

Channel Shaping Using Reconfigurable Intelligent Surfaces: From Diagonal to Beyond

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I. ASSUMPTION

We introduce Beyond-Diagonal (BD) Reconfigurable Intelligent Surface (RIS) in Multiple-Input Multiple-Output (MIMO) Point-to-point Channel (PC) and Interference Channel (IC). All proposals are based on assumption of *asymmetric* passive BD RIS, i.e., symmetry constraint $\Theta_g = \Theta_g^T$ is relaxed. This is feasible when asymmetric passive components (e.g., ring hybrids and branch-line hybrids) [1] are available. This assumption was also made in Hongyu's papers [2], [3]. For quadratic problems, the proposed algorithms may be extended to symmetric BD RIS by replacing singular value decomposition with Takagi factorization [4].

II. MIMO-PC

A. Channel Power Maximization

Consider a BD RIS with N^S elements, which is divided into G groups of equal L elements.

$$\max_{\Theta} \left\| \mathbf{H}^D + \sum_g \mathbf{H}_g^B \Theta_g \mathbf{H}_g^F \right\|_F^2 \quad (1a)$$

$$\text{s.t.} \quad \Theta_g^H \Theta_g = \mathbf{I}, \quad \forall g \in \mathcal{G} \triangleq \{1, \dots, G\}. \quad (1b)$$

For *symmetric* BD-RIS, the problem has been solved in

- Matteo's paper [5]: SISO and equivalent¹;
- Ignacio's paper [6]: SISO and directless MISO/SIMO.

Remark 1. The difficulty of (1) is that the RIS needs to balance the additive (direct-indirect) and multiplicative (forward-backward) eigenspace alignment. Interestingly, it has the same form as the weighted orthogonal Procrustes problem [7]:

$$\min_{\Theta} \left\| \mathbf{C} - \mathbf{A} \Theta \mathbf{B} \right\|_F^2 \quad (2a)$$

$$\text{s.t.} \quad \Theta^H \Theta = \mathbf{I}. \quad (2b)$$

There exists no trivial solution to (2). One lossy transformation, by moving Θ to one side [8], formulates a standard orthogonal Procrustes problem:

$$\min_{\Theta} \left\| \mathbf{A}^\dagger \mathbf{C} - \Theta \mathbf{B} \right\|_F^2 \quad (3a)$$

$$\text{s.t.} \quad \Theta^H \Theta = \mathbf{I}. \quad (3b)$$

(3) has a global optimal solution $\Theta^* = \mathbf{U} \mathbf{V}^H$, where \mathbf{U} and \mathbf{V} are left and right singular matrix of $\mathbf{A}^\dagger \mathbf{C} \mathbf{B}^H$ [9]. This low-complexity solution will be compared with the one proposed later.

¹Single-stream MIMO with given precoder and combiner.

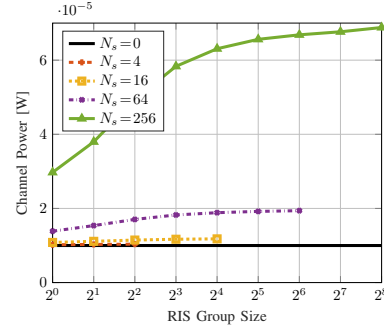


Fig. 1. Average channel power versus RIS elements N^S and group size L for $(N^T, N^R) = (8, 4)$, $(\Lambda^D, \Lambda^F, \Lambda^B) = (65, 54, 46)$ dB.

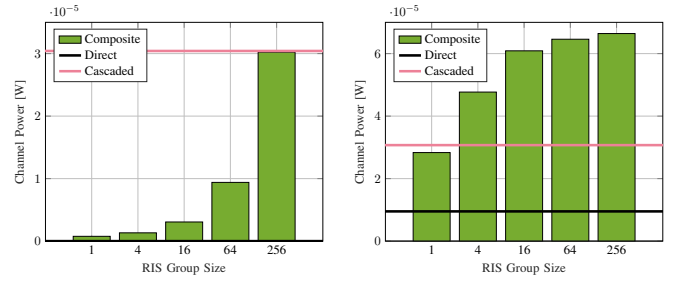


Fig. 2. Average channel power versus RIS group size L for $(N^T, N^S, N^R) = (8, 256, 4)$, $(\Lambda^D, \Lambda^F, \Lambda^B) = (65, 54, 46)$ dB.

Inspired by [10], we propose an iterative algorithm to solve (1). The idea is to successively approximate the quadratic objective with a sequence of affine functions and solve the resulting subproblems in closed form.

Proposition 1. Start from any $\Theta^{(0)}$, the sequence

$$\Theta_g^{(r+1)} = \mathbf{U}_g^{(r)} \mathbf{V}_g^{(r)}, \quad \forall g \quad (4)$$

converges to a stationary point of (1), where $\mathbf{U}_g^{(r)}$ and $\mathbf{V}_g^{(r)}$ are left and right singular matrix of

$$\begin{aligned} \mathbf{M}_g^{(r)} = & \mathbf{H}_g^B \mathbf{H}^D \mathbf{H}_g^H + \sum_{g' < g} \mathbf{H}_{g'}^B \mathbf{H}_{g'}^H \Theta_{g'}^{(r+1)} \mathbf{H}_{g'}^F \mathbf{H}_{g'}^H \\ & + \sum_{g' \geq g} \mathbf{H}_{g'}^B \mathbf{H}_{g'}^H \Theta_{g'}^{(r)} \mathbf{H}_{g'}^F \mathbf{H}_{g'}^H. \end{aligned} \quad (5)$$

Proof. To be added. \square

Fig. 1 shows that, apart from adding reflecting elements N^S , increasing the group size L also improves the channel

power. This behavior is more pronounced for a large RIS. For example, the gain of pairwise connection is 2.8 % for $N^S = 16$ and 28 % for $N^S = 256$. It implies that the channel shaping capability of BD RIS scales with group size L .

Fig. 2b and 2a compare the average channel power without and with direct link. ‘‘Cascaded’’ means the sum of element-wise product of first $N = \min(N^T, N^S, N^R)$ eigenvalues (i.e., element-wise power product) of the forward and backward channels. We observe that diagonal RIS wastes substantial cascaded power and struggles to align the direct-indirect eigenspace. When the direct link is absent, only 2.6 % of available power is utilized by diagonal RIS while 100 % power is recycled by fully-connected RIS. When the direct link is present, the proposed BD RIS design can balance the direct-indirect and forward-backward eigenspace alignment for an optimal channel boost. It is worth noting that, when L is sufficiently large, the composite channel power surpasses the power sum of direct and cascaded channels, thanks to the constructive *amplitude superposition* of direct and cascaded channels. This again emphasizes the advantage of in-group connection of BD RIS.

B. Rate Maximization

The problem is formulated w.r.t. precoder (instead of transmit covariance matrix) for reference:

$$\max_{\mathbf{W}, \Theta} \log \det \left(\mathbf{I} + \frac{\mathbf{W}^H \mathbf{H}^H \mathbf{H} \mathbf{W}}{\sigma_n^2} \right) \quad (6a)$$

$$\text{s.t.} \quad \|\mathbf{W}\|_F^2 \leq P, \quad (6b)$$

$$\Theta_g^H \Theta_g = \mathbf{I}, \quad \forall g. \quad (6c)$$

(6) is jointly non-convex and solved by Alternating Optimization (AO). For a given Θ , the optimal precoder is given by

$$\mathbf{W}^* = \mathbf{V} \mathbf{S}^{*1/2}, \quad (7)$$

where \mathbf{V} is right singular matrix of \mathbf{H} and \mathbf{S}^* is a diagonal matrix of the water-filling power allocation. For a given \mathbf{W} , we update Θ by Riemannian Conjugate Gradient (RCG) method along the geodesics [11].

Remark 2. A geodesic refers to the shortest path between two points in a Riemannian manifold. Unitary constraint (6c) translates to a Stiefel manifold where the geodesics have simple expressions described by the exponential map [12].

For general optimization problems with block unitary constraint, the adapted RCG method at iteration r for block g is summarized below, where $f(\Theta_g^{(r)})$ is the objective function also evaluated over $\{\{\Theta_{g'}^{(r+1)}\}_{g' < g}, \{\Theta_{g'}^{(r)}\}_{g' > g}\}$.

- 1) Compute the Euclidean gradient

$$\nabla_g^E(r) = \frac{\partial f(\Theta_g^{(r)})}{\partial \Theta_g^*}; \quad (8)$$

- 2) Translate to the Riemannian gradient

$$\nabla_g^R(r) = \nabla_g^E(r) \Theta_g^{(r)H} - \Theta_g^{(r)} \nabla_g^E(r)^H; \quad (9)$$

Algorithm 1: RCG Method for RIS MIMO-PC Rate Maximization

Input: $\mathbf{H}^D, \mathbf{H}^F, \mathbf{H}^B, \mathbf{W}, L, \sigma_n^2$

Output: Θ^*

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1:  $r \leftarrow 0, \Theta^{(0)}$ 
2: Repeat
3:    $r \leftarrow r + 1$ 
4:   For  $g \leftarrow 1$  to  $G$ 
5:      $\Theta_g^{(r)} \leftarrow (14), (9)-(13)$ 
6:   End For
7: Until  $|R^{(r)} - R^{(r-1)}|/R^{(r-1)} \leq \epsilon$ 
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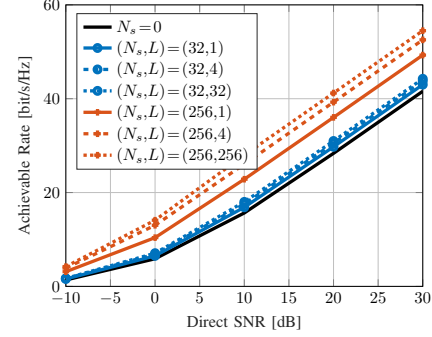


Fig. 3. Average achievable rate versus RIS elements N^S and group size L for $(N^T, N^R) = (8, 4)$, $(\Lambda^D, \Lambda^F, \Lambda^B) = (65, 54, 46)$ dB.

- 3) Determine the weight factor

$$\gamma_g^{(r)} = \frac{\text{tr} \left((\nabla_g^R(r) - \nabla_g^R(r-1)) \nabla_g^R(r)^H \right)}{\text{tr} \left(\nabla_g^R(r-1) \nabla_g^R(r-1)^H \right)}; \quad (10)$$

- 4) Compute the conjugate direction

$$\mathbf{D}_g^{(r)} = \nabla_g^R(r) + \gamma_g^{(r)} \mathbf{D}_g^{(r-1)}; \quad (11)$$

- 5) Determine the Armijo step size²

$$\mu_g^{(r)} = \arg \max_{\mu_g} f \left(\exp(\mu_g \mathbf{D}_g^{(r)}) \Theta_g^{(r)} \right); \quad (12)$$

- 6) Perform rotational update along local geodesics

$$\Theta_g^{(r+1)} = \exp \left(\mu_g^{(r)} \mathbf{D}_g^{(r)} \right) \Theta_g^{(r)}. \quad (13)$$

Remark 3. The adapted RCG method leverages the fact that block unitary matrices are closed under multiplication (but not necessarily under addition). Its advantage over universal manifold optimization [13], [14] is trifold:

- No retraction is involved;
- Lower computational complexity per iteration [12];
- Faster convergence thanks appropriate operational space.

The complex derivative of (6a) w.r.t. RIS block g is

$$\frac{\partial R}{\partial \Theta_g^*} = \frac{1}{\sigma_n^2} \mathbf{H}_g^B \mathbf{H}_g^H \mathbf{H} \mathbf{W} \left(\mathbf{I} + \frac{\mathbf{W}^H \mathbf{H}^H \mathbf{H} \mathbf{W}}{\sigma_n^2} \right)^{-1} \mathbf{W}^H \mathbf{H}_g^F. \quad (14)$$

Algorithm 1 summarizes the adapted RCG method for the RIS rate maximization subproblem.

²To double the step size, simply square the argument instead of recomputing the matrix exponential, i.e., $\exp(2\mu_g \mathbf{D}_g) = \exp^2(\mu_g \mathbf{D}_g)$.

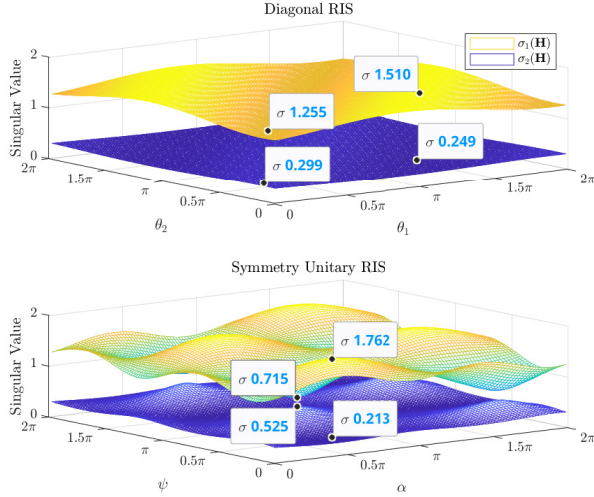


Fig. 4. Channel singular value shaping by diagonal and symmetry unitary RIS for $(N^T, N^S, N^R) = (2, 2, 2)$ without direct link.

Fig. 3 illustrates how RIS configuration influences the MIMO PC achievable rate. To ensure a 20 bit/s/Hz transmission, an Signal-to-Noise Ratio (SNR) of 13.5 dB is required for a 8T4R system. This value decreases to 12.5 dB (resp. 8 dB) when 32- (resp. 256-) element diagonal RIS is present. If tetrads can be formed in BD RIS, the SNR can be reduced by another 20 % (resp. 44 %). Further increase in L yields a marginal gain and incurs $\mathcal{O}(L^2)$ connections. We thus conclude dyadic or tetradic BD RIS usually strike a good balance between performance and complexity.

C. Channel Singular Value Redistribution

We first show the channel shaping benefit of BD RIS by a toy example. Consider $(N^T, N^S, N^R) = (2, 2, 2)$ and assume the direct link is absent. The diagonal RIS is $\Theta^D = \text{diag}(e^{j\theta_1}, e^{j\theta_2})$ while the unitary RIS has 4 independent angular parameters

$$\Theta^U = e^{j\phi} \begin{bmatrix} e^{j\alpha} \cos \psi & e^{j\beta} \sin \psi \\ -e^{-j\beta} \sin \psi & e^{-j\alpha} \cos \psi \end{bmatrix}. \quad (15)$$

When the direct link is absent, ϕ has no impact on the singular value because $\text{sv}(e^{j\phi} \mathbf{A}) = \text{sv}(\mathbf{A})$. For a fair comparison, we enforce symmetry with $\beta = \pi/2$. Fig. 4 illustrates all possible channel singular values achieved by diagonal and symmetry unitary RIS. Despite using the same number of elements and parameters, BD RIS provides much wider dynamic ranges of $\sigma_1(\mathbf{H})$ and $\sigma_2(\mathbf{H})$ than diagonal RIS. Larger gaps are expected when the symmetry constraint can be relaxed.

We then analyze the channel shaping capability of BD RIS under specific setups.

1) *Rank-Deficient Channel*: In rank-deficient channels, BD RIS Θ^B cannot achieve a higher Degree of Freedom (DoF) than diagonal RIS Θ^D . This is because $\text{sv}(\Theta^B) = \text{sv}(\Theta^D) = \mathbf{1}$ and

$$\begin{aligned} \text{rank}(\mathbf{H}) &\leq \text{rank}(\mathbf{H}^D) + \text{rank}(\mathbf{H}^B \Theta \mathbf{H}^F) \\ &\leq \text{rank}(\mathbf{H}^D) + \min(\text{rank}(\mathbf{H}^B), \text{rank}(\Theta), \text{rank}(\mathbf{H}^F)). \end{aligned} \quad (16)$$

Note BD RIS can still provide a higher indirect SNR as shown in Fig. 1 and 2.

2) *Rank-1 Indirect Channel*: The indirect channel is rank-1 iff the forward or backward channel is rank-1. Let $\mathbf{H}^F = \sigma^F \mathbf{u}^F \mathbf{v}^{FH}$ without loss of generality. In this case, the channel Gram matrix can be written as Hermitian-plus-rank-1:

$$\mathbf{G} \triangleq \mathbf{H} \mathbf{H}^H = \mathbf{Y} + \mathbf{z} \mathbf{z}^H, \quad (17)$$

where $\mathbf{Y} \triangleq \mathbf{H}^D (\mathbf{I} - \mathbf{v}^F \mathbf{v}^{FH}) \mathbf{H}^D = \mathbf{T} \mathbf{T}^H$ and $\mathbf{z} \triangleq \sigma^F \mathbf{H}^B \Theta \mathbf{u}^F + \mathbf{H}^D \mathbf{v}^F$. Regardless of RIS size and structure³, its n -th ($n \geq 2$) eigenvalues are bounded by the Cauchy interlacing formula [9]

$$\lambda_1(\mathbf{Y}) \geq \lambda_2(\mathbf{G}) \geq \lambda_2(\mathbf{Y}) \geq \dots \geq \lambda_{N-1}(\mathbf{Y}) \geq \lambda_N(\mathbf{G}) \geq \lambda_N(\mathbf{Y}). \quad (18)$$

The equivalent singular value inequality is

$$\sigma_1(\mathbf{T}) \geq \sigma_2(\mathbf{H}) \geq \sigma_2(\mathbf{T}) \geq \dots \geq \sigma_{N-1}(\mathbf{T}) \geq \sigma_N(\mathbf{H}) \geq \sigma_N(\mathbf{T}). \quad (19)$$

(19) implies that, if the indirect channel is rank-1, then the RIS can at most enlarge the n -th ($n \geq 2$) channel singular value to the $(n-1)$ -th singular value of \mathbf{T} . Note that the largest channel singular value is unbounded with a sufficiently large RIS.

3) *Fully-Connected RIS Without Direct Link*: Denote the singular value decomposition of forward / backward channels as $\mathbf{H}^{B/F} = \mathbf{U}^{B/F} \Sigma^{B/F} \mathbf{V}^{B/FH}$. The composite channel is

$$\mathbf{H} = \mathbf{H}^B \Theta \mathbf{H}^F = \mathbf{U}^B \Sigma^B \mathbf{X} \Sigma^F \mathbf{V}^{FH}, \quad (20)$$

where $\mathbf{X} = \mathbf{V}^{BH} \Theta \mathbf{U}^F$.

Proposition 2. In this case, the singular value bounds on \mathbf{H} are equivalent to the singular value bounds on $\mathbf{B}\mathbf{F}$, where \mathbf{B} and \mathbf{F} are arbitrary matrices with singular values Σ^B and Σ^F .

Proof. We first observe that singular value control problem can be solved w.r.t. unitary \mathbf{X} and retrieved by $\Theta = \mathbf{V}^B \mathbf{X} \mathbf{U}^{FH}$. Also, $\text{sv}(\mathbf{U}^B \Sigma^B \mathbf{X} \Sigma^F \mathbf{V}^{FH}) = \text{sv}(\bar{\mathbf{U}}^B \Sigma^B \bar{\mathbf{V}}^{BH} \bar{\mathbf{U}}^F \Sigma^F \bar{\mathbf{V}}^{FH}) = \text{sv}(\mathbf{B}\mathbf{F})$ where $\bar{\mathbf{U}}^{B/F}$ and $\bar{\mathbf{V}}^{B/F}$ are arbitrary unitary matrices. \square

The problem now becomes, given Σ^B and Σ^F , what can we say about the singular value of $\mathbf{B}\mathbf{F}$. One comprehensive answer is Horn's inequality [16]: for all admissible triples (I, J, K) ,

$$\prod_{k \in K} \sigma_k(\mathbf{B}\mathbf{F}) \leq \prod_{i \in I} \sigma_i(\mathbf{B}) \prod_{j \in J} \sigma_j(\mathbf{F}). \quad (21)$$

It gives upper bound on the largest singular value and lower bound on the smallest singular value:

$$\sigma_1(\mathbf{B}\mathbf{F}) \leq \sigma_1(\mathbf{B}) \sigma_1(\mathbf{F}) \quad (22)$$

$$\sigma_N(\mathbf{B}\mathbf{F}) \geq \sigma_N(\mathbf{B}) \sigma_N(\mathbf{F}). \quad (23)$$

Another useful result is introduced in [17]: for all $p > 0$,

$$\sum_n \sigma_n^p(\mathbf{B}\mathbf{F}) \leq \sum_n \sigma_n^p(\mathbf{B}) \sigma_n^p(\mathbf{F}). \quad (24)$$

When $p=2$, it implies the channel energy is upper bounded by the sum of element-wise power product of the forward and

³A similar conclusion was made for diagonal RIS in [15].

backward channels, as illustrated in Fig. 2(a). Interestingly, (22)–(24) are simultaneously tight when $\mathbf{X} = \mathbf{I}$ and $\mathbf{\Theta} = \mathbf{V}^B \mathbf{U}^{FH}$. This solution was claimed in [18] to achieve channel capacity, but it is not true at moderate SNR.

Finally, we characterize the *Pareto front* of channel singular values via optimization approach.

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