Correction on Clicker question for Lecture 25

Given the LP problem $\max_{A\vec{x}\le\vec{b},\vec{x}\ge\vec{0}} \vec{c}\cdot\vec{x}$, consider its primal and dual dictionaries determined by $B, N, \vec{b}, \vec{c}$:

$$\textit{primal}\qquad \begin{array}{ccccc} z & = & \vec{c}_B^T B^{-1}\vec{b} & +[\vec{c}_N^T - \vec{c}_B^T B^{-1}N]\vec{x}_N \\ \hline \vec{x}_B & = & B^{-1}\vec{b} & -B^{-1}N\vec{x}_N \end{array}$$

$$\textit{dual}\qquad \begin{array}{ccccc} -w & = & -\vec{c}_B^T B^{-1}\vec{b} & -\left(B^{-1}\vec{b}\right)^T \vec{y}_B \\ \hline \vec{y}_N & = & -[\vec{c}_N^T - \vec{c}_B^T B^{-1}N]^T & +\left(B^{-1}N\right)^T \vec{y}_B \end{array}$$

Let $\vec{x}^*, \vec{y}^*$ be the corresponding **primal/dual basic solutions**, respectively. Find a wrong statement:

A. If $\vec{x}^*$, $\cancel{\vec{y}^*}$ ~~are~~ is primal/~~dual~~ optimal, ~~respectively~~, then $\vec{x}_B^* \ge 0$ and $\vec{y}_N^* \ge 0$. **[I didn't mean to include $\vec{y}^*$ in this sentence. After this correction, A. is the answer.]**

B. If $\vec{x}_B^* \ge 0$ and $\vec{y}_N^* \ge 0$, then $\vec{x}^*, \vec{y}^*$ are primal/dual optimal, respectively.

C. $\vec{x}_N^* = \vec{0}$, and $\vec{y}_B^* = \vec{0}$.

D. One of A, B, and C is wrong. **[In the original problem before the correction, D was the answer.]**

Comparison with strong duality theorem

primal | *dual*

$$\begin{array}{rcll} z & = & \vec{c}_B^T \vec{x}_B^* & -(\vec{y}_N^*)^T \vec{x}_N \\ \hline \vec{x}_B & = & \vec{x}_B^* & -B^{-1} N \vec{x}_N \end{array} \qquad \begin{array}{rcll} -w & = & -(\vec{y}^*)^T \vec{b} & -\left(\vec{x}_B^*\right)^T \vec{y}_B \\ \hline \vec{y}_N & = & \vec{y}_N^* & +\left(B^{-1} N\right)^T \vec{y}_B \end{array}.$$

- At the feasible and dual feasible dictionary (that is, $\vec{x}_B^* \geq \vec{0}$, $\vec{y}_N^* \geq \vec{0}$), the corresponding primal/dual basic solutions $\vec{x}^*, \vec{y}^*$ are primal/dual optimal, respectively.
- Optimality of the primal basic solution $\vec{x}^*$ **does not necessarily** imply dual optimality of the dual basic solution $\vec{y}^*$ as it may not be dual feasible (it may happen $\vec{y}_N^* \not\geq \vec{0}$).
- **Strong duality** says if there is a primal optimal solution then there is a dual optimal solution.
  - It **does not mean** that for an optimal basic solution, the corresponding dual basic solution is dual optimal. The corresponding dual basic solution may not be feasible, **unless** the dictionary is **final**, that is, both primal/dual feasible.
  - In the strong duality theorem, you can read off the dual optimal solution from the dictionary, **only when the dictionary is final, where the corresponding dual basic solution has $\vec{y}_N^* \geq \vec{0}$.**

Basic idea for sensitivity analysis

How can we find the new optimal solution in the new situation, using the old optimal solution of the problem, **without having to solve the new problem from scratch**?

- Changing $\vec{c}$
  - affects $\vec{c}_B, \vec{c}_N$ (so $[\vec{c}_N^T - \vec{c}_B^T B^{-1} N]$),
  - but it does not affect $B, N, \vec{x}_B^* = B^{-1}\vec{b}$.
  - It does not affect feasibility of the primal basic solution.
- Changing $\vec{b}$
  - affects $\vec{x}_B^* = B^{-1}\vec{b}$
  - but it does not affect $B, N, \vec{c}_B, \vec{c}_N, [\vec{c}_N^T - \vec{c}_B^T B^{-1} N]$, & $(\vec{y}^*)^T = \vec{c}_B^T B^{-1}$.
  - It does not affect dual feasibility of the dual basic solution.
- Under the changes, we may **recycle unaffected items.**

Example

$$\begin{array}{llrrrcll}
\text{Maximize} & z = & x_1 & +x_2 & & & & \\
\text{subject to} & & x_1 & +2x_2 & +x_3 & \leq & 2 & \quad \& \quad x_1, x_2, x_3 \geq 0, \\
& & 2x_1 & +x_2 & +x_3 & \leq & 2 &
\end{array}$$

We got an optimal feasible dictionary with

$$\vec{x}_B^* = \begin{bmatrix} x_2^* \\ x_1^* \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}, \quad N = \begin{array}{c} \begin{array}{ccc} x_5 & x_4 & x_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}, \quad B = \begin{array}{c} \begin{array}{cc} x_2 & x_1 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array},$$

$$\vec{c}_N^T = \begin{array}{c} \begin{array}{ccc} x_5 & x_4 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \vec{c}_B^T = \begin{array}{c} \begin{array}{cc} x_2 & x_1 \end{array} \\ \begin{bmatrix} 1 & 1 \end{bmatrix} \end{array},$$

and $(\vec{y}^*)^T = \vec{c}_B^T B^{-1} = \begin{bmatrix} y_1^* & y_2^* \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \end{bmatrix}$ is the optimal dual solution.

Question 2: In the same problem, how much can we change $\vec{b}$ to $\vec{b} = (2, 2+s)$ still keeping the same dual optimal $\vec{y}^*$? Find the range for s. What happens to the primal optimal solution?

- Can recycle $B, N, \vec{c}_B, \vec{c}_N$, $[\vec{c}_N^T - \vec{c}_B^T B^{-1} N]$, & $(\vec{y}^*)^T = \vec{c}_B^T B^{-1}$.
- $\vec{y}^*$ is dual feasible, its dual feasibility is not affected by the change of $\vec{b}$.
- So, it is still dual optimal if the new $\vec{x}_B^* = B^{-1}\vec{b}$ is primal feasible, i.e. $\vec{x}_B^* \geq \vec{0}$.
- In fact, the last line is "an if and only if statement" that is, if the new $\vec{x}_B^* = B^{-1}\vec{b}$ is primal feasible, i.e. $\vec{x}_B^* \geq \vec{0}$, then the same $\vec{y}^*$ is still dual optimal.
  It is because $\vec{y}^*$ is **nondegenerate** and from the result of the theorem in Lecture 25.
  -

$$(\vec{y}_N^*)^T = -[\vec{c}_N^T - (\vec{y}^*)^T N]$$

$$= -\left[\begin{array}{c} \begin{array}{ccc} x_5 & x_4 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{array} - \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 1/3 & 1/3 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_4 & x_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \right] > \vec{0}$$

- This gives us range of s. The rest is matter of computation.
- Note that for such a range of s, the corresponding $\vec{x}_B^*$ is primal optimal.

Dual Simplex Method

Recall **primal /dual correspondence.**
For $\max_{A\vec{x}\leq\vec{b},\vec{x}\geq 0} \vec{c}^T\vec{x}$, with A is $m \times n$:

- $x_j \longleftrightarrow y_{m+j}, \quad j = 1, ..., n$
- $x_{n+i} \longleftrightarrow y_i, \quad i = 1, ..., m$
- $\vec{x}_B \longleftrightarrow \vec{y}_B$
- $\vec{x}_N \longleftrightarrow \vec{y}_N$

primal   *dual*

$$\begin{array}{ccccc} z & = & \vec{c}_B^T\vec{x}_B^* & -(\vec{y}_N^*)^T\vec{x}_N \\ \hline \vec{x}_B & = & \vec{x}_B^* & -B^{-1}N\vec{x}_N \end{array} \qquad \begin{array}{ccccc} -w & = & -(\vec{y}^*)^T\vec{b} & -\left(\vec{x}_B^*\right)^T\vec{y}_B \\ \hline \vec{y}_N & = & \vec{y}_N^* & +\left(B^{-1}N\right)^T\vec{y}_B \end{array}.$$

Dual Simplex Method:Dual Pivot

$$z = -2x_1 - 2x_2 \quad \leftarrow \text{dual feasible}$$

$$\begin{aligned}
x_3 &= -1 + x_1 + 2x_2 \\
x_4 &= -1 + 2x_1 + x_2 \\
x_5 &= -0 + x_1 + x_2
\end{aligned}$$

x_3 leaving
x_2 entering

dual pivot

$$z = -1 - x_1 - x_3$$

$$\begin{aligned}
x_2 &= \tfrac{1}{2} - \tfrac{1}{2}x_1 + \tfrac{1}{2}x_3 \\
x_4 &= -\tfrac{1}{2} + \tfrac{3}{2}x_1 + \tfrac{1}{2}x_3 \\
x_5 &= \tfrac{1}{2} + \tfrac{1}{2}x_1 + \tfrac{1}{2}x_3
\end{aligned}$$

x_4 leaving
x_1 entering

dual pivot

$$z = -\tfrac{4}{3} - \tfrac{2}{3}x_4 - \tfrac{2}{3}x_3$$

$$\begin{aligned}
x_2 &= \tfrac{1}{3} + \tfrac{1}{3}x_4 + \tfrac{2}{3}x_3 \\
x_1 &= \tfrac{1}{3} + \tfrac{2}{3}x_4 - \tfrac{1}{3}x_3 \\
x_5 &= \tfrac{2}{3} + \tfrac{1}{3}x_4 + \tfrac{1}{3}x_3
\end{aligned}$$

optimal

$$\vec{x}^* = (\boxed{\tfrac{1}{3}, \tfrac{1}{3}}, 0, 0, \tfrac{2}{3})$$

mirror

$x_2 + 1 = x_3 \longleftrightarrow y_1$

$x_2 \longleftrightarrow y_5 = y_3 + 2$

mirror

$x_4 = x_2 + 2 \longleftrightarrow y_2$

$x_1 \longleftrightarrow y_4 = y_3 + 1$

$$-w = y_1 + y_2 + 0$$

$$\begin{aligned}
y_4 &= 2 - y_1 - 2y_2 - y_3 \\
y_5 &= 2 - 2y_1 - y_2 - y_3
\end{aligned}$$

pivot

y_1 entering
y_5 leaving

$$-w = 1 - \tfrac{1}{2}y_5 + \tfrac{1}{2}y_2 - \tfrac{1}{2}y_3$$

$$\begin{aligned}
y_4 &= 1 + \tfrac{1}{2}y_5 - \tfrac{3}{2}y_2 - \tfrac{1}{2}y_3 \\
y_1 &= 1 - \tfrac{1}{2}y_5 - \tfrac{1}{2}y_2 - \tfrac{1}{2}y_3
\end{aligned}$$

pivot

y_2 entering
y_4 leaving

$$-w = \tfrac{4}{3} - \tfrac{1}{3}y_5 - \tfrac{1}{3}y_4 - \tfrac{2}{3}y_3$$

$$\begin{aligned}
y_2 &= \tfrac{2}{3} - \tfrac{1}{3}y_5 - \tfrac{2}{3}y_4 - \tfrac{1}{3}y_3 \\
y_1 &= \tfrac{2}{3} - \tfrac{2}{3}y_5 + \tfrac{1}{3}y_4 - \tfrac{1}{3}y_3
\end{aligned}$$

optimal

$$\vec{y}^* = (\boxed{\tfrac{2}{3}, \tfrac{2}{3}, 0}, 0, 0)$$

Dual Simplex Method: Dual Pivot

primal

$$\begin{array}{ccccc} z & = & \vec{c}_B^T \vec{x}_B^* & -(\vec{y}_N^*)^T \vec{x}_N \\ \hline \vec{x}_B & = & \vec{x}_B^* & -B^{-1} N \vec{x}_N \end{array}$$

dual

$$\begin{array}{ccccc} -w & = & -(\vec{y}^*)^T \vec{b} & -\left(\vec{x}_B^*\right)^T \vec{y}_B \\ \hline \vec{y}_N & = & \vec{y}_N^* & +\left(B^{-1} N\right)^T \vec{y}_B \end{array}.$$

- Dual Pivot
  = the operation **on the primal dictionary** corresponding to the usual pivot on the dual dictionary.
- From a feasible dictionary ($\vec{x}_B^* \geq \vec{0}$),
  - Pivot tries to achieve dual feasibility ($\vec{y}_N^* \geq \vec{0}$).
  - Pivot keeps feasibility of the dictionary.
- From a dual feasible dictionary (i.e. $\vec{y}_N^* \geq \vec{0}$, dual dictionary is feasible).
  - Dual pivot tries to achieve primal feasibility ($\vec{x}_B^* \geq \vec{0}$).
  - Dual pivot keeps dual feasibility of the dictionary.

Primal and dual simplex methods

primal

$$\begin{array}{ccll} z & = & \vec{c}_B^T \vec{x}_B^* & -(\vec{y}_N^*)^T \vec{x}_N \\ \hline \vec{x}_B & = & \vec{x}_B^* & -B^{-1} N \vec{x}_N \end{array}$$

dual

$$\begin{array}{ccll} -w & = & -(\vec{y}^*)^T \vec{b} & -\left(\vec{x}_B^*\right)^T \vec{y}_B \\ \hline \vec{y}_N & = & \vec{y}_N^* & +\left(B^{-1} N\right)^T \vec{y}_B \end{array}.$$

For the primal problem:

- Starting from a feasible dictionary ($\vec{x}_B^* \geq \vec{0}$), can apply the simplex method to arrive at a dual feasible dictionary ($\vec{y}_N^* \geq \vec{0}$) (or get unboundedness in the end).
- Starting from a dual feasible dictionary ($\vec{y}_N^* \geq \vec{0}$), can apply the dual simplex method to arrive at a feasible dictionary ($\vec{x}_B^* \geq \vec{0}$) (or get infeasibility in the end).
- From a dictionary neither feasible or dual feasible, can apply two phase method either to the primal or the dual problem.

Dual Pivot. Example without optimal solution

primal

$$
\begin{array}{rcrrr}
z & = & 3 & -2x_1 & -x_2 \\
\hline
x_4 & = & 1 & -x_1 & -x_2 \\
x_3 & = & -2 & -3x_1 & -2x_2
\end{array}
$$

x_3 leaving but no one entering

the primal LP is infeasible

dual

$$
\begin{array}{rcrrr}
-w & = & -3 & -y_2 & +2y_1 \\
\hline
y_3 & = & 2 & +y_2 & +3y_1 \\
y_4 & = & 1 & +y_2 & +2y_1
\end{array}
$$

y_1 entering but no one leaving

the dual LP is unbounded

Clicker question for Lecture 26

For **Dual Pivot**, find a correct statement:

A. For

$$\begin{array}{rcrrr} z & = & -4/3 & -2/3x_4 & -2/3x_3 \\ \hline x_2 & = & 1/3 & +1/3x_4 & +2/3x_3 \\ x_1 & = & -1/3 & +2/3x_4 & -1/3x_3 \\ x_5 & = & 2/3 & +1/3x_4 & +1/3x_3 \end{array}$$

x_1 is leaving and x_3 is entering.

B. For

$$\begin{array}{rcrrr} z & = & 0 & -2x_1 & -2x_2 \\ \hline x_3 & = & -1 & +x_1 & +2x_2 \\ x_4 & = & -1 & +2x_1 & +x_2 \\ x_5 & = & -0 & +x_1 & +x_2 \end{array}$$

x_3 is leaving and x_1 is entering.

C. For

$$\begin{array}{rcrrr} z & = & -1 & -x_1 & -x_3 \\ \hline x_2 & = & 1/2 & -1/2x_1 & +1/2x_3 \\ x_4 & = & -1/2 & +3/2x_1 & +1/2x_3 \\ x_5 & = & 1/2 & +1/2x_1 & +1/2x_3 \end{array}$$

x_4 is leaving and x_1 is entering.