Practice midterm questions — do not hand in

1. Use the two-phase method to solve the following LP problem

$$\begin{array}{lrrrcr} \text{Maximise} & x_1 & +3x_2 & -6x_3 & & \\ \text{Subject to} & x_1 & -x_2 & -x_3 & \leq & 2 \\ & -x_1 & & +x_3 & \leq & -1 \\ & & x_2 & -x_3 & \leq & 2 \\ & x_1, & x_2 & \geq 0 & & \end{array}$$

Solution:

- Axillary problem

$$\begin{array}{rcrrrrr} x_4 & = & 2 & -x_1 & +x_2 & +x_3 & +x_0 \\ x_5 & = & -1 & +x_1 & & -x_3 & +x_0 \\ x_6 & = & 2 & & -x_2 & +x_3 & \\ \hline w & = & & & & & -x_0 \end{array}$$

- Solve it to get

$$\begin{array}{rcrrrrr} x_1 & = & 1 & -x_0 & & +x_3 & +x_5 \\ x_4 & = & 1 & +2x_0 & +x_2 & & -x_5 \\ x_6 & = & 2 & +x_0 & -x_2 & +x_3 & \\ \hline w & = & & -x_0 & & & \end{array}$$

- Hence original problem is feasible. Second phase dictionary is therefore

$$\begin{array}{rcrrrr} x_1 & = & 1 & & +x_3 & +x_5 \\ x_4 & = & 1 & +x_2 & & -x_5 \\ x_6 & = & 2 & -x_2 & +x_3 & \\ \hline z & = & 1 & +3x_2 & -5x_3 & +x_5 \end{array}$$

- Solve this to get

$$\begin{array}{rcrrrr} x_1 & = & 4 & +2x_3 & -x_4 & -x_6 \\ x_2 & = & 2 & +x_3 & -x_6 & \\ x_5 & = & 3 & +x_3 & -x_4 & -x_6 \\ \hline z & = & 10 & -x_3 & -x_4 & -4x_6 \end{array}$$

- So optimal value is $z = 10$ and optimal solution is

$$x_1 = 4 \qquad x_2 = 2 \qquad x_3 = 0 \qquad x_4 = 0 \qquad x_5 = 3 \qquad x_6 = 0$$

2. Consider the following LP problem:

$$\begin{array}{lrrrcr} \text{Maximise} & 7x_1 & +5x_2 & +2x_3 & & \\ \text{Subject to} & x_1 & +x_2 & +x_3 & \leq & 5 \\ & x_1 & +3x_2 & +x_3 & \leq & 10 \\ & 3x_1 & +x_2 & +x_3 & \leq & 6 \\ & x_1, & x_2, & x_3 & \geq 0 & \end{array}$$

Use complementary slackness to see if $(x_1^*, x_2^*, x_3^*) = (1, 3, 0)$ is an optimal solution.

Solution:

- The primal objective function is $7 + 15 = 22$.
- Subsituting $(x_1^*, x_2^*, x_3^*) = (1, 3, 0)$ into the primal constraints gives

$$4 < 5$$
$$10 = 10$$
$$6 = 6$$

  So complementary slackness implies that $y_1^* = 0$.

- The dual problem is

$$\begin{array}{lrrrcr} \text{Minimise} & 5y_1 & +10y_2 & +6y_3 & & \\ \text{ST} & y_1 & +y_2 & +3y_3 & \geq & 7 \\ & y_1 & +3y_2 & +y_3 & \geq & 5 \\ & y_1 & +y_2 & +y_3 & \geq & 2 \end{array}$$

- Since $x_1^*, x_2^* > 0$, Complementary slackness implies that the first and second dual constraints are equalities:

$$y_1 + y_2 + 3y_3 = 7$$
$$y_1 + 3y_2 + y_3 = 5$$

- Since $y_1^* = 0$ we have two equations to solve

$$y_2^* + 3y_3^* = 7$$
$$3y_2^* + y_3^* = 5$$

  These have solution $y_2^* = 1, y_3^* = 2$.

- Check that $(y_1^*, y_2^*, y_3^*) = (0, 1, 2)$ is feasible and gives same objective function value:

$$\begin{aligned} 0 + 10 + 12 = z = 22 && \checkmark \\ 0 + 1 + 6 = 7 && \checkmark \\ 0 + 6 + 1 > 5 && \checkmark \\ 0 + 2 + 1 > 2 && \checkmark \end{aligned}$$

- Since this is feasible and gives same objective function value, we conclude that $(1, 3, 0)$ is indeed an optimal solution.

3. (a) State the fundamental theorem of linear programming.

Solution: Every linear programming problem in standard form has the following three properties:

- If it has no optimal solution then it is either infeasible or unbounded.
- If it has a feasible solution then it has a basic feasible solution.
- If it has an optimal solution then it has a basic optimal solution.

(b) Explain (referring to relevant theorems) why if a primal problem is unbounded then the corresponding dual problem must be infeasible.

Solution:

- The weak duality theorem says that if $\vec{x}^*$ is any feasible solution to the primal problem and $\vec{y}^*$ is any feasible solution to the corresponding dual problem then

$$\vec{c}.\vec{x}^* \leq \vec{b}.\vec{y}^*$$

ie the value of the objective function of the primal problem at $\vec{x}^*$ is less than or equal to the objective function of the dual problem at $\vec{y}^*$.

- Say that the primal problem is unbounded but that the dual problem has a feasible solution $\vec{y}^*$.
- The objective function of the dual problem at $\vec{y}^*$ takes some value — call it M.

- The weak duality theorem implies that for any feasible solution $\vec{x}^*$ of the primal problem we have

$$\vec{c}.\vec{x}^* \leq M$$

- Hence the objective function of the primal problem is bounded above by M.
- This contradicts the assumption that the primal problem is unbounded.
- Hence no feasible solution to the dual problem can exist. ie the dual problem is infeasible.

4. At Café Sunfrancs a cappuccino is made from one shot of espresso, three ounces of milk, and six ounces of foam. A latté is made from one shot of espresso, seven ounces of milk, and two ounces of foam. A café sells only cappucinos and lattés, and makes one dollar profit on each drink it sells. Today the café has materials to produce 50 shots of espresso, 20 ounces of milk, and 30 ounces of foam.

   Write down a linear program to maximize the profit the café will make. Write down the dual LP. For all variables involved (objective, decision, and slack, both in the primal and dual), state in what units they are given.

Solution:

- Let $x_1 =\#$ cappucinos made, and $x_2 =\#$ lattés made.
- Let $z =$ dollars of profit.
- The primal problem is

$$\begin{array}{llllll} \text{maximise} & x_1 & +x_2 & & & \\ \text{subject to} & x_1 & +x_2 & \leq & 50 & \text{(espresso bound)} \\ & 3x_1 & +7x_2 & \leq & 20 & \text{(milk bound)} \\ & 6x_1 & +2x_2 & \leq & 30 & \text{(foam bound)} \\ & x_1, & x_2, & \geq & 0 & \text{non-negativity} \end{array}$$

- Since the right-hand side of the constraints are units of espresso, milk and foam (respectively), it follows that the slack variables x_3, x_4, x_5 are in units of espressos, milk and foam (respectively).

- The dual problem is

$$
\begin{array}{llllcll}
\text{minimise} & 50y_1 & +20y_2 & +30y_3 & & & \\
\text{subject to} & y_1 & +3y_2 & +6y_3 & \geq & 1 & \text{(dollars per cappucino)} \\
 & y_1 & +7y_2 & +2y_3 & \geq & 1 & \text{(dollars per latté)} \\
 & y_1, & y_2, & y_3 & \geq & 0 & \text{non-negativity}
\end{array}
$$

- The objective function is measured in dollars.
- y_1 is dollars per espresso, y_2 is dollars per unit of milk and y_3 is dollars per unit of foam.
- Since the right-hand side of the dual constraints are dollars per cappucino and dollars per latté, it follows that the dual slack variables, y_4 and y_5 are in units of dollars per cappucino and dollars per latté (respectively).