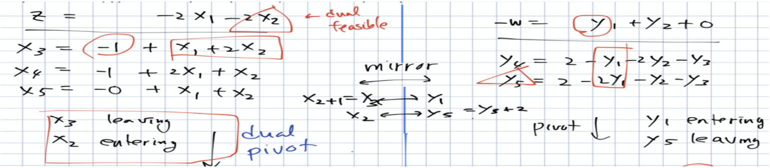
Dual pivot algorithm



- Start from a **dual feasible** dictionary.
- Choose a leaving variable the one with $-$ value.
  - If none then the dictionary is feasible, thus both primal and dual feasible. Thus, optimal.
- If the pivot row (the row of the leaving variable) has no positive coefficient, then the dual problem is unbounded, and the primal problem is infeasible.
- Otherwise, compare the ratio for the each nonbasic variable:

$$\frac{\text{the corresponding coefficient of the objective row}}{\text{positive coefficient of the pivot row}}$$

- Choose the entering variable such that the ratio is **the least** negative **(smallest absolute value)**.

Dual Pivot: Example

$$
\begin{array}{rcrrr}
z & = & 0 & -2x_1 & -2x_2 \\
\hline
x_3 & = & -1 & +x_1 & +2x_2 \\
x_4 & = & -1 & +2x_1 & +x_2 \\
x_5 & = & -0 & +x_1 & +x_2
\end{array}
\qquad
\begin{array}{cc}
\frac{-2}{1} & \frac{-2}{2} \\
x_1 & x_2
\end{array}
$$

x_2 entering

$$
\begin{array}{rcrrr}
z & = & -1 & -x_1 & -x_3 \\
\hline
x_2 & = & 1/2 & -1/2x_1 & +1/2x_3 \\
x_4 & = & -1/2 & +3/2x_1 & +1/2x_3 \\
x_5 & = & 1/2 & +1/2x_1 & +1/2x_3
\end{array}
\qquad
\begin{array}{cc}
\frac{-1}{3/2} & \frac{-1}{1/2} \\
x_1 & x_3
\end{array}
$$

x_1 entering

$$
\begin{array}{rcrrr}
z & = & -4/3 & -2/3x_4 & -2/3x_3 \\
\hline
x_2 & = & 1/3 & +1/3x_4 & +2/3x_3 \\
x_1 & = & 1/3 & +2/3x_4 & -1/3x_3 \\
x_5 & = & 2/3 & +1/3x_4 & +1/3x_3
\end{array}
$$

primal and dual feasible,
so optimal!

Lecture 27. Parametric LP

Basic idea for sensitivity analysis How can we find the new optimal solution in the new situation, using the old optimal solution of the problem, **without having to solve the new problem from scratch**?

- Changing $\vec{c}$
  - affects $\vec{c}_B, \vec{c}_N$ (so $[\vec{c}_N^T - \vec{c}_B^T B^{-1} N]$),
  - but it does not affect B, N, $\vec{x}_B^* = B^{-1}\vec{b}$.
  - It does not affect feasibility of the primal basic solution.
- Changing $\vec{b}$
  - affects $\vec{x}_B^* = B^{-1}\vec{b}$
  - but it does not affect $B, N, \vec{c}_B, \vec{c}_N$, $[\vec{c}_N^T - \vec{c}_B^T B^{-1} N]$, & $(\vec{y}^*)^T = \vec{c}_B^T B^{-1}$.
  - It does not affect dual feasibility of the dual basic solution.
- Under the changes, we may **recycle unaffected items.**

Parametric LP: Example with changing $\vec{b}$.

$$
\begin{array}{lrrrcrl}
\text{Maximize} \quad z = & x_1 & +x_2 & & & & \\
\text{subject to} & x_1 & +2x_2 & +x_3 & \leq & 2 & \quad \& \quad x_1, x_2, x_3 \geq 0, \\
 & 2x_1 & +x_2 & +x_3 & \leq & 2+s &
\end{array}
$$

Question: Find the optimal basic solution $\vec{x}^*(s)$ and the optimal value $z^*(s)$ for all s.

Step1: $s = 0$.

Solve it for $s = 0$, that is, for $\vec{b} = (2, 2)$.
For example, we applied the revised simplex method and in the final iteration, and we got an optimal feasible dictionary with

$$\vec{x}_B^* = \begin{bmatrix} x_2^* \\ x_1^* \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}, \quad N = \begin{array}{c} \begin{array}{ccc} x_5 & x_4 & x_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}, \quad B = \begin{array}{c} \begin{array}{cc} x_2 & x_1 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array},$$

$$\vec{c}_N^T = \begin{array}{c} \begin{array}{ccc} x_5 & x_4 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \vec{c}_B^T = \begin{array}{c} \begin{array}{cc} x_2 & x_1 \end{array} \\ \begin{bmatrix} 1 & 1 \end{bmatrix} \end{array},$$

and $(\vec{y}^*)^T = \vec{c}_B^T B^{-1} = \begin{bmatrix} y_1^* & y_2^* \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \end{bmatrix}$ is the optimal dual solution.

Remaining steps for parametric LP with changing $\vec{b}$:

- From $s = 0$ case, can recycle $B, N, \vec{c}_B, \vec{c}_N$, $[\vec{c}_N^T - \vec{c}_B^T B^{-1} N]$, & $(\vec{y}^*)^T = \vec{c}_B^T B^{-1}$.
- $\vec{y}^*$ is dual feasible, its dual feasibility is not affected by the change of $\vec{b}$.
- Can use $\vec{x}_B^* = B^{-1}\vec{b}$, to get $\vec{x}_B^*(s)$.
- $\vec{x}_B^*(s)$ is optimal for those s with $\vec{x}_B^*(s) \geq \vec{0}$.
- Until now the dictionary (B, N, etc) is the result from $s = 0$ case, **except** $\vec{x}_B^*(s)$. We have a dictionary with parameter s.
- For the other range of s, this basic solution $\vec{x}^*(s)$ is not feasible.
- For this other range of s, dual pivot on the dictionary. This will change $\vec{y}^*$ but dual feasibility remains valid.
- Dual pivot will change the dictionary (B, N, etc).
- In the new dictionary, will get a range of s with optimality.
- For other s, dual pivot again, get a range of s with optimality.
- Repeat until to cover all s.

Parametric LP with changing $\vec{b}$

$$
\boxed{\begin{aligned}
z &= \tfrac{4}{3} + \tfrac{1}{3}s - \tfrac{1}{3}x_5 - \tfrac{1}{3}x_4 - \tfrac{2}{3}x_3 \\
x_2 &= \boxed{\tfrac{2}{3} - \tfrac{1}{3}s} + \tfrac{1}{3}x_5 - \tfrac{2}{3}x_4 - \tfrac{1}{3}x_3 \\
x_1 &= \boxed{\tfrac{2}{3} + \tfrac{2}{3}s} - \tfrac{2}{3}x_5 + \tfrac{1}{3}x_4 - \tfrac{1}{3}x_3
\end{aligned}}
$$

≥ 0 for $-1 \leq s \leq 2$. optimal.

$\vec{x}^* = (\tfrac{2}{3} + \tfrac{2}{3}s,\ \tfrac{2}{3} - \tfrac{1}{3}s,\ 0,\ 0,\ 0)$

$\vec{y}^* = (\tfrac{1}{3}\ \ \tfrac{1}{3},\ 0,\ 0,\ \tfrac{2}{3})$ $\qquad z^* = \tfrac{4}{3} + \tfrac{1}{3}s$

$s < -1$ dual pivot

$$
\boxed{\begin{aligned}
z &= 2 + s - x_5 - x_1 - \tfrac{2}{3}x_3 \\
x_2 &= \boxed{2 + s} - x_5 - 2x_1 - x_3 \\
x_4 &= \boxed{-2 - 2s} + 2x_5 + 3x_1 + x_3
\end{aligned}}
$$

≥ 0 for $-2 \leq s \leq -1$. optimal.

$\vec{x}^* = (0,\ 2+s,\ 0,\ -2-2s,\ 0)$

$\vec{y}^* = (0\ \ 1,\ 1,\ 0,\ \tfrac{2}{3})$ $\qquad z^* = 2 + s$

$s < -2$ dual pivot

$$
\boxed{\begin{aligned}
z &= 2 + s - x_5 - x_1 - \tfrac{2}{3}x_3 \\
x_2 &= \underbrace{2 + s}_{<0} - x_5 - 2x_1 - x_3 \\
x_4 &= \underbrace{-2 - 2s}_{>0} + 2x_5 + 3x_1 + x_3
\end{aligned}}
$$

"infeasible" for $s < -2$.

no ≥ 0 coefficient in the pivot row
dual is unbounded
so primal is infeasible

dual pivot

$s > 2$

$$
\boxed{\begin{aligned}
z &= 2 - x_2 - x_4 - \tfrac{2}{3}x_3 \\
x_5 &= \boxed{-2 + s} + 3x_2 + 2x_4 + x_3 \\
x_1 &= \boxed{2} - 2x_2 - x_4 - x_3
\end{aligned}}
$$

≥ 0 for $s > 2$.
so optimal.

$\vec{x}^* = (2,\ 0,\ 0,\ 0,\ -2+s)$

$\vec{y}^* = (1\ \ 0,\ 0,\ 1,\ \tfrac{2}{3})$ $\qquad z^* = 2$