Lecture 31: Matrix Games.

Lecture 31 Wed. Nov 20.

- fairness of a game.
- suggested exercises Vanderbei 11.1, 11.2, 11.3 Chvatal 15.2

Unfair games

- If $\max_{\vec{x} \text{ stoch.}} \left[\min_{\vec{y} \text{ stoch.}} \vec{x}^T A \vec{y} \right] = \min_{\vec{y} \text{ stoch.}} \left[\max_{\vec{x} \text{ stoch.}} \vec{x}^T A \vec{y} \right] > 0$ then
  - Player 1 gains at least this amount by choosing an optimal strategy $\vec{x}^*$, and
  - in this case, Player 2 loses at least this amount regardless what he chooses.

  This is **unfair** to Player 2.
- If $\max_{\vec{x} \text{ stoch.}} \left[\min_{\vec{y} \text{ stoch.}} \vec{x}^T A \vec{y} \right] = \min_{\vec{y} \text{ stoch.}} \left[\max_{\vec{x} \text{ stoch.}} \vec{x}^T A \vec{y} \right] < 0$ then
  - Player 1 loses at least the absolute value of this amount if Player 2 chooses an optimal strategy $\vec{y}^*$, and
  - Player 2 gains at least the absolute value of this amount.

  This is **unfair** to Player 1.

Unfair game example:

For the payoff matrix for Player 1: $A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$.

- Got the final dictionary for **the Player 1 problem**:

$$\begin{array}{ccccc} z & = & -4 + \frac{42}{10} & -\frac{6}{10}x_4 & -\frac{4}{10}x_3 \\ \hline x_1 & = & \frac{7}{10} & -\frac{1}{10}x_4 & +\frac{1}{10}x_3 \\ x_2 & = & \frac{3}{10} & +\frac{1}{10}x_4 & -\frac{1}{10}x_3 \end{array}$$

- So, $x_1^* = 7/10$, $x_2^* = 3/10$, $x_3^* = 0, x_4^* = 0$.
- Optimal objective value for Player 1 is $z^* = -4 + 42/10 = 1/5 > 0$.
- No matter what Player 2 does, he loses at least $1/5$ to Player 1, **IF** Player 1 is smart enough and chooses the optimal strategy $\vec{x}^*$.
- This game is unfair to Player 2.

Fair game

Definition

We say that the game is **fair**

if $\max\limits_{\vec{x}\text{ stoch.}} \left[\min\limits_{\vec{y}\text{ stoch.}} \vec{x}^T A \vec{y} \right] = 0$

(or equivalently, $\min\limits_{\vec{y}\text{ stoch.}} \left[\max\limits_{\vec{x}\text{ stoch.}} \vec{x}^T A \vec{y} \right] = 0$.)

Fair game example:

Example (Rock-Paper-Scissors)

The payoff matrix for the row player:

$$
\text{row player} \quad
\begin{array}{c}
 \\ r \\ p \\ s
\end{array}
\overset{\text{column player}}{
\begin{array}{c}
\begin{array}{ccc} \;r\; & \;p\; & \;s\; \end{array} \\
\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}
\end{array}}
$$

Fair game example:

Example (Rock-Paper-Scissors)

The payoff matrix for the row player:

$$\begin{array}{cc} & \text{column player} \\ & \begin{array}{ccc} r & p & s \end{array} \\ \text{row player} & \begin{array}{c} r \\ p \\ s \end{array} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

- For this game, the payoff matrix A satisfies $A^T = -A$ "anti-symmetric".

Theorem

If $A^T = -A$,

then $\displaystyle \max_{\vec{x} \text{ stoch.}} \left[\min_{\vec{y} \text{ stoch.}} \vec{x}^T A \vec{y} \right] = 0$,

so the corresponding game is fair.

If $A^T = -A$, then $\max\limits_{\vec{x}\text{ stoch.}} \left[\min\limits_{\vec{y}\text{ stoch.}} \vec{x}^T A \vec{y} \right] = 0$. Proof:

Assume $A^T = -A$. Note that $m = n$. Then,

- $\vec{x}^T A\vec{y} = (\vec{x}^T A\vec{y})^T = \vec{y}^T A^T \vec{x} = \vec{y}^T (-A)\vec{x} = -\vec{y}^T A\vec{x}$.

$$\max_{\vec{x}\text{ stoch.}} \left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T A\vec{y} \right] = \max_{\vec{x}\text{ stoch.}} \left[\min_{\vec{y}\text{ stoch.}} -\vec{y}^T A\vec{x} \right]$$

$$= \max_{\vec{x}\text{ stoch.}} \left[-\max_{\vec{y}\text{ stoch.}} \vec{y}^T A\vec{x} \right] \qquad (\text{using } \min(-\alpha) = -\max\alpha)$$

$$= -\min_{\vec{x}\text{ stoch.}} \left[\max_{\vec{y}\text{ stoch.}} \vec{y}^T A\vec{x} \right] \qquad (\text{using } \min(-\alpha) = -\max\alpha)$$

$$= -\min_{\vec{y}\text{ stoch.}} \left[\max_{\vec{x}\text{ stoch.}} \vec{x}^T A\vec{y} \right] \qquad (\text{can exchange } \vec{x} \leftrightarrow \vec{y} \text{ as } m = n)$$

- From mini-max theorem, this implies that .

$$\max_{\vec{x}\text{ stoch.}} \left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T A\vec{y} \right] = -\max_{\vec{x}\text{ stoch.}} \left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T A\vec{y} \right].$$

- Therefore, $\max\limits_{\vec{x}\text{ stoch.}} \left[\min\limits_{\vec{y}\text{ stoch.}} \vec{x}^T A\vec{y} \right] = 0$. (If $\alpha = -\alpha$ then $\alpha = 0$.)

Fair games **without** $A^T = -A$.

There are many fair games without anti-symmetry.

For the payoff matrix for Player 1: $A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$.

We saw that this is an unfair game.

- Player 1 has advantage of $1/5$ and Player 2 has disadvantage of $1/5$.

Can modify the game:

- **To make it fair**, give a bonus of$1/5$ to Player 2, to compensate the disadvantage, **in each cases**.
- This means we modify A by **substracting** $1/5$ from each entry.

New payoff matrix

$$A' = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} 1/5 & 1/5 \\ 1/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 2-1/5 & -1-1/5 \\ -4-1/5 & 3-1/5 \end{bmatrix}$$

- This new payofff matrix is **not** anti-symmetric. $(A')^T \neq -(A')$.
- Still, the new game is fair.
- The next slide has more mathematical explanation.

Let U be an $m \times n$ matrix given by

$$U = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ldots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \qquad \text{(Every entry is 1)}$$

Then, for each stochastic vectors $\vec{x} \in \mathbb{R}^m$, $\vec{y} \in \mathbb{R}^n$

$$\vec{x}^T U \vec{y} = \vec{x}^T \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ldots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \vec{y} = \vec{x}^T \begin{bmatrix} y_1 + \cdots + y_n \\ \vdots \\ y_1 + \cdots + y_n \end{bmatrix} = \vec{x}^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= x_1 + \cdots x_m = 1$$

Therefore,

- $\vec{x}^T(A+\lambda U)\vec{y} = \vec{x}^T A\vec{y} + \lambda\vec{x}^T U\vec{y} = \vec{x}^T A\vec{y} + \lambda.$
  This is like giving bonus $-\lambda$ to the player 2, equivalently, giving bonus λ to player 1.
- $$\max_{\vec{x}\text{ stoch.}}\left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T(A+\lambda U)\vec{y}\right] = \max_{\vec{x}\text{ stoch.}}\left[\min_{\vec{y}\text{ stoch.}} (\vec{x}^T A\vec{y}+\lambda)\right]$$
  $$= \max_{\vec{x}\text{ stoch.}}\left[\left(\min_{\vec{y}\text{ stoch.}} \vec{x}^T A\vec{y}\right)+\lambda\right]$$
  $$= \left[\max_{\vec{x}\text{ stoch.}}\left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T A\vec{y}\right]\right]+\lambda$$
- We can put $\lambda = -\max_{\vec{x}\text{ stoch.}}\left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T A\vec{y}\right]$ to make the game fair for the modified game with the new payoff matrix $(A+\lambda U)$.