- You will *not* be able to use a calculator or computer for either the midterm or the final exam, so please do not use one for this assignment. You may use one to *check* your answer, but please do not use one to solve the problem.
- Only part of the problems may be graded. But, you have to submit all the problems.
- The deadline is *by 6pm on Friday*, October 11. You have a grace period until 9am the next day. The grace period is to take care of any technical issues you have while submitting the file. The grace period should give ample time for handling any issues, so No late HW after the grace period will be accepted, regardless of your technical issues.
- Submit only pdf files.
- Remember to use Anstee's rule.

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1. $\boxed{\text{4 marks}}$ Give an example of a dictionary for which the current basic feasible solution is optimal and yet the coefficients of the non-basic variables in the z row are not all negative.

2. $\boxed{\text{5 marks}}$ Prove the following:

   Theorem Let $\mathbf{A}$ and $\vec{b}$ be given. Then either

   - there exists an $\vec{x}$ st $\vec{x} \geq 0$ and $\mathbf{A}\vec{x} \leq \vec{b}$, or
   - there exists a $\vec{y}$ st $\mathbf{A}^T y \geq 0, \vec{y} \geq 0$ and $\vec{b}.\vec{y} < 0$

   but not both. Note the strict inequality in the second.

   Hint: Use both weak and strong duality theorems. You will also need the fundamental theorem of linear programming.

3. $\boxed{\text{3 marks}}$ Consider the LP:

$$\begin{array}{lrrrrl} \text{Maximize} & 12x_1 & +20x_2 & +21x_3 & +18x_4 & \\ & 24x_1 & +40x_2 & +46x_3 & +44x_4 & \leq 1200 \\ & x_1 & +x_2 & +x_3 & +x_4 & \leq 30 \\ & 3x_1 & +6x_2 & +6x_3 & +6x_4 & \leq 150 \end{array} \qquad x_1, x_2, x_3, x_4 \geq 0$$

   Someone claims the final dictionary has

$$z = 540 - x_2 - 3x_4 - 4x_6 - 3x_7$$

   Explain what optimal solution to the dual this implies and explain why there must have been an error in the final row for z.

   Hint: This question is related to the Strong Duality Theorem given in the class (see also Chvatal, proof of Theorem 5.1 pp58-59).

4. . **Setting up an LP from a practical problem.** In the fine tradition of bad puns on mathematical assignments, my colleagues and I are starting the soon-to-be-famous Opple Rubber Company. Our company makes three different products:

   - fashionable rubberised slippers sold under the "iMoc" name
   - rubberised protectors for fruit called "oPods", and
   - a small annoying musical instrument called the "oPhone".

   Our company receives the rubber it needs in 200cm long ribbons. Each product requires a certain amount of rubber.

   - 1 pair of iMoc slippers needs 90cm of a ribbon
   - 1 oPod needs 70cm of a ribbon
   - 1 oPhone needs 50cm of ribbon

   A large order has come in and the company needs to make at least

   - 300 pairs of iMocs
   - 400 oPods
   - 1000 oPhones

   We would like to work out how to cut up the sheets so as to minimise waste. This problem can be broken down into smaller parts:

   (a) $\boxed{\text{3 marks}}$ There are 6 ways to cut a 200cm rubber sheet into pieces of length 90cm, 70cm and 50 cm with minimal waste — what are they and how much rubber does each one waste? Please list them in order of most waste to least. *Note* Do not include ways such as $(70, 70)$ since this leaves 60cm and one could cut a 50cm segment from it.

   (b) $\boxed{\text{4 marks}}$ Each of the ways of cutting a sheet wastes a certain amount of rubber. Obviously we would like to minimise this waste while still producing enough iMocs, oPods and oPhones. For some reason our cutting machine is unable to cut the ribbon in four equal pieces, so ignore this possibility — this leaves the other five cutting options. Write this as a linear programming problem. *Note* Please label your variables $y_1, \ldots, y_5$ so that the corresponding amount ribbon wasted is ordered from greatest to least.

5. $\boxed{\text{5 marks}}$ Solve the linear programming problem

$$
\begin{array}{rlllllll}
\text{minimize } w = 40y_1 + 30y_2 + 20y_3 + 10y_4 + 10y_5 & & & & & & & \\
 & y_1 & & +2y_3 & +y_4 & & \geq & 300 \\
\text{subject to} & y_1 & +y_2 & & & +2y_5 & \geq & 400 \\
 & & 2y_2 & & +2y_4 & +y_5 & \geq & 1000 \\
 & y_1, \ldots, y_5 \geq 0 & & & & & &
\end{array}
$$

by solving the dual problem and then using Complementary Slackness. Don't worry if your answer comes out to be non-integer.

6. $\boxed{\text{5 marks}}$ The optimal solution to the linear program:

$$\begin{array}{lrrrl}
\text{Maximize} & 10x_1 & +14x_2 & +20x_3 & \\
 & 2x_1 & +3x_2 & +4x_3 & \leq 220 \\
 & 4x_1 & +2x_2 & -x_3 & \leq 385 \\
 & x_1 & & +4x_3 & \leq 160
\end{array} \qquad x_1, x_2, x_3 \geq 0$$

is $x_1 = 60, x_2 = 0, x_3 = 25$. Write down the dual problem. Use this information above to find an optimal solution to the dual (**don't use the simplex algorithm**) explaining your work (name theorems used). Explain how this confirms that the optimal solution to the primal I claimed is in fact an optimal solution.

The following two problems are OPTIONAL for your practice with complementary slackness. Do not hand in.

7. $\boxed{\text{5 marks}}$ Question 5.3(a) from Chvatal.

$$\begin{array}{lrrrrrcl}
\text{Maximise} & 7x_1 & +6x_2 & +5x_3 & -2x_4 & +3x_5 & & \\
\text{subject to} & x_1 & +3x_2 & +5x_3 & -2x_4 & +2x_5 & \leq & 4 \\
 & 4x_1 & +2x_2 & -2x_3 & +x_4 & +x_5 & \leq & 3 \\
 & 2x_1 & +4x_2 & +4x_3 & -2x_4 & +5x_5 & \leq & 5 \\
 & 3x_1 & +x_2 & +2x_3 & -x_4 & -2x_5 & \leq & 1 \\
 & & & & \multicolumn{4}{r}{x_1, x_2, x_3, x_4, x_5 \geq 0}
\end{array}$$

Is $(0, 4/3, 2/3, 5/3, 0)$ optimal?

8. $\boxed{\text{5 marks}}$ Question 5.3(b) from Chvatal.

maximise	$4x_1$	$+5x_2$	$+x_3$	$+3x_4$	$-5x_5$	$+8x_6$		
st non-neg and	1	0	-4	3	1	1	$\leq$	1
	5	3	1	0	-5	3	$\leq$	4
	4	5	-3	3	-4	1	$\leq$	4
	0	-1	0	2	1	-5	$\leq$	5
	-2	1	1	1	2	2	$\leq$	7
	2	-3	2	-1	4	5	$\leq$	5

Is $(0, 0, 5/2, 7/2, 0, 1/2)$ optimal?