Math 340 Practice problems. Solutions.

1. (a) *Solve the following linear programming problem, using our standard two phase method and using Anstee's rule. It will take three pivots in phase 1 (including special pivot to feasibility) and one pivot in phase 2.*

$$\begin{array}{rrrrcr}
\text{Maximize} & -3x_1 & +2x_2 & -2x_3 \\
& & x_2 & -x_3 & \leq & 4 \\
& x_1 & -x_2 & & \leq & -3 \\
& x_1 & & -x_3 & \leq & -1
\end{array} \qquad x_1, x_2, x_3 \geq 0$$

SOLUTION.

Phase 1 initial dictionary:

$$\begin{array}{rrrrrr}
x_4 = & 4 & & -x_2 & +x_3 & +x_0 \\
x_5 = & -3 & -x_1 & +x_2 & & +x_0 \\
x_6 = & -1 & -x_1 & & +x_3 & +x_0 \\
w = & & & & & -x_0
\end{array}$$

Special pivot to feasibility, x_0 enters, x_5 exits:

$$\begin{array}{rrrrrr}
x_0 = & 3 & +x_1 & -x_2 & & +x_5 \\
x_4 = & 7 & +x_1 & -2x_2 & +x_3 & +x_5 \\
x_6 = & 2 & & -x_2 & +x_3 & +x_5 \\
w = & -3 & -x_1 & +x_2 & & -x_5
\end{array}$$

Now x_2 enters, x_6 exits:

$$\begin{array}{rrrrrr}
x_0 = & 1 & +x_1 & -x_3 & & +x_6 \\
x_2 = & 2 & & +x_3 & +x_5 & -x_6 \\
x_4 = & 3 & +x_1 & -x_3 & -x_5 & +2x_6 \\
w = & -1 & -x_1 & +x_3 & & -x_6
\end{array}$$

Next x_3 enters, x_0 exits:

$$\begin{array}{rrrrrr}
x_2 = & 3 & +x_1 & +x_5 & & -x_0 \\
x_3 = & 1 & +x_1 & & +x_6 & -x_0 \\
x_4 = & 2 & & -x_5 & +x_6 & +x_0 \\
w = & & & & & -x_0
\end{array}$$

Phase 1 ends, initial dictionary for phase 2:

$$\begin{array}{rrrrr}
x_2 = & 3 & +x_1 & +x_5 \\
x_3 = & 1 & +x_1 & & +x_6 \\
x_4 = & 2 & & -x_5 & +x_6 \\
z = & 4 & -3x_1 & +2x_5 & -2x_6
\end{array}$$

Then x_5 enters, x_4 exits:

$$\begin{array}{rrrrr}
x_2 = & 5 & +x_1 & -x_4 & +x_6 \\
x_3 = & 1 & +x_1 & & +x_6 \\
x_5 = & 2 & & -x_4 & +x_6 \\
z = & 8 & -3x_1 & -2x_4
\end{array}$$

And we've reached optimality. The optimal solution is $x_1 = 0$, $x_2 = 5$, $x_3 = 1$, $x_4 = 0$, $x_5 = 2$, $x_6 = 0$ which makes $z = 8$.

(b) *Either give two optimal solutions or show the optimal solution is unique.*

SOLUTION. In any optimal solution $x_1 = x_4 = 0$ because they appear with negative coefficient in the z-row of the final dictionary (recall the argument: since $x_1, x_4 \geq 0$, we have $z = 8 - 3x_1 - 2x_4 \leq 8$; equality can only hold if $x_1 = x_4 = 0$). But x_6 is a non-basic variable with zero coefficient in the z-row, this has the potential for multiple solutions. Looking at the top part of the dictionary, we see that x_6 can take any non-negative value without any other variable becoming zero, so *all* solutions are given by $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 5 + t, 1 + t, 0, 2 + t, t)$ for $t \geq 0$. (The problem, of course, just asks for one additional solution, so we could have just set $x_6 = 3$ or whatever.)

2. *Consider the following linear program:*

$$
\begin{array}{rrrrl}
\text{Maximize} & -2x_1 & +3x_2 & +7x_3 & \\
& & +x_2 & +2x_3 & \leq 2 \\
& -x_1 & +x_2 & +4x_3 & \leq 3 \\
& x_1 & +x_2 & +4x_3 & \leq 4
\end{array}
\qquad x_1, x_2, x_3 \geq 0
$$

(a) *Give the Dual Linear Program of the above Primal Linear Program.*

SOLUTION.

$$
\begin{array}{rrrrl}
\text{Minimize} & 2y_1 & +3y_2 & +4y_3 & \\
& & -y_2 & +y_3 & \geq -2 \\
& y_1 & +y_2 & +y_3 & \geq 3 \\
& 2y_1 & +4y_2 & +4y_3 & \geq 7
\end{array}
\qquad y_1, y_2, y_3 \geq 0
$$

(b) *You are given that an optimal primal solution has $x_1^* = 0$, $x_2^* = 1$, $x_3^* = \frac{1}{2}$. Determine an optimal dual solution (without pivoting), stating which theorems you have used.*

SOLUTION. We will use Complementary Slackness: since $x_2^*, x_3^* > 0$, the second and third constraint of dual have no slack:

$$
y_1^* + y_2^* + y_3^* = 3
$$
$$
2y_1^* + 4y_2^* + 4y_3^* = 7
$$

Now we check the primal constraints for slack. Only the third constraint has slack, so again by Complementary Slackness, we have $y_3^* = 0$.

Solving the equations we have found, we get that there is only one optimal dual solution, namely: $y_1^* = \frac{5}{2}, y_2^* = \frac{1}{2}, y_3^* = 0$

(c) *Consider changing the objective function of the primal by raising the coefficient of x_1 from -2 to -1. Does the primal solution $x_1^* = 0$, $x_2^* = 1$, $x_3^* = \frac{1}{2}$ and the optimal dual solution determined in b) remain optimal to their new LP's? Explain.*

SOLUTION. Yes, they remain optimal. One way we can check is by using weak duality: if you have feasible primal and dual solutions which have equal objective function values, both must be optimal. So consider the old optimal primal and dual solutions for the new primal and new dual. The old optimal primal soltion remains feasible for the new primal because the primal's constraints haven't changed. The old optimal dual solution remains feasible because the only dual constraint that changes is the first one, which becomes $-y_2^* + y_3^* \geq -1$ and is still true. Finally, the objective functions agree: the one for the dual hasn't changed; the primal objective function has changed, but the only change is in the coefficient of x_1 and since $x_1^* = 0$ the value doesn't change.

3. *Given $A^{\text{aug}}, \mathbf{b}, \mathbf{c}^{\text{aug}}$, current basis (and A_B^{-1} for your computational ease), use our Revised Simplex method and Anstee's rule to determine the next entering variable (if there is one), the next leaving variable (if there is one), and the new basic feasible solution after the pivot*

as well as the new basis matrix A_B (if there is both an entering and leaving variable). The current basis is $\{x_7, x_3, x_2\}$.

$$A^{\text{aug}} = \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}\begin{pmatrix} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 2 & -1 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 & 0 & 0 & 1 \end{array} \end{pmatrix}, \mathbf{b} = \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}\begin{pmatrix} -3 \\ -2 \\ -5 \end{pmatrix}, A_B^{-1} = \begin{array}{c} \\ x_7 \\ x_3 \\ x_2 \end{array}\begin{pmatrix} \begin{array}{ccc} x_5 & x_6 & x_7 \\ -3 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{array} \end{pmatrix}$$

$$(\mathbf{c}^{\text{aug}})^\top = \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ (1 & -1 & 1 & 2 & 0 & 0 & 0) \end{array}$$

SOLUTION. The coefficients in the z-row are given by $\mathbf{c}_N^\top - \mathbf{c}_B^\top A_B^{-1} A_N$, which is:

$$\begin{array}{cccc} x_1 & x_4 & x_5 & x_6 \\ (1 & 2 & 0 & 0) \end{array} - \begin{array}{ccc} x_7 & x_3 & x_2 \\ (0 & 1 & -1) \end{array} \begin{array}{c} \\ x_7 \\ x_3 \\ x_2 \end{array}\begin{pmatrix} \begin{array}{ccc} x_5 & x_6 & x_7 \\ -3 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{array} \end{pmatrix} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}\begin{pmatrix} \begin{array}{cccc} x_1 & x_4 & x_5 & x_6 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array} \end{pmatrix} =$$

$$\begin{array}{cccc} x_1 & x_4 & x_5 & x_6 \\ (1 & 2 & 0 & 0) \end{array} - \begin{array}{ccc} x_5 & x_6 & x_7 \\ (0 & 1 & 0) \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}\begin{pmatrix} \begin{array}{cccc} x_1 & x_4 & x_5 & x_6 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array} \end{pmatrix} =$$

$$\begin{array}{cccc} x_1 & x_4 & x_5 & x_6 \\ (1 & 2 & 0 & 0) \end{array} - \begin{array}{cccc} x_1 & x_4 & x_5 & x_6 \\ (1 & 1 & 0 & 1) \end{array} = \begin{array}{cccc} x_1 & x_4 & x_5 & x_6 \\ (0 & 1 & 0 & -1) \end{array}.$$

So the entering variable is x_4. To determine the exiting variable we need to find the largest $t \geq 0$ such that $A_B^{-1}\mathbf{b} - tA_B^{-1}A_4 \geq 0$.

We have: $A_B^{-1}\mathbf{b} = \begin{array}{c} x_7 \\ x_3 \\ x_2 \end{array}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $A_B^{-1}A_4 = \begin{array}{c} x_7 \\ x_3 \\ x_2 \end{array}\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$, so $A_B^{-1}\mathbf{b} - tA_B^{-1}A_4 = \begin{array}{c} x_7 \\ x_3 \\ x_2 \end{array}\begin{pmatrix} 2 - 5t \\ 1 - 2t \\ 3 - t \end{pmatrix}$,

which is ≥ 0 for $t \leq 2/5$. The exiting variable is x_7, and the next basic solution is $x_4 = \max t = 2/5$, $x_3 = 1 - 2(2/5) = 1/5$, $x_2 = 3 - 2/5 = 13/5$.

By simply consulting A^{aug}, we see the new A_B will be $\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}\begin{pmatrix} \begin{array}{ccc} x_4 & x_3 & x_2 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & -2 \end{array} \end{pmatrix}$. If the

question had asked instead for the new A_B^{-1}, probably the best way to get it is from

$(\text{new } A_B^{-1}) = E^{-1}(\text{old } A_B^{-1})$, where $E = \begin{pmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ (the first column is $A_B^{-1}A_4$), whose

inverse is $E^{-1} = \begin{pmatrix} 1/5 & 0 & 0 \\ -2/5 & 1 & 0 \\ -1/5 & 0 & 1 \end{pmatrix}$. Multiplying out gives new $A_B^{-1} = \begin{array}{c} \\ x_4 \\ x_3 \\ x_2 \end{array}\begin{pmatrix} \begin{array}{ccc} x_5 & x_6 & x_7 \\ -\frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{array} \end{pmatrix}.$

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4. *A student wants to take on extra curricular activities that will boost their chance of finding a job after graduation. We quantify the boost in terms of career points. There are limits on the students time, money and loss of self esteem. The activities of networking, taking professional exams and doing sports will all contribute to landing a job after graduation; each unit of the activities will result in some number of career points. The student wishes to maximize the number of career points. The activities (for this student) have different costs in terms of time, money and self esteem. The student has 34 hours to spend on these activities and has $1800 to spend (the costs are given in units of $100 below). A student can only lose 9 units of self esteem before affecting their success in UBC courses.*

	networking	exams	sports	available
hours	5	6	4	34
money	3	3	2	18
self esteem	1	2	1	9
career points	7	8	5	

Let x_1 denote the units of networking, x_2 denote the units of professional exams and x_3 denote the units of sports and let x_{3+i} denote the ith slack for $i = 1,2,3$. The final dictionary is:

$$\begin{aligned}
x_1 &= 2 + x_4 - 2x_5 \\
x_2 &= 2 + x_4 - x_5 - 2x_6 \\
x_3 &= 3 - 3x_4 + 4x_5 + 3x_6 \\
z &= 45 - 2x_5 - x_6
\end{aligned}
\qquad
A_B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}
\begin{array}{ccc} x_4 & x_5 & x_6 \\ \left(\begin{array}{ccc} -1 & 2 & 0 \\ -1 & 1 & 2 \\ 3 & -4 & -3 \end{array} \right) \end{array}$$

***NOTE**: All questions are independent of one another.*

(a) *Give the marginal values for each of the resources: hours, money and esteem. What are the units for the marginal values?*

SOLUTION. The marginal values are minus the coefficients of the slack variables x_4, x_5 and x_6 in the z-row:

Hours: 0 career points per hour.

Money: 2 career points per hundred dollars (you could also say 0.02 career points per dollar).

Self esteem: 1 career point per unit of self esteem.

(b) *Give the range on c_1 (career points for unit of networking) so that the current solution remains optimal.*

SOLUTION. We need $c_N^\top - c_B^\top A_B^{-1} A_N$ to stay ≤ 0. Let's calculate! Since the basis in the final dictionary is x_1, x_2, x_3, we have $c_N = \mathbf{0}, c_B = c, A_N = I$.

We have $c_N^\top - c_B^\top A_B^{-1} A_N = -c^\top A_B^{-1} = (c_1 - 7 \quad -2c_1 + 12 \quad -1)$, which is ≤ 0 for $6 \le c_1 \le 7$.

(c) *Consider the possibility of changing maximum loss of esteem to $9 + p$ and the maximum money to $18 + p$. Determined the range on p so that the current basis $\{x_1, x_2, x_3\}$ remains optimal and report the profit as a function of p.*

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SOLUTION. The basis remains optimal as long as $A_B^{-1}(b + \Delta b) \geq 0$, where $\Delta b = \begin{pmatrix} 0 & p & p \end{pmatrix}^\top$. We get:

$$A_B^{-1}(b + \Delta b) = A_B^{-1}b + A_B^{-1}\Delta b = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2p \\ 3p \\ -7p \end{pmatrix} = \begin{pmatrix} 2 + 2p \\ 2 + 3p \\ 3 - 7p \end{pmatrix}.$$

This is ≥ 0 for $-2/3 \leq p \leq 3/7$. In that range the objective function is

$$z^* + \mathbf{y}^* \cdot \Delta b = 45 + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p \\ p \end{pmatrix} = 45 + 3p.$$

(d) *Change the availability of hours, money and esteem to 13, 6 and 4 respectively De-termine the new optimal solution using the Dual Simplex method. Report the new solution as well as the new marginal values.*

SOLUTION. The new $x_B^* = A_B^{-1} \begin{pmatrix} 13 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$, so the optimal basis of the old

problem is not feasible for the new one. The new z is $(-1)(7) + (1)(8) + (3)(5) = 16$. We perform a dual simplex pivot. Since $x_1 = -1 < 0$ is the only negative variable, it is the one that exits the basis. To fidn the entering variable, we want the largest $t \geq 0$ such that $\begin{pmatrix} 0 & -2 & -1 \end{pmatrix} + t \begin{pmatrix} 1 & -2 & 0 \end{pmatrix} \leq 0$. This is $t = 0$, so x_4 is the entering variable. The next dictionary is then:

$$\begin{array}{rrrrr} x_2 = & 2 & +x_1 & +x_5 & -2x_6 \\ x_3 = & & -3x_1 & -2x_5 & +3x_6 \\ x_4 = & 1 & +x_1 & +2x_5 & \\ z = & 16 & & -2x_5 & -x_6 \end{array}$$

So an optimal solution for these new resource limits is given by $x_1 = 0 = x_3$, $x_2 = 2$ which produces $z = 16$. In words, under these limits, you're best bet is to forgo net-working and sports and just take exams until you simultaneously run out of money and self esteem. The marginal values happen to be the same as before: $0, 2, 1$ for hours, money and self esteem, respectively.

5. *Let A be a given $m \times n$ matrix. Let b be a $m \times 1$ vector. Show that either*

   i) There exist an x with $Ax = b$, $x \geq 0$,

   or

   ii) There exists y with $A^T y \geq 0$, $b \cdot y < 0$,

   but not both.

SOLUTION.

Consider the following pair of LPs:

Primal	Dual
$\max \mathbf{0} \cdot \mathbf{x}$	$\min \mathbf{b} \cdot \mathbf{y}$
subject to	subject to
$A\mathbf{x} = \mathbf{b}$	$A^\top \mathbf{y} \geq \mathbf{0}$
$\mathbf{x} \geq \mathbf{0}$	y free

They are dual to each other: the variables in the vector $\mathbf{y}$ correspond to the *equality* constraints $A\mathbf{x} = \mathbf{b}$, so they are free.

Notice that the primal cannot be unbounded because it's objective function is 0 for any feasible solution. So, by the fundamental theorem of linear programming, the primal is either infeasible or has an optimal solution.

Case of an infeasible primal. Here *i)* does *not* occur, let's show *ii)* does. Notice that the dual is feasible, because $\mathbf{y} = \mathbf{0}$ satisfies the constraints. But the dual cannot have an optimal solution since then, by strong duality, so would the primal. This means that the dual must be unbounded and so, in particular, must have some feasible solution with $\mathbf{b} \cdot \mathbf{y} < 0$.

Case where the primal has an optimal solution. Then *i)* does occur, lets show *ii)* does not. By weak duality any feasible dual solution satisfies $\mathbf{0} \cdot \mathbf{x} \leq \mathbf{b} \cdot \mathbf{y}$, so we cannot have $\mathbf{b} \cdot \mathbf{y} < 0$.

6. *Let a, b be given numbers with $a, b > 0$. Consider the game given by payoff matrix A below (the payoff to the row player).*

$$A = \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

Give optimal strategies (which may involve expressions in a, b) for both the row player and the column player and indicate how they give bounds on the value of the game $v(A)$ and show that $v(A) = 0$.

SOLUTION. To confirm that the value of the game is 0, we just need to produce stochastic vectors $\mathbf{x}$ and $\mathbf{y}$ such that the smallest entry of $\mathbf{x}^\top A$ and the largest entry of $A\mathbf{y}$ are both 0. It's easy to spot that $\mathbf{y}^\top = (1/2 \quad 1/2)^\top$ gives $A\mathbf{y} = (0 \quad 0)^\top$. This shows that player two can ensure that player one wins no more than 0 on average per round, so $v(A) \leq 0$.

For $\mathbf{x}$, we have $\mathbf{x}^\top A = (ax_1 - bx_2 \quad -ax_1 + bx_2)$ and, of course, $x_1 + x_2 = 1$. Solving we get $\mathbf{x}^\top = \left(\frac{b}{a+b} \quad \frac{a}{a+b} \right)$ and then $\mathbf{x}^\top A = (0 \quad 0)$. This shows that player one can ensure she gets at least 0 on average per round, so that $v(A) \geq 0$ and therefore $v(A) = 0$ as desired.

7. *Let A, $\mathbf{b}$ and $\mathbf{c}$ be given. Assume that our standard LP:* $\max \ \mathbf{c} \cdot \mathbf{x}$ *subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ has an optimal solution. Assume the LP:* $\max \ (-\mathbf{c}) \cdot \mathbf{x}$ *subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ has an optimal solution. Show that for any $\mathbf{u}$ satisfying $A\mathbf{u} \leq \mathbf{0}$ and $\mathbf{u} \geq \mathbf{0}$ that we may deduce that $\mathbf{u}$ also satisfies $\mathbf{c} \cdot \mathbf{u} = 0$.*

SOLUTION. Assume that $\mathbf{u}$ satisfies $A\mathbf{u} \leq \mathbf{0}$ and $\mathbf{u} \geq \mathbf{0}$. We need to prove that $\mathbf{c} \cdot \mathbf{u} = 0$.

Let $\mathbf{x}$ be some optimal solution for the LP $\max \mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Consider the vector $\mathbf{x} + \mathbf{u}$. It is also feasible for this LP:

- Adding $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{u} \geq \mathbf{0}$, we get $\mathbf{x} + \mathbf{u} \geq \mathbf{0}$.

- Adding $A\mathbf{x} \leq \mathbf{b}$ and $A\mathbf{u} \leq \mathbf{0}$, we get $A(\mathbf{x} + \mathbf{u}) \leq \mathbf{b}$.

Therefore the objective function for $\mathbf{x} + \mathbf{u}$ must be less than or equal to the objective function on the optimal solution $\mathbf{x}$. That is, $\mathbf{c} \cdot (\mathbf{x} + \mathbf{u}) \leq \mathbf{c} \cdot \mathbf{x}$, which tells us that $\mathbf{c} \cdot \mathbf{u} \leq 0$.

An analogous argument with the other LP (the one whose objective function is $(-\mathbf{c}) \cdot \mathbf{x}$) will show that $(-\mathbf{c}) \cdot \mathbf{u} \geq 0$, that is, that $\mathbf{c} \cdot \mathbf{u} \geq 0$.