- You will *not* be able to use a calculator or computer for either the midterm or the final exam, so please do not use one for this assignment. You may use one to *check* your answer, but please do not use one to solve the problem.
- Only part of the problems may be graded. But, you have to submit all the problems.
- The deadline is *by 6pm on Friday*, October 11. You have a grace period until 9am the next day. The grace period is to take care of any technical issues you have while submitting the file. The grace period should give ample time for handling any issues, so No late HW after the grace period will be accepted, regardless of your technical issues.
- Submit only pdf files.
- Remember to use Anstee's rule.

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1. $\boxed{\text{4 marks}}$ Give an example of a dictionary for which the current basic feasible solution is optimal and yet the coefficients of the non-basic variables in the z row are not all negative.

Solution: An example of an LP and an associated dictionary where the basic solution is the optimal solution and yet some coefficients in the z row are strictly positive must correspond to a case where you are some number of degenerate pivots away from optimality.

An easy example is

$$\begin{array}{llll} \max & x_1 & & \\ & x_1 \quad +x_2 & \leq 0 & \qquad x_1, x_2 \geq 0 \\ & x_1 \quad -2x_2 & \leq 1 & \end{array}$$

which has the initial dictionary

$$\begin{array}{rcrrr} x_3 & = & 0 & -x_1 & -x_2 \\ x_4 & = & 1 & -x_1 & +2x_2 \\ z & = & & x_1 & \end{array}$$

One can easily argue that the current solution (all original variables 0) is optimal and is in fact the only feasible solution. Many examples can be constructed which are only one degenerate pivot away from being seen to be optimal.

2. $\boxed{\text{5 marks}}$ Prove the following:

Theorem Let $\mathbf{A}$ and $\vec{b}$ be given. Then either

- there exists an $\vec{x}$ st $\vec{x} \geq 0$ and $\mathbf{A}\vec{x} \leq \vec{b}$, or
- there exists a $\vec{y}$ st $\mathbf{A}^T y \geq 0, \vec{y} \geq 0$ and $\vec{b}.\vec{y} < 0$

but not both. Note the strict inequality in the second.

Hint: Use both weak and strong duality theorems. You will also need the fundamental theorem of linear programming.

Solution:

Wrote this as an LP problem and use duality. Now we have constraints, but no objective function, so let us just use $z = \vec{0}.\vec{x}$.

$$\begin{aligned} &\max \vec{0}.\vec{x} \\ &\mathbf{A}\vec{x} \leq \vec{b} \\ &\vec{x} \geq 0 \end{aligned} \qquad\qquad \begin{aligned} &\min \vec{b}.\vec{y} \\ &\mathbf{A}^T\vec{y} \geq 0 \\ &\vec{y} \geq 0 \end{aligned}$$

Since the primal problem is bounded (the objective function is zero), the fundamental theorem of linear programming tells us that the primal problem either has an optimal solution or is infeasible.

- If the primal has an optimal solution (option 1), $\vec{x}^*$ then the objective function is 0. Weak duality tells us that any feasible solution of the dual problem must give $0 \leq \vec{b}.\vec{y}$. Hence option 2 cannot happen if option 1 does happen.
- Assume the primal is infeasible. We see that the dual is always feasible since $\vec{y} = 0$ is always a solution. Again, the fundamental theorem of LP tells that the dual is either has an optimal solution or unbounded.
  - If the dual has an optimal solution, $\vec{y}^*$ then strong duality implies that there is an optimal and so feasible solution to the primal — which we have assumed does not exist.
  - Hence the dual must be unbounded. Since it is unbounded, then there exists a feasible solution of the dual, $\vec{y}'$ with objective function $\vec{b}.\vec{y}' = -1$. Hence there exists a $\vec{y}$ such that $\vec{b}\vec{y} < 0$.
- The fundamental theorem of linear programming tells us that either the primal has an optimal solution or is infeasible. If optimal, then option 1 happens but not option 2. If the primal is infeasible, option 1 cannot happen, but option 2 must.

3. $\boxed{\text{3 marks}}$ Consider the LP:

$$\begin{array}{lrrrrrl} \text{Maximize} & 12x_1 & +20x_2 & +21x_3 & +18x_4 & & \\ & 24x_1 & +40x_2 & +46x_3 & +44x_4 & \leq 1200 & \\ & x_1 & +x_2 & +x_3 & +x_4 & \leq 30 & x_1, x_2, x_3, x_4 \geq 0 \\ & 3x_1 & +6x_2 & +6x_3 & +6x_4 & \leq 150 & \end{array}$$

Someone claims the final dictionary has

$$z = 540 - x_2 - 3x_4 - 4x_6 - 3x_7$$

Explain what optimal solution to the dual this implies and explain why there must have been an error in the final row for z.

Hint: This question is related to the Strong Duality Theorem given in the class (see also Chvatal, proof of Theorem 5.1 pp58-59).

Solution: Following our proof of the Strong Duality Theorem, an optimal solution to the dual has

$$y_i^* = -\text{coefficient of the } i\text{th slack of the primal}$$

and so $y_1^* = 0, y_2^* = 4, y_3^* = 3$. But our supposedly optimal solution to the primal has $z = 540$ and yet $1200y_1^* + 30y_2^* + 150y_3^* = 570$ which violates Strong Duality and so some error must have been made.

4. . **Setting up an LP from a practical problem.** In the fine tradition of bad puns on mathematical assignments, my colleagues and I are starting the soon-to-be-famous Opple Rubber Company. Our company makes three different products:

- fashionable rubberised slippers sold under the "iMoc" name
- rubberised protectors for fruit called "oPods", and
- a small annoying musical instrument called the "oPhone".

Our company receives the rubber it needs in 200cm long ribbons. Each product requires a certain amount of rubber.

- 1 pair of iMoc slippers needs 90cm of a ribbon
- 1 oPod needs 70cm of a ribbon
- 1 oPhone needs 50cm of ribbon

A large order has come in and the company needs to make at least

- 300 pairs of iMocs
- 400 oPods

- 1000 oPhones

We would like to work out how to cut up the sheets so as to minimise waste. This problem can be broken down into smaller parts:

(a) $\boxed{\text{3 marks}}$ There are 6 ways to cut a 200cm rubber sheet into pieces of length 90cm, 70cm and 50 cm with minimal waste — what are they and how much rubber does each one waste? Please list them in order of most waste to least. *Note* Do not include ways such as $(70, 70)$ since this leaves 60cm and one could cut a 50cm segment from it.

Solution: The 6 different ways are

1. $(90, 70)$ wasting 40
2. $(70, 50, 50)$ wasting 30
3. $(90, 90)$ wasting 20
4. $(90, 50, 50)$ wasting 10
5. $(70, 70, 50)$ wasting 10
6. $(50, 50, 50, 50)$ wasting 0

(b) $\boxed{\text{4 marks}}$ Each of the ways of cutting a sheet wastes a certain amount of rubber. Obviously we would like to minimise this waste while still producing enough iMocs, oPods and oPhones. For some reason our cutting machine is unable to cut the ribbon in four equal pieces, so ignore this possibility — this leaves the other five cutting options. Write this as a linear programming problem. *Note* Please label your variables $y_1, \ldots, y_5$ so that the corresponding amount ribbon wasted is ordered from greatest to least.

Solution:

- Let $y_1, y_2, \ldots, y_5$ be the number of ribbons that are cut according to the 5 different ways above (excluding the possibility of $(50, 50, 50, 50)$.
- The wasted rubber is therefore

$$w = 40y_1 + 30y_2 + 20y_3 + 10y_4 + 10y_5$$

- The number of 90cm segments needs to be at least 300, so

$$y_1 + 2y_3 + y_4 \geq 300.$$

- The number of 70cm segments needs to be at least 400 so

$$y_1 + y_2 + 2y_5 \geq 200.$$

- The number of 50cm segments needs to be at least 1000 so

$$2y_2 + 2y_4 + y_5 \geq 1000$$

- So the LP problem is

$$\begin{array}{lrrrrrcr} \text{minimize } w = 40y_1 + 30y_2 + 20y_3 + 10y_4 + 10y_5 \\ & y_1 & & +2y_3 & +y_4 & & \geq & 300 \\ \text{subject to} & y_1 & +y_2 & & & +2y_5 & \geq & 400 \\ & & 2y_2 & & +2y_4 & +y_5 & \geq & 1000 \\ & y_1, \dots, y_5 \geq 0 \end{array}$$

5. $\boxed{\text{5 marks}}$ Solve the linear programming problem

$$\begin{array}{lrrrrrcr} \text{minimize } w = 40y_1 + 30y_2 + 20y_3 + 10y_4 + 10y_5 \\ & y_1 & & +2y_3 & +y_4 & & \geq & 300 \\ \text{subject to} & y_1 & +y_2 & & & +2y_5 & \geq & 400 \\ & & 2y_2 & & +2y_4 & +y_5 & \geq & 1000 \\ & y_1, \dots, y_5 \geq 0 \end{array}$$

by solving the dual problem and then using Complementary Slackness. Don't worry if your answer comes out to be non-integer.

Solution: The dual problem is

$$\begin{array}{lrrrcr} \text{maximise } z = 300x_1 + 400x_2 + 1000x_3 \\ & x_1 & +x_2 & & \leq & 40 \\ & & +x_2 & +2x_3 & \leq & 30 \\ \text{subject to non-negativity and} & 2x_1 & & & \leq & 20 \\ & x_1 & & +2x_3 & \leq & 10 \\ & & 2x_2 & +x_3 & \leq & 10 \end{array}$$

Write in dictionary form:

$$\begin{array}{rclrrr}
x_4 & = & 40 & -x_1 & -x_2 & \\
x_5 & = & 30 & & -x_2 & -2x_3 \\
x_6 & = & 20 & -2x_1 & & \\
x_7 & = & 10 & -x_1 & & -2x_3 \\
x_8 & = & 10 & & -2x_2 & -x_3 \\ \hline
z & = & 0 & +300x_1 & +400x_2 & +1000x_3
\end{array}$$

So x_3 enters and x_7 leaves

$$\begin{array}{rclrrr}
x_3 & = & 5 & -(1/2)x_1 & & -(1/2)x_7 \\
x_4 & = & 40 & -x_1 & -x_2 & \\
x_5 & = & 20 & +x_1 & -x_2 & +x_7 \\
x_6 & = & 20 & -2x_1 & & \\
x_8 & = & 5 & +(1/2)x_1 & -2x_2 & +(1/2)x_7 \\ \hline
z & = & 5000 & -200x_1 & +400x_2 & -500x_7
\end{array}$$

So x_2 enters and x_8 leaves

$$\begin{array}{rclrrr}
x_2 & = & (5/2) & +(1/4)x_1 & +(1/4)x_7 & -(1/2)x_8 \\
x_3 & = & 5 & -(1/2)x_1 & -(1/2)x_7 & \\
x_4 & = & (75/2) & -(5/4)x_1 & -(1/4)x_7 & +(1/2)x_8 \\
x_5 & = & (35/2) & +(3/4)x_1 & +(3/4)x_7 & +(1/2)x_8 \\
x_6 & = & 20 & -2x_1 & & \\ \hline
z & = & 6000 & -100x_1 & -400x_7 & -200x_8
\end{array}$$

Hence the solution is $z = 6000$ and

$$(x_1, x_2, x_3) = (0, 5/2, 5)$$
$$(x_4, \ldots, x_8) = (75/2, 35/2, 20, 0, 0)$$

We need to map this back to the original problem. We use complementary slackness to do so. It tells us that

$$x_j^* > 0 \implies \sum a_{ij} y_i^* = c_j$$
$$\sum a_{ij} x_j^* < b_i \implies y_i^* = 0$$

So since $x_2, x_3 > 0$ we have

$$y_1^* + y_2^* + 2y_5^* = 400$$
$$2y_2^* + 2y_4^* + y_5^* = 1000$$

Subsituting the x_j^* into the inequalities we see (in order)

$$\begin{aligned} 2.5 &< 40 \\ 12.5 &< 30 \\ 0 &< 20 \\ 10 &= 10 \\ 10 &= 10 \end{aligned}$$

Hence $y_1^* = y_2^* = y_3^* = 0$. The two equations above then imply $y_5^* = 200$ and $y_4^* = 400$. We can check that this does also give $w = 6000$.

6. $\boxed{\text{5 marks}}$ The optimal solution to the linear program:

$$\begin{array}{lrrrl} \text{Maximize} & 10x_1 & +14x_2 & +20x_3 & \\ & 2x_1 & +3x_2 & +4x_3 & \leq 220 \\ & 4x_1 & +2x_2 & -x_3 & \leq 385 \\ & x_1 & & +4x_3 & \leq 160 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

is $x_1 = 60, x_2 = 0, x_3 = 25$. Write down the dual problem. Use this information above to find an optimal solution to the dual (**don't use the simplex algorithm**) explaining your work (name theorems used). Explain how this confirms that the optimal solution to the primal I claimed is in fact an optimal solution.

Solution:

$$\text{Dual:} \qquad \begin{array}{lrrrl} \text{Minimize} & 220y_1 & +385y_2 & +160y_3 & \\ & 2y_1 & +4y_2 & +y_3 & \geq 10 \\ & 3y_1 & +2y_2 & & \geq 14 \\ & 4y_1 & -y_2 & +4y_3 & \geq 20 \end{array} \qquad y_1, y_2, y_3 \geq 0$$

Now $x_1 = 60 > 0$ implies $2y_1^* + 4y_2^* + y_3^* = 10$ by Complementary Slackness.

Also $x_3 = 25 > 0$ implies $4y_1^* - y_2^* + 4y_3^* = 20$ by Complementary Slackness.

Also $4x_1 + 2x_2 - x_3 = 215 < 385$ implies $y_2^* = 0$ by Complementary Slackness.

The optimal solution to the dual can be determined by solving three equations in 3 unknowns to obtain

$$y_1^* = 5, y_2^* = 0, y_3^* = 0$$

We check feasibilty of our primal and dual solutions and then, since Complementary Slackness is satisfied, the Theorem of Complementary Slackness shows that the primal (and dual) solution is optimal. An alternate way is to note you have a feasible solution to the primal $(60, 0, 25)$ with objective function value $10 \times 60 + 14 \times 0 + 20 \times 25 = 1100$ and a feasible solution to the dual $(5, 0, 0)$ with objective function value $220 \times 5 + 385 \times 0 + 160 \times 0 = 1100$ and so by Weak Duality, both must be optimal.

Comment: It is somewhat lucky that the three equations determine an optimal dual solution but that is how the question was chosen. The value of the question is in testing your hands on understanding of Complementary Slackness.

The following two problems are OPTIONAL for your practice with complementary slackness. Do not hand in.

7. $\boxed{\text{5 marks}}$ Question 5.3(a) from Chvatal.

$$\begin{array}{lrrrrrcl}
\text{Maximise} & 7x_1 & +6x_2 & +5x_3 & -2x_4 & +3x_5 & & \\
\text{subject to} & x_1 & +3x_2 & +5x_3 & -2x_4 & +2x_5 & \leq & 4 \\
 & 4x_1 & +2x_2 & -2x_3 & +x_4 & +x_5 & \leq & 3 \\
 & 2x_1 & +4x_2 & +4x_3 & -2x_4 & +5x_5 & \leq & 5 \\
 & 3x_1 & +x_2 & +2x_3 & -x_4 & -2x_5 & \leq & 1 \\
 & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4, x_5 \geq 0}
\end{array}$$

Is $(0, 4/3, 2/3, 5/3, 0)$ optimal?

Solution:

- The dual problem is

$$\begin{array}{lrrrrl}
\text{minimise} & 4y_1 & +3y_2 & +5y_3 & +y_4 & \\
\text{subject to} & y_1 & +4y_2 & +2y_3 & +3y_4 & \geq 7 \\
 & 3y_1 & +2y_2 & +4y_3 & +y_4 & \geq 6 \\
 & 5y_1 & -2y_2 & +4y_3 & +2y_4 & \geq 5 \\
 & -2y_1 & +y_2 & -2y_3 & -y_4 & \geq -2 \\
 & 2y_1 & +y_2 & +5y_3 & -2y_4 & \geq 3 \\
 & & & \multicolumn{3}{r}{y_1, y_2, y_3, y_4 \geq 0}
\end{array}$$

- Since $x_2^*, x_3^*, x_4^* > 0$ it follows that the corresponding dual constraints are equalities:

$$\begin{aligned}
3y_1 + 2y_2 + 4y_3 + y_4 &= 6 \\
5y_1 - 2y_2 + 4y_3 + 2y_4 &= 5 \\
-2y_1 + y_2 - 2y_3 - y_4 &= -2
\end{aligned}$$

- Substitute x^* into the primal constraints gives

$$\begin{aligned} 4 + 10/3 - 10/3 &= 4 \\ 8/3 - 4/3 + 5/3 &= 3 \\ 16/3 + 8/3 - 10/3 &= 14/3 < 5 \\ 4/3 + 4/3 - 5/3 &= 1 \end{aligned}$$

  Since the third constraint is a strict inequality, $y_3^* = 0$.

- So we have to solve the following 3 equations

$$\begin{aligned} 3y_1 + 2y_2 + 4y_3 + y_4 &= 6 \\ 5y_1 - 2y_2 + 4y_3 + 2y_4 &= 5 \\ -2y_1 + y_2 - 2y_3 - y_4 &= -2 \end{aligned}$$

  These have solution $y_1^* = y_2^* = y_3^* = 1$.

- So complementary slackness implies the dual optimal solution is $(1, 1, 0, 1)$.
- This gives optimal value $4 + 3 + 1 = 8$. The optimal value of the primal was $24/3 + 10/3 - 10/3 = 8$.
- All the $y_i^* > 0$, sub into the constraints gives

$$\begin{aligned} 1 + 4 + 3 &= 8 > 7 \\ 3 + 2 + 1 &= 6 \\ -2 + 4 + 2 &= 4 < 5 \qquad \text{problem} 1 + 5 - 2 = 4 > 3 \end{aligned}$$

- The dual solution implied by complementary slackness is not feasible. Hence the orginal values of x^* is not optimal.

8. $\boxed{\text{5 marks}}$ Question 5.3(b) from Chvatal.

$$\begin{array}{l|rrrrrr|ll} \text{maximise} & 4x_1 & +5x_2 & +x_3 & +3x_4 & -5x_5 & +8x_6 & & \\ \hline \text{st non-neg and} & 1 & 0 & -4 & 3 & 1 & 1 & \leq & 1 \\ & 5 & 3 & 1 & 0 & -5 & 3 & \leq & 4 \\ & 4 & 5 & -3 & 3 & -4 & 1 & \leq & 4 \\ & 0 & -1 & 0 & 2 & 1 & -5 & \leq & 5 \\ & -2 & 1 & 1 & 1 & 2 & 2 & \leq & 7 \\ & 2 & -3 & 2 & -1 & 4 & 5 & \leq & 5 \end{array}$$

Is $(0, 0, 5/2, 7/2, 0, 1/2)$ optimal?

Solution:

- Since $x_3, x_4, x_6 > 0$ the corresponding dual constraints are equalities:

$$\begin{aligned} -4y_1 + y_2 - 3y_3 + y_5 + 2y_6 &= 1 \\ 3y_1 + 3y_3 + 2y_4 + y_5 - y_6 &= 3 \\ y_1 + 3y_2 + y_3 - 5y_4 + 2y_5 + 5y_6 &= 8 \end{aligned}$$

- Sub the x^* into the constraints to get

$$\begin{aligned} -20/2 + 21/2 + 1/2 &= 1 \\ 5/2 + 3/2 &= 4 \\ -15/2 + 21/2 + 1/2 &= 7/2 < 4 \\ 14/2 - 5/2 &= 9/2 < 5 \\ 5/2 + 7/2 + 2/2 &= 7 \\ 10/2 - 7/2 + 5/2 &= 4 < 5 \end{aligned}$$

  Since the third, fourth and sixth constraints are strict inequalities, it follows that $y_3^*, y_4^*, y_6^* = 0$.

- Hence we need to solve

$$\begin{aligned} -4y_1 + y_2 + y_5 &= 1 \\ 3y_1 + y_5 &= 3 \\ y_1 + 3y_2 + 2y_5 &= 8 \end{aligned}$$

  This solve to give $y_1^* = 1/2, y_2^* = 3/2, y_5^* = 3/2$.

- The dual objective function is $z = 1/2 + 12/2 + 21/2 = 17$. The primal is $5/2 + 21/2 + 8/2 = 17$.

- The dual variables are all non-negative. Need to check dual constraints.

$$\begin{aligned} [1,5,4,0,-2,2][1/2,3/2,0,0,3/2,0]^T &= 1/2 + 15/2 - 6/2 = 5 > 4 \checkmark \\ [0,3,5,-1,1,-3][1/2,3/2,0,0,3/2,0]^T &= 9/2 + 3/2 = 12/2 > 1 \checkmark \\ [-4,1,-3,0,1,2][1/2,3/2,0,0,3/2,0]^T &= -4/2 + 3/2 + 3/2 = 1 \checkmark \\ [3,0,3,2,1,-1][1/2,3/2,0,0,3/2,0]^T &= 3/2 + 3/2 = 3 \checkmark \\ [1,-5,-4,1,2,4][1/2,3/2,0,0,3/2,0]^T &= 1/2 - 15/2 + 6/2 = -8/2 > -5 \checkmark \\ [1,3,1,-5,2,5][1/2,3/2,0,0,3/2,0]^T &= 1/2 + 9/2 + 6/2 = 8 \checkmark \end{aligned}$$

- So the dual solution is feasible and gives same objective function.

- Hence the original solution is optimal.