# Lecture 28: Matrix Games

Game theory has many applications: economics, computer science, physics, etc.
Founders: J. von Neumann, ...., J. Nash (Nobel prize and Abel prize), ....

John von Neumann

John Nash

# Matrix Games

**Example (Rock-Paper-Scissors)**

The payoff matrix for the row player:

$$
\begin{array}{cc}
 & \text{column player} \\
\text{row player} &
\begin{array}{c}
\\ r \\ p \\ s
\end{array}
\begin{array}{c}
\begin{array}{ccc} r & p & s \end{array} \\
\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}
\end{array}
\end{array}
$$

$$\begin{array}{cc} & \text{column player} \\ & \begin{array}{ccc} r & p & s \end{array} \\ \text{row player} & \begin{array}{c} r \\ p \\ s \end{array} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

**[Two person zero-sum game]**

- Player 1 (row player) has strategies $1, 2, ..., m$.
- Player 2 (column player) has strategies $1, 2, ..., n$.
- Payoff matrix $A = [a_{ij}]$.
  - $$a_{ij} = \begin{array}{l} \text{payoff to the row player if the row player plays strategy } i \\ \text{and column player plays strategy } j. \end{array}$$
    $$= -\text{ payoff to the column player}$$
  - $$\text{payoff for the row player } + \text{payoff for the column player} = 0.$$

# [Morra] (Ch. 15 in [Chvatal])

- Each player hides 1 or 2 dollars and guess how much the other player has hidden
- If neither player guess correctly, then no one wins. Repeat.
- If both player guess correctly, then no one wins. Repeat.
- If only one player guess correctly, then the player wins **from the other player**, an amount the same as the total hidden $$.

**Payoff matrix (for the row player).**
$[hide, guess]$

$$
\begin{array}{cc}
 & \text{column player} \\
 & \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\
\text{row player} \begin{array}{c} [1,1] \\ [1,2] \\ [2,1] \\ [2,2] \end{array} & \begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}
\end{array}
$$

# Mixed strategies and Stochastic vectors

Mixed strategies:

- May give weights (frequencies) on each strategy, say when they play the game **many many times**.
  - At a given time, players choose their strategy **randomly**.
  - Player 1 (row player) chooses strategy $i$ for $x_i$ fraction of times;
  - Player 2 (column player) chooses strategy $j$ for $y_j$ fraction of times;

  - $x_i =$ probability for player 1 (row player) to choose $i$;
  - $y_j =$ probability for player 2 (column player) to choose $j$.

$\vec{x} = (x_1, ..., x_m)$ is **stochastic**, that is,

$$x_1, ..., x_m \geq 0 \quad \& \quad x_1 + ... + x_m = 1.$$

$\vec{y} = (y_1, ..., y_n)$ is **stochastic**, that is,

$$y_1, ..., y_n \geq 0 \quad \& \quad y_1 + ... + y_n = 1.$$

- Player 1 chooses a stochastic vector $\vec{x}$.
- Player 2 chooses a stochastic vector $\vec{y}$.

**Assume** Player 1 and Player 2 choose their strategies **independently**:

$$\Pr[row = i \ \& \ column = j] = \Pr[row = i] \times \Pr[column = j]$$
$$= x_i y_j$$

**Then,** for

- $\vec{x} = (x_1, ..., x_m)$: player 1's strategy
- $\vec{y} = (y_1, ..., y_n)$: player 2's strategy
- $A = [a_{ij}]$: the payoff matrix for row player (player 1),

**the average payoff for Player 1** (row player) is

$$= \sum \text{"payoff for } (i,j)\text{"} \times \text{"probability for } (i,j)''$$
$$= \sum_{i=}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$
$$= \vec{x}^T A \vec{y}$$

and **the average payoff for Player 2** (column player)$= -\vec{x}^T A \vec{y}$.

# Mixed strategies: Rock-Payer-Scissors Example

If Player 1 chooses $\vec{x}^T = \begin{array}{c} \begin{array}{ccc} r & p & s \end{array} \\ \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$

and Player 2 chooses $\vec{y}^T = \begin{array}{c} \begin{array}{ccc} r & p & s \end{array} \\ \begin{bmatrix} 1/2 & 0 & 1/2 \end{bmatrix} \end{array}$

Then, the payoff for play 1 is

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} r & p & s \end{array} \\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$= 1/6 - 1/6 = 0.$$

# Optimization problems

- Player 1 (row player)
  tries to maximize $\vec{x}^T A \vec{y}$ by choosing good $\vec{x}$.
- Player 2 (coliumn player)
  tries to maximize $-\vec{x}^T A \vec{y}$, equivalently, minimize $\vec{x}^T A \vec{y}$ by choosing good $\vec{y}$,
- Each player does not affect/control the other player's decision: They are independent!

# Remarks

$$\min_{\vec{y} \text{ stochastic}} (\vec{x}^*)^T A\vec{y} \quad \leq \quad (\vec{x}^*)^T A\vec{y}^* \quad \leq \quad \max_{\vec{x} \text{ stochastic}} \vec{x}^T A\vec{y}^*$$

- The lefft inequality will be strict if $\vec{y}^*$ is not the solution of the min. problem.
- The right inequality will be strict if $\vec{x}^*$ is not the solution of the max. problem.

## Remark

- *Such strategies* $(\vec{x}^*, \vec{y}^*)$ *with*

$$\min_{\vec{y} \text{ stochastic}} (\vec{x}^*)^T A\vec{y} = \max_{\vec{x} \text{ stochastic}} \vec{x}^T A\vec{y}^*$$

*is called a* **Nash equilibrium**. *These are special strategies.*

# Optimization problems

Given strategy $\vec{x}$ (a stochastic vector),

- $$\text{Player 1's payoff} \geq \min_{\vec{y}\ \text{stochastic}} \vec{x}^T A \vec{y}.$$

  The right is “the worst-case scenario” for player 1.
- The right is the best-case scenario for player 2, which she can choose.
- Player 1 cannot avoid the possibility of such a worst-case scenario.

**Objective for player 1** : " Find the best of the worst scenarios".

## Definition (Optimization problem for player 1 (row player))

- To find a mixed strategy $\vec{x}$ (a stochastic vector) that maximizes $\min_{\vec{y} \text{ stochastic}} \vec{x}^T A \vec{y}$.
- That is,

$$\max_{\vec{x} \text{ stochastic}} \left[ \min_{\vec{y} \text{ stochastic}} \vec{x}^T A \vec{y} \right].$$

Remark:

- If the player 1 chooses the optimal strategy $\vec{x}^*$ then **no matter what the player 2 chooses** the player 1 will **gain at least** $\max_{\vec{x} \text{ stochastic}} \left[ \min_{\vec{y} \text{ stochastic}} \vec{x}^T A \vec{y} \right]$.

# Optimization problems for player 2's point of view

Given strategy $\vec{y}$ (a stochastic vector),

- $$\text{Player 2's payoff} \quad \geq \quad -\max_{\vec{x}\ \text{stochastic}} \vec{x}^T A\vec{y}.$$

  The right is "the worst-case scenario" for player 2.

- The right is the best-case scenario for player 1, which she can choose.
- Player 2 cannot avoid the possibility of such a worst-case scenario.

**Objective for player 2** : " Find the best of the worst scenarios".

## Definition (Optimization problem for player 2 (column player))

- To find a mixed strategy $\vec{y}$ (a stochastic vector) that minimizes $\max_{\vec{x}\text{ stochastic}} \vec{x}^T A\vec{y}$.
- That is,

$$\min_{\vec{y}\text{ stochastic}} \left[ \max_{\vec{x}\text{ stochastic}} \vec{x}^T A\vec{y} \right].$$

Remark:

- If the player 2 chooses the optimal strategy $\vec{y}^*$ then **no matter what the player 1 chooses** the player 2 will **loose at most** $\min_{\vec{y}\text{ stochastic}} \left[ \max_{\vec{x}\text{ stochastic}} \vec{x}^T A\vec{y} \right]$.