Complimentary Slackness (CS)

Theorem

Let $\vec{x}^* = (x_1^*, \cdots, x_n^*)$ be feasible for the primal problem, and let $\vec{y}^* = (y_1^*, \cdots, y_m^*)$ be feasible for the dual problem.
Then the following are equivalent:

- $\vec{x}^*$ *and* $\vec{y}^*$ *are optimal for the primal and for the dual, resp.*
-
$$\textbf{CS}: \quad \begin{array}{ll} x_j^* y_{m+j}^* = 0 & \text{for } j = 1, \cdots, n. \\ y_i^* x_{n+i}^* = 0 & \text{for } i = 1, \cdots, m. \end{array}$$

Here,

x_{n+i} is the i-th slack variable fo the primal;

y_{m+j} is the j-th slack variable fo the dual.

Example from Lecture 15

$$\begin{array}{lrrcl}
\text{Maximize} & 4x_1 & +5x_2 & & \\
\text{subject to} & x_1 & +x_2 & \leq & 5 \\
 & x_1 & +2x_2 & \leq & 8 \\
 & 2x_1 & +x_2 & \leq & 8
\end{array} \quad \& \quad x_1, x_2 \geq 0,$$

Is $(x_1^*, x_2^*) = (2, 3)$ **an optimal solution?**
Check it without solving LP from the simplex algorithm.

Solution

Consider the dual problem:

$$\begin{array}{lrrrcl}
\text{Minimize} & 5y_1 & +8y_2 & +8y_3 & & \\
\text{subject to} & y_1 & +y_2 & +2y_3 & \geq & 4 \\
 & y_1 & +2y_2 & +y_3 & \geq & 5
\end{array} \quad \& \quad x_1, x_2 \geq 0,$$

- **If** $\vec{x}^*$ **were** an optimal solution, then from the strong duality an optimal solution $\vec{y}^* = (y_1^*, y_2^*, y_3^*)$ for dual should exist.
- **We use the complimentary slackness (CS) to find the dual optimal solution** $\vec{y}^*$ **assuming the given** $(x_1^*, x_2^*) = (2, 3)$ **is optimal. (If** $\vec{x}^*$ **was optimal then** $\vec{y}^*$ **should be optimal as well.)**
- Note that here the number of primal variables $n = 2$ and the number of dual variables $m = 3$.

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- Note that here the number of primal variables $n = 2$ and the number of dual variables $m = 3$.

Example continued

From $(x_1^*, x_2^*) = (2, 3)$ compute the slack:

$$x_{2+1}^* = 5 - (2+3) = 0, \quad x_{2+2}^* = 8 - (2 + 2 \cdot 3) = 0,$$
$$x_{2+3}^* = 8 - (2 \cdot 2 + 3) = 1 > 0.$$

($\vec{x}^*$ is feasible as all original and slack variables are ≥ 0.)

$$\text{Use CS:} \begin{pmatrix} x_1^* y_{3+1}^* = 0 \\ x_2^* y_{3+2}^* = 0 \\ x_{2+1}^* y_1^* = 0 \\ x_{2+2}^* y_2^* = 0 \\ x_{2+3}^* y_3^* = 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} y_{3+1}^* = 0 \text{ for } x_1^* = 2 > 0 \\ y_{3+2}^* = 0 \text{ for } x_2^* = 3 > 0 \\ \text{no info for } y_1^* \text{ as } x_{2+1}^* = 0 \\ \text{no info for } y_2^* \text{ as } x_{2+2}^* = 0 \\ y_3^* = 0 \text{ for } x_{2+3}^* > 0 \end{pmatrix}$$

Now,

$$\begin{array}{l} y_{3+1}^* = 0 \\ y_{3+2}^* = 0 \\ y_3^* = 0 \end{array} \Longrightarrow \begin{array}{l} y_1^* + y_2^* + 2y_3^* = 4 \\ y_1^* + 2y_2^* + y_3^* = 5 \\ y_3^* = 0 \end{array} \Longrightarrow (y_1^*, y_2^*, y_3^*) = (3, 1, 0)$$

Note that we used the **full** set of CS equations to get $\vec{y}^*$; therefore, $(\vec{x}^*, \vec{y}^*)$ satisfies CS.

Example continued

So, from the CS theorem, to check optimality, it remains to check feasibility of $\vec{x}^*$ and $\vec{y}^*$.

- Already checked feasibility of $\vec{x}^*$.
- For feasibility of $\vec{y}^*$:
  - $y_1^* = 3, y_2^* = 1, y_3^* = 0$ so ≥ 0.
  - We had $y_{3+1}^* = 0, y_{3+2}^* = 0$ so ≥ 0.

  Thus $\vec{y}^*$ is feasible.

Therefore, $(\vec{x}^*, \vec{y}^*)$ is optimal for (primal, dual). $\square$

Remark: After checking the feasibility, one can check the optimality by looking

$$\vec{c} \cdot \vec{x}^* = 4 \cdot 2 + 5 \cdot 3 = 23$$

$$\vec{b} \cdot \vec{y}^* = 5 \cdot 3 + 8 \cdot 1 + 8 \cdot 0 = 23$$

They are equal (for the feasible $(\vec{x}^*.\vec{y}^*)$), so weak duality implies that $(\vec{x}^*.\vec{y}^*)$ is optimal.

Different Example

$$\begin{array}{lrrcl} \text{Maximize} & 4x_1 & +5x_2 & & \\ \text{subject to} & x_1 & +x_2 & \leq & 5 \\ & x_1 & +2x_2 & \leq & 8 \\ & 2x_1 & +x_2 & \leq & 8 \end{array} \quad \& \quad x_1, x_2 \geq 0,$$

Is $(x_1^*, x_2^*) = (3, 2)$ an optimal solution?
Check it without solving LP from the simplex algorithm.

Solution
Consider the dual problem:

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Example continued

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Now,

$$\begin{matrix} y_{3+1}^* = 0 \\ y_{3+2}^* = 0 \\ y_2^* = 0 \end{matrix} \implies \begin{matrix} y_1^* + y_2^* + 2y_3^* = 4 \\ y_1^* + 2y_2^* + y_3^* = 5 \\ y_2^* = 0 \end{matrix} \implies (y_1^*, y_2^*, y_3^*) = (6, 0, -1)$$

Note that $\vec{y}^*$ is not feasible (to the dual) because $y_3^* < 0$; so it is not optimal. This **violates** the CS theorem. Thus, $\vec{x}^*$ cannot be optimal.