Some practice problems for Math 340

1 (10 points)**.** Use a graphical method to find the optimal solution to

$$
\begin{array}{lrcrcr}
\text{maximize} & 3x_1 & + & 4x_2 & & \\
\text{subject to} & x_1 & + & 2x_2 & \leq & 60 \\
 & 2x_1 & + & x_2 & \leq & 80 \\
 & -x_1 & + & x_2 & \leq & 20 \\
 & & x_1, & x_2 & \geq & 0
\end{array}
$$

2 (15 points)**.** Consider the dictionary below.

$$
\begin{array}{rcrcrcr}
x_1 & = & \boldsymbol{(4-2\alpha)} & - & s_1 & + & s_2 \\
x_2 & = & \boldsymbol{\beta} & + & s_1 & + & 2s_2 \\
s_3 & = & 4 & - & 2s_1 & - & 3s_2 \\
z & = & \boldsymbol{\gamma} & + & \boldsymbol{(\alpha-3)}s_1 & + & \boldsymbol{\delta}s_2
\end{array}
$$

This dictionary is repeated overleaf for your convenience.

(a) (4 points) Find the range of values for each of $\alpha, \beta, \gamma, \delta$ such that the dictionary is feasible.

(b) (4 points) Find the range of values for each of $\alpha, \beta, \gamma, \delta$ such that the dictionary is dual feasible.

(c) (7 points) Find the range of values for each of $\alpha, \beta, \gamma, \delta$ such that the dictionary is dual feasible but *not* feasible.

3 (15 points). (a) (5 points) State the Weak Duality Theorem.

(b) (10 points) Consider a primal LP problem with a feasible solution x^* that yields z^* when substituted into the objective function z. Let its dual LP problem have a feasible solution y^* that yields w^* when substituted into its objective function w.

Prove that if $z^* = w^* = \alpha$ then α is the optimal value for *both* problems.

4 (15 points)**.** Consider the problem

$$\begin{array}{lrrrcl} \text{Maximize} & 10x_1 & +18x_2 & +10x_3 & & \\ \text{subject to} & 2x_1 & +2x_2 & +x_3 & \leq & 7 \\ & x_1 & +3x_2 & +2x_3 & \leq & 5 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

(a) (5 points) A friend claims she solved it using the simplex method and got optimal value 43. Prove she is wrong *without* using the simplex method or complimentary slackness.

(b) (10 points) Another friend says

"I got optimal solution $\frac{11}{4}, 0, \frac{3}{4}$ and optimal value 41."

Use complimentary slackness to answer the following.

Is $x_1 = \frac{11}{4}, x_2 = 0, x_3 = \frac{3}{4}$ the optimal solution? Is 41 the optimal value?

5 (20 points)**.** Consider the problem

$$\begin{array}{lrrrcl} \text{Maximize} & 2x_1 & +3x_2 & +3x_3 & & \\ \text{subject to} & 3x_1 & +x_2 & & \leq & 60 \\ & -x_1 & +x_2 & +4x_3 & \leq & 10 \\ & 2x_1 & -2x_2 & +5x_3 & \leq & 15 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

Solve this problem using the *revised simplex method* and eta factorization. Use the largest coefficient rule to select your entering and leaving variables. (You should find you are stopped during the third iteration.)

6 (25 points)**.** We run a small smoothie company with four flavours: Apple-Acai, Banana-Berry, Citrus Cooler, and Durian Delight. To make a carton of Apple-Acai we require 1 kg mixed fruit and 1 kg yogurt. To make a carton of Banana-Berry we require 2 kg mixed fruit and 1 kg yogurt. To make a carton of Citrus Cooler we require 3 kg mixed fruit and 2 kg yogurt. To make a carton of Durian Delight we require 1 kg mixed fruit and 3 kg yogurt. On each carton of Apple-Acai we make \$ 5 profit, each carton of Banana-Berry \$ 6 profit, each carton of Citrus Cooler \$ 9 profit, each carton of Durian Delight \$ 8.

We can afford 50 kg mixed fruit, and 30 kg yogurt per day. If x_1, x_2, x_3 are the 10's of cartons we sell per day we get the following Linear Program.

$$\text{maximise } 5x_1 + 6x_2 + 9x_3 + 8x_4 \text{ subject to } x_1, x_2, x_3, x_4 \geq 0 \text{ and}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &\leq 5 \\ x_1 + x_2 + 2x_3 + 3x_4 &\leq 3 \end{aligned}$$

After applying the simplex method we get the final dictionary

$$\begin{array}{rcrrrrr} x_1 & = & 1 & -x_3 & -5x_4 & +s_1 & -2s_2 \\ x_2 & = & 2 & -x_3 & +2x_4 & -s_1 & +s_2 \\ z & = & 17 & -2x_3 & -5x_4 & -s_1 & -4s_2 \end{array}$$

(a) (5 points) Compute B for the revised simplex method for this final dictionary and label your columns appropriately. Calculate B^{-1}.

(b) (5 points) A carton of Mango Madness can be made from 1 kg mixed fruit and 2 kg yogurt at a profit of \$ 7. Is it profitable to produce it?

(c) (5 points) Determine the range on b_2 (yogurt) so that the basis $\{x_1, x_2\}$ remains optimal.

(d) (5 points) Determine the range on c_1 (profit on Apple-Acai) so that the basis $\{x_1, x_2\}$ remains optimal.

(e) (5 points) If we can afford 70 kg of yogurt per day what is the new optimal value, and optimal solution?

Mathematics 340 Very Brief Solutions (Your actual solutions should be much more detailed!)

1.

(x_1, x_2)	z
$(0,0)$	0
$(0,20)$	80
$(6\frac{2}{3}, 26\frac{2}{3})$	$126\frac{2}{3}$
$(33\frac{1}{3}, 13\frac{1}{3})$	$153\frac{1}{3}$
$(40,0)$	120

2. (a) $\alpha \leq 2$, $0 \leq \beta$, γ anything, δ anything.

(b) $\alpha \leq 3$, β anything, γ anything, $\delta \leq 0$.

(c) $\alpha \leq 2$, $\beta < 0$ OR $2 < \alpha \leq 3$, β anything, THEN γ anything, $\delta \leq 0$.

3. (a) See the notes.

(b) By the WDT if the primal optimal value is Z then $\alpha = z^* \leq Z \leq w^* = \alpha$. By the WDT if the dual optimal value is W then $\alpha = z^* \leq W \leq w^* = \alpha$. So $Z = W = \alpha$.

4. (a) Add 3 times the first and 4 times the second constraint to get

$$10x_1 + 18x_2 + 10x_3 \leq 10x_1 + 18x_2 + 11x_3 \leq 41$$

so 41 is an upper bound, and the answer can't be 43.

(b) NO: The 1st and 2nd constraint aren't maximized, so $y_1 = y_2 = 0$. Since $x_1, x_3 > 0$ this means $2y_1 + y_2 = 10 = y_1 + 2y_2$ so $y_1, y_2 \neq 0$. So this isn't the optimal solution.

YES: Letting $x_1 = \frac{11}{4}, x_2 = 0, x_3 = \frac{3}{4}$ we get $y_1 = 3, y_2 = 4$, the constraints are satisfied and in both cases $z = 41$.

5. *First iteration:*

- Form matrices.

$$b = \begin{bmatrix} 60 \\ 10 \\ 15 \end{bmatrix} \quad x_B^* = \begin{bmatrix} 60 \\ 10 \\ 15 \end{bmatrix} \quad x_B = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad x_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad c_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad c_N = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$$

$$B_0 = I = \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \quad A_N = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 4 \\ 2 & -2 & 5 \end{pmatrix} \end{array}$$

- Solve $yB = c_B$ to get $y = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. Then $c_N - yA_N = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$. Hence x_2 enters and $d = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. So $t = 10$. Since the second component $= 0$, it follows that s_2 leaves.

- Update things

$$x_B^* = \begin{bmatrix} 50 \\ 10 \\ 35 \end{bmatrix} \quad x_B = \begin{bmatrix} s_1 \\ x_2 \\ s_3 \end{bmatrix} \quad x_N = \begin{bmatrix} x_1 \\ s_2 \\ x_3 \end{bmatrix} \quad c_B = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix} \quad c_N = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}$$

$$A_N = \begin{array}{c} \begin{matrix} x_1 & s_2 & x_3 \end{matrix} \\ \begin{pmatrix} 3 & 0 & 0 \\ -1 & 1 & 4 \\ 2 & 0 & 5 \end{pmatrix} \end{array} \quad E_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Second iteration:

- Solve $yE_1 = c_B$ to get $y = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}$. Then $c_N - yA_N = \begin{bmatrix} 5 & -3 & -9 \end{bmatrix}$. Hence x_1 enters and $d = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$. So $t = 25/2$. Since the first component $= 0$, it follows that s_1 leaves.

- Update things

$$x_B^* = \begin{bmatrix} 25/2 \\ 45/2 \\ 35 \end{bmatrix} \quad x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_2 \\ x_3 \end{bmatrix} \quad c_B = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix} \quad c_N = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}$$

$$A_N = \begin{array}{c} \begin{matrix} s_1 & s_2 & x_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{pmatrix} \end{array} \quad E_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Third iteration:

- Solve $yE_1E_2 = c_B$.

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix} \qquad \Longrightarrow u = \begin{bmatrix} 5/4 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 5/4 & 3 & 0 \end{bmatrix} \qquad \Longrightarrow y = \begin{bmatrix} 5/4 & 7/4 & 0 \end{bmatrix}$$

- Choose entering variable.

$$c_N - yA_N = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 5/4 & 7/4 & 0 \end{bmatrix} \begin{array}{c} \begin{matrix} s_1 & s_2 & x_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{pmatrix} \end{array} = \begin{bmatrix} -5/4 & -7/4 & -4 \end{bmatrix}$$

Since all are negative, there are no entering columns. The current BFS is optimal.

- The solution is

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = x_B^* = \begin{bmatrix} 25/2 \\ 45/2 \\ 35 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad z = 185/2.$$

6. (a) $B = \begin{array}{c} \begin{matrix} x_1 & x_2 \end{matrix} \\ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \end{array}, B^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

(b) $1.1 + 2.4 = 9 > 7$ so no.

(c) $\frac{5}{2} \leq \alpha \leq 5$.

(d) $4 \leq \beta \leq 6$.

(e) Dual pivot to get dictionary

$$\begin{array}{rcrllll} x_1 & = & 4 & -\frac{5}{2}x_2 & -\frac{7}{2}x_3 & -\frac{3}{2}s_1 & -\frac{9}{2}s_2 \\ x_4 & = & 1 & +\frac{1}{2}x_2 & +\frac{1}{2}x_3 & +\frac{1}{2}s_1 & -\frac{1}{2}s_2 \\ x_4 & = & 28 & -\frac{5}{2}x_2 & -\frac{9}{2}x_3 & -\frac{3}{2}s_1 & -\frac{3}{2}s_2 \end{array}$$

and solution $x_1 = 4, x_2 = 0, x_3 = 0, x_4 = 1, s_1 = s_2 = 0$ and optimal value 28.