wound.

2. For the evaluated frequency of Fourier Transform being at f Hz, f ∈ ℝ, each subsegment with length of $\frac{1}{f}$ units ***on the time axis*** corresponds to 1 full wind (2π radians) around the **complex plane**.

3. Furthering Point 2, any point (***c, h(c)***) on the curve of h(t), if being wound around the complex plane at frequency of f Hz, will have an argument of $\frac{-2\pi c}{f}$ radians provided that process of winding starts at t = 0. The negative sign in front is due to the winding being clockwise.

We will now move to a more complex example. Fig. 1e below demonstrates the winding of the function h(t) = $\sin^2(t)$:

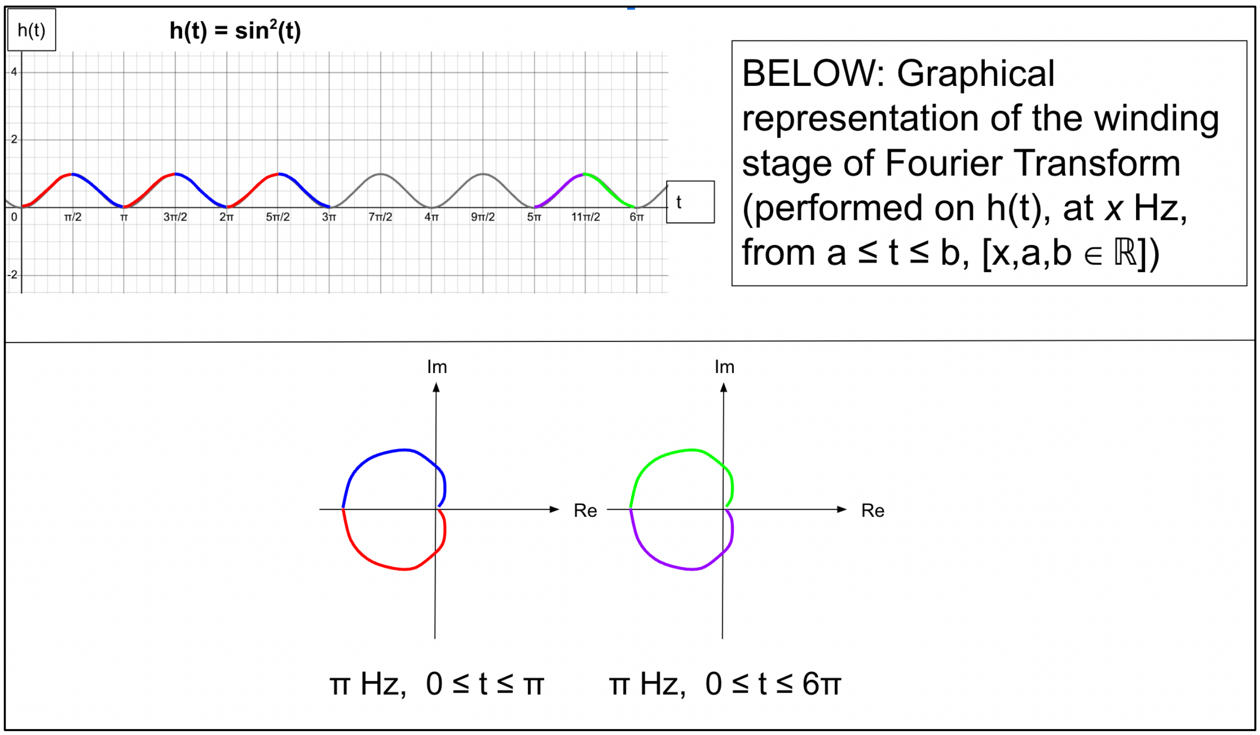


Fig. 1e

The inherent frequency of $\sin^2(t)$ is $\frac{1}{\pi}$ Hz. As every cycle of $\sin^2(t)$ corresponds to 1 full wind around the complex plane, the shape is independent to the length of evaluated domain, since any two cycles' shapes are identical, **leading to perfect overlapping as shown by Fig. 1e** (green & purple sections completely overlap previous red & blue sections). This is not true if the evaluated frequency (i.e. winding frequency) isn't $\frac{1}{\pi}$ Hz.

This same principle applies to other mathematical functions as well. If the evaluated frequency matches one of the constituent frequencies, the **center-of-mass** of the resulting wound-up curve will be significantly non-zero, since the shape is independent from evaluated domain length due to perfect overlapping. Otherwise, the **center-of-mass** tends to zero as evaluated domain length tends to infinity, **due to imperfect overlapping**. This is a fundamental concept of Fourier Transform.

Below, I am going to explain how Fourier Transform mathematically achieves the "winding" mathematically, and how its output relates to the "center of mass" of the wound-up curve.

Mathematical Foundations of Fourier Transform

Fourier Transform exploits Euler's formula, which states that:

$$e^{it} = \cos(t) + isin(t) \qquad [t \in \mathbb{R}]$$

Below, the Fourier Transform formula is built up through multiple intermediate steps, namely "a(t)", "b(t)", "c(t)", "d(t)" and "CM" (i.e. center-of-mass). Given that a function, h(t), is evaluated from $a \leq t \leq b$, the expression below, a(t), covers an arc of 2(b-a)π radians with radius 1, around the complex plane origin:

$$a(t) = e^{2\pi it}$$

Since the winding is of clockwise fashion, the expression becomes:

$$b(t) = e^{-2\pi it}$$

The expression below scales the arc by the amplitude of h(t), effectively winding the function around the origin:

$$c(t) = h(t)e^{-2\pi it}$$

Since the function is wound at a frequency, hence the expression becomes:

$$d(t) = h(t)e^{-2\pi ift}$$

The previous expression establishes that every segment of h(t) spanning a domain length of $\frac{1}{f}$ units correspond to a full "wind" (2π radians) around the origin, $f \in \mathbb{R},\ f \neq 0$. Evaluating frequency at 0 is meaningless, as a 0 Hz sinusoid is simply blank. As mentioned previously, center-of-mass is an important concept in Fourier Transform. To find the center-of-mass, simply integrate the expression from the designated bounds of h(t), and divide by the length of the evaluated signal:

$$\mathrm{CM} = \frac{1}{\mathrm{b-a}}\int_{\mathrm{a}}^{\mathrm{b}} h(t)e^{-2\pi ift}\,dt\ \ [a, b\ \in\ \mathbb{R}\ \cup\ \pm\infty]$$

The actual Fourier Transform does not divide the integral by the signal length, effectively magnifying the values by the length of evaluated signal:

$$\hat{h}(f) = \int_{\mathrm{a}}^{\mathrm{b}} h(t)e^{-2\pi ift}\,dt = (\mathrm{b}-\mathrm{a})\mathrm{CM}$$

The classic Fourier Transform equation is in the form below, where *t* is replaceable with any variable:

$$\hat{h}(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi ift}\,dt\ =\ (\mathrm{b}-\mathrm{a})\mathrm{CM}$$

Practically speaking, as time cannot be negative, the maximum boundaries for a Fourier Transform applied on a time-domain function, h(t) is:

$$\hat{h}(f) = \int_{0}^{\infty} h(t)e^{-2\pi ift}\,dt$$

Evaluating Fourier Transform at a frequency produces a complex-valued output, which is called a “Fourier coefficient”, specific to the evaluated frequency **(3Blue1Brown)**. The output is labeled $\hat{h}(f)$, the circumflex being a convention and f being the evaluated frequency. For example, $\hat{h}(11)$ represents the Fourier coefficient corresponding to an evaluated frequency of 11 Hz. **By evaluating Fourier Transform across a range of frequencies and plotting the resulting Fourier coefficients moduli with respect to their evaluated frequency, a frequency-domain plot is produced.**

For any real-valued function h(t), it will equal to its complex conjunction h*(t), as the imaginary part must be 0, so - 0*i* and + 0*i* makes no difference. Hence:

$$\hat{h}(f) = \int_{0}^{\infty} h(t)e^{-2\pi ift}\,dt$$

$$\hat{h}(f) = \int_{0}^{\infty} h^{*}(t)e^{-2\pi ift}\,dt$$

Since $e^{2\pi ift} = \cos(2\pi ft) + i\sin(2\pi ft)$, and $e^{-2\pi ift} = \cos(2\pi ft) - i\sin(2\pi ft)$

$$\hat{h}(f) = \int_{0}^{\infty} h^{*}(t)e^{-2\pi ift}\,dt$$

$$\hat{h}(f) = \int_{0}^{\infty} h^{*}(t)\left(\cos(2\pi ft) - isin(2\pi ft)\right)dt$$

$$\hat{h}(f) = \int_{0}^{\infty} [h^*(t)\cos(2\pi f t) - h^*(t)\, i\sin(2\pi f t)]\, dt$$

Since $\cos(-2\pi f t) = \cos(2\pi f t)$, and $\sin(-2\pi f t) = -\sin\,(2\pi f t)$:

$$\hat{h}(f) = \int_{0}^{\infty} [h^*(t)\cos(-2\pi f t) + h^*(t)\, i\sin(-2\pi f t)]\, dt$$

$$\hat{h}(f) = \int_{0}^{\infty} h^*(t)\, e^{2\pi i(-f)t}\, dt = \hat{h}^*(-f)$$

Hence it is shown that $\hat{h}(f) = \hat{h}^*(-f)$. This identity is important for forming the equation for the inverse Fourier Transform which will be shown later.

As Fourier Coefficients are complex-valued, there are multiple ways a frequency-domain plot can be produced. **I will plot the complex moduli, as that takes both the complex and real parts into account.**

It is easy to recreate the time-domain signal, using a method called "Inverse Fourier Transform". Below I will show how its formula is derived.

Fig. 1f shows the winding of 3 cosine functions, with varying magnitudes of horizontal shift. center-of-mass positions of the wound-up graphs are denoted using black dots. As shown, the argument of the center-of-mass correspond to the magnitude of leftwards horizontal shift. The integral of the wound-up function is a multiple of the center-of-mass, hence its argument remains unchanged.

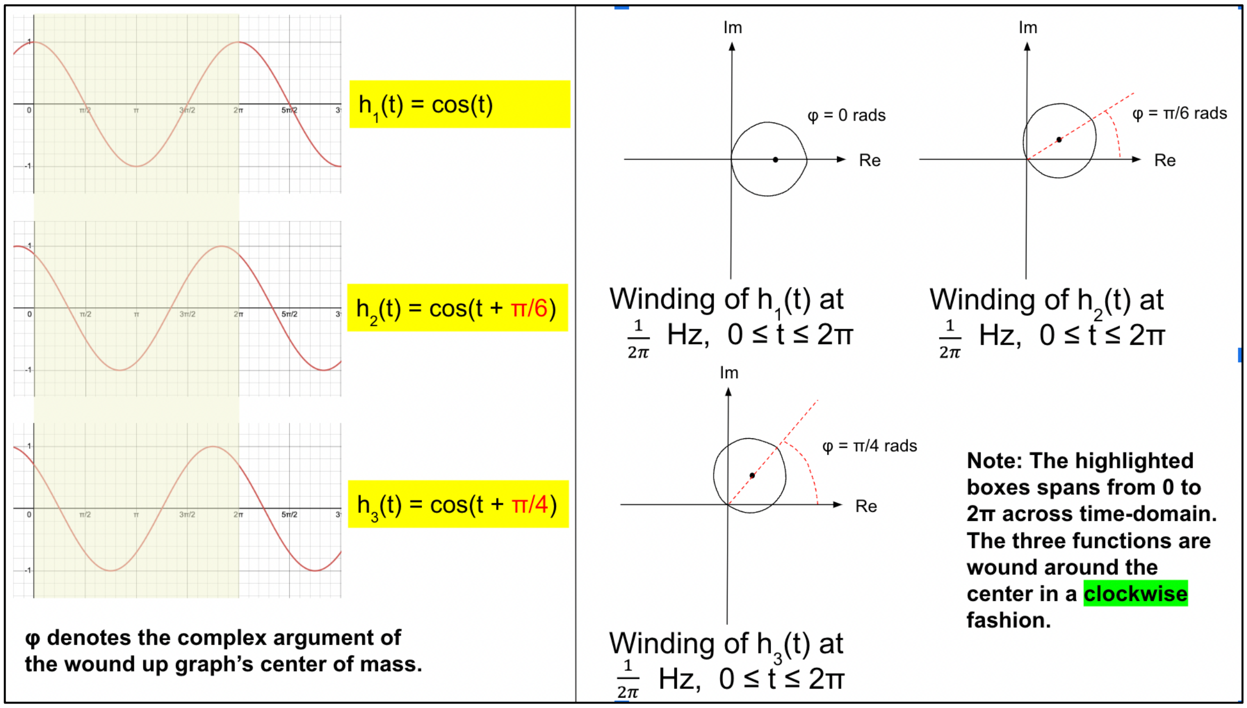


Fig. 1f

One can recreate the constituent sinusoid for a frequency *f*, $c_f(t)$, across the time domain, using insights from the Fourier Coefficient of that respective frequency.

For $\hat{h}(f) = \hat{h}_{RE}(f) + \hat{h}_{IM}(f)i$: ($\hat{h}_{RE}(f)$ and $\hat{h}_{IM}(f)$ are both real valued)

$$c_f(t) = \sqrt{\hat{h}_{RE}(f)^2 + \hat{h}_{IM}(f)^2}\cos\left(2\pi f t + arctan\left(\frac{\hat{h}_{IM}(f)}{\hat{h}_{RE}(f)}\right)\right)$$

The amplitude of the sinusoid is highlighted in red, whilst the phase shift in blue. The amplitude can further be normalized through multiplying a factor of $\frac{1}{b-a}$ in front, *b* and *a* being the upper and lower bounds of the original Fourier Transform, as:

$$\hat{h}(f) = \int_a^b h(t)e^{-2\pi i f t}\,dt = (\mathbf{b} - \mathbf{a})\mathrm{CM}$$

One can, in principle, recreate the signal, h(t), by aggregating all of the constituent sinusoids across an unlimited range of frequencies through insights from the frequency-domain plot. One may *subjectively* neglect Fourier coefficients of certain "noise" and "redundant" frequencies in the process, thus recreating a relatively "denoised" signal. This idea can be expressed mathematically:

$$h(t) = \int_{-\infty}^{\infty} \mathrm{c_f}(t)\, df$$

$$h(t) = \int_{-\infty}^{\infty} \sqrt{\hat{\mathrm{h}}_{RE}(f)^2 + \hat{\mathrm{h}}_{IM}(f)^2} \cos\left(2\pi f t + arctan\left(\frac{\hat{h}_{IM}(f)}{\hat{h}_{RE}(f)}\right)\right) df$$

One can greatly simplify the above using the below trigonometric identity, proven in Appendix III:

$$Acos(cx) - Bsin(cx) = \sqrt{A^2 + B^2} cos\left(cx - \arctan\left(\frac{B}{A}\right)\right)$$

Hence for:

$$h(t) = \int_{-\infty}^{\infty} \sqrt{\hat{\mathrm{h}}_{RE}(f)^2 + \hat{\mathrm{h}}_{IM}(f)^2} \cos\left(2\pi f t + arctan\left(\frac{\hat{h}_{IM}(f)}{\hat{h}_{RE}(f)}\right)\right) df$$

let $A = \hat{h}_{RE}(f), B = \hat{h}_{IM}(f),\ c = 2\pi f$:

$$h(t) = \int_{-\infty}^{\infty} \left[\hat{\mathrm{h}}_{RE}(f)\cos(2\pi f t) - \hat{\mathrm{h}}_{IM}(f)\sin(2\pi f t)\right] df$$

In previous sections it has been proven that $\hat{h}(f) = \hat{h}^*(-f)$. For $\hat{h}(f) = \hat{\mathrm{h}}_{RE}(f) + \hat{\mathrm{h}}_{IM}(f)i$, that means $\hat{\mathrm{h}}_{RE}(f)$ is an even function, and $\hat{\mathrm{h}}_{IM}(f)$ is an odd function. Therefore, it is possible to further encapsulate the above formula:

$$h(t) = \int_{-\infty}^{\infty} \hat{h}(f) e^{2\pi i f t}\, df$$

The above is the standard Inverse Fourier Transform expression **(Weisstein)**. It assumes a **continuous, boundless spectrum** of frequencies. Rewriting equation yields:

$$h(t) = \int_{-\infty}^{\infty} \hat{h}(f)\big(\cos(2\pi f t) + i\sin(2\pi f t)\big)\, df$$

$$h(t) = \int_{-\infty}^{\infty} \left[\hat{\mathrm{h}}_{RE}(f)\cos(2\pi f t) + i\hat{\mathrm{h}}_{IM}(f)\cos(2\pi f t) + i\hat{\mathrm{h}}_{RE}(f)\sin(2\pi f t) - \hat{\mathrm{h}}_{IM}(f)\sin(2\pi f t)\right] df$$

Which the integral of the parts highlighted in red will equal to zero, since $\cos(2\pi f t)$ even, and $\sin(2\pi f t)$ is odd. Multiplying 1 even and 1 odd function together yields an odd function. By their very definition, integrating odd functions result in zero, if the absolute value of the lower and upper bounds are of the same. When the upper and lower bounds do not have the same magnitude, the form below is much more useful (as the red parts will not be zero when integrated):

$$h(t) = \int_{a}^{b} \left[\hat{\mathrm{h}}_{RE}(f)\cos(2\pi f t) - \hat{\mathrm{h}}_{IM}(f)\sin(2\pi f t)\right] df \qquad [a, b \in \mathbb{R}]$$

I am going to visually showcase a slightly more complex example below. I will showcase the application of Fourier Transform on the function $\boldsymbol{h(t) = sin^2(t) + sin^2(\frac{t}{2})}$, with evaluated frequency of $\frac{1}{1.5}$ Hz and designated bounds $0 \le t \le 3$; in mathematical notation, $\hat{h}(f) = \int_0^3 h(t) e^{-2\pi i f t}\, dt$. This particular function is used for demonstration because of the following reasons:

1. There are 2 components of two different frequencies. In real world, functions are often not perfectly linear/ sinusoidal. Fourier Transform is built on the basis that every

integrable function can be re-expressed as a sum of different sinusoids of different amplitude, phase and frequency, so this function acts as a very simplified example.

2. The two sinusoidal components are squared, so the wound-up segment stays above the x-axis through its entirety, making my subsequent visualization neater. Fig. 1g demonstrates the effects of squaring:

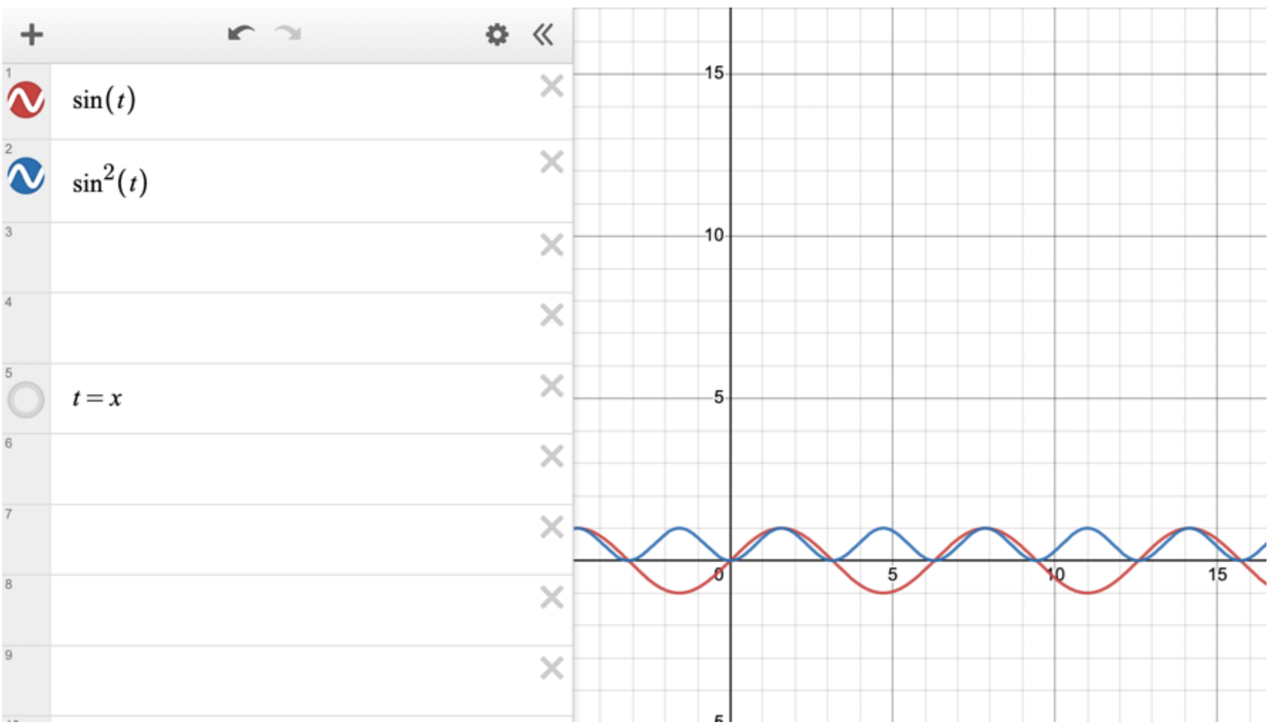
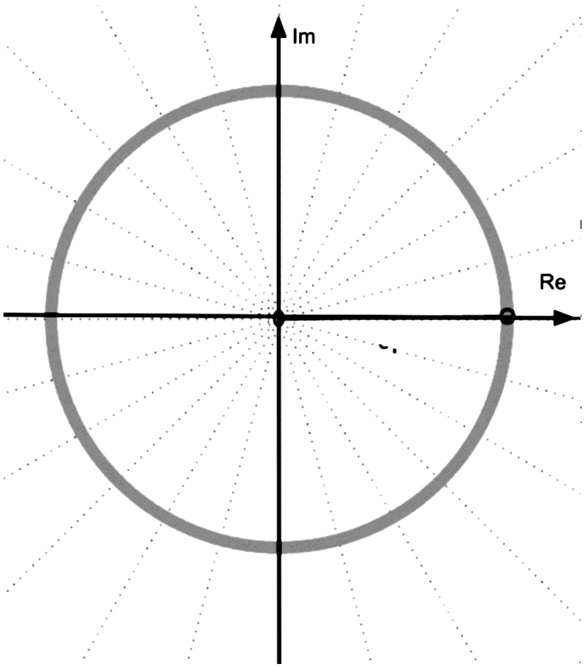
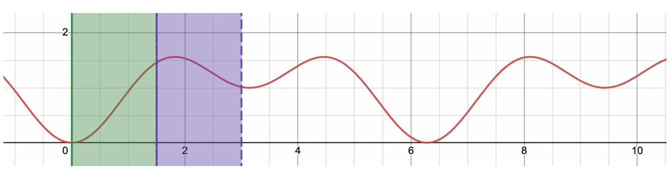


Fig. 1g

It is important to note that $\boldsymbol{sin^2(t)}$ has a frequency of $\frac{1}{\pi}$ Hz, whilst $sin^2(\frac{t}{2})$ has a frequency of $\frac{1}{2\pi}$ Hz. The chart of figures below elucidates steps of constructing the complex output of $\int_0^3 h(t)e^{-2\pi i f t}$:

Graph	Expression & Explanation
 *Fig. 2*	$$e^{-3\pi i t}$$ $$0 \leq t \leq 3$$ An arc of radius 1 that spans 4π radians is covered.
 *Fig. 3*	$$\boldsymbol{sin^2(t) + sin^2\left(\frac{t}{2}\right)}$$ $$0 \leq t \leq 3$$ The winding of curve sections in the green and blue boxes (each with length of 1.5 units in time-domain) is shown in Fig. 4. Figure 3 delineates the example function, h(t), in its unrestricted domain.

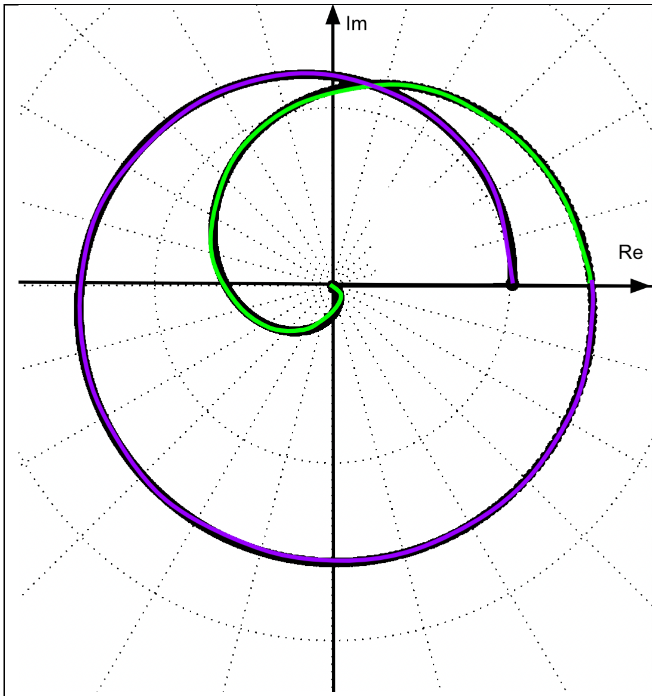


Fig. 4

$$(\boldsymbol{sin^2(t) + sin^2\left(\frac{t}{2}\right)) e^{-2\pi i(\frac{1}{1.5})t}}$$

$$0 \leq t \leq 3$$

As Fourier transform is evaluated at a frequency of 1/1.5 Hz, one full wrap (2π radians) around the origin in clockwise corresponds to a section of h(t) that spans 1.5 units across the t-axis. As h(t) in this case is restricted to $0 \leq t \leq 3$ due to integration bounds, only that section is processed.

Segment (see Fig. 3) in green section is wrapped around across 2π radians, followed by that of the purple section, also across 2π radians. Each colored section in Fig. 3 spans a length of 1.5 across the time-domain.

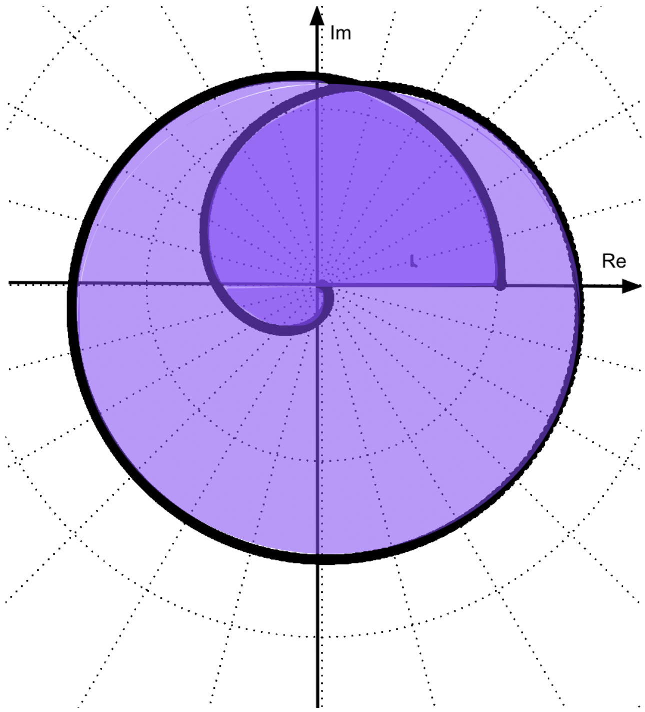


Fig. 5

$$\int_0^3 h(t)e^{-2\pi i(\frac{1}{1.5})t}\,dt$$

$$0 \le t \le 3$$

The colored section represents the integral of the wound-up curve of h(t).

As seen, there is a section that is integrated twice (darker purple) due to section overlaps. The center-of-mass can then be calculated by dividing the integral by evaluated domain length, and in this case, by 3.

Effects of Signal Length on Fourier Transform's Effectiveness

I will use the same example function, $\boldsymbol{h(t) = \cos(2t) + \cos(t)}$, to analytically perform Fourier Transform, at frequency f Hz, $[f \in \mathbb{R}]$, and evaluated domain $0 \le t \le b$, $[b \in \mathbb{R}]$. $\boldsymbol{\cos(2t)}$ has a frequency of $\frac{1}{\pi}$ Hz, whilst $\boldsymbol{\cos(t)}$ has a frequency of $\frac{1}{2\pi}$ Hz. I will analytically derive the general solution in terms of parameters *b* and *f*. Afterwards I will substitute different values of *b* to investigate the effects of evaluated domain length – an important real-life factor.

Firstly, substitute h(t)'s expression into the general formula of Fourier Transform:

$$\hat{h}(f) = \int_{0}^{b} h(t)\, e^{-2\pi i f t}\, dt$$

$$\hat{h}(f) = \int_{0}^{b} (\cos(2t) + \cos(t)) e^{-2\pi i f t}\, dt$$

Now, rewrite $cos(2t)$ and $cos(t)$ with accordance to Euler's equation for simplification:

$\because\ e^{it} = \cos(t) + isin(t)\ ;\ e^{-it} = \cos(t) - isin(t)$

$\therefore\ e^{-it} + e^{it} = (\cos(t) - isin(t)) + \cos(t) + i\sin(t)$

$2\cos\,(t) = e^{-it} + e^{it}$

$cos(t) = \frac{e^{-it}+e^{it}}{2}$

$cos(2t) = \frac{e^{-2it}+e^{2it}}{2}$

Rewriting equation yields:

$$\hat{h}(f) = \int_{0}^{b} (\cos(2t) + \cos(t)) e^{-2\pi i f t}\, dt$$

$$\hat{h}(f) = \int_{0}^{b} \left(\frac{e^{-2it} + e^{2it}}{2} + \frac{e^{-it} + e^{it}}{2}\right) e^{-2\pi i f t}\, dt$$

Perform u-substitution - let $u = e^{it}$, d$u = ie^{it}dt$:

$$\hat{h}(f) = \int_{e^{0i}}^{e^{bi}} [\frac{u^{-2} + u^{2}}{2} + \frac{u^{-1} + u}{2}] u^{-2\pi f}\ \frac{du}{ie^{it}}$$

$$\hat{h}(f) = \int_{1}^{e^{bi}} \left(\frac{u^{-2}+u^{2}}{2} + \frac{u^{-1}+u}{2}\right) u^{-2\pi f} \ \frac{du}{iu}$$

$$\hat{h}(f) = \frac{1}{i}\int_{1}^{e^{bi}} \left(\frac{u^{-2}+u^{2}}{2} + \frac{u^{-1}+u}{2}\right) u^{-2\pi f-1} \ du$$

$$\hat{h}(f) = -i\int_{1}^{e^{bi}} \left(\frac{u^{-2}+u^{2}}{2} + \frac{u^{-1}+u}{2}\right) u^{-2\pi f-1} \ du$$

$$\hat{h}(f) = \frac{-i}{2}\int_{1}^{e^{bi}} (u^{-2}+u^{2}+u^{-1}+u)u^{-2\pi f-1} \ du$$

$$\hat{h}(f) = \frac{-i}{2}\int_{1}^{e^{bi}} (u^{-2-2\pi f-1}+u^{2-2\pi f-1}+u^{-1-2\pi f-1}+u^{1-2\pi f-1}) \ du$$

$$\hat{h}(f) = \frac{-i}{2}\int_{1}^{e^{bi}} (u^{-3-2\pi f}+u^{1-2\pi f}+u^{-2-2\pi f}+u^{-2\pi f}) \ du$$

Further processing the equation yields three different equations, valid for three different scenarios:

$$\hat{h}(f) = \begin{cases} \dfrac{ie^{2(-1-\pi f)ib}-i}{4(1+\pi f)} + \dfrac{ie^{2(1-\pi f)ib}-i}{4(-1+\pi f)} + \dfrac{ie^{(-1-2\pi f)ib}-i}{2(1+2\pi f)} + \dfrac{ie^{(1-2\pi f)ib}-i}{2(-1+2\pi f)}, & \text{for } f \neq \dfrac{1}{\pi}, \dfrac{1}{2\pi} \\ \dfrac{ie^{-4bi}-i}{8} + \dfrac{b}{2} + \dfrac{ie^{-3bi}-i}{6} + \dfrac{ie^{-bi}-i}{2}, & \text{for } f = \dfrac{1}{\pi} \\ -\dfrac{ie^{-3bi}-i}{12} + \dfrac{e^{bi}-i}{4} - \dfrac{e^{-2bi}-i}{8} - \dfrac{\mathrm{b}}{4}, & \text{for } f = \dfrac{1}{2\pi} \end{cases}$$

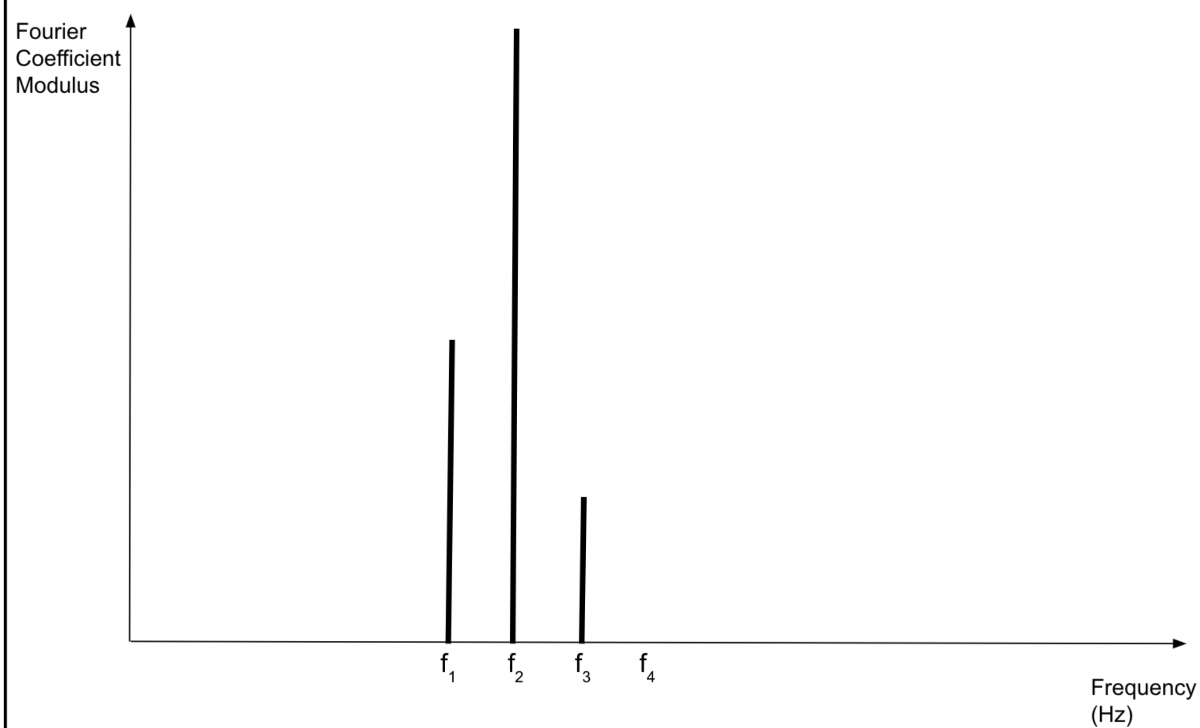
For the derivation of the above equations, please check Section (I) of Appendix II. **From inspection, it can already be seen that the value of $\hat{h}\left(\frac{1}{2\pi}\right)$ and $\hat{h}\left(\frac{1}{\pi}\right)$ will fastly diverge as b increases, due to the $\frac{b}{2}$ and $-\frac{b}{4}$ terms, respectively**.

I am going to evaluate $|\hat{h}(f)|$ for frequency values: f = 1.5Hz, f = $\frac{110}{111\pi}$ Hz, f = $\frac{1}{\pi}$ Hz and f = $\frac{1}{2\pi}$ Hz. $\frac{1}{\pi}$ Hz and $\frac{1}{2\pi}$ Hz are the **2 constituent frequencies** of h(t), whilst $\frac{110}{111\pi}$ Hz is a frequency close $\frac{1}{\pi}$ Hz. Meanwhile, 1.5Hz is far from any of the two constituent frequencies. For each f the following values of b (i.e. length of signal) are going to be evaluated: b = 1, 10, 100, 1000, 10000, 100000, all of which are integer powers of 10.

Below are the general equations for $\hat{h}(1.5)$, $\hat{h}\left(\frac{110}{111\pi}\right)$, $\hat{h}\left(\frac{1}{\pi}\right)$ and $\hat{h}\left(\frac{1}{2\pi}\right)$ in terms of b, using the three equations above **(check entire derivation in Section (II), Appendix II)** :

Value of f in $\hat{h}(f)$	Equation
1.5	$\boldsymbol{\hat{h}}(\mathbf{1.5}) = \frac{ie^{3(-1-\pi f)i}-i}{4(1+\pi f)} + \frac{ie^{3(1-\pi f)i}-i}{4(-1+\pi f)} + \frac{ie^{1.5(-1-2\pi f)i}-i}{2(1+2\pi f)} + \frac{ie^{1.5(1-2\pi f)i}-i}{2(-1+2\pi f)}$
$\frac{110}{111\pi}$	$\boldsymbol{\hat{h}}\left(\frac{\mathbf{110}}{\mathbf{111\pi}}\right) = \frac{ie^{\frac{220}{222\pi}(-1-\pi f)i}-i}{4(1+\pi f)} + \frac{ie^{\frac{220}{222\pi}(1-\pi f)i}-i}{4(-1+\pi f)} + \frac{ie^{\frac{110}{111\pi}(-1-2\pi f)i}-i}{2(1+2\pi f)} + \frac{ie^{\frac{110}{111\pi}(1-2\pi f)i}-i}{2(-1+2\pi f)}$
$\frac{1}{\pi}$	$\hat{h}\left(\frac{1}{\pi}\right) = \frac{ie^{-4bi}-i}{8} + \frac{b}{2} + \frac{ie^{-3bi}-i}{6} + \frac{ie^{-bi}-i}{2}$
$\frac{1}{2\pi}$	$\hat{h}\left(\frac{1}{2\pi}\right) = -\frac{ie^{-3bi}-i}{12} + \frac{e^{bi}-i}{4} - \frac{e^{-2bi}-i}{8} - \frac{\mathrm{b}}{4}$

I will investigate how evaluated domain length affects the ratio between Fourier Coefficients moduli of different frequencies. Fig. 6a below visually elucidates how the ratio between the moduli of Fourier coefficients relates to the shape of frequency-domain plot.



Above shows the shape of the frequency-domain plot given that:

1. $|\hat{h}(f_1)| \,/\, |\hat{h}(f_2)| < 1$
2. $|\hat{h}(f_1)| \,/\, |\hat{h}(f_3)| > 1$
3. $|\hat{h}(f_4)| \,/\, |\hat{h}(f_1)|$ **tends to 0**

Fig. 6a

Below, Table 1 showcases the **Fourier Coefficient** calculated with respect to parameters *b* and *f*, whilst Table 2 showcases the **complex moduli** of values from Table 1:

	b = 1	b = 10	b = 100	b = 1000	b = 10000	b = 100000
$f = 1.5\ Hz$	0.0310 – 0.230i	-0.0153 – 0.263i	0.0264 – 0.0718i	-0.0313 – 0.199i	-0.0102 – 0.230i	0.00128 – 0.215i
$f = \frac{110}{111\pi}\ Hz$	1.17 – 0.452i	4.74 + 0.825i	43.4 + 20.4i	22.5 + 107i	49.2 + 80.1i	54.9 + 52.0i
$f = \frac{1}{\pi}\ Hz$	0.850 – 0.768i	4.66 – 1.27i	49.5 – 0.430i	500 – 0.764i	5000 – 1.33i	50000 – 1.33i
$f = \frac{1}{2\pi}\ Hz$	1.17 – 0.456i	4.79 + 0.63i	49.4 – 0.230i	501 – 0.452i	5000 + 0.663i	50000 – 0.667i

Table 1

	b = 1	b = 10	b = 100	b = 1000	b = 10000	b = 100000
f = 1.5 Hz	0.232	0.263	0.0765	0.201	0.230	0.215
$f = \frac{110}{111\pi}$ *Hz*	1.25	4.81	48.0	109	94.0	75.6
$f = \frac{1}{\pi}$ *Hz*	1.15	4.83	49.5	500	5000	50000
$f = \frac{1}{2\pi}$ *Hz*	1.26	4.83	49.4	501	5000	50000

Table 2

To visualize the above data, I utilized Logger Pro to plot two graphs:

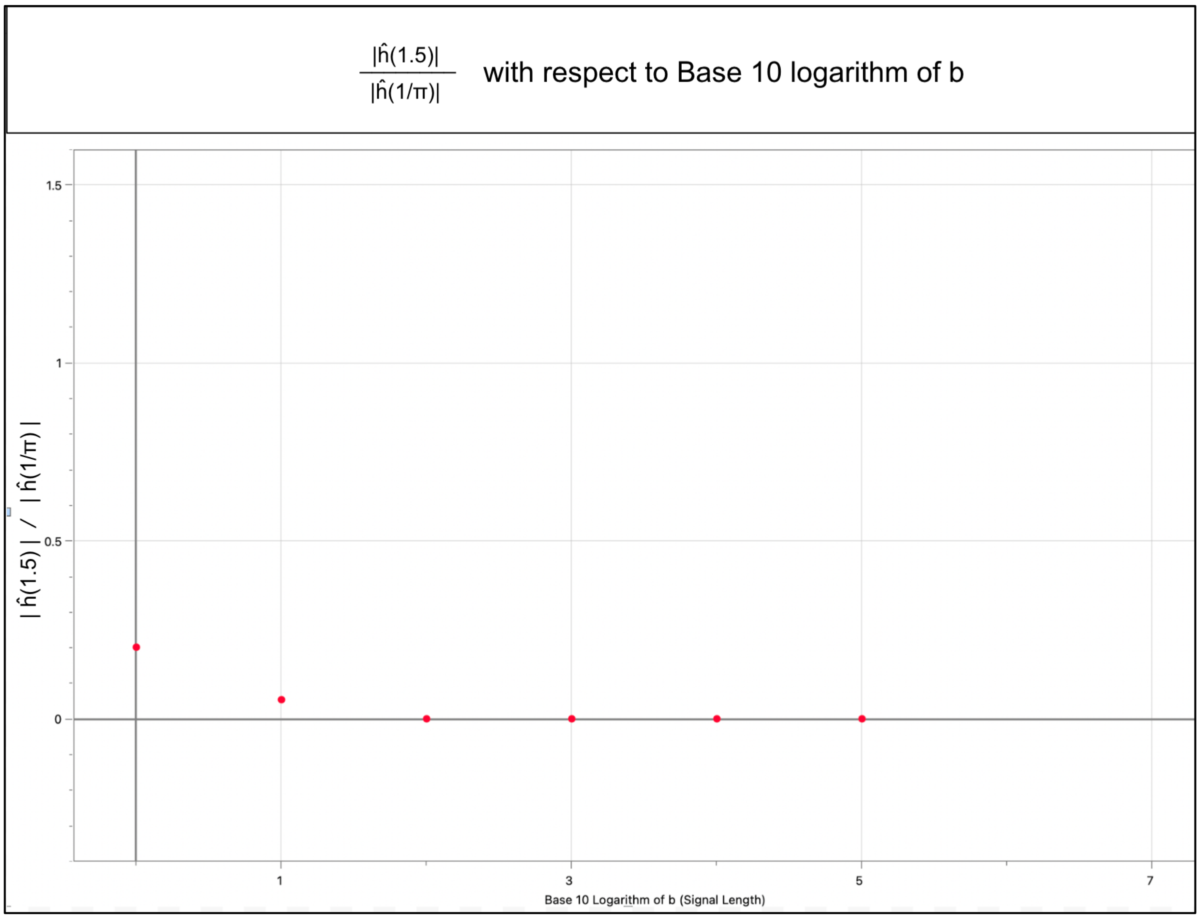


Fig. 6b

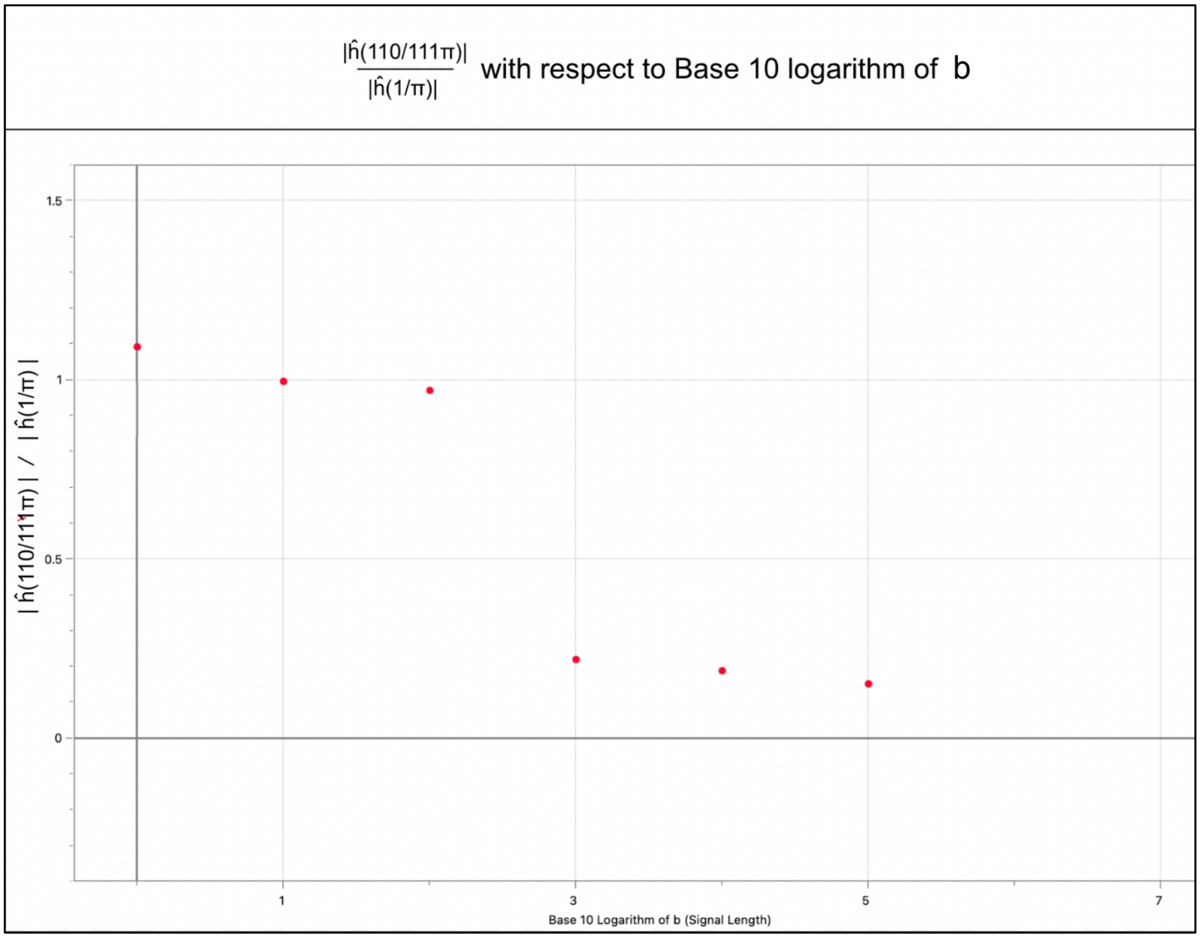


Fig. 6c

Fig. 6b and 6c plots $|\hat{h}(1.5)|$ as a percentage of $|\hat{h}(\frac{1}{\pi})|$, and $|\hat{h}(\frac{110}{111\pi})|$ as a percentage $|\hat{h}(\frac{1}{\pi})|$ against the Base 10 logarithm of b (signal length), respectively. As shown in Fig. 6b, the proportion of $|\hat{h}(1.5)|$ with respect to $|\hat{h}(\frac{1}{\pi})|$ drops as value of log(b) increases. The proportion of $|\hat{h}(\frac{110}{111\pi})|$ with respect to $|\hat{h}(\frac{1}{\pi})|$ also drops as log(b) increases, as shown in Fig. 6c, **despite the two frequencies being very close to each other**. In fact, one can use the squeeze theorem to prove that $\frac{|\hat{h}(f)|}{|\hat{h}(\frac{1}{\pi})|}$ always tend to 0 when $b \to \infty$, f being **any non-constituent frequencies**.

Consequently values of $|\hat{h}(f)|$ become increasingly negligible as evaluated length increases. Squeeze Theorem proofs are not shown as it is beyond this Extended Essay's scope.

Note that although both the proportion of | $\hat{h}(\frac{110}{111\pi})$ | and | $\hat{h}(1.5)$ | with respect to | $\hat{h}(\frac{1}{\pi})$ | tends towards zero as b tends to infinity, the initial proportion of | $\hat{h}(\frac{110}{111\pi})$ | is significantly higher compared to that of | $\hat{h}(1.5)$ |, as the two frequencies are of greater proximity. Furthermore, | $\hat{h}(\frac{110}{111\pi})$ | has a greater value than | $\hat{h}(\frac{1}{\pi})$ | for log(b) = 0, and the two moduli are very close also for log(b) =1 and log(b) = 2.

Interestingly, the ratio between | $\hat{h}(\frac{1}{\pi})$ | and | $\hat{h}(\frac{1}{2\pi})$ | tends to 1 as $b \to \infty$ (i.e. | $\hat{h}(\frac{1}{\pi})$ | = | $\hat{h}(\frac{1}{2\pi})$ |), which exactly represents the ratio between the strength of the two composite frequencies in the original time-domain signal, h(t). This can again be proven via Squeeze Theorem.

The above calculations have demonstrated that **besides the presence of noise, short evaluated domain length, resulted by the real-life factor of limited signal length, can also play a factor.** With the knowledge on Inverse Fourier Transform from previous sections, we can draw a few conclusions:

1. It is impossible to certainly and accurately pinpoint the constituent frequencies of a signal through performing Fourier Transform at relatively short evaluated lengths. Real-world signals can consist of an assortment of different frequencies at various strengths, **so at relatively short evaluate lengths a visible peak on the frequency-domain plot can either be a constituent frequency, an unrelated frequency, or a noise frequency. Performing Fourier Transform on such signal and then performing the inverse will actually add *more noise* into the recreated signal**, as sinusoids of unrelated or noise frequencies with nonzero amplitudes will be aggregated during the IFT stage (as the values of $\left|\hat{h}(f)\right|$ $[f \neq f_{constituent}]$ are significantly nonzero). **This renders Fourier Transform ineffective as a tool of denoising in that case.**

2. As evaluated length increases, Fourier Transform as a denoising tool becomes increasingly effective. The ratio $\frac{|\hat{h}(f)|}{|\hat{h}(f_{constituent})|}$ tends to 0 as evaluated length tends to ∞, and thus in that case sinusoids of unrelated frequencies have negligible impact to the

recreated signal. **In other words, visible peaks on the frequency-domain plot will most either be a noise or constituent frequency.**

Fig. 7a and 7b further contextualizes my findings. **Note that the frequency-domain plots were *artificially* created for the sole purpose of contextualization**. The scales of both figures are different and are adjusted to the modulus of $f_{constituent}$.

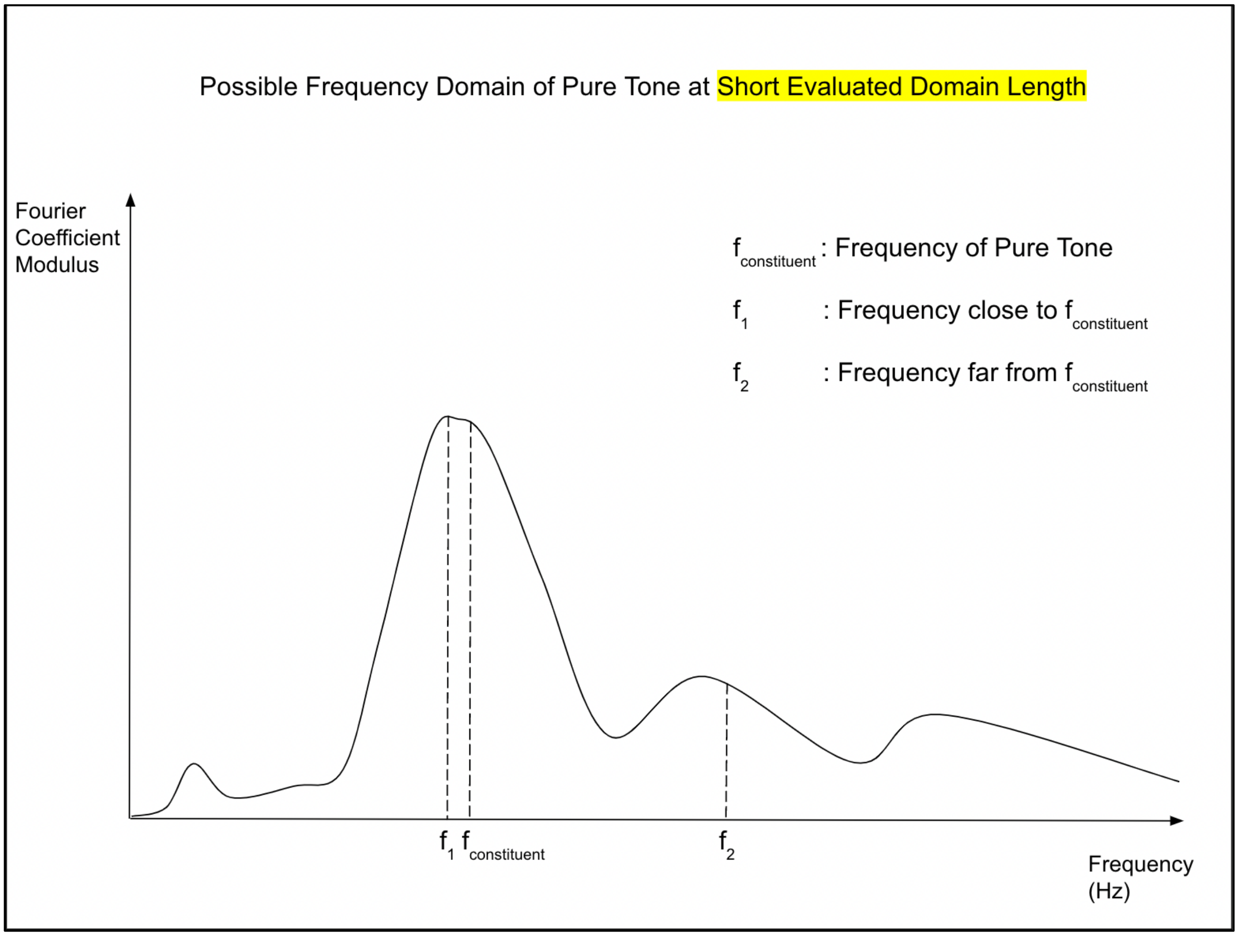


Fig. 7a: Short Evaluated Length

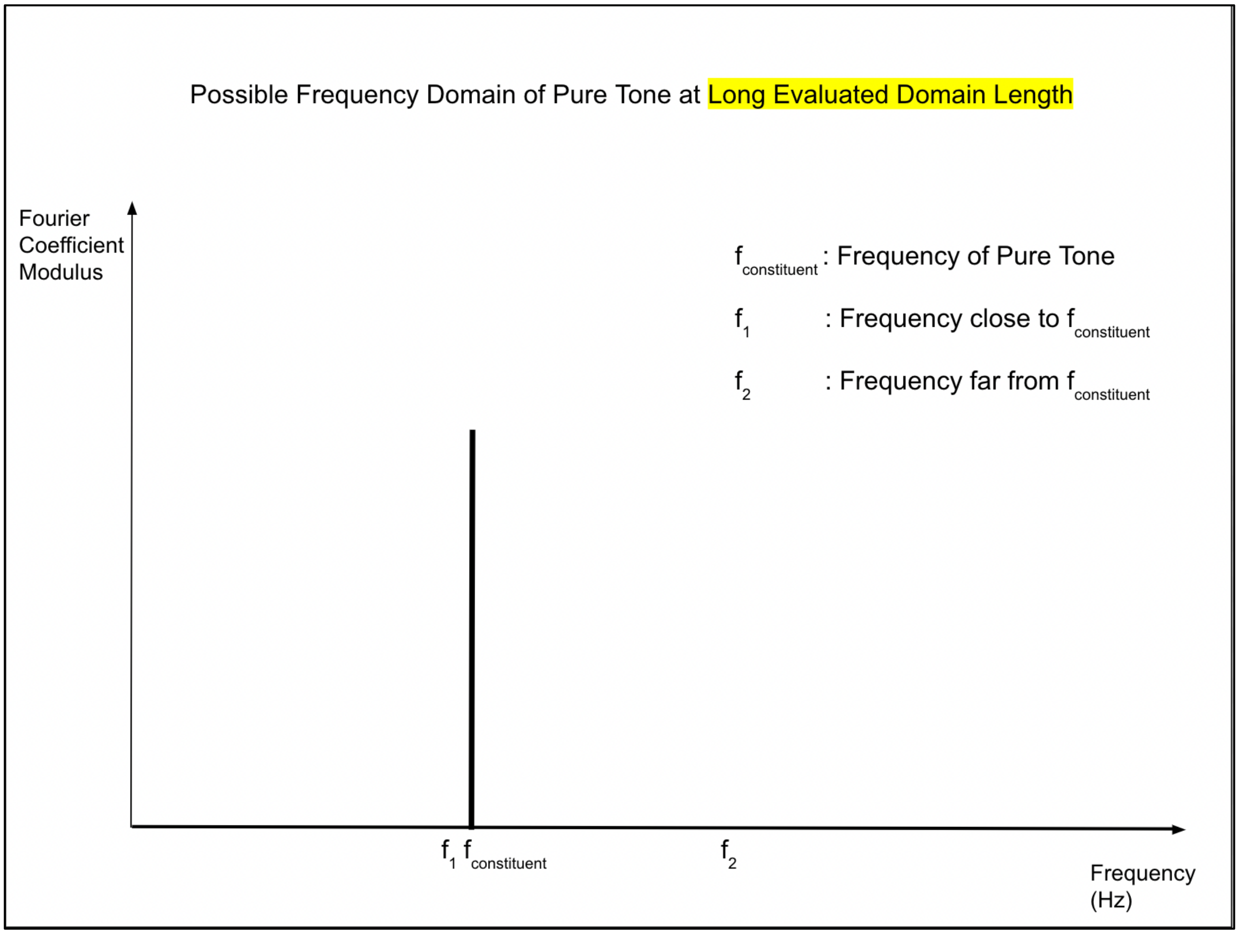


Fig. 7b: Long Evaluated Length

Discrete Fourier Transform

In real life, the sampling of sound signals is not continuous; rather, devices discretely collect air pressure variations values at a sampling frequency and then convert them into amplitude values **(HyClassProject)**. Hence the continuous Fourier Transform doesn't apply, and we utilize a modified, discrete form of Fourier Transform (DFT):

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i k n}{N}}$$

Where:

- N is the number of data points (samples) in the time-domain array being Fourier-transformed

- x_n is the array of discrete time-domain values (time-domain array) being Fourier-transformed.

- n is the location of the data point on time domain, ranging from integers between 0 to (N – 1)

- X_k is the DFT output array at the *k*th frequency, the value of *k* ranging from integers between 0 to (N – 1).

(Xu)

The working principle is the same between the continuous and discrete forms of Fourier Transform. Instead of integrating the wound-up function, DFT uses summation to add all the complex coordinates of all the wound-up discrete data points. The $\frac{k}{N}$ in DFT is analogous to the frequency (*f*) term in the continuous Fourier Transform formula, whilst *n* is analogous to the time (*t*) term. It is just that individual points – instead of a continuous line segment – are being wound.

In other words, only specific frequencies values can be evaluated; the separation between each subsequent specific frequency is called a **frequency bin**, calculated by $\frac{f_{sampling}}{N}$. The evaluated frequency of the *k*th frequency is hence given by $k \cdot \frac{f_{sampling}}{N}$. For example, when there are 100 data points, and $f_{sampling} = 100$ Hz, the frequency bin is $\frac{100}{100}$ = 1 Hz.

Fig. 8 below graphically elucidates the idea of sampling frequency. Simply put, sampling frequency means how frequent data points are being recorded. For example, a sampling frequency of 60Hz indicates that 60 data points are sampled **per second**. Physicist Harry Nyquist once stated that frequencies above ½ of the sampling frequency cannot be evaluated **(Weisstein)**, and the frequencies before that should be multiplied by a factor of 2; this is because the unprocessed DFT frequency-domain plot reflects along the Nyquist frequency (f = ½ $f_{sampling}$) **(Xu)**. will not explain its derivation and proof as it isn't this EE's focus. **With a higher sampling frequency, the real sound signal being recorded can more accurately be modeled, leading to more accurate representations of constituent frequencies and thus more effective denoising.**

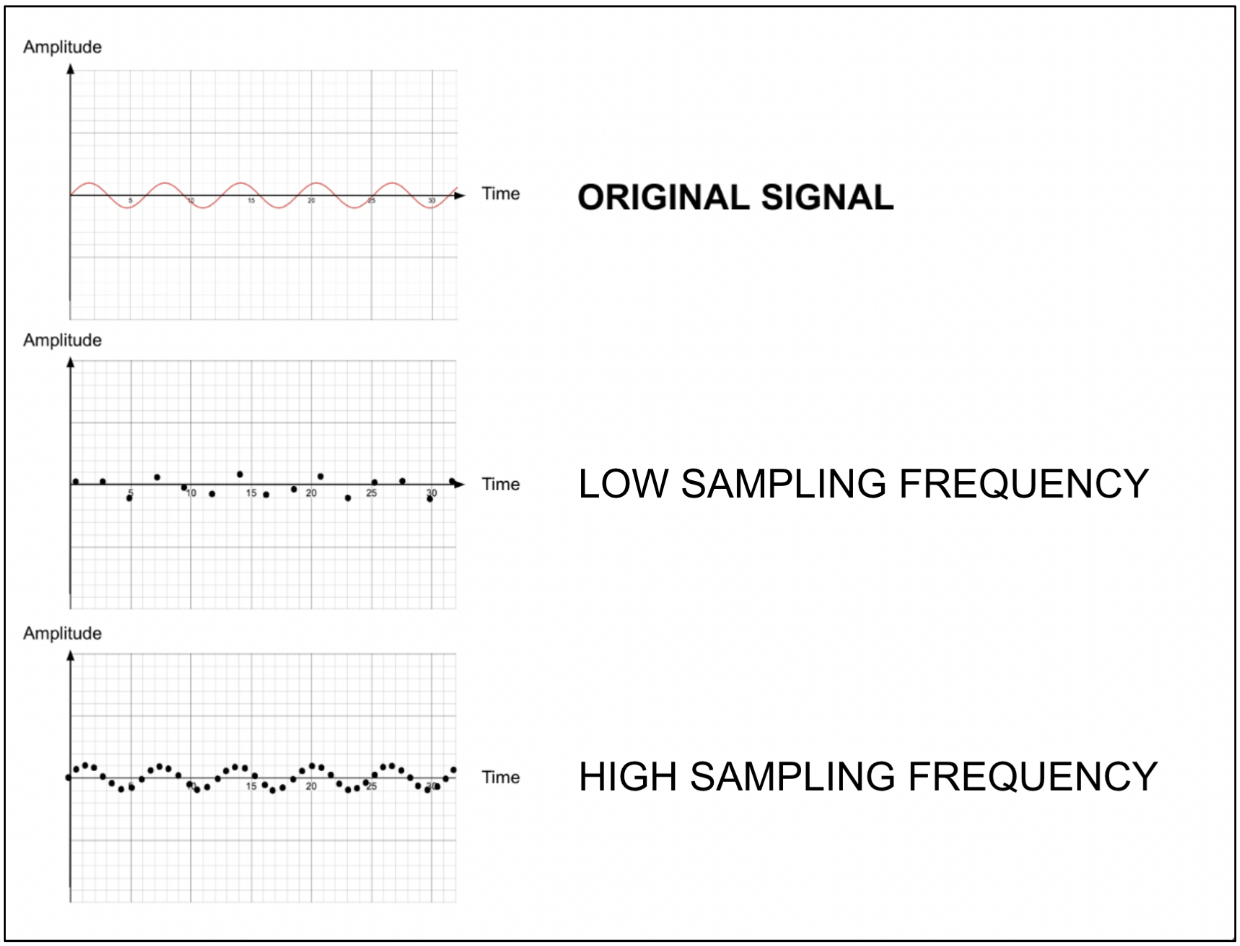


Fig. 8

The reconstruction of signal is the same with continuous Fourier Transform. The only difference is that summation is used instead of integration, as only Fourier Coefficients of certain frequencies can be yielded through DFT.

For $X_k = X_{RE(k)} + X_{IM(k)} i$: ($\hat{h}_{RE}(f)$ and $\hat{h}_{IM}(f)$ both yield real valued coefficients)

$$x_{recreated}(t) = \sum_{k=0}^{\frac{N}{2}-1} [X_{RE(k)} \cos\left(2\pi k \cdot \frac{f_{sampling}}{N} t\right) - X_{IM(k)} \sin\left(2\pi k \cdot \frac{f_{sampling}}{N} t\right)]$$

In which $k \cdot \frac{f_{sampling}}{N}$ is the evaluated frequency of DFT, and the frequency of the reconstructed sinusoid. If some specific frequencies are perceived as noise, those Fourier coefficients can simply be omitted during the reconstruction phase, thus achieving denoising.

Real Life Exploration using Python Program

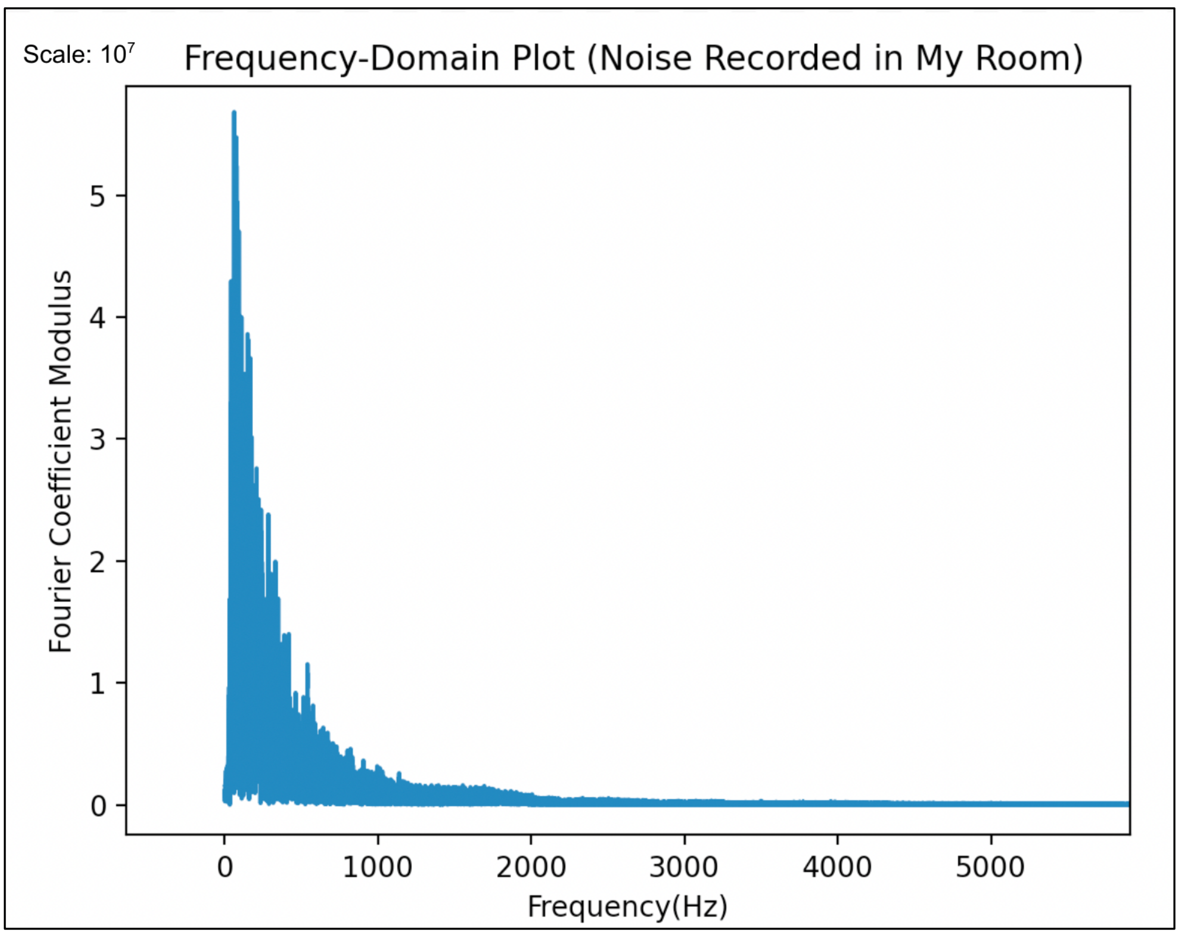


Fig. 9

Fig. 9 shows that noise can consist of many different frequencies. This makes it hard to be filtered out using insights from frequency-domain plots, shedding light on a real-life limiting factor for Fourier Transform-assisted denoising.

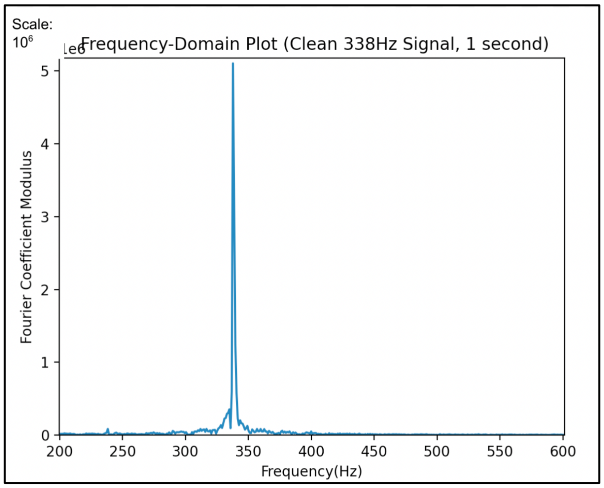


Fig. 10a.

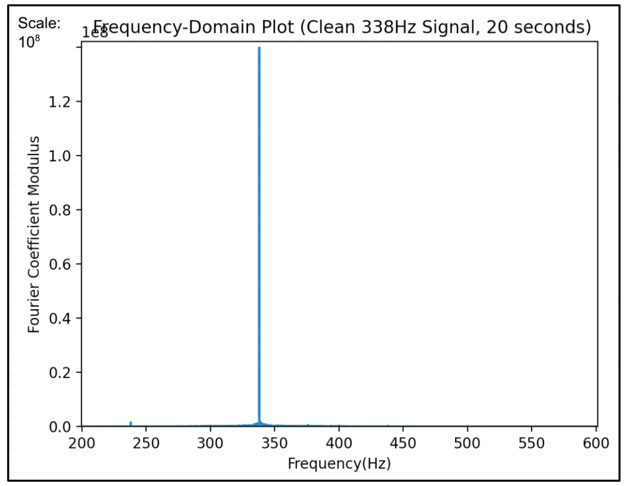


Fig. 10b

As aligned to my previous hypothesis, longer time domain allows the production of more pronounced frequency-domain plots (see Fig. 10a and 10b).

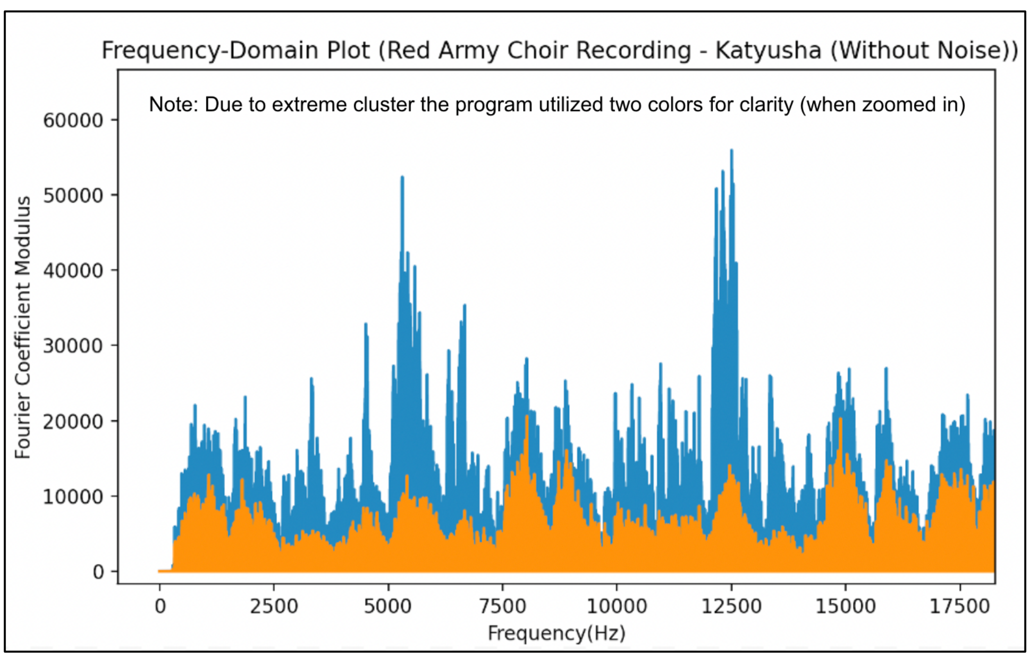


Fig. 11

The Russian/ Soviet folk song “Katyusha” consists of many different frequencies, due to its rather sophisticated rhythm and lyrics, even when there is no noise. It is easy to imagine the sheer difficulty identifying noise frequencies, in the case a noisy recording of “Katyusha” is being discretely Fourier Transformed. This shows overtly complex sound signals can hinder Fourier Transform’s effectiveness in denoising, especially when paired with complex noises like that in Fig. 9.

Conclusion

The type of noise, as a real-life factor, affects Fourier Transform’s effectiveness in denoising sound signals. If the noise is of a single frequency, it can more easily be identified in the frequency domain plot and be filtered out during the reconstruction phase. But if the noise is composed of many frequencies, like that of white noise (which composes of all frequencies with equal amplitudes, similar to white light), then filtering that out based off insights from the frequency-domain plot becomes near impossible. **This becomes even more problematic if the sound signal is complicated**, as then there will already be many constituent frequencies with varying amplitudes.

The real-life factor of signal length (i.e. evaluated domain length) also affects Fourier Transform’s effectiveness. As the evaluated length of sound signal increases, frequency domain plots will have more pronounced peaks, since ratio of unrelated frequencies to constituent frequencies will decrease. This leads to devreased uncertainty during the denoising process, as Fourier Coefficients of unrelated frequencies will have less weight in the reconstructed signal. In the context of DFT, increasing the signal length also allows more samples to be taken, assuming the same sampling frequency. This decreases the frequency bin length, leading to a relatively more “continuous” frequency spectrum and thus allows more accurate signal reconstructions.

Finally, a larger sampling frequency allows a wider range of frequencies to be evaluated with respect to the Nyquist limit, allowing a wider range of frequencies amongst the sinusoids used to reconstruct the signal; this enhances the accuracy of Fourier Transform in recreating the clean part of the original signal.

To conclude, the optimal real-life conditions for Fourier Transform as a denoising tool is:

1. Long signal length & evaluated domain length

2. Relatively uncomplex signal

3. High sampling frequency

4. Absence of non-monotonic noise

Further Research

One may consider pursuing further research in the following directions:

1. Exploration of the extent of evaluated time frame/ sampling frequency on the efficiency and accuracy of Doppler radars/ speed detectors.

2. Writing a Fourier Transform program that can denoise real-life sound signals.

3. Comparison of Fourier Transform-assisted denoising to other methods.

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Appendices

Appendix I - Python Program

```python
import scipy as sp
import numpy as np
from scipy.io import *
from scipy import fftpack
from matplotlib import pyplot as plt
from decimal import Decimal
sample_rate,data = wavfile.read("")
raw_fft = fftpack.fft(data)
FFT = abs(raw_fft)

#PSD is power spectral density: power is amplitude squared

PSD = raw_fft * np.conj(raw_fft) / (len(raw_fft)/sample_rate)
freq_vector = fftpack.fftfreq(len(FFT),(1.0/sample_rate))


plt.plot(freq_vector[range(len(FFT)//2)],FFT[range(len(FFT)//2)])
plt.title(" Frequency-Domain Plot")
plt.xlabel("Frequency(Hz)")
plt.ylabel("Fourier Coefficient Modulus")
plt.show()
```

Appendix II - Mathematical Derivation

Section (I):

For $f \neq \frac{1}{\pi}, \frac{1}{2\pi}$ the integral can be evaluated entirely using the power rule, as in these cases none of u 's exponent will equal -1:

$$\hat{h}(f) = \frac{-i}{2}\int_{1}^{e^{bi}} \left(u^{-3-2\pi f} + u^{1-2\pi f} + u^{-2-2\pi f} + u^{-2\pi f}\right)\, du$$

$$\hat{h}(f) = \frac{-i}{2}\left[\frac{u^{-2-2\pi f}}{-2-2\pi f} + \frac{u^{2-2\pi f}}{2-2\pi f} + \frac{u^{-1-2\pi f}}{-1-2\pi f} + \frac{u^{-2\pi f+1}}{-2\pi f+1}\right]_{1}^{e^{bi}}$$

Resubstituting $u = e^{it}$ and reapplying original bounds:

$$\hat{h}(f) = i\left[\frac{u^{-2-2\pi f}}{4+4\pi f} + \frac{u^{2-2\pi f}}{-4+4\pi f} + \frac{u^{-1-2\pi f}}{2+4\pi f} + \frac{u^{-2\pi f+1}}{4\pi f-2}\right]_{0}^{b}$$

$$\hat{h}(f) = i\left[\frac{e^{2(-1-\pi f)it}}{4(1+\pi f)} + \frac{e^{2(1-\pi f)it}}{4(-1+\pi f)} + \frac{e^{(-1-2\pi f)it}}{2(1+2\pi f)} + \frac{e^{(1-2\pi f)it}}{2(-1+2\pi f)}\right]_{0}^{b} \qquad \textbf{(Equation. 1)}$$

Alternatively, Equation 1 can be expanded:

$$\hat{h}(f) = i\left[\frac{e^{2(-1-\pi f)it}}{4(1+\pi f)} + \frac{e^{2(1-\pi f)it}}{4(-1+\pi f)} + \frac{e^{(-1-2\pi f)it}}{2(1+2\pi f)} + \frac{e^{(1-2\pi f)it}}{2(-1+2\pi f)}\right]_{0}^{b}$$

$$\hat{h}(f) = \frac{ie^{2(-1-\pi f)ib} - ie^{0}}{4(1+\pi f)} + \frac{ie^{2(1-\pi f)ib} - ie^{0}}{4(-1+\pi f)} + \frac{ie^{(-1-2\pi f)ib} - ie^{0}}{2(1+2\pi f)} + \frac{ie^{(1-2\pi f)ib} - ie^{0}}{2(-1+2\pi f)}$$

$$\hat{h}(f) = \frac{ie^{2(-1-\pi f)ib} - i}{4(1+\pi f)} + \frac{ie^{2(1-\pi f)ib} - i}{4(-1+\pi f)} + \frac{ie^{(-1-2\pi f)ib} - i}{2(1+2\pi f)} + \frac{ie^{(1-2\pi f)ib} - i}{2(-1+2\pi f)}$$

(Equation 1)

And for $f = \frac{1}{\pi}$:

$$\hat{h}\left(\frac{1}{\pi}\right) = \frac{-i}{2}\int_{1}^{e^{bi}} \left(u^{-3-2\pi\left(\frac{1}{\pi}\right)} + u^{1-2\pi\left(\frac{1}{\pi}\right)} + u^{-2-2\pi\left(\frac{1}{\pi}\right)} + u^{-2\pi\left(\frac{1}{\pi}\right)}\right)\, du$$

$$\hat{h}\left(\frac{1}{\pi}\right) = \frac{-i}{2}\int_{1}^{e^{bi}} \left(u^{-3-2} + u^{1-2} + u^{-2-2} + u^{-2}\right)\, du$$

Resubstituting $u = e^{it}$ and reapplying original bounds:

$$\hat{h}\left(\frac{1}{\pi}\right) = \frac{-i}{2}\left[\frac{u^{-4}}{-4} + \ln(u) + \frac{u^{-3}}{-3} + -u^{-1}\right]_{1}^{e^{bi}}$$

$$\hat{h}\left(\frac{1}{\pi}\right) = i\left[\frac{u^{-4}}{8} - \frac{\ln(u)}{2} + \frac{u^{-3}}{6} + \frac{u^{-1}}{2}\right]_{1}^{e^{bi}}$$

$$\hat{h}\left(\frac{1}{\pi}\right) = \frac{ie^{-4bi} - i}{8} - \frac{i\ln\left(e^{bi}\right) - i\ln(1)}{2} + \frac{ie^{-3bi} - i}{6} + \frac{ie^{-bi} - i}{2}$$

$$\hat{h}\left(\frac{1}{\pi}\right) = \frac{ie^{-4bi} - i}{8} + \frac{b}{2} + \frac{ie^{-3bi} - i}{6} + \frac{ie^{-bi} - i}{2}$$

(Equation 2)

Finally, for $f = \frac{1}{2\pi}$:

$$\hat{h}\left(\frac{1}{2\pi}\right) = \frac{-i}{2}\int_{1}^{e^{bi}} \left(u^{-3-2\pi\left(\frac{1}{2\pi}\right)} + u^{1-2\pi\left(\frac{1}{2\pi}\right)} + u^{-2-2\pi\left(\frac{1}{2\pi}\right)} + u^{-2\pi\left(\frac{1}{2\pi}\right)}\right)\, du$$

$$\hat{h}\left(\frac{1}{2\pi}\right) = \frac{i}{4}\int_{1}^{e^{bi}} \left(u^{-3-1} + u^{1-1} + u^{-2-1} + u^{-1}\right)\, du$$

Resubstituting $u = e^{it}$ and reapplying original bounds:

$$\hat{h}\left(\frac{1}{2\pi}\right) = \frac{i}{4}\left[\frac{u^{-3}}{-3} + u + \frac{u^{-2}}{-2} + \ln(u)\right]_{1}^{e^{bi}}$$

$$\hat{h}\left(\frac{1}{2\pi}\right) = i\left[\frac{u^{-3}}{-12} + \frac{u}{4} + \frac{u^{-2}}{-8} + \frac{\ln(u)}{4}\right]_{1}^{e^{bi}}$$

$$\hat{h}\left(\frac{1}{2\pi}\right) = \frac{ie^{-3bi} - i}{-12} + \frac{e^{bi} - i}{4} + \frac{e^{-2bi} - i}{-8} + \frac{i\ln\left(e^{bi}\right) - i\ln(1)}{4}$$

$$\hat{h}\left(\frac{1}{2\pi}\right) = -\frac{ie^{-3bi} - i}{12} + \frac{e^{bi} - i}{4} - \frac{e^{-2bi} - i}{8} - \frac{\mathrm{b}}{4}$$

(Equation 3)

Equation 1 is the general Fourier Transform output of the example function, $\hat{\mathrm{h}}(\mathrm{f})$, applicable for $[f \in \mathbb{R}, f \neq \frac{1}{\pi}, \frac{1}{2\pi}]$, as those two values would result in at least one fraction with a 'zero' denominator; this is because Equation 1 is obtained through integrating all the *u* exponentials with power rule, with the prerequisite that **none of *u* 's exponents will be equal to negative one**, in which is clearly invalid when $f = \frac{1}{\pi}$ or when $f = \frac{1}{2\pi}$. Equation 2 and Equation 3 covers $f = \frac{1}{\pi}$ and $f = \frac{1}{2\pi}$, respectively. Therefore, when $f \in \mathbb{R}, f \neq \frac{1}{\pi}, \frac{1}{2\pi}$, the general form, **Equation 1**, applies. Otherwise, **Equation 2** applies for $f = \frac{1}{\pi}$, and **Equation 3** for $f = \frac{1}{2\pi}$.

Appendix III - Proof of Trigonometric Identity

Let us construct the below equation:

$$Acos(cx) - Bsin(cx) = rcos\big(c(x-\theta)\big) \quad [A, B, c, r, \theta \in \mathbb{R}]$$

$$Acos(cx) - Bsin(cx) = rcos(cx + c\theta)$$

$$Acos(cx) - Bsin(cx) = rcos(cx)rcos(c\theta) - rsin(cx)\sin(c\theta)$$

$$Acos(cx) - Bsin(cx) = rcos(cx)rcos(c\theta) - rsin(cx)\sin(c\theta)$$

Hence, we can reasonably get the following equations:

$$Acos(cx) = rcos(cx)rcos(c\theta)$$

$$-Bsin(cx) = rsin(cx)rsin(c\theta)$$

We can hence derive the following from the above two equations:

$$A = rcos(c\theta)$$

$$B = rsin(c\theta)$$

$$\frac{B}{A} = \tan\,(c\theta)$$

$$c\theta = \arctan\left(\frac{B}{A}\right)$$

Substituting those into the original equation:

$$Acos(cx) - Bsin(cx) = rcos\left(cx - \arctan\left(\frac{B}{A}\right)\right)$$

From the equations $A = rcos(c\theta)$ and $B = rsin(c\theta)$:

$$r^2cos^2(c\theta) + r^2sin^2(c\theta) = A^2 + B^2$$

$$r^2 = A^2 + B^2$$

$$r = \sqrt{A^2 + B^2}$$

Hence we get the following identity:

$$Acos(cx) - Bsin(cx) = \sqrt{A^2 + B^2}cos\left(cx - \arctan\left(\frac{B}{A}\right)\right)$$