2019 Math 340:101. Midterm Friday, 2019 October 18 IN CLASS **Time $\leq$50min**

Last name ....

First name ....

Student number ....

Grade: out of 32.

Student's signature: ....

There are total 6 problems.

This is a closed-book examination. ONLY pen/pencil/eraser are allowed.

1. A company produces three types of drones: A-type, B-type, and C-type. To produce the drones the company needs cameras, CPUs, and propellers.

   - Each A-type drone requires 3 cameras, 1 CPU, and 4 propellers.
   - Each B-type drone requires 2 cameras, 2 CPUs, and 3 propellers.
   - Each C-type drone requires 4 cameras, 2 CPUs, and 4 propellers.

   There is no limit how many drones the company can assemble per each day, as long as the parts are available, however, for each day, the company can get only up to 200 cameras, 100 CPUs, and 300 propellers. Suppose that

   - Producing each A-type drone will bring 5 dollar profit to the company.
   - Producing each B-type drone will bring 3 dollar profit to the company.
   - Producing each C-type drone will bring 7 dollar profit to the company.

   (a) (2 points) Write down a linear program problem to maximize the profit the company will make per day. Do not solve the problem.

   Solution: Let x_1 = number of A-type drones to be produced; x_2 = number of B-type drones to be produced; x_3 = number of C-type drones to be produced. The LP problem is

$$\begin{array}{lrrrcl} \text{Maximize} & 5x_1 & +3x_2 & +7x_7 & & \\ \text{Subject to} & 3x_1 & +2x_2 & +4x_3 & \leq & 200 \\ & x_1 & +2x_2 & +2x_3 & \leq & 100 \\ & 4x_1 & +3x_2 & +4x_3 & \leq & 300 \end{array} \quad \& \quad x_1, x_2, x_3 \geq 0.$$

   (b) (2 points) Write down the corresponding dual LP problem.

   Solution:

   - The dual is

$$\begin{array}{lrrrcl} \text{Minimize} & 200y_1 & +100y_2 & +300y_3 & & \\ \text{Subject to} & 3y_1 & +y_2 & +4y_3 & \geq & 5 \\ & 2y_1 & +2y_2 & +3y_3 & \geq & 3 \\ & 4y_1 & +2y_2 & +4y_3 & \geq & 7 \end{array} \quad \& \quad y_1, y_2, y_3 \geq 0.$$

   (c) (2 points) What does **each** of your **dual variables** represent for and in what unit? A brief statement is sufficient as an answer.

   Solution:

- y_1 is dollars per a camera, y_2 is dollars per a CPU and y_3 is dollars per a propeller. They represent possible net values (or marginal value, or shadow price) of each item.

2. (4 points) Suppose you got the following dictionary from the auxiliary problem (in the two phase method) of an LP problem:

$$\begin{array}{rcrrrr} x_4 & = & 1 & -x_0 & -x_1 & \\ x_2 & = & 2 & & -2x_1 & -x_5 \\ x_3 & = & & +x_0 & +x_1 & +x_5 \\ \hline w & = & & -x_0 & & \end{array}$$

The objective function of the original problem was $z = x_1 + x_2$. Solve the **original problem**, that is, either

- find an optimal solution $\vec{x}^* = (x_1^*, x_2^*)$ to the original problem,
- or show that it does not exist.

Solution:

- This is a feasible dictionary and since the coefficients in the objective function are all ≤ 0, it is a final dictionary for the auxiliary problem. The optimal value $w = 0$. Thus the original LP problem is feasible.
- Now, to get a feasible dictionary for the original LP problem set $x_0 = 0$, and write $z = x_1 + x_2 = x_1 + (2 - 2x_1 - x_5) = 2 - x_1 - x_5$. We get

$$\begin{array}{rcrrr} x_4 & = & 1 & -x_1 & \\ x_2 & = & 2 & -2x_1 & -x_5 \\ x_3 & = & & +x_1 & +x_5 \\ \hline z & = & 2 & -x_1 & -x_5 \end{array}$$

- This is a final dictionary and gives the optimal solution $\vec{x}^* = (x_1^*, x_2^*) = (0, 2)$.

3. (4 points) An LP problem has the following dictionary after several iterations of the simplex method:

$$\begin{array}{rcrrrr} x_6 & = & 1 & -x_2 & -x_4 & +x_5 \\ x_3 & = & 7 & & -x_4 & -x_5 \\ x_1 & = & 8 & +x_2 & -x_4 & -x_5 \\ \hline z & = & 15 & +x_2 & -2x_4 & -2x_5 \end{array}$$

Find an optimal solution to this problem **as well as** an optimal solution to the dual problem.

Solution:

- This is a feasible dictionary. Let x_2 enter and x_6 leave, and get
-

$$\begin{array}{rcrrrr} x_2 & = & 1 & -x_6 & -x_4 & +x_5 \\ x_3 & = & 7 & & -x_4 & -x_5 \\ x_1 & = & 9 & -x_6 & -2x_4 & \\ \hline z & = & 16 & -x_6 & -3x_4 & -x_5 \end{array}$$

- Since it is feasible and all coefficients in z are ≤ 0, this is an optimal dictionary, with the optimal solution $\vec{x}^* = (x_1^*, x_2^*, x_3^*) = (9, 1, 7)$.
- Apply the strong duality theorem to the final dictionary and get the optimal dual solution, $\vec{y}^* = (y_1^*, y_2^*, y_3^*) = (3, 1, 1)$. (Here we applied the identity $y_i^* = -c_{n+i}^*$ from the strong duality statement.)

4. For a given number α, consider the following LP problem:

$$\begin{array}{lrrrcl} \text{Maximize } z = & x_1 & +\alpha x_2 & +2x_3 & & \\ \text{Subject to} & x_1 & +x_2 & +x_3 & \leq & 4 \\ & x_1 & -2x_2 & +x_3 & \leq & 3 \\ & x_1 & +x_2 & +2x_3 & \leq & 4 \end{array} \quad \& \quad x_1, x_2, x_3 \geq 0.$$

Suppose this problem has an optimal solution:

$$x_1^* = 2, \qquad x_2^* = 0, \qquad x_3^* = 1.$$

(a) (4 points) Find the optimal solution to its **dual problem**.

Solution:

- We use complementary slackness.
- Notice that for this optimal solution $\vec{x}^*$ the slack variables are $x_4^* = 4-(2+0+1) = 1 > 0, x_5^* = 3-(2-0+1) = 0, x_6^* = 4-(2+0+2) = 0$.
- Let $\vec{y}^*$ be the dual optimal solution. Apply the complimentary slackness:

$$\begin{array}{ll} y_1^* x_4^* = 0 & \text{so } y_1^* = 0 \\ y_2^* x_5^* = 0 & \text{no info} \\ y_3^* x_6^* = 0 & \text{no info} \\ x_1^* y_4^* = 0 & \text{so } y_4^* = 0. \\ x_2^* y_5^* = 0 & \text{no info} \\ x_3^* y_6^* = 0 & \text{so } y_6^* = 0 \end{array}$$

- We got $y_1^* = 0$. Moreover, from $y_4^* = 0$, $y_6^* = 0$, we have $y_1^* + y_2^* + y_3^* = 1$, $y_1^* + y_2^* + 2y_3^* = 2$. From this we get $y_2^* = 0, y_3^* = 1$. So $\vec{y}^* = (0, 0, 1)$.

(b) (2 points) In the above situation, is it possible to have $\alpha \geq 2$? To earn credit, you have to justify your answer.

Solution:

- In the dual problem, we have the constraint $y_1 - 2y_2 + y_3 \geq \alpha$.
- As $\vec{y}^* = (0, 0, 1)$ is feasible, we have $0 - 0 + 1 \geq \alpha$, so that $\alpha \leq 1$.
- So, NO it is not possible.

5. (a) (6 points) Check whether each of the following statements is correct or not. Circle True or False, and you do not need to explain your answer. (1 mark for each correct answer.)

   i. **True / False.** If an LP problem is feasible then its dual problem must be feasible.

   ii. **True / False.** Given an LP problem, it is possible to have an optimal solution, while its dual problem has no optimal solution.

   iii. **True / False.** Given an LP problem, if you apply Anstee's rule, it is possible that the simplex algorithm may not terminate after a finite number of iterations.

   iv. **True / False.** It is always possible to determine after a finite number of calculations, whether a given standard form LP problem is feasible or not.

   v. **True / False.** For given $\vec{a} \in \mathbf{R}^n$ and $b \in \mathbf{R}$, if $\vec{a} \cdot \vec{x} = b$ for any $\vec{x} \in \mathbf{R}^n$, then it must hold that $\vec{a} = \vec{0}$ and $b = 0$.

   vi. **True / False.** For every $\vec{y} \geq \vec{0}$, it holds that $\max_{A\vec{x} \leq \vec{0}, \vec{x} \geq \vec{0}} \vec{c} \cdot \vec{x} \leq \max_{\vec{x} \geq \vec{0}} [\vec{c} \cdot \vec{x} - \vec{y} \cdot A\vec{x})]$.

Solution:

i. False, because unboundedness may happen.

ii. False, because of strong duality.

iii. True, cycling may happen under Anstee's rule.

iv. True, by applying the phase 1 of two phase method and the Bland's rule; under Bland's rule the simplex algorithm terminates in finite number of iterations.

v. True. We have seen this in the class.

vi. True. This is what we have seen in the class while discussing weak duality.

(b) (1 point) How many are true among the three statements i., ii., and iii.?

Solution: 1

(c) (1 point) How many are true among the three statements iv., v., and vi.?

Solution: 3.

6. (4 points) Suppose that the LP problem $\max_{A\vec{x}\le\vec{0}\,\&\,\vec{x}\ge\vec{0}} \vec{c}\cdot\vec{x}$ has an optimal solution $\vec{x}^*$. Evaluate $\vec{c}\cdot\vec{x}^*$. To earn credit, you have to justify your answer. (You have to quote correctly what theorems you are using.)

Solution:

- Note that from the strong duality theorem, as the optimal solution to the primal problem exists by the assumption, the optimal dual solution exists and the optimal value of the primal problem is the same as the optimal value of the dual problem.
- The dual problem reads $\min_{A^T\vec{y}\ge\vec{c},\,\vec{y}\ge\vec{0}} 0$.
- This is feasible as we already know the optimal dual solution exists. Obviously the optimal value for the dual is 0.
- This implies that the optimal value $\vec{c}\cdot\vec{x}^*$ of the primal problem is 0.

Extra Page 1

Extra Page 2

Extra Page 3