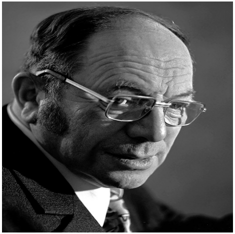
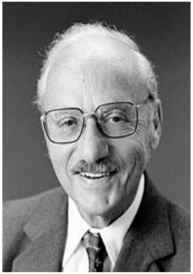
Lecture 30: Matrix Games. Duality

John von Neumann



Leonid Kantorovich



George Dantzig

Lecture 30-Monday. Nov 18. Lecture 31. Wed..

- duality between player 1 and player 2 problems.
- minimax theorem
- fairness of a game.
- suggested exercises Chvatal 15.6, 15.7, 15.8

Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$ be the payoff matrix for Player 1.

Player 1's problem

$$\begin{array}{ll} \text{Max} & v \\ \text{subj.} & v \leq x_1 - 3x_2 \\ & v \leq -2x_1 + 4x_2 \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{ll} \text{Max} & v \\ \text{subj.} & v\vec{\mathbf{e}}_n \leq A^T\vec{x} \\ & \vec{\mathbf{e}}_m^T\vec{x} = 1 \\ & \vec{x} \geq \vec{0}_m \end{array}$$

Player 2's problem

$$\begin{array}{ll} \text{Min} & u \\ \text{subj.} & u \geq y_1 - 2y_2, \\ & u \geq -3y_1 + 4y_2 \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{array}$$

$$\begin{array}{ll} \text{Min} & u \\ \text{subj.} & u\vec{\mathbf{e}}_m \geq A\vec{y} \\ & \vec{\mathbf{e}}_n^T\vec{y} = 1 \\ & \vec{y} \geq \vec{0}_n \end{array}$$

The LP problem for Player 1 $\max_{\vec{x} \text{ stoch.}} \left[\min_{\vec{y} \text{ stoch.}} \vec{x}^T A \vec{y} \right]$

The LP problem for Player 1:

$$\begin{array}{rr} \text{Maximize} & v \\ \text{subject to} & v\vec{\mathbf{e}}_n \leq A^T\vec{x} \\ & \vec{\mathbf{e}}_m^T\vec{x} = 1 \\ & \vec{x} \geq \vec{0}_m \end{array}$$

Standard form:

$$\begin{array}{rr} \text{Maximize} & v_+ - v_- \\ \text{subject to} & -A^T\vec{x} + (v_+ - v_-)\vec{\mathbf{e}}_n \leq \vec{0}_n \\ & \vec{\mathbf{e}}_m^T\vec{x} \leq 1 \\ & -\vec{\mathbf{e}}_m^T\vec{x} \leq -1 \\ & \vec{x} \geq \vec{0}_m \;\&\; v_+, v_- \geq 0 \end{array}$$

LP problem for Player 1 $\max_{\vec{x}\text{ stoch.}} \left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$ in Standard Form:

$$
\begin{array}{ll}
\text{Maximize} & v_+ - v_- \\
\text{subject to} & -A^T\vec{x} + (v_+ - v_-)\vec{\mathbf{e}}_n \le \vec{0}_n \\
& \vec{\mathbf{e}}_m^T\vec{x} \le 1 \\
& -\vec{\mathbf{e}}_m^T\vec{x} \le -1 \\
& \vec{x} \ge \vec{0}_m \ \& \ v_+, v_- \ge 0
\end{array}
$$

In matrices:

Maximise $\begin{bmatrix} \vec{0}_m^T & 1 & -1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ v_+ \\ v_- \end{bmatrix}$

Subject to

$$
\begin{bmatrix} -A^T & \vec{e}_n & -\vec{e}_n \\ \vec{e}_m^T & 0 & 0 \\ -\vec{e}_m^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \\ v_+ \\ v_- \end{bmatrix} \le \begin{bmatrix} \vec{0}_n \\ 1 \\ -1 \end{bmatrix}
$$

$$
\begin{bmatrix} \vec{x}^T & v_+ & v_- \end{bmatrix} \ge \vec{0}^T
$$

LP problem for Player 2 $\min_{\vec{y} \text{ stoch.}} \left[\max_{\vec{x} \text{ stoch.}} \vec{x}^T A \vec{y} \right]$

The LP problem for Player 2:

$$\begin{array}{rr} \text{Minimize} & u \\ \text{subject to} & u\vec{\mathbf{e}}_m \geq A\vec{y} \\ & \vec{\mathbf{e}}_n^T \vec{y} = 1 \\ & \vec{y} \geq \vec{0} \end{array}$$

Standard minimization form:

$$\begin{array}{rr} \text{Minimize} & u_+ - u_- \\ \text{subject to} & -A\vec{y} + (u_+ - u_-)\vec{\mathbf{e}}_m \geq \vec{0}_m \\ & \vec{\mathbf{e}}_n^T \vec{y} \geq 1 \\ & -\vec{\mathbf{e}}_n^T \vec{y} \geq -1 \\ & \vec{y} \geq \vec{0}_n \ \& \ u_+, u_- \geq 0 \end{array}$$

LP problem for Player 2 $\min\limits_{\vec{y}\text{ stoch.}} \left[\max\limits_{\vec{x}\text{ stoch.}} \vec{x}^T A\vec{y}\right]$ in Standard Form:

$$\begin{array}{rr} \text{Minimize} & u_+ - u_- \\ \text{subject to} & -A\vec{y} + (u_+ - u_-)\vec{\mathbf{e}}_m \geq \vec{0}_m \\ & \vec{\mathbf{e}}_n^T \vec{y} \geq 1 \\ & -\vec{\mathbf{e}}_n^T \vec{y} \geq -1 \\ & \vec{y} \geq \vec{0}_n \ \&\ u_+, u_- \geq 0 \end{array}$$

In matrices:

Minimize $\begin{bmatrix} \vec{0}_n^T & 1 & -1 \end{bmatrix} \begin{bmatrix} \vec{y} \\ u_+ \\ u_- \end{bmatrix}$

Subject to

$$\begin{bmatrix} -A & \vec{e}_m & -\vec{e}_m \\ \vec{e}_n^T & 0 & 0 \\ -\vec{e}_n^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{y} \\ u_+ \\ u_- \end{bmatrix} \geq \begin{bmatrix} \vec{0}_m \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \vec{y}^T & u_+ & u_- \end{bmatrix} \geq \vec{0}^T$$

These two problems are **Dual to each other**!

$$\max_{\vec{x}\text{ stoch.}} \left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$$

$$Max. \begin{bmatrix} \vec{0}_m^T & 1 & -1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ v_+ \\ v_- \end{bmatrix}$$

$$Subj.to \begin{bmatrix} -A^T & \vec{e}_n & -\vec{e}_n \\ \vec{e}_m^T & 0 & 0 \\ -\vec{e}_m^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \\ v_+ \\ v_- \end{bmatrix} \leq \begin{bmatrix} \vec{0}_n \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \vec{x}^T & v_+ & v_- \end{bmatrix} \geq \vec{0}^T$$

$$\min_{\vec{y}\text{ stoch.}} \left[\max_{\vec{x}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$$

$$Min. \begin{bmatrix} \vec{0}_n^T & 1 & -1 \end{bmatrix} \begin{bmatrix} \vec{y} \\ u_+ \\ u_- \end{bmatrix}$$

$$Subj.to \begin{bmatrix} -A & \vec{e}_m & -\vec{e}_m \\ \vec{e}_n^T & 0 & 0 \\ -\vec{e}_n^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{y} \\ u_+ \\ u_- \end{bmatrix} \geq \begin{bmatrix} \vec{0}_m \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \vec{y}^T & u_+ & u_- \end{bmatrix} \geq \vec{0}^T$$

- **These two LP problems are dual to each other.**

Theorem (Min-Max theorem. von Neumann 1928)

$$\max_{\vec{x}\ \text{stochastic}} \left[\min_{\vec{y}\ \text{stochastic}} \vec{x}^T A \vec{y} \right] = \min_{\vec{y}\ \text{stochastic}} \left[\max_{\vec{x}\ \text{stochastic}} \vec{x}^T A \vec{y} \right]$$

Proof.

Note that the two problems can be written as LP problems which are dual to each other. Therefore, strong duality implies the equality. $\square$

- Jon von Neumann [1928]

John von Neumann

- “As far as I can see, there could be no theory of games ... without that theorem ..."
- "I thought there was nothing worth publishing until the Minimax Theorem was proved”

The min-max theorem is not obvious and the special properties of the feasible set and the function are necessary for it to hold.

Example

$f(x, y) = |x - y|$ defined on $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.
Given $-1 \leq y \leq 1$,

$$\max_{-1 \leq x \leq 1} |x - y| = \begin{cases} y + 1 & \text{if } y \geq 0, \\ 1 - y & \text{if } y < 0 \end{cases}$$

$$\geq 1 \quad (\text{ here equality holds when } y = 0)$$

So,

$$\min_{-1 \leq y \leq 1} \max_{-1 \leq x \leq 1} |x - y| = 1.$$

However, given x, $\min_{-1 \leq y \leq 1} |x - y| = 0$, so

$$\max_{-1 \leq x \leq 1} \min_{-1 \leq y \leq 1} |x - y| = 0.$$

For this function $f(x, y)$ minimax theorem does not hold.

Primal problem: $\max_{\vec{x}\text{ stoch.}} \left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$

Primal (Player 1)

$$
\begin{array}{lr}
\text{Max.} & v_+ - v_- \\
\text{subj.} & -A^T\vec{x} + (v_+ - v_-)\vec{\mathbf{e}}_n \leq \vec{0}_n \\
 & \vec{\mathbf{e}}_m^T\vec{x} \leq 1 \\
 & -\vec{\mathbf{e}}_m^T\vec{x} \leq -1 \\
 & \vec{x} \geq \vec{0}_m \;\&\; v_+, v_- \geq 0
\end{array}
$$

- original $x_1, ..., x_m, v_+, v_-$
- Slack variables

$$
\begin{bmatrix} x_{m+1} \\ \vdots \\ x_{m+n} \end{bmatrix} = -\begin{bmatrix} -A^T & \vec{e}_n & -\vec{e}_n \end{bmatrix} \begin{bmatrix} \vec{x} \\ v_+ \\ v_- \end{bmatrix}
$$

$$
\begin{aligned}
s_+ &= 1 - \vec{\mathbf{e}}_m^T\vec{x} \\
s_- &= -1 + \vec{\mathbf{e}}_m^T\vec{x}
\end{aligned}
$$

Dual problem: $\min_{\vec{y}\text{ stoch.}} \left[\max_{\vec{x}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$

Dual (Player 2)

$$\begin{array}{lr} \text{Min.} & u_+ - u_- \\ \text{subj.} & -A\vec{y} + (u_+ - u_-)\vec{\mathbf{e}}_m \geq \vec{0}_m \\ & \vec{\mathbf{e}}_n^T \vec{y} \geq 1 \\ & -\vec{\mathbf{e}}_n^T \vec{y} \geq -1 \\ & \vec{y} \geq \vec{0}_n \ \& \ u_+, u_- \geq 0 \end{array}$$

- original $x_1, \ldots, x_m, v_+, v_-$
- Slack variables

$$\begin{bmatrix} y_{n+1} \\ \vdots \\ y_{n+m} \end{bmatrix} = \begin{bmatrix} -A & \vec{e}_m & -\vec{e}_m \end{bmatrix} \begin{bmatrix} \vec{y} \\ u_+ \\ u_- \end{bmatrix}$$

$$t_+ = \vec{\mathbf{e}}_n^T \vec{y} - 1$$

$$t_- = -\vec{\mathbf{e}}_n^T \vec{y} + 1$$

Primal / Dual variables

Primal (Player 1)

$$\begin{array}{ll} \text{Max.} & v_+ - v_- \\ \text{subj.} & -A^T\vec{x} + (v_+ - v_-)\vec{\mathbf{e}}_n \le \vec{0}_n \\ & \vec{\mathbf{e}}_m^T\vec{x} \le 1 \\ & -\vec{\mathbf{e}}_m^T\vec{x} \le -1 \\ & \vec{x} \ge 0 \ \&\ v_+, v_- \ge 0 \end{array}$$

- original $x_1, ..., x_m, v_+, v_-$
- slack $x_{m+1}, ..., x_{m+n}, s_+, s_-$

Dual (Player 2)

$$\begin{array}{ll} \text{Min.} & u_+ - u_- \\ \text{subj.} & -A\vec{y} + (u_+ - u_-)\vec{\mathbf{e}}_m \ge \vec{0}_m \\ & \vec{\mathbf{e}}_n^T\vec{y} \ge 1 \\ & -\vec{\mathbf{e}}_n^T\vec{y} \ge -1 \\ & \vec{y} \ge 0 \ \&\ u_+, u_- \ge 0 \end{array}$$

- original $y_1, ..., y_n, u_+, u_-$
- slack $y_{n+1}, ..., y_{n+m}, t_+, t_-$

- $$\begin{array}{rclll} \textit{original} & x_i & \longleftrightarrow & \textit{slack} & y_{n+i} \quad \text{for } i = 1, .., m \\ \textit{slack} & x_{m+j} & \longleftrightarrow & \textit{original} & y_j \quad \text{for } j = 1, .., n \end{array}$$
- $$\begin{array}{rclll} \textit{original} & v_+ & \longleftrightarrow & \textit{slack} & t_+ \\ \textit{original} & v_- & \longleftrightarrow & \textit{slack} & t_- \\ \textit{slack} & s_+ & \longleftrightarrow & \textit{original} & u_+ \\ \textit{slack} & s_- & \longleftrightarrow & \textit{original} & u_- \end{array}$$

Application of duality

From the final dictionary of Player 1's problem, we can read off an optimal strategy for Player 2, vice versa, using the correspondence between the primal and dual variables.

For the payoff matrix for Player 1: $A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$.

Got the final dictionary:

$$\begin{array}{rcllll} z & = & -4 + \frac{42}{10} & -\frac{6}{10}x_4 & -\frac{4}{10}x_3 \\ \hline x_1 & = & \frac{7}{10} & -\frac{1}{10}x_4 & +\frac{1}{10}x_3 \\ x_2 & = & \frac{3}{10} & +\frac{1}{10}x_4 & -\frac{1}{10}x_3 \end{array}$$

So, $x_1^* = 7/10$, $x_2^* = 3/10$, $x_3^* = 0$, $x_4^* = 0$, with optimal objective value $z^* = -4 + 42/10 = 0.2$.
Use

$$x_3 = x_{2+1} \longleftrightarrow y_1$$
$$x_4 = x_{2+2} \longleftrightarrow y_2$$

and get dual basic solution $y_1^* = 4/10 = 2/5$, $y_2^* = 6/10 = 3/5$.

- Note that this dictionary is primal and dual feasible.
- So, the dual basic solution (y_1^*, y_2^*) is dual optimal, that is, an optimal strategy of Player 2.
- This is unique dual optimal solution because of nondegeneracy of $\vec{x}^*$.