Additional notes for the modified standard form (the standard equality form)

Matrix notation:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & & \\ \vdots & & \ddots & & & 1 & \\ & & & & & & \ddots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & & & & 1 \end{bmatrix}$$

$m \times (m+n)$ matrix

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n+m}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix} \in \mathbb{R}^{n+m} \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^{m} \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n+m}$$

Can rewrite the LP

$$\boxed{\begin{array}{ll} \text{maximize} & \vec{c} \cdot \vec{x} \\ \text{subject to} & \begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \geq \vec{0} \end{cases} \end{array}}$$

Note $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$\vec{a} \geq \vec{b}$ means $a_i \geq b_i$ for all i.

e.g. $\vec{x} \geq \vec{0}$ means $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$.

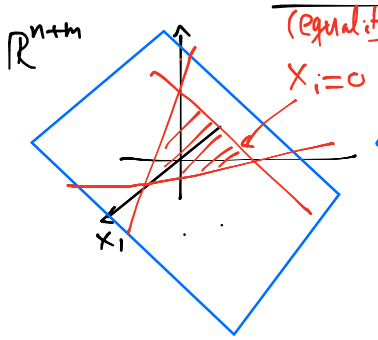
e.g. maximize $(1,1,1,0,0) \cdot (x_1, x_2, x_3, x_4, x_5)$

subject to

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Terminology: hyperplane in an n-dimensional space is an $(n-1)$-dimensional plane.

- Geometry of the feasible region $\{\vec{x} \in \mathbb{R}^{n+m} \mid A\vec{x} = \vec{b},\ \vec{x} \geq 0\}$ of the modified standard form (equality form)



The feasible set is contained in $\{\vec{x} \mid A\vec{x} = \vec{b}\}$, an n-dim'l subspace

intersection of the hyperplanes

$$a_{11}x_1 + \cdots + a_{1n}x_1 + x_{n+1} = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_1 + x_{n+m} = b_m$$

m of those.

Recall:

Thm Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \in \mathbb{R}^n$ (n-dim'l vector space in general.), $b_1, \ldots, b_m \in \mathbb{R}$

Let $n \geq m$.

Define hyperplanes: $L_k = \{\vec{x} \in \mathbb{R}^n \mid \vec{a}_k \cdot \vec{x} = b_k\}$

Suppose $\vec{a}_1, \ldots, \vec{a}_m$ are linearly independent.

(i.e. $y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_m\vec{a}_m = \vec{0} \leftarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ implies $y_1 = y_2 = \cdots = y_m = 0$)

Then the intersection $L_1 \cap L_2 \cap \cdots \cap L_m$

i.e. $\{\vec{x} \in \mathbb{R}^n \mid \vec{a}_k \cdot \vec{x}_k = b_k \ \forall k = 1, 2, \ldots, m\}$

is an $(n-m)$-dimensional subspace in $\mathbb{R}^n$.

Exercise Use the Thm A above to show

$$\dim \{\vec{x} \mid A\vec{x} = \vec{b}\} = (m+n) - m = n$$

for the matrix A from the modified standard form

(the 'equality' form)

Hint: Use the fact that $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 1 & & \\ \vdots & \ddots & & & \ddots & \\ a_{m1} & & a_{mn} & & & 1 \end{bmatrix}$

and show that the row vectors of A are linearly independent

This part is $n \times n$ identity matrix.