2019 Math 340:101. Quiz 4 Friday, 2019 October 25 IN CLASS

Last name ....

First name ....
(Your preferred first name as in the Canvas)

Student number ....

Grade

1. (2 points) Consider the LP problem:

$$\begin{array}{lrrrl} \text{Maximize} & x_1 & +x_2 & +3x_3 & \\ \text{subject to} & 2x_1 & +x_2 & & \leq 5 \\ & x_1 & & +2x_3 & \leq 2 \\ & x_1 & +x_2 & +2x_3 & \leq 7 \end{array} \qquad x_1, x_2, x_3 \geq 0.$$

Suppose the basic variable is $\vec{x}_B = (x_2, x_5, x_6)$. Find the corresponding $\vec{x}_N$ and the matrices B, N and the vectors $\vec{c}_B$, and $\vec{c}_N$.

Solution: Notice that

$$\vec{c}^T = \begin{bmatrix} 1 & 1 & 3 & 0 & 0 & 0 \end{bmatrix} \qquad [A|I] = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

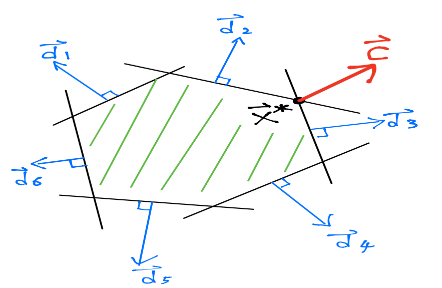
Therefore,

$$\vec{x}_B = \begin{bmatrix} x_2 \\ x_5 \\ x_6 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_1 \\ x_4 \\ x_3 \end{bmatrix}$$

$$\vec{c}_B^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad \vec{c}_N^T = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$$

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array} \qquad \mathbf{N} = \begin{array}{c} \begin{array}{ccc} x_1 & x_4 & x_3 \end{array} \\ \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{pmatrix} \end{array}$$

2. (4 points) Consider $\displaystyle\max_{A\vec{x}\leq\vec{b},\, \vec{x}\geq\vec{0}} \vec{c}\cdot\vec{x}$.



- A has 2 columns and 4 rows.
- The picture is in 2D.
- The shaded region in the picture is the feasible region of this LP problem, and
- $\vec{d}_i$'s are orthogonal to the corresponding hyperplanes.

Under the given condition, check whether each of the following statement is correct or not. Circle True or False. (1 mark per two correct answers, and 1 mark for giving all six correct answers).

(A) Changing $\vec{c}$ to another vector must change the optimal solution. **True** / **False**

(B) Changing $\vec{b}$ to another vector must change the optimal solution. **True** / **False**

(C) One and only one optimal dual solution exists. **True** / **False**

(D) Every row vector of A is one of $\vec{d}_1, \cdots, \vec{d}_6$. **True** / **False**

(E) If $\vec{y}^* = (y_1^*, \cdots, y_6^*)$ is an optimal dual solution (given in both original and slack variables), then four of $y_1^*, \cdots, y_6^*$ must be zero. **True** / **False**

(F) Changing $\vec{b}$ to another vector must change the optimal dual solution. **True** / **False**

Solution:

A) False.

B) False.

C) True.

D) True.

E) True

F) False.

3. (4 points) Let A be a given 3×3 matrix, and $\vec{a} = (-1, 0, 0)$, $\vec{b} = (1, 2, 3)$, $\vec{c} = (1, 1, 1)$, and $\vec{d} = (-2, 0, -5)$ are vectors in $\mathbf{R}^3$. Let $\vec{x}^*$ be an optimal solution of the LP problem $\max\limits_{A\vec{x} \leq \vec{b}, \vec{x} \geq \vec{0}} \vec{c} \cdot \vec{x}$. Assume that

$$\vec{c} \cdot \vec{x} = \vec{c} \cdot \vec{x}^* + \vec{a} \cdot \vec{x} + \vec{d} \cdot \vec{b} - \vec{d} \cdot A\vec{x} \qquad \textbf{for all } x \in \mathbf{R}^3\textbf{.}$$

Find an optimal solution $\vec{y}^*$ of the **dual problem**. To earn credit, you have to clearly justify your answer.

Solution: Can write $\vec{d} \cdot A\vec{x} = A^T \vec{d} \cdot \vec{x}$. Therefore,

$$\vec{c} \cdot \vec{x} = \vec{c} \cdot \vec{x}^* + \vec{a} \cdot \vec{x} + \vec{d} \cdot \vec{b} - A^T \vec{d} \cdot \vec{x} \qquad \text{for all } x \in \mathbf{R}^3.$$

Rearrange

$$(\vec{c} - \vec{a} + A^T \vec{d}) \cdot \vec{x} = \vec{c} \cdot \vec{x}^* + \vec{d} \cdot \vec{b} \qquad \text{for all } x \in \mathbf{R}^3.$$

As this holds for all $\vec{x} \in \mathbf{R}^3$, it implies that $(\vec{c} - \vec{a} + A^T \vec{d}) = \vec{0}$ and $\vec{c} \cdot \vec{x}^* + \vec{d} \cdot \vec{b} = 0$. Therefore, $\vec{c} + A^T \vec{d} = \vec{a}$ and $\vec{c} \cdot \vec{x}^* = -\vec{d} \cdot \vec{b}$. Let $\vec{y}^* = -\vec{d} = (2, 0, 5)$. Note that $\vec{y}^* \geq \vec{0}$. And $\vec{c} - A^T \vec{y}^* = \vec{a} \leq \vec{0}$. So, $\vec{y}^*$ is dual feasible. Moreovere $\vec{c} \cdot \vec{x}^* = \vec{y}^* \cdot \vec{b}$. Therefore, from the weak duality, $\vec{y}^*$ is dual optimal.

Alternative method: One can rewrite the expression

$$\begin{aligned}
&\vec{c} \cdot \vec{x}^* + \vec{a} \cdot \vec{x} + \vec{d} \cdot \vec{b} - \vec{d} \cdot A\vec{x} \\
&= \vec{c} \cdot \vec{x}^* + \vec{a} \cdot \vec{x} + \vec{d} \cdot (\vec{b} - \vec{A}\vec{x}) \\
&= \vec{c} \cdot \vec{x}^* + \vec{a} \cdot \vec{x} + \vec{d} \cdot \vec{v}
\end{aligned}$$

where $\vec{v}$ denotes the slack variable. The last line is an expression of the objective function that appears in the final dictionary as $\vec{a} \leq \vec{0}$ and $\vec{d} \leq \vec{0}$. Then, the strong duality implies that the dual optimal solution $\vec{y}^* = -\vec{d}$ (that is, $-$ the coefficient of the slack variable).