- You will *not* be able to use a calculator or computer for either the midterm or the final exam, so please do not use one for this assignment. You may use one to *check* your answer, but please do not use one to solve the problem.
- Only part of the problems may be graded. But, you have to submit all the problems.
- The deadline is *by 6pm on Friday*, September 27. You have a grace period until 9am Saturday, September 29. The grace period is to take care of any technical issues you have while submitting the file. The grace period should give ample time for handling any issues, so No late HW after the grace period will be accepted, regardless of your technical issues.
- **Submit only pdf files.**
- Remember to use

Anstee's rule:
- Choose the entering variable with the largest positive coefficient.
- If there is a tie, then choose the one with the smaller subscript.
- If there is a choice of leaving variable, then choose the one with the smallest subscript.

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1. Solve the following LP problems using the simplex method. At each step please state the entering and leaving variables and the current basic feasible solution. Clearly state the optimal solution and the optimal value.

   (a) $\boxed{\text{3 marks}}$ Maximise $z = 2x_1 + 3x_2 + 3x_3$, subject to

$$\begin{array}{rrrcl} 3x_1 & +x_2 & & \leq & 60 \\ -x_1 & +x_2 & +4x_3 & \leq & 10 \\ 2x_1 & -2x_2 & +5x_3 & \leq & 15 \end{array}$$

   and $x_1, x_2, x_3 \geq 0$. *This problem requires 2 pivots.*

Solution: We first introduce slack variables x_4, x_5 and x_6 and write our first dictionary:

$$\begin{array}{rcrrrr} x_4 & = & 60 & -3x_1 & -x_2 & \\ x_5 & = & 10 & +x_1 & -x_2 & -4x_3 \\ x_6 & = & 15 & -2x_1 & +2x_2 & -5x_3 \\ \hline z & = & & 2x_1 & +3x_2 & +3x_3 \end{array}$$

A quick check shows that this is feasible. The current basic feasible solution is

$$x_1 = x_2 = x_3 = 0 \qquad x_4 = 60 \qquad x_5 = 10 \qquad x_6 = 15 \qquad z = 0$$

We have a choice of 3 entering variables, and x_2 and x_3 both have equally large coefficients, so we pick the one with the smaller subscript — x_2 is the entering variable. Now, x_2 is restricted to be less than 60 and 10 by the first and second equations (the last equation does not constrain it). Hence the leaving variable is x_5.

$$\begin{array}{rcrrrr} x_2 & = & 10 & +x_1 & -4x_3 & -x_5 \\ x_4 & = & 50 & -4x_1 & +4x_3 & +x_5 \\ x_6 & = & 35 & & -13x_3 & -2x_5 \\ \hline z & = & 30 & +5x_1 & -9x_3 & -3x_5 \end{array}$$

So the current feasible solution is

$$x_1 = x_3 = x_5 = 0 \qquad x_2 = 10 \qquad x_4 = 50 \qquad x_6 = 35 \qquad z = 30$$

Now, since there is a positive coefficient in the expression for z we are not yet done. There is only 1 choice of entering variable, x_1. The first and third equations do not restrict x_1, and the second restricts $x_1 \leq 50/4 = 25/2$. Hence x_4 is the leaving variable. Pivoting gives:

$$\begin{array}{rcrrrr} x_1 & = & \frac{25}{2} & +x_3 & -\frac{1}{4}.x_4 & +\frac{1}{4}.x_5 \\ x_2 & = & \frac{45}{2} & -3x_3 & -\frac{1}{4}.x_4 & -\frac{3}{4}.x_5 \\ x_6 & = & 35 & -13x_3 & & -2x_5 \\ \hline z & = & \frac{185}{2} & -4x_3 & -\frac{5}{4}x_4 & -\frac{7}{4}.x_5 \end{array}$$

The current feasible solution is

$$x_1 = \frac{25}{2} \qquad x_2 = \frac{45}{2} \qquad x_3 = x_4 = x_5 = 0 \qquad x_6 = 35 \qquad z = \frac{185}{2}$$

Since there are no longer any positive coefficients in z, this is the optimal solution and $z = \frac{185}{2}$ is the optimal value.

(b) $\boxed{\text{4 marks}}$ Maximise $z = 3x_1 + 2x_2 + 4x_3$, subject to

$$\begin{array}{rrrcl} x_1 & +x_2 & +2x_3 & \leq & 4 \\ 2x_1 & & +3x_3 & \leq & 5 \\ 2x_1 & +x_2 & +3x_3 & \leq & 7 \end{array}$$

and $x_1, x_2, x_3 \geq 0$. *This problem requires 3 pivots.*
Note: this problem is Q2.1a from Chávtal.

Solution: Introduce slack variables and write in dictionary form

$$\begin{array}{rcrrrr} x_4 & = & 4 & -x_1 & -x_2 & -2x_3 \\ x_5 & = & 5 & -2x_1 & & -3x_3 \\ x_6 & = & 7 & -2x_1 & -x_2 & -3x_3 \\ \hline z & = & & 3x_1 & +2x_2 & +4x_3 \end{array}$$

All the variables are potentially entering, but we pick the one with the largest coefficient — so x_3 is the entering variable. It is restricted to be at most $2, 5/3, 7/3$ (respectively). So x_5 is the leaving variable.

Pivoting gives

$$\begin{array}{rcrrrr} x_3 & = & \frac{5}{3} & -\frac{2}{3}x_1 & & -\frac{1}{3}x_5 \\ x_4 & = & \frac{2}{3} & +\frac{1}{3}x_1 & -x_2 & +\frac{2}{3}x_5 \\ x_6 & = & 2 & & -x_2 & +x_5 \\ \hline z & = & \frac{20}{3} & +\frac{1}{3}x_1 & +2x_2 & -\frac{4}{3}x_5 \end{array}$$

The current feasible solution is

$$x_1 = x_2 = x_5 = 0 \qquad x_3 = \frac{5}{3} \qquad x_4 = \frac{2}{3} \qquad x_6 = 2 \qquad z = \frac{20}{3}$$

Choose x_2 to be the entering variable. It is constrained to be at most $2/3, 2$ — so x_4 is the leaving variable. Pivoting gives

$$\begin{array}{rcrrrr} x_2 & = & \frac{2}{3} & +\frac{1}{3}x_1 & -x_4 & +\frac{2}{3}x_5 \\ x_3 & = & \frac{5}{3} & -\frac{2}{3}x_1 & & -\frac{1}{3}x_5 \\ x_6 & = & \frac{4}{3} & -\frac{1}{3}x_1 & +x_4 & +\frac{1}{3}x_5 \\ \hline z & = & 8 & +x_1 & -2x_4 & \end{array}$$

So the current feasible solution is

$$x_1 = x_4 = x_5 = 0 \qquad x_2 = \frac{2}{3} \qquad x_3 = \frac{5}{3} \qquad x_6 = \frac{4}{3} \qquad z = 8$$

We only have 1 choice of entering variable — x_1. It is constrained to be at most $5/2$ or 4, so x_3 is the leaving variable. Pivoting gives

$$\begin{array}{rcrrrr} x_1 & = & \frac{5}{2} & -\frac{3}{2}x_3 & & -\frac{1}{2}x_5 \\ x_2 & = & \frac{3}{2} & -\frac{1}{2}x_3 & -x_4 & +\frac{1}{2}x_5 \\ x_6 & = & \frac{1}{2} & +\frac{1}{2}x_3 & +x_4 & +\frac{1}{2}x_5 \\ \hline z & = & \frac{21}{2} & -\frac{3}{2}x_3 & -2x_4 & -\frac{1}{2}x_5 \end{array}$$

Since there are no negative coefficients in the expression for z we are done. The optimal solution is therefore

$$x_1 = \frac{5}{2} \qquad x_2 = \frac{3}{2} \qquad x_3 = x_4 = x_5 = 0 \qquad x_6 = \frac{1}{2}$$

and the optimal value is $z = \frac{21}{2}$.

2. $\boxed{\text{3 marks}}$ Show that the three inequalities

$$-x + 2y \le -2 \qquad 2x + y \ge 1 \qquad -3x + y \ge -4$$

have no solution x, y with $x, y \ge 0$ by using our two phase method.

Solution: The three inequalities $-x+2y \le -2$, $2x+y \ge 1$, $-3x+y \ge -4$ have no solution x, y with $x, y \ge 0$:

We first transform the inequalities to standard form $-x+2y \le -2$, $-2x-y \le -1$, $+3x-y \le 4$. Then apply Phase One:

$$\begin{array}{rcrrrr} x_1 & = & -2 & +x & -2y & +x_0 \\ x_2 & = & -1 & +2x & +y & +x_0 \\ x_3 & = & 4 & -3x & +y & +x_0 \\ w & = & & & & -x_0 \end{array}$$

Fake pivot to feasibility, x_0 enters and x_1 leaves.

$$\begin{array}{rcrrrr} x_0 & = & 2 & -x & +2y & +x_1 \\ x_2 & = & 1 & +x & +3y & +x_1 \\ x_3 & = & 6 & -4x & +3y & +x_1 \\ w & = & -2 & +x & +y & -x_1 \end{array}$$

(I used the standard pivoting rule with x before y) x enters and x_3 leaves.

$$\begin{array}{rcrrrr} x_0 & = & 1/2 & +(1/4)x_3 & +(5/4)y & +(3/4)x_1 \\ x_2 & = & 5/2 & -(1/4)x_3 & +(15/4)y & +(5/4)x_1 \\ x & = & 3/2 & -(1/4)x_3 & +(3/4)y & +(1/4)x_1 \\ w & = & -1/2 & -(1/4)x_3 & -(5/4)y & -(3/4)x_1 \end{array}$$

We are at optimality with $w = -1/2$. Thus we cannot drive x_0 to 0. Using the magic coefficients, the negatives of the coefficients of the slack variables x_1, x_2, x_3, namely $\frac{3}{4}, 0, \frac{1}{4}$ we apply these to the three inequalities:

$$\frac{3}{4}(-x+2y \le -2) + 0(-2x - y \le -1) + \frac{1}{4}(+3x - y \le 4)$$

to get the contradiction to $y \ge 0$:

$$(5/4)y \le \frac{-1}{2}.$$

Thus the three inequalities only yield solutions with $y < 0$.

3. $\boxed{\text{4 marks}}$ Use the two-phase method to find the solution of the following LP problem:

$$\begin{array}{lrrcl} \text{maximise } z = 3x_1 + x_2 & & & & \\ & x_1 & +x_2 & \leq & 1 \\ \text{subject to} & -2x_1 & +x_2 & \geq & 1 \\ & 3x_1 & +2x_2 & \geq & 4 \\ \text{and} & & x_1, x_2 \geq 0 & & \end{array}$$

This requires around 3 pivots.

Solution: Write in dictionary form:

$$\begin{array}{rcrrr} x_3 & = & 1 & -x_1 & -x_2 \\ x_4 & = & -1 & -2x_1 & +x_2 \\ x_5 & = & -4 & +3x_1 & +2x_2 \\ \hline z & = & 3x_1 & +x_2 & \end{array}$$

This is not feasible — so we use the two-phase method. Introduce x_0 and maximise $w = -x_0$:

$$\begin{array}{rcrrrr} x_3 & = & 1 & -x_1 & -x_2 & +x_0 \\ x_4 & = & -1 & -2x_1 & +x_2 & +x_0 \\ x_5 & = & -4 & +3x_1 & +2x_2 & +x_0 \\ \hline w & = & & & & -x_0 \end{array}$$

The most negative variable is x_5 — so it leaves and x_0 enters. This is our "fake pivot to feasibility":

$$\begin{array}{rcrrrr} x_0 & = & 4 & -3x_1 & -2x_2 & +x_5 \\ x_3 & = & 5 & -4x_1 & -3x_2 & +x_5 \\ x_4 & = & 3 & -5x_1 & -x_2 & +x_5 \\ \hline w & = & -4 & +3x_1 & +2x_2 & -x_5 \end{array}$$

Still not finished — x_1 enters and x_4 leaves:

$$\begin{array}{rcrrrr} x_0 & = & 11/5 & -(7/5)x_2 & +(3/5)x_4 & +(2/5)x_5 \\ x_1 & = & 3/5 & -(1/5)x_2 & -(1/5)x_4 & +(1/5)x_5 \\ x_3 & = & 13/5 & -(11/5)x_2 & +(4/5)x_4 & +(1/5)x_5 \\ \hline w & = & -11/5 & +(7/5)x_2 & -(3/5)x_4 & -(2/5)x_5 \end{array}$$

Still not finished — x_2 enters and x_3 leaves

$$\begin{array}{rcrrrr} x_0 & = & 6/11 & +(7/11)x_3 & +(1/11)x_4 & +(3/11)x_5 \\ x_1 & = & 4/11 & +(1/11)x_3 & -(3/11)x_4 & +(2/11)x_5 \\ x_2 & = & 13/11 & -(5/11)x_3 & +(4/11)x_4 & +(1/11)x_5 \\ \hline w & = & -6/11 & -(7/11)x_3 & -(1/11)x_4 & -(3/11)x_5 \end{array}$$

So we now stop. The maximum value of $w = -6/11$. Hence the minimum value of $x_0 = 6/11$. Since this is non-zero, we conclude that the original LP problem is not feasible.

4. $\boxed{\text{4 marks}}$ Use the two-phase method to solve the following LP problem:

$$\text{maximise } z = 3x_1 + 2x_2 + 3x_3$$
$$\text{subject to} \quad \begin{array}{rrrcr} 2x_1 & +x_2 & +x_3 & \leq & 2 \\ 3x_1 & +4x_2 & +2x_3 & \geq & 8 \end{array}$$
$$\text{and } x_1, x_2, x_3 \geq 0$$

This requires around 3 pivots.

Solution: First put in standard form and write as a dictionary:

$$\begin{array}{rcrrrr} x_4 & = & 2 & -2x_1 & -x_2 & -x_3 \\ x_5 & = & -8 & +3x_1 & +4x_2 & +2x_3 \\ \hline z & = & & 3x_1 & +2x_2 & +3x_3 \end{array}$$

This is not a feasible dictionary — use the 2 phase method. The auxillary problem is to maximise w with

$$\begin{array}{rcrrrrr} x_4 & = & 2 & -2x_1 & -x_2 & -x_3 & +x_0 \\ x_5 & = & -8 & +3x_1 & +4x_2 & +2x_3 & +x_0 \\ \hline w & = & & & & & -x_0 \end{array}$$

Since x_5 is the most-negative basic variable, the "fake pivot to feasibility" swaps x_0 and x_5:

$$\begin{array}{rcrrrrr} x_0 & = & 8 & -3x_1 & -4x_2 & -2x_3 & +x5 \\ x_4 & = & 10 & -5x_1 & -5x_2 & -3x_3 & +x_5 \\ \hline w & = & -8 & +3x_1 & +4x_2 & +2x_3 & -x_5 \end{array}$$

Pivot again — x_2 enters and x_0 leaves

$$\begin{array}{rcrrrrr} x_2 & = & 2 & -(1/4)x_0 & -(3/4)x_1 & -(1/2)x_3 & +(1/4)x_5 \\ x_4 & = & 0 & (5/4)x_0 & -(5/4)x_1 & -(1/2)x_3 & -(1/4)x_5 \\ \hline w & = & & -x_0 & & & \end{array}$$

So we have a feasible solution! To get to the feasible dictionary of the original problem we delete x_0 and rewrite z in terms of x_1, x_3 and x_5. Rewrite z:

$$\begin{aligned} z &= 3x_1 + 2x_2 + 3x_3 \\ &= 3x_1 + 3x_3 + 2(2 - (3/4)x_1 - (1/2)x_3 + (1/4)x_5) \\ &= 4 + (3/2)x_1 + 2x_3 + (1/2)x_5 \end{aligned}$$

So the dictionary is:

$$\begin{array}{rcrrrr} x_2 & = & 2 & -(3/4)x_1 & -(1/2)x_3 & +(1/4)x_5 \\ x_4 & = & 0 & -(5/4)x_1 & -(1/2)x_3 & -(1/4)x_5 \\ \hline z & = & 4 & +(3/2)x_1 & +2x_3 & +(1/2)x_5 \end{array}$$

So we pivot again. x_3 enters and x_4 must leave:

$$\begin{array}{rcrrrr} x_2 & = & 2 & +(1/2)x_1 & +x_4 & +(1/2)x_5 \\ x_3 & = & & -(5/2)x_1 & -2x_4 & -(1/2)x_5 \\ \hline z & = & 4 & -(7/2)x_1 & -4x_4 & -(1/2)x_5 \end{array}$$

So we stop here. The optimum solution is

$$x_1 = x_4 = x_5 = 0 \qquad x_2 = 2 \qquad x_3 = 0 \qquad z = 4$$

5. (a) $\boxed{\text{3 marks}}$ We say a set C of points in $\mathbf{R}^n$ is convex if for every pair $\mathbf{x}, \mathbf{y} \in C$, all points on the line segment joining $\mathbf{x}$ and $\mathbf{y}$ are in C. Thus C is a convex set if for every pair $\mathbf{x}, \mathbf{y} \in C$ and any $\lambda \in (0, 1)$. then $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C$. Let A be an $m \times n$ matrix and $\mathbf{b}$ a given vector in $\mathbf{R}^m$. Show that

$$F = \{\mathbf{x} \in \mathbf{R}^n \,:\, A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}\}$$

is a convex set.

Solution:
We check that for $\lambda \in [0, 1]$, that $\lambda \geq 0$ and $(1-\lambda) \geq 0$. Let $\mathbf{x}.\mathbf{y} \in F$. Then we have $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, $A\mathbf{y} \leq \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$.

We note that if $\mathbf{u}, \mathbf{v}$ are vectors and $c \geq 0$ a scalar, then if $\mathbf{u} \geq \mathbf{v}$, we have $c\mathbf{u} \geq c\mathbf{v}$. You can check entry by entry to see this. Now $\mathbf{x} \geq \mathbf{0}$ implies $\lambda\mathbf{x} \geq \lambda\mathbf{0} = \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$ implies $(1-\lambda)\mathbf{y} \geq (1-\lambda)\mathbf{0} = \mathbf{0}$. Then $\lambda\mathbf{x}+(1-\lambda)\mathbf{y} \geq \mathbf{0}$. Also $A\mathbf{x} \leq \mathbf{b}$ implies $A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) \leq \lambda\mathbf{b}$ and $A\mathbf{y} \leq \mathbf{b}$ implies $A((1-\lambda)\mathbf{y}) = (1-\lambda)(A\mathbf{y}) \leq (1-\lambda)\mathbf{b}$. Then we obtain

$$A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = A(\lambda\mathbf{x}) + A((1-\lambda)\mathbf{y}) \leq \lambda\mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b}$$

We now conclude $(\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in F$ and so F is convex.

(b) $\boxed{\text{2 marks}}$ Consider an LP: max $\mathbf{c} \cdot \mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Assume the LP has two optimal solutions $\mathbf{u}$ and $\mathbf{v}$. Show that for any choice of $\lambda \in [0, 1]$ (i.e. $0 \leq \lambda \leq 1$), that $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ is also an optimal solution. First show that $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ has the same value of the objective function as $\mathbf{u}$. Then show $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ is a feasible solution to the LP.

Solution:
By the previous question, $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ is a feasible solution to the LP. Since $\mathbf{u}$ and $\mathbf{v}$ are both optimal, then $\mathbf{c} \cdot \mathbf{u} = \mathbf{c} \cdot \mathbf{v}$. We compute

$$\mathbf{c} \cdot (\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) = \lambda\mathbf{c} \cdot \mathbf{u} + (1-\lambda)\mathbf{c} \cdot \mathbf{v} = (\lambda + (1-\lambda))\mathbf{c} \cdot \mathbf{u} = \mathbf{c} \cdot \mathbf{u}$$

using $\mathbf{c} \cdot \mathbf{u} = \mathbf{c} \cdot \mathbf{v}$. We conclude that $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ is an optimal solution since it is feasible and has the appropriate value of the objective function.

6. We say that a function $f : \mathbf{R}^n \to \mathbf{R}$ is convex if for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\lambda \in [0,1]$, we have $f((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) \leq (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y})$.

   (a) $\boxed{\text{2 marks}}$ Give an example of a convex function. To get credits, you have to justify why it is convex.

   Solution: For example, any function $f(\mathbf{x}) = \mathbf{c}\cdot\mathbf{x}$ while $\mathbf{c} \in \mathbf{R}^n$ is a given vector, is a convex function; it is a linear function in fact, and any linear function is a particular example of a convex function. It is easy to see why this is convex: for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\lambda \in [0,1]$, $f((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) = \mathbf{c} \cdot ((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) = (1-\lambda)\mathbf{c} \cdot \mathbf{x} + \lambda\mathbf{c} \cdot \mathbf{y} = (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y})$. (Note that it even satisfies the equality which then certainly satisfies the inequality $\leq$.)

   (b) $\boxed{\text{3 marks}}$ Let f_1, f_2 be two given convex functions on $\mathbf{R}^n$. Then, consider the function $g : \mathbf{R}^n \to \mathbf{R}$ defined as

$$g(\mathbf{x}) = \max[f_1(\mathbf{x}), f_2(\mathbf{x})];$$

   this means that given $\mathbf{x}$, $g(\mathbf{x})$ takes the larger value between $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$, that is,

$$g(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \text{if } f_1(\mathbf{x}) \geq f_2(\mathbf{x}), \\ f_2(\mathbf{x}) & \text{if } f_2(\mathbf{x}) \geq f_1(\mathbf{x}). \end{cases}$$

   Prove that g is a convex function. [**Note that you have to prove this for ANY convex functions f_1 and f_2. You will get zero mark if you do this only for a particular pair of functions.**]

   Solution:
   Let $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, and $\lambda \in [0,1]$. Consider $(1-\lambda)\mathbf{x} + \lambda\mathbf{y}$.
   Two cases:
   If $f_1((1-\lambda)\mathbf{x}+\lambda\mathbf{y}) \geq f_2((1-\lambda)\mathbf{x}+\lambda\mathbf{y})$, then $g((1-\lambda)\mathbf{x}+\lambda\mathbf{y}) = f_1(1-\lambda)\mathbf{x}+\lambda\mathbf{y})$ from the definition, which by convexity of f_1, is $\leq (1-\lambda)f_1(\mathbf{x}) + \lambda f_1(\mathbf{y})$. This is $\leq (1-\lambda)g(\mathbf{x}) + \lambda g(\mathbf{y})$ as $f_1 \leq g$ from the definition of g. This verifies $g((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) \leq (1-\lambda)g(\mathbf{x}) + \lambda g(\mathbf{y})$ in this case.
   The other case is similar. Namely, if $f_2((1-\lambda)\mathbf{x}+\lambda\mathbf{y}) \geq f_1((1-\lambda)\mathbf{x}+\lambda\mathbf{y})$, then $g((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) = f_2(1-\lambda)\mathbf{x} + \lambda\mathbf{y})$ from the definition, which by convexity of f_2, is $\leq (1-\lambda)f_2(\mathbf{x}) + \lambda f_2(\mathbf{y})$. This is $\leq (1-\lambda)g(\mathbf{x}) + \lambda g(\mathbf{y})$ as $f_2 \leq g$ from the definition of g. This verifies $g((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) \leq (1-\lambda)g(\mathbf{x}) + \lambda g(\mathbf{y})$ in this case.

These two cases together imply the desired inequality:
$g((1-\lambda)\mathbf{x}+\lambda\mathbf{y}) \leq (1-\lambda)g(\mathbf{x})+\lambda g(\mathbf{y})$, completing the proof.