- You will *not* be able to use a calculator or computer for either the midterm or the final exam, so please do not use one for this assignment. You may use one to *check* your answer, but please do not use one to solve the problem.
- Only part of the problems may be graded. But, you have to submit all the problems.
- The deadline is *by 6pm on Friday*, Nov. 1. You have a grace period until 9am the next day. The grace period is to take care of any technical issues you have while submitting the file. The grace period should give ample time for handling any issues, so No late HW after the grace period will be accepted, **regardless** your technical issues.
- Submit only pdf files.

1. $\boxed{\text{5 marks}}$ Find all the solutions of the following LP problem:

$$\text{maximise } z = 3x_1 + x_2 + 0x_3$$

$$\begin{array}{rrrrl} \text{subject to} & x_1 & +2x_2 & & \leq 5 \\ & x_1 & +x_2 & -x_3 & \leq 2 \\ & 7x_1 & +3x_2 & -5x_3 & \leq 20 \end{array}$$

$$\text{and } x_1, x_2, x_3 \geq 0$$

Solution: Write in dictionary form:

$$\begin{array}{rcrrrr} x_4 & = & 5 & -x_1 & -2x_2 & \\ x_5 & = & 2 & -x_1 & -x_2 & +x_3 \\ x_6 & = & 20 & -7x_1 & -3x_2 & +5x_3 \\ \hline z & = & & 3x_1 & +x_2 & \end{array}$$

Pivot — x_1 enters and x_5 leaves:

$$\begin{array}{rcrrrr} x_1 & = & 2 & -x_2 & +x_3 & -x_5 \\ x_4 & = & 3 & -x_2 & -x_3 & +x_5 \\ x_6 & = & 6 & +4x_2 & -2x_3 & +7x_5 \\ z & = & 6 & -2x_2 & +3x_3 & -3x_5 \end{array}$$

Pivot again — x_3 enters and x_4 or x_6 might leave — choose x_4:

$$\begin{array}{rcrrrr} x_1 & = & 5 & -2x_2 & -x_4 & \\ x_3 & = & 3 & -x_2 & -x_4 & +x_5 \\ x_6 & = & 0 & 6x_2 & +2x_4 & +5x_5 \\ \hline z & = & 15 & -5x_2 & -3x_4 & \end{array}$$

So we stop here. The solution is

$$x_2 = x_4 = x_5 = 0 \qquad x_1 = 5 \qquad x_3 = 3 \qquad x_6 = 0 \qquad z = 15$$

But the expression for z is independent of x_5 — so we can possibly set it to values other than 0. So set $x_2 = x_4 = 0, x_5 = t$:

$$x_1 = 5 \qquad x_3 = 3 + t \qquad x_6 = 5t \qquad z = 15$$

So this is valid for any $t \geq 0$. So the full solution is

$$\begin{aligned} x_1 &= 5 & x_2 &= 0 \\ x_3 &= 3 + t & x_4 &= 0 \\ x_5 &= t & x_6 &= 5t \\ z &= 15 \end{aligned}$$

for $t \geq 0$. One can subsitute this into the original constraints to confirm that it is indeed feasible.

2. $\boxed{\text{5 marks}}$ The company "Le chocolat délicieux" has you solve an LP problem, the objective being measured in dollars of profit. You solve the LP problem, arriving at a final dictionary that is non-degenerate. Currently they are getting 3000 kilos of cocoa ingredients from the contractor "Cocoa-good" at the price \$5 per kilo. The company wants to increase their production, but, the contractor "Cocoa-good" cannot provide additional amount. Then, another supplier "Cocoa-better" offers to provide a tiny bit of cocoa ingredients at the price \$7 per kilo. Everything else in the LP problem will remain the same. How do you determine whether the company should buy a tiny bit of such ingredients to make more production? In other words, using what criteria would you advise the company buy or not a tiny bit of such ingredient? Explain your answer clearly using relevant theorems.

Solution:

- When the primal problem is solved we can find the optimal solution to the corresponding dual problem. (By strong duality.)
- Let y^* be the component of the optimal dual solution, corresponding to the constraint $(0.7)x_1 + (0.76)x_2 \leq 3000$.
- We know that y^* is the marginal value of the corresponding resource — cocoa. So if that resource is changed to $3000+t$ then the objective function, the profit, increases by y^*t, assuming that they are paying \$5 per kilo of cocoa – the price with the current contractor.
- But, this is under the assumption that we pay \$5 per kilo. If we pay \$ 7 per kilo, then we are paying \$ (7-5)=\$2 additionally per kilo. Therefore, the net additional profit will be $\$(y^*t - 2t)$.

- Clearly this is only worth doing if $y^*t - 2t > 0$ i.e. $y^* > 2$.
- If $y^* = 2$ then it makes no difference and if $y^* < 2$ then we will loose money.

3. $\boxed{\text{5 marks}}$ Consider the following dictionary from a standard form LP problem.

$$\begin{array}{rcrrrr} x_4 & = & 5 & -x_1 & -2x_2 & \\ x_5 & = & 0 & -x_1 & -x_2 & +x_3 \\ x_6 & = & 20 & -7x_1 & -3x_2 & +5x_3 \\ \hline z & = & & -3x_1 & -x_2 & -x_3 \end{array}$$

Find **all** the optimal **dual** solutions. (Hint: you may want to use the equation in Lecture 19 and complimentary slackness.)

Solution: This dictionary is in fact an optimal dictionary as the coefficients for z are all negative. Note that the for the optimal solution $\vec{x}^* = (0, 0, 0)$ is the unique optimal solution. but it is degenerate because of the constant 0 in the second row of the above dictionary. This dictionary is in fact the initial dictionary as the nonbasic variables are all original variables. Therefore in the standard form LP, maximize $\vec{c}\cdot\vec{x}$ subject to $A\vec{x} \le \vec{b}$ and $\vec{x} \ge \vec{0}$, we see that $\vec{c} = (-3, -1, -1)$ and $\vec{b} = (5, 0, 20)$ and the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 7 & 3 & -5 \end{bmatrix}$. For the optimal solution $\vec{x}^* = (0, 0, 0)$, the slack variable values are $x_4^* = 5, x_5^* = 0.x_6^* = 20$, so, complimentary slackness implies $y_1^* = 0$, $y_3^* = 0$. Moreover, from the identity discussed in Lecture 19, we see that for the dual optimal solution $\vec{y}^*$ and its slack variable vector $\vec{w}^*$,

$$\vec{c} = \begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix} = y_2^* \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + w_1^* \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + w_2^* \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + w_3^* \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Here, the first vector on the right is the second row vector of the matrix A, and the rest two are the unit basis vectors $-\vec{e}_1$, $-\vec{e}_2$, and $-\vec{e}_3$.

Solving the above vector equation for $y_2^*, w_1^*, w_2^*, w_3^* \ge 0$ we get all the dual optimal solutions $\vec{y}^*$. From the vector equation we see that

$$\begin{aligned} -3 &= y_2^* - w_1^* \\ -1 &= y_2^* - w_2^* \\ -1 &= -y_2^* - w_3^* \end{aligned}$$

Note that for the slack values w_1^*, w_2^*, w_3^* they only need to satisfy ≥ 0, therefore, we get constrains on y_2^* as

$$\begin{aligned} -3 &\le y_2^* \\ -1 &\le y_2^* \\ -1 &\le -y_2^* \end{aligned}$$

Considering all these, we get $-1 \le y_2^* \le 1$. But, also we had the condition $y_2^* \ge 0$ from the second line before the second last display (or from feasibility). So, $0 \le y_2^* \le 1$ Therefore, we found all the optimal dual solutions $y_2^* = 0, y_2 = t$, $0 \le t \le 1$, and $y_3^* = 0$.

4. Consider the standard form LP problem: Maximise $\vec{c} \cdot \vec{x}$ subject to $A\vec{x} \le \vec{b}$ and $\vec{x} \ge \vec{0}$, where $c = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$, and $b = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$. It has the final dictionary

$$\begin{array}{rcrrrr} x_2 & = & 4 & -x_1 & -x_3 & -x_4 \\ x_5 & = & 7 & & -3x_3 & -2x_4 \\ x_6 & = & 2 & -x_1 & & -x_4 \\ z & = & 8 & -x_1 & -x_3 & -2x_4 \end{array}$$

(a) $\boxed{\text{5 marks}}$ Let $\vec{x}^*$ be an optimal solution to this (primal) problem. Find the largest ϵ such that for any t, with $0 \le t \le \epsilon$, the vector $\vec{x}^*$ is still an optimal solution to the new primal problem obtained by changing $\vec{c}$ to $\vec{c} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$. (Hint: you may want to use the equation in Lecture 19 among others.)

Solution: From the last dictionary, we see that optimal solution $\vec{x}^* = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$ with slack values $\begin{bmatrix} x_4^* \\ x_5^* \\ x_6^* \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$. Notice that at $\vec{x}^*$, the following hyperplanes from the equality cases of constraints pass through: $x_1 = 0$, $x_3 = 0$, $x_4 = 0$. For $\vec{x}^*$ still to be the optimal solution to the new problem with $\vec{c} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$, it should happen that $\vec{c} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+t \\ 2 \\ 1 \end{bmatrix}$ can be written as (see Lecture 19)

$$\begin{bmatrix} 1+t \\ 2 \\ 1 \end{bmatrix} = y_1^{**} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + w_1^{**} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + w_3^{**} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

for some nonnegative values (in fact from the dual optimal solution to the new problem) $y_1^{**}, w_1^{**}, w_2^{**} \ge 0$. Here, the first vector on the right is the first row vector of the matrix A, and the rest two are the unit basis vectors $-\vec{e}_1$ and $-\vec{e}_3$. From the last display, we see that

$$\begin{array}{rcrrr} t & = & -1 & +y_1^{**} & -w_1^{**} \\ 2 & = & & y_1^{**} & \\ 1 & = & & y_1^{**} & -w_3^{**} \end{array}$$

We can reduce the number of variables by using the second and third line, so that $y_1^{**} = 2$ and $w_3^{**} = 1$, and no condition on w_1^{**} but $w_1^{**} \geq 0$. Therefore,

$$t = -1 + 2 - w_1^{**} = 1 - w_1^{**} \leq 1 \text{ as } w_1^{**} \geq 0.$$

So, $0 \leq t \leq 1$, thus the largest value for ϵ is 1.

(b) $\boxed{\text{5 marks}}$ * (This problem is more subtle than (a). Be careful.) Let $\vec{y}^*$ be an optimal solution to the dual problem. Find the largest ϵ such that for any t, with $-\epsilon \leq t \leq \epsilon$, the vector $\vec{y}^*$ is still an optimal solution to the new **dual** problem obtained by changing $\vec{b}$ to $\vec{b} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$. (Hint: First turn the dual problem into a standard form LP. You may also want to use the equation in Lecture 19 among others.)

Solution: Note that the dual problem is equivalent to the following standard form problem: Maximize $-\vec{b} \cdot \vec{y}$ subject to $-A^T\vec{y} \leq -\vec{c}$ and $\vec{y} \geq \vec{0}$. We will use this, just to apply the same reasoning as in the problem (a).

From the last dictionary, we see an optimal dual solution is $\vec{y}^* = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ with slack values $\begin{bmatrix} w_1^* \\ w_2^* \\ w_3^* \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Notice that at $\vec{y}^*$, the following hyperplanes from the equality cases of constraints pass through: $y_2 = 0$, $y_3 = 0$ and $w_2 = 0$. Note that the last dictionary of the primal problem is non-degenerate, so is the primal optimal solution $\vec{x}^*$ which is also the basic solution of the last dictionary. Therefore, from the theorem in the Lecture 18 proved in Lecture 19, the dual optimal solution $\vec{y}^*$ is unique.

For $\vec{y}^*$ still to be the optimal solution to the new problem with $\vec{b} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$, it should happen that $-\left(\vec{b}+\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4-t \\ 1 \\ 2 \end{bmatrix}$ can be written as (see Lecture 19)

$$\begin{bmatrix} -4-t \\ 1 \\ 2 \end{bmatrix} = x_2^{**} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + v_2^{**} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + v_3^{**} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

for some nonnegative values (in fact from the dual optimal solution to the new problem) $x_2^{**}, v_2^{**}, v_3^{**} \geq 0$. Here, the first vector on the right is the second row vector of the matrix $-A^T$ (so the second column of $-A$), and the rest two are the unit basis vectors $-\vec{e}_1$ and $-\vec{e}_3$. From the last display, we see that

$$\begin{aligned} -4-t &= -x_2^{**} \\ 1 &= 2x_2^{**} - v_2^{**} \\ 2 &= x_2^{**} - v_3^{**} \end{aligned}$$

From the first line $x_2^{**} = 4+t$. Plug in this to the second and the third line, to get $v_2^{**} = 7+2t$ and $v_3^{**} = 2+t$. The only conditions remained are $x_2^{**}, v_2^{**}, v_3^{**} \geq 0$, therefore we get

$$4+t \geq 0, 7+2t \geq 0, 2+t \geq 0$$

To satisfy all these, $t \geq -2$. Therefore, the largest eligible ϵ is $\epsilon = 2$.