

Overview

■ What does this paper investigate?

The capability of Diagonal (D)- and Beyond Diagonal (BD)-Reconfigurable Intelligent Surface (RIS) to redistribute the singular values of MIMO channels.

■ How does it differ from previous work?

It derives analytical singular value bounds for specific channel conditions. And proposes a novel BD-RIS optimization framework for general problems.

■ What are the benefits?

BD-RIS improves the dynamic range of individual channel singular values and the trade-off in manipulating them. This boosts channel power gain and capacity.

BD-RIS model

Consider an $N_T \times N_S \times N_R$ setup with BD-RIS divided into G groups of L elements each. Define $N = \min(N_T, N_R)$ and $\mathbf{H}_{B/F} \stackrel{\text{svd}}{=} \mathbf{U}_{B/F} \mathbf{\Sigma}_{B/F} \mathbf{V}_{B/F}^H$.

$$\mathbf{\Theta} = \text{diag}(\mathbf{\Theta}_1, \dots, \mathbf{\Theta}_G), \quad \mathbf{\Theta}_g^H \mathbf{\Theta}_g = \mathbf{I}_L \quad \forall g, \quad \mathbf{H} = \mathbf{H}_D + \sum_g \overbrace{\mathbf{H}_{B,g} \mathbf{\Theta}_g \mathbf{H}_{F,g}}^{\substack{\text{direct-indirect:} \\ \text{inter-group, additive}}} \quad \underbrace{\mathbf{H}_{B,g} \mathbf{\Theta}_g \mathbf{H}_{F,g}}_{\substack{\text{backward-forward:} \\ \text{intra-group, multiplicative}}}.$$

- **Branch matching:** Pairing and combining the entries of $\mathbf{H}_{B,g}$ and $\mathbf{H}_{F,g}$ through unitary transformation $\mathbf{\Theta}_g$.
- **Mode alignment:** Aligning and ordering the singular vectors of $\{\mathbf{H}_g\}$ with those of \mathbf{H}_D through unitary transformations $\{\mathbf{\Theta}_g\}$.

Example 1: SISO channel gain maximization

SISO mode alignment reduces to phase matching and any L (incl. D-RIS) suffices by

$$\mathbf{\Theta}_g^{\text{SISO}} = \frac{h_D}{|h_D|} \mathbf{V}_{B,g} \mathbf{U}_{F,g}^H \quad \forall g,$$

where $\mathbf{V}_{B,g} = [\mathbf{h}_{B,g}/\|\mathbf{h}_{B,g}\|, \mathbf{N}_{B,g}]$, $\mathbf{U}_{F,g} = [\mathbf{h}_{F,g}/\|\mathbf{h}_{F,g}\|, \mathbf{N}_{F,g}]$, and $\mathbf{N}_{B,g}, \mathbf{N}_{F,g}$ are orthonormal bases of null spaces of $\mathbf{h}_{B,g}, \mathbf{h}_{F,g}$. The channel gain is a function of L

$$\max_{\mathbf{\Theta}_{BD}} |h| = |h_D| + \sum_g \sum_l |h_{B,g,\pi_{B,g}(l)}| |h_{F,g,\pi_{F,g}(l)}|,$$

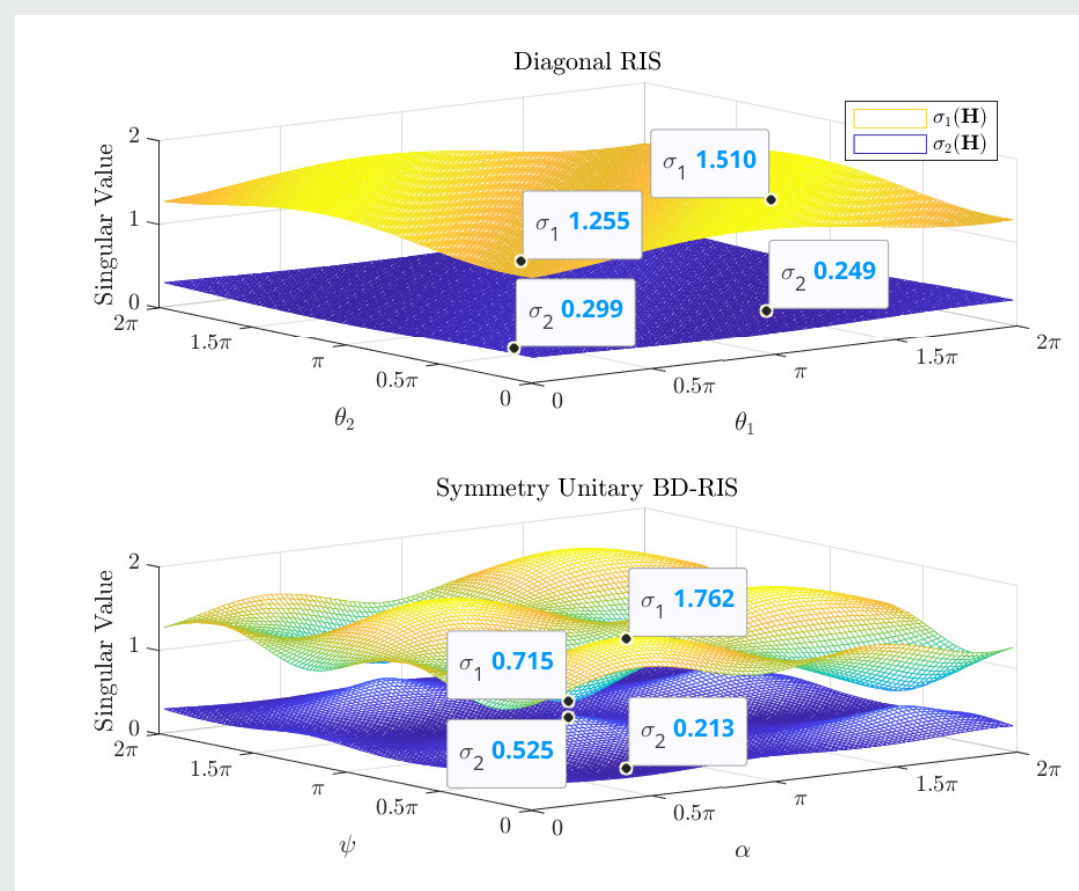
which generalizes $\max_{\mathbf{\Theta}_D} |h| = |h_D| + \sum_{n=1}^{N_S} |h_{B,n}| |h_{F,n}|$ using permutations $\pi_{B,g}, \pi_{F,g}$ that pair the l -th strongest backward and forward branches.

Example 2: $2 \times 2 \times 2$ shaping

D-RIS and fully-connected BD-RIS can be modeled by 2 and 4 angular parameters:

$$\mathbf{\Theta}_D = \text{diag}(e^{j\theta_1}, e^{j\theta_2}), \quad \mathbf{\Theta}_{BD} = e^{j\phi} \begin{bmatrix} e^{j\alpha} \cos \psi & e^{j\beta} \sin \psi \\ -e^{-j\beta} \sin \psi & e^{-j\alpha} \cos \psi \end{bmatrix}.$$

Assume the BD-RIS is symmetric (i.e., $\beta = \pi/2$) and the direct channel is negligible (i.e., $\text{sv}(e^{j\phi} \mathbf{A}) = \text{sv}(\mathbf{A})$). For one channel realization, we can reveal channel singular values achieved by D- and BD-RIS by grid search over (θ_1, θ_2) and (α, ψ) .



Here, both singular values are manipulated up to $\pm 9\%$ by D-RIS and $\pm 42\%$ by symmetric fully-connected BD-RIS, using 2 and 3 circuit components respectively.

Proposition 1: Degrees of Freedom (DoF)

In point-to-point MIMO, BD-RIS cannot achieve a larger number of DoF than D-RIS.

Proposition 2: Rank-deficient channels

If the minimum rank of backward and forward channels is k , then for D-RIS or BD-RIS of arbitrary number of elements, the n -th channel singular value is bounded by

$$\sigma_n(\mathbf{H}) \leq \sigma_{n-k}(\mathbf{T}), \quad \text{if } n > k, \\ \sigma_n(\mathbf{H}) \geq \sigma_n(\mathbf{T}), \quad \text{if } n < N - k + 1,$$

where \mathbf{T} is arbitrary auxiliary matrix satisfying

$$\mathbf{T} \mathbf{T}^H = \begin{cases} \mathbf{H}_D (\mathbf{I} - \mathbf{V}_F \mathbf{V}_F^H) \mathbf{H}_D^H, & \text{if } \text{rank}(\mathbf{H}_F) = k, \\ \mathbf{H}_D^H (\mathbf{I} - \mathbf{U}_B \mathbf{U}_B^H) \mathbf{H}_D, & \text{if } \text{rank}(\mathbf{H}_B) = k. \end{cases}$$

Corollary 2.1: Line-of-Sight (LoS) channel

If one of backward and forward channels is LoS, then a D-RIS or BD-RIS can only manipulate the channel singular values up to

$$\sigma_1(\mathbf{H}) \geq \sigma_1(\mathbf{T}) \geq \sigma_2(\mathbf{H}) \geq \dots \geq \sigma_{N-1}(\mathbf{T}) \geq \sigma_N(\mathbf{H}) \geq \sigma_N(\mathbf{T}).$$

As $N_S \rightarrow \infty$, N out of those $2N$ bounds can be simultaneously tight.

Proposition 3: Negligible direct channel

If the direct channel is negligible, then a fully-connected BD-RIS can manipulate the channel singular values up to

$$\text{sv}(\mathbf{H}) = \text{sv}(\mathbf{B}\mathbf{F}),$$

where \mathbf{B} and \mathbf{F} are arbitrary matrices with $\text{sv}(\mathbf{B}) = \text{sv}(\mathbf{H}_B)$ and $\text{sv}(\mathbf{F}) = \text{sv}(\mathbf{H}_F)$.

Corollary 3.3: Individual singular value

If the direct channel is negligible, then the n -th channel singular value is bounded by

$$\max_{i+j=N_S} \sigma_i(\mathbf{H}_B) \sigma_j(\mathbf{H}_F) \leq \sigma_n(\mathbf{H}) \leq \min_{i+j=n+1} \sigma_i(\mathbf{H}_B) \sigma_j(\mathbf{H}_F),$$

which are attained respectively at

$$\mathbf{\Theta}_{sv-n-max}^{\text{MIMO-ND}} = \mathbf{V}_B \mathbf{P} \mathbf{U}_F^H, \quad \mathbf{\Theta}_{sv-n-min}^{\text{MIMO-ND}} = \mathbf{V}_B \mathbf{Q} \mathbf{U}_F^H,$$

where \mathbf{P}, \mathbf{Q} are permutation matrices of dimension N_S satisfying:

- The (i, j) -th entry is 1, where

$$(i, j) = \begin{cases} \arg \min_{i+j=n+1} \sigma_i(\mathbf{H}_B) \sigma_j(\mathbf{H}_F) & \text{for } \mathbf{P}, \\ \arg \max_{i+j=n+N_S} \sigma_i(\mathbf{H}_B) \sigma_j(\mathbf{H}_F) & \text{for } \mathbf{Q}, \end{cases}$$

and ties may be broken arbitrarily;

- After deleting the i -th row and j -th column, the resulting submatrix \mathbf{Y} is arbitrary permutation matrix of dimension $N_S - 1$ satisfying

$$\sigma_{n-1}(\hat{\Sigma}_B \mathbf{Y} \hat{\Sigma}_F) \geq \min_{i+j=n+1} \sigma_i(\mathbf{H}_B) \sigma_j(\mathbf{H}_F) \quad \text{for } \mathbf{P},$$

$$\sigma_{n+1}(\hat{\Sigma}_B \mathbf{Y} \hat{\Sigma}_F) \leq \max_{i+j=n+N_S} \sigma_i(\mathbf{H}_B) \sigma_j(\mathbf{H}_F) \quad \text{for } \mathbf{Q},$$

where $\hat{\Sigma}_B, \hat{\Sigma}_F$ are Σ_B, Σ_F with both i -th row and j -th column deleted.

Corollary 3.4: Channel power gain

If the direct channel is negligible, then the channel power gain is bounded by

$$\sum_{n=1}^N \sigma_n^2(\mathbf{H}_B) \sigma_{N_S-n+1}^2(\mathbf{H}_F) \leq \|\mathbf{H}\|_F^2 \leq \sum_{n=1}^N \sigma_n^2(\mathbf{H}_B) \sigma_n^2(\mathbf{H}_F),$$

which are attained respectively at

$$\mathbf{\Theta}_{P-max}^{\text{MIMO-ND}} = \mathbf{V}_B \mathbf{U}_F^H, \quad \mathbf{\Theta}_{P-min}^{\text{MIMO-ND}} = \mathbf{V}_B \mathbf{J} \mathbf{U}_F^H,$$

where \mathbf{J} is the backward identity matrix.

Corollary 3.5: Channel capacity at very low and high SNR

If the direct channel is negligible, then the channel capacity at very low and high SNR are approximately bounded from above by

$$C_{\rho \downarrow} \lesssim \sigma_1^2(\mathbf{H}_B) \sigma_1^2(\mathbf{H}_F),$$

$$C_{\rho \uparrow} \lesssim N \log \frac{\rho}{N} + 2 \log \prod_{n=1}^N \sigma_n(\mathbf{H}_B) \sigma_n(\mathbf{H}_F).$$

Their upper bounds can be attained at, for example, $\mathbf{\Theta}_{P-max}^{\text{MIMO-ND}}$.

Algorithm 1: Group-wise geodesic optimization for BD-RIS

For BD-RIS optimization problem of the form

$$\max_{\mathbf{\Theta}} f(\mathbf{\Theta}) \\ \text{s.t.} \quad \mathbf{\Theta}_g^H \mathbf{\Theta}_g = \mathbf{I}, \quad \forall g,$$

we propose a *geodesic* Riemannian Conjugate Gradient (RCG) algorithm below.

- 1 Compute the Euclidean gradient:

$$\nabla_{E,g}^{(r)} = \frac{\partial f(\mathbf{\Theta}_g^{(r)})}{\partial \mathbf{\Theta}_g^{(r)*}};$$

- 2 Translate to the Riemannian gradient evaluated at the identity:

$$\tilde{\nabla}_{R,g}^{(r)} = \nabla_{E,g}^{(r)} \mathbf{\Theta}_g^{(r)H} - \mathbf{\Theta}_g^{(r)} \nabla_{E,g}^{(r)H};$$

- 3 Determine the conjugate direction:

$$\mathbf{D}_g^{(r)} = \tilde{\nabla}_{R,g}^{(r)} + \tilde{\gamma}_g^{(r)} \mathbf{D}_g^{(r-1)}, \quad \tilde{\gamma}_g^{(r)} = \frac{\text{tr}((\tilde{\nabla}_{R,g}^{(r)} - \tilde{\nabla}_{R,g}^{(r-1)}) \tilde{\nabla}_{R,g}^{(r)H})}{\text{tr}(\tilde{\nabla}_{R,g}^{(r-1)} \tilde{\nabla}_{R,g}^{(r-1)H})};$$

- 4 Perform *multiplicative* update along geodesic:

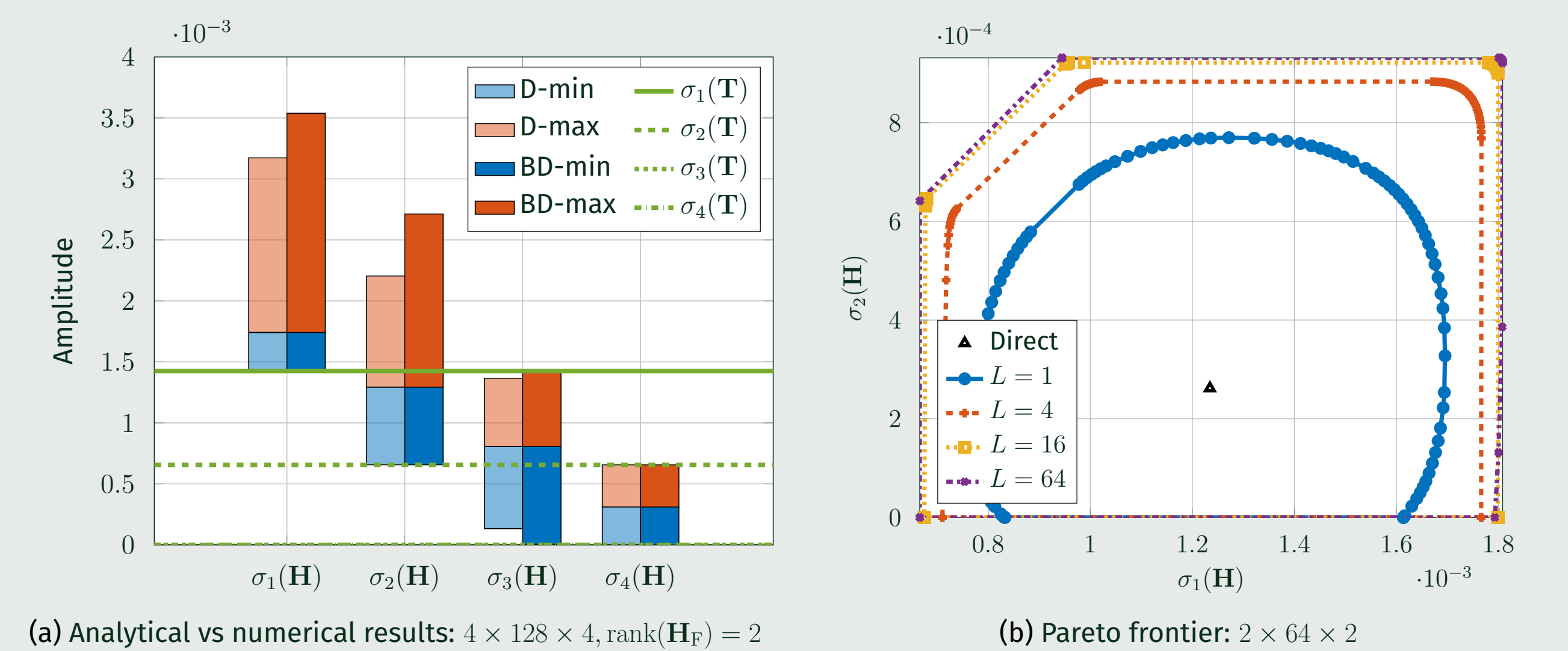
$$\mathbf{\Theta}_g^{(r+1)} = \mathbf{G}_g^{(r)}(\mu) = \exp(\mu \mathbf{D}_g^{(r)}) \mathbf{\Theta}_g^{(r)},$$

where μ is refinable by the Armijo rule. To double the step size, one can simply square the rotation matrix instead of recomputing the matrix exponential.

Result 1: Algorithm evaluation

RCG path	$N_S = 16$			$N_S = 256$		
	Objective	Iterations	Time [s]	Objective	Iterations	Time [s]
Geodesic	4.359×10^{-3}	11.59	1.839×10^{-2}	1.163×10^{-2}	25.58	3.461
Non-geodesic	4.329×10^{-3}	30.92	5.743×10^{-2}	1.116×10^{-2}	61.40	13.50

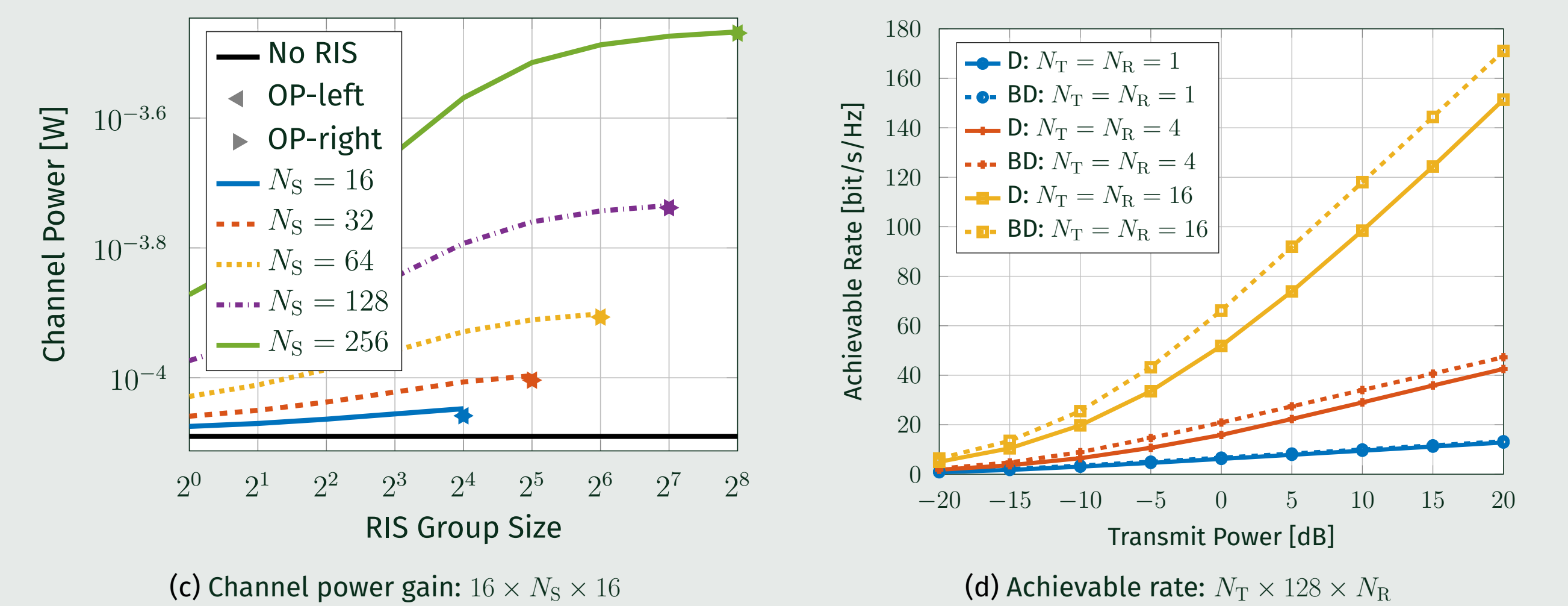
Result 2: Bounds and Pareto frontier of singular values



(a) Analytical vs numerical results: $4 \times 128 \times 4$, $\text{rank}(\mathbf{H}_F) = 2$

(b) Pareto frontier: $2 \times 64 \times 2$

Result 3: Channel power gain and achievable rate



(c) Channel power gain: $16 \times N_S \times 16$

(d) Achievable rate: $N_T \times 128 \times N_R$