Math 340 Practice Midterm Time 45min

This is a closed-book examination. ONLY pen/pencil/eraser are allowed.

1. (4 points) You are given an LP problem and are also given a feasible solution to it and a feasible solution to its dual. Explain (referring to relevant theorems) why both problems must have optimal solutions.

Solution:

- The fundamental theorem of LP tells us that an LP problem is either infeasible, unbounded or has an optimal solution.
- Since the primal problem and the dual problem have feasible solutions they must be either unbounded or have optimal solutions.
- Weak duality implies that the optimal value of the primal problem is bounded above by the optimal value of the dual problem. *Alternatively* The strong duality theorem tells us that the optimal value of the primal problem is equal to that of the dual problem.
- Hence if the primal problem were unbounded, then there could not be a feasible solution to the dual. And if the dual problem were unbounded, then there cannot be a solution to the primal problem.
- Hence both problems have optimal solutions.

2. A colleague hands you the following LP problem

$$\begin{array}{lrrrrcr} \text{Maximize} & 9x_1 & +6x_2 & +x_3 & +9x_4 & & \\ \text{Subject to} & 2x_1 & +x_2 & +x_3 & +x_4 & \leq & 8 \\ & 2x_1 & +x_2 & & +4x_4 & \leq & 12 \\ & 10x_1 & +7x_2 & +4x_3 & -2x_4 & \leq & 45 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

and tells you that they used the simplex method to get to the following dictionary:

$$\begin{array}{rcrrrrr} x_2 & = & \frac{20}{3} & -2x_1 & -\frac{4}{3}x_3 & -\frac{4}{3}x_5 & +\frac{1}{3}x_6 \\ x_4 & = & \frac{4}{3} & +\frac{1}{3}x_3 & +\frac{1}{3}x_5 & -\frac{1}{3}x_6 & \\ x_7 & = & 1 & +4x_1 & +6x_3 & +10x_5 & -3x_6 \\ \hline z & = & 52 & -3x_1 & -6x_3 & -5x_5 & -x_6 \end{array}$$

(a) (2 points) Find the optimal solution.

Solution:

- Since all coefficients in z are negative this is an optimal dictionary.
- The optimal solution is therefore

$$\begin{array}{llll} x_1 = 0 & x_2 = \dfrac{20}{3} & x_3 = 0 & x_4 = \dfrac{4}{3} \\ x_5 = 0 & x_6 = 0 & x_7 = 1 & \\ z = 52 & & & \end{array}$$

(b) (3 points) Write down the dual of the original problem.

Solution:

- The dual is

$$\begin{array}{lrrrcl} \text{Minimize} & 8y_1 & +12y_2 & +45y_3 & & \\ \text{Subject to} & 2y_1 & +2y_2 & +10y_3 & \geq & 9 \\ & y_1 & +y_2 & +7y_3 & \geq & 6 \\ & y_1 & & +4y_3 & \geq & 1 \\ & y_1 & +4y_2 & -2y_3 & \geq & 9 \\ & y_1, & y_2, & y_3 & \geq & 0 \end{array}$$

(c) (2 points) Find the optimal solution to the dual problem.

Solution:

- Strong duality tells us we can read off the optimal solution of the dual problem from the coefficients of optimal dictionary.
- In particular $y_i^* = -\bar{c}_{n+i}$
- Hence the optimal solution of the dual is

$$y_1^* = 5 \qquad y_2^* = 1 \qquad y_3^* = 0$$

- This gives objective function value $8 \times 5 + 12 \times 1 + 0 = 52$ ✓
- Substitute into dual constraints to check feasibility

$$\begin{aligned} 10 + 2 + 0 &> 9 \\ 5 + 1 + 0 &> 6 \\ 5 + 0 &> 1 \\ 5 + 4 - 0 &> 9 \end{aligned}$$

So all are satisfied and all $y_i^* \geq 0$.

3. (6 points) Use the two-phase simplex method (and Anstee's rule) to solve the following linear programming problem:

$$\begin{array}{lrrcr} \text{Maximize} & 5x_1 & +3x_2 & & \\ \text{Subject to} & 2x_1 & +4x_2 & \leq & 4 \\ & -x_1 & -x_2 & \leq & -6 \\ & x_1, & x_2 & \geq 0 & \end{array}$$

Solution:

- Write in dictionary form

$$\begin{array}{rcrrr} x_3 & = & 4 & -2x_1 & -4x_2 \\ x_4 & = & -6 & +x_1 & +x_2 \\ \hline z & = & 0 & +5x_1 & +3x_2 \end{array}$$

- This is not a feasible dictionary, so we must use the two-phase method.
- Form the auxiliary problem. We add x_0 to each row and replace the objective function $w = -x_0$:

$$\begin{array}{rcrrrr} x_3 & = & 4 & -2x_1 & -4x_2 & +x_0 \\ x_4 & = & -6 & +x_1 & +x_2 & +x_0 \\ \hline w & = & & & & -x_0 \end{array}$$

- The most negative row is x_4 so we do a "fake pivot to feasibility" where x_0 enters and x_4 leaves:

$$\begin{array}{rcrrrr} x_0 & = & 6 & -x_1 & -x_2 & +x_4 \\ x_3 & = & 10 & -3x_1 & -5x_2 & +x_4 \\ \hline w & = & -6 & +x_1 & +x_2 & -x_4 \end{array}$$

- By Anstee's rule — x_1 enters and x_3 leaves

$$\begin{array}{rcrrrr} x_0 & = & (8/3) & +(2/3)x_2 & +(1/3)x_3 & +(2/3)x_4 \\ x_1 & = & (10/3) & -(5/3)x_2 & -(1/3)x_3 & +(1/3)x_4 \\ \hline w & = & -(8/3) & -(2/3)x_2 & -(1/3)x_3 & -(2/3)x_4 \end{array}$$

- There are no entering variables because all the coefficients in the objective function are negative.
- Hence the optimal value of auxiliary problem is $-8/3$. Hence the minimum value of $x_0 = 8/3$.
- Since the optimal value of $x_0 \neq 0$ we conclude that the original problem is not feasible.

Extra work space for Question 3

4. (8 points) Check whether each of the following statements is correct or not. Circle True or False, and you do not need to explain your answer. Each correct answer will earn 1 mark.

(a) i. **True / False.** If a standard form linear programming problem is feasible but unbounded, then its dual problem is infeasible.

ii. **True / False.** Given a linear programming problem, it is possible to have no optimal solution, while there is an optimal solution to its dual problem.

iii. **True / False.** There is a linear programming problem for which both primal and dual problems are not feasible simulataneously.

iv. **True / False.** For each vector $\vec{y} \in \mathbb{R}^n$, it holds that $\max_{\vec{x} \in \mathbb{R}^n} [\vec{y} \cdot \vec{x}] = +\infty$.

Solution:

a. True by weak duality

b. False by strong duality

c. True.

d. False. If $\vec{y} = \vec{0}$, then $\max_{\vec{x} \in \mathbb{R}^n} [\vec{y} \cdot \vec{x}] = 0$.

(b) **(Read carefully.)** Suppose that Prof. Anstee is following his rule to perform the simplex method to solve an LP problem. At a certain step he gets a feasible dictionary D_1, and by continuing iterations ("pivotings"), he gets subsequent dictionaries D_2, D_3, D_4, D_5, D_6, and D_7.

i. **True** / **False**. The dictionaries $D_2, D_3, D_4, D_5, D_6, D_7$ must be feasible.

ii. **True** / **False**. In some cases, it is possible to have an optimal basic solution to D_1 but, non-optimal basic solution to D_3.

iii. **True** / **False**. If D_7 is identical to D_1, then this LP problem has no optimal solution.

iv. **True** / **False**. If D_7 is not identical to D_1, then the dictionaries D_1, D_2, D_3, D_4, D_5, D_6, and D_7 all have different basic solutions from each other.

Solution:

a. True. If started from a feasible dictionary, the simplex methods gives only feasible dictionaries.

b. False. Each pivot from a feasible dictionary only improves the objective function value. Once optimal, it stays to be optimal, as you cannot improved the optimal value.

c. False. Cycling does not imply that there is no optimal solution.

d. False. It is possible to be degenerate without cycling, thus having the same basic solutions between two consecutive dictionaries.

5. A colleague hands you the following LP problem

$$\begin{array}{llllll} \text{Maximize } z = & 2x_1 & +x_2 & +2x_3 & & \\ \text{Subject to} & x_1 & +x_2 & +2x_3 & \leq & 6 \\ & 3x_1 & +6x_2 & +3x_3 & \leq & 18 \\ & 4x_1 & +2x_2 & +2x_3 & \leq & 12 \\ & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

and tells you that they found the following optimal solution.

$$x_1^* = 2 \qquad x_2^* = 0 \qquad x_3^* = 2 \qquad z = 8$$

(a) (2 points) Check that their solution is feasible.

Solution:

- All are non-negative. ✓
- Substitute into the inequalities

$$\begin{array}{ll} 2 + 4 = 6 & \checkmark \\ 6 + 6 < 18 & \checkmark \\ 8 + 4 = 12 & \checkmark \end{array}$$

(b) (6 points) Check that their solution is optimal using complementary slackness.

Solution:

- Write down the dual problem:

$$\begin{array}{llllll} \text{Minimise} & 6y_1 & +18y_2 & +12y_3 & & \\ \text{subject to} & y_1 & +3y_2 & +4y_3 & \geq & 2 \\ & y_1 & +6y_2 & +2y_3 & \geq & 1 \\ & 2y_1 & +3y_2 & +2y_3 & \geq & 2 \\ & y_1, & y_2, & y_3 & \geq & 0 \end{array}$$

- Since $x_1^*, x_3^* > 0$ it follows that the first and third dual constraints are equalities:

$$y_1 + 3y_2 + 4y_3 = 2$$
$$2y_1 + 3y_2 + 2y_3 = 2$$

- The second primal constraint is a strict inequality, so $y_2^* = 0$.
- Hence

$$y_1 + 4y_3 = 2$$
$$2y_1 + 2y_3 = 2$$

  which solve to give $y_1^* = \frac{2}{3}$ and $y_3 = \frac{1}{3}$.
- This gives objective function $12/3 + 12/3 = 8$ ✓
- These are all non-negative, and substituting into the dual constraints gives

$$2/3 + 4/3 = 2 \checkmark$$
$$2/3 + 2/3 > 1 \checkmark$$
$$4/3 + 2/3 = 2 \checkmark$$

- The dual solution implied by complementary slackness is feasible and gives the same objective function. Hence the original solution is optimal.

Extra Page 1

Extra Page 2

Extra Page 3