MATH 340: Linear Programming. Practice problems

1. [12 marks]

a) [10 marks] Solve the following linear programming problem, using our standard two phase method and using Anstee's rule. It will take three pivots in phase 1 (including special pivot to feasibility) and one pivot in phase 2.

$$\begin{array}{lrrrcr} \text{Maximize} & -3x_1 & +2x_2 & -2x_3 & & \\ & & x_2 & -x_3 & \leq & 4 \\ & x_1 & -x_2 & & \leq & -3 \\ & x_1 & & -x_3 & \leq & -1 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

b) [2 marks] Either give two optimal solutions or show the optimal solution is unique.

2. [10 marks] Consider the following linear program:

$$\begin{array}{lrrrl} \text{Maximize} & -2x_1 & +3x_2 & +7x_3 & \\ & & +x_2 & +2x_3 & \leq 2 \\ & -x_1 & +x_2 & +4x_3 & \leq 3 \\ & +x_1 & +x_2 & +4x_3 & \leq 4 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

a) [2 marks] Give the Dual Linear Program of the above Primal Linear Program.

b) [6 marks] You are given that an optimal primal solution has $x_1^* = 0$, $x_2^* = 1$, $x_3^* = \frac{1}{2}$. Determine an optimal dual solution (without pivoting), stating which theorems you have used.

c) [2 marks] Consider changing the objective function of the primal by raising the coefficient of x_1 from -2 to -1. Does the primal solution $x_1^* = 0$, $x_2^* = 1$, $x_3^* = \frac{1}{2}$ and the optimal dual solution determined in b) remain optimal to their new LP's? Explain.

3. [10 marks] Given $A, \mathbf{b}, \mathbf{c}$, current basis (and B^{-1} for your computational ease), use our Revised Simplex method and Anstee's rule to determine the next entering variable (if there is one), the next leaving variable (if there is one), and the new basic feasible solution after the pivot as well as the new basis matrix B (if there is both an entering and leaving variable). The current basis is $\{x_7, x_3, x_2\}$.

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \left(\begin{array}{ccccccc} 2 & -1 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \mathbf{b} \\ \left(\begin{array}{c} -3 \\ -2 \\ -5 \end{array}\right) \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_7 \\ x_3 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \left(\begin{array}{ccc} -3 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{array}\right) \end{array}$$

$$\begin{array}{c} \\ \mathbf{c}^T \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \left(\begin{array}{ccccccc} 1 & -1 & 1 & 2 & 0 & 0 & 0 \end{array}\right) \end{array}$$

4. A student wants to take on extra curricular activities that will boost their chance of finding a job after graduation. We quantify the boost in terms of *career points.* There are limits on the students time, money and loss of self esteem. The activities of networking, taking professional exams and doing sports will all contribute to landing a job after graduation; each unit of the activities will result in some number of career points. The student wishes to maximize the number of career points. The activities (for this student) have different costs in terms of time, money and self esteem. The student has 34 hours to spend on these activities and has $1800 to spend (the costs are given in units of $100 below). A student can only lose 9 units of self esteem before affecting their success in UBC courses.

	networking	exams	sports	total available
hours	5	6	4	34
money	3	3	2	18
esteem	1	2	1	9
career points	7	8	5	

Let x_1 denote the units of networking, x_2 denote the units of professional exams and x_3 denote the units of sports and let x_{3+i} denote the ith slack for $i = 1, 2, 3$. The final dictionary is:

$$\begin{array}{rcrrrr} x_1 & = & 2 & +x_4 & -2x_5 & \\ x_2 & = & 2 & +x_4 & -x_5 & -2x_6 \\ x_3 & = & 3 & -3x_4 & +4x_5 & +3x_6 \\ z & = & 45 & & -2x_5 & -x_6 \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 2 \\ 3 & -4 & -3 \end{pmatrix} \end{array}$$

NOTE: All questions are independent of one another.

a) [3 marks] Give the marginal values (shadow prices) for each of the resources: hours, money and esteem. What are the units for the marginal values?

b) [5 marks] Give the range on c_1 (career points for unit of networking) so that the current solution remains optimal.

c) [5 marks] Consider the possibility of changing maximum loss of esteem to $9 + p$ and the maximum money to $18 + p$. Determined the range on p so that the current basis $\{x_1, x_2, x_3\}$ remains optimal and report the profit as a function of p.

Hint for d), e),f): You need not complete all of the very final dictionary, merely the variables in the basis and the constants and **all** the entries in the z row.

d) [5 marks] Change the availability of hours, money and esteem to 13, 6 and 4 respectively Determine the new optimal solution using the Dual Simplex method. Report the new solution as well as the new marginal values.

5. [10 marks] Let A be a given $m \times n$ matrix. Let $\mathbf{b}$ be a $m \times 1$ vector. Show that either

   i) There exist an $\mathbf{x}$ with $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,

      or

   ii) There exists $\mathbf{y}$ with $A^T\mathbf{y} \geq \mathbf{0}$, $\mathbf{b} \cdot \mathbf{y} < 0$,

      but not both.

6. [10 marks] Let a, b be given numbers with $a, b > 0$. Consider the game given by payoff matrix A below (the payoff to the row player).

$$A = \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

   Give optimal strategies (which may involve expressions in a, b) for both the row player and the column player and indicate how they give bounds on the value of the game $v(A)$ and show that $v(A) = 0$.

7. [5 marks] Let A, $\mathbf{b}$ and $\mathbf{c}$ be given. Assume that our standard LP: max $\mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ has an optimal solution. Assume the LP: max $(-\mathbf{c}) \cdot \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ has an optimal solution. Show that for any $\mathbf{u}$ satisfying $A\mathbf{u} \leq \mathbf{0}$ and $\mathbf{u} \geq \mathbf{0}$ that we may deduce that $\mathbf{u}$ also satisfies $\mathbf{c} \cdot \mathbf{u} = 0$.