- You will *not* be able to use a calculator or computer for either the midterm or the final exam, so please do not use one for this assignment. You may use one to *check* your answer, but please do not use one to solve the problem.
- Only part of the problems may be graded. But, you have to submit all the problems.
- The deadline is *by 6pm on Friday*, Nov. 22. You have a grace period until 9am in the morning the next day. The grace period is to take care of any technical issues you have while submitting the file. The grace period should give ample time for handling any issues, so No late HW after the grace period will be accepted, **regardless** your technical issues.
- Submit only pdf files.

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1. $\boxed{\text{5 marks}}$ Consider the following LP problem.

$$\begin{array}{llrrl} \text{Minimise} & w = & 3x_1 & +2x_2 & \\ \text{Subject to} & & -x_1 & +2x_2 & \geq 4 \\ & & 2x_1 & -3x_2 & \geq 6 \\ & & x_1, & x_2 & \geq 0 \end{array}$$

(a) Rewrite the above LP in standard form (*do not* give the dual of this problem).

Solution:

- This becomes

$$\begin{array}{llrrl} \text{Maximise} & z = & -3x_1 & -2x_2 & \\ \text{Subject to} & & x_1 & -2x_2 & \leq -4 \\ & & -2x_1 & +3x_2 & \leq -6 \\ & & x_1, & x_2 & \geq 0 \end{array}$$

where $z = -w$.

(b) What is dictionary corresponding to the LP in your answer to part (a)?

Solution:

- In dictionary form this is

$$\begin{array}{rcrrr} x_3 & = & -4 & -x_1 & +2x_2 \\ x_4 & = & -6 & +2x_1 & -3x_2 \\ \hline z & = & & -3x_1 & -2x_2 \end{array}$$

(c) Your answer to part (b) should be a dictionary that is not feasible, but is dual-feasible. Use the dual-simplex method to find the optimal dictionary and so solve the original LP problem. (Remember to choose the most negative basic variable to be the leaving variable.)

Solution:

- The most negative basic variable is x_4 — it leaves
- Rewrite this as $0 = -6 + 2x_1 - 3x_2 - x_4$ and add $q\times$ this equation to the equation for z:

$$z = -6q + x_1(-3 + 2q) + x_2(-2 - 3q) - qx_4$$

Hence $q = 3/2$ and x_1 enters.

- Pivoting gives

$$\begin{array}{rcrrr} x_1 & = & 3 & +(3/2)x_2 & +(1/2)x_4 \\ x_3 & = & -7 & +(1/2)x_2 & -(1/2)x_4 \\ \hline z & = & -9 & -(13/2)x_2 & -(3/2)x_4 \end{array}$$

- Still not feasible. Now x_3 leaves.
- Rewrite pivot row as $0 = -7 + (1/2)x_2 - (1/2)x_4 - x_3$ and add $q\times$ this equation to z:

$$z = -9 - 7q + x_2(-13/2 + q/2) + x_4(-3/2 - q/2) - qx_3$$

Hence $q = 13$ and x_2 enters.

- Pivoting gives

$$\begin{array}{rcrrr} x_1 & = & 24 & +3x_3 & +2x_4 \\ x_2 & = & 14 & +2x_3 & +x_4 \\ \hline z & = & -100 & -13x_3 & -8x_4 \end{array}$$

- This is now dual-feasible and feasible, and hence optimal.
- The optimal solution is

$$x_1 = 24 \qquad x_2 = 14 \qquad z = -100 \qquad w = 100.$$

2. $\boxed{\text{5 marks}}$ There are six independent questions below. What can you say about the dual if you already know: (answer for each of the following considered individually)

a) a feasible solution to the primal exists?

b) an optimal solution to the primal exists?

c) an optimal solution to the primal exists with $x_1 > 0$?

d) an optimal solution to the primal exists with $x_1 = 0$?

e) there is no feasible solution to the primal?

f) the primal is unbounded?

Solution: a) If a feasible solution to the primal exists, the dual has to be bounded.

b) If an optimal solution to the primal exists, then by Strong Duality, there is an optimal solution to the dual with the same value of the objective function.

c) If an optimal solution to the primal exists with $x_1 > 0$, then we have the same conclusion as b), but using the Theorem of Complementary Slackness, we deduce that any optimal solution to the dual has the first constraint being equality (first dual slack is 0).

d) If an optimal solution to the primal exists with $x_1 = 0$, then we cannot make any grander observations than that of b). In some instances, $x_1 = 0$ may force other facts.

e) If there is no feasible solution to the primal, then by Strong Duality, the dual cannot have an optimal solution. Thus the dual is either unbounded or is infeasible.

f) If the primal is unbounded then the dual is infeasible.

3. $\boxed{\text{5 marks}}$ Our famous game theory protagonists, Claude and Trucula, get sick of playing Morra and decide on a simpler game a bit like "rock-paper-scissors". Trucula being a gentleman, but not too bright, agrees to the following set of rules proposed by Claude:

- Trucula plays one of "rock" or "paper".
- Claude has a coin and can choose one of "heads" or "tails"
- The payoff matrix (to Trucula) is

$$\mathbf{A} = \begin{array}{c} \\ R \\ P \end{array} \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \end{array}$$

(a) What is Trucula's optimal strategy?

Solution:

- The corresponding LP is

$$\text{maximise } z = v$$
$$\text{subject to} \begin{cases} v & -3x_1 & +2x_2 & \leq & 0 \\ v & +2x_1 & -x_2 & \leq & 0 \\ & x_1, & x_2, & \geq & 0 \end{cases}$$

  where x_1, x_2 is the probability of playing "rock" and "scissors" respectively.

- Set $x_2 = 1 - x_1$ gives

$$\begin{array}{rrcr} v & -5x_1 & \leq & -2 \\ v & +3x_1 & \leq & 1 \end{array}$$

- The corresponding dictionary is

$$\begin{array}{rcrrr} x_3 & = & -2 & -v & +5x_1 \\ x_4 & = & 1 & -v & -3x_1 \\ \hline z & = & & v & \end{array}$$

- Do a "fake pivot" to make things feasible — v enters and x_3 leaves:

$$\begin{array}{rcrrr} v & = & -2 & +5x_1 & -x_3 \\ x_4 & = & 3 & -8x_1 & +x_3 \\ \hline z & = & -2 & +5x_1 & -x_3 \end{array}$$

  This is not yet optimal.

- Pivot again — x_1 enters and x_4 leaves:

$$\begin{array}{rcrrr} v & = & -(1/8) & -(3/8)x_3 & -(5/8)x_4 \\ x_1 & = & (3/8) & +(1/8)x_3 & -(1/8)x_4 \\ \hline z & = & -(1/8) & -(3/8)x_3 & -(5/8)x_4 \end{array}$$

  This is optimal.

- Hence Trucula's optimal strategy is

  * Play "Rock" with probability 3/8.
  * Play "Paper" with probability 5/8.

  and the value of the game is $-1/8$ — ie Trucula looses at least an average of 1/8 per round.

(b) Verify that Claude can use a similar optimal strategy to that of Trucula — ie Claude plays "heads" with the same probability that Trucula plays "rock" and that Claude plays "tails" with the same probability that Trucula plays "paper".

Solution:

– If Claude uses the same mixed strategy as Trucula then the payoff to Trucula is:

$$\begin{aligned}\vec{x}^{*T}\mathbf{A}\vec{y}^* &= \begin{bmatrix}3/8 & 5/8\end{bmatrix}\begin{bmatrix}3 & -2\\ -2 & 1\end{bmatrix}\begin{bmatrix}3/8\\ 5/8\end{bmatrix}\\ &= \begin{bmatrix}3/8 & 5/8\end{bmatrix}\begin{bmatrix}-(1/8)\\ -(1/8)\end{bmatrix} = -1/8\end{aligned}$$

– Hence this combination of strategies gives payoff to Trucula equal to the optimal computed above. By the minimax-theorem (strong duality) this is also optimal for Claude.

(c) How can the payoff matrix be altered so that the game is fair?

Solution:

– The value of the game as described above is $-1/8$ — ie Trucula looses 1/8 per round.

– Trucula can even things up (as we proved in class) by demanding 1/8 per round. This gives payoff matrix:

$$\begin{bmatrix}25/8 & -15/8\\ -15/8 & 9/8\end{bmatrix}$$

4. $\boxed{\text{5 marks}}$ Let $A = (a_{ij})$ be an $m \times n$ payoff matrix of a two person zero sum game.

a) Assume that the average entry in column j of A is at least 5 for each $j = 1, 2, \ldots, n$. Show that the assured (expected) winnings for the row player (player 1) is at least 5.

Solution:

- Assume that the average entry in column j of A is at least 5 for each $j = 1, 2, \ldots, n$.

- If we consider the (mixed) strategy for player 1 $\mathbf{x} = (1/m, 1/m, \ldots, 1/m)^T$, then it is easy to see that each entry of $\mathbf{x}^T A$ is at least 5.

b) Find an example such that the average entry in row i of A is at least 5 for each $i = 1, 2, .., m$ and yet the game is fair.

Solution:

- A simple example is

$$A = \begin{bmatrix} 10 & 0 \\ 10 & 0 \end{bmatrix}$$

where the average entry of each row is 5. As the entries of A are all nonnegative, the payoff for the row player is always ≥ 0. On the other hand, the column player can choose $\mathbf{y} = (0, 1)^T$ to make the resulting product $A\mathbf{y} = \vec{0}$ so that the winnings for the row player are at most 0. Therefore, this is a fair game.

5. [This problem is similar to Chvátal, page 237, Problem 15.2.]

(a) $\boxed{\text{5 marks}}$ Consider a 'battleship' type game played on a 4×4 board. Player 1 secretly chooses a location for a domino (there are 24 possibilities but that is not so crucial to answering this question). Player 2 secretly chooses a position (among the 16 different possibilities). Player 1 wins \$1 (and player 2 loses \$1) if the domino does not occupy a position chosen by player 2 else player 2 wins \$1 (and player 1 loses \$1). One would quess that the value of the game for Player 1 is $12/16 = 3/4$. Give a proof of this fact. Explicitly considering the 24×16 payoff matrix would probably be unproductive but you can use properties of the payoff matrix. **Here, the value of the game of Player 1 is the maximum possible outcome gain that Player 1 is to be able to make no matter what Player 2 does. It is also defined in Chvatal p233 or Vanderbei p155.**

Solution:

- Let A be the 24×16 payoff matrix for the player 1.
- If we consider a covering of the 4×4 board by 8 dominoes (8 dominoes can be arranged to cover all squares) and then choose a strategy $\mathbf{x}^*$ for the row player of $1/8$ for each of the strategies corresponding to

the position of the dominoes and 0's for other 16 strategies/domino positions.

- Then we may compute $(\mathbf{x}^*)^T A = (6/8, 6/8, \ldots, 6/8)$ (each sum will have one $-1/8$ and seven $+1/8$ terms and the rest 0).
- Thus $v(A) \geq 6/8 = 3/4$.
- Now we note that each row of A will have precisely two -1's and and 14 1's. Thus if we choose the strategy $\mathbf{y}^*$ for player 2 of $(1/16, 1/16, \ldots, 1/16)^T$ and so the 24×1 vector $A\mathbf{y}^* = (12/16, 12/16, \ldots, 12/16)^T$.
- Thus $v(A) \leq 12/16$. Thus $v(A) = 12/16 = 3/4$.

(b) $\boxed{\text{5 marks}}$ Now for the same game, change the board to be an $N \times N$ board, where N is a large integer. What happens to the value of the game as $N \to \infty$? Justify your answer.

Solution:

- Note that the number of positions of the $N \times N$ board is N^2.
- Let $D(N)$ be the number of locations of a domino. Clearly $D(N) \to \infty$ as $N \to \infty$.
- Let $v(N)$ be the value of the game.
- Let the column player choose $\vec{y}^* = (1/N^2, ..., 1/N^2)$.
- Observe that each row of the payoff matrix A has two entries of value -1 and $N^2 - 2$ entries of value 1.
- Then, $A\vec{y}^* = (N^2 - 4)\frac{1}{N^2} = (N^2 - 4)\frac{1}{N^2}$. So, $\vec{x}^T Ay^* = (N^2 - 4)\frac{1}{N^2}$ for any stochastic vector $\vec{x}$.
- This means $v(N) \leq (N^2 - 4)\frac{1}{N^2}$.
- Let the row player choose $\vec{x}^* = (1/D(N), ..., 1/D(N))$.
- For each column of the payoff matrix, the number of entries of -1 is either 2, 3, or 4, depending whether the location is at the corner, along the sides, or at a nonboundary location in the board. Accordingly, the number of entries of 1 is $D(N) - 2$, $D(N) - 3$, or $D(N) - 4$.
- Therefore $\vec{x}^* A$ is a row vector of N^2 entries, while each entry is either $(D(N) - 4)/D(N)$, $(D(N) - 6)/D(N)$, or $(D(N) - 8)/D(N)$.

- So, for any stochastic vector y, the value $\vec{x}^* A\vec{y}$ is between $(D(N) - 4)/D(N)$ and $(D(N) - 8)/D(N)$, i.e.

$$(D(N) - 4)/D(N) \leq \vec{x}^* A\vec{y} \leq (D(N) - 8)/D(N).$$

- This means the value $v(N) \geq (D(N) - 4)/D(N)$ as the row player can guarantee the payoff at least $(D(N) - 4)/D(N)$ by choosing $\vec{x}^*$, regardless of the choice $\vec{y}$ of the coluimn player.

- In summary we have shown that

$$\frac{D(N) - 4}{D(N)} \leq v(N) \leq \frac{N^2 - 4}{N^2}.$$

- As $N \to \infty$, both sides go to 1, thus $v(N) \to 1$.

* Note that in fact, using the same reasoning as in the solution of (a), one can verify that $v(N) = \frac{N^2-4}{N^2}$, which obviously tends to 1 as $N \to \infty$.