2019 Math 340:101. Quiz 5 Friday, 2019 November 8 IN CLASS

Last name . . . .

First name . . . .

Student number . . . .

Grade

1. (3 points) Given the following dictionary of a primal problem, find the corresponding **dual dictionary**.

$$\begin{array}{rcrrr} x_2 & = & 1 & -x_1 & -x_3 \\ x_4 & = & 2 & -2x_1 & +x_3 \\ x_5 & = & 2 & +x_1 & +x_3 \\ \hline z & = & 3 & -x_1 & +x_3 \end{array}$$

Solution:

- First put the objective function at the first row:

$$\begin{array}{rcrrr} z & = & 3 & -x_1 & +x_3 \\ \hline x_2 & = & 1 & -x_1 & -x_3 \\ x_4 & = & 2 & -2x_1 & +x_3 \\ x_5 & = & 2 & +x_1 & +x_3 \end{array}$$

- Recall the correspondence between the primal and dual variables. As there are 2 original variables and 3 slack variables, we have

$$x_j \longleftrightarrow y_{3+j} \text{ for } j = 1, 2,$$
$$x_{2+i} \longleftrightarrow y_i \text{ for } i = 1, 2, 3.$$

- Then, use the "$-$ transpose" property of the dual dictionary

$$\begin{array}{rcrrrr} -w & = & -3 & -y_5 & -2y_2 & -2y_3 \\ \hline y_4 & = & 1 & +y_5 & +2y_2 & -y_3 \\ y_1 & = & -1 & +y_5 & -y_2 & -y_3 \end{array}$$

2. (2 points) (1 mark per two correct answers) We use the notation in the class. You are given $A, \vec{b}, \vec{c}, \vec{x}_B, \vec{x}_N, B$ and N. Let $\vec{x}^*$ be the basic solution of the corresponding primal dictionary, and let $\vec{y}^*$ be the corresponding dual basic solution (i.e. the basic solution of the dual dictionary). Check whether the following statements are true or false.

   (a) **True** / **False** If $\vec{x}^*$, $\vec{y}^*$ are primal/dual feasible, respectively, then they are primal/dual optimal, respectively.

   (b) **True** / **False** It is possible in some examples, that $\vec{x}^*$ is not primal feasible and $\vec{y}^*$ is not dual feasible.

   (c) **True** / **False** If $\vec{x}^*$ is the unique primal optimal solution, then $\vec{y}^*$ is the unique dual optimal solution.

   (d) **True** / **False** If we change both $\vec{b}$ and $\vec{c}$ to different vectors $\vec{b}'$ and $\vec{c}'$, respectively, while $A, \vec{x}_B, \vec{x}_N, B$ and N remain the same, then the resulting primal and dual basic solutions $\vec{x}^{**}$, $\vec{y}^{**}$ will have to be different from $\vec{x}^*$, $\vec{y}^*$, respectively.

Solution.

(a) True.

(b) True.

(c) False.

(d) False.

3. We use the notation as in the class. You are given $A, \vec{b}, \vec{c}$, $\vec{x}_B$, B and B^{-1}.

$$[A|I] = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}, \qquad \vec{b} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \qquad \vec{x}_B = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix}$$

$$\vec{c}^T = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad B = \begin{array}{c} \begin{array}{ccc} x_3 & x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \qquad B^{-1} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $\vec{x}^*$ be the basic solution of the corresponding primal dictionary, and let $\vec{y}^*$ be the corresponding dual basic solution (i.e. the basic solution of the dual dictionary).

(i) (3 points) Find $x_1^*, x_2^*, x_3^*, x_4^*, x_5^*$ and $y_1^*, y_2^*, y_3^*, y_4^*, y_5^*$.

Solution: As $\vec{x}^*$ is the basic solution

$$\vec{x}_B^* = \begin{bmatrix} x_3^* \\ x_1^* \\ x_2^* \end{bmatrix} = B^{-1}\vec{b} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\vec{x}_N^* = \begin{bmatrix} x_4^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As $\vec{y}^*$ is the dual basic solution, $(\vec{y}^*)^T B = \vec{c}_B^T$, so

$$(\vec{y}^*)^T = \vec{c}_B^T B^{-1} = \begin{array}{c} \begin{array}{ccc} x_3 & x_1 & x_2 \end{array} \\ \begin{bmatrix} 0 & 3 & 2 \end{bmatrix} \end{array} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0 & 3 & -4 \end{bmatrix} \end{array}$$

So, $(y_1^*, y_2^*, y_3^*) = (0, 3, -4)$. Note that $\vec{y}_B = (y_1, y_4, y_5)$ using the primal/dual variable correspondence. Note that because $\vec{y}^*$ is dual basic, $\vec{y}_B^* = (y_1^*, y_4^*, y_5^*) = \vec{0}$. Therefore, $(y_1^*, y_2^*, y_3^*, y_4^*, y_5^*) = (0, 3, -4, 0, 0)$.

(ii) (2 points) Is $\vec{x}^*$ optimal? To earn credit, you have to justify your answer.

Solution.

Note that $\vec{x}_B^* > \vec{0}$ so $\vec{x}^*$ is non-degenerate. if it was optimal then the dual basic solution $\vec{y}^*$ must be dual feasible. However, $(y_1^*, y_2^*, y_3^*, y_4^*, y_5^*) \not\geq \vec{0}$, so not dual feasible.

Blank page