Lecture 29: Matrix Games

Vanderbei Ch.11. sections 1–3. Chvatal p228–233 matrix games

- formulating the corresponding LP problem
- how to solve the LP problem
- suggested exercises: Chvatal 15.1, 15.4,

Matrix Games: Player 1's problem $\max_{\vec{x} \text{ stoch.}} \left[\min_{\vec{y} \text{ stoch.}} \vec{x}^T A \vec{y} \right]$

How to find $\vec{x}^*$ optimal strategy for $\max_{\vec{x} \text{ stochastic}} \left[\min_{\vec{y} \text{ stochastic}} \vec{x}^T A \vec{y} \right]$?

- This is not an LP in this form, as
- the function $x \mapsto \min_{\vec{y} \text{ stochastic}} \vec{x}^T A \vec{y}$ is not a linear function.

Set up the corresponding LP: Player 1's problem

Consider $\min_{\vec{y} \text{ stochastic}} \vec{x}^T A \vec{y}$

- Constraint: $\vec{y}$ is stochastic
  - the feasible region is give by $y_1 + ... + y_n = 1$ and $y_1, ..., y_n \geq 0$.
- the corner points are $\vec{e}^1, ...\vec{e}^n \in \mathbb{R}^n$ while

$$\vec{e}^j = (0, .., 0, 1, 0, ..., 0) \qquad \text{(all 0 except 1 at the } j\text{-th component)}$$

- The minimum of an LP occurs at a corner point. Therefore,

$$\min_{\vec{y} \text{ stochastic}} \vec{x}^T A \vec{y} = \min_{\vec{e}^1, \ldots, \vec{e}^n} \vec{x}^T A \vec{e}^j$$

- Then can write

$$\min_{\vec{e}^1, \ldots, \vec{e}^n} \vec{x}^T A \vec{e}^j = \max_{v \leq \vec{x}^T A \vec{e}^j, j=1, \ldots, n} v$$

LP problem for Player 1 $\max_{\vec{x} \text{ stoch.}} \left[\min_{\vec{y} \text{ stoch.}} \vec{x}^T A \vec{y} \right]$

Then can write

$$
\begin{aligned}
& \max_{\vec{x} \text{ stochastic}} \left[\min_{\vec{y} \text{ stochastic}} \vec{x}^T A \vec{y} \right] \\
= & \max_{\vec{x} \text{ stochastic}} \left[\min_{\vec{e}^1, \ldots, \vec{e}^n} \vec{x}^T A \vec{e}^j \right] \\
= & \max_{\vec{x} \text{ stochastic}} \left[\max_{v \leq \vec{x}^T A \vec{e}^j, j=1,\ldots,n} v \right] \\
= & \max_{\vec{x} \text{ stochastic}, v \leq \vec{x}^T A \vec{e}^j, j=1,\ldots,n} v
\end{aligned}
$$

The latter is an LP!

The LP problem for Player 1:

$$
\begin{array}{lrl}
\text{Maximize} & & v \\
\text{subject to} & v \leq \vec{x}^T A \vec{e}^j, & j = 1, \ldots, n \\
& & x_1 + \ldots + x_m = 1 \\
& & x_1, \ldots, x_m \geq 0
\end{array}
$$

$$\max_{\vec{x}\text{ stoch.}} \left[\min_{\vec{y}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$$

Example (Rock-Paper-Scissors)

The payoff matrix for the row player:

$$\text{row player} \quad \begin{array}{c} \\ \\ \end{array} \begin{array}{c c} & \text{column player} \\ & \begin{array}{ccc} r & p & s \end{array} \\ \begin{array}{c} r \\ p \\ s \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

The corresponding LP problem is: [See the board.]

The LP problem for Player 1: $\max\limits_{\vec{x}\ \textbf{stoch.}} \left[\min\limits_{\vec{y}\ \textbf{stoch.}} \vec{x}^T A \vec{y} \right]$

$$
\begin{array}{ll}
\text{Maximize} & v \\
\text{subject to} & v \le \vec{x}^T A \vec{e}^j, \quad j = 1, ..., n \\
 & x_1 + ... + x_m = 1 \\
 & x_1, ..., x_m \ge 0
\end{array}
$$

Use notaation $\vec{\mathbf{e}}_m = (1, ..., 1) \in \mathbb{R}^m$, in particular, $\vec{\mathbf{e}}_n = (1, ..., 1) \in \mathbb{R}^n$.
Then, can write
The LP problem for Player 1:

$$
\begin{array}{ll}
\text{Maximize} & v \\
\text{subject to} & v\vec{\mathbf{e}}_n \le A^T \vec{x} \\
 & \vec{\mathbf{e}}_m^T \vec{x} = 1 \\
 & \vec{x} \ge 0
\end{array}
$$

LP problem for Player 1 $\max\limits_{\vec{x}\text{ stoch.}} \left[\min\limits_{\vec{y}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$

Example

The payoff matrix for the row player:

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

- Turn it into an LP for Player 1. See the board
- Solving this LP will need some considerations as the LP is not in a standard form.

Example continued

Solve the Player 1 problem in the previous example.
See the board.

Matrix Games: Player 2's problem $\min_{\vec{y} \text{ stoch.}} \left[\max_{\vec{x} \text{ stoch.}} \vec{x}^T A \vec{y} \right]$

How to find $\vec{x}^*$ optimal strategy for $\min_{\vec{y} \text{ stochastic}} \left[\max_{\vec{x} \text{ stochastic}} \vec{x}^T A \vec{y} \right]$?

- This is not an LP in this form as
- the function $y \mapsto \max_{\vec{x} \text{ stochastic}} \vec{x}^T A \vec{y}$ is not a linear function.

Set up the corresponding LP: Player 2's problem

$$\min_{\vec{y}\ \textbf{stoch.}} \left[\max_{\vec{x}\ \textbf{stoch.}} \vec{x}^T A \vec{y} \right]$$

Consider $\displaystyle\max_{\vec{x}\ \text{stochastic}} \vec{x}^T A \vec{y}$

- Constraint: $\vec{x}$ is stochastic
  - the feasible region is give by $x_1 + ... + x_m = 1$ and $x_1, ..., x_m \geq 0$.
- the corner points are $\vec{e}^1, ...\vec{e}^m \in \mathbb{R}^m$ while

$$\vec{e}^i = (0, .., 0, 1, 0, ..., 0) \qquad \text{(all 0 except 1 at the } i\text{-th component)}$$

- The minimum of an LP occurs at a corner point. Therefore,

$$\max_{\vec{x}\ \text{stochastic}} \vec{x}^T A \vec{y} = \max_{\vec{e}^1, \ldots, \vec{e}^m} (\vec{e}^i)^T A \vec{y}$$

- Then can write

$$\max_{\vec{e}^1, \ldots, \vec{e}^m} (\vec{e}^i)^T A \vec{x} = \min_{u \geq (\vec{e}^i)^T A \vec{y}, i=1, \ldots, m} u$$

$$\min_{\vec{y}\text{ stoch.}} \left[\max_{\vec{x}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$$

Then can write

$$\begin{aligned}
&\min_{\vec{y}\text{ stochastic}} \left[\max_{\vec{x}\text{ stochastic}} \vec{x}^T A \vec{y} \right] \\
&= \min_{\vec{y}\text{ stochastic}} \left[\max_{\vec{e}^1,\dots,\vec{e}^m} (\vec{e}^i)^T A \vec{y} \right] \\
&= \min_{\vec{y}\text{ stochastic}} \left[\min_{u \geq (\vec{e}^i)^T A \vec{x}, i=1,\dots,n} u \right] \\
&= \min_{\vec{y}\text{ stochastic}, u \geq (\vec{e}^i)^T A \vec{x}, i=1,\dots,n} u
\end{aligned}$$

The latter is an LP!

The LP problem for Player 2:

$$\begin{array}{lrl}
\text{Minimize} & & u \\
\text{subject to} & u \geq (\vec{e}^i)^T A \vec{y}, & i = 1, \dots, m \\
& & y_1 + \dots + y_n = 1 \\
& & y_1, \dots, y_n \geq 0
\end{array}$$

LP problem for Player 2 $\min\limits_{\vec{y}\text{ stoch.}} \left[\max\limits_{\vec{x}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$

Example (Rock-Paper-Scissors)

The payoff matrix for the row player:

$$
\begin{array}{cc}
 & \text{column player} \\
 & \begin{array}{ccc} r & p & s \end{array} \\
\text{row player} & \begin{array}{c} r \\ p \\ s \end{array} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}
\end{array}
$$

The corresponding LP problem for the column player (Player 2) is:
Student exercise

The LP problem for Player 2 $\min\limits_{\vec{y}\ \textbf{stoch.}} \left[\max\limits_{\vec{x}\ \textbf{stoch.}} \vec{x}^T A \vec{y} \right]$:

$$
\begin{array}{rl}
\text{Minimize} & \qquad\qquad u \\
\text{subject to} & u \geq (\vec{e}^i)^T A\vec{y}, \quad i = 1, \ldots, m \\
& y_1 + \ldots + y_n = 1 \\
& y_1, \ldots, y_n \geq 0
\end{array}
$$

Use notation $\vec{\mathbf{e}}_n = (1, \ldots, 1) \in \mathbb{R}^n$, in particular, $\vec{\mathbf{e}}_m = (1, \ldots, 1) \in \mathbb{R}^m$.
Then, can write
The LP problem for Player 2:

$$
\begin{array}{rl}
\text{Minimize} & u \\
\text{subject to} & u\vec{\mathbf{e}}_m \geq A\vec{y} \\
& \vec{\mathbf{e}}_n^T \vec{y} = 1 \\
& \vec{y} \geq 0
\end{array}
$$

LP problem for Player 2 $\min_{\vec{y}\text{ stoch.}} \left[\max_{\vec{x}\text{ stoch.}} \vec{x}^T A \vec{y} \right]$

Example

The payoff matrix for the row player:

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

Students' exercise

- Turn it into an LP for Player 2.
- Solving this LP will need some considerations as the LP is not in a standard form.

Solve the Player 2's problem in the previous example.
Students' exercise