Gaussian Naive Bayes and Linear Regression

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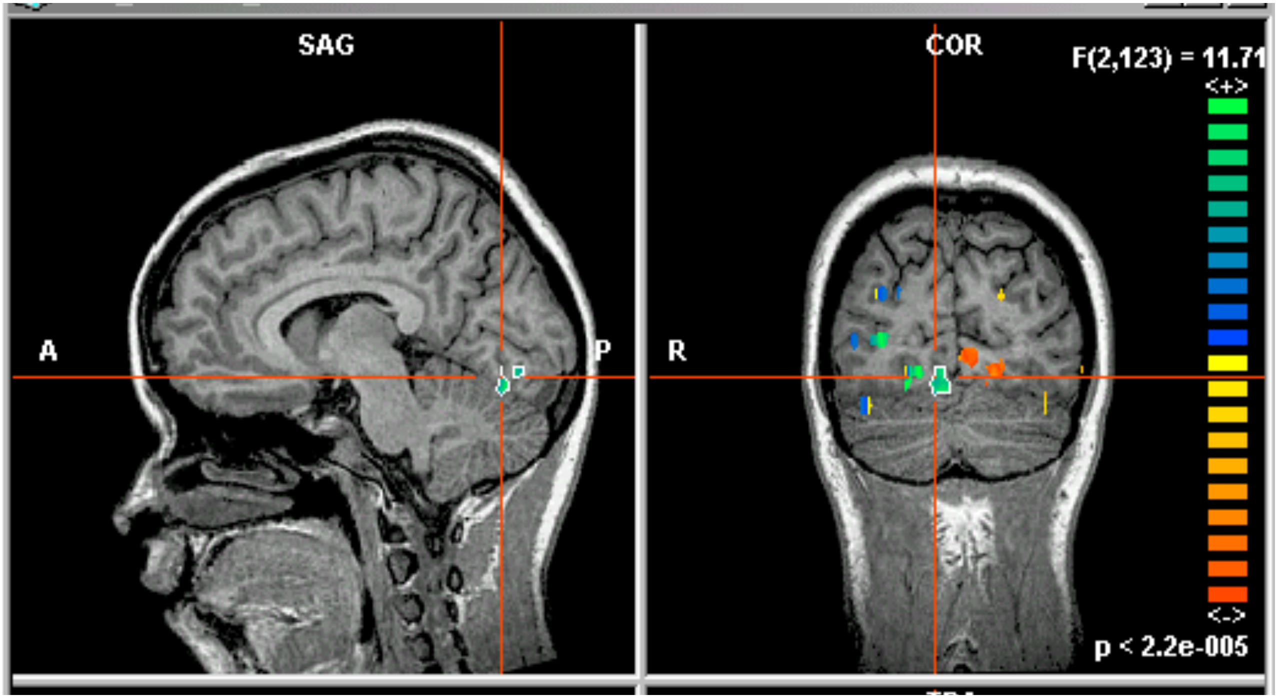
Many Slides from Tom Mitchell

What if we have continuous X_i ?

Eg., image classification: X_i is real-valued i[th] pixel



What if we have continuous X_i?

Eg., image classification: X_i is real-valued ith pixel

Naïve Bayes requires $P(X_i \mid Y=y_k)$, but X_i is real (continuous)

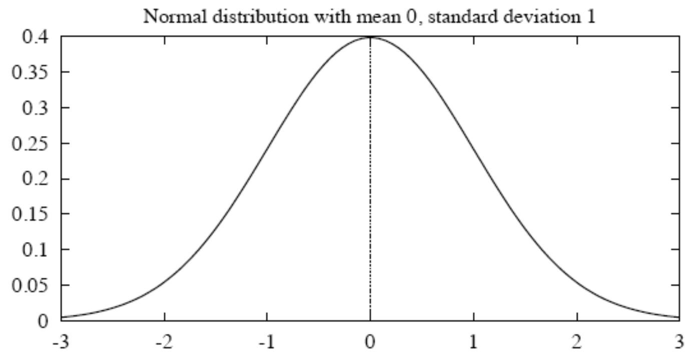
$$P(Y = y_k | X_1 \ldots X_n) = \frac{P(Y = y_k) \prod_i P(X_i | Y = y_k)}{\sum_j P(Y = y_j) \prod_i P(X_i | Y = y_j)}$$

Common approach: assume $P(X_i \mid Y=y_k)$ follows a Normal (Gaussian) distribution

Gaussian Distribution

(also called "Normal")

p(x) is a *probability density function*, whose integral (not sum) is 1



$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

The probability that X will fall into the interval (a, b) is given by

$$\int_a^b p(x)dx$$

- Expected, or mean value of X, $E[X]$, is

$$E[X] = \mu$$

- Variance of X is

$$Var(X) = \sigma^2$$

- Standard deviation of X, σ_X, is

$$\sigma_X = \sigma$$

Gaussian Naïve Bayes Algorithm – continuous X_i (but still discrete Y)

- Train Naïve Bayes (examples)

  for each value y_k

  estimate* $\pi_k \equiv P(Y = y_k)$

  for each attribute X_i estimate $P(X_i|Y = y_k)$

  - class conditional mean μ_{ik}, variance σ_{ik}

- Classify (X^{new})

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new}|Y = y_k)$$

$$Y^{new} \leftarrow \arg\max_{y_k} \; \pi_k \prod_i \mathcal{N}(X_i^{new}; \mu_{ik}, \sigma_{ik})$$

* probabilities must sum to 1, so need estimate only n-1 parameters...

Estimating Parameters: Y discrete, X_i continuous

Maximum likelihood estimates:

jth training example

$$\widehat{\mu}_{ik} = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j X_i^j \delta(Y^j = y_k)$$

ith feature

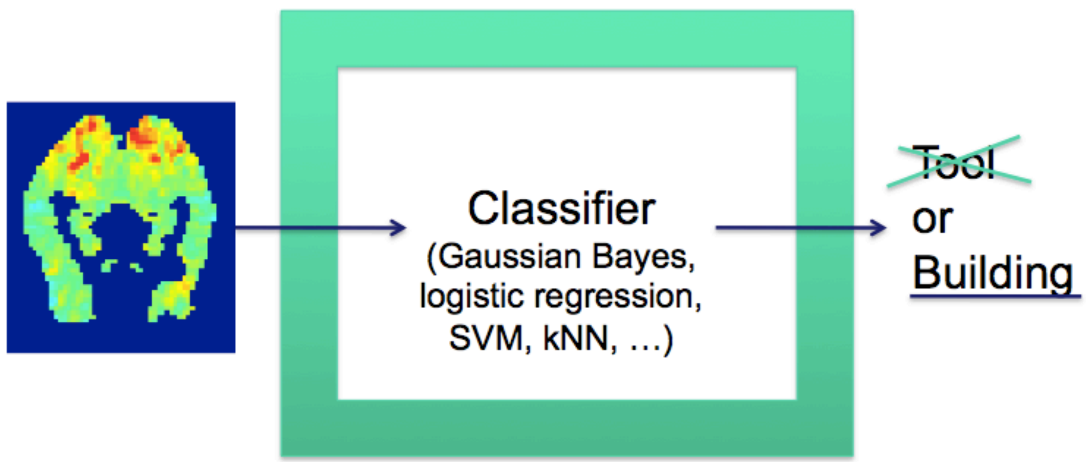
kth class

δ()=1 if (Y^j=y_k) else 0

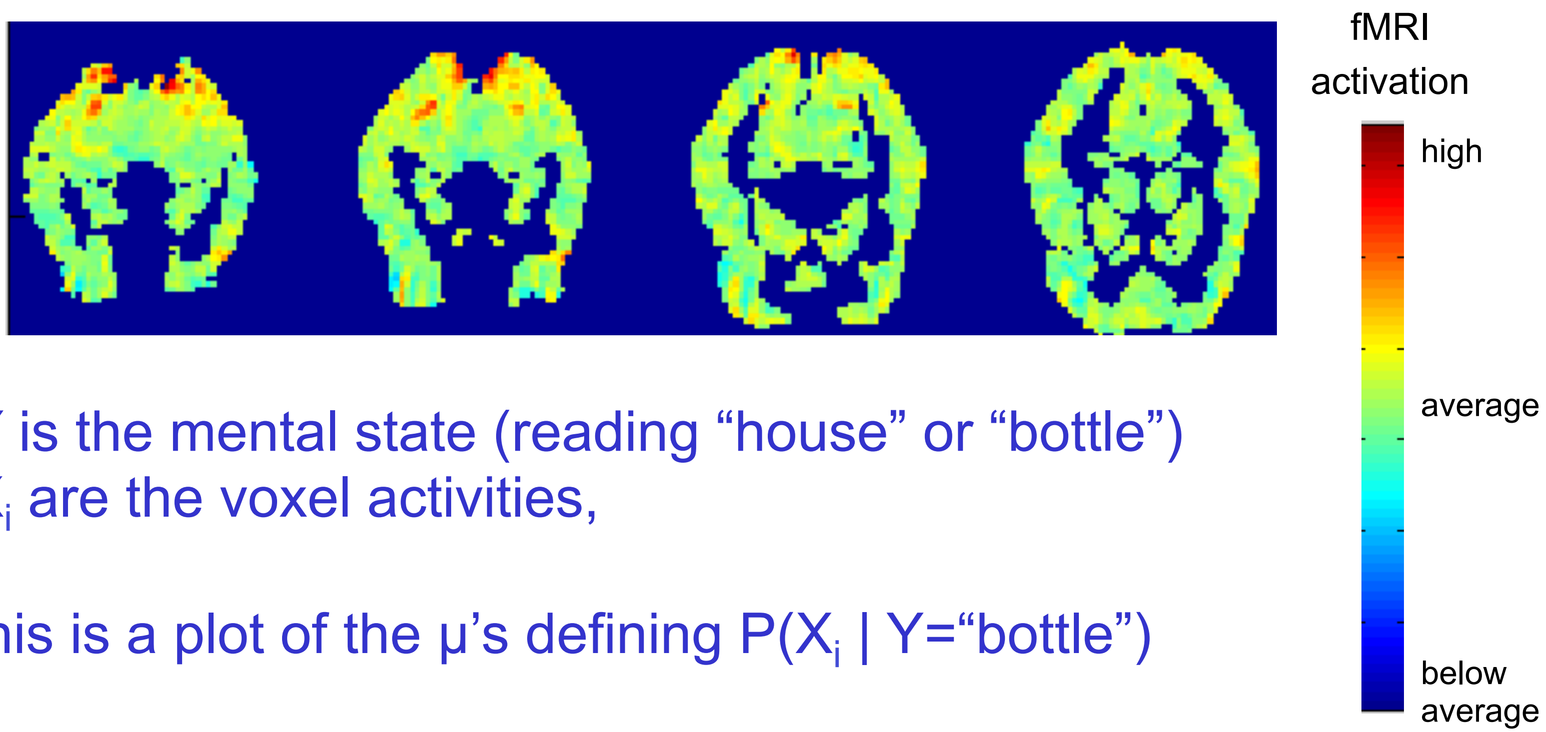
$$\widehat{\sigma}_{ik}^2 = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j (X_i^j - \widehat{\mu}_{ik})^2 \delta(Y^j = y_k)$$

GNB Example: Classify a person's cognitive state, based on brain image

- reading a sentence or viewing a picture?
- reading the word describing a "Tool" or "Building"?
- answering the question, or getting confused?



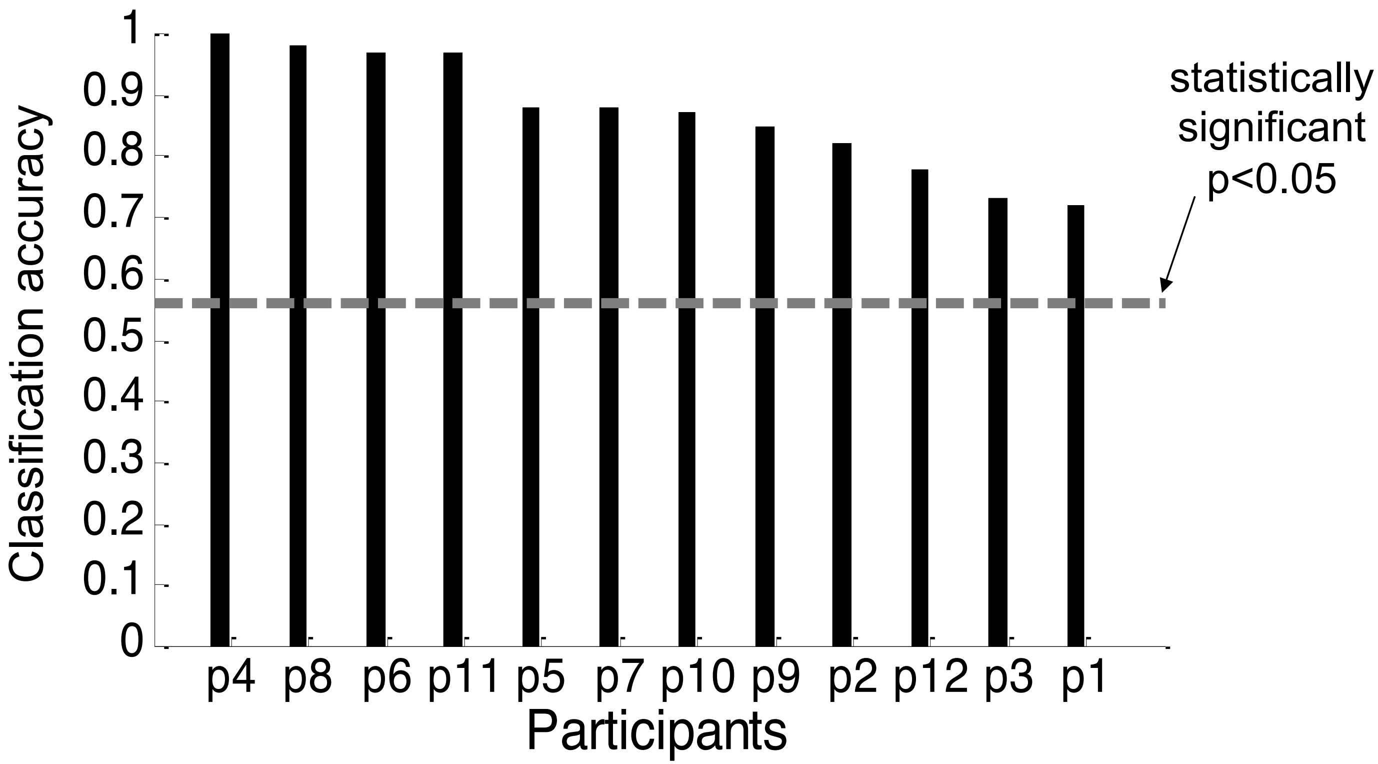
Mean activations over all training examples for Y="bottle"



Y is the mental state (reading "house" or "bottle")
X_i are the voxel activities,

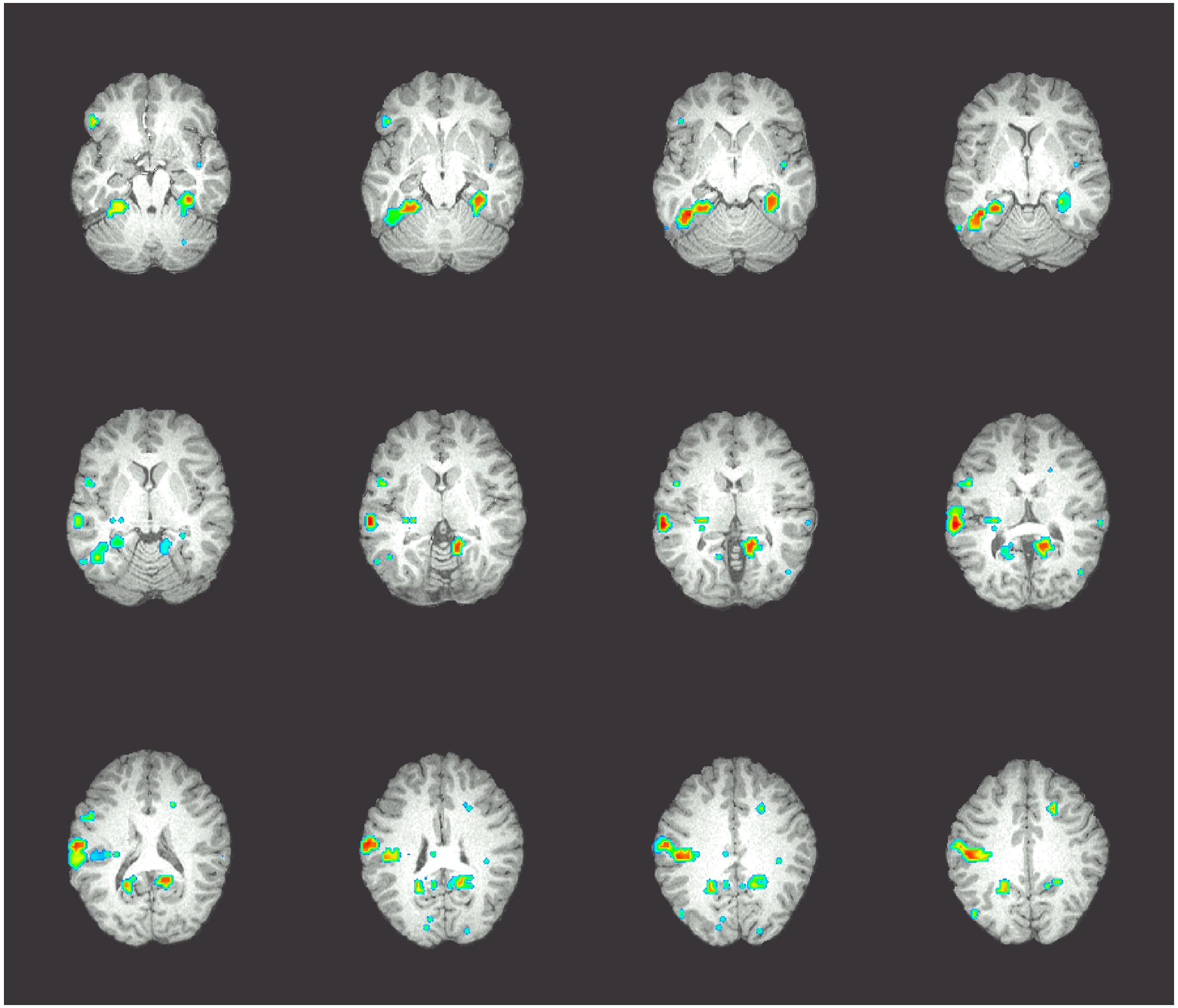
this is a plot of the μ's defining $P(X_i \mid Y=\text{"bottle"})$

Classification task: is person viewing a “tool” or “building”?



Where is information encoded in the brain?

Accuracies of cubical 27-voxel classifiers centered at each significant voxel [0.7-0.8]



Naïve Bayes: What you should know

- Designing classifiers based on Bayes rule
- Conditional independence
  - What it is
  - Why it's important
- Naïve Bayes assumption and its consequences
  - Which (and how many) parameters must be estimated under different generative models (different forms for $P(X|Y)$)
    - and why this matters
- How to train Naïve Bayes classifiers
  - MLE and MAP estimates
  - with discrete and/or continuous inputs X_i

Naïve Bayes with Log Probabilities

$$
\begin{aligned}
c_{MAP} &= \operatorname{argmax}_c P(c|x_1, \ldots, x_n) \\
&= \operatorname{argmax}_c P(c) \prod_{i=1}^{n} P(x_i|c) \\
&= \operatorname{argmax}_c \log\left(P(c) \prod_{i=1}^{n} P(x_i|c) \right) \\
&= \operatorname{argmax}_c \log P(c) + \sum_{i=1}^{n} \log P(x_i|c)
\end{aligned}
$$

What if we want to calculate posterior log-probabilities?

$$P(c|x_1,\ldots,x_n) = \frac{P(c)\prod_{i=1}^{n} P(x_i|c)}{\sum_{c'} P(c')\prod_{i=1}^{n} P(x_i|c')}$$

What if we want to calculate posterior log-probabilities?

$$P(c|x_1, \ldots, x_n) = \frac{P(c) \prod_{i=1}^{n} P(x_i|c)}{\sum_{c'} P(c') \prod_{i=1}^{n} P(x_i|c')}$$

$$\log P(c|x_1, \ldots, x_n) = \log \frac{P(c) \prod_{i=1}^{n} P(x_i|c)}{\sum_{c'} P(c') \prod_{i=1}^{n} P(x_i|c')}$$

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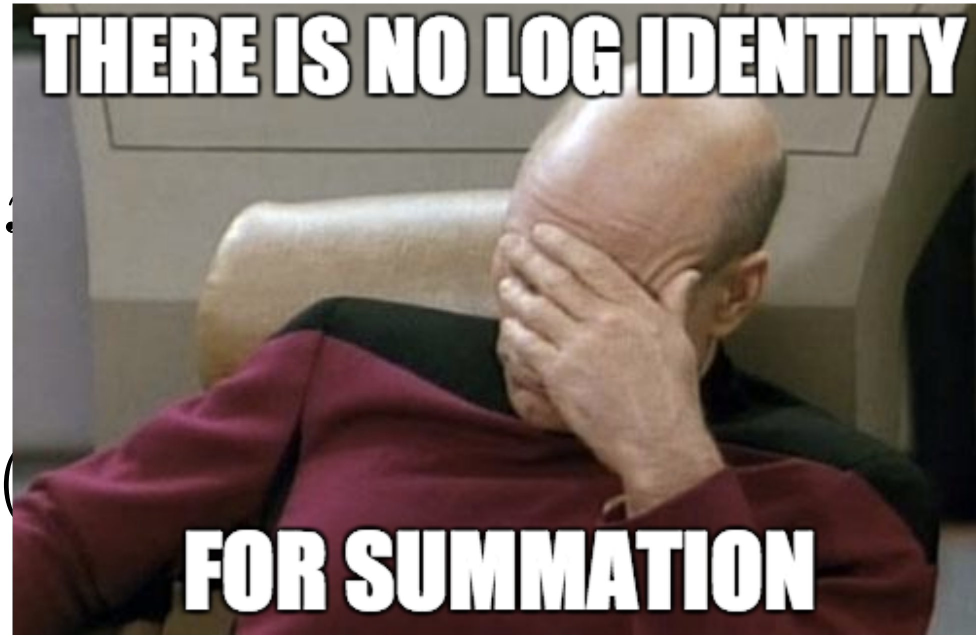
$$= \log P(c) + \sum_{i=1}^n \log P(x_i|c) - \log\left[\sum_{c'} P(c')\prod_{i=1}^n P(x_i|c')\right]$$

What if we want to calculate posterior log-probabilities?



$$P(c| \quad \frac{\prod_{i=1}^{n} P(x_i|c)}{\prod_{i=1}^{n} P(x_i|c')}$$

$$\log P(\quad \frac{\prod_{i=1}^{n} P(x_i|c)}{c') \prod_{i=1}^{n} P(x_i|c')}$$

$$= \log P(c) + \sum_{i=1}^{n} \log P(x_i|c) - \log \left[\sum_{c'} P(c') \prod_{i=1}^{n} P(x_i|c') \right]$$

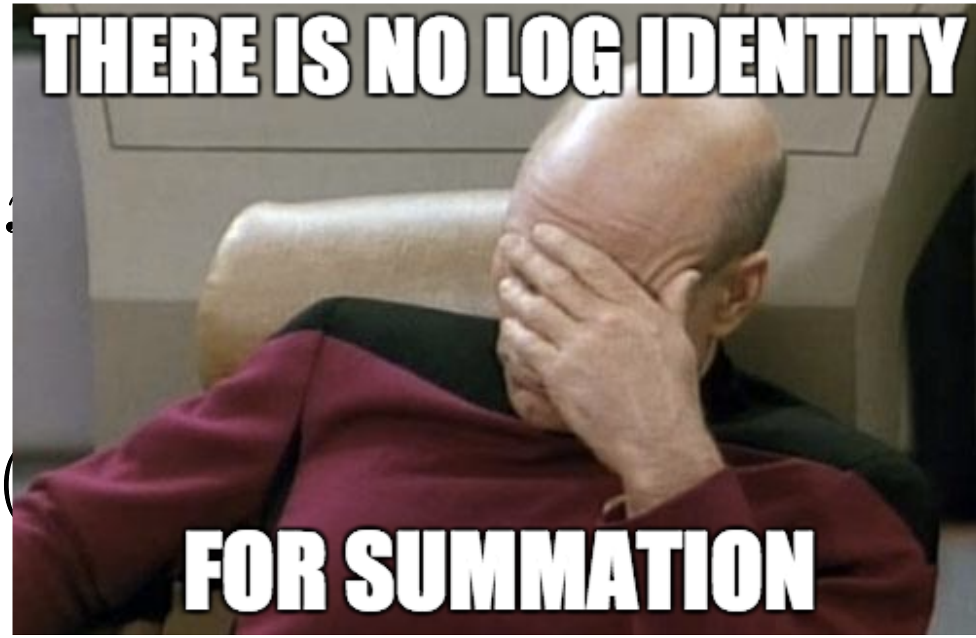
Log Exp Sum Trick

- We have: a bunch of log probabilities.
  - log(p1), log(p2), log(p3), … log(pn)
- We want: log(p1 + p2 + p3 + … pn)

Log Exp Sum Trick:

$$\log[\sum_i \exp(x_i)] = x_{max} + \log[\sum_i \exp(x_i - x_{max})]$$

What if we want to calculate posterior log-probabilities?



$$P(c| \ldots \prod_{i=1}^{n} P(x_i|c) \Big/ \ldots \prod_{i=1}^{n} P(x_i|c')$$

$$\log P(\ldots \prod_{i=1}^{n} P(x_i|c) \Big/ \ldots c') \prod_{i=1}^{n} P(x_i|c')$$

$$= \log P(c) + \sum_{i=1}^{n} \log P(x_i|c) - \log \left[\sum_{c'} P(c') \prod_{i=1}^{n} P(x_i|c') \right]$$

Linear Regression

Regression

So far, we've been interested in learning P(Y|X) where Y has discrete values (called 'classification')

What if Y is continuous? (called 'regression')

- predict weight from gender, height, age, …
- predict Google stock price today from Google, Yahoo, MSFT prices yesterday
- predict each pixel intensity in robot's current camera image, from previous image and previous action

Regression

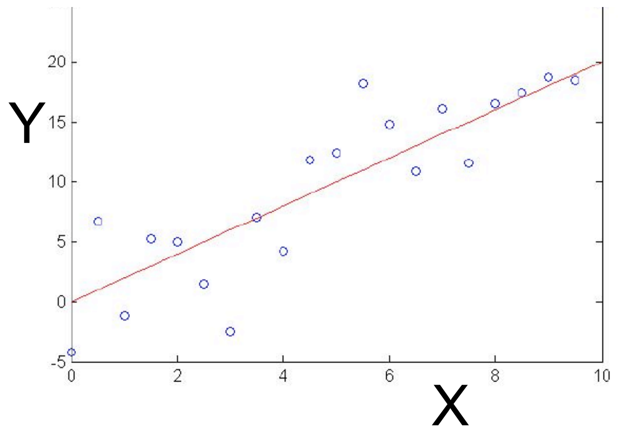
Wish to learn $f:X \rightarrow Y$, where Y is real, given $\{<x^1,y^1>\ldots<x^n,y^n>\}$

Approach:

1. choose some parameterized form for $P(Y|X; \theta)$
   (θ is the vector of parameters)

2. derive learning algorithm as MCLE or MAP estimate for θ

1. Choose parameterized form for P(Y|X; θ)



Assume Y is some deterministic f(X), plus random noise

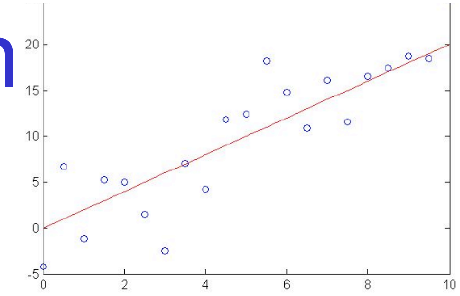
$$y = f(x) + \epsilon \qquad \text{where} \quad \epsilon \sim N(0, \sigma)$$

Therefore Y is a random variable that follows the distribution

$$p(y|x) = N(f(x), \sigma)$$

and the expected value of y for any given x is f(x)

Training Linear Regression



$$p(y|x;W) = N(w_0 + w_1 x, \sigma)$$

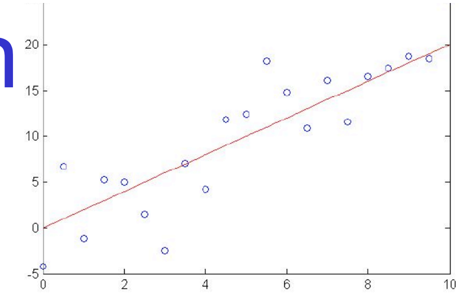
How can we learn W from the training data?

Maximum Likelihood Estimation Recipe

1. Use the log-likelihood
2. Differentiate with respect to the parameters
3. *Equate to zero and solve

*Often requires numerical approximation (no closed form solution)

Training Linear Regression



$$p(y|x;W) = N(w_0 + w_1 x, \sigma)$$

How can we learn W from the training data?

Learn Maximum Conditional Likelihood Estimate!

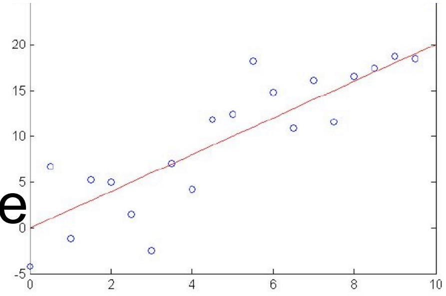
$$W_{MCLE} = \arg\max_W \prod_l p(y^l|x^l, W)$$

$$W_{MCLE} = \arg\max_W \sum_l \ln p(y^l|x^l, W)$$

where

$$p(y|x;W) = \frac{1}{\sqrt{2\pi\sigma^2}} \; e^{-\frac{1}{2}\left(\frac{y-f(x;W)}{\sigma}\right)^2}$$

Training Linear Regression

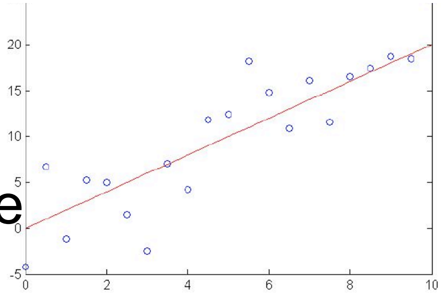


Learn Maximum Conditional Likelihood Estimate

$$W_{MCLE} = \arg\max_{W} \sum_{l} \ln p(y^l | x^l, W)$$

where $$p(y|x;W) = \frac{1}{\sqrt{2\pi\sigma^2}} \; e^{-\frac{1}{2}(\frac{y-f(x;W)}{\sigma})^2}$$

Training Linear Regression



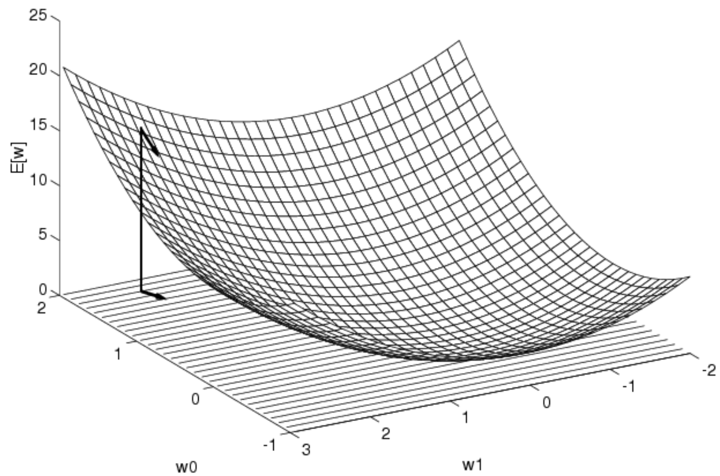
Learn Maximum Conditional Likelihood Estimate

$$W_{MCLE} = \arg\max_W \sum_l \ln p(y^l | x^l, W)$$

where $$p(y|x; W) = \frac{1}{\sqrt{2\pi\sigma^2}} \; e^{-\frac{1}{2}(\frac{y - f(x;W)}{\sigma})^2}$$

so: $$W_{MCLE} = \arg\min_W \sum_l (y - f(x; W))^2$$

Gradient Descent



Gradient

$$\nabla E[\vec{w}] \equiv \left[\frac{\partial E}{\partial w_0}, \frac{\partial E}{\partial w_1}, \cdots \frac{\partial E}{\partial w_n}\right]$$

Training rule:

$$\Delta \vec{w} = -\eta \nabla E[\vec{w}]$$

i.e.,

$$\Delta w_i = -\eta \frac{\partial E}{\partial w_i}$$

Gradient Descent:

Batch gradient: use error $E_D(\mathbf{w})$ over entire training set D

Do until satisfied:

1. Compute the gradient $\nabla E_D(\mathbf{w}) = \left[\frac{\partial E_D(\mathbf{w})}{\partial w_0} \dots \frac{\partial E_D(\mathbf{w})}{\partial w_n}\right]$
2. Update the vector of parameters: $\mathbf{w} \leftarrow \mathbf{w} - \eta \nabla E_D(\mathbf{w})$

Stochastic gradient: use error $E_d(\mathbf{w})$ over single examples $d \in D$

Do until satisfied:

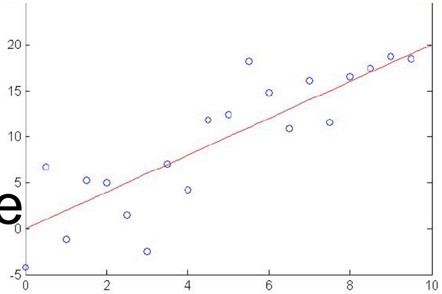
1. Choose (with replacement) a random training example $d \in D$
2. Compute the gradient just for d: $\nabla E_d(\mathbf{w}) = \left[\frac{\partial E_d(\mathbf{w})}{\partial w_0} \dots \frac{\partial E_d(\mathbf{w})}{\partial w_n}\right]$
3. Update the vector of parameters: $\mathbf{w} \leftarrow \mathbf{w} - \eta \nabla E_d(\mathbf{w})$

Stochastic approximates Batch arbitrarily closely as $\eta \to 0$

Stochastic can be much faster when D is very large

Intermediate approach: use error over subsets of D

Training Linear Regression



Learn Maximum Conditional Likelihood Estimate

$$W_{MCLE} = \arg\min_{W} \sum_{l} (y - f(x;W))^2$$

Can we derive gradient descent rule for training?

$$\frac{\partial \sum_l (y - f(x;W))^2}{\partial w_i} = \sum_l 2(y - f(x;W)) \frac{\partial (y - f(x;W))}{\partial w_i}$$

$$= \sum_l -2(y - f(x;W)) \frac{\partial f(x;W)}{\partial w_i}$$

Normal Equation

$$w^* = (X^T X)^{-1} X^T y$$

How About MAP instead of MLE?

$$w^* = \arg\max_W \sum_l \ln P(Y^l | X^l; W) + \ln N(W | 0, I)$$

$$= \arg\max_W \sum_l \ln P(Y^l | X^l; W) - c \sum_i w_i^2$$

Regression - What you should know

- MLE -> Sum of Squared Errors
- MAP -> Sum of Squared Errors minus sum of squared weights
- Learning is an optimization problem once we choose objective function
  - Maximize Data Likelihood
  - Maximize Posterior Prob of Weights
- Can use Gradient Descent as General Learning Algorithm