Overview

- **What does this paper investigate?**
  The capability of Diagonal (D) and Beyond Diagonal (BD) Reconfigurable Intelligent Surface (RIS) to redistribute the singular values of MIMO channels.

- **How does it differ from previous work?**
  It derives analytical singular value bounds for specific channel conditions. And proposes a novel BD-RIS optimization framework for general problems.

- **What are the benefits?**
  BD-RIS improves the dynamic range of individual channel singular values and the trade-off in manipulating them. This boosts channel power gain and capacity.

BD-RIS model

Consider an $N_T \times N_S \times N_R$ setup with BD-RIS divided into G groups of L elements each. Define $N = \min(N_T, N_R)$, $\bar{N} = \max(N_T, N_S, N_R)$, and $\mathbf{H}_{B/F} \overset{svd}{=} \mathbf{U}_{B/F}\boldsymbol{\Sigma}_{B/F}\mathbf{V}_{B/F}^H$.

$$\boldsymbol{\Theta} = \operatorname{diag}(\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_G), \quad \boldsymbol{\Theta}_g^H \boldsymbol{\Theta}_g = \mathbf{I}_L \; \forall g, \quad \mathbf{H} = \underbrace{\mathbf{H}_D}_{} + \sum_g \underbrace{\mathbf{H}_{B,g}\boldsymbol{\Theta}_g\mathbf{H}_{F,g}}_{}.$$

direct-indirect: inter-group, additive
backward-forward: intra-group, multiplicative

- **Branch matching:** Pairing and combining the entries of $\mathbf{H}_{B,g}$ and $\mathbf{H}_{F,g}$ through unitary transformation $\boldsymbol{\Theta}_g$.
- **Mode alignment:** Aligning and ordering the singular vectors of $\{\mathbf{H}_g\}$ with those of $\mathbf{H}_D$ through unitary transformations $\{\boldsymbol{\Theta}_g\}$.

Example 1: SISO channel gain maximization

SISO mode alignment reduces to phase matching and any L (incl. D-RIS) suffices by

$$\boldsymbol{\Theta}_g^{SISO} = \frac{h_D}{|h_D|}\mathbf{V}_{B,g}\mathbf{U}_{F,g}^H \; \forall g,$$

where $\mathbf{V}_{B,g} = [\mathbf{h}_{B,g}/\|\mathbf{h}_{B,g}\|, \mathbf{N}_{B,g}]$, $\mathbf{U}_{F,g} = [\mathbf{h}_{F,g}/\|\mathbf{h}_{F,g}\|, \mathbf{N}_{F,g}]$, and $\mathbf{N}_{B,g}, \mathbf{N}_{F,g}$ are orthonormal bases of null spaces of $\mathbf{h}_{B,g}, \mathbf{h}_{F,g}$. The channel gain is a function of L

$$\max_{\boldsymbol{\Theta}_{BD}} |h| = |h_D| + \sum_g \sum_l |h_{B,g,\pi_{B,g}(l)}||h_{F,g,\pi_{F,g}(l)}|,$$

which generalizes $\max_{\boldsymbol{\Theta}_D} |h| = |h_D| + \sum_{n=1}^{N_S} |h_{B,n}||h_{F,n}|$ using permutations $\pi_{B,g}, \pi_{F,g}$ that pair the l-th strongest backward and forward branches.

Example 2: $2 \times 2 \times 2$ shaping

D-RIS and fully-connected BD-RIS can be modeled by 2 and 4 angular parameters:

$$\boldsymbol{\Theta}_D = \operatorname{diag}(e^{j\theta_1}, e^{j\theta_2}), \quad \boldsymbol{\Theta}_{BD} = e^{j\phi}\begin{bmatrix} e^{j\alpha}\cos\psi & e^{j\beta}\sin\psi \\ -e^{-j\beta}\sin\psi & e^{-j\alpha}\cos\psi \end{bmatrix}.$$

Assume the BD-RIS is symmetric (i.e., $\beta = \pi/2$) and the direct channel is negligible (i.e., $sv(e^{j\phi}\mathbf{A}) = sv(\mathbf{A})$). The plot relies on grid search for one channel realization.

Diagonal RIS
Symmetry Unitary BD-RIS
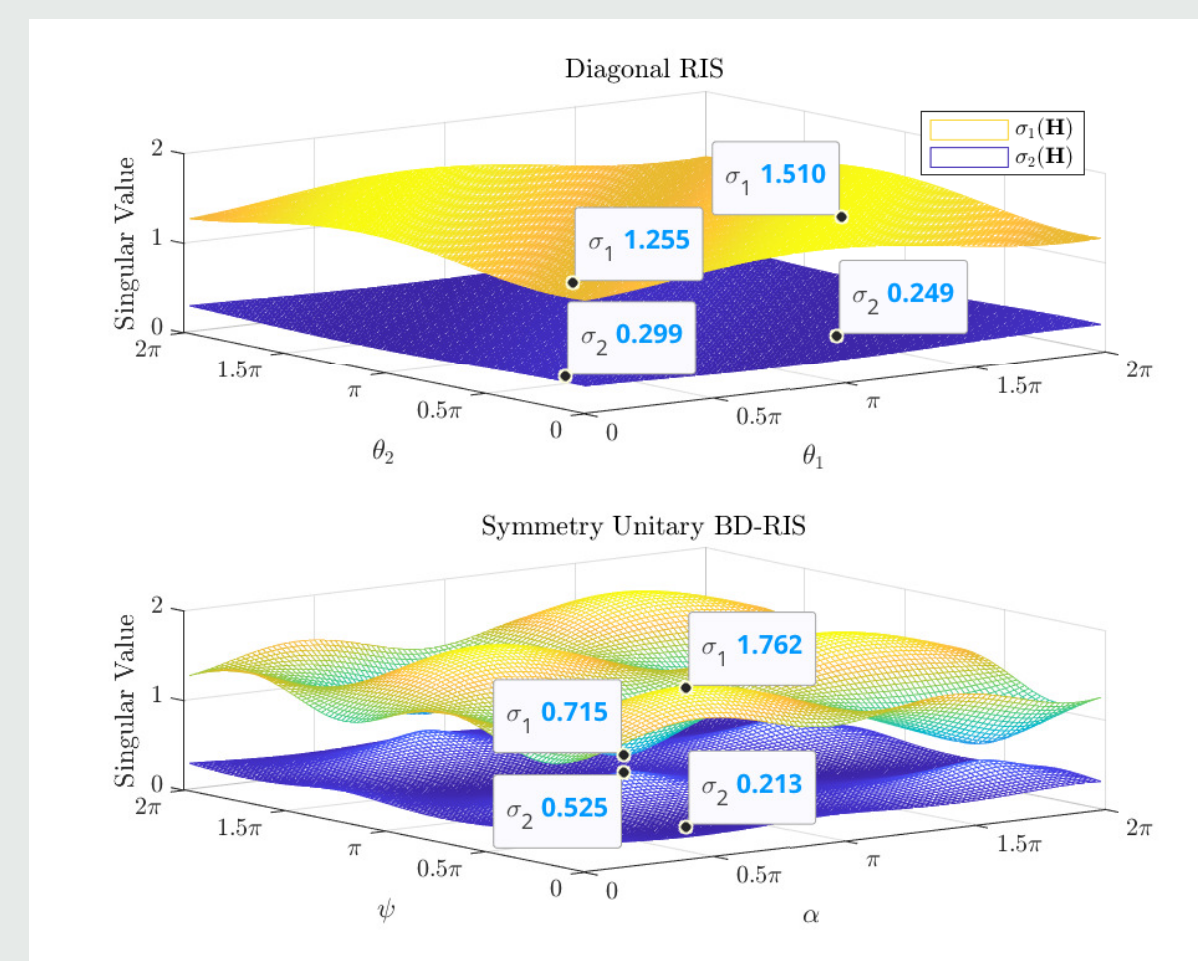
Here, both singular values are manipulated up to $\pm 9\%$ by D-RIS and $\pm 42\%$ by symmetric fully-connected BD-RIS, using 2 and 3 circuit components respectively.

Proposition 1: Degrees of Freedom (DoF)

In point-to-point MIMO, BD-RIS cannot achieve a larger number of DoF than D-RIS.

Proposition 2: Rank-deficient channels

If the minimum rank of backward and forward channels is k, then for D-RIS or BD-RIS of arbitrary number of elements, the n-th channel singular value is bounded by

$$\sigma_n(\mathbf{H}) \leq \sigma_{n-k}(\mathbf{T}), \qquad if\ n > k,$$
$$\sigma_n(\mathbf{H}) \geq \sigma_n(\mathbf{T}), \qquad if\ n < N - k + 1,$$

where $\mathbf{T}$ is arbitrary auxiliary matrix satisfying

$$\mathbf{T}\mathbf{T}^H = \begin{cases} \mathbf{H}_D(\mathbf{I} - \mathbf{V}_F\mathbf{V}_F^H)\mathbf{H}_D^H, & if\ \operatorname{rank}(\mathbf{H}_F) = k, \\ \mathbf{H}_D^H(\mathbf{I} - \mathbf{U}_B\mathbf{U}_B^H)\mathbf{H}_D, & if\ \operatorname{rank}(\mathbf{H}_B) = k. \end{cases}$$

Corollary 2.1: Line-of-Sight (LoS) channel

If one of backward and forward channels is LoS, then a D-RIS or BD-RIS can only manipulate the channel singular values up to

$$\sigma_1(\mathbf{H}) \geq \sigma_1(\mathbf{T}) \geq \sigma_2(\mathbf{H}) \geq \ldots \geq \sigma_{N-1}(\mathbf{T}) \geq \sigma_N(\mathbf{H}) \geq \sigma_N(\mathbf{T}).$$

As $N_S \to \infty$, N out of those $2N$ bounds can be simultaneously tight.

Proposition 3: Negligible direct channel

If the direct channel is negligible, then a fully-connected BD-RIS can manipulate the channel singular values up to

$$sv(\mathbf{H}) = sv(\mathbf{BF}),$$

where $\mathbf{B}$ and $\mathbf{F}$ are arbitrary matrices with $sv(\mathbf{B}) = sv(\mathbf{H}_B)$ and $sv(\mathbf{F}) = sv(\mathbf{H}_F)$.

Corollary 3.3: Individual singular value

If the direct channel is negligible, then the n-th channel singular value is bounded by

$$\max_{i+j=n+N_S} \sigma_i(\mathbf{H}_B)\sigma_j(\mathbf{H}_F) \leq \sigma_n(\mathbf{H}) \leq \min_{i+j=n+1} \sigma_i(\mathbf{H}_B)\sigma_j(\mathbf{H}_F),$$

which are attained respectively at

$$\boldsymbol{\Theta}_{sv\text{-}n\text{-}max}^{MIMO\text{-}ND} = \mathbf{V}_B\mathbf{P}\mathbf{U}_F^H, \quad \boldsymbol{\Theta}_{sv\text{-}n\text{-}min}^{MIMO\text{-}ND} = \mathbf{V}_B\mathbf{Q}\mathbf{U}_F^H,$$

where $\mathbf{P}, \mathbf{Q}$ are permutation matrices of dimension N_S satisfying:
- The (i,j)-th entry is 1, where

$$(i,j) = \begin{cases} \arg\min_{i+j=n+1} \sigma_i(\mathbf{H}_B)\sigma_j(\mathbf{H}_F) & for\ \mathbf{P}, \\ \arg\max_{i+j=n+N_S} \sigma_i(\mathbf{H}_B)\sigma_j(\mathbf{H}_F) & for\ \mathbf{Q}, \end{cases}$$

and ties may be broken arbitrarily;
- After deleting the i-th row and j-th column, the resulting submatrix $\mathbf{Y}$ is arbitrary permutation matrix of dimension $N_S - 1$ satisfying

$$\sigma_{n-1}(\hat{\boldsymbol{\Sigma}}_B\mathbf{Y}\hat{\boldsymbol{\Sigma}}_F) \geq \min_{i+j=n+1} \sigma_i(\mathbf{H}_B)\sigma_j(\mathbf{H}_F) \quad for\ \mathbf{P},$$
$$\sigma_{n+1}(\hat{\boldsymbol{\Sigma}}_B\mathbf{Y}\hat{\boldsymbol{\Sigma}}_F) \leq \max_{i+j=n+N_S} \sigma_i(\mathbf{H}_B)\sigma_j(\mathbf{H}_F) \quad for\ \mathbf{Q},$$

where $\hat{\boldsymbol{\Sigma}}_B, \hat{\boldsymbol{\Sigma}}_F$ are $\boldsymbol{\Sigma}_B, \boldsymbol{\Sigma}_F$ with both i-th row and j-th column deleted.

Corollary 3.4: Channel power gain

If the direct channel is negligible, then the channel power gain is bounded by

$$\sum_{n=1}^N \sigma_n^2(\mathbf{H}_B)\sigma_{N_S-n+1}^2(\mathbf{H}_F) \leq \|\mathbf{H}\|_F^2 \leq \sum_{n=1}^N \sigma_n^2(\mathbf{H}_B)\sigma_n^2(\mathbf{H}_F),$$

which are attained respectively at

$$\boldsymbol{\Theta}_{P\text{-}max}^{MIMO\text{-}ND} = \mathbf{V}_B\mathbf{U}_F^H, \quad \boldsymbol{\Theta}_{P\text{-}min}^{MIMO\text{-}ND} = \mathbf{V}_B\mathbf{J}\mathbf{U}_F^H,$$

where $\mathbf{J}$ is the backward identity matrix.

Corollary 3.5: Channel capacity at very low and high SNR

If the direct channel is negligible, then the channel capacity at very low and high SNR are approximately bounded from above by

$$C_{\rho_\downarrow} \lesssim \sigma_1^2(\mathbf{H}_B)\sigma_1^2(\mathbf{H}_F),$$
$$C_{\rho_\uparrow} \lesssim N\log\frac{\rho}{N} + 2\log\prod_{n=1}^N \sigma_n(\mathbf{H}_B)\sigma_n(\mathbf{H}_F).$$

Their upper bounds can be attained at, for example, $\boldsymbol{\Theta}_{P\text{-}max}^{MIMO\text{-}ND}$.

Algorithm 1: Group-wise geodesic optimization

For BD-RIS optimization problem of the form

$$\max_{\boldsymbol{\Theta}} \quad f(\boldsymbol{\Theta})$$
$$\text{s.t.} \quad \boldsymbol{\Theta}_g^H\boldsymbol{\Theta}_g = \mathbf{I}, \quad \forall g,$$

we propose a *geodesic* Riemannian Conjugate Gradient (RCG) algorithm below.

1. Compute the Euclidean gradient:
$$\nabla_{E,g}^{(r)} = \frac{\partial f(\boldsymbol{\Theta}_g^{(r)})}{\partial \boldsymbol{\Theta}_g^*};$$

2. Translate to the Riemannian gradient evaluated at the identity:
$$\tilde{\nabla}_{R,g}^{(r)} = \nabla_{E,g}^{(r)}\boldsymbol{\Theta}_g^{(r)H} - \boldsymbol{\Theta}_g^{(r)}\nabla_{E,g}^{(r)H};$$

3. Determine the conjugate direction:
$$\mathbf{D}_g^{(r)} = \tilde{\nabla}_{R,g}^{(r)} + \tilde{\gamma}_g^{(r)}\mathbf{D}_g^{(r-1)}, \quad \tilde{\gamma}_g^{(r)} = \frac{\operatorname{tr}\left((\tilde{\nabla}_{R,g}^{(r)} - \tilde{\nabla}_{R,g}^{(r-1)})\tilde{\nabla}_{R,g}^{(r)H}\right)}{\operatorname{tr}\left(\tilde{\nabla}_{R,g}^{(r-1)}\tilde{\nabla}_{R,g}^{(r-1)H}\right)};$$

4. Perform *multiplicative* update along geodesic:
$$\boldsymbol{\Theta}_g^{(r+1)} = \mathbf{G}_g^{(r)}(\mu) = \exp(\mu\mathbf{D}_g^{(r)})\boldsymbol{\Theta}_g^{(r)},$$

where μ is refinable by the Armijo rule.

Result 1: Algorithm evaluation

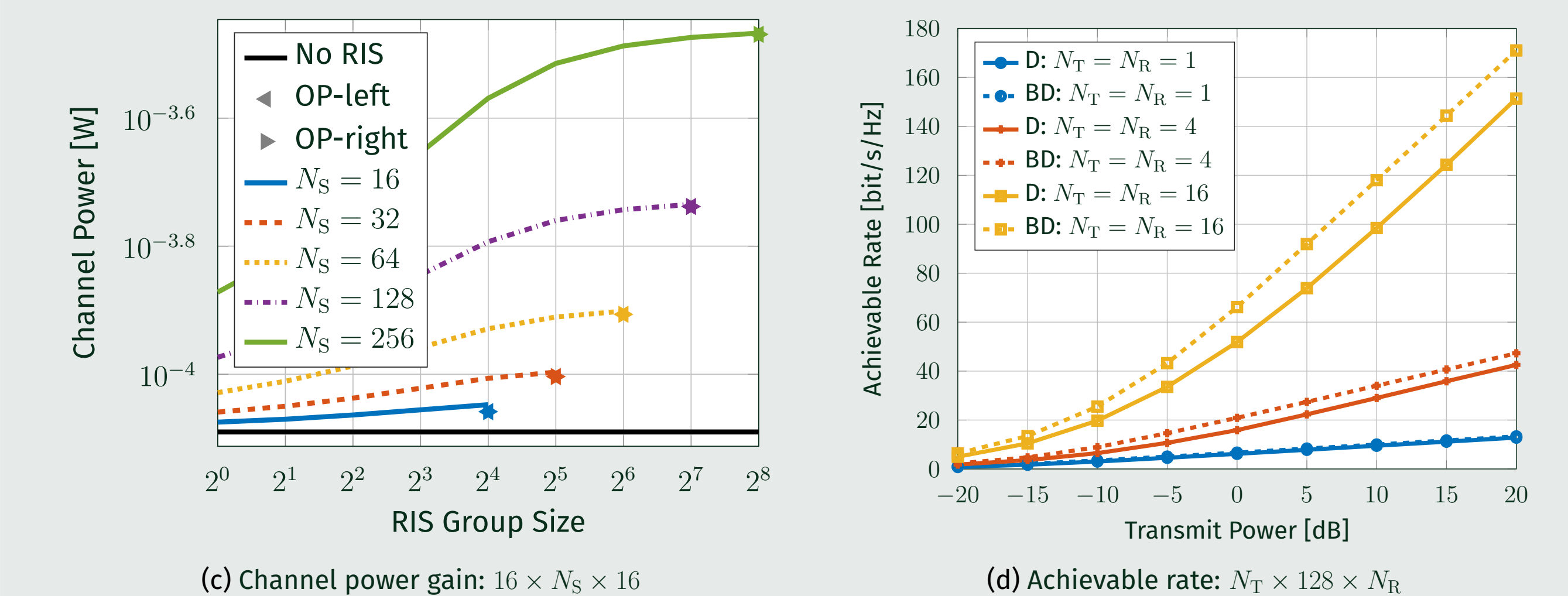| RCG path	$N_S = 16$			$N_S = 256$		
	Objective	Iterations	Time [s]	Objective	Iterations	Time [s]
Geodesic	4.359×10^{-3}	11.59	1.839×10^{-2}	1.163×10^{-2}	25.58	3.461
Non-geodesic	4.329×10^{-3}	30.92	5.743×10^{-2}	1.116×10^{-2}	61.40	13.50

Result 2: Bounds and Pareto frontier of singular values
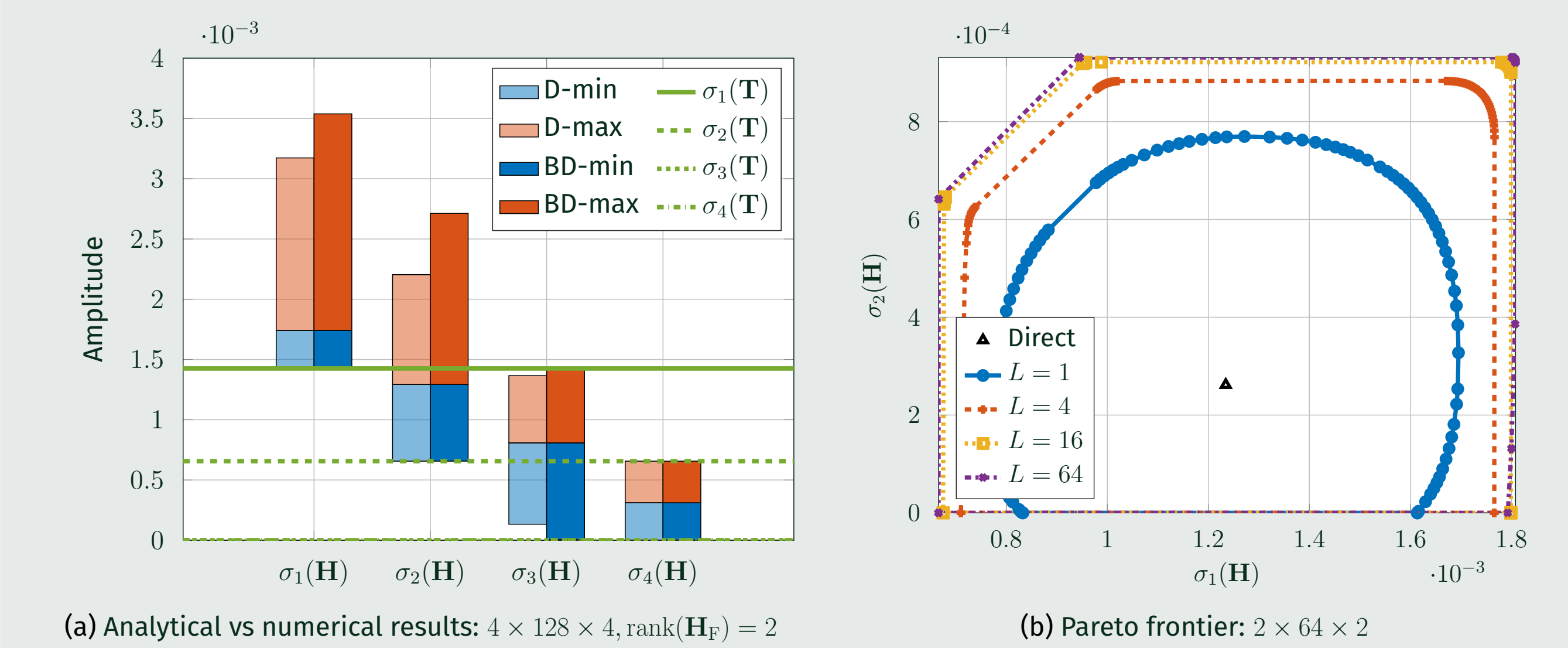
(a) Analytical vs numerical results: $4 \times 128 \times 4$, $\operatorname{rank}(\mathbf{H}_F) = 2$

(b) Pareto frontier: $2 \times 64 \times 2$

Result 3: Channel power gain and achievable rate

(c) Channel power gain: $16 \times N_S \times 16$

(d) Achievable rate: $N_T \times 128 \times N_R$