- You will *not* be able to use a calculator or computer for either the midterm or the final exam, so please do not use one for this assignment. You may use one to *check* your answer, but please do not use one to solve the problem.
- Only part of the problems may be graded. But, you have to submit all the problems.
- The deadline is *by 6pm on Friday*, September 27. You have a grace period until 9am Saturday, September 28. The grace period is to take care of any technical issues you have while submitting the file. The grace period should give ample time for handling any issues, so No late HW after the grace period will be accepted, regardless of your technical issues.
- **Submit only pdf files.**
- Remember to use **Anstee's rule**.

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1. Solve the following LP problems using the simplex method. At each step please state the entering and leaving variables and the current basic feasible solution. Clearly state the optimal solution and the optimal value.

   (a) $\boxed{\text{3 marks}}$ Maximise $z = 2x_1 + 3x_2 + 3x_3$, subject to

$$\begin{array}{llll} 3x_1 & +x_2 & & \leq & 60 \\ -x_1 & +x_2 & +4x_3 & \leq & 10 \\ 2x_1 & -2x_2 & +5x_3 & \leq & 15 \end{array}$$

   and $x_1, x_2, x_3 \geq 0$. *This problem requires 2 pivots.*

   (b) $\boxed{\text{4 marks}}$ Maximise $z = 3x_1 + 2x_2 + 4x_3$, subject to

$$\begin{array}{lllll} x_1 & +x_2 & +2x_3 & \leq & 4 \\ 2x_1 & & +3x_3 & \leq & 5 \\ 2x_1 & +x_2 & +3x_3 & \leq & 7 \end{array}$$

   and $x_1, x_2, x_3 \geq 0$. *This problem requires 3 pivots.*
   Note: this problem is Q2.1a from Chávtal.

2. $\boxed{\text{3 marks}}$ Show that the three inequalities

$$-x + 2y \leq -2 \qquad 2x + y \geq 1 \qquad -3x + y \geq -4$$

   have no solution x, y with $x, y \geq 0$ by using our two phase method.

3. $\boxed{\text{4 marks}}$ Use the two-phase method to find the solution of the following LP problem:

$$\begin{array}{rllll} & \multicolumn{4}{l}{\text{maximise } z = 3x_1 + x_2} \\ & x_1 & +x_2 & \leq & 1 \\ \text{subject to} & -2x_1 & +x_2 & \geq & 1 \\ & 3x_1 & +2x_2 & \geq & 4 \\ \text{and} & \multicolumn{4}{l}{x_1, x_2 \geq 0} \end{array}$$

   This requires around 3 pivots.

4. $\boxed{\text{4 marks}}$ Use the two-phase method to solve the following LP problem:

$$\begin{array}{ll} & \text{maximise } z = 3x_1 + 2x_2 + 3x_3 \\ \text{subject to} & \begin{array}{rrrcl} 2x_1 & +x_2 & +x_3 & \leq & 2 \\ 3x_1 & +4x_2 & +2x_3 & \geq & 8 \end{array} \\ & \text{and } x_1, x_2, x_3 \geq 0 \end{array}$$

This requires around 3 pivots.

5. (a) $\boxed{\text{3 marks}}$ We say a set C of points in $\mathbf{R}^n$ is convex if for every pair $\mathbf{x}, \mathbf{y} \in C$, all points on the line segment joining $\mathbf{x}$ and $\mathbf{y}$ are in C. Thus C is a convex set if for every pair $\mathbf{x}, \mathbf{y} \in C$ and any $\lambda \in (0, 1)$. then $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C$. Let A be an $m \times n$ matrix and $\mathbf{b}$ a given vector in $\mathbf{R}^m$. Show that

$$F = \{\mathbf{x} \in \mathbf{R}^n \,:\, A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}\}$$

is a convex set.

(b) $\boxed{\text{2 marks}}$ Consider an LP: max $\mathbf{c} \cdot \mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Assume the LP has two optimal solutions $\mathbf{u}$ and $\mathbf{v}$. Show that for any choice of $\lambda \in [0, 1]$ (i.e. $0 \leq \lambda \leq 1$), that $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ is also an optimal solution. First show that $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ has the same value of the objective function as $\mathbf{u}$. Then show $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ is a feasible solution to the LP.

6. We say that a function $f : \mathbf{R}^n \to \mathbf{R}$ is convex if for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\lambda \in [0, 1]$, we have $f((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) \leq (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y})$.

(a) $\boxed{\text{2 marks}}$ Give an example of a convex function. To get credits, you have to justify why it is convex.

(b) $\boxed{\text{3 marks}}$ Let f_1, f_2 be two given convex functions on $\mathbf{R}^n$. Then, consider the function $g : \mathbf{R}^n \to \mathbf{R}$ defined as

$$g(\mathbf{x}) = \max[f_1(\mathbf{x}), f_2(\mathbf{x})];$$

this means that given $\mathbf{x}$, $g(\mathbf{x})$ takes the larger value between $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$, that is,

$$g(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \text{if } f_1(\mathbf{x}) \geq f_2(\mathbf{x}), \\ f_2(\mathbf{x}) & \text{if } f_2(\mathbf{x}) \geq f_1(\mathbf{x}). \end{cases}$$

Prove that g is a convex function. [**Note that you have to prove this for ANY convex functions f_1 and f_2. You will get zero mark if you do this only for a particular pair of functions.**]