- You will *not* be able to use a calculator or computer for either the midterm or the final exam, so please do not use one for this assignment. You may use one to *check* your answer, but please do not use one to solve the problem.
- Only part of the problems may be graded. But, you have to submit all the problems.
- The deadline is *by 6pm on Friday*, Nov. 15. You have a grace period until 9am the next day. The grace period is to take care of any technical issues you have while submitting the file. The grace period should give ample time for handling any issues, so No late HW after the grace period will be accepted, **regardless** your technical issues.
- Submit only pdf files.

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1. $\boxed{\text{5 marks}}$ Solve the following LP problem using the revised simplex method:

$$\begin{array}{lrrrcl} \text{Maximize} & 2x_1 & +3x_2 & +3x_3 & & \\ \text{subject to} & 3x_1 & +x_2 & & \leq & 60 \\ & -x_1 & +x_2 & +4x_3 & \leq & 10 \\ & 2x_1 & -2x_2 & +5x_3 & \leq & 15 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

Hint: requires ≤ 3 iterations.

Solution: First iteration

- Form matrices:

$$\vec{b} = \begin{bmatrix} 60 \\ 10 \\ 15 \end{bmatrix} \qquad \vec{x}_B^* = \begin{bmatrix} 60 \\ 10 \\ 15 \end{bmatrix}$$

$$\vec{x}_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \mathbf{N} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 4 \\ 2 & -2 & 5 \end{bmatrix} \end{array}$$

$$\vec{c}_B^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \vec{c}_N^T = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$$

- Solve $\vec{y}^T\mathbf{B} = \vec{c}_B^T$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \Longrightarrow \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

- Choose entering column:

$$\vec{c}_N^T - y\mathbf{N} = \vec{c}_N^T - 0 = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$$

We pick column 2. Hence x_2 enters and

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

- Solve $\mathbf{B}\vec{d} = \vec{a}$:

$$\begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \vec{d} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{d} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

- Find largest t such that $\vec{x}_B^* - t\vec{d} \geq 0$

$$\begin{bmatrix} 60 \\ 10 \\ 15 \end{bmatrix} - t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

So $t = 10$. Since the second component $= 0$, it follows that x_5 leaves.

- Update things

$$\vec{x}_B^* = \begin{bmatrix} 50 \\ 10 \\ 35 \end{bmatrix}$$

$$\vec{x}_B = \begin{bmatrix} x_4 \\ x_2 \\ x_6 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_1 \\ x_5 \\ x_3 \end{bmatrix}$$

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} x_4 & x_2 & x_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{array} \qquad \mathbf{N} = \begin{array}{c} \begin{array}{ccc} x_1 & x_5 & x_3 \end{array} \\ \begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 4 \\ 2 & 0 & 5 \end{bmatrix} \end{array}$$

$$\vec{c}_B^T = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix} \qquad \vec{c}_N^T = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}$$

Second iteration

- Solve $\vec{y}^T\mathbf{B} = \vec{c}_B^T$:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} x_4 & x_2 & x_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix} \implies \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}$$

- Choose entering column:

$$\begin{aligned} \vec{c}_N^T - y\mathbf{N} &= \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 0 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} x_1 & x_5 & x_3 \end{array} \\ \begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 4 \\ 2 & 0 & 5 \end{bmatrix} \end{array} \\ &= \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} -3 & 3 & 12 \end{bmatrix} = \begin{bmatrix} 5 & -3 & -9 \end{bmatrix} \end{aligned}$$

So we pick column 1. Hence x_1 enters and

$$\vec{a} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

- Solve $\mathbf{B}\vec{d} = \vec{a}$:

$$\begin{array}{c} \begin{array}{ccc} x_4 & x_2 & x_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{array} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \implies \vec{d} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

- Find largest t such that $\vec{x}_B^* - t\vec{d} \geq 0$

$$\begin{bmatrix} 50 \\ 10 \\ 35 \end{bmatrix} - t \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

So $t = 25/2$. Since the first component $= 0$, it follows that x_4 leaves.

- Update things

$$\vec{x}_B^* = \begin{bmatrix} 25/2 \\ 45/2 \\ 35 \end{bmatrix}$$

$$\vec{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_6 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_4 \\ x_5 \\ x_3 \end{bmatrix}$$

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_6 \end{array} \\ \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \end{array} \qquad \mathbf{N} = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \end{array}$$

$$\vec{c}_B^T = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix} \qquad \vec{c}_N^T = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}$$

Third iteration

- Solve $\vec{y}^T\mathbf{B} = \vec{c}_B^T$:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} x_4 & x_2 & x_6 \end{array} \\ \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix} \implies \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 5/4 & 7/4 & 0 \end{bmatrix}$$

- Choose entering variable

$$\vec{c}_N^T - y\mathbf{N} = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 5/4 & 7/4 & 0 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \end{array} = \begin{bmatrix} -5/4 & -7/4 & -4 \end{bmatrix}$$

  Since all are negative, there are no entering columns. The current basic feasible solution is optimal.

- The current solution is

$$\vec{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_6 \end{bmatrix} = \vec{x}_B^* = \begin{bmatrix} 25/2 \\ 45/2 \\ 35 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_4 \\ x_5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z = \vec{c}_B^T\vec{x}_B^* = 25 + 135/2 = 185/2$$

2. $\boxed{\text{5 marks}}$ Solve the following LP problem using the revised simplex method.

$$\begin{array}{lrrcl} \text{Maximize} & 2x_1 & +3x_2 & & \\ \text{subject to} & x_1 & +2x_2 & \leq & 6 \\ & 2x_1 & +x_2 & \leq & 8 \\ & -x_1 & +x_2 & \leq & 1 \\ & & 2x_2 & \leq & 1 \\ & x_1, & x_2, & \geq & 0. \end{array}$$

Hint: requires ≤ 3 iterations.

Solution: First iteration

- Form matrices

$$\vec{x}_B^* = \begin{bmatrix} 6 \\ 8 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\vec{c}_B^T = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{c}_N^T = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$\mathbf{N} = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \end{array} \qquad \mathbf{B}_0 = \begin{array}{c} \begin{array}{cccc} x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- Solve $\vec{y}^T\mathbf{B}_0 = \vec{c}_B^T$. Since $\mathbf{B}_0 = I$ and $\vec{c}_B^T = 0$ we have $\vec{y}^T = [0\ 0\ 0\ 0]$.
- Look at $\vec{c}_N^T - \vec{y}^T\mathbf{N}$ to choose entering column:

$$\vec{c}_N^T - \vec{y}^T\mathbf{N} = [2\ 3] \qquad \text{since } \vec{y}^T = 0$$

  We choose the second component — hence x_2 enters and $\vec{a} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$.

- Solve $\mathbf{B}_0\vec{d} = \vec{a}$. Since $\mathbf{B}_0 = I$ we have $\vec{d} = \vec{a}$.
- Find largest t such that $\vec{x}_B^* - t\vec{d} \geq 0$:

$$\begin{bmatrix} 6 \\ 8 \\ 1 \\ 1 \end{bmatrix} - t\begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \geq 0 \qquad \Longrightarrow t = 1/2 \text{ and } \begin{bmatrix} 6 \\ 8 \\ 1 \\ 1 \end{bmatrix} - (1/2)\begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15/2 \\ 1/2 \\ 0 \end{bmatrix}$$

  The fourth component is zero — hence x_6 leaves.

- Update, x_2 swaps with x_6

$$\vec{x}_B^* = \begin{bmatrix} 5 \\ 15/2 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad \vec{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_2 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_1 \\ x_6 \end{bmatrix}$$

$$\vec{c}_B^T = \begin{bmatrix} 0 & 0 & 0 & 3 \end{bmatrix} \qquad \vec{c}_N^T = \begin{bmatrix} 2 & 0 \end{bmatrix}$$

$$\mathbf{N} = \begin{array}{c} \begin{matrix} x_1 & x_6 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Second iteration

- Solve $\vec{y}^T \mathbf{E}_1 = \vec{c}_B^T$:

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 3 \end{bmatrix} \implies \vec{y}^T = \begin{bmatrix} 0 & 0 & 0 & 3/2 \end{bmatrix}$$

- Look at $\vec{c}_N^T - \vec{y}^T \mathbf{N}$:

$$\begin{bmatrix} 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 3/2 \end{bmatrix} \begin{array}{c} \begin{matrix} x_1 & x_6 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 2 & -3/2 \end{bmatrix}$$

We pick the first component — x_1 enters and $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$.

- Solve $\mathbf{E}_1 \vec{d} = \vec{a}$:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \implies \vec{d} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

- Find largest t such that $\vec{x}_B^* - t\vec{d} \geq 0$:

$$\begin{bmatrix} 5 \\ 15/2 \\ 1/2 \\ 1/2 \end{bmatrix} - t \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \geq 0 \implies t = 15/4 \text{ and } \begin{bmatrix} 5 \\ 15/2 \\ 1/2 \\ 1/2 \end{bmatrix} - (15/4) \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 0 \\ 17/4 \\ 1/2 \end{bmatrix}$$

  The second component is zero — x_4 leaves.

- Update — x_1 swaps with x_4:

$$\vec{x}_B^* = \begin{bmatrix} 5/4 \\ 15/4 \\ 17/4 \\ 1/2 \end{bmatrix} \qquad \vec{x}_B = \begin{bmatrix} x_3 \\ x_1 \\ x_5 \\ x_2 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_4 \\ x_6 \end{bmatrix}$$

$$\vec{c}_B^T = \begin{bmatrix} 0 & 2 & 0 & 3 \end{bmatrix} \qquad \vec{c}_N^T = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\mathbf{N} = \begin{array}{c} \begin{matrix} x_4 & x_6 \end{matrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \mathbf{E}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Third iteration

- Solve $\vec{y}^T \mathbf{E}_1 \mathbf{E}_2 = \vec{c}_B^T$:

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 3 \end{bmatrix} \implies \vec{u}^T = \begin{bmatrix} 0 & 1 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 3 \end{bmatrix} \implies \vec{y}^T = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$$

- Look at $\vec{c}_N^T - \vec{y}^T \mathbf{N}$:

$$\begin{bmatrix} 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{array}{c} \begin{matrix} x_4 & x_6 \end{matrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array} = [-1, -1]$$

  There are no entering variables or columns, so current solution is optimal.

- Optimal solution is

$$\vec{x}_B = \begin{bmatrix} x_3 \\ x_1 \\ x_5 \\ x_2 \end{bmatrix} = \vec{x}_B^* = \begin{bmatrix} 5/4 \\ 15/4 \\ 17/4 \\ 1/2 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_4 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$z = \vec{c}_B^T \vec{x}_B = 2 \times (15/4) + 3 \times (1/2) = 9$$

3. $\boxed{\text{5 marks}}$ Your colleague is working on the following LP problem:

$$\begin{array}{lrrrcl} \text{Maximize } z = & -x_1 & +x_2 & +2x_3 & & \\ \text{subject to} & x_1 & +2x_2 & -x_3 & \leq & 2 \\ & 2x_1 & +3x_2 & +x_3 & \leq & 6 \\ & -2x_1 & +4x_2 & +2x_3 & \leq & 8 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

He tells you that he has found an optimal solution when x_2, x_5, x_6 are non-basic and x_1, x_3, x_4 are basic.

(a) Write down the original dictionary and the dictionary that is thought to be optimal: Give $\mathbf{B}, \mathbf{N}, \vec{c_N}^T, \vec{c}_N^T, \vec{x}_B, \vec{x}_N, \vec{b}$, and remember to write in the basis headings (that is, label the columns of matrices with the corresponding variables).

Solution:

- The original dictionary is given by

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \mathbf{N} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ -2 & 4 & 2 \end{bmatrix} \end{array}$$

$$\vec{x}_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$$

$$\vec{c}_B^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \vec{c}_N^T = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}$$

- So if we make x_2, x_5, x_6 non-basic and the other variables basic, we get

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} x_1 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \end{array} \qquad \mathbf{N} = \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \end{array}$$

$$\vec{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_2 \\ x_5 \\ x_6 \end{bmatrix}$$

$$\vec{c}_B^T = \begin{bmatrix} -1 & 2 & 0 \end{bmatrix} \qquad \vec{c}_N^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

(b) What are the values of the basic and non-basic variables in your colleague's dictionary? Is this a feasible solution?

Solution:

- So the current basic feasible solution is given by $\vec{x}_N = 0$ and $\vec{x}_B^* = \mathbf{B}^{-1}\vec{b}$. To find the latter we solve $\mathbf{B}\vec{x}_B^* = \vec{b}$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$$

  which gives $x_4^* = 6, x_3^* = \frac{14}{3}, x_1^* = \frac{2}{3}$. The other variables are zero.

- Since all the variables are non-negative, it follows that the dictionary is feasible.

(c) Is your colleague's dictionary optimal?

Solution:

- To check optimality we need to verify that $\vec{c}_N^T - \vec{c}_B^T\mathbf{B}^{-1}\mathbf{N} \leq 0$.
- First compute $\vec{y}^T = \vec{c}_B^T\mathbf{B}^{-1}$ by solving $\vec{y}^T\mathbf{B} = \vec{c}_B^T$:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} x_1 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} -1 & 2 & 0 \end{bmatrix}$$

This gives $y_1 = 0$, $2y_2 - 2y_3 = -1$, $y_2 + 2y_3 = 2$. Hence $\vec{y}^T = [0, \frac{1}{3}, \frac{5}{6}]$

- So $\vec{c}_N^T - \vec{y}^T\mathbf{N}$ is:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{3} & \frac{5}{6} \end{bmatrix} \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} -\frac{10}{3} & -\frac{1}{3} & -\frac{5}{6} \end{bmatrix}$$

Hence the current dictionary is optimal.

(d) What is the optimal value?

Solution:

- The objective function is therefore $z = -1 \times \frac{2}{3} + 0 + 2 \times \frac{14}{3} = \frac{26}{3}$.

4. $\boxed{\text{5 marks}}$ [This is continuation of the problem 3] Your colleague is working on the following LP problem:

$$\begin{array}{lrrrcl} \text{Maximise } z = & -x_1 & +x_2 & +2x_3 & & \\ \text{subject to} & x_1 & +2x_2 & -x_3 & \leq & 2 \\ & 2x_1 & +3x_2 & +x_3 & \leq & 6 \\ & -2x_1 & +4x_2 & +2x_3 & \leq & 8 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

He tells you that he has found an optimal solution when x_2, x_5, x_6 are non-basic and x_1, x_3, x_4 are basic. **[You can use the results of Problem 3.]**

(a) Your colleague then decides to change the third constraint to

$$-2x_1 + 4x_2 + 2x_3 \leq 8 + \beta$$

Over what range of β is your colleague's dictionary feasible and optimal. What is the corresponding solution + optimal value?

Solution:

- Changing $\vec{b}$ does not effect the coefficients of the non-basic variables in z. Hence they all remain negative.

- So we only need to check feasibility, we do this by seeing if $\vec{x}_B^* \geq 0$

- To do this we solve $\mathbf{B}\vec{x}_B^* = \vec{b}$:

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 8+\beta \end{bmatrix}$$

which gives

$$\vec{x}_B^* = \begin{bmatrix} \frac{4-\beta}{6} \\ \frac{14+\beta}{3} \\ \frac{12+\beta}{2} \end{bmatrix}$$

- In order for this to be non-negative, we need

$$-12 \leq \beta \leq 4$$

- And in this range the optimal solution is

$$x_1^* = \frac{4-\beta}{6} \qquad x_2^* = 0 \qquad x_3^* = \frac{14+3\beta}{3}$$
$$x_4^* = \frac{12+\beta}{2} \qquad x_5^* = x_6^* = 0$$

and the objective function is

$$\begin{aligned} z &= -x_1 + x_2 + 2x_3 = -\frac{4-\beta}{6} + 2\frac{14+\beta}{3} \\ &= \frac{26}{3} + \frac{5\beta}{6} \end{aligned}$$

(b) Your (increasingly indecisive) colleague then decides to leave the third constraint as it was originally, but instead changes the objective function to

$$z = (\gamma - 1)x_1 + x_2 + 2x_3$$

Over what range of γ is the dictionary feasible and optimal. What is the corresponding solution and optimal value?

Solution:

- Changing $\vec{c}$ does not change the current value of $\vec{x}_B^*$ and so the dictionary remains feasible.

- Hence we only need to check optimality, which we do by checking that $\vec{c}_N^T - \vec{c}_B^T \mathbf{B}^{-1}\mathbf{N} \leq 0$.
- First solve $\vec{y}^T \mathbf{B} = \vec{c}_B^T$:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} x_1 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} \gamma - 1 & 2 & 0 \end{bmatrix}$$

$$\vec{y}^T = \begin{bmatrix} 0 & \frac{1+\gamma}{3} & \frac{5-\gamma}{6} \end{bmatrix}$$

- Now check $\vec{c}_N^T - \vec{y}^T \mathbf{A}_N$:

$$\vec{c}_N^T - \vec{y}^T \mathbf{A}_N = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1+\gamma}{3} & \frac{5-\gamma}{6} \end{bmatrix} \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \end{array}$$
$$= \begin{bmatrix} \frac{-10-\gamma}{3} & \frac{-1-\gamma}{3} & \frac{\gamma-5}{6} \end{bmatrix}$$

- Now in order for $\vec{c}_N^T$ to be non-positive we require

$$-1 \leq \gamma \leq 5$$

- In this range the optimal solution is

$$x_1^* = \frac{2}{3} \qquad x_2^* = 0 \qquad x_3^* = \frac{14}{3} \qquad x_4^* = 6 \qquad x_5^* = x_6^* = 0$$

  and the optimal value is

$$z = (\gamma - 1)x_1^* + x_2^* + 2x_3^* = \frac{2\gamma - 2}{3} + 0 + \frac{28}{3}$$
$$= \frac{26}{3} + \frac{2}{3}\gamma$$

5. $\boxed{\text{5 marks}}$ Consider the following LP problem in which the objective function depends

on a parameter γ:

$$\begin{array}{rllll} & \max\ (4+\gamma)x_1 + 5x_2 & & & \\ & x_1 & +x_2 & \leq & 5 \\ \text{such that} & x_1 & +2x_2 & \leq & 8 \\ & 2x_1 & +x_2 & \leq & 8 \\ & x_1, & x_2 & \geq 0 & \end{array}$$

When $-\frac{3}{2} \leq \gamma \leq 1$ this has optimal dictionary

$$\begin{array}{rcrrr} x_1 & = & 2 & -2x_3 & +x_4 \\ x_2 & = & 3 & +x_3 & -x_4 \\ x_5 & = & 1 & +3x_3 & -x_4 \\ \hline z & = & (23+2\gamma) & -(3+2\gamma)x_3 & +(\gamma-1)x_4 \end{array}$$

<u>Starting from this dictionary</u> find the optimal solution and the optimal value as a function of γ,

(a) when $-\frac{3}{2} \leq \gamma \leq 1$,

Solution: The initial dictionary is

$$\begin{array}{rcrrr} x_1 & = & 2 & -2x_3 & +x_4 \\ x_2 & = & 3 & +x_3 & -x_4 \\ x_5 & = & 1 & +3x_3 & -x_4 \\ \hline z & = & (23+2\gamma) & -(3+2\gamma)x_3 & -(1-\gamma)x_4 \end{array}$$

This is feasible and optimal while $-\frac{3}{2} \leq \gamma \leq 1$. This gives

$$x_1 = 2 \qquad x_2 = 3 \qquad x_3 = x_4 = 0 \qquad x_5 = 1 \qquad z = 23 + 2\gamma$$

(b) when $-\infty \leq \gamma \leq -\frac{3}{2}$,

Solution: When $\gamma \leq -3/2$ then the then the coefficient of x_3 becomes positive and so x_3 must enter, and x_1 must leave. Pivoting gives:

$$\begin{array}{rcrrr} x_3 & = & 1 & -(1/2)x_1 & +(1/2)x_4 \\ x_2 & = & 4 & -(1/2)x_1 & -(1/2)x_4 \\ x_5 & = & 4 & -(3/2)x_1 & +(1/2)x_4 \\ \hline z & = & 20 & +(3/2+\gamma)x_1 & -(5/2)x_4 \end{array}$$

We see that this is feasible optimal for all $\gamma < -3/2$. This gives

$$x_1 = 0 \qquad x_2 = 4 \qquad x_3 = 1 \qquad x_4 = 0 \qquad x_5 = 4 \qquad z = 20$$

(c) when $1 \leq \gamma \leq 6$ and

Solution: If $\gamma > 1$ then the coefficient of x_4 becomes positive and so it must enter the basis. The leaving variable is x_5. Pivoting gives:

$$\begin{array}{rcrrr} x_1 & = & 3 & +x_3 & -x_5 \\ x_2 & = & 2 & -2x_3 & +x_5 \\ x_4 & = & 1 & +3x_3 & -x_5 \\ \hline z & = & (22+3\gamma) & +(\gamma-6)x_3 & +(1-\gamma)x_5 \end{array}$$

this is clearly feasible + optimal while $1 \leq \gamma \leq 6$. This gives

$$x_1 = 3 \qquad x_2 = 2 \qquad x_3 = 0 \qquad x_4 = 1 \qquad x_5 = 0 \qquad z = 22 + 3\gamma$$

(d) when $6 \leq \gamma \leq \infty$.

Solution: If $\gamma > 6$ the coefficient of x_3 becomes positive and it must enter the basis, and we see that x_2 leaves. Pivoting gives

$$\begin{array}{rcrrr} x_1 & = & 4 & -(1/2)x_2 & -(1/2)x_5 \\ x_3 & = & 1 & -(1/2)x_2 & +(1/2)x_5 \\ x_4 & = & 4 & -(3/2)x_2 & +(1/2)x_5 \\ \hline z & = & (16+4\gamma) & +(3-\gamma/2)x_2 & +(2-\gamma/2)x_5 \end{array}$$

This is feasible + optimal for all $\gamma \geq 6$.

$$x_1 = 4 \qquad x_2 = 0 \qquad x_3 = 1 \qquad x_4 = 4 \qquad x_5 = 0 \qquad z = 16 + 4\gamma$$

Do not solve the problem from scratch each time. You will find it easier to do this problem using dictionaries rather than the revised simplex method.

Solution: So to summarise we have

$$\begin{array}{rccc} \gamma \leq -3/2 & x_1^* = 0 & x_2^* = 4 & z = 20 + 0\gamma \\ -3/2 \leq \gamma \leq 1 & x_1^* = 2 & x_2^* = 3 & z = 23 + 2\gamma \\ 1 \leq \gamma \leq 6 & x_1^* = 3 & x_2^* = 2 & z = 22 + 3\gamma \\ 6 \leq \gamma & x_1^* = 4 & x_2^* = 0 & z = 16 + 4\gamma \end{array}$$

6. $\boxed{\text{5 marks}}$ Consider the following LP problem in which one of the constraints bounds

depends on a parameter β:

$$\max\ 4x_1 + 5x_2$$
$$\text{such that}\quad \begin{array}{llcr} x_1 & +x_2 & \le & 5 \\ x_1 & +2x_2 & \le & 8+\beta \\ 2x_1 & +x_2 & \le & 8 \\ x_1, & x_2 & \ge 0 & \end{array}$$

When $-1 \le \beta \le 2$ the optimal dictionary is

$$\begin{array}{rcrrr} x_1 & = & 2-\beta & -2x_3 & +x_4 \\ x_2 & = & 3+\beta & +x_3 & -x_4 \\ x_5 & = & 1+\beta & +3x_3 & -x_4 \\ \hline z & = & 23+\beta & -3x_3 & -x_4 \end{array}$$

Starting from this dictionary find the optimal solution and the optimal value as a function of β,

(a) when $-1 \le \beta \le 2$,

Solution: The initial dictionary is

$$\begin{array}{rcrrr} x_1 & = & 2-\beta & -2x_3 & +x_4 \\ x_2 & = & 3+\beta & +x_3 & -x_4 \\ x_5 & = & 1+\beta & +3x_3 & -x_4 \\ \hline z & = & 23+\beta & -3x_3 & -x_4 \end{array}$$

This is dual-feasible and feasible when $-1 \le \beta \le 2$. This gives

$$x_1 = 2-\beta \qquad x_2 = 3+\beta \qquad x_3 = x_4 = 0 \qquad x_5 = 1+\beta \qquad z = 23+\beta$$

(b) when $2 \le \beta \le \infty$,

Solution: When $\beta \ge 2$ the dictionary remains dual feasible, but stops being feasible. In particular, $x_1 < 0$. Hence x_1 is the leaving variable. Rewriting the pivot row gives $0 = (2-\beta) - x_1 - 2x_3 + x_4$. Adding $q\times$ this equation to z gives

$$z = 23 + 2q - 2\beta q - qx_1 + x_3(-3-2q) + x_4(-1+q)$$

The largest q that keeps dual-feasibility is $q = 1$. Hence x_4 is the entering variable. Pivoting gives

$$\begin{array}{rcrrr} x_4 & = & -2+\beta & +x_1 & +2x_3 \\ x_2 & = & 5 & -x_1 & -x_3 \\ x_5 & = & 3 & -x_1 & +x_3 \\ \hline z & = & 25 & -x_1 & -5x_3 \end{array}$$

This is feasible and dual-feasible for all $\beta \geq 2$, and gives

$$x_1 = 0 \qquad x_2 = 5 \qquad x_3 = 0 \qquad x_4 = -2 + \beta \qquad x_5 = 3 \qquad z = 25$$

(c) when $-4 \leq \beta \leq -1$,

Solution: In this range x_5 becomes negative — so x_5 is the leaving variable. Rewriting the pivot row gives $0 = 1+\beta-x_5+3x_3-x_4$. Adding $q\times$ this equation to z gives

$$z = (23+\beta) + q(1+\beta) - x_5 q + x_3(-3+3q) + x_4(-1-q)$$

Hence $q = 1$ and x_3 is the leaving variable. Pivoting gives

$$\begin{array}{rcrrr} x_1 & = & 8/3-(1/3)\beta & +(1/3)x_4 & -(2/3)x_5 \\ x_2 & = & 8/3+(2/3)\beta & -(2/3)x_4 & +(1/3)x_5 \\ x_3 & = & -1/3-(1/3)\beta & +(1/3)x_4 & +(1/3)x_5 \\ \hline z & = & 24+2\beta & -2x_4 & -x_5 \end{array}$$

This is feasible and optimal for $-4 \leq \beta \leq -1$, and gives

$$x_1 = \frac{8-\beta}{3} \qquad x_2 = \frac{8+2\beta}{3} \qquad x_3 = -\frac{1+\beta}{3} \qquad x_4 = 0 \qquad x_5 = 0$$
$$z = 24 + 2\beta$$

(d) and when $-8 \leq \beta \leq -4$.

Solution: In this range x_2 becomes negative — so x_2 is the leaving variable. Rewrite the pivot row to get $0 = (8/3)+(2/3)\beta-x_2-(2/3)x_4+(1/3)x_5$. Adding $q\times$ this row to z gives:

$$z = 24 + 2\beta + q((8/3)+(2/3)\beta) - qx_2 + x_4(-2-(2/3)q) + x_5((-1+(1/3)q).$$

Hence $q = 3$ and x_5 is the entering variable.

$$\begin{array}{rcrrr} x_1 & = & 8+\beta & -2x_2 & -x_4 \\ x_3 & = & -3-\beta & +x_2 & +x_4 \\ x_5 & = & -8-2\beta & +3x_2 & +2x_4 \\ \hline z & = & 32+4\beta & -3x_2 & -4x_4 \end{array}$$

This is feasible and optimal for $-8 \leq \beta \leq -4$, and gives

$$x_1^* = 8+\beta \qquad x_2^* = 0 \qquad x_3 = -3-\beta \qquad x_4 = 0 \qquad x_5 = -8-2\beta$$
$$z = 32 + 4\beta$$

(e) What happens to this problem when $\beta < -8$?

Solution: It suffices to see that the relevant constraint becomes

$$x_1 + 2x_2 \leq 8 + \beta < 0$$

Since x_1, x_2 are non-negative, this constraint cannot be satisfied and the problem becomes infeasible.

If we instead try to apply the dual-simplex method we find that x_1 becomes negative, and so the pivot row is $0 = 8 + \beta - x_1 - 2x_2 - x_4$. Adding $q\times$ this row to z gives:

$$z = 32 + 4\beta + q(8 + \beta) - qx_1 + x_2(-3 - 2q) + x_4(-4 - q)$$

The coefficients of the variables stay negative for all $q > 0$. Hence there is no entering variable. So we cannot pivot. Hence the problem is infeasible (the dual problem is unbounded).

<u>Do not</u> solve the problem from scratch each time. You will find it easier to do this problem using dictionaries and the dual-simplex method.

Solution: In summary

$\beta < -8$	infeasible		
$-8 \leq \beta \leq -4$	$x_1^* = 8 + \beta$	$x_2^* = 0$	$z^* = 32 + 4\beta$
$-4 \leq \beta \leq -1$	$x_1^* = \dfrac{8 - \beta}{3}$	$x_2^* = \dfrac{8 + 2\beta}{3}$	$z^* = 24 + 2\beta$
$-1 \leq \beta \leq 2$	$x_1^* = 2 - \beta$	$x_2^* = 3 + \beta$	$z^* = 23 + \beta$
$2 \leq \beta \leq \infty$	$x_1^* = 0$	$x_2^* = 5$	$z^* = 25$