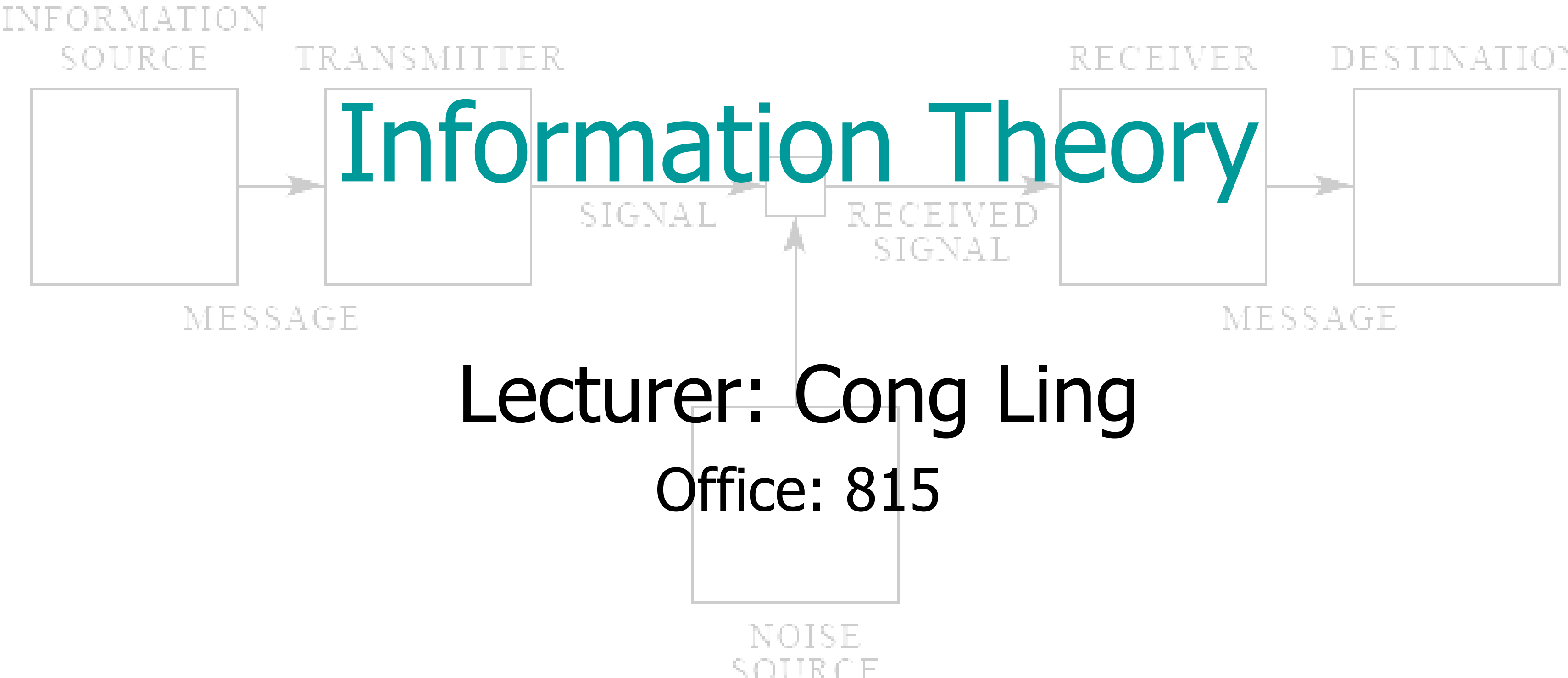
# Information Theory

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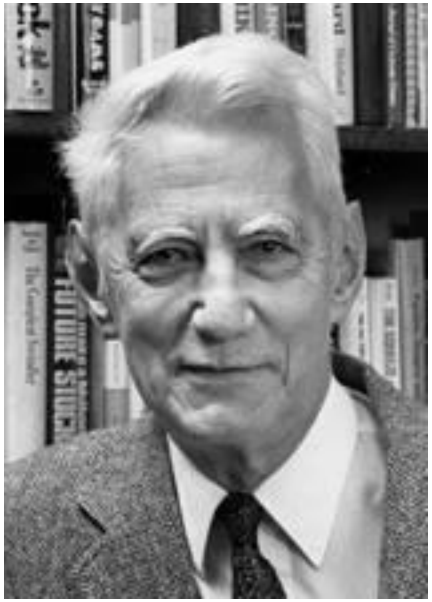
# Lectures

# Claude Shannon

- C. E. Shannon, ”A mathematical theory of communication,” Bell System Technical Journal, 1948.
- Two fundamental questions in communication theory:
- Ultimate limit on data compression
  - entropy
- Ultimate transmission rate of communication
  - channel capacity
- Almost all important topics in information theory were initiated by Shannon

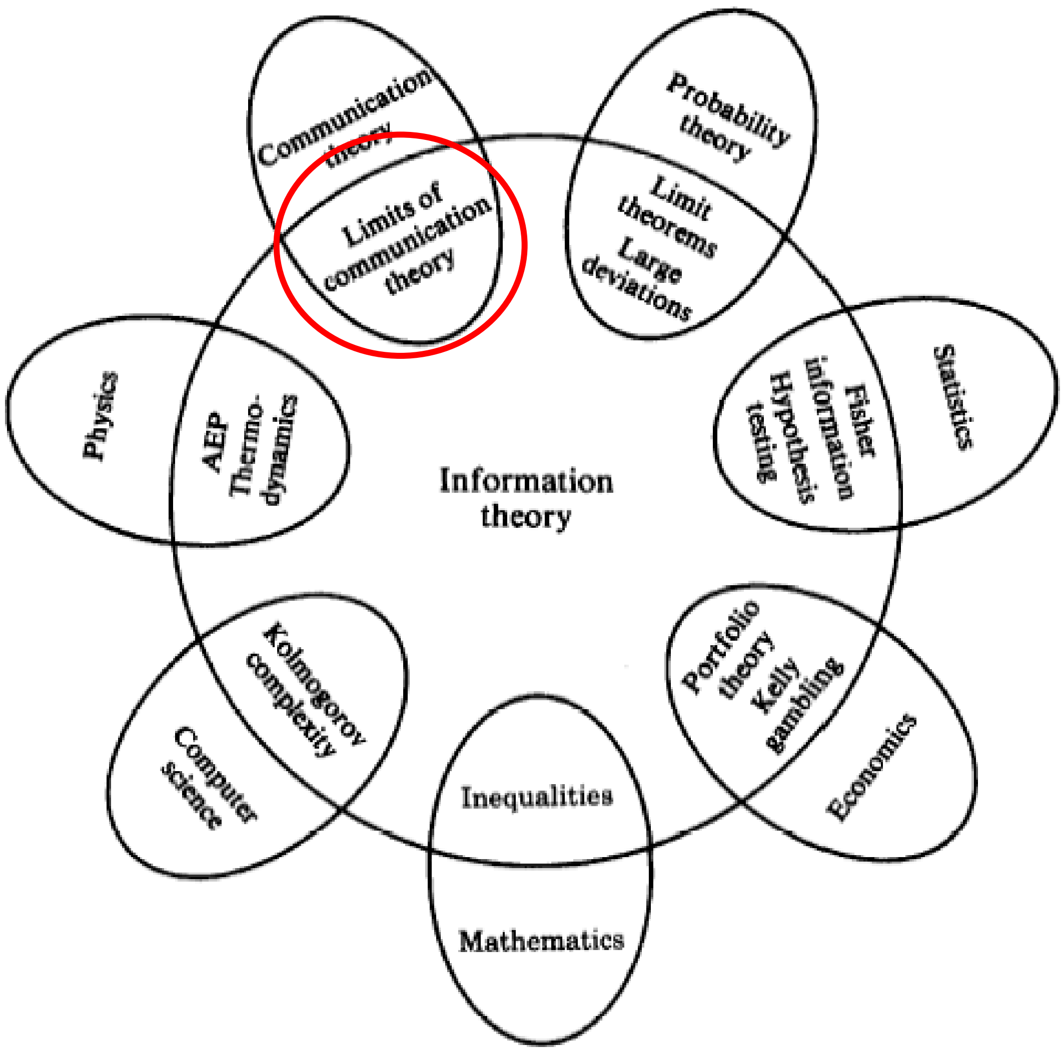


1916 - 2001

# Origin of Information Theory

- Common wisdom in 1940s:
  - It is impossible to send information error-free at a positive rate
  - Error control by using retransmission: rate $\rightarrow$ 0 if error-free
- Still in use today
  - ARQ (automatic repeat request) in TCP/IP computer networking
- Shannon showed reliable communication is possible for all rates below channel capacity
- As long as source entropy is less than channel capacity, asymptotically error-free communication can be achieved
- And anything can be represented in bits
  - Rise of digital information technology

# Relationship to Other Fields



# Course Objectives

- In this course we will (focus on communication theory):
  - Define what we mean by information.
  - Show how we can compress the information in a source to its theoretically minimum value and show the tradeoff between data compression and distortion.
  - Prove the channel coding theorem and derive the information capacity of different channels.
  - Generalize from point-to-point to network information theory.

# Relevance to Practice

- Information theory suggests means of achieving ultimate limits of communication
  - Unfortunately, these theoretically optimum schemes are computationally impractical
  - So some say "little info, much theory" (wrong)
- Today, information theory offers useful guidelines to design of communication systems
  - Polar code (achieves channel capacity)
  - CDMA (has a higher capacity than FDMA/TDMA)
  - Channel-coding approach to source coding (duality)
  - Network coding (goes beyond routing)

# Books/Reading

Book of the course:

- *Elements of Information Theory* by T M Cover & J A Thomas, Wiley, £39 for 2nd ed. 2006, or £14 for 1st ed. 1991 (Amazon)

Free references

- *Information Theory and Network Coding* by R. W. Yeung, Springer http://iest2.ie.cuhk.edu.hk/~whyeung/book2/

- *Information Theory, Inference, and Learning Algorithms* by D MacKay, Cambridge University Press http://www.inference.phy.cam.ac.uk/mackay/itila/

- *Lecture Notes on Network Information Theory* by A. E. Gamal and Y.-H. Kim, (Book is published by Cambridge University Press) http://arxiv.org/abs/1001.3404

- C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal,* Vol. 27, pp. 379–423, 623–656, July, October, 1948.

# Other Information

- Course webpage: http://www.commsp.ee.ic.ac.uk/~cling
- Assessment: Exam only – no coursework.
- Students are encouraged to do the problems in problem sheets.
- Background knowledge
  - Mathematics
  - Elementary probability
- Needs intellectual maturity
  - Doing problems is not enough; spend some time thinking

# Notation

- Vectors and matrices
  - $\mathbf{v}$=vector, $\mathbf{V}$=matrix
- Scalar random variables
  - *x* = R.V, $x$ = specific value, X = alphabet
- Random column vector of length $N$
  - **x** = R.V, $\mathbf{x}$ = specific value, X$^N$ = alphabet
  - *x*$_i$ and $x_i$ are particular vector elements
- Ranges
  - $a$:$b$ denotes the range $a$, $a$+1, …, $b$
- Cardinality
  - |X| = the number of elements in set X

# Discrete Random Variables

- A random variable $x$ takes a value $x$ from the alphabet X with probability $p_X(x)$. The vector of probabilities is $\mathbf{p}_X$.

Examples:

X = [1;2;3;4;5;6], $\mathbf{p}_X = [{}^1/_6; {}^1/_6; {}^1/_6; {}^1/_6; {}^1/_6; {}^1/_6]$

$\mathbf{p}_X$ is a "probability mass vector"

"english text"

X = [a; b;…, y; z; <space>]

$\mathbf{p}_X$= [0.058; 0.013; …; 0.016; 0.0007; 0.193]

Note: we normally drop the subscript from $p_X$ if unambiguous

# Expected Values

- If $g(x)$ is a function defined on X then

$$E_X\, g(X) = \sum_{x \in \mathrm{X}} p(x) g(x)$$

often write $E$ for $E_X$

Examples:

X = [1;2;3;4;5;6], $\mathbf{p}_X$ = [$^1/_6$; $^1/_6$; $^1/_6$; $^1/_6$; $^1/_6$; $^1/_6$]

$$E\, X = 3.5 = \mu$$

$\sigma^2 = E(x^2) - (E(x))^2 = 15.17 - 12.25 = 2.92$

$$E\, X^2 = 15.17 = \sigma^2 + \mu^2$$

$$E \sin(0.1X) = 0.338$$

$-\log_2(\frac{1}{6}) = 2.58$

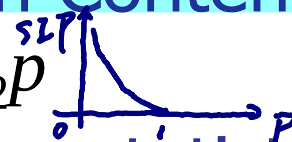
$$E - \log_2(p(X)) = 2.58$$

This is the "entropy" of $X$

# Shannon Information Content

SIP: the amount of info. associated with an event with probability P.

- The Shannon Information Content of an outcome with probability $p$ is $-\log_2 p$




smaller probability
↓
more information . uncommon cases.

- Shannon's contribution – a statistical view
  - Messages, noisy channels are random
  - Pre-Shannon era: deterministic approach (Fourier…)

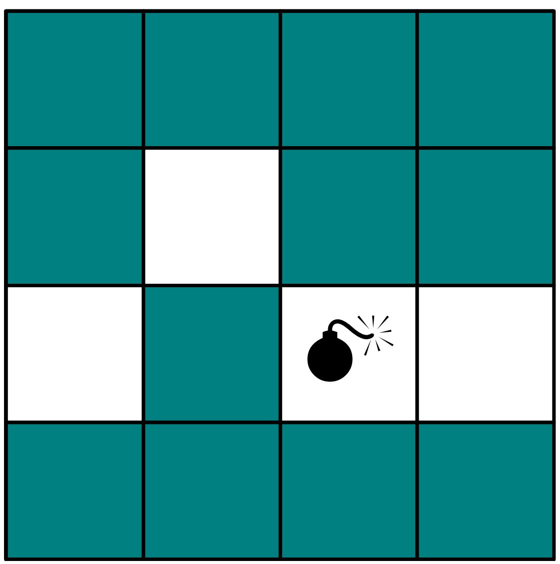
- Example 1: Coin tossing
  - X = [Head; Tail], $\mathbf{p} = [½; ½]$, SIC $= [1; 1]$ bits
- Example 2: Is it my birthday ?
  - X = [No; Yes], $\mathbf{p} = [{}^{364}/_{365}; {}^{1}/_{365}]$, SIC $= [0.004; 8.512]$ bits

Unlikely outcomes give more information

# Minesweeper

- Where is the bomb ?
- 16 possibilities – needs 4 bits to specify

$4 = \log_2 16$



| Guess | | Prob | SIC |
|---|---|---|---|
| 1. | No | $^{15}/_{16}$ | 0.093 bits |
| 2. | No | $^{14}/_{15}$ | 0.100 bits |
| 3. | No | $^{13}/_{14}$ | 0.107 bits |
| 4. | Yes | $^{1}/_{13}$ | 3.700 bits |
| | | Total | 4.000 bits |

$\text{SIC} = -\log_2 p$

$H'(X) = \log_2 |X| \geq H(X)$

'=' apply at uniform distribution.

entropy: a measurement of uncertainty; bits numbers required to describe a r.v..; expectation of information

# Entropy

$$H(x) = E - \log_2(p_x(x)) = -\sum_{x \in X} p_x(x) \log_2 p_x(x)$$

|| transmission cost or verification

- $H(x)$ = the average Shannon Information Content of $x$
- $H(x)$ = the average information gained by knowing its value
- the average number of "yes-no" questions needed to find $x$ is in the range $[H(X), H(X)+1)$
- $H(x)$ = the amount of uncertainty before we know its value

We use $\log(x) \equiv \log_2(x)$ and measure $H(X)$ in bits

- if you use $\log_e$ it is measured in nats
- 1 nat = $\log_2(e)$ bits = 1.44 bits

- $$\log_2(x) = \frac{\ln(x)}{\ln(2)} \qquad \frac{d \log_2 x}{dx} = \frac{\log_2 e}{x}$$

$H(X)$ depends only on the probability vector $\mathbf{p}_X$ not on the alphabet $X$, so we can write $H(\mathbf{p}_X)$

$H(x) = -(1-p)\log(1-p) - p\log p$

$H'(x) = \log(1-p) + 1 - \log p - 1 = \log(1-p) - \log p$

$H'(x) = 0 \Rightarrow p = \frac{1}{2}$

$H(x) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1$

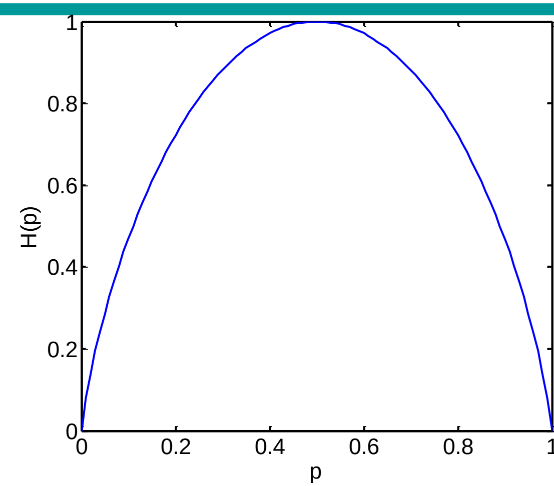
# Entropy Examples

## (1) Bernoulli Random Variable

X = [0;1], $\mathbf{p}_X = [1-p; p]$

$$H(x) = -(1-p)\log(1-p) - p\log p$$

Very common – we write $H(p)$ to mean $H([1-p; p])$.




$$H(p) = -(1-p)\log(1-p) - p\log p$$

$$H'(p) = \log(1-p) - \log p$$

$$H''(p) = -p^{-1}(1-p)^{-1}\log e$$

Maximum is when $p$=1/2

## (2) Four Coloured Shapes

X = [●; ■; ◆; ✿], $\mathbf{p}_X = [½; ¼; ^1/_8; ^1/_8]$

$$H(x) = H(\mathbf{p}_x) = \sum -\log(p(x))\,p(x)$$

$$= 1 \times ½ + 2 \times ¼ + 3 \times ⅛ + 3 \times ⅛ = 1.75 \text{ bits}$$

entropy $\begin{cases} \text{monotonic non-decreasing} \\ \text{additive} \\ \text{non-negative} \end{cases}$ $\Rightarrow H(X) = -c \sum_{x \in X} p(x) \log p(x) \quad (c > 0 \to c = 1)$

# Comments on Entropy

- Entropy plays a central role in information theory
- Origin in thermodynamics
  - $S = k \ln\Omega$, $k$: Boltzman's constant, $\Omega$: number of microstates
  - The second law: entropy of an isolated system is non-decreasing
- Shannon entropy
  - Agrees with intuition: additive, monotonic, continuous
  - Logarithmic measure could be derived from an axiomatic approach (Shannon 1948)

# Lecture 2

- Joint and Conditional Entropy
  - Chain rule
- Mutual Information
  - If $x$ and $y$ are correlated, their mutual information is the average information that $y$ gives about $x$
    - E.g. Communication Channel: $x$ transmitted but $y$ received
    - It is the amount of information transmitted through the channel
- Jensen's Inequality

# Joint and Conditional Entropy

## Joint Entropy: $H(x,y)$

$H(x,y) = E - \log P(x,y)$

$= -\sum_{x,y} P(x,y) \log P(x,y)$

$= \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = 1.5$ bits

| $p(x,y)$ | $y$=0 | $y$=1 |
|---|---|---|
| $x$=0 | ½ | ¼ |
| $x$=1 | 0 | ¼ |

$$H(x,y) = E - \log p(x,y)$$

$$= -\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{4}\log\tfrac{1}{4} - 0\log 0 - \tfrac{1}{4}\log\tfrac{1}{4} = 1.5 \text{ bits}$$

Note: $0 \log 0 = 0$

$0 \log 0 = \lim_{x \to 0} x \log x$

$= \lim_{x \to 0} \frac{\log x}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} (-x)$

$= 0$

## Conditional Entropy: $H(y\,|\,x)$

$H(y|x) = E - \log P(y|x)$

$= -\sum_{x,y} P(x,y) \log P(y|x)$

$= -\frac{1}{2} \times \log\frac{2}{3} - \frac{1}{4} \times \log\frac{1}{3} - \frac{1}{4} \times \log 1$

$= 0.689$ bits

| $p(y\|x)$ | $y$=0 | $y$=1 |
|---|---|---|
| $x$=0 | $^2/_3$ | $^1/_3$ |
| $x$=1 | 0 | 1 |

Note: rows sum to 1

$$H(y\,|\,x) = E - \log p(y\,|\,x)$$

$$= -\sum_{x,y} p(x,y) \log p(y\,|\,x)$$

$$= -\tfrac{1}{2}\log\tfrac{2}{3} - \tfrac{1}{4}\log\tfrac{1}{3} - 0\log 0 - \tfrac{1}{4}\log 1 = 0.689 \text{ bits}$$

# Conditional Entropy – View 1

$= \sum_x \sum_y P(x,y) \log P(x)$

$= \sum_x P(x) \log P(x)$

$P(Y|X) = \frac{P(X,Y)}{P(X)}$   $H(Y|X) = E - \log P(Y|X)$

$= E - \log \frac{P(X,Y)}{P(X)}$

## Additional Entropy:

| $p(x, y)$ | $y=0$ | $y=1$ | $p(x)$ |
|---|---|---|---|
| $x=0$ | ½ | ¼ | ¾ |
| $x=1$ | 0 | ¼ | ¼ |

$$p(y \mid x) = p(x, y) \div p(x) \quad = E - \log P(x,y) - E - \log P(x)$$

$$H(y \mid x) = E - \log p(y \mid x) = -\sum_{x,y} P(x,y) \log P(x,y) + \sum_{x,y} P(x,y) \log P(x)$$

$$= E\{-\log p(x, y)\} - E\{-\log p(x)\} = H(x,y) - H(x)$$

$$= H(x, y) - H(x) = H(½, ¼, 0, ¼) - H(¾, ¼) = 0.689 \text{ bits}$$

remaining uncertainty

*H(Y|X)* is the average additional information in *Y* when you know *X*

$H(x,y) = -\sum_{x,y} P(x,y) \log P(x,y)$

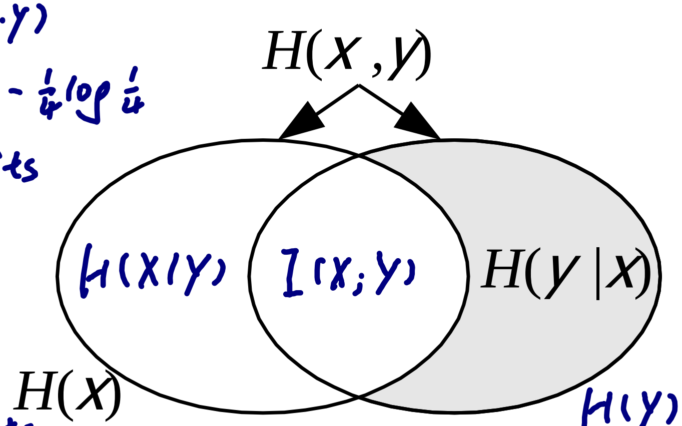
$= -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{4}\log\frac{1}{4}$

$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1.5$ bits

$H(x) = -\sum_x P(x) \log P(x)$

$= -\frac{3}{4}\log\frac{3}{4} - \frac{1}{4}\log\frac{1}{4}$

$= 0.311 + 0.5 = 0.811$ bits




$H(Y|X) = H(X,Y) - H(X) = 0.689$ bits

# Conditional Entropy – View 2

| $p(x, y)$ | $y$=0 | $y$=1 | $H(y \mid x=x)$ | $p(x)$ |
|---|---|---|---|---|
| $x$=0 | ½ | ¼ | H(1/3) | ¾ |
| $x$=1 | 0 | ¼ | H(1) | ¼ |

## Average Row Entropy:

$H(Y|X) = -\sum_{x,y} P(x,y) \log P(y|x)$

$= -\sum_{x,y} P(y|x) P(x) \log P(y|x)$

$= -\sum_{x} P(x) \sum_{y} P(y|x) \log P(y|x) = \underbrace{\sum_{x} P(x) H(Y|X=x)}$

Conditional entropy $H(Y|X)$

remaining uncertainty of $Y$ known $X$

$H(Y|X) = H(X,Y) - H(X)$

weighted average row entropy

$H(Y|X) = \sum_{x} P(x) H(Y|X=x)$

$$H(y \mid x) = E - \log p(y \mid x) = \sum_{x,y} - p(x,y) \log p(y \mid x)$$

$$= \sum_{x,y} - p(x) p(y \mid x) \log p(y \mid x) = \sum_{x \in X} p(x) \sum_{y \in Y} - p(y \mid x) \log p(y \mid x)$$

$$= \sum_{x \in X} p(x) H\left(y \mid x = x\right) = ¾ \times H(1/3) + ¼ \times H(0) = 0.689 \text{ bits}$$

$H(Y|X=0) = -\sum_{y} P(y|X=0) \log P(y|X=0) = -\frac{2}{3}\log\frac{2}{3} - \frac{1}{3}\log\frac{1}{3} = 0.918$ bits

$H(Y|X=1) = -\sum_{y} P(y|X=1) \log P(y|X=1) = -1\log 1 = 0$ bits

Take a weighted average of the entropy of each row using $p(x)$ as weight

$\therefore H(Y|X) = \sum_{x} P(x) H(Y|X=x) = \frac{3}{4} \times 0.918 + \frac{1}{4} \times 0 = 0.689$ bits

$$H(x,y,z) = -\sum_{x,y,z} p(x,y,z)\log p(x,y,z) = -\sum_{x,y,z} p(x,y,z)\log p(z|x,y)\,p(y|x)\,p(x)$$

$$= -\sum_{x,y,z} p(x,y,z)\log p(z|x,y) - \sum_{x,y,z} p(x,y,z)\log p(y|x) - \sum_{x,y,z} p(x,y,z)\log p(x)$$

# Chain Rules

$$= -\sum_{z} p(z)\log p(z|x,y) - \sum_{y} p(y)\log p(y|x) - \sum_{x} p(x)\log p(x)$$

$$= H(z|x,y) + H(y|x) + H(x)$$

- Probabilities

$$p(x,y,z) = p(z \mid x,y)\,p(y \mid x)\,p(x)$$

- Entropy

$$\boxed{\begin{aligned} &H(x,y,z) = H(z \mid x,y) + H(y \mid x) + H(x) \\ &H(x_{1:n}) = \sum_{i=1}^{n} H(x_i \mid x_{1:i-1}) \end{aligned}}$$

The log in the definition of entropy converts products of probability into sums of entropy

Mutual information: the reduction of uncertainty in x given y.

↕

# Mutual Information

Conditional entropy: the remaining uncertainty of x given y.

Mutual information is the average amount of information that you get about *x* from observing the value of *y*

- Or the reduction in the uncertainty of *x* due to knowledge of *y*

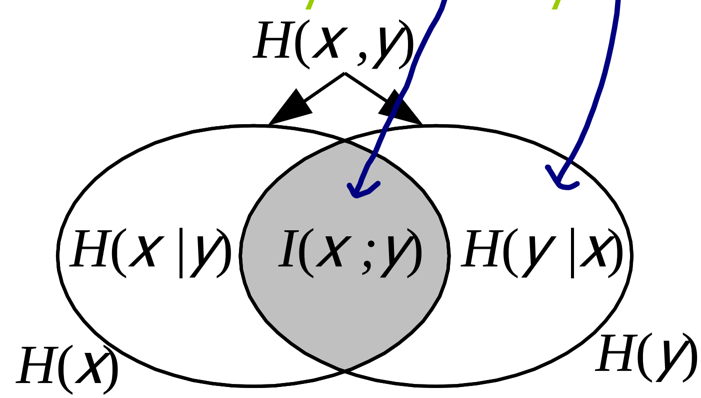
$$\boxed{I(x;y) = H(x) - H(x \mid y) = H(x) + H(y) - H(x,y)}$$

target

Information in *x* — Information in *x* when you already know *y*

Mutual information is symmetrical

$$I(x;y) = I(y;x)$$




Use ";" to avoid ambiguities between $I(x;y,z)$ and $I(x,y;z)$

$H(X) = -\sum_x P(x)\log P(x) = -\frac{3}{4}\log\frac{3}{4} - \frac{1}{4}\log\frac{1}{4} = 0.811$ bits

$H(Y) = -\sum_y P(y)\log P(y) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1$ bit

$H(X,Y) = -\sum_{x,y} P(x,y)\log P(x,y) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{4}\log\frac{1}{4} = 1.5$ bits

# Mutual Information Example

$H(X,Y)$

$H(X|Y)$ $I(X;Y)$ $H(Y|X)$

$H(X)$ $H(Y)$

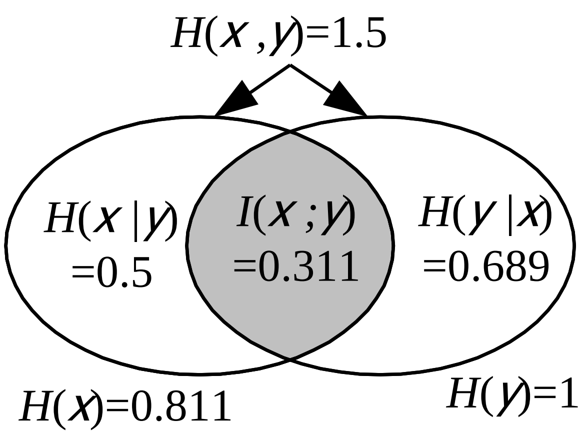
$H(X|Y) = H(X,Y) - H(Y) = 0.5$ bits

$H(Y|X) = H(X,Y) - H(X) = 0.689$ bits

$I(X;Y) = H(X) + H(Y) - H(X,Y) = 0.311$ bits

| $p(x,y)$ | $y=0$ | $y=1$ |
|---|---|---|
| $x=0$ | ½ | ¼ |
| $x=1$ | 0 | ¼ |

- If you try to guess $y$ you have a 50% chance of being correct.
- However, what if you know $x$?
  - Best guess: choose $y = x$
  - If $x=0$ ($p=0.75$) then 66% correct prob
  - If $x=1$ ($p=0.25$) then 100% correct prob
  - Overall 75% correct probability




$$I(x;y) = H(x) - H(x|y)$$
$$= H(x) + H(y) - H(x,y)$$
$$H(x) = 0.811, \quad H(y) = 1, \quad H(x,y) = 1.5$$
$$I(x;y) = 0.311$$

$I(X;Y) = H(X) - H(X|Y)$

$I(X_1, X_2 \cdots X_n; Y) = H(X_1, X_2 \cdots X_n) - H(X_1, X_2 \cdots X_n | Y)$

# Conditional Mutual Information

$= \sum_{i=1}^{n} H(X_i | X_{1:i-1}) - \sum_{i=1}^{n} H(X_i | X_{1:i-1}, Y)$

$= \sum_{i=1}^{n} I(X_i ; Y | X_{1:i-1})$

$I(X_1, X_2; Y) = I(X_1; Y) + I(X_2; Y | X_1)$
$= H(X_1) - H(X_1|Y) +$
$H(X_2|X_1) - H(X_2|X_1, Y)$
$= H(X_1, X_2) - H(X_1, X_2 | Y)$

## Conditional Mutual Information

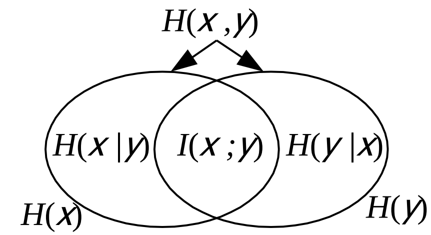
$$\boxed{\begin{aligned} I(x;y\,|\,z) &= H(x\,|\,z) - H(x\,|\,y,z) \\ &= H(x\,|\,z) + H(y\,|\,z) - H(x,y\,|\,z) \end{aligned}}$$

**Note**: $Z$ conditioning applies to both $X$ and $Y$

## Chain Rule for Mutual Information

$$I(x_1, x_2, x_3; y) = I(x_1; y) + I(x_2; y\,|\,x_1) + I(x_3; y\,|\,x_1, x_2)$$

$$I(x_{1:n}; y) = \sum_{i=1}^{n} I(x_i; y\,|\,x_{1:i-1})$$




# Review/Preview

- Entropy: $H(x) = \sum_{x \in \mathcal{X}} -\log_2(p(x))\, p(x) = E - \log_2(p_X(x))$
  - Positive and bounded $0 \leq H(x) \leq \log|\mathcal{X}|$ ♦
    fixed pattern uniform distribution
- Chain Rule: $H(x, y) = H(x) + H(y \mid x) \leq H(x) + H(y)$ ♦
  - Conditioning reduces entropy $H(y \mid x) \leq H(y)$ ♦
- Mutual Information:

$$I(y; x) = H(y) - H(y \mid x) = H(x) + H(y) - H(x, y)$$

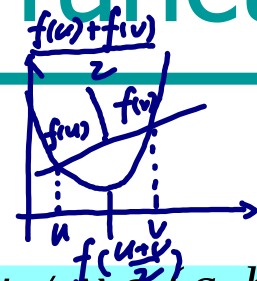
  - Positive and Symmetrical $I(x; y) = I(y; x) \geq 0$ ♦
  - $x$ and $y$ independent $\Leftrightarrow$ $H(x, y) = H(y) + H(x)$ ♦

$$\Leftrightarrow I(x; y) = 0$$

♦ = inequalities not yet proved

# Convex & Concave functions

$f(x)$ is strictly convex over $(a,b)$ if




definition

$$f(\lambda u+(1-\lambda)v)<\lambda f(u)+(1-\lambda)f(v) \quad \forall u\neq v\in(a,b),\ 0<\lambda<1$$

– every chord of $f(x)$ lies above $f(x)$

– $f(x)$ is concave $\Leftrightarrow$ $-f(x)$ is convex

- Examples
  – Strictly Convex: $x^2, x^4, e^x, x\log x\ [x\geq 0]$
  – Strictly Concave: $\log x, \sqrt{x}\quad [x\geq 0]$
  – Convex and Concave: $x$

– Test: $\dfrac{d^2 f}{dx^2}>0 \quad \forall x\in(a,b) \qquad \Rightarrow f(x)$ is strictly convex

Concave is like this

"convex" (not strictly) uses "≤" in definition and "≥" in test

# Jensen's Inequality

Jensen's Inequality: (a) $f(x)$ convex $\Rightarrow Ef(X) \geq f(EX)$

(b) $f(x)$ strictly convex $\Rightarrow Ef(X) > f(EX)$ unless $x$ constant

## Proof by induction on |X|

These sum to 1

– |X|=1: $E\, f(X) = f(E\, X) = f(x_1)$

– |X|=$k$: $E\, f(X) = \sum_{i=1}^{k} p_i f(x_i) = p_k f(x_k) + (1-p_k) \sum_{i=1}^{k-1} \frac{p_i}{1-p_k} f(x_i)$

convex: $Ef(x) \geq f(Ex)$

$$\geq p_k f(x_k) + (1-p_k) f\left( \sum_{i=1}^{k-1} \frac{p_i}{1-p_k} x_i \right)$$

Assume JI is true for |X|=$k$–1

definition: $f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda) f(v)$

$$\geq f\left( p_k x_k + (1-p_k) \sum_{i=1}^{k-1} \frac{p_i}{1-p_k} x_i \right) = f(E\, x)$$

Follows from the definition of convexity for two-mass-point distribution

# Jensen's Inequality Example

## Mnemonic example:

$f(x) = x^2$ : strictly convex

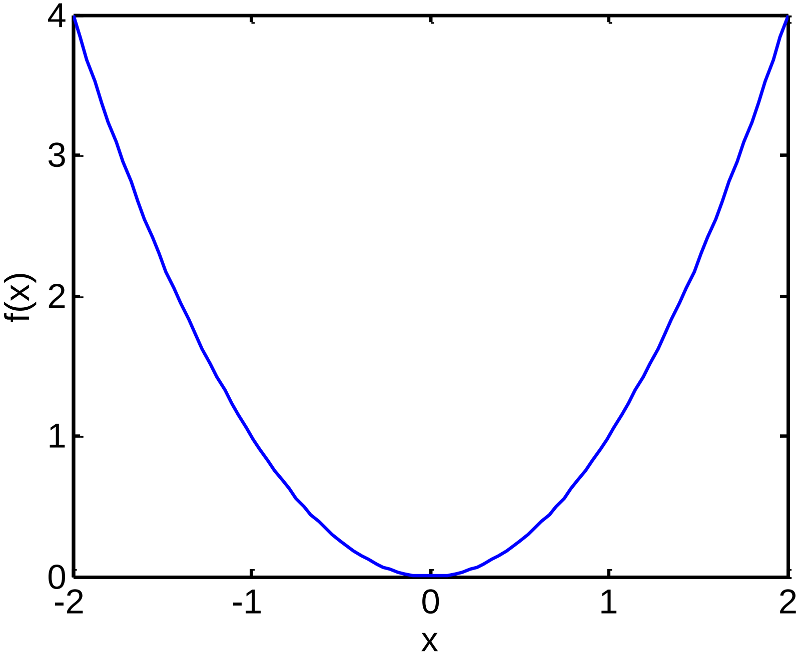
X $= [-1; +1]$

$\mathbf{p} = [½; ½]$

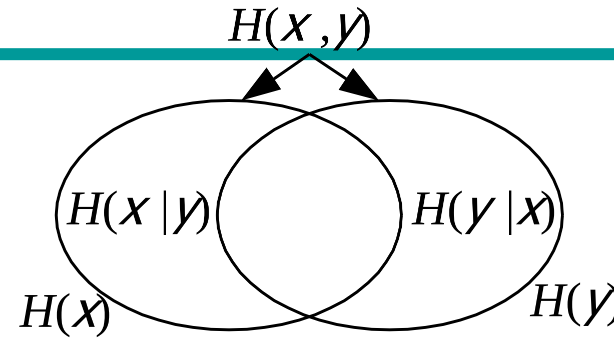
$E\ X = 0$

$f(E\ X)=0$

$E\ f(X) = 1 > f(E\ X)$



# Summary




- Chain Rule:

$$H(x,y) = H(y \mid x) + H(x)$$

- Conditional Entropy:

$$H(y \mid x) = H(x,y) - H(x) = \sum_{x \in X} p(x) H(y \mid x)$$

  – Conditioning reduces entropy

$$H(y \mid x) \le H(y)$$ ♦

- Mutual Information $I(x;y) = H(x) - H(x \mid y) \le H(x)$ ♦
  – In communications, mutual information is the amount of information transmitted through a noisy channel

- Jensen's Inequality $f(x)$ convex $\Rightarrow Ef(x) \ge f(Ex)$

♦ = inequalities not yet proved

# Lecture 3

- Relative Entropy
  - A measure of how different two probability mass vectors are
- Information Inequality and its consequences
  - Relative Entropy is always positive
    - Mutual information is positive
    - Uniform bound
    - Conditioning and correlation reduce entropy
- Stochastic Processes
  - Entropy Rate
  - Markov Processes

entropy $H(X) = -\sum_x P(x)\log_2 P(x) = \sum_x P(x)\log_2 \frac{1}{P(x)}$ : a measurement of uncertainty

cross entropy $H(P,Q) = -\sum_x P(x)\log_2 Q(x) = \sum_x P(x)\log_2 \frac{1}{Q(x)}$ : the cost of selected scheme to eliminate the system uncertainty.

relative entropy $D(P\|Q) = \sum_x P(x)\log_2 \frac{P(x)}{Q(x)}$ : the cost difference of schemes corresponding to different distributions

→ the invalidity of the system when the real distribution is $P(x)$ and the assumption is $Q(x)$

# Relative Entropy

Relative Entropy or Kullback-Leibler Divergence between two probability mass vectors $\mathbf{p}$ and $\mathbf{q}$

$$D(\mathbf{p}\,\|\,\mathbf{q}) = \sum_{x\in X} p(x)\log\frac{p(x)}{q(x)} = E_{\mathbf{p}}\log\frac{p(X)}{q(X)} = E_{\mathbf{p}}\left(-\log q(X)\right) - H(X)$$

where $E_{\mathbf{p}}$ denotes an expectation performed using probabilities $\mathbf{p}$

$D(P\|Q) \geq 0$ : the assumed distribution $Q(x)$ cannot be more accurate than the real case.

$D(\mathbf{p}\|\mathbf{q})$ measures the "distance" between the probability mass functions $\mathbf{p}$ and $\mathbf{q}$.

We must have $p_i$=0 whenever $q_i$=0 else $D(\mathbf{p}\|\mathbf{q})=\infty$

Beware: $D(\mathbf{p}\|\mathbf{q})$ is not a true distance because:

- (1) it is asymmetric between $\mathbf{p}$, $\mathbf{q}$ and
- (2) it does not satisfy the triangle inequality.

# Relative Entropy Example

$D(p\|q) = E_p \log_2 \frac{p(x)}{q(x)} = E_p \log_2 \frac{1}{q(x)} - H(p(x))$

$X = [1\ 2\ 3\ 4\ 5\ 6]^T$

$H(p) = -\sum_x p(x)\log_2 p(x) = 2.585$ . $H(q) = -\sum q(x)\log_2 q(x) = 2.161$

$D(p\|q) = E_p \log_2 \frac{1}{q(x)} - H(p(x)) = 2.935 - 2.585 = 0.35$

$$\mathbf{p} = \left[\tfrac{1}{6}\quad \tfrac{1}{6}\quad \tfrac{1}{6}\quad \tfrac{1}{6}\quad \tfrac{1}{6}\quad \tfrac{1}{6}\right] \Rightarrow \quad H(\mathbf{p}) = 2.585$$

$D(q\|p) = E_q \log_2 \frac{1}{p(x)} - H(q(x)) = 2.585 - 2.161 = 0.424$

$$\mathbf{q} = \left[\tfrac{1}{10}\quad \tfrac{1}{10}\quad \tfrac{1}{10}\quad \tfrac{1}{10}\quad \tfrac{1}{10}\quad \tfrac{1}{2}\right] \Rightarrow \quad H(\mathbf{q}) = 2.161$$

$$D(\mathbf{p}\,\|\,\mathbf{q}) = E_{\mathbf{p}}(-\log q_x) - H(\mathbf{p}) = 2.935 - 2.585 = 0.35$$

$$D(\mathbf{q}\,\|\,\mathbf{p}) = E_{\mathbf{q}}(-\log p_x) - H(\mathbf{q}) = 2.585 - 2.161 = 0.424$$

# Information Inequality

The assumed distribution $q(x)$ cannot be more accurate than the real case.

## Information (Gibbs') Inequality: $D(\mathbf{p} \| \mathbf{q}) \geq 0$

- Define $\mathsf{A} = \{x : p(x) > 0\} \subseteq \mathsf{X}$
- Proof $-D(\mathbf{p} \| \mathbf{q}) = -\sum_{x \in \mathsf{A}} p(x) \log \frac{p(x)}{q(x)} = \sum_{x \in \mathsf{A}} p(x) \log \frac{q(x)}{p(x)}$

log: concave function

$Ef(x) \leq f(Ex)$

Jensen's inequality

$$\leq \log\left(\sum_{x \in \mathsf{A}} p(x) \frac{q(x)}{p(x)}\right) = \log\left(\sum_{x \in \mathsf{A}} q(x)\right) \leq \log\left(\sum_{x \in \mathsf{X}} q(x)\right) = \log 1 = 0$$

If $D(\mathbf{p}||\mathbf{q})=0$: Since $\log(\ )$ is strictly concave we have equality in the proof only if $q(x)/p(x)$, the argument of $\log$, equals a constant.

But $\sum_{x \in \mathsf{X}} p(x) = \sum_{x \in \mathsf{X}} q(x) = 1$ so the constant must be $1$ and $\mathbf{p} \equiv \mathbf{q}$

# Information Inequality Corollaries

- Uniform distribution has highest entropy
  - Set $\mathbf{q} = [|X|^{-1}, \ldots, |X|^{-1}]^T$ giving $H(\mathbf{q}) = \log|X|$ bits

$$D(\mathbf{p} \| \mathbf{q}) = E_{\mathbf{p}}\{-\log q(x)\} - H(\mathbf{p}) = \log|X| - H(\mathbf{p}) \geq 0$$

- Mutual Information is non-negative

$D(P \| Q) = E_P \left(\log_2 \frac{P(x)}{Q(x)}\right)$

$$I(y; x) = H(x) + H(y) - H(x, y) = E \log \frac{p(x, y)}{p(x)p(y)}$$

$I(X; Y) = E \log_2 \frac{P(X,Y)}{P(X)P(Y)} = D(P(x,y) \| P(x)P(y))$

$$= D(p(x, y) \| p(x)p(y)) \geq 0$$

with equality only if $p(x,y) \equiv p(x)p(y) \Leftrightarrow x$ and $y$ are independent.

# More Corollaries

- Conditioning reduces entropy

$$0 \le I(x;y) = H(y) - H(y \mid x) \quad \Rightarrow \quad H(y \mid x) \le H(y)$$

with equality only if $x$ and $y$ are independent.

- Independence Bound

$$H(x_{1:n}) = \sum_{i=1}^{n} H(x_i \mid x_{1:i-1}) \le \sum_{i=1}^{n} H(x_i)$$

with equality only if all $x_i$ are independent.

E.g.: If all $x_i$ are identical $H(x_{1:n}) = H(x_1)$

independence bound $H(X_{1:n}) = \sum_{i=1}^{n} H(X_i | X_{1:i-1}) \le \sum_{i=1}^{n} H(X_i)$

conditional independence bound $H(X_{1:n}|Y_{1:n}) = \sum_{i=1}^{n} H(X_i | X_{1:i-1}, Y_{1:n}) \le \sum_{i=1}^{n} H(X_i|Y_i)$

mutual information independence bound $I(X_{1:n}|Y_{1:n}) = H(X_{1:n}) - H(X_{1:n}|Y_{1:n})$

$\ge \sum_{i=1}^{n} H(X_i) - \sum_{i=1}^{n} H(X_i|Y_i) = \sum_{i=1}^{n} I(X_i;Y_i)$

# Conditional Independence Bound

- Conditional Independence Bound

$$H(x_{1:n} \mid y_{1:n}) = \sum_{i=1}^{n} H(x_i \mid x_{1:i-1}, y_{1:n}) \le \sum_{i=1}^{n} H(x_i \mid y_i)$$

- Mutual Information Independence Bound

If all $x_i$ are independent or, by symmetry, if all $y_i$ are independent:

$$I(x_{1:n}; y_{1:n}) = H(x_{1:n}) - H(x_{1:n} \mid y_{1:n})$$

$$\ge \sum_{i=1}^{n} H(x_i) - \sum_{i=1}^{n} H(x_i \mid y_i) = \sum_{i=1}^{n} I(x_i; y_i)$$

E.g.: If $n=2$ with $x_i$ i.i.d. Bernoulli ($p=0.5$) and $y_1=x_2$ and $y_2=x_1$, then $I(x_i;y_i)=0$ but $I(x_{1:2}; y_{1:2}) = 2$ bits.

# Stochastic Process

## Stochastic Process $\{x_i\} = x_1, x_2, \ldots$

**Entropy:** $H(\{x_i\}) = H(x_1) + H(x_2 \mid x_1) + \ldots \overset{\text{often}}{=} \infty$

**Entropy Rate:** $\boxed{H(\mathrm{X}) = \lim_{n \to \infty} \frac{1}{n} H(x_{1:n})}$ if limit exists

the increasing rate of entropy w.r.t. $n$.

- Entropy rate estimates the additional entropy per new sample.
- Gives a lower bound on number of code bits per sample.

**Examples:**

- Typewriter with $m$ equally likely letters each time: $H(\mathrm{X}) = \log m$
- $x_i$ i.i.d. random variables: $H(\mathrm{X}) = H(x_i)$

# Stationary Process

Stochastic Process $\{x_i\}$ is stationary iff

$$p(x_{1:n} = a_{1:n}) = p(x_{k+(1:n)} = a_{1:n}) \quad \forall k, n, a_i \in \mathcal{X}$$

If $\{x_i\}$ is stationary then $H(\mathcal{X})$ exists and

$$\boxed{H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(x_{1:n}) = \lim_{n\to\infty} H(x_n \mid x_{1:n-1})}$$

**Proof:**

$$0 \le H(x_n \mid x_{1:n-1}) \overset{(a)}{\le} H(x_n \mid x_{2:n-1}) \overset{(b)}{=} H(x_{n-1} \mid x_{1:n-2})$$

(a) conditioning reduces entropy, (b) stationarity

Hence $H(x_n|x_{1:n-1})$ is positive, decreasing $\Rightarrow$ tends to a limit, say $b$

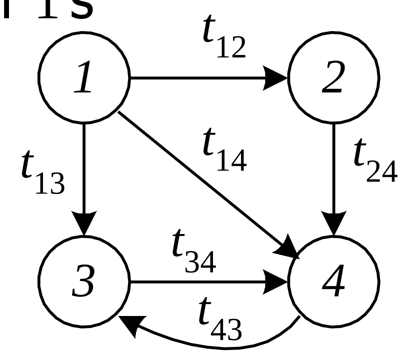
Hence

$$H(x_k \mid x_{1:k-1}) \to b \quad \Rightarrow \quad \frac{1}{n} H(x_{1:n}) = \frac{1}{n} \sum_{k=1}^{n} H(x_k \mid x_{1:k-1}) \to b = H(\mathcal{X})$$

# Markov Process (Chain)

Discrete-valued stochastic process $\{x_i\}$ is

- Independent iff $p(x_n|x_{0:n-1})=p(x_n)$
- Markov iff $p(x_n|x_{0:n-1})=p(x_n|x_{n-1})$
  - time-invariant iff $p(x_n=b|x_{n-1}=a) = p_{ab}$ indep of $n$
  - States
  - Transition matrix: $\mathbf{T} = \{t_{ab}\}$
    - Rows sum to 1: $\mathbf{T1} = \mathbf{1}$ where $\mathbf{1}$ is a vector of 1's
    - $\mathbf{p}_n = \mathbf{T}^T\mathbf{p}_{n-1}$
    - Stationary distribution: $\mathbf{p}_\$ = \mathbf{T}^T\mathbf{p}_\$$




Independent Stochastic Process is easiest to deal with, Markov is next easiest

# Stationary Markov Process

If a Markov process is

a) irreducible: you can go from any state $a$ to any $b$ in a finite number of steps

b) aperiodic: $\forall$ state $a$, the possible times to go from $a$ to $a$ have highest common factor = 1

then it has exactly one stationary distribution, $\mathbf{p}_\$$.

– $\mathbf{p}_\$$ is the eigenvector of $\mathbf{T}^T$ with $\lambda = 1$: $\mathbf{T}^T \mathbf{p}_\$ = \mathbf{p}_\$$

$\mathbf{T}^n \underset{n \to \infty}{\to} \mathbf{1}\mathbf{p}_\$^T$ where $\mathbf{1} = [1 \quad 1 \quad \cdots \quad 1]^T$

– Initial distribution becomes irrelevant (asymptotically stationary) $(\mathbf{T}^T)^n \mathbf{p}_0 = \mathbf{p}_\$ \mathbf{1}^T \mathbf{p}_0 = \mathbf{p}_\$, \quad \forall \mathbf{p}_0$

# Chess Board

$H(X) = \lim_{n\to\infty} H(X_n | X_{n-1})$

$= \lim_{n\to\infty} -\sum P(X_n, X_{n-1}) \log_2 P(X_n | X_{n-1})$

## Random Walk

$= -\sum P(X_{n-1}) P(X_n | X_{n-1}) \log_2 P(X_n | X_{n-1})$

$= -\sum P_{\$,i}\, t_{ij} \log_2(t_{ij})$

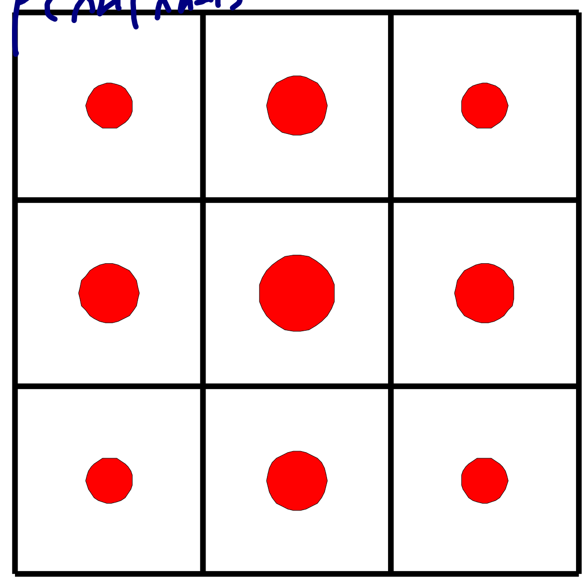
- Move ⬄ ⇳ ⬁ ⬀ ⬃ ⬂ equal prob

$P(X_n | X_{n-1}) = \dfrac{P(X_n, X_{n-1})}{P(X_{n-1})}$

- $\mathbf{p}_1 = [1\ 0\ \ldots\ 0]^T$
  - $H(\mathbf{p}_1) = 0$
- $\mathbf{p}_\$ = {}^1/_{40} \times [3\ 5\ 3\ 5\ 8\ 5\ 3\ 5\ 3]^T$
  - $H(\mathbf{p}_\$) = 3.0855$
- $H(X) = \lim_{n\to\infty} H(x_n | x_{n-1})$

$$= \lim_{n\to\infty} \sum -p(x_n, x_{n-1}) \log p(x_n | x_{n-1}) = \sum_{i,j} -p_{\$,i} t_{i,j} \log(t_{i,j}) = 2.2365$$

H($p_8$)=3.0827,    H($p_8$ | $p_7$)=2.23038



# Summary

- Relative Entropy: $D(\mathbf{p} \| \mathbf{q}) = E_{\mathbf{p}} \log \frac{p(x)}{q(x)} \geq 0$
  - $D(\mathbf{p}\|\mathbf{q}) = 0$ iff $\mathbf{p} \equiv \mathbf{q}$
- Corollaries
  - Uniform Bound: Uniform $\mathbf{p}$ maximizes $H(\mathbf{p})$
  - $I(x\,;\,y) \geq 0 \Rightarrow$ Conditioning reduces entropy
  - Indep bounds: $H(x_{1:n}) \leq \sum_{i=1}^{n} H(x_i)$ $\quad H(x_{1:n} \mid y_{1:n}) \leq \sum_{i=1}^{n} H(x_i \mid y_i)$

$$I(x_{1:n}; y_{1:n}) \geq \sum_{i=1}^{n} I(x_i; y_i) \quad \text{if } x_i \text{ or } y_i \text{ are indep}$$

- Entropy Rate of stochastic process:
  - $\{x_i\}$ stationary: $H(\mathrm{X}) = \lim_{n \to \infty} H(x_n \mid x_{1:n-1})$
  - $\{x_i\}$ stationary Markov:

$$H(\mathrm{X}) = H(x_n \mid x_{n-1}) = \sum_{i,j} -p_{\$,i} t_{i,j} \log(t_{i,j})$$

# Lecture 4

- Source Coding Theorem
  - $n$ i.i.d. random variables each with entropy $H$(X) can be compressed into more than $nH$(X) bits as $n$ tends to infinity
- Instantaneous Codes
  - Symbol-by-symbol coding
  - Uniquely decodable
- Kraft Inequality
  - Constraint on the code length
- Optimal Symbol Code lengths
  - Entropy Bound

# Source Coding

- Source Code: $C$ is a mapping $X \rightarrow D^+$
  - X a random variable of the message
  - $D^+$ = set of all finite length strings from D
  - D is often binary

    an extension contains all codewords derived from the basis set.
  - e.g. {E, F, G} $\rightarrow \{0,1\}^+$: $C$(E)=0, $C$(F)=10, $C$(G)=11
- Extension: $C^+$ is mapping $X^+ \rightarrow D^+$ formed by concatenating $C(x_i)$ without punctuation
  - e.g. $C^+$(EFEEGE) =01000110

# Desired Properties

- Non-singular: $x_1 \neq x_2 \Rightarrow C(x_1) \neq C(x_2)$
  - Unambiguous description of a single letter of X
- Uniquely Decodable: $C^+$ is non-singular

  extended message
  - The sequence $C^+(x^+)$ is unambiguous
  - A stronger condition
  - Any encoded string has only one possible source string producing it
  - However, one may have to examine the entire encoded string to determine even the first source symbol
  - One could use punctuation between two codewords but inefficient

# Instantaneous Codes


- Instantaneous (or Prefix) Code
  - No codeword is a prefix of another
  - Can be decoded instantaneously without reference to future codewords
- Instantaneous $\Rightarrow$ Uniquely Decodable $\Rightarrow$ Non-singular

Examples:

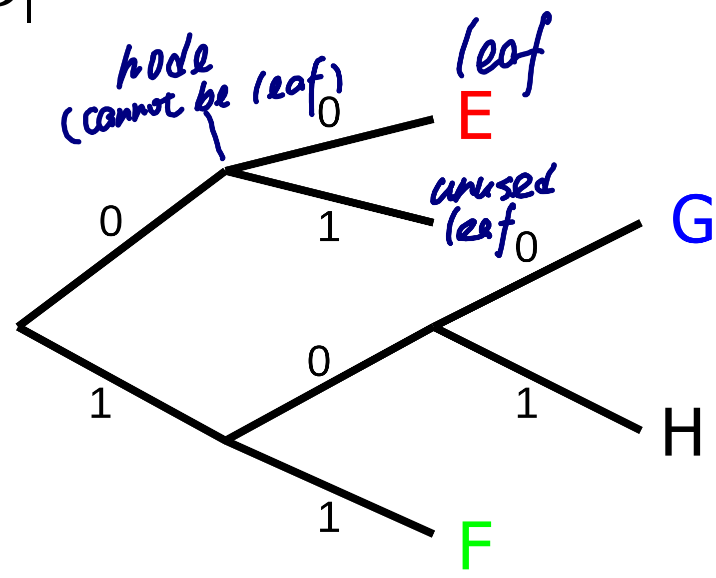
- $C(E,F,G,H) = (0, 1, 00, 11)$ u ✓ I ✗ $\bar{I}\bar{U}$
- $C(E,F) = (0, 101)$ u ✓ I ✓ $IU$
- $C(E,F) = (1, 101)$ u ✓ I ✗ $\bar{I}U$
- $C(E,F,G,H) = (00, 01, 10, 11)$ u ✓ I ✓ $IU$
- $C(E,F,G,H) = (0, 01, 011, 111)$ u ✓ I ✗ $\bar{I}U$

# Code Tree

Instantaneous code: $C$(E,F,G,H) = (00, 11, 100, 101)

Form a $D$-ary tree where $D = |\mathrm{D}|$

- $D$ branches at each node
- Each codeword is a leaf
- Each node along the path to a leaf is a prefix of the leaf
  $\Rightarrow$ can't be a leaf itself
- Some leaves may be unused




11101100000 $\rightarrow$ FHGEE

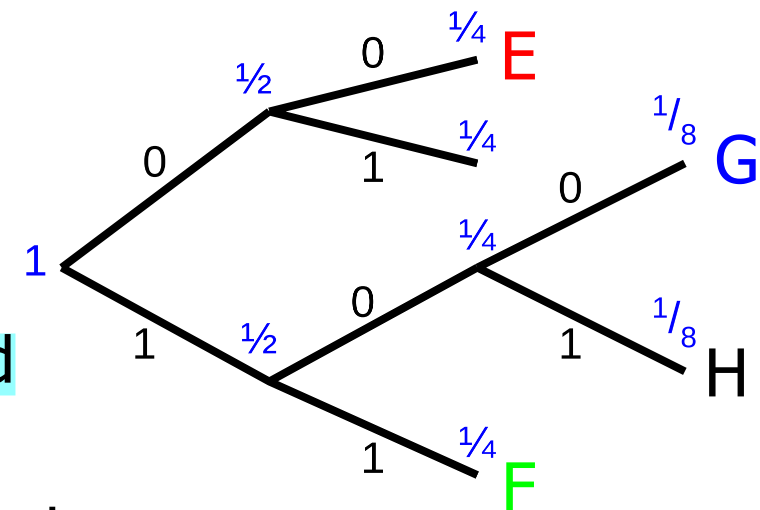
Kraft inequality (for instantaneous codes)

$\sum_{i=1}^{|X|} 2^{-l_i} \leq 1$ (budget)

# Kraft Inequality (instantaneous codes)

code cost $\propto \frac{1}{\text{code length}}$

- Limit on codeword lengths of instantaneous codes
  - Not all codewords can be too short
- Codeword lengths $l_1, l_2, \ldots, l_{|X|} \Rightarrow$ $\boxed{\sum_{i=1}^{|X|} 2^{-l_i} \leq 1}$
- Label each node at depth $l$ with $2^{-l}$
- Each node equals the sum of all its leaves
- Equality iff all leaves are utilised
- Total code budget = 1
  Code 00 uses up ¼ of the budget
  Code 100 uses up $^1/_8$ of the budget




Same argument works with D-ary tree

McMillan inequality (for uniquely decodable codewords)

$\sum_{i=1}^{|X|} D^{-l_i} \le 1$ (same with Kraft ineq.)

# McMillan Inequality (uniquely decodable codes)

$S^N = \left(\sum_{i=1}^{|X|} D^{-l_i}\right)^N = \sum_{i_1=1}^{|X|}\sum_{i_2=1}^{|X|}\cdots\sum_{i_N=1}^{|X|} D^{-(l_{i_1}+l_{i_2}+\cdots+l_{i_N})} = \sum_{x\in X} D^{-\text{length}(C^+(x))}$

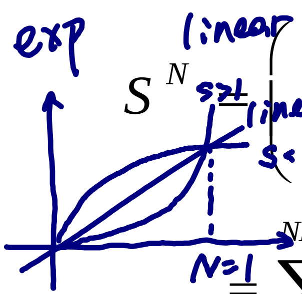
If uniquely decodable $C$ has codeword lengths

$l_1, l_2, \ldots, l_{|X|}$, then $\sum_{i=1}^{|X|} D^{-l_i} \le 1$ every codebook length sum

The same

Proof: Let $S = \sum_{i=1}^{|X|} D^{-l_i}$ and $\boxed{M = \max l_i}$ then for ~~any $N$~~ $N \ge 1$,

$S^N \le NM$ for $N \ge 1 \Rightarrow S \le 1$.




$$S^N = \left(\sum_{i=1}^{|X|} D^{-l_i}\right)^N = \sum_{i_1=1}^{|X|}\sum_{i_2=1}^{|X|}\cdots\sum_{i_N=1}^{|X|} D^{-\left(l_{i_1}+l_{i_2}+\ldots+l_{i_N}\right)} = \sum_{\mathbf{x}\in X^N} D^{-\text{length}\{C^+(\mathbf{x})\}}$$

sum over all possible codeword length

$$= \sum_{l=1}^{NM} D^{-l}\,|\mathbf{x} : l = \text{length}\{C^+(\mathbf{x})\}| \le \sum_{l=1}^{NM} D^{-l} D^{l} = \sum_{l=1}^{NM} 1 = NM$$

re-order sum by total length

Sum over all sequences of length $N$

If $S > 1$ then $S^N > NM$ for some $N$. Hence $S \le 1$.

max number of distinct sequences of length $l$

Implication: uniquely decodable codes doesn't offer further reduction of codeword lengths than instantaneous codes

# McMillan Inequality (uniquely decodable codes)

If uniquely decodable $C$ has codeword lengths

$l_1, l_2, \ldots, l_{|\mathcal{X}|}$ , then $\sum_{i=1}^{|\mathcal{X}|} D^{-l_i} \le 1$ The same

Proof: Let $S = \sum_{i=1}^{|\mathcal{X}|} D^{-l_i}$ and $M = \max l_i$ then for any $N$,

$$S^N = \left( \sum_{i=1}^{|\mathcal{X}|} D^{-l_i} \right)^N = \sum_{i_1=1}^{|\mathcal{X}|} \sum_{i_2=1}^{|\mathcal{X}|} \cdots \sum_{i_N=1}^{|\mathcal{X}|} D^{-\left(l_{i_1} + l_{i_2} + \ldots + l_{i_N}\right)} = \sum_{\mathbf{x} \in \mathcal{X}^N} D^{-\text{length}\{C^+(\mathbf{x})\}}$$

$$= \sum_{l=1}^{NM} D^{-l} \, |\mathbf{x} : l = \text{length}\{C^+(\mathbf{x})\}| \le \sum_{l=1}^{NM} D^{-l} D^{l} = \sum_{l=1}^{NM} 1 = NM$$

If $S > 1$ then $S^N > NM$ for some $N$. Hence $S \le 1$.

Implication: uniquely decodable codes doesn't offer further reduction of codeword lengths than instantaneous codes

# How Short are Optimal Codes?

If $l(x) = \text{length}(C(x))$ then $C$ is optimal if

$L = E\, l(x)$ is as small as possible.

We want to minimize $\sum_{x \in \mathcal{X}} p(x) l(x)$ subject to

1. $\sum_{x \in \mathcal{X}} D^{-l(x)} \le 1$
2. all the $l(x)$ are integers

Simplified version:

Ignore condition 2 and assume condition 1 is satisfied with equality.

optimistical mistakes

less restrictive so lengths may be shorter than actually possible $\Rightarrow$ lower bound

# Optimal Codes (non-integer $l_i$)

- Minimize $\sum_{i=1}^{|X|} p(x_i) l_i$ subject to $\sum_{i=1}^{|X|} D^{-l_i} = 1$

optimum at bound

$$D^{-l_i} = \frac{p(x_i)}{\lambda \ln D}$$

$$\underbrace{\sum_{i=1}^{|X|} D^{-l_i}}_{1} = \underbrace{\sum_{i=1}^{|X|} p(x_i)}_{1} \cdot \frac{1}{\lambda \ln D} \Rightarrow \lambda = \frac{1}{\ln D}$$

Use Lagrange multiplier:

Define $J = \sum_{i=1}^{|X|} p(x_i) l_i + \lambda \sum_{i=1}^{|X|} D^{-l_i}$ and set $\frac{\partial J}{\partial l_i} = 0$

$$\frac{\partial D^x}{\partial x} = \ln D \cdot D^x$$

$$\therefore D^{-l_i} = p(x_i)$$

$$\therefore l_i = -\log_D p(x_i) = H_D(X)$$

$$\frac{\partial J}{\partial l_i} = p(x_i) - \lambda \ln(D) D^{-l_i} = 0 \quad \Rightarrow \quad D^{-l_i} = p(x_i) / \lambda \ln(D)$$

also $\sum_{i=1}^{|X|} D^{-l_i} = 1 \quad \Rightarrow \quad \lambda = 1/\ln(D) \quad \Rightarrow \quad l_i = -\log_D(p(x_i))$

Information content
||
codeword length

$$E\, l(X) = E - \log_D(p(X)) = \frac{E - \log_2(p(X))}{\log_2 D} = \frac{H(X)}{\log_2 D} = H_D(X)$$

entropy of $x$ base $D$.

no uniquely decodable code can do better than this

# Bounds on Optimal Code Length

Round up optimal code lengths: $l_i = \lceil -\log_D p(x_i) \rceil$

- $l_i$ are bound to satisfy the Kraft Inequality (since the optimum lengths do) $\sum D^{-l_i} \le 1$ ? $\lceil l_i \rceil$ ✓ $\lfloor l_i \rfloor$ × $l_i$ ×
- For this choice, $-\log_D(p(x_i)) \le l_i \le -\log_D(p(x_i)) + 1$
- Average shortest length:

$$\boxed{H_D(x) \le L^* < H_D(x) + 1}$$

(since we added <1 to optimum values)

- We can do better by encoding blocks of $n$ symbols

normalise the length

$$n^{-1} H_D(x_{1:n}) \le n^{-1} E\, l(x_{1:n}) \le n^{-1} H_D(x_{1:n}) + n^{-1}$$

$n \nearrow \Rightarrow$ tighter bound $\Rightarrow$ avg. symbol length $\rightarrow$ entropy.

- If entropy rate of $x_i$ exists ($\Leftarrow x_i$ is stationary process)

$$n^{-1} H_D(x_{1:n}) \rightarrow H_D(\mathrm{X}) \quad \Rightarrow \quad n^{-1} E\, l(x_{1:n}) \rightarrow H_D(\mathrm{X})$$

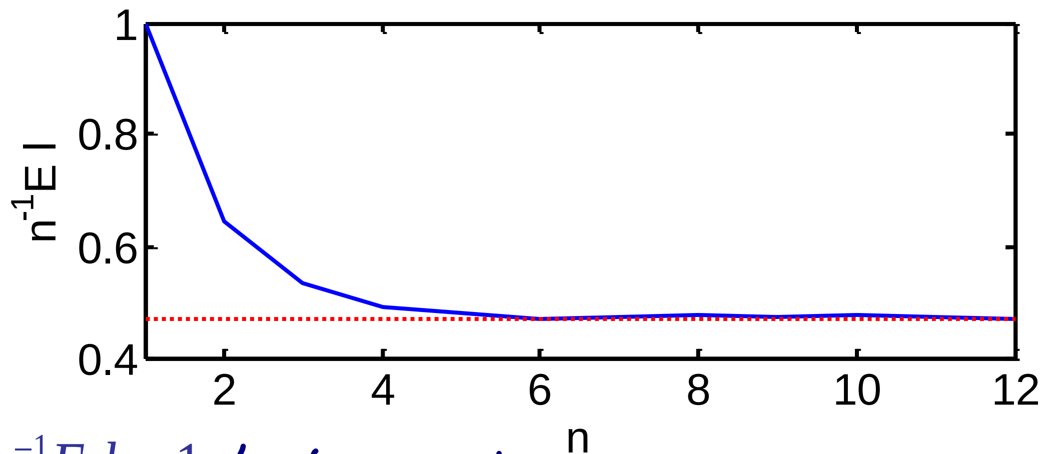
Also known as source coding theorem

# Block Coding Example

X = [A;B], $\mathbf{p}_x$ = [0.9; 0.1]

$H(x_i) = 0.469$

Huffman coding:




$n^{-1}El = \frac{0.9\times1+0.1\times1}{1} = 1$

- $n$=1

| sym | A | B |
|---|---|---|
| prob | 0.9 | 0.1 |
| code | 0 | 1 |

$n^{-1}E\,l = 1$ block length $n$

$n^{-1}El = \frac{0.81\times1+0.09\times2+0.1\times3}{2} = 0.645$

$\Rightarrow$ avg. symbol length $\rightarrow$ entropy

$n^{-1}El$ (longer block, less uncertainty)

- $n$=2

| sym | AA | AB | BA | BB |
|---|---|---|---|---|
| prob | 0.81 | 0.09 | 0.09 | 0.01 |
| code | 0 | 11 | 100 | 101 |

$n^{-1}E\,l = 0.645$

- $n$=3

| sym | AAA | AAB | … | BBA | BBB |
|---|---|---|---|---|---|
| prob | 0.729 | 0.081 | … | 0.009 | 0.001 |
| code | 0 | 101 | … | 10010 | 10011 |

$n^{-1}E\,l = 0.583$

The extra 1 bit inefficiency becomes insignificant for large blocks

# Summary

- McMillan Inequality for D-ary codes:
  - any uniquely decodable C has $\sum_{i=1}^{|\mathcal{X}|} D^{-l_i} \leq 1$
- Any uniquely decodable code:

$$E\, l(x) \geq H_D(x)$$

- Source coding theorem
  - Symbol-by-symbol encoding

$$H_D(x) \leq E\, l(x) \leq H_D(x) + 1$$

  - Block encoding $n^{-1} E\, l(x_{1:n}) \to H_D(\mathcal{X})$

# Lecture 5

- Source Coding Algorithms
- Huffman Coding
- Lempel-Ziv Coding

# Huffman Code

An optimal binary instantaneous code must satisfy:

1. $p(x_i) > p(x_j) \quad \Rightarrow \quad l_i \leq l_j$ (else swap codewords)
2. The two longest codewords have the same length (else chop a bit off the longer codeword)

3. $\exists$ two longest codewords differing only in the last bit (else chop a bit off all of them)

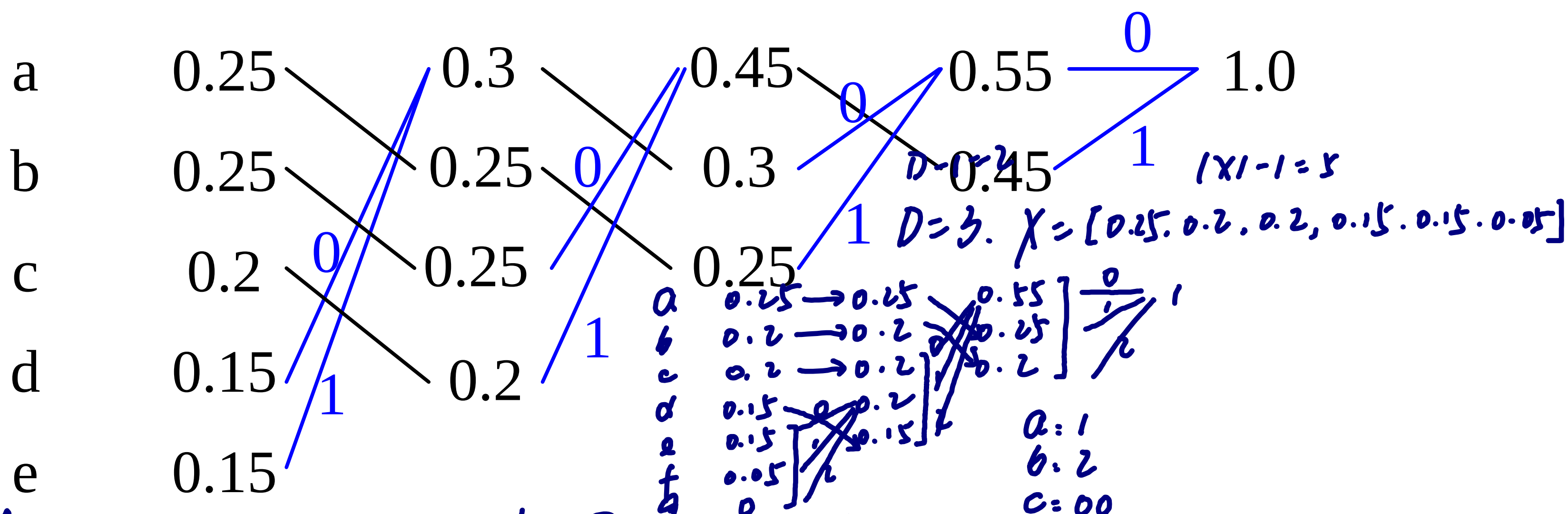
Huffman Code construction

1. Take the two smallest $p(x_i)$ and assign each a different last bit. Then merge into a single symbol.
2. Repeat step 1 until only one symbol remains

Used in JPEG, MP3…

# Huffman Code Example

X = [a, b, c, d, e], $p_X$ = [0.25  0.25  0.2  0.15  0.15]




① go from symbol to the end ② read backwards

Read diagram backwards for codewords:

a: 1
b: 2
c: 00
d: 02
e: 010
f: 011
g: 012 (X)

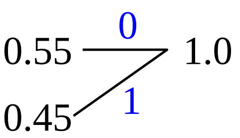
$C(X)$ = [01  10  11  000  001], $L$ = 2.3, $H(X)$ = 2.286

For D-ary code, first add extra zero-probability symbols until $|X|-1$ is a multiple of $D-1$ and then group D symbols at a time

# Huffman Code is Optimal Instantaneous Code

Huffman traceback gives codes for progressively larger alphabets:

a 0.55 0 1 a:0
b 0.45 1 b:1
$L_2 = 0.55 + 0.45 = 1$




$\mathbf{p}_2$=[0.55 0.45],

$\mathbf{c}_2$=[0 1], $L_2$=1

a 0.45 0.55 0 1 a: 1
b 0.3 0 0.45 1 b: 00
c 0.25 1 c: 01
$L_3 = 0.45 \times 1 + 0.55 \times 2 = 1.55$

$\mathbf{p}_3$=[0.45 0.3 0.25],

$\mathbf{c}_3$=[1 00 01], $L_3$=1.55

a 0.3 0.45 0.55 0 1 a: 00
b 0.25 0.3 0 0.45 1 b: 01
c 0.25 0 0.25 1 c: 10
d 0.2 1 d: 11
$L_4 = 2$

$\mathbf{p}_4$=[0.3 0.25 0.25 0.2],

$\mathbf{c}_4$=[00 01 10 11], $L_4$=2

a 0.25 0.3 0.45 0.55 0 1 a: 01
b 0.25 0.25 0.3 0 0.45 1 b: 10
c 0.2 0.25 0 0.25 1 c: 11
d 0.15 0 0.2 1 d: 000
e 0.15 1 e: 001

$\mathbf{p}_5$=[0.25 0.25 0.2 0.15 0.15],

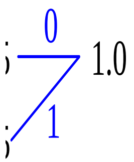
$\mathbf{c}_5$=[01 10 11 000 001], $L_5$=2.3

$L_5 = 2 \times 0.7 + 3 \times 0.3 = 2.3$

We want to show that all these codes are optimal including $C_5$

# Huffman Code is Optimal Instantaneous Code

Huffman traceback gives codes for progressively larger alphabets:




$\mathbf{p}_2$=[0.55 0.45],

$\mathbf{c}_2$=[0 1], $L_2$=1

$\mathbf{p}_3$=[0.45 0.3 0.25],

$\mathbf{c}_3$=[1 00 01], $L_3$=1.55

$\mathbf{p}_4$=[0.3 0.25 0.25 0.2],

$\mathbf{c}_4$=[00 01 10 11], $L_4$=2

$\mathbf{p}_5$=[0.25 0.25 0.2 0.15 0.15],

$\mathbf{c}_5$=[01 10 11 000 001], $L_5$=2.3

We want to show that all these codes are optimal including $C_5$

# Huffman Optimality Proof

Suppose one of these codes is sub-optimal:

- $\exists\ m>2$ with $\mathbf{c}_m$ the first sub-optimal code (note $\mathbf{c}_2$ is definitely optimal)
- An optimal $\mathbf{c}'_m$ must have $L_{C'm} < L_{Cm}$
- Rearrange the symbols with longest codes in $\mathbf{c}'_m$ so the two lowest probs $p_i$ and $p_j$ differ only in the last digit (doesen't change optimality)
- Merge $x_i$ and $x_j$ to create a new code $\mathbf{c}'_{m-1}$ as in Huffman procedure
- $L_{C'm-1} = L_{C'm} - p_i - p_j$ since identical except 1 bit shorter with prob $p_i + p_j$
- But also $L_{Cm-1} = L_{Cm} - p_i - p_j$ hence $L_{C'm-1} < L_{Cm-1}$ which contradicts assumption that $\mathbf{c}_m$ is the first sub-optimal code

Hence, Huffman coding satisfies $H_D(x) \le L < H_D(x) + 1$

Note: Huffman is just one out of many possible optimal codes

# Shannon-Fano Code

Fano code    Fano: split

1. Put probabilities in decreasing order
2. Split as close to 50-50 as possible; repeat with each half

source of information loss

largest probability: shortest. all zeros

| | | | | | | |
|---|---|---|---|---|---|---|
| a | 0.20 | 0 | 0 | | | 00 |
| b | 0.19 | 0 | 1 | 0 | | 010 |
| c | 0.17 | 0 | 1 | 1 | | 011 |
| d | 0.15 | 1 | 0 | 0 | | 100 |
| e | 0.14 | 1 | 0 | 1 | | 101 |
| f | 0.06 | 1 | 1 | 0 | | 110 |
| g | 0.05 | 1 | 1 | 1 | 0 | 1110 |
| h | 0.04 | 1 | 1 | 1 | 1 | 1111 |

smallest probability. longest. all ones.

$H(\boldsymbol{x}) = 2.81$ bits

$L_{SF} = 2.89$ bits

Not necessarily optimal: the best code for this $\mathbf{p}$ actually has $L = 2.85$ bits

# Shannon versus Huffman

## Shannon

$$F_i = \sum_{k=1}^{i-1} p(x_k), \qquad p(x_1) \ge p(x_2) \ge \cdots \ge p(x_m)$$

encoding : round the number $F_i \in [0,1]$ to $\lceil -\log p(x_i) \rceil$ bits

$$H_D(x) \le L_{SF} \le H_D(x) + 1 \qquad \text{(excercise)}$$

Shannon: $\lceil -\log p(x_i) \rceil$

$$\mathbf{p}_x = [0.36 \quad 0.34 \quad 0.25 \quad 0.05] \quad \Rightarrow \quad H(x) = 1.78 \text{ bits}$$

$$-\log_2 \mathbf{p}_x = [1.47 \quad 1.56 \quad 2 \quad 4.32]$$

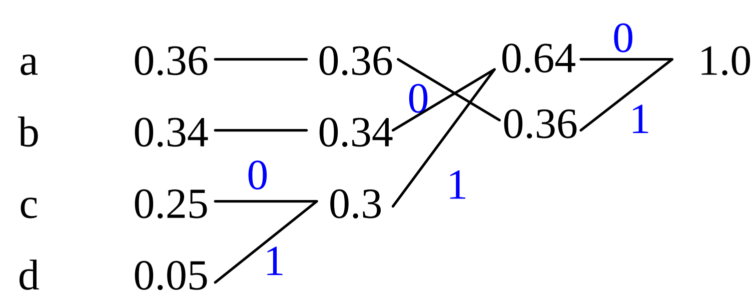
$$\mathbf{l}_S = \lceil -\log_2 \mathbf{p}_x \rceil = [2 \quad 2 \quad 2 \quad 5]$$

$$L_S = 2.15 \text{ bits}$$

## Huffman

$$\mathbf{l}_H = [1 \quad 2 \quad 3 \quad 3]$$

$$L_H = 1.94 \text{ bits}$$




Individual codewords may be longer in Huffman than Shannon but not the average

# Issues with Huffman Coding

- Requires the probability distribution of the source
  - Must recompute entire code if any symbol probability changes
    - A block of $N$ symbols needs $|X|^N$ pre-calculated probabilities
- For many practical applications, however, the underlying probability distribution is unknown
  - Estimate the distribution
    - Arithmetic coding: extension of Shannon-Fano coding; can deal with large block lengths
  - Without the distribution
    - Universal coding: Lempel-Ziv coding

# Universal Coding

- Does not depend on the distribution of the source
- Compression of an individual sequence
- Run length coding
  - Runs of data are stored (e.g., in fax machines)

    Example: WWWWWWWWWBBWWWWWWWBBBBBBWW

    white black

    9W2B7W6B2W
- Lempel-Ziv coding encode strings into phrases
  - Generalization that takes advantage of runs of strings of characters (such as WWWWWWWWWBB)
  - Adaptive dictionary compression algorithms
  - Asymptotically optimum: achieves the entropy rate for any stationary ergodic source

# Lempel-Ziv Coding (LZ78)

Memorize previously occurring substrings in the input data

– parse input into the shortest possible distinct 'phrases', i.e., each phrase is the shortest phrase not seen earlier

| phrases | # | codewords (head location + tail) |
|---|---|---|
| A | 1 | 0A |
| B | 2 | 0B |
| AA | 3 | 1A |
| BA | 4 | 2A |
| BAB | 5 | 4B |
| BB | 6 | 2B |
| AB | 7 | 1B |

– number the phrases starting from 1 (0 is the empty string)

ABAABABABBBAB… new phrase = old phrase (head) + tail

12 3 4 5 6 7

Look up a dictionary

– each phrase consists of a previously occurring phrase (head) followed by an additional A or B (tail)

– encoding: give location of head followed by the additional symbol for tail

0A0B1A2A4B2B1B…

– decoder uses an identical dictionary

locations are underlined

# Lempel-Ziv Example

Input = 1011010100010010001001010010

| Dictionary | | Send | Decode |
|---|---|---|---|
| 0000 | φ | 1 | 1 |
| 0001 | 1 | 00 | 0 |
| 0010 | 0 | 011 | 11 |
| 0011 | 11 | 101 | 01 |
| 0100 | 01 | 1000 | 010 |
| 0101 | 010 | 0100 | 00 |
| 0110 | 00 | 0010 | 10 |
| 0111 | 10 | 1010 | 0100 |
| 1000 | 0100 | 10001 | 01001 |
| 1001 | 01001 | 10010 | 010010 |

location

No need to always send 4 bits

Remark:

- No need to send the dictionary (imagine zip and unzip!)
- Can be reconstructed
- Need to send 0's in 01, 010 and 001 to avoid ambiguity (i.e., instantaneous code)

# Lempel-Ziv Comments

Dictionary $D$ contains $K$ entries $D(0), \ldots, D(K-1)$. We need to send $M$=ceil(log $K$) bits to specify a dictionary entry. Initially $K$=1, $D(0)= \phi$ = null string and $M$=ceil(log $K$) = 0 bits.

| Input | Action |
|---|---|
| 1 | "1" $\notin D$ so send "1" and set $D(1)$="1". Now $K=2 \Rightarrow M=1$. |
| 0 | "0" $\notin D$ so split it up as "$\phi$"+"0" and send location "0" (since $D(0)= \phi$) followed by "0". Then set $D(2)$="0" making $K=3 \Rightarrow M=2$. |
| 1 | "1" $\in D$ so don't send anything yet – just read the next input bit. |
| 1 | "11" $\notin D$ so split it up as "1" + "1" and send location "01" (since $D(1)$= "1" and $M$=2) followed by "1". Then set $D(3)$="11" making $K=4 \Rightarrow M=2$. |
| 0 | "0" $\in D$ so don't send anything yet – just read the next input bit. |
| 1 | "01" $\notin D$ so split it up as "0" + "1" and send location "10" (since $D(2)$= "0" and $M$=2) followed by "1". Then set $D(4)$="01" making $K=5 \Rightarrow M=3$. |
| 0 | "0" $\in D$ so don't send anything yet – just read the next input bit. |
| 1 | "01" $\in D$ so don't send anything yet – just read the next input bit. |
| 0 | "010" $\notin D$ so split it up as "01" + "0" and send location "100" (since $D(4)$= "01" and $M$=3) followed by "0". Then set $D(5)$="010" making $K=6 \Rightarrow M=3$. |

So far we have sent 1000111011000 where dictionary entry numbers are in red.

# Lempel-Ziv Properties

- Simple to implement
- Widely used because of its speed and efficiency
  - applications: compress, gzip, GIF, TIFF, modem …
  - variations: LZW (considering last character of the current phrase as part of the next phrase, used in Adobe Acrobat), LZ77 (sliding window)
  - different dictionary handling, etc
- Excellent compression in practice
  - many files contain repetitive sequences
  - worse than arithmetic coding for text files

# Asymptotic Optimality

- Asymptotically optimum for stationary ergodic source (i.e. achieves entropy rate)
- Let $c(n)$ denote the number of phrases for a sequence of length $n$
- Compressed sequence consists of $c(n)$ pairs (location, last bit)
- Needs $\boxed{c(n)[\log c(n)+1]}$ bits in total
- $\{X_i\}$ stationary ergodic $\Rightarrow$

$$\limsup_{n\to\infty} n^{-1} l(X_{1:n}) = \limsup_{n\to\infty} \frac{c(n)[\log c(n)+1]}{n} \leq H(\mathrm{X}) \text{ with probability } 1$$

  - Proof: C&T chapter 12.10
  - may only approach this for an enormous file

# Summary

- Huffman Coding: $H_D(x) \le E\, l(x) \le H_D(x) + 1$
  - Bottom-up design
  - Optimal $\Rightarrow$ shortest average length
- Shannon-Fano Coding: $H_D(x) \le E\, l(x) \le H_D(x) + 1$
  - Intuitively natural top-down design
- Lempel-Ziv Coding
  - Does not require probability distribution
  - Asymptotically optimum for stationary ergodic source (i.e. achieves entropy rate)

# Lecture 6

- Markov Chains
  - Have a special meaning
  - Not to be confused with the standard definition of Markov chains (which are sequences of discrete random variables)
- Data Processing Theorem
  - You can't create information from nothing
- Fano's Inequality
  - Lower bound for error in estimating $X$ from $Y$

# Markov Chains

If we have three random variables: $x$, $y$, $z$

$$p(x,y,z) = p(z \mid x,y)\,p(y \mid x)\,p(x)$$

they form a Markov chain $x \to y \to z$ if

$$p(z \mid x,y) = p(z \mid y) \Leftrightarrow p(x,y,z) = p(z \mid y)\,p(y \mid x)\,p(x)$$

y: agent. translator.

A Markov chain $x \to y \to z$ means that

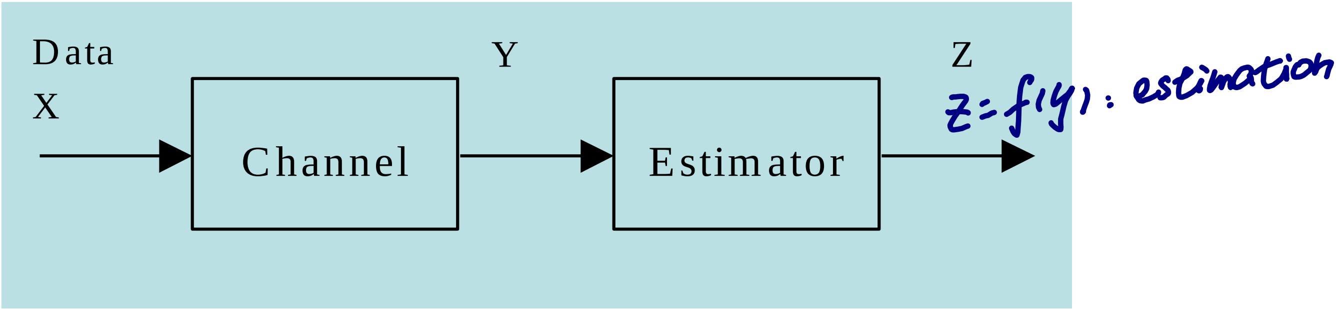
- the only way that $x$ affects $z$ is through the value of $y$
- if you already know $y$, then observing $x$ gives you no additional information about $z$, i.e. $I(x;z \mid y) = 0 \Leftrightarrow H(z \mid y) = H(z \mid x,y)$
- if you know $y$, then observing $z$ gives you no additional information about $x$.

$H(x \mid y) = H(x \mid y,z)$

$I(x;z \mid y) = H(z \mid y) - H(z \mid x,y)$

# Data Processing

- Estimate $z = f(y)$, where $f$ is a function
- A special case of a Markov chain $x \rightarrow y \rightarrow f(y)$




- Does processing of $y$ increase the information that $y$ contains about $x$? No. even less!

# Markov Chain Symmetry

$x \to y \to z$ MC $\Rightarrow$ $z \to y \to x$ MC

If $x \to y \to z$

$$p(x,z\,|\,y) = \frac{p(x,y,z)}{p(y)} \overset{(a)}{=} \frac{p(x,y)\,\underset{}{\overset{p(z|x,y)}{\overset{\shortparallel}{p(z\,|\,y)}}}}{p(y)} = p(x\,|\,y)\,p(z\,|\,y)$$

given $y$, $x$, $z$ are independent.

(a) $p(z\,|\,x,y) = p(z\,|\,y)$

Hence $x$ and $z$ are conditionally independent given $y$

$p(z|y) = \frac{p(y,z)}{p(y)}$

Also $x \to y \to z$ iff $z \to y \to x$ since

$p(x|y)p(z|y) = p(x,z|y)$

$$p(x\,|\,y) = p(x\,|\,y)\frac{p(z\,|\,y)\,p(y)}{p(y,z)} \overset{(a)}{=} \frac{p(x,z\,|\,y)\,p(y)}{p(y,z)} = \frac{p(x,y,z)}{p(y,z)}$$

$$= p(x\,|\,y,z)$$

given $y$, $z$ does not provide additional infomation

(a) $p(x,z\,|\,y) = p(x\,|\,y)\,p(z\,|\,y)$

Conditionally indep.

Markov chain property is symmetrical

# Data Processing Theorem

If $x \to y \to z$ then $\boxed{I(x;y) \ge I(x;z)}$ extract useful info do not lose mutual info

(even decrease)

– processing $y$ cannot add new information about $x$

If $x \to y \to z$ then $\boxed{I(x;y) \ge I(x;y \mid z)}$

– Knowing $z$ does not increase the amount $y$ tells you about $x$

(even less)

Proof:

Apply chain rule in different ways

$$I(\underline{x};\underline{y,z}) = I(x;y) + \underbrace{I(x;z \mid y)}_{=0} = I(x;z) + I(x;y \mid z)$$

(a)

$x \to y$: observation

$y \to z$: data processing

but $I(x;z \mid y) = 0$

hence $I(x;y) = I(x;z) + I(x;y \mid z)$

so $I(x;y) \ge I(x;z)$ and $I(x;y) \ge I(x;y \mid z)$

(a) $I(x;z)=0$ iff $x$ and $z$ are independent; Markov $\Rightarrow p(x,z \mid y) = p(x \mid y)p(z \mid y)$

# So Why Processing?

- One can not create information by manipulating the data
- But no information is lost if equality holds
- Sufficient statistic
  - z contains all the information in y about x
  - Preserves mutual information $I(x\,;y) = I(x\,;\,z)$
- The estimator should be designed in a way such that it outputs sufficient statistics
- Can the estimation be arbitrarily accurate?

- what if $|X| \to \inf$?

# Fano's Inequality

Pe has a lower bound.

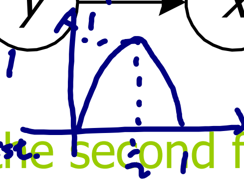
If we estimate *x* from *y*, what is $p_e = p(\hat{x} \neq x)$ ?

$$x \longrightarrow y \xrightarrow{H(P_e)} \hat{x}$$

$$H(x \mid y) \le H(p_e) + p_e \log |\mathcal{X}|$$

$H(P_e) = -P_e \log P_e - (1-P_e)\log(1-P_e) \le 1$

$=1$ when uniform dist.



$$\Rightarrow p_e \ge \frac{(H(x \mid y) - H(p_e))}{\log |\mathcal{X}|} \overset{(a)}{\ge} \frac{(H(x \mid y) - 1)}{\log |\mathcal{X}|}$$

(a) the second form is weaker but easier to use

$P_e \sim H(x \mid y)$

Proof: Define a random variable $e = \begin{cases} 1 & \hat{x} \neq x \ (P_e) \\ 0 & \hat{x} = x \ (1-P_e) \end{cases}$

$$H(e, x \mid \hat{x}) = H(x \mid \hat{x}) + \underline{H(e \mid x, \hat{x})} = H(e \mid \hat{x}) + H(x \mid e, \hat{x})$$ chain rule

no entropy if $x, \hat{x}$ are known

$$\Rightarrow H(x \mid \hat{x}) + 0 \le H(e) + H(x \mid e, \hat{x})$$ remove restraint $H \ge 0$; $H(e \mid y) \le H(e)$

$$= H(e) + \underline{H(x \mid \hat{x}, e=0)}(1-p_e) + \underline{H(x \mid \hat{x}, e=1)}\, p_e$$

no entropy if no error at all $(x = \hat{x})$

$x \neq \hat{x}$. still uncertainty $\le \log(|X|-1)$

$$\le H(p_e) + 0 \times (1-p_e) + p_e \log |\mathcal{X}|$$ $H(e) = H(p_e)$

$$H(x \mid y) \le H(x \mid \hat{x}) \ \text{since} \ I(x;\hat{x}) \le I(x;y)$$ Markov chain

(e=1: at least 1 error)

# Implications

$P_e \geq \frac{H(Y|X) - H(P_e)}{\log(|X|-1)} \geq \frac{H(Y|X) - 1}{\log|X|}$

- Zero probability of error $p_e = 0 \Rightarrow H(x \mid y) = 0$
- Low probability of error if $H(x|y)$ is small
- If $H(x|y)$ is large then the probability of error is high
- Could be slightly strengthened to

$$H(x \mid y) \leq H(p_e) + p_e \log(|\mathcal{X}| - 1)$$

- Fano's inequality is used whenever you need to show that errors are inevitable
  - E.g., Converse to channel coding theorem

MAP (maximum a posteriori prob)

$x \to y \to \hat{x}$

$\hat{x} = \arg\max_x P(x|y) = \begin{cases} 1, & y=1 \\ 2, & y=2 \end{cases}$

| $x \backslash y$ | 1 | 2 |
|---|---|---|
| 1 | 0.3 | 0.05 |
| 2 | 0.05 | 0.3 |
| 3 | 0.05 | 0.05 |
| 4 | 0.05 | 0.05 |
| 5 | 0.05 | 0.05 |

choose $\hat{x} = y$

$P_e = 1 - 0.6 = 0.4$

(select largest prob. in col.)

# Fano Example

$X = \{1:5\}$, $\mathbf{p}_X = [0.35, 0.35, 0.1, 0.1, 0.1]^T$

$Y = \{1:2\}$ if $x \le 2$ then $y = x$ with probability 6/7 while if $x > 2$ then $y = 1$ or $2$ with equal prob.

Our best strategy is to guess $\hat{x} = y \quad (x \to y \to \hat{x})$

– $\mathbf{p}_{x|y=1} = [0.6, 0.1, 0.1, 0.1, 0.1]^T$

– actual error prob: $p_e = 0.4$

$H(y|x) = -\sum_{x,y} P(x,y)\log_2 P(y|x) = -0.3\log_2\frac{6}{7} \times 2 - 0.05\log_2\frac{1}{7} \times 2 - 0.05\log_2\frac{1}{2} \times 6 = 0.714$ bits

$H(x|y) = -\sum_{x,y} P(x,y)\log_2 P(x|y) = -0.3\log_2 0.6 \times 2 - 0.05\log_2 0.1 \times 8 = 1.771$ bits

Fano bound: $p_e \ge \frac{H(x|y)-1}{\log(|X|-1)} = \frac{1.771-1}{\log(4)} = 0.3855$ (exercise)

Main use: to show when error free transmission is impossible since $p_e > 0$

# Summary

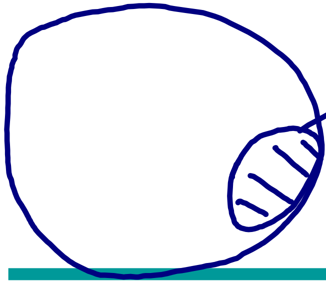
- Markov: $x \to y \to z \Leftrightarrow p(z \mid x, y) = p(z \mid y) \Leftrightarrow I(x; z \mid y) = 0$
- Data Processing Theorem: if $x \to y \to z$ then
  - $I(x\,;\,y) \geq I(x\,;\,z),\ I(y\,;\,z) \geq I(x\,;\,z)$
  - $I(x\,;\,y) \geq I(x\,;\,y \mid z)$ can be false if not Markov
  - Long Markov chains: If $x_1 \to x_2 \to x_3 \to x_4 \to x_5 \to x_6,$ then Mutual Information increases as you get closer together:
    - e.g. $I(x_3, x_4) \geq I(x_2, x_4) \geq I(x_1, x_5) \geq I(x_1, x_6)$
- Fano's Inequality: if $x \to y \to \hat{x}$ then

$$p_e \geq \frac{H(x \mid y) - H(p_e)}{\log\left(|\mathcal{X}| - 1\right)} \geq \frac{H(x \mid y) - 1}{\log\left(|\mathcal{X}| - 1\right)} \geq \frac{H(x \mid y) - 1}{\log |\mathcal{X}|}$$

weaker but easier to use since independent of $p_e$

# Lecture 7

- Law of Large Numbers
  - Sample mean is close to expected value
- Asymptotic Equipartition Principle (AEP)
  - $-\log P(x_1, x_2, \ldots, x_n)/n$ is close to entropy $H$
- The Typical Set
  - Probability of each sequence close to $2^{-nH}$
  - Size ($\sim 2^{nH}$) and total probability ($\sim 1$)
- The Atypical Set
  - Unimportant and could be ignored



typical set

# Typicality: Example

typical: $\log p(x) = nH(x)$

not typical: $\log p(x) \neq nH(p)$

X = {a, b, c, d}, $\mathbf{p} = [0.5\ 0.25\ 0.125\ 0.125]$

$-\log \mathbf{p} = [1\ 2\ 3\ 3] \Rightarrow H(\mathbf{p}) = 1.75$ bits

Sample eight i.i.d. values

- typical $\Rightarrow$ correct proportions

  adbabaac $-\log p(\mathbf{x}) = 14 = 8\times 1.75 = nH(\mathbf{x})$

- not typical $\Rightarrow \log p(\mathbf{x}) \neq nH(\mathbf{x})$

  dddddddd $-\log p(\mathbf{x}) = 24$

# Convergence of Random Variables

- Convergence

$$x_n \xrightarrow[n\to\infty]{} y \quad \Rightarrow \quad \forall \varepsilon > 0, \exists m \text{ such that } \forall n > m, |x_n - y| < \varepsilon$$

Example: $x_n = \pm 2^{-n}, \quad y = 0$

choose $m = -\log \varepsilon$

- Convergence in probability (weaker than convergence)

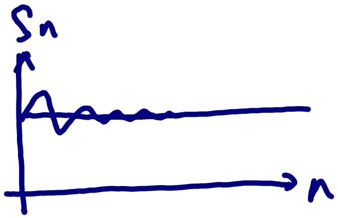
$$x_n \xrightarrow{\text{prob}} y \quad \Rightarrow \quad \forall \varepsilon > 0, \quad P(|x_n - y| > \varepsilon) \to 0$$

Example: $x_n \in \{0;1\}, \quad p = [1-n^{-1}; n^{-1}]$

for any small $\varepsilon$, $p(|x_n| > \varepsilon) = n^{-1} \xrightarrow{n\to\infty} 0$

so $x_n \xrightarrow{\text{prob}} 0$ (but $x_n \not\to 0$)

Note: $y$ can be a constant or another random variable




# Law of Large Numbers

Given i.i.d. $\{x_i\}$, sample mean $s_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

– $E\ s_n = E\ x = \mu$ $\quad$ $\mathrm{Var}\ s_n = n^{-1}\ \mathrm{Var}\ x = n^{-1}\sigma^2$

As $n$ increases, $\mathrm{Var}\ s_n$ gets smaller and the values become clustered around the mean

LLN: $\quad s_n \xrightarrow{\text{prob}} \mu$

$$\Leftrightarrow \quad \forall \varepsilon > 0, \quad P(|s_n - \mu| > \varepsilon) \underset{n\to\infty}{\to} 0$$

The expected value of a random variable is equal to the long-term average when sampling repeatedly.

# Asymptotic Equipartition Principle

- $\mathbf{x}$ is the i.i.d. sequence $\{x_i\}$ for $1 \le i \le n$
  - Prob of a particular sequence is $p(\mathbf{x}) = \prod_{i=1}^{n} p(x_i)$
  - Average $E - \log\ p(\mathbf{x}) = n\ E - \log\ p(x_i) = nH(x)$
- AEP: average SIC of a certain content

$$-\frac{1}{n}\log p(\mathbf{x}) \xrightarrow{\text{prob}} H(x)$$

deterministic

- Proof:

$$-\frac{1}{n}\log p(\mathbf{x}) = -\frac{1}{n}\sum_{i=1}^{n}\log p(x_i)$$

long-term average

$n \uparrow$

mean

law of large numbers $\xrightarrow{\text{prob}}$ $E - \log p(x_i) = H(x)$

$N=1$ $\log 0.2$
$\log 0.8$

# Typical Set

$|-\frac{1}{n}\log P(x)-H(X)|\le\varepsilon$

$|-\log P(x)-nH(X)|\le n\varepsilon$

$\log P(x)=nH(X)\pm n\varepsilon$

## Typical set (for finite n)

size $n$: sequence length

\# i.i.d.

$$T_\varepsilon^{(n)}=\left\{\mathbf{x}\in \mathrm{X}^n : \left|-n^{-1}\log\ p(\mathbf{x})-H(X)\right|<\varepsilon\right\}$$

$\leftarrow$ deviation

typical set: average SIC close to the entropy divided by $n$.

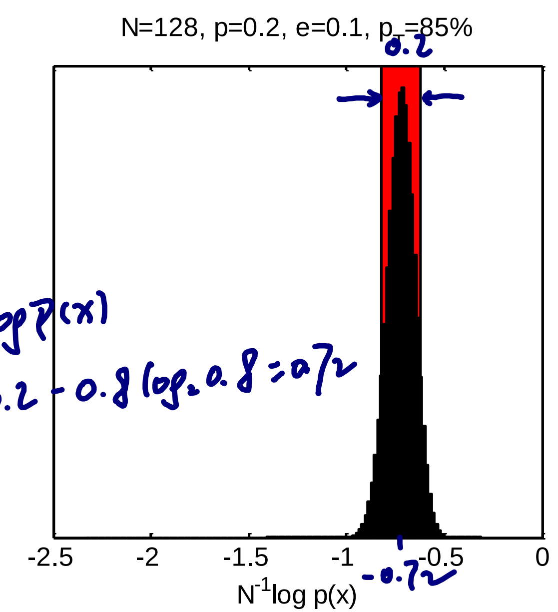
$n\uparrow \Rightarrow$ typicality $\uparrow$

## Example:

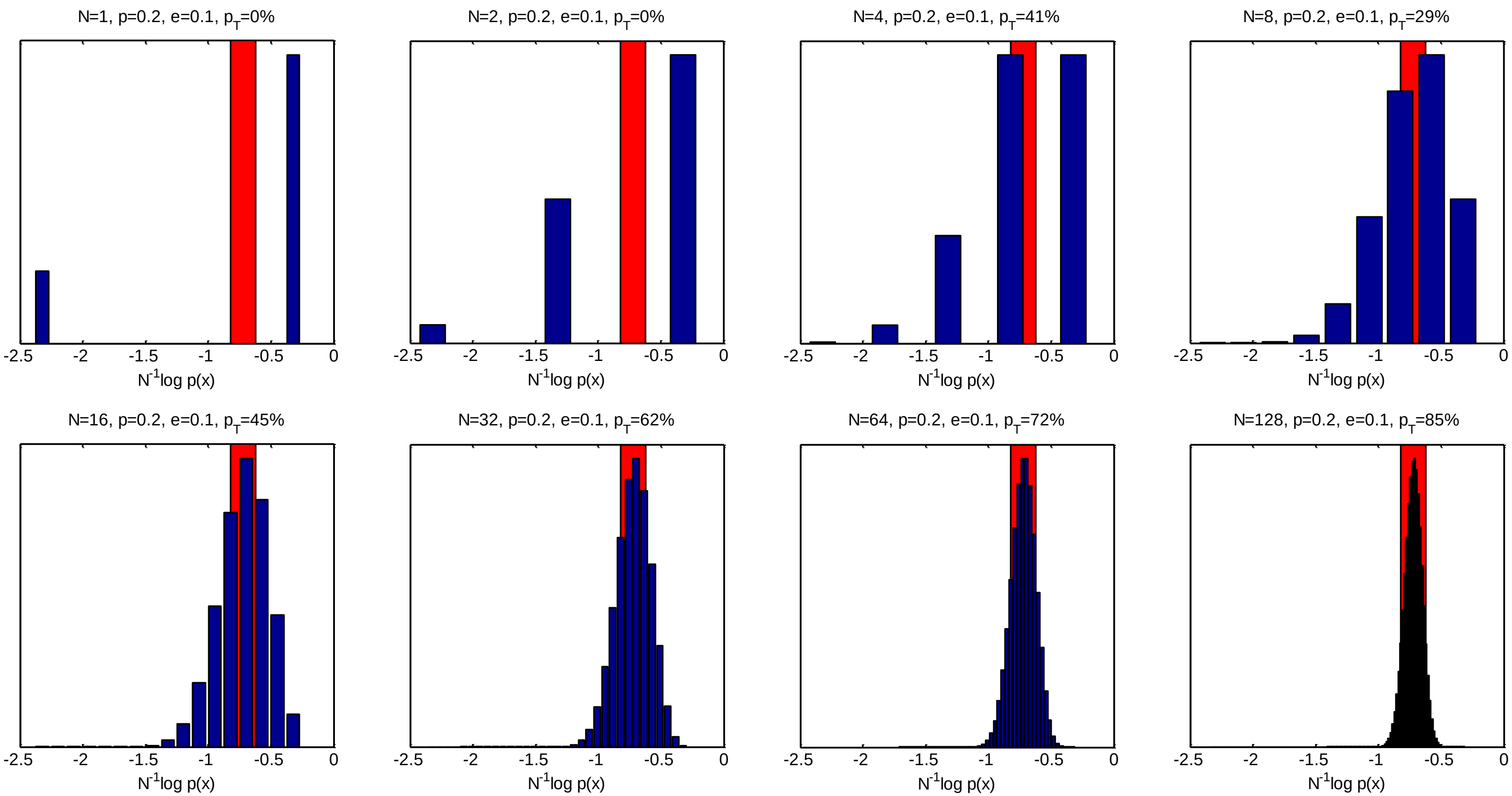
- $x_i$ Bernoulli with $p(x_i=1)=p$
- *e.g.* $p([0\ 1\ 1\ 0\ 0\ 0])=p^2(1-p)^4$
- For $p=0.2$, $H(X)=0.72$ bits
- Red bar shows $T_{0.1}^{(n)}$

$H(X)=-\sum_x P(x)\log P(x)$

$=-0.2\log_2 0.2-0.8\log_2 0.8=0.72$



# Typical Set Frames




01,01,00,1101,0011,01000,0000001,0010100,00,010000010000,
0000100001000000, 0000000010000000

$$-\frac{1}{n}\log P(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n} -\log P(x_i) \xrightarrow{\text{prob}} E(-\log P(x_i)) = H(X)$$

$$\forall n > N_\varepsilon,\ P\left[\left|\left(-\frac{1}{n}\log P(\bar{x})\right) - H(X)\right| > \varepsilon\right] < \varepsilon \Rightarrow P(\bar{x}\in T_\varepsilon^{(n)}) > 1-\varepsilon \text{ for } n > N_\varepsilon$$

# Typical Set: Properties

$$P(\bar{x}\in T_\varepsilon^{(n)}) \xrightarrow[P(x)\le 2^{-nH(X)+n\varepsilon}]{\log P(x)\le -nH(X)+n\varepsilon} \le \sum_{\bar{x}\in T_\varepsilon^{(n)}} 2^{-n(H(X)-\varepsilon)} = 2^{-n(H(X)-\varepsilon)}\,|T_\varepsilon^{(n)}|$$

$\vee$ $1-\varepsilon$

$$-nH(X)+n\varepsilon \ge \log P(x) \ge -nH(X)-n\varepsilon$$

$$\mathbf{x}\in T_\varepsilon^{(n)} \Rightarrow \log\ p(\mathbf{x}) = -nH(X) \pm n\varepsilon$$

$$P(\bar{x}\in T_\varepsilon^{(n)}) \xrightarrow[P(x)\ge 2^{-nH(X)-n\varepsilon}]{\log P(x)\ge -nH(X)-n\varepsilon} \ge \sum_{\bar{x}\in T_\varepsilon^{(n)}} 2^{-n(H(X)+\varepsilon)} = 2^{-n(H(X)+\varepsilon)}\cdot|T_\varepsilon^{(n)}|$$

$\wedge$ $1$

$$\boxed{p(\mathbf{x}\in T_\varepsilon^{(n)}) > 1-\varepsilon \quad \text{for} \quad n > N_\varepsilon}$$

$$\therefore 2^{n(H(X)+\varepsilon)} \ge |T_\varepsilon^{(n)}| > (1-\varepsilon)\,2^{n(H(X)-\varepsilon)}$$

(for $n > N_\varepsilon$)

$$\boxed{(1-\varepsilon)2^{n(H(X)-\varepsilon)} \overset{n>N_\varepsilon}{<} \left|T_\varepsilon^{(n)}\right| \le 2^{n(H(X)+\varepsilon)}}$$

size $\approx 2^{nH(X)}$ # numbers in source coding

**Proof 2:**

$$-n^{-1}\log\ p(\mathbf{x}) = n^{-1}\sum_{i=1}^{n} -\log\ p(x_i) \xrightarrow{\text{prob}} E -\log\ p(x_i) = H(X)$$

Hence $\forall\varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall n > N_\varepsilon \quad p\left(\left|-n^{-1}\log\ p(\mathbf{x}) - H(X)\right| > \varepsilon\right) < \varepsilon$

**Proof 3a:**

$$\text{f.l.e. } n,\quad 1-\varepsilon < p(\mathbf{x}\in T_\varepsilon^{(n)}) \overset{\log P(x)\le -nH(X)+n\varepsilon}{\le} \sum_{\mathbf{x}\in T_\varepsilon^{(n)}} 2^{-n(H(X)-\varepsilon)} = 2^{-n(H(X)-\varepsilon)}\underbrace{\left|T_\varepsilon^{(n)}\right|}_{\text{size}}$$

**Proof 3b:**

$$1 = \sum_{\mathbf{x}} p(\mathbf{x}) \ge \sum_{\mathbf{x}\in T_\varepsilon^{(n)}} p(\mathbf{x}) \overset{\log P(x)\ge -nH(X)-n\varepsilon}{\ge} \sum_{\mathbf{x}\in T_\varepsilon^{(n)}} 2^{-n(H(X)+\varepsilon)} = 2^{-n(H(X)+\varepsilon)}\left|T_\varepsilon^{(n)}\right|$$

# Consequence

- for any $\varepsilon$ and for $n > N_\varepsilon$

  "Almost all events are almost equally surprising"

- $p(\mathbf{x} \in T_\varepsilon^{(n)}) > 1-\varepsilon$ and $\log p(\mathbf{x}) = -nH(X) \pm n\varepsilon$

$2^{n(H(x)+\varepsilon)} \geq |T_\varepsilon^{(n)}| > (1-\varepsilon)\, 2^{n(H(x)-\varepsilon)}$

## Coding consequence

$\lceil n(H+\varepsilon) \rceil$

– $\mathbf{x} \in T_\varepsilon^{(n)}$ : '0' + at most 1+$n(H+\varepsilon)$ bits

typical

– $\mathbf{x} \notin T_\varepsilon^{(n)}$ : '1' + at most 1+$n\log|\mathrm{X}|$ bits

not typical    small prob. (uniform)

– $L$ = Average code length

$$\leq p(\mathbf{x} \in T_\varepsilon^{(n)})[2 + n(H+\varepsilon)]$$

$2+n(H+\varepsilon)$

$+\ \varepsilon[2+n\log|X|]$

$$+\, p(\mathbf{x} \notin T_\varepsilon^{(n)})[2 + n\log|\mathrm{X}|]$$

$= 2 + nH + n\varepsilon + 2\varepsilon + n\varepsilon\log|X|$

$= n\left(H + \varepsilon + \varepsilon\log|X| + \frac{2}{n}(1+\varepsilon)\right)$

$1 \cdot$ 1st term + $\varepsilon \cdot$ 2nd term

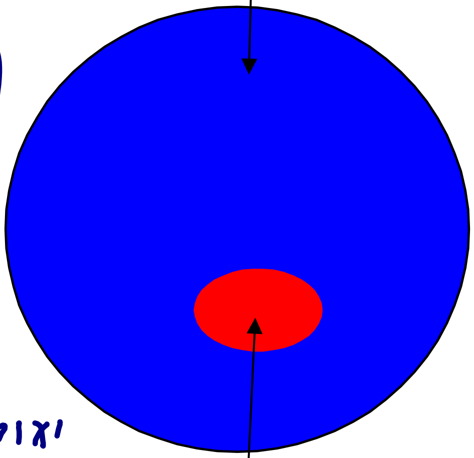
$$\leq n(H+\varepsilon) + \varepsilon\left(n\log|\mathrm{X}|\right) + 2\varepsilon + 2$$

$= n(H+\varepsilon')$

$$= n\left(H + \varepsilon + \varepsilon\log|\mathrm{X}| + 2(\varepsilon+2)n^{-1}\right) = n(H+\varepsilon')$$

$|\mathrm{X}|^n$ elements

|full set|



|min typical set|

$\leq 2^{n(H(x)+\varepsilon)}$ elements

# Source Coding & Data Compression

For any choice of $\varepsilon > 0$, we can, by choosing block size, $n$, large enough, do the following:

$n(H+\varepsilon') \approx L \Rightarrow H+\varepsilon' \approx \frac{L}{n}$ ( expectation of typical codelength $\approx nH$ )

- make a lossless code using only $H(X)+\varepsilon$ bits per symbol on average:

$$\frac{L}{n} \le H + \varepsilon$$

typical avg. code length

block size (# i.i.d bits)

fixed

n small: lossy (not typical)

n large: complex

express $X^n$ by $nH$ bits.

- The coding is one-to-one and decodable
  - However impractical due to exponential complexity
- Typical sequences have short descriptions of length $\approx nH$
  - Another proof of source coding theorem (Shannon's original proof)
- However, encoding/decoding complexity is exponential in $n$

# Smallest high-probability Set

$T_\varepsilon^{(n)}$ is a small subset of $X^n$ containing most of the probability mass. Can you get even smaller ?

For any $0 < \varepsilon < 1$, choose $N_0 = -\varepsilon^{-1}\log \varepsilon$ , then for any $n > \max(N_0, N_\varepsilon)$ and any subset $S^{(n)}$ satisfying $\left|S^{(n)}\right| < 2^{n(H(X)-2\varepsilon)}$

$$p\left(\mathbf{x} \in S^{(n)}\right) = p\left(\mathbf{x} \in S^{(n)} \cap T_\varepsilon^{(n)}\right) + p\left(\mathbf{x} \in S^{(n)} \cap \overline{T_\varepsilon^{(n)}}\right)$$

$$< \left|S^{(n)}\right| \max_{\mathbf{x} \in T_\varepsilon^{(n)}} p(\mathbf{x}) + p\left(\mathbf{x} \in \overline{T_\varepsilon^{(n)}}\right)$$

$$< 2^{n(H-2\varepsilon)} 2^{-n(H-\varepsilon)} + \varepsilon \qquad \text{for } n > N_\varepsilon$$

$$= 2^{-n\varepsilon} + \varepsilon < 2\varepsilon \qquad \text{for } n > N_{0,} \quad 2^{-n\varepsilon} < 2^{\log \varepsilon} = \varepsilon$$

Answer: No

# Summary

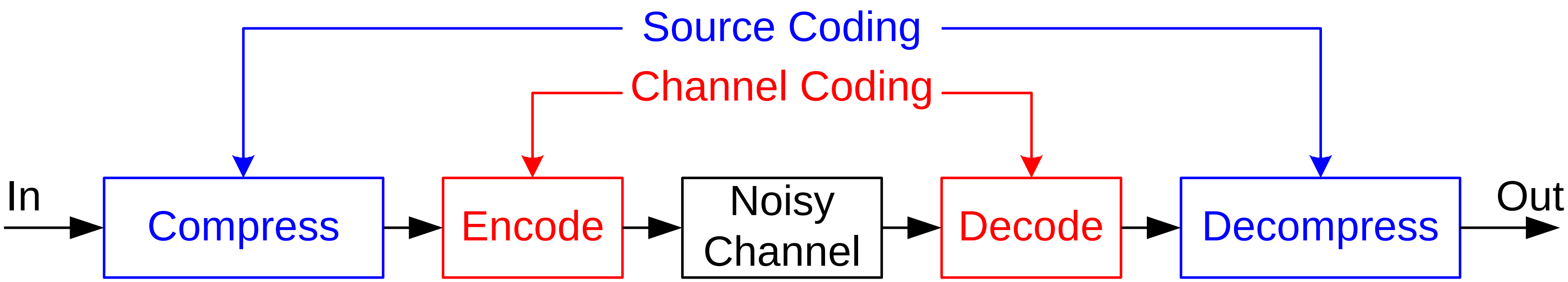
- Typical Set
  - Individual Prob $\mathbf{x} \in T_\varepsilon^{(n)} \Rightarrow \log p(\mathbf{x}) = -nH(x) \pm n\varepsilon$
  - Total Prob $p(\mathbf{x} \in T_\varepsilon^{(n)}) > 1 - \varepsilon \text{ for } n > N_\varepsilon$
  - Size $(1-\varepsilon)2^{n(H(x)-\varepsilon)} \overset{n>N_\varepsilon}{<} \left|T_\varepsilon^{(n)}\right| \le 2^{n(H(x)+\varepsilon)}$
- No other high probability set can be much smaller than $T_\varepsilon^{(n)}$
- Asymptotic Equipartition Principle
  - Almost all event sequences are equally surprising
- Can be used to prove source coding theorem

# Lecture 8

- Channel Coding
- Channel Capacity
  - The highest rate in bits per channel use that can be transmitted reliably ♦
  - The maximum mutual information
- Discrete Memoryless Channels
  - Symmetric Channels
  - Channel capacity
    - Binary Symmetric Channel
    - Binary Erasure Channel
    - Asymmetric Channel

♦ = proved in channel coding theorem

# Model of Digital Communication




- Source Coding
  - Compresses the data to remove redundancy
- Channel Coding
  - Adds redundancy/structure to protect against channel errors

# Discrete Memoryless Channel

(i/o discrete)

- Input: $x \in$ X, Output $y \in$ Y


- Time-Invariant Transition-Probability Matrix

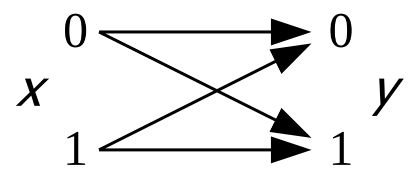
$$\left(\mathbf{Q}_{y|x}\right)_{i,j} = p\left(y = y_j \mid x = x_i\right)$$

  – Hence $\mathbf{p}_y = \mathbf{Q}_{y|x}^T \mathbf{p}_x$
  – $\mathbf{Q}$: each row sum = $1$, average column sum = $|X||Y|^{-1}$
- Memoryless: $\mathbf{p}(y_n | x_{1:n}, y_{1:n-1}) = \mathbf{p}(y_n | x_n)$ current output current input
- DMC = Discrete Memoryless Channel

# Binary Channels

- Binary Symmetric Channel
  (BSC)
  - X = [0 1], Y = [0 1]

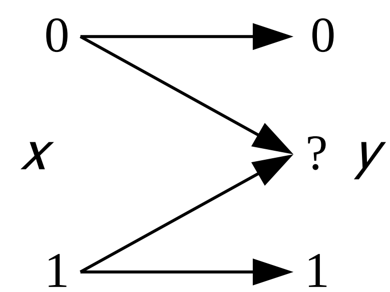
$$\begin{pmatrix} 1-f & f \\ f & 1-f \end{pmatrix} S$$




$f$ : error prob.
$1-f$ : correct prob.

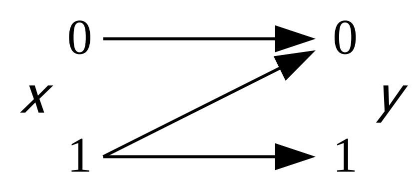
- Binary Erasure Channel
  (BEC)
  - X = [0 1], Y = [0 ? 1]

$$\begin{array}{c|ccc} x \backslash y & 0 & ? & 1 \\ \hline 0 & & & \\ 1 & & & \end{array} \quad \begin{pmatrix} 1-f & f & 0 \\ 0 & f & 1-f \end{pmatrix}$$




- Z Channel
  - X = [0 1], Y = [0 1]

$$\begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & & \\ 1 & & \end{array} \quad \begin{pmatrix} 1 & 0 \\ f & 1-f \end{pmatrix}$$




Symmetric: rows are permutations of each other; columns are permutations of each other

$\Sigma = \frac{2}{3}$ (same sum)

Weakly Symmetric: rows are permutations of each other; columns have the same sum

WS $\begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$ $\Sigma = 1$ (def.)

Capacity $\begin{cases} S \\ WS \end{cases}$

# Weakly Symmetric Channels

x\Y

Weakly Symmetric: Q =

1. All columns of $\mathbf{Q}$ have the same sum $= |X||Y|^{-1}$
   – If $x$ is uniform (i.e. $p(x) = |X|^{-1}$) then $y$ is uniform

Column

$$p(y) = \sum_{x \in X} p(y \mid x)\, p(x) = |X|^{-1} \sum_{x \in X} p(y \mid x) = |X|^{-1} \times |X||Y|^{-1} = |Y|^{-1}$$

2. All rows are permutations of each other
   – Each row of Q has the same entropy so

$$H(y \mid x) = \sum_{x \in X} p(x) H(y \mid x = x) = H(\mathbf{Q}_{1,:}) \sum_{x \in X} p(x) = H(\mathbf{Q}_{1,:})$$

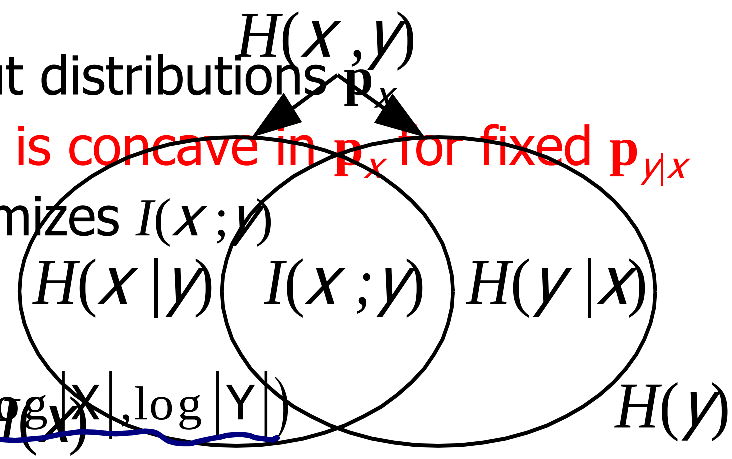
where $\mathbf{Q}_{1,:}$ is the entropy of the first (or any other) row of the $\mathbf{Q}$ matrix

Symmetric: 1. All rows are permutations of each other
2. All columns are permutations of each other
Symmetric $\Rightarrow$ weakly symmetric

# Channel Capacity

- Capacity of a DMC channel: $\boxed{C = \max_{\mathbf{p}_x} I(x;y)}$
  - Mutual information (not entropy itself) is what could be transmitted through the channel
  - Maximum is over all possible input distributions $\mathbf{p}_x$ ♦
  - ∃ only one maximum since $I(x;y)$ is concave in $\mathbf{p}_x$ for fixed $\mathbf{p}_{y|x}$
  - We want to find the $\mathbf{p}_x$ that maximizes $I(x;y)$
  - Limits on $C$:

$$0 \le C \le \min\left(H(x), H(y)\right) \le \min\left(\log|\mathrm{X}|, \log|\mathrm{Y}|\right)$$

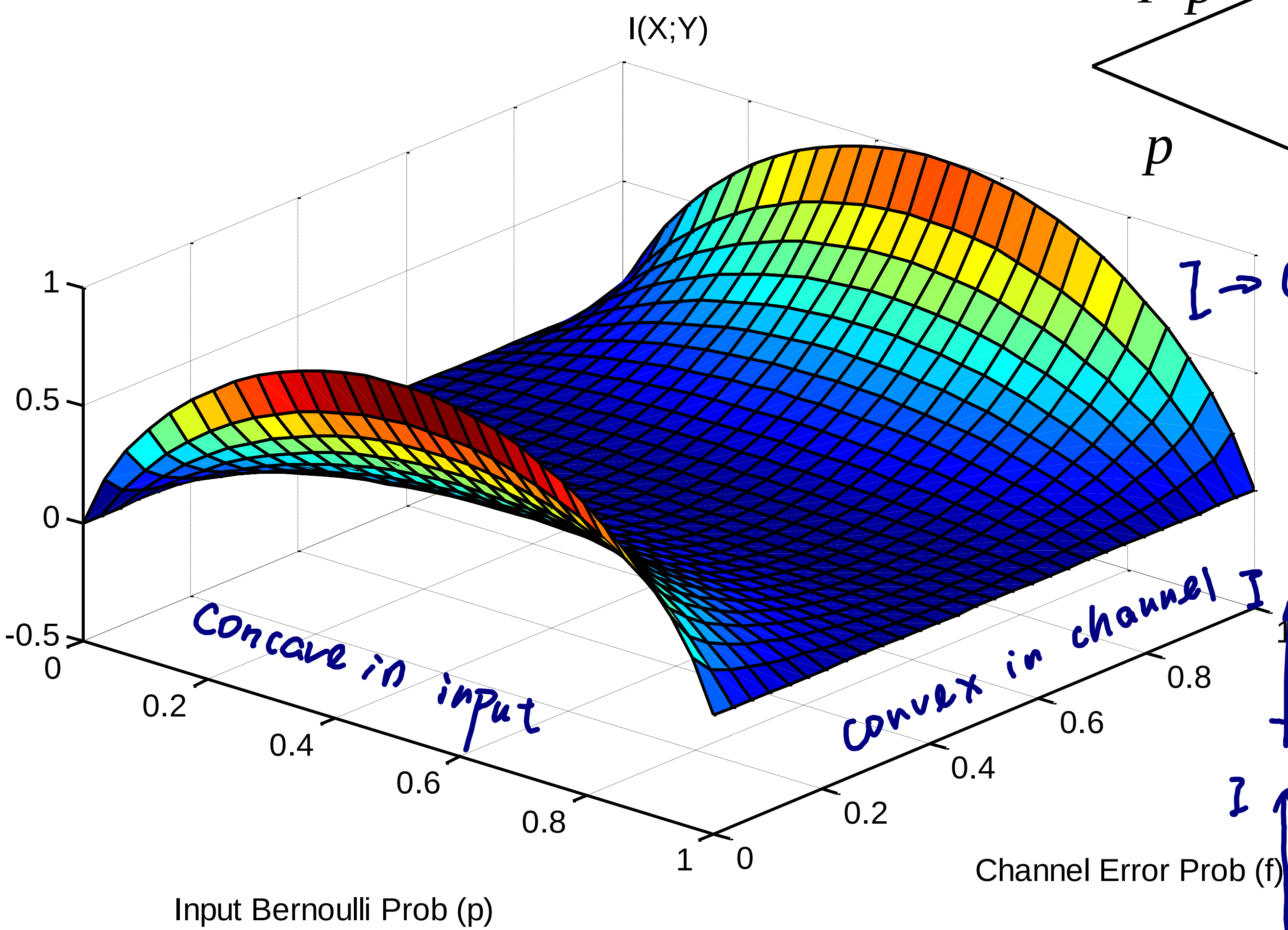


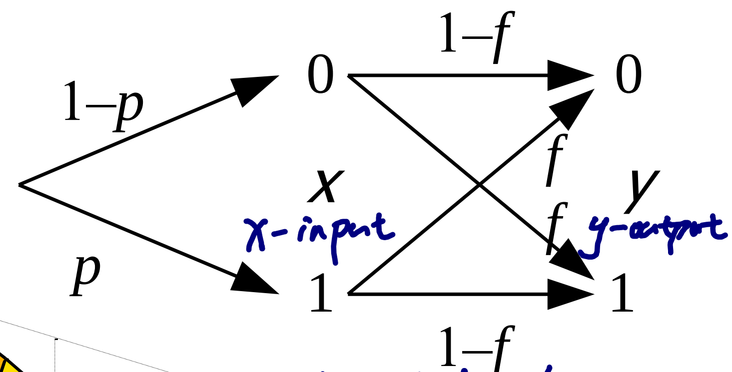

- Capacity for $n$ uses of channel:

$$C^{(n)} = \frac{1}{n} \max_{\mathbf{p}_{x_{1:n}}} I(x_{1:n}; y_{1:n})$$

♦ = proved in two pages time

# Mutual Information Plot




I → C | channel fixed, input uniform

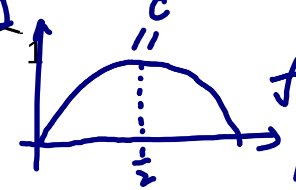
$H(y|x)$

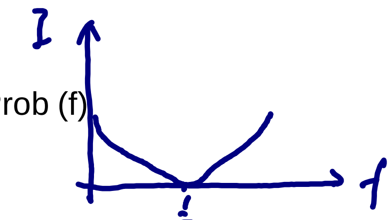
$pf + p) - H(f)$

$I(x;y) = H(f - 2pf + p) - H(f)$

$= C$

fix f



$I(X,Z;Y) = I(X;Y) + I(Z;Y|X) = I(Z;Y) + I(X;Y|Z)$

$I(Z;Y|X) = H(Y|X) - H(Y|X.Z)$ fixed $P_{Y|X}$ 0

# Mutual Information Concave in $\mathbf{p}_X$

$\therefore I(X;Y) \geq I(X;Y|Z)$

$= \lambda I(X;Y|Z=1) + (1-\lambda) I(X;Y|Z=0)$

$= \lambda I(U;Y) + (1-\lambda) I(V;Y)$

Mutual Information $I(x;y)$ is concave in $\mathbf{p}_x$ for fixed $\mathbf{p}_{y|x}$

(input)

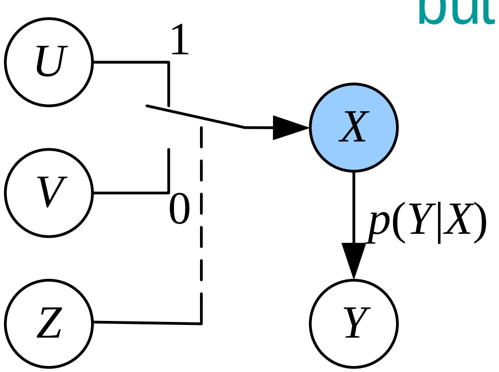
Proof: Let $u$ and $v$ have prob mass vectors $\mathbf{p}_\mathbf{u}$ and $\mathbf{p}_\mathbf{v}$

X: input
Y: output

- Define $z$: bernoulli random variable with $p(1) = \lambda$
- Let $x = u$ if $z=1$ and $x=v$ if $z=0$ $\Rightarrow$ $\mathbf{p}_x = \lambda\,\mathbf{p}_\mathbf{u} + (1-\lambda)\,\mathbf{p}_\mathbf{v}$

$$I(x,z;y) = I(x;y) + I(z;y\,|\,x) = I(z;y) + I(x;y\,|\,z)$$

but $I(z;y\,|\,x) = H(y\,|\,x) - H(y\,|\,x,z) = 0$ so




$$I(x;y) \geq I(x;y\,|\,z)$$

$$= \lambda I(x;y\,|\,z=1) + (1-\lambda) I(x;y\,|\,z=0)$$

$$= \lambda I(u;y) + (1-\lambda) I(v;y)$$

Special Case: $y=x \Rightarrow I(x;x)=H(x)$ is concave in $\mathbf{p}_x$

$I(X;Y,Z) = I(X;Y) + I(X;Y|Z) = I(X;Z) + I(X;Z|Y)$

$I(X;Z) \overset{\text{fixed } P_X}{=\!=} 0$

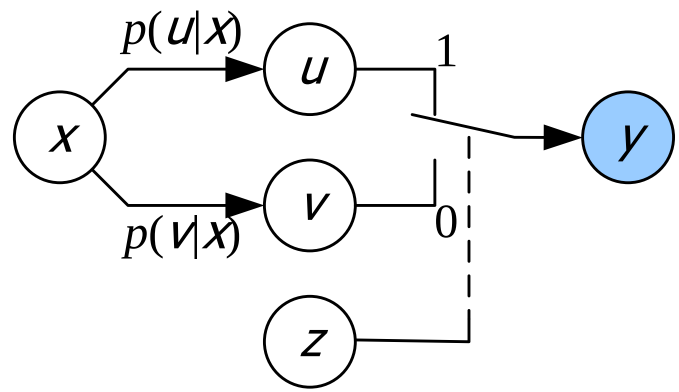
# Mutual Information Convex in $\mathbf{p}_{Y|X}$

$I(X;Y) \le I(X;Z|Y)$

$= \lambda I(X;Y|Z=1) + (1-\lambda) I(X;Y|Z=0) = \lambda I(X;U) + (1-\lambda) I(X;V)$

Mutual Information $I(x\,;y)$ is convex in $\mathbf{p}_{y|x}$ (channel) for fixed $\mathbf{p}_x$

Proof: define $u$, $v$, $x$ etc:




– $$\mathbf{p}_{y|x} = \lambda \mathbf{p}_{u|x} + (1-\lambda)\mathbf{p}_{v|x}$$

$$I(x;y,z) = I(x;y\,|\,z) + I(x;z)$$

$$= I(x;y) + I(x;z\,|\,y)$$

but $I(x;z)=0$ and $I(x;z\,|\,y) \ge 0$ so

$$I(x;y) \le I(x;y\,|\,z)$$

$$= \lambda I(x;y\,|\,z=1) + (1-\lambda) I(x;y\,|\,z=0)$$

$$= \lambda I(x;u) + (1-\lambda) I(x;v)$$

# $n$-use Channel Capacity

## For Discrete Memoryless Channel:

$$I(x_{1:n};y_{1:n}) = H(y_{1:n}) - H(y_{1:n} \mid x_{1:n})$$

$$= \sum_{i=1}^{n} H(y_i \mid y_{1:i-1}) - \sum_{i=1}^{n} H(y_i \mid x_i)$$

Chain; Memoryless

$$\le \sum_{i=1}^{n} H(y_i) - \sum_{i=1}^{n} H(y_i \mid x_i) = \sum_{i=1}^{n} I(x_i;y_i)$$

Conditioning Reduces Entropy

with equality if $y_i$ are independent $\Rightarrow$ $x_i$ are independent

We can maximize $I(\mathbf{x};\mathbf{y})$ by maximizing each $I(x_i;y_i)$ independently and taking $x_i$ to be i.i.d.

- We will concentrate on maximizing $I(x; y)$ for a single channel use
- The elements of $X_i$ are not necessarily i.i.d.

BSC $\begin{pmatrix} 1-f & f \\ f & 1-f \end{pmatrix}$ $I(X;Y) = H(Y) - H(Y|X)$

$= H(Y) - H(Q_{1,:}) \le \log|Y| - H(Q_{1,:})$

# Capacity of Symmetric Channel

$I(X;Y) \le \log_2 2 - H(f) = \underbrace{1-H(f)}_{C}$

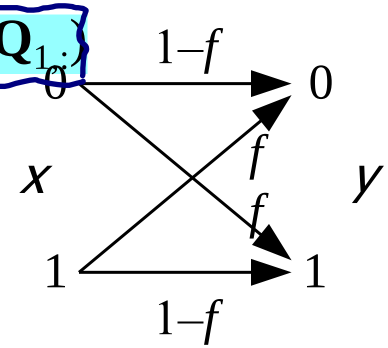
If channel is weakly symmetric: $I \to C$ $\left\{ \begin{array}{l} \text{channel fixed} \\ \text{input uniform} \end{array} \right.$

$$I(x;y) = H(y) - \underbrace{H(y \mid x)}_{\text{row entropy}} = H(y) - H(\mathbf{Q}_{1,:}) \le \log|\mathrm{Y}| - H(\mathbf{Q}_{1,:})$$

with equality iff input distribution is uniform

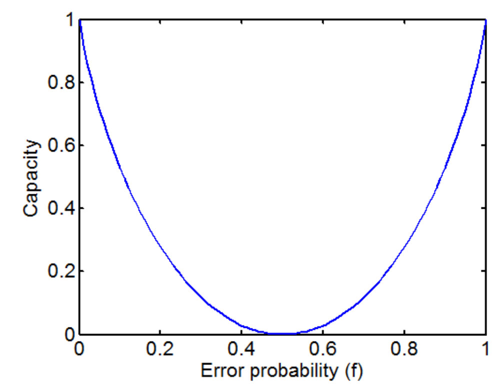
$\therefore$ Information Capacity of a WS channel is $\boxed{C = \log|\mathrm{Y}| - H(\mathbf{Q}_{1,:})}$

For a binary symmetric channel (BSC): $\begin{pmatrix} 1-f & f \\ f & 1-f \end{pmatrix}$ $H(f)$




- $|\mathrm{Y}| = 2$
- $H(\mathbf{Q}_{1,:}) = H(f)$
- $I(x;y) \le 1 - H(f)$

$\therefore$ Information Capacity of a BSC is $1-H(f)$



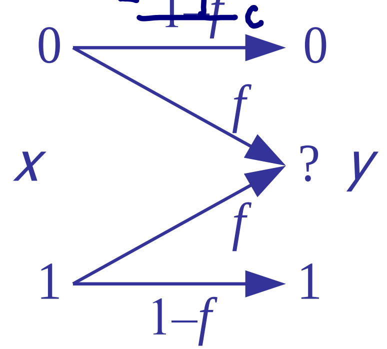
# Binary Erasure Channel (BEC)

BEC

$$\begin{array}{c|ccc} x \backslash y & 0 & ? & 1 \\ \hline 0 & 1-f & f & 0 \\ 1 & 0 & f & 1-f \end{array}$$

$$I(x;y) = H(x) - H(x|y)$$
$$= H(x) - P(y=0)H(x|y=0) - P(y=?)H(x|y=?) - P(y=1)H(x|y=1)$$
$$= H(x) - P(y=0)\cdot 0 - P(y=?)H(x) - P(y=1)\cdot 0 = H(x)(1-f)$$
$$\leq 1-f \quad C$$

$$\begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{ccc} 0 & ? & 1 \\ \left(\begin{array}{c} 1-f \\ \underset{(H=0)}{0} \end{array}\right. & \begin{array}{c} f \\ f \end{array} & \left.\begin{array}{c} 0 \ (H=0) \\ 1-f \end{array}\right) \end{array}$$




$$I(x;y) = H(x) - H(x|y)$$
$$= H(x) - p(y=0)\times 0 - p(y=?)H(x) - p(y=1)\times 0$$
$$= H(x) - H(x)f$$
$$= (1-f)H(x)$$
$$\leq 1-f$$
$$C = 1-f$$

$H(x|y) = 0$ when $y=0$ or $y=1$

mutual info ⇒ capacity
- ≤
- = when uniform distribution

since max value of $H(x) = 1$
with equality when x is uniform

since a fraction $f$ of the bits are lost, the capacity is only $1-f$ and this is achieved when $x$ is uniform

$I(X;Y) = H(Y) - H(Y|X)$

$H(Y) = 2\cdot(-a\log a) - (1-2a)\log(1-2a)$

$H(Y|X) = 2aH(f) + (1-2a)\cdot 0$

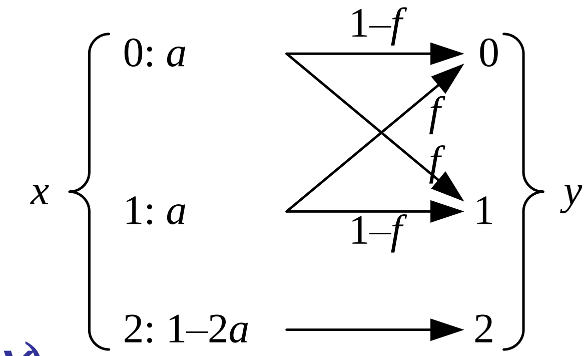
| | $x\backslash y$ | 0 | 1 | 2 |
|---|---|---|---|---|
| $a$ | 0 | $1-f$ | $f$ | 0 |
| $a$ | 1 | $f$ | $1-f$ | 0 |
| $1-2a$ | 2 | 0 | 0 | 1 |

# Asymmetric Channel Capacity

Let $\mathbf{p}_x = [a \;\; a \;\; 1-2a]^T \Rightarrow \mathbf{p}_y = \mathbf{Q}^T\mathbf{p}_x = \mathbf{p}_x$

$$H(y) = -2a\log a - (1-2a)\log(1-2a)$$

$$H(y\,|\,x) = 2aH(f) + (1-2a)H(1) = 2aH(f)$$




To find $C$, maximize $I(x\,;y) = H(y) - H(y\,|x)$

$$\mathbf{Q} = \begin{pmatrix} 1-f & f & 0 \\ f & 1-f & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I = -2a\log a - (1-2a)\log(1-2a) - 2aH(f)$$

$$\frac{dI}{da} = -2\log e - 2\log a + 2\log e + 2\log(1-2a) - 2H(f) = 0$$

$$\log\frac{1-2a}{a} = \log\left(a^{-1} - 2\right) = H(f) \quad \Rightarrow \quad a = \left(2 + 2^{H(f)}\right)^{-1}$$

$$\Rightarrow \quad C = -2a\log\left(a2^{H(f)}\right) - (1-2a)\log(1-2a) \;\; = -\log(1-2a)$$

Note:
$d(\log x) = x^{-1}\log e$

Examples: $f = 0 \Rightarrow H(f) = 0 \Rightarrow a = {}^1/_3 \Rightarrow C = \log 3 = 1.585$ bits/use
$f = ½ \Rightarrow H(f) = 1 \Rightarrow a = ¼ \Rightarrow C = \log 2 = 1$ bits/use

# Summary

- Given the channel, mutual information is concave in input distribution
- Channel capacity $C = \max_{\mathbf{p}_x} I(x;y)$
  - The maximum exists and is unique
- DMC capacity
  - Weakly symmetric channel: $\log|\mathrm{Y}| - H(\mathbf{Q}_{1,:})$
  - BSC: $1 - H(f)$
  - BEC: $1 - f$
  - In general it very hard to obtain closed-form; numerical method using convex optimization instead

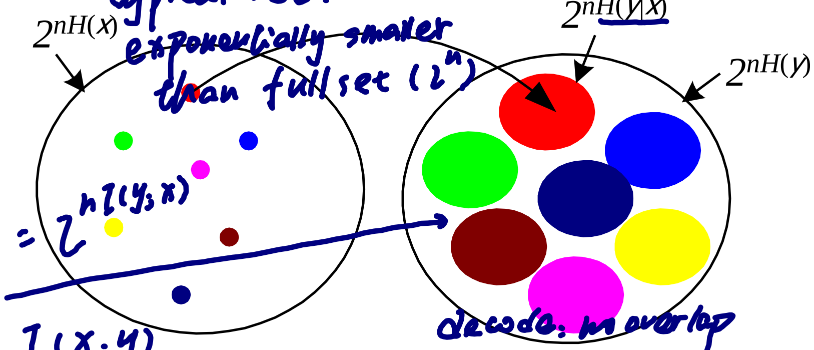
# Lecture 9

- Jointly Typical Sets
- Joint AEP
- Channel Coding Theorem
  - Ultimate limit on information transmission is channel capacity
  - The central and most successful story of information theory
  - Random Coding
  - Jointly typical decoding

# Intuition on the Ultimate Limit

- Consider blocks of $n$ symbols:


# codewords $= \frac{2^{nH(y)}}{2^{nH(y|x)}} = 2^{nI(y;x)}$




rate = $\frac{\log \# \text{codewords}}{\text{codeword length}} = I(x;y)$

  - For large $n$, an average input sequence $x_{1:n}$ corresponds to about $2^{nH(y|x)}$ typical output sequences
  - There are a total of $2^{nH(y)}$ typical output sequences
  - For nearly error free transmission, we select a number of input sequences whose corresponding sets of output sequences hardly overlap
  - The maximum number of distinct sets of output sequences is $2^{n(H(y)-H(y|x))} = 2^{nI(y;x)}$
  - One can send $I(y;x)$ bits per channel use

for large n can transmit at any rate < $C$ with negligible errors

# Jointly Typical Set

**x,y** is the i.i.d. sequence $\{x_i, y_i\}$ for $1 \le i \le n$

- Prob of a particular sequence is $p(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{N} p(x_i, y_i)$
- $E - \log p(\mathbf{x},\mathbf{y}) = n\ E - \log p(x_i, y_i) = nH(x, y)$
- Jointly Typical set: $\begin{cases} x \text{ typical} \\ y \text{ typical} \\ x.y \text{ typical} \end{cases}$

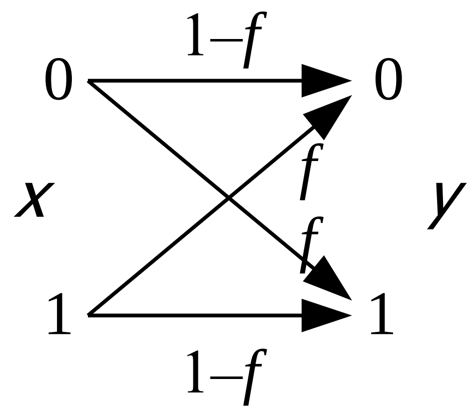
$$J_\varepsilon^{(n)} = \Big\{ \mathbf{x},\mathbf{y} \in \mathrm{XY}^n : \left| -n^{-1} \log p(\mathbf{x}) - H(x) \right| < \varepsilon,$$

$$\left| -n^{-1} \log p(\mathbf{y}) - H(y) \right| < \varepsilon,$$

$$\left| -n^{-1} \log p(\mathbf{x},\mathbf{y}) - H(x, y) \right| < \varepsilon \Big\}$$

# Jointly Typical Example

## Binary Symmetric Channel




$$f = 0.2, \quad \mathbf{p}_x = (0.75 \quad 0.25)^T$$

$$\mathbf{p}_y = (0.65 \quad 0.35)^T, \quad \mathbf{P}_{xy} = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.6 \ (0.25\times0.8) & 0.15 \ (0.75\times0.2) \\ 1 & 0.05 \ (0.25\times0.2) & 0.2 \ (0.25\times0.8) \end{array}$$

## Jointly typical example (for any $\varepsilon$):

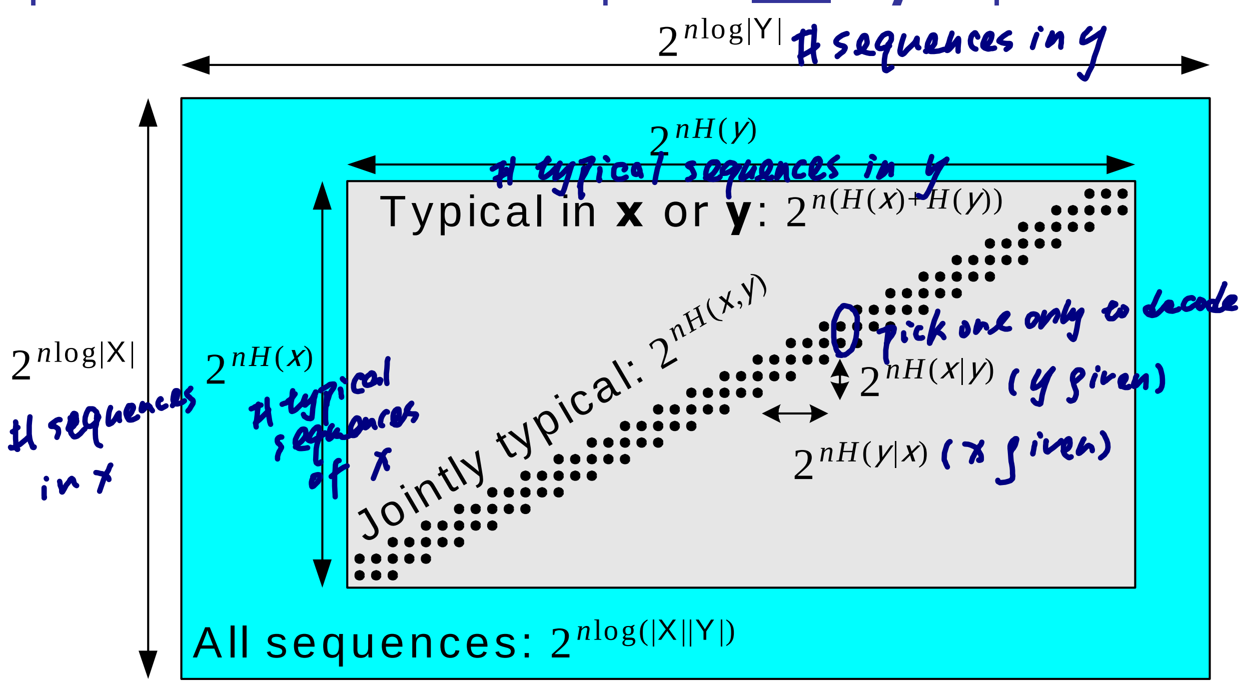
**x** = 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

**y** = 1 1 1 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0

jointly typical set: all combinations of *x* and *y* have exactly the right frequencies

# Jointly Typical Diagram

Each point defines both an **x** sequence and a **y** sequence




Dots represent jointly typical pairs (**x**,**y**)

Inner rectangle represents pairs that are typical in **x** or **y** but not necessarily jointly typical

- There are about $2^{nH(X)}$ typical $\mathbf{x}$'s in all
- Each typical $\mathbf{y}$ is jointly typical with about $2^{nH(X|Y)}$ of these typical $\mathbf{x}$'s
- The jointly typical pairs are a fraction $2^{-nI(X;Y)}$ of the inner rectangle
- Channel Code: choose $\mathbf{x}$'s whose J.T. $\mathbf{y}$'s don't overlap; use J.T. for decoding
- There are $2^{nI(X;Y)}$ such codewords $\mathbf{x}$'s

# Joint Typical Set Properties

1. Indiv Prob: $\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)} \quad \Rightarrow \quad \log p(\mathbf{x},\mathbf{y}) = -\underline{n}H(x,y) \pm \underline{n}\varepsilon$
2. Total Prob: $p\left(\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)}\right) > 1-\varepsilon$ for $n > N_\varepsilon$
3. Size: $(1-\varepsilon)2^{n(H(x,y)-\varepsilon)} \overset{n>N_\varepsilon}{<} \left|J_\varepsilon^{(n)}\right| \le 2^{n(H(x,y)+\varepsilon)}$

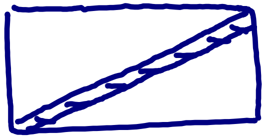
Proof 2: (use weak law of large numbers)

Choose $N_1$ such that $\forall n > N_1, \quad p\left(\left|-n^{-1}\log p(\mathbf{x}) - H(x)\right| > \varepsilon\right) < \frac{\varepsilon}{3}$

Similarly choose $N_2, N_3$ for other conditions and set $N_\varepsilon = \max(N_1, N_2, N_3)$

Proof 3: $1-\varepsilon < \sum_{\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)}} p(\mathbf{x},\mathbf{y}) \le \left|J_\varepsilon^{(n)}\right| \max_{\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)}} p(\mathbf{x},\mathbf{y}) = \left|J_\varepsilon^{(n)}\right| 2^{-n(H(x,y)-\varepsilon)}$ $n>N_\varepsilon$

$1 \ge \sum_{\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)}} p(\mathbf{x},\mathbf{y}) \ge \left|J_\varepsilon^{(n)}\right| \min_{\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)}} p(\mathbf{x},\mathbf{y}) = \left|J_\varepsilon^{(n)}\right| 2^{-n(H(x,y)+\varepsilon)}$ $\forall n$





# Properties

4. If $\mathbf{p}_{x'}=\mathbf{p}_x$ and $\mathbf{p}_{y'}=\mathbf{p}_y$ with $x'$ and $y'$ independent:

$$(1-\varepsilon)2^{-n(I(X,Y)+3\varepsilon)} \le p\left(\mathbf{x}',\mathbf{y}' \in J_\varepsilon^{(n)}\right) \le 2^{-n(I(X,Y)-3\varepsilon)} \text{ for } n > N_\varepsilon$$

Proof: |J| × (Min Prob) ≤ Total Prob ≤ |J| × (Max Prob)

$$p\left(\mathbf{x}',\mathbf{y}' \in J_\varepsilon^{(n)}\right) = \sum_{\mathbf{x}',\mathbf{y}' \in J_\varepsilon^{(n)}} p(\mathbf{x}',\mathbf{y}') = \sum_{\mathbf{x}',\mathbf{y}' \in J_\varepsilon^{(n)}} p(\mathbf{x}')p(\mathbf{y}')$$

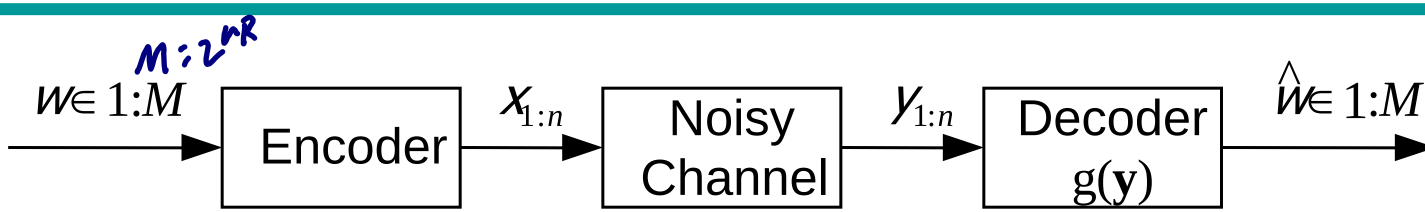
$$p\left(\mathbf{x}',\mathbf{y}' \in J_\varepsilon^{(n)}\right) \le \left|J_\varepsilon^{(n)}\right| \max_{\mathbf{x}',\mathbf{y}' \in J_\varepsilon^{(n)}} p(\mathbf{x}')p(\mathbf{y}')$$

$$\le 2^{n(H(X,Y)+\varepsilon)}2^{-n(H(X)-\varepsilon)}2^{-n(H(Y)-\varepsilon)} = 2^{-n(I(X;Y)-3\varepsilon)}$$

$$p\left(\mathbf{x}',\mathbf{y}' \in J_\varepsilon^{(n)}\right) \ge \left|J_\varepsilon^{(n)}\right| \min_{\mathbf{x}',\mathbf{y}' \in J_\varepsilon^{(n)}} p(\mathbf{x}')p(\mathbf{y}')$$

$$\ge (1-\varepsilon)2^{-n(I(X;Y)+3\varepsilon)} \quad \text{for} \quad n > N_\varepsilon$$

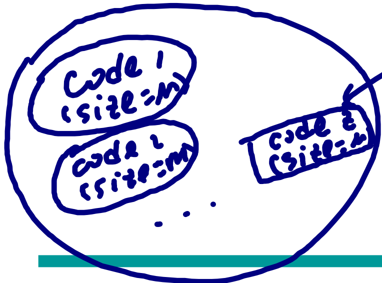
# Channel Coding




- Assume Discrete Memoryless Channel with known $\mathbf{Q}_{y|x}$
- An $(M, n)$ code is
  (# codewords; code length)
  - A fixed set of $M$ codewords $\mathbf{x}(w) \in \mathcal{X}^n$ for $w=1{:}M$
  - A deterministic decoder $g(\mathbf{y}) \in 1{:}M$

  $M = 2^{nR}$
- The rate of an $(M,n)$ code: $R = (\log M)/n$ bits/transmission (normalised)
- Error probability $\lambda_w = p\left(g\left(\mathbf{y}(w)\right) \neq w\right) = \sum_{\mathbf{y} \in \mathcal{Y}^n} p\left(\mathbf{y} \mid \mathbf{x}(w)\right) \delta_{g(\mathbf{y}) \neq w}$
  - Maximum Error Probability $\lambda^{(n)} = \max_{1 \le w \le M} \lambda_w$
  - Average Error probability $P_e^{(n)} = \frac{1}{M} \sum_{w=1}^{M} \lambda_w$

$\delta_C = 1$ if $C$ is true or $0$ if it is false




small error prob.

# Shannon's ideas

- Channel coding theorem: the basic theorem of information theory
  - Proved in his original 1948 paper
- How do you correct all errors?
- Shannon's ideas
  - Allowing arbitrarily small but nonzero error probability
  - (by sending successive bits
  - Using the channel many times in succession so that AEP holds $AEP: -\frac{1}{n}\log P(X_1, X_2 \ldots X_n) \to H(X)$
  - Consider a randomly chosen code and show the expected average error probability is small
    - Use the idea of typical sequences
    - Show this means ∃ at least one code with small max error prob
    - Sadly it doesn't tell you how to construct the code

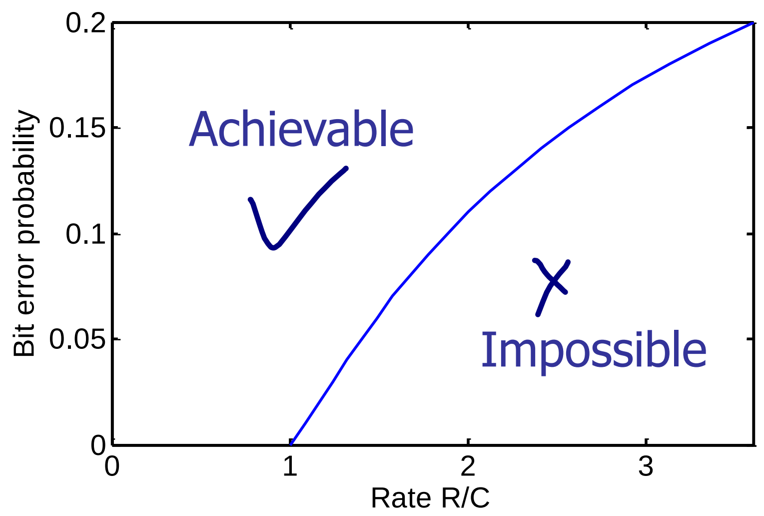
# Channel Coding Theorem

- A rate $R$ is achievable if $R<C$ and not achievable if $R>C$
  - If $R<C$, $\exists$ a sequence of $(2^{nR},n)$ codes with max prob of error $\lambda^{(n)}\to 0$ as $n\to\infty$
  - Any sequence of $(2^{nR},n)$ codes with max prob of error $\lambda^{(n)}\to 0$ as $n\to\infty$ must have $R \le C$

A very counterintuitive result:

Despite channel errors you can get arbitrarily low bit error rates provided that $R<C$

don't need to sacrify data rate when $R<C$.



# Summary

- Jointly typical set

$$-\log p(\mathbf{x},\mathbf{y}) = nH(X,Y) \pm n\varepsilon$$

$$p\left(\mathbf{x},\mathbf{y} \in J_{\varepsilon}^{(n)}\right) > 1-\varepsilon$$

$$\left|J_{\varepsilon}^{(n)}\right| \le 2^{n\left(H(X,Y)+\varepsilon\right)}$$

$$(1-\varepsilon)2^{-n\left(I(X,Y)+3\varepsilon\right)} \le p\left(\mathbf{x}',\mathbf{y}' \in J_{\varepsilon}^{(n)}\right) \le 2^{-n\left(I(X,Y)-3\varepsilon\right)}$$

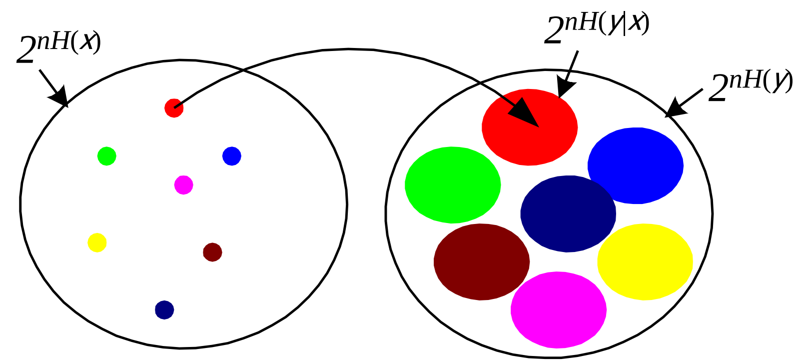
- Machinery to prove channel coding theorem

# Lecture 10

- Channel Coding Theorem
  - Proof
    - Using joint typicality
    - Arguably the simplest one among many possible ways
    - Limitation: does not reveal $P_e \sim e^{-nE(R)}$
  - Converse (next lecture)

# Channel Coding Principle

- Consider blocks of $n$ symbols:


  - An average input sequence $x_{1:n}$ corresponds to about $2^{nH(y|x)}$ typical output sequences
  - Random Codes: Choose $M = 2^{nR}$ ( $R \le I(x;y)$ ) random codewords $\mathbf{x}(w)$
    - their typical output sequences are unlikely to overlap much.
  - Joint Typical Decoding: A received vector $\mathbf{y}$ is very likely to be in the typical output set of the transmitted $\mathbf{x}(w)$ and no others. Decode as this $w$.

Channel Coding Theorem: for large $n$, can transmit at any rate $R < C$ with negligible errors

# Random $(2^{nR},n)$ Code

$$\begin{array}{c|cccc} w \backslash n & 1 & 2 & \cdots & n \\ \hline X(1) & X(1,1) & X(1,2) & \cdots & X(1,n) \\ X(2) & X(2,1) & X(2,2) & \cdots & X(2,n) \\ \vdots & & & & \\ X(M) & X(M,1) & X(M,2) & \cdots & X(M,n) \end{array}$$

$w=1$, $w=2$, …, $w=M$; $M=2^{nR}$

$X_{\cdot,j} \sim P_x$

code: $M \times n$ i.i.d.

↳ completely random → can generate many!

- Choose $\varepsilon \approx$ error prob, joint typicality $\Rightarrow N_\varepsilon$, choose $n>N_\varepsilon$
- Choose $\mathbf{p}_x$ so that $I(x;y)=C$, the information capacity
- Use $\mathbf{p}_x$ to choose a code C with random $\mathbf{x}(w)\in X^n$, $w=1:2^{nR}$
  - the receiver knows this code and also the transition matrix $\mathbf{Q}$
- Assume the message $W\in 1:2^{nR}$ is uniformly distributed
- If received value is $\mathbf{y}$; decode the message by seeing how many $\mathbf{x}(w)$'s are jointly typical with $\mathbf{y}$
  - if $\mathbf{x}(k)$ is the only one then $k$ is the decoded message
  - if there are $0$ or $\geq 2$ possible $k$'s then declare an error message 0
  - we calculate error probability averaged over all C and all $W$

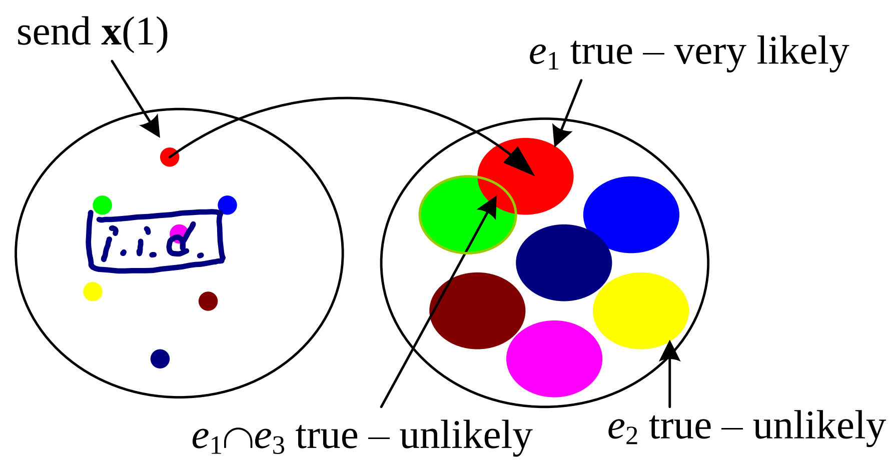
$$p(E)=\underbrace{\sum_C p(C)2^{-nR}\underbrace{\sum_{w=1}^{2^{nR}}\lambda_w(C)}_{\text{all codewords for certain code}}}_{\text{ensemble average over all codes}}=2^{-nR}\sum_{w=1}^{2^{nR}}\sum_C p(C)\lambda_w(C)\overset{(a)}{\underset{w.\ i.i.d.}{=}}\sum_C p(C)\lambda_1(C)=p(E\mid w=1)$$

error prob. of sending 1st codeword

(a) since error averaged over all possible codes is independent of $w$

# Decoding Errors

- Assume we transmit $\mathbf{x}(1)$ and receive $\mathbf{y}$
- Define the J.T. events $e_w = \left\{ (\mathbf{x}(w), \mathbf{y}) \in J_\varepsilon^{(n)} \right\}$ for $w \in 1:2^{nR}$


- Decode using joint typicality
- We have an error if either $e_1$ false or $e_w$ true for $w \geq 2$
- The $\mathbf{x}(w)$ for $w \neq 1$ are independent of $\mathbf{x}(1)$ and hence also independent of $\mathbf{y}$. So $p(e_w \text{ true}) < 2^{-n(I(X,Y)-3\varepsilon)}$ for any $w \neq 1$

Joint AEP

# Error Probability for Random Code

$\varepsilon + 2^{-n(\overbrace{C-R-3\varepsilon}^{>0})}$

$\le \varepsilon + 2^{+\frac{\log\varepsilon}{C-R-3\varepsilon}(C-R-3\varepsilon)}$

$= \varepsilon + 2^{+\log\varepsilon}$

$= 2\varepsilon$

- Upper bound

Union bound: $P(A \cup B) \le P(A) + P(B)$

p(A∪B)≤p(A)+p(B)

$$p(E) = p(E \mid W = 1) = p\left(\overline{e_1} \cup e_2 \cup e_3 \cup \cdots \cup e_{2^{nR}}\right) \le p\left(\overline{e_1}\right) + \sum_{w=2}^{2^{nR}} p(e_w)$$

(1) Joint typicality

$$\le \varepsilon + \sum_{i=2}^{2^{nR}} 2^{-n(I(X;Y)-3\varepsilon)} < \varepsilon + 2^{nR}2^{-n(I(X;Y)-3\varepsilon)}$$

(2) Joint AEP

$$\le \varepsilon + 2^{-n(C-R-3\varepsilon)} \le 2\varepsilon \quad \text{for} \quad R < C - 3\varepsilon \text{ and } n > -\frac{\log\varepsilon}{C-R-3\varepsilon}$$

$P(E) \le 2\varepsilon$: the error prob. can be arbitrarily small if

$R < C - 3\varepsilon$ : rate < capacity

$n > -\frac{\log\varepsilon}{C-R-3\varepsilon}$ : n large enough

we have chosen $p(x)$ such that $I(X;Y) = C$

- Since average of $P(E)$ over all codes is $\le 2\varepsilon$ there must be at least one code for which this is true: this code has $2^{-nR}\sum_{w}\lambda_w \le 2\varepsilon$ ♦
- Now throw away the worst half of the codewords; the remaining ones must all have $\boxed{\lambda_w \le 4\varepsilon}$. The resultant code has rate $R - n^{-1} \cong R$. ♦

♦ = proved on next page

# Code Selection & Expurgation

$2\varepsilon \ge \frac{1}{M}\sum_w \lambda_w = \frac{1}{M}\Big(\underbrace{\sum_{w=1}^{M/2}\lambda_w}_{\text{bad (v)}} + \underbrace{\sum_{w=\frac{M}{2}+1}^{M}\lambda_w}_{\text{good (x)}}\Big) \ge \frac{1}{M}\sum_{w=1}^{M/2}\lambda_w \ge \frac{1}{M}\frac{M}{2}\lambda_{\frac{M}{2}} = \frac{1}{2}\lambda_{\frac{M}{2}}$

$\Rightarrow \lambda_{\frac{M}{2}} \le 4\varepsilon.$

rate: $R' = \frac{\log\frac{M}{2}}{n} = \frac{\log M - \log 2}{n} = \frac{\log M}{n} - \frac{\log 2}{n} = R - \frac{1}{n} \xrightarrow{n\to\infty} R$

- Since average of $P(E)$ over all codes is $\le 2\varepsilon$ there must be at least one code for which this is true.

  Proof:
  $$2\varepsilon \ge K^{-1}\sum_{i=1}^{K} P_{e,i}^{(n)} \ge K^{-1}\sum_{i=1}^{K}\min_i\left(P_{e,i}^{(n)}\right) = \min_i\left(P_{e,i}^{(n)}\right)$$

  $K$ = num of codes

- Expurgation: Throw away the worst half of the codewords; the remaining ones must all have $\lambda_w \le 4\varepsilon$.

  code: size=M

  throw away bad half

  sort by error prob. $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_M$

  Proof: Assume $\lambda_w$ are in descending order

  $$2\varepsilon \ge M^{-1}\sum_{w=1}^{M}\lambda_w \ge M^{-1}\sum_{w=1}^{½M}\lambda_w \ge M^{-1}\sum_{w=1}^{½M}\lambda_{½M} \ge ½\,\lambda_{½M}$$

  $$\Rightarrow \quad \lambda_{½M} \le 4\varepsilon \quad \Rightarrow \quad \lambda_w \le 4\varepsilon \quad \forall\ w > ½M$$

$M' = ½ \times 2^{nR}$ messages in $n$ channel uses $\Rightarrow \quad R' = n^{-1}\log M' = R - n^{-1}$

# Summary of Procedure

sufficient large

- For any $R < C - 3\varepsilon$ set $n = \max\left\{N_\varepsilon, -(\log \varepsilon)/(C - R - 3\varepsilon), \varepsilon^{-1}\right\}$

see (a),(b),(c) below

- Find the optimum $\mathbf{p}_X$ so that $I(x\,;\,y) = C$

$M = 2^{nR}$

- Choosing codewords randomly (using $\mathbf{p}_X$) to construct codes with $2^{nR}$ codewords and using joint typicality as the decoder (a)
- Since average of $P(E)$ over all codes is $\leq 2\varepsilon$ there must be at least one code for which this is true. (b)
- Throw away the worst half of the codewords. Now the worst codeword has an error prob $\leq 4\varepsilon$ with rate $= R - n^{-1} > R - \varepsilon$ (c)
- The resultant code transmits at a rate as close to $C$ as desired with an error probability that can be made as small as desired (but $n$ unnecessarily large).

Note: $\varepsilon$ determines both error probability and closeness to capacity

# Remarks

- Random coding is a powerful method of proof, not a method of signaling
- Picking randomly will give a good code
- But $n$ has to be large (AEP)
- Without a structure, it is difficult to encode/decode
  - Table lookup requires exponential size
- Channel coding theorem does not provide a practical coding scheme
- Folk theorem (but outdated now):
  - Almost all codes are good, except those we can think of

# Lecture 11

- Converse of Channel Coding Theorem
  - Cannot achieve R>C
- Capacity with feedback
  - No gain for DMC but simpler encoding/ decoding
- Joint Source-Channel Coding
  - No point for a DMC

$w_{1:nR}$ i.i.d. $\Rightarrow H(w_i) = 1 \Rightarrow H(\underline{w}) = nR$ (by summation)

# Converse of Coding Theorem

- Fano's Inequality: if $P_e^{(n)}$ is error prob when estimating *w* from **y**,

$$H(w\,|\,\mathbf{y}) \le 1 + P_e^{(n)} \underbrace{\log|W|}_{H(w)} = 1 + \underbrace{nR}_{H(w)} P_e^{(n)}$$

- Hence $nR = H(w) = H(w\,|\,\mathbf{y}) + I(w\,;\mathbf{y})$ Definition of $I$

$w \to \mathbf{x}(w)$

$$\le H(w\,|\,\mathbf{y}) + I(\mathbf{x}(w);\mathbf{y})$$ Markov : $w \to \mathbf{x} \to \mathbf{y} \to \hat{w}$

$$\le 1 + nRP_e^{(n)} + I(\mathbf{x};\mathbf{y})$$ Fano

$$\le 1 + nRP_e^{(n)} + nC$$ $n$-use DMC capacity

$$\Rightarrow P_e^{(n)} \ge \frac{R - C - n^{-1}}{R} \underset{n\to\infty}{\to} 1 - \frac{C}{R} > 0 \text{ if } R > C$$

lower bound of $P_e$

$nR \le 1 + nRP_e^{(n)} + nC$

$P_e^{(n)} \ge \dfrac{R - C - \frac{1}{n}}{R}$

$\underline{\underline{n\to\infty}}$ $1 - \frac{C}{R}$ $\begin{cases} > 0 \text{ if } R > C \\ \le 0 \text{ (can be 0) if } R \le C \end{cases}$

cannot achieve zero error rate!

- For large (hence for all) $n$, $P_e^{(n)}$ has a lower bound of $(R-C)/R$ if *w* equiprobable
  – If achievable for small $n$, it could be achieved also for large $n$ by concatenation.

# Minimum Bit-Error Rate


## Suppose

$H(w)=nR$

- $w_{1:nR}$ is i.i.d. bits with $H(w_i)=1$
- The bit-error rate is $P_b = \underset{i}{E}\left\{ p(w_i \neq \hat{w}_i)\right\} \overset{\Delta}{=} \underset{i}{E}\left\{ p(e_i)\right\}$

## Then

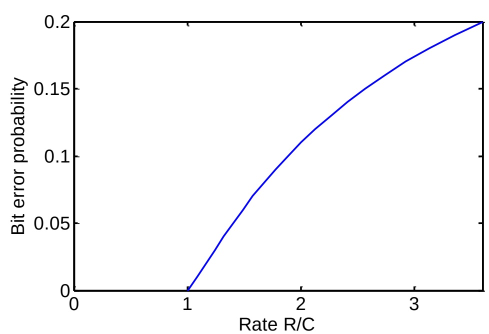
$$nC \overset{(a)}{\geq} I(x_{1:n};y_{1:n}) \overset{(b)}{\geq} I(w_{1:nR};\hat{w}_{1:nR}) = H(w_{1:nR}) - H(w_{1:nR} \mid \hat{w}_{1:nR})$$

$$= nR - \sum_{i=1}^{nR} H(w_i \mid \hat{w}_{1:nR}, w_{1:i-1}) \overset{(c)}{\geq} nR - \sum_{i=1}^{nR} H(w_i \mid \hat{w}_i) = nR\left(1 - \underset{i}{E}\left\{H(w_i \mid \hat{w}_i)\right\}\right)$$

$$\overset{(d)}{=} nR\left(1 - \underset{i}{E}\left\{H(e_i \mid \hat{w}_i)\right\}\right) \overset{(c)}{\geq} nR\left(1 - \underset{i}{E}\left\{H(e_i)\right\}\right) \overset{(e)}{\geq} nR\left(1 - H\left(\underset{i}{E}\, P(e_i)\right)\right) = nR\left(1 - H(P_b)\right)$$

## Hence

$$\boxed{\begin{aligned} R &\leq C\left(1 - H(P_b)\right)^{-1} \\ P_b &\geq H^{-1}(1 - C/R) \end{aligned}}$$




(a) n-use capacity

(b) Data processing theorem

(c) Conditioning reduces entropy

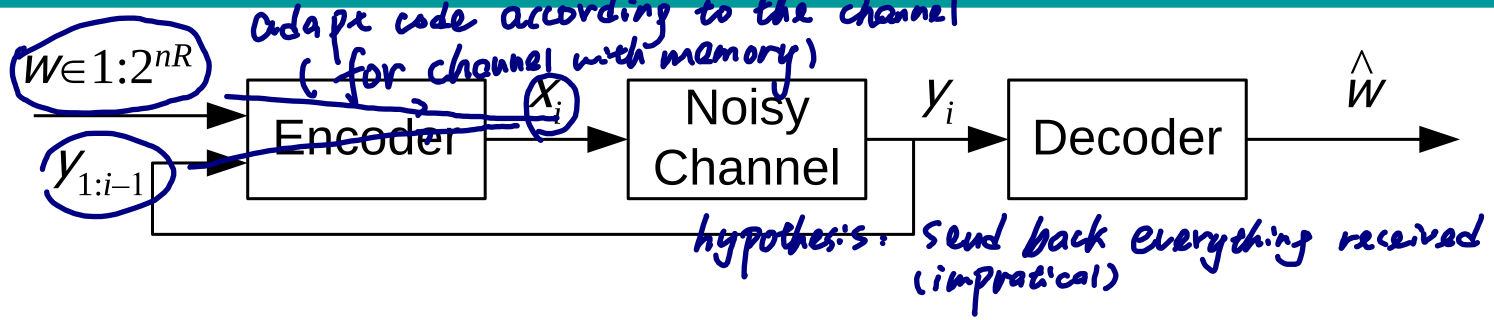
(d) $e_i = w_i \oplus \hat{w}_i$

(e) Jensen: $E\,H(x) \leq H(E\,x)$

# Coding Theory and Practice

- Construction for good codes
  - Ever since Shannon founded information theory
  - Practical: Computation & memory $\propto n^k$ for some $k$
- Repetition code: rate → 0
- Block codes: encode a block at a time
  - Hamming code: correct one error
  - Reed-Solomon code, BCH code: multiple errors (1950s)
- Convolutional code: convolve bit stream with a filter
- Concatenated code: RS + convolutional
- Capacity-approaching codes:
  - Turbo code: combination of two interleaved convolutional codes (1993)
  - Low-density parity-check (LDPC) code (1960)
  - Dream has come true for some channels today

# Channel with Feedback



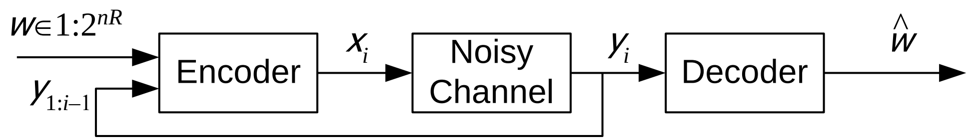

- Assume error-free feedback: does it increase capacity ?
- A $(2^{nR}, n)$ feedback code is
  - A sequence of mappings $x_i = x_i(w, y_{1:i-1})$ for $i=1:n$ (depends)
  - A decoding function $\hat{w} = g(y_{1:n})$
- A rate R is achievable if ∃ a sequence of $(2^{nR}, n)$ feedback codes such that $P_e^{(n)} = P(\hat{w} \neq w) \xrightarrow[n\to\infty]{} 0$
- Feedback capacity, $C_{FB} \geq C$, is the sup of achievable rates

intuition: $C_{FB} = C$ if abandon all feedbacks

Fano's ineq. ⇒ prove impossible



# Feedback Doesn't Increase Capacity




$$I(W;\mathbf{y}) = H(\mathbf{y}) - H(\mathbf{y}\,|\,W)$$

$$= H(\mathbf{y}) - \sum_{i=1}^{n} H(y_i\,|\,\underbrace{y_{1:i-1}, W}_{})$$

$x_i \sim \begin{cases} W \\ y_{1:n-1} \end{cases}$

↓ no new conditions

$$= H(\mathbf{y}) - \sum_{i=1}^{n} H(y_i\,|\,y_{1:i-1}, W, x_i)$$ since $x_i = x_i(w, y_{1:i-1})$

↓ $y_i$ only depends on $x_i$

$$= H(\mathbf{y}) - \sum_{i=1}^{n} H(y_i\,|\,x_i)$$ since $y_i$ only directly depends on $x_i$

$H(y) = \sum_{i=1}^{n} H(y_i\,|\,y_{1:i-1}) \le \sum_{i=1}^{n} H(y_i)$

$$\le \sum_{i=1}^{n} H(y_i) - \sum_{i=1}^{n} H(y_i\,|\,x_i) = \sum_{i=1}^{n} I(x_i;y_i) \le nC$$

cond reduces ent

DMC

Hence

$H(W|\mathbf{y}) \le 1 + nRP_e^{(n)}$

$$nR = H(W) = \underline{H(W\,|\,\mathbf{y})} + I(W;\mathbf{y}) \le \underline{1 + nRP_e^{(n)}} + nC$$ Fano

$$\Rightarrow P_e^{(n)} \ge \frac{R - C - n^{-1}}{R}$$ ➔ Any rate > $C$ is unachievable

The DMC does not benefit from feedback: $C_{FB} = C$

$E\{k\} = (1-f) + 2(f - f^2) + 3(f^2 - f^3) + \cdots + k(f^{n-1} - f)$

$= 1 + f + f^2 + \cdots = \frac{1}{1-f}$

# Example: BEC with feedback

k: # transmissions required to recover one bit.

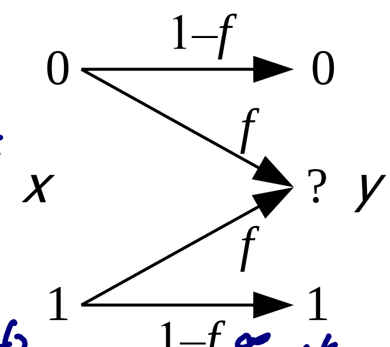
$k=1 \quad P_1 = 1-f$

$k=2 \quad P_2 = f(1-f) = f - f^2$

$k=3 \quad P_3 = f^2(1-f) = f^2 - f^3$

$k \quad P_k = f^{k-1}(1-f) = f^{k-1} - f^k$

- Capacity is $1-f$
- Encode algorithm
  - If $y_i = ?$, tell the sender to retransmit bit $i$
  - Average number of transmissions per bit:




Capacity: number of successful transmission bits per transmission

$$\underbrace{1 + f + f^2 + \cdots = \frac{1}{1-f}}$$

$E\{k\} = \sum_{k=1}^{\infty} k f^{k-1}(1-f)$

$h(f) = H'(f)$

$h(f) = \sum_{k=1}^{\infty} k f^{k-1}$

$H(f) = \sum_{k=1}^{\infty} f^k$

$= \sum_{k=0}^{\infty} f^k - 1$

$= \frac{1}{1-f} - 1 = \frac{f}{1-f}$

- Average number of successfully recovered bits per transmission $= 1-f$
  - Capacity is achieved!

$H'(f) = \frac{1-f+f}{(1-f)^2} = \frac{1}{(1-f)^2}$

$E\{k\} = \frac{1}{(1-f)^2} \cdot (1-f) = \frac{1}{1-f}$

- Capacity unchanged but encoding/decoding algorithm much simpler.

# Joint Source-Channel Coding

$$w_{1:n} \longrightarrow \boxed{\text{Encoder}} \xrightarrow{x_{1:n}} \boxed{\text{Noisy Channel}} \xrightarrow{y_{1:n}} \boxed{\text{Decoder}} \xrightarrow{\hat{w}_{1:n}}$$

- Assume $w_i$ satisfies AEP and $|W|<\infty$
  - Examples: i.i.d.; Markov; stationary ergodic
- Capacity of DMC channel is $C$
  - if time-varying: $C = \lim_{n\to\infty} n^{-1} I(\mathbf{x};\mathbf{y})$

source code  channel code

$H(W) < \bar{R} < C$

- Joint Source-Channel Coding Theorem: ♦

$\exists$ codes with $P_e^{(n)} = P(\hat{w}_{1:n} \neq w_{1:n}) \xrightarrow[n\to\infty]{} 0$ iff $H(W) < C$

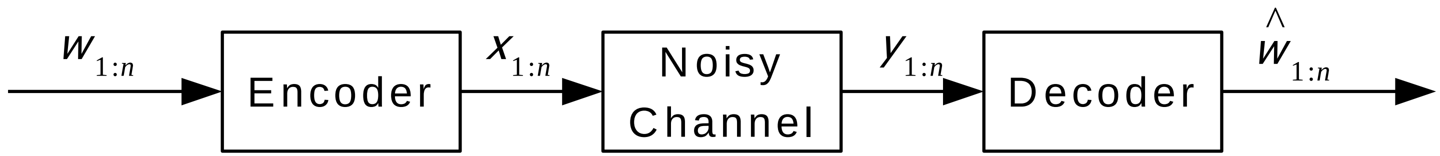
  - errors arise from two reasons
    - Incorrect encoding of **w** (source coding)
    - Incorrect decoding of **y** (channel coding)

♦ = proved on next page

# Source-Channel Proof ($\Leftarrow$)

- Achievability is proved by using two-stage encoding
  - Source coding
  - Channel coding
- For $n > N_\varepsilon$ there are only $2^{n(H(\mathrm{W})+\varepsilon)}$ **w**'s in the typical set: encode using $n(H(W)+\varepsilon)$ bits
  - encoder error < $\varepsilon$
- Transmit with error prob less than $\varepsilon$ so long as $H(W)+\varepsilon < C$
- Total error prob $< 2\varepsilon$

# Source-Channel Proof ($\Rightarrow$)




**Fano's Inequality:** $H(\mathbf{w} \mid \hat{\mathbf{w}}) \le 1 + P_e^{(n)} n \log |\mathrm{W}|$

$$H(W) \le n^{-1} H(w_{1:n})$$ entropy rate of stationary process

$$= n^{-1} H(w_{1:n} \mid \hat{w}_{1:n}) + n^{-1} I(w_{1:n}; \hat{w}_{1:n})$$ definition of $I$

$$\le n^{-1}\left(1 + P_e^{(n)} n \log |\mathrm{W}|\right) + n^{-1} I(x_{1:n}; y_{1:n})$$ Fano + Data Proc Inequ

$$\le n^{-1} + P_e^{(n)} \log |\mathrm{W}| + C$$ Memoryless channel

Let $n \to \infty \quad \Rightarrow \quad P_e^{(n)} \to 0 \quad \Rightarrow \quad H(W) \le C$

# Separation Theorem

- Important result: source coding and channel coding might as well be done separately since same capacity
  - Joint design is more difficult (for linear channels)
- Practical implication: for a DMC we can design the source encoder and the channel coder separately
  - Source coding: efficient compression
  - Channel coding: powerful error-correction codes
- Not necessarily true for
  - Correlated channels
  - Multiuser channels
- Joint source-channel coding: still an area of research
  - Redundancy in human languages helps in a noisy environment

# Summary

- Converse to channel coding theorem
  - Proved using Fano's inequality
  - Capacity is a clear dividing point:
    - If R < C, error prob. → 0
    - Otherwise, error prob. → 1
- Feedback doesn't increase the capacity of DMC
  - May increase the capacity of memory channels (e.g., ARQ in TCP/IP)
- Source-channel separation theorem for DMC and stationary sources

# Lecture 12

- Polar codes
  - Channel polarization
  - How to construct polar codes
  - Encoding and decoding
- Polar source coding
- Extension

# About Polar Codes

- Provably capacity-achieving
- Encoding complexity $O(N \log N)$
- Successive decoding complexity $O(N \log N)$
- Probability of error $\approx 2^{-\sqrt{N}}$
- Main idea: channel polarization

# What Is Channel Polarization?

- Normal channel
- Extreme channel

channel

sometimes cute,
sometimes lazy,
hard to manage

**Polarization**

Useless channel

Perfect channel

# Channel Polarization

- Among all channels, there are two classes which are easy to communicate optimally
  - The perfect channels

    the output Y determines the input X
  - The useless channels

    Y is independent of X
- Polarization is a technique to convert noisy channels to a mixture of extreme channels
  - The process is information-conserving

$F_2 \otimes F_2 = \begin{bmatrix} F_2 & 0 \\ F_2 & F_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$F_8 = F_2 \otimes F_2 \otimes F_2 = F_2 \otimes F_4$

# Generator Matrix

- Generator Matrix

$$\mathbf{F}_N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes n}, N = 2^n$$

$\otimes n$ denotes the $n$-fold Kronecker product.

- Example

$$\mathbf{F}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{F}_4 = \begin{bmatrix} \mathbf{F}_2 & 0 \\ \mathbf{F}_2 & \mathbf{F}_2 \end{bmatrix} \text{ and so on.}$$

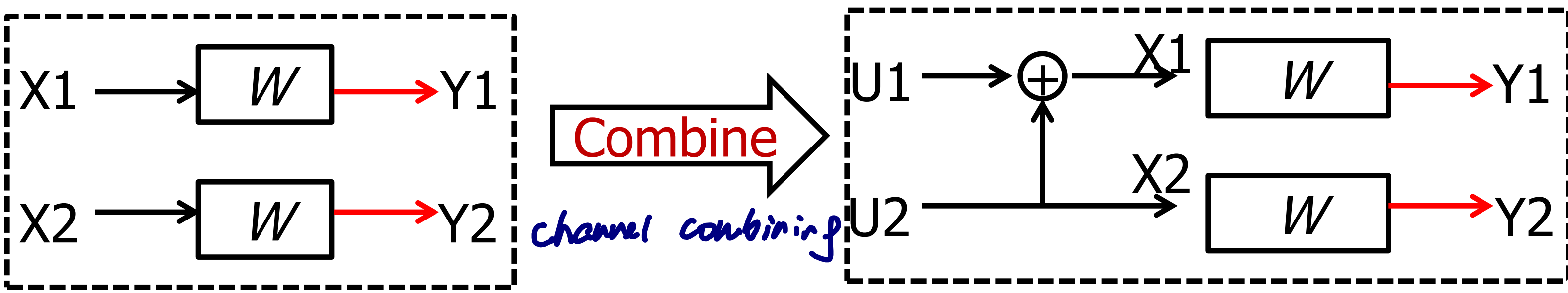
- Encoding

Let $\mathbf{u}$ be the length-$N$ input to the encoder, then $\mathbf{x} = \mathbf{u}\mathbf{F}_N$ is the codeword.

# Channel Combining and Splitting

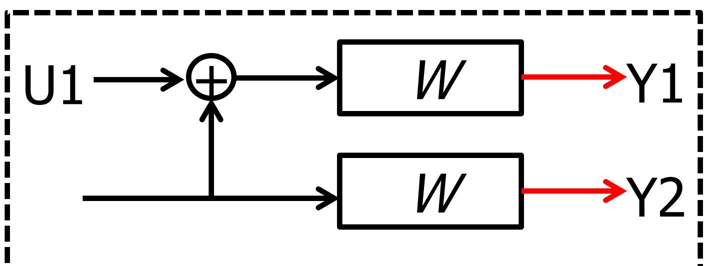
$\underline{x} = \underline{u} F_N$ $\quad [x_1\ x_2] = [u_1\ u_2] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = [u_1+u_2\ \ u_2]$

- Basic operation ($N = 2$)

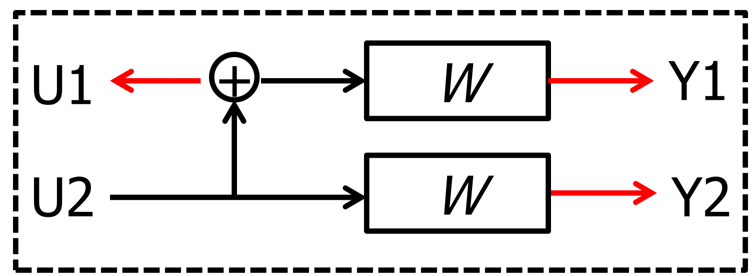



(X1, X2)=(U1, U2)$\mathbf{F}_2$

Channel splitting




channel splitting




$W^-$: U1→(Y1, Y2) $\qquad$ $W^+$: U2→(Y1, Y2, U1)

# What Happens?

$W^-$

$(y_1, y_2) =$ ① $(u_1+u_2, u_2)$ $(1-p)^2$ decode $u_1$? ①

② $(?, u_2)$ $p(1-p)$ $W^-$: erasure prob. $1-(1-p)^2 = 2p-p^2 > p$ ($p<1$) (gets worse!)

③ $(u_1+u_2, ?)$ $p(1-p)$

④ $(?, ?)$ $p^2$

- Suppose $W$ is a BEC($p$), i.e., Y=X with probability $1-p$, and Y=? (erasure) with probability $p$.

$W^+$

$(y_1, y_2)$ ① $(u_1+u_2, u_2)$ $(1-p)^2$ decode $u_2$ (assume $u_1$ known)? ① ② ③

② $(?, u_2)$ $p(1-p)$ $W^+$: erasure prob. $p^2 < p$ ($p<1$) (gets better!)

③ $(u_1+u_2, ?)$ $p(1-p)$

④ $(?, ?)$ $p^2$

$u_1$ is known

  - $W^-$ has input U1 and output (Y1, Y2)=(U1+U2, U2) or (?, U2) or (U1+U2, ?) or (?, ?).
  - $W^-$ is a BEC($2p-p^2$)

$C(W^-) = 1-(2p-p^2) = 1-2p+p^2 = (1-p)^2$

$C(W^+) = 1-p^2$

  - $W^+$ has input U2 and output (Y1, Y2, U1)=(U1+U2, U2, U1) or (?, U2, U1) or (U1+U2, ?, U1) or (?, ?, U1).

$C(W^+) - C(W^-) = 1-p^2-(1-2p+p^2) = -2p^2+2p = 2p(1-p) \geq 0$

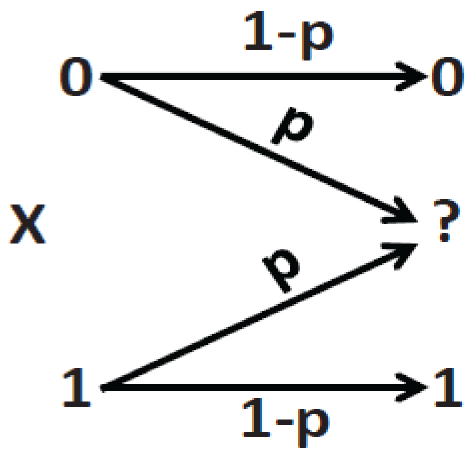
$C(W^-)+C(W^+) = 2(1-p) = 2C(W)$

  - $W^+$ is a BEC($p^2$)

- $W^-$ is worse than $W$, and $W^+$ is better (recall capacity $C(W)=1-p$).
  - $C(W^-)+C(W^+) = 2C(W)$
  - $C(W^-) \leq C(W) \leq C(W^+)$

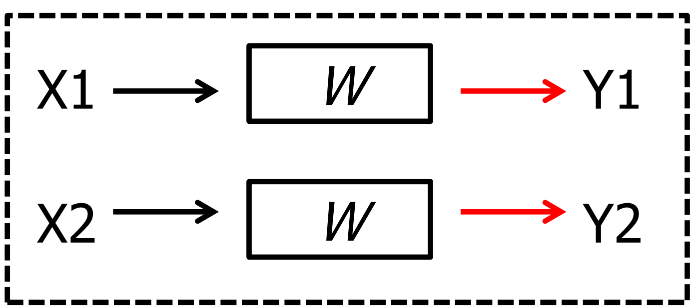
# Example: BEC(0.5)

- $W$ is a BEC with erasure probability p = 0.5.




$$\begin{array}{cc} & Y \\ X & \boxed{\begin{matrix} 1-p & p & 0 \\ 0 & p & 1-p \end{matrix}} \end{array}$$

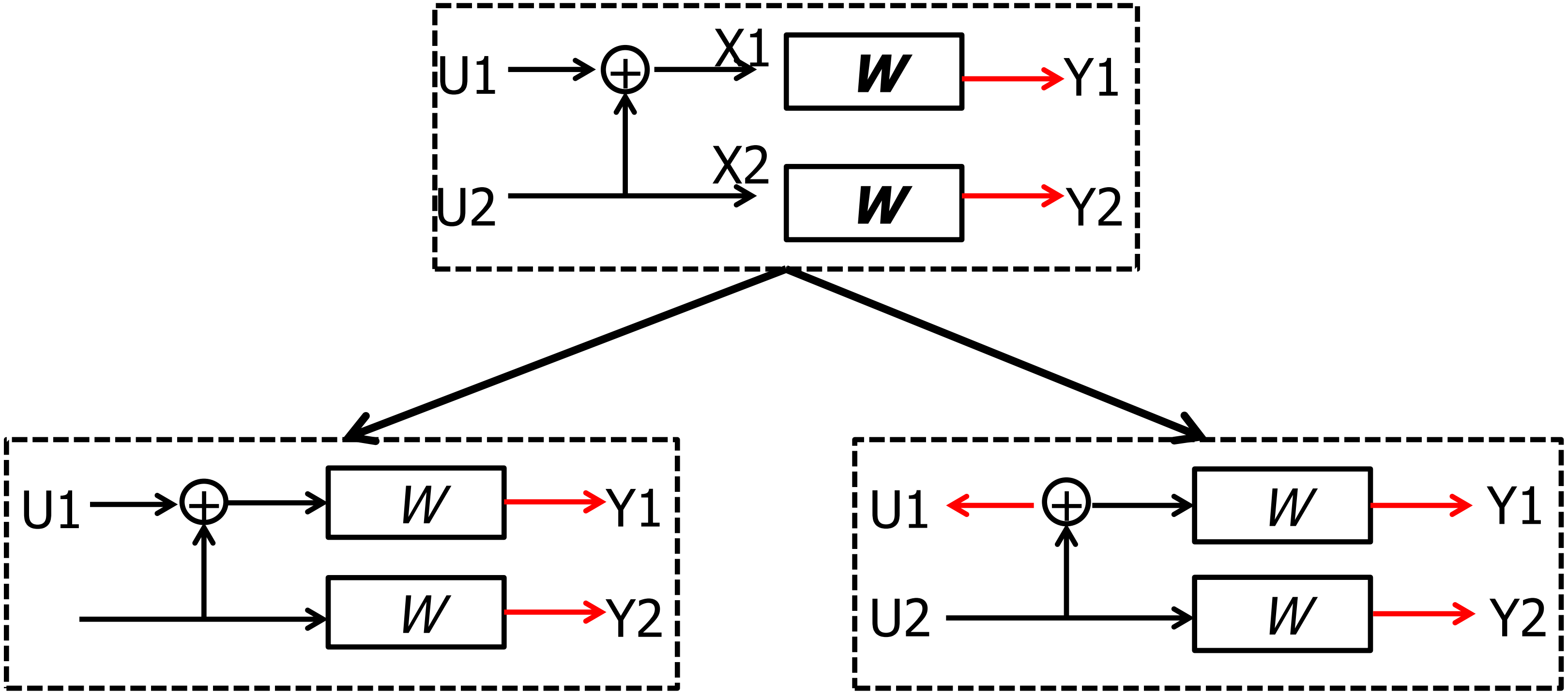
- If we use two copies of $W$ separately




$$2C(W) = 2 \times 0.5 = 1$$

# Example: BEC(0.5)

- Channel combining and splitting



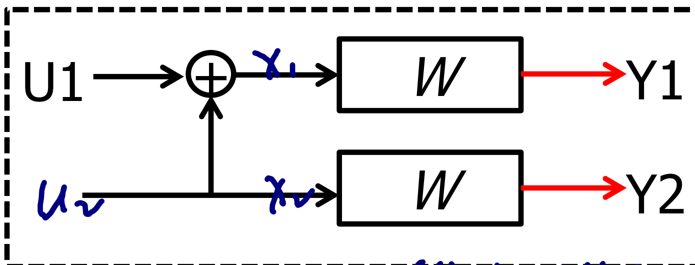
# Example: BEC(0.5)

$[X_1\ X_2] = [u_1\ u_2]\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$X_1 = U_1 + U_2 \rightarrow y_1$

$X_2 = U_2 \rightarrow y_2$

- Channel $W^-$




$$C(W^-) = 4 \times \frac{1}{16} \underline{\log 2} = 0.25$$

Transitional probabilities

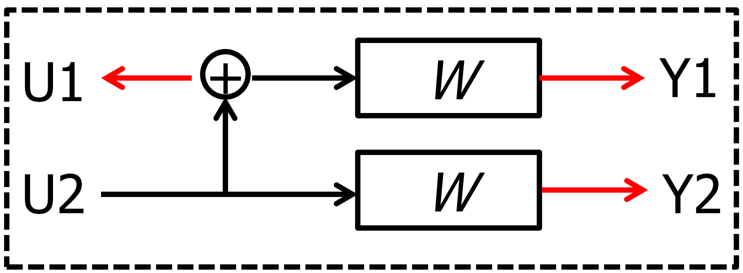
(Y1,Y2)

| | 00 | 0? | 01 | ?0 | ?? | ?1 | 10 | 1? | 11 |
|---|---|---|---|---|---|---|---|---|---|
| | $\begin{cases} u_1+u_2=0 \\ u_2=0 \end{cases} \Rightarrow \begin{cases} u_1=0 \\ u_2=0 \end{cases}$ | $\begin{cases} u_1+u_2=0 \\ u_2=? \end{cases} \Rightarrow \begin{cases} u_1=0 \\ u_2=0 \end{cases}$ | $\begin{cases} u_1+u_2=0 \\ u_2=1 \end{cases}$ ✗ | $\begin{cases} u_1+u_2=? \\ u_2=0 \end{cases} \Rightarrow \begin{cases} u_1=? \\ u_2=0 \end{cases}$ | $\begin{cases} u_1+u_2=? \\ u_2=? \end{cases} \Rightarrow \begin{cases} u_1=? \\ u_2=? \end{cases}$ | $\begin{cases} u_1+u_2=? \\ u_2=1 \end{cases} \Rightarrow \begin{cases} u_1=? \\ u_2=1 \end{cases}$ | $\begin{cases} u_1+u_2=1 \\ u_2=0 \end{cases} \Rightarrow \begin{cases} u_1=1 \\ u_2=0 \end{cases}$ | $\begin{cases} u_1+u_2=1 \\ u_2=? \end{cases} \Rightarrow \begin{cases} u_1=? \\ u_2=? \end{cases}$ | $\begin{cases} u_1+u_2=1 \\ u_2=1 \end{cases} \Rightarrow \begin{cases} u_1=? \\ u_2=1 \end{cases}$ |
| U1 (sent) 0 | $u_2$ ↓ ch 1/8 $\frac{1}{2} \times \frac{1}{4}$ | 1/8 $\frac{1}{2} \times \frac{1}{4}$ | 0 | 1/8 $\frac{1}{2} \times \frac{1}{4}$ | 1/4 $1 \times \frac{1}{4}$ | 1/8 $\frac{1}{2} \times \frac{1}{4}$ | 0 | 1/8 $\frac{1}{2} \times \frac{1}{4}$ | 1/8 $\frac{1}{2} \times \frac{1}{4}$ |
| 1 | 0 | 1/8 | 1/8 | 1/8 | 1/4 | 1/8 | 1/8 | 1/8 | 0 |
| | ✓ | | ✓ | | | | ✓ | | ✓ |

only 1 possible input given output ⇒ valid transmission

# Example: BEC(0.5)

- Channel $W^+$




$\frac{1}{8}\log_2\frac{1}{4}$

$$C(W^+) = 12 \times \frac{1}{16}\log 2 = 0.75$$

$$C(W^-) + C(W^+) = 2C(W)$$

$$C(W^-) < C(W) < C(W^+)$$

(Y1,Y2,U1) Transitional probabilities

U2

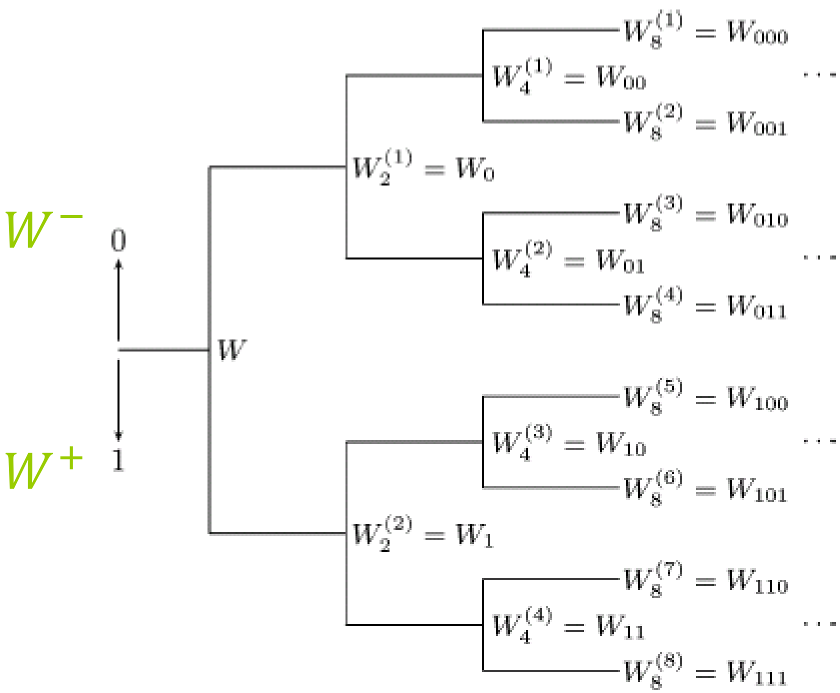
| | 000 | 0?0 | 010 | ?00 | ??0 | ?10 | 100 | 1?0 | 110 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1/8 | 1/8 | 0 | 1/8 | 1/8 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1/8 | 1/8 | 0 | 1/8 | 1/8 |

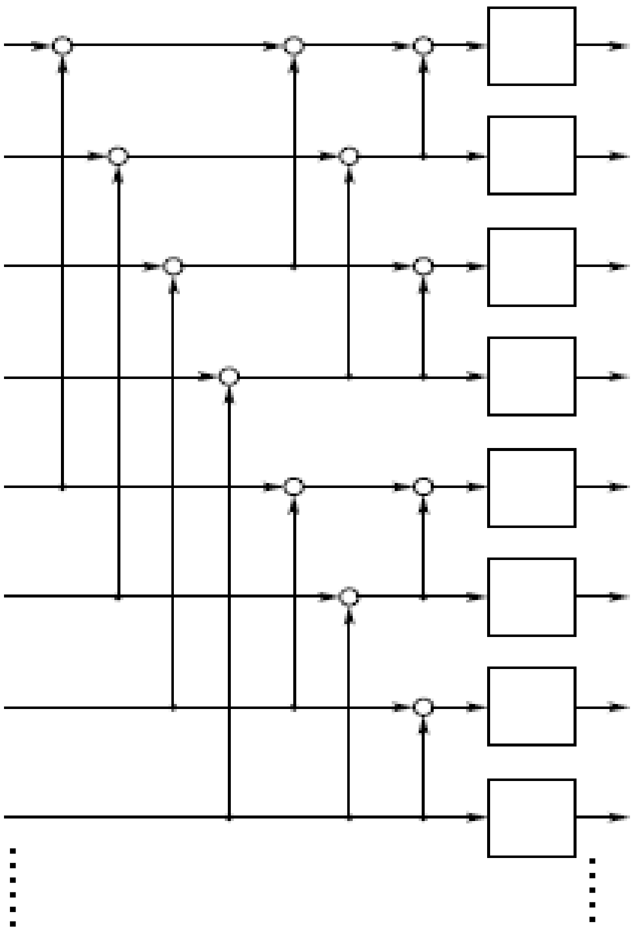
impossible confused!

| | 001 | 0?1 | 011 | ?01 | ??1 | ?11 | 101 | 1?1 | 111 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1/8 | 1/8 | 0 | 1/8 | 1/8 | 0 |
| 1 | 0 | 1/8 | 1/8 | 0 | 1/8 | 1/8 | 0 | 0 | 0 |

# More Polarization

- Repeating this, we obtain *N* 'bit channels' at the *n*-th step.
- More conveniently, this process can be described as a binary tree.
  - –Note how the 'bit channels' $W_{b_1 b_2 \cdots b_n}$ are labelled in the tree.



# Martingale

- Now pick a 'bit channel' uniformly at random on the *n*-th level of the tree, which is equivalent to a random traverse on the tree, namely, at each step the r.v $b_i$ takes the value of 0 or 1 with equal probability.
- We claim capacity $C_n$ at the *n*-th step is a martingale.
- Proof: By information-preserving

$$E[C_{n+1}|b_1,\ldots,b_n] = \frac{1}{2}C\left(W_{b_1 b_2 \ldots b_n 0}\right) + \frac{1}{2}C\left(W_{b_1 b_2 \ldots b_n 1}\right)$$
$$= C\left(W_{b_1 b_2 \ldots b_n}\right) = C_n$$

- By the martingale convergence theorem, $C_n$ converges to a random variable $C_\infty$ such that $E[C_\infty] = E[C_0] = C_0 = C(W)$.
- In fact, the limit $C_\infty = 0$ or 1 is a binary random variable (these are the fixed points of the polar transform).

# Review of Martingales

- Let $\{X_n, n \geq 0\}$ be a random process. If
$$E[X_{n+1}|X_n, \ldots, X_1, X_0] = X_n$$
then $\{X_n\}$ is referred to as a martingale.
- Martingale convergence theorem: Let $\{X_n, n \geq 0\}$ be a martingale with finite means. Then there exists a random variable $X_\infty$ such that
$$X_n \to X_\infty \quad \text{almost surely}$$
as $n \to \infty$.

# How to construct polar codes

- To achieve $C(W)$, we need to identify the indices of those bit channels (branches in tree) with capacity $\approx 1$.
- For BEC, this can be computed recursively

$$C\left(W_{b_1b_2\dots b_n0}\right) = C\left(W_{b_1b_2\dots b_n}\right)^2$$

$$C\left(W_{b_1b_2\dots b_n1}\right) = 2C\left(W_{b_1b_2\dots b_n}\right) - C\left(W_{b_1b_2\dots b_n}\right)^2$$

- For other types of channels, it is difficult to obtain closed-form formulas. So numerical computation is often used.

perfect useless

$F_N$

$R = \frac{NC(W)}{N} = C(W)$

# Polarization Speed

- For any positive real number $\beta < 0.5$,

$$\lim_{n\to\infty} \frac{1}{N} \# \left\{(b_1 \cdots b_n) : C\left(W_{b_1 b_2 \ldots b_n}\right) \geq 1 - 2^{-N^{\beta}}\right\}$$

the fraction of good channels equals the capacity.

$$= C(W).$$

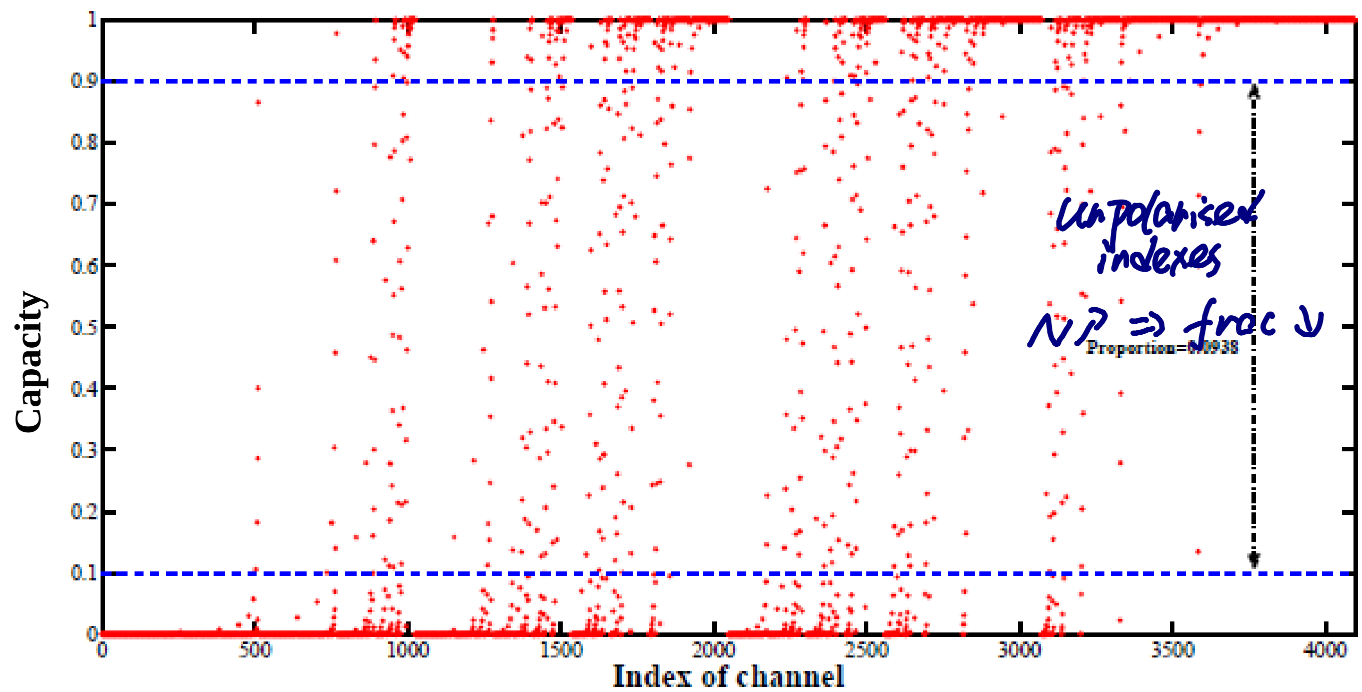
$$\lim_{n\to\infty} \frac{1}{N} \# \left\{(b_1 \cdots b_n) : C\left(W_{b_1 b_2 \ldots b_n}\right) < 1 - 2^{-N^{\beta}}\right\}$$

$$= 1 - C(W).$$

- The above statements do not hold for $\beta > 0.5$.
- Thus, the polarization speed is roughly $2^{-\sqrt{N}}$.

# Convergence

- The portion of almost prefect bit channels is $C(W)$, meaning that the capacity is achieved.
- Example: capacities for $N = 2^{12}$ for BEC(0.5)



# Encoding

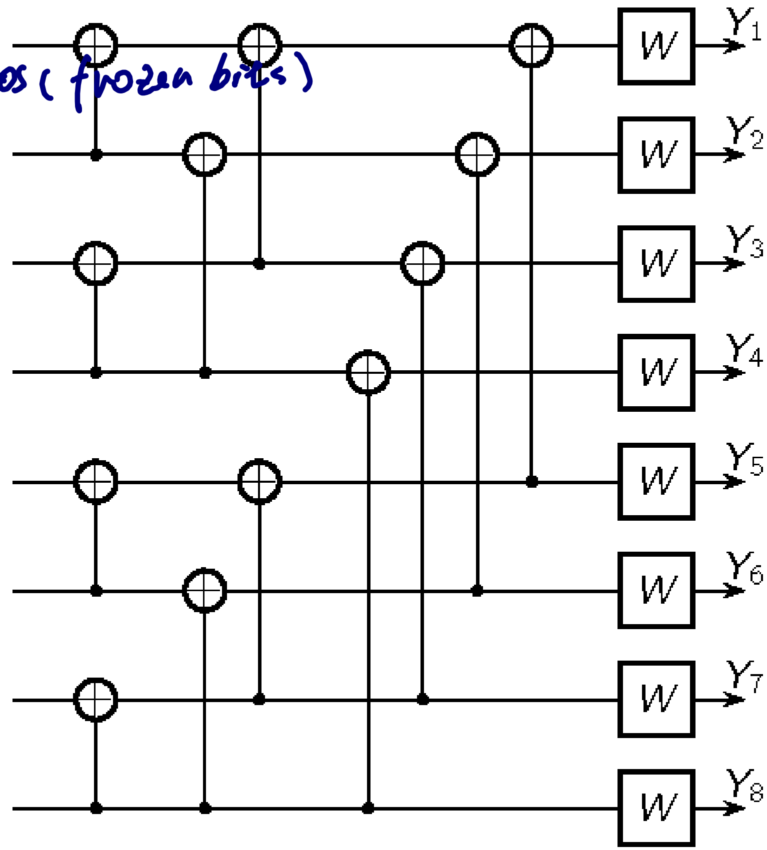
- Given $N = 2^n$, calculate $C\left(W_{b_1 b_2 \ldots b_n}\right)$ for all synthetic bit channels.
- Given rate $R < 1$ and $K = NR$, sort $C\left(W_{b_1 b_2 \ldots b_n}\right)$ in descending order and define the union of the indices of the first $K$ elements as the information set $\Omega$.
- Choose the information bits $u^{\Omega}$ and freeze $u^{\Omega^c}$ to be all-zero. Obtain the codeword $\mathbf{x} = \left(u^{\Omega}, u^{\Omega^c}\right) \cdot \mathbf{F}_N$.

# Construction Example

- $W$ is a BEC(0.5), $N = 8$, $R$=0.5.

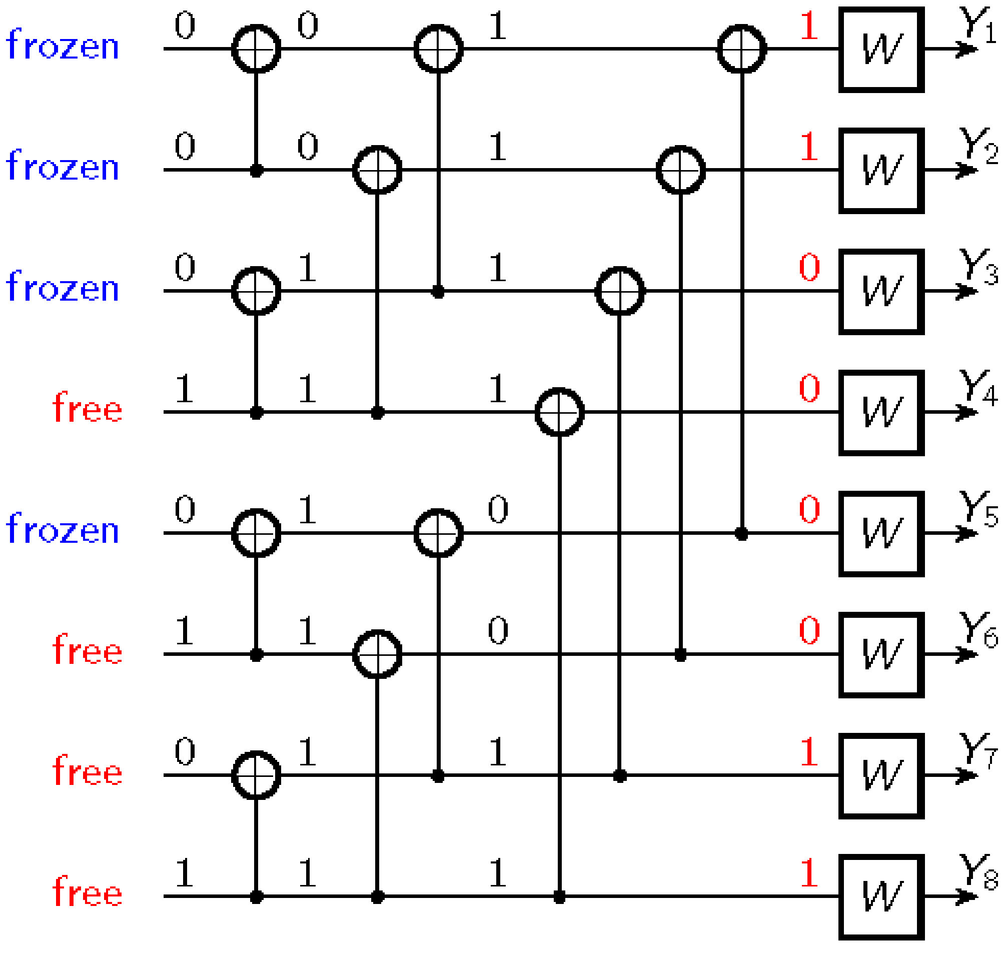
| $C(W_i)$ | Rank | | |
|---|---|---|---|
| 0.0039 | 8 | frozen | 0 |
| 0.1211 | 7 | frozen | 0 |
| 0.1914 | 6 | frozen | 0 |
| 0.6836 | 4 | data | $U_4$ |
| 0.3164 | 5 | frozen | 0 |
| 0.8086 | 3 | data | $U_6$ |
| 0.8789 | 2 | data | $U_7$ |
| 0.9961 | 1 | data | $U_8$ |

all zeros ( frozen bits )



# Construction Example

- $W$ is a BEC(0.5), $N = 8$, $R$=0.5.




Encoding complexity $O(N \log N)$

# Successive decoding

- For the decoding we need to compute the likelihood ratio for $u_i, i = (b_1 \cdots b_n)$

$$LR(u_i) = \frac{W_{b_1 b_2 \ldots b_n}(\cdot\,|1)}{W_{b_1 b_2 \ldots b_n}(\cdot\,|0)}$$

  If $i \in \Omega$, $\hat{u}_i = 1$ if $LR(u_i) > 1$; otherwise, $\hat{u}_i = 0$.

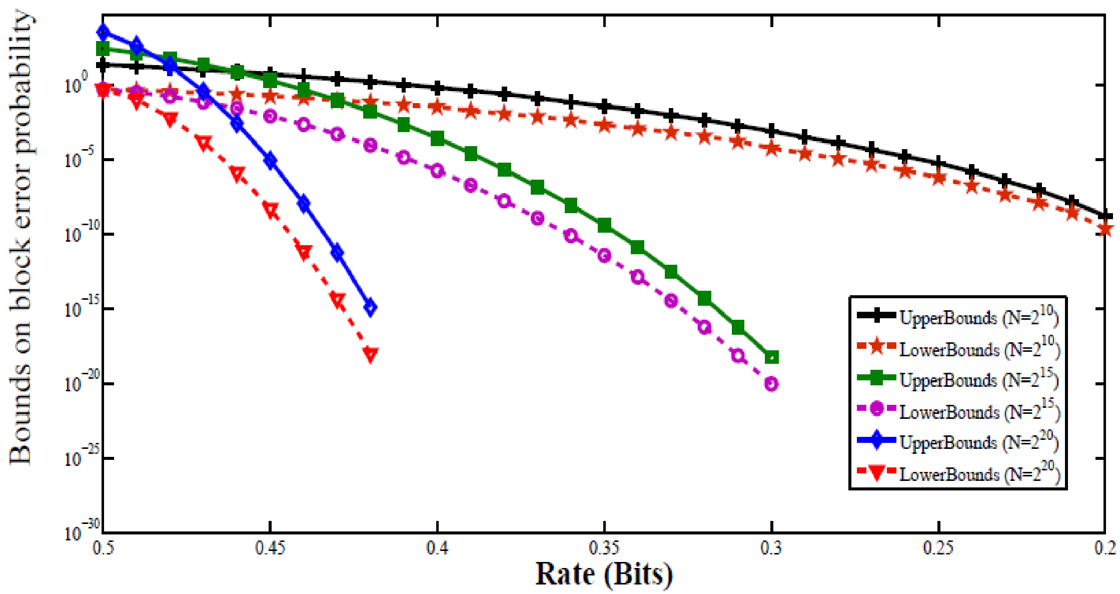
- Similar to $C\left(W_{b_1 b_2 \ldots b_n}\right)$, $LR(u_i)$ can also be calculated recursively.
- For more details of the decoding, see

E. Arikan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *IEEE Trans. on Information Theory,*, vol. 55, no. 7, pp. 3051–3073, 2009.

# Probability of Error

- For a polar code with block length $N$ and rate $R < C(W)$, the error probability under the successive cancellation decoding is given by

$$P_e \leq O(2^{-N^{\beta}}) \qquad \beta < 0.5$$




**Error Bound of the SC decoding for BEC(0.5)**

Polar code
↙ source coding
↘ channel coding



# Polar Source Coding

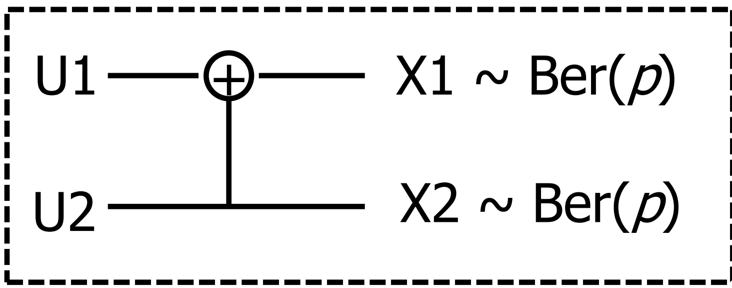
- Let $x$ be a random variable generated by a Bernoulli source Ber($p$), i.e.,
$$\Pr(x=0)=p \text{ and } \Pr(x=1)=1-p.$$
- The entropy (in bits) of $x$ is
$$H(x) = -p\log_2 p - (1-p)\log_2(1-p)$$
- If $H(x) = 0$, i.e., $p = 0$ or $1$, $x$ is a constant, no need for compression.
- If $H(x) = 1$, i.e., $p = 0.5$, $x$ is totally random, we cannot do any compression.
- In other cases, can the polarization technique be used to achieve rate $H(x)$?

# Source Polarization

- Similar idea applies to source coding:

  general sources $\xrightarrow{\text{polarization}}$ extreme sources

- Basic source polarization




$$(\mathrm{U1}, \mathrm{U2}) = (\mathrm{X1}, \mathrm{X2})\mathbf{F}_2$$

$$H(\mathrm{U1}) + H(\mathrm{U2}|\mathrm{U1}) = H(\mathrm{U1}, \mathrm{U2})$$
$$= H(\mathrm{X1}, \mathrm{X2}) = 2H(x)$$

$$H(\mathrm{U1}) \geq H(x) \geq H(\mathrm{U2}|\mathrm{U1})$$

The process is entropy-conserving, but we obtain two new sources with higher and lower entropy than the original one.

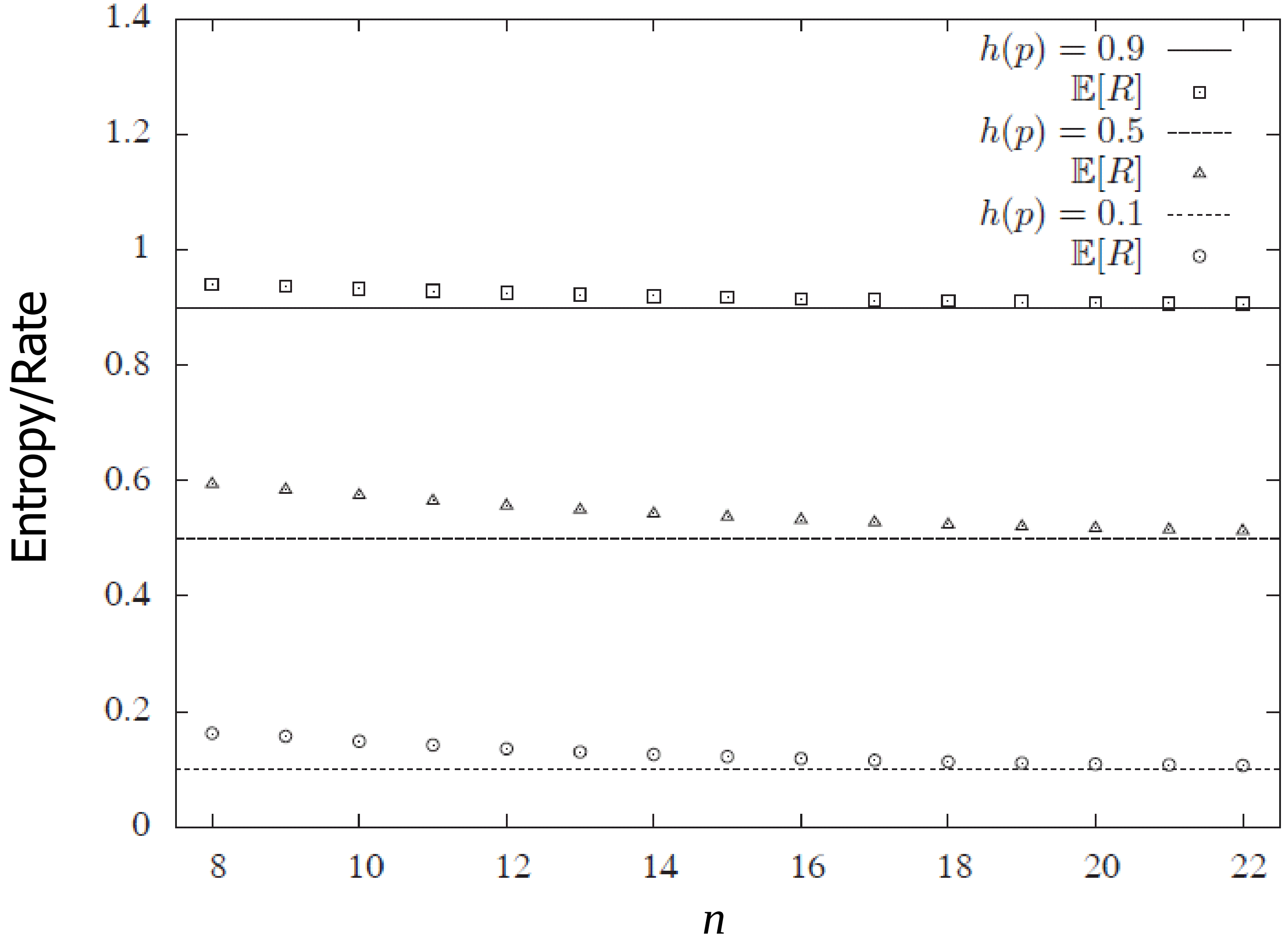
- Example: when $p = 0.11$, $H(x) = 0.5$, $H(\mathrm{U1}) = 0.713$, $H(\mathrm{U2}|\mathrm{U1}) = 0.287$.

# Source Coding

- Keep polarizing by increasing $N$, the entropy of the synthetic sources tends to 0 or 1.
- Again, by the property of the martingale, the proportion of those sources with entropy close to 1 is close to $H(x)$.
- Source coding is realized by recording the indexes with entropy close to 1, while the rest bits can be recovered with high probability because their associated entropy is almost 0.
- For me details, see

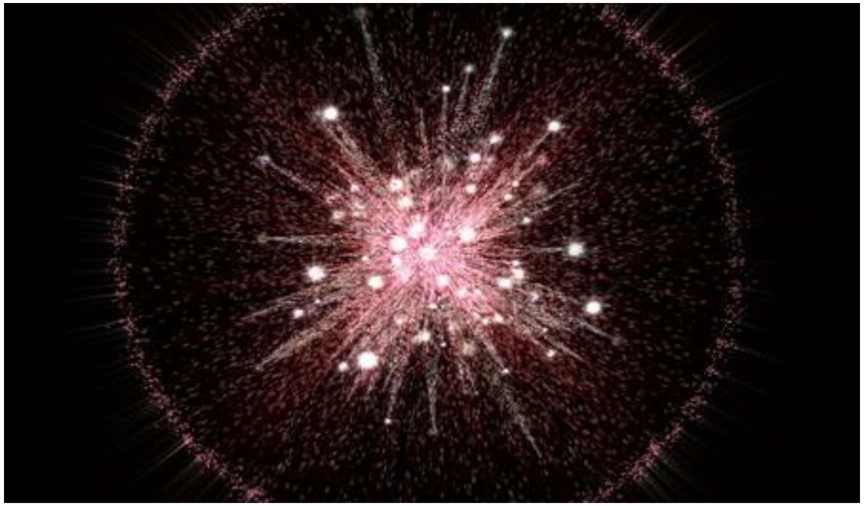
  E. Arikan, "Source polarization," *IEEE* ISIT 2010, pp. 899-903.

# Performance



# Extensions

- Polar codes also achieve capacity of other types of channels (discrete or continuous).
- Achieve entropy bound of other types of sources (lossless or lossy).
- Quantum polar codes, network information theory…



**Big bang in information theory**

# Lecture 13

- Continuous Random Variables
- Differential Entropy
  - can be negative
  - not really a measure of the information in $x$
  - coordinate-dependent
- Maximum entropy distributions
  - Uniform over a finite range
  - Gaussian if a constant variance

# Continuous Random Variables

## Changing Variables

- pdf: $f_X(x)$ CDF: $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$
- For $g(x)$ monotonic: $y = g(x) \Leftrightarrow x = g^{-1}(y)$

$$F_Y(y) = F_X\left(g^{-1}(y)\right) \quad \text{or} \quad 1 - F_X\left(g^{-1}(y)\right)$$ according to slope of $g(x)$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right| = f_X(x)\left|\frac{dx}{dy}\right|$$ where $x = g^{-1}(y)$

- Examples:

$y = 4x \Rightarrow x = \frac{y}{4}$. $\left|\frac{dx}{dy}\right| = \frac{1}{4}$

$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| = 0.5 \times \frac{1}{4} = \frac{1}{8}$ $y \in (0.8)$

$z = x^4 \Rightarrow x = z^{\frac{1}{4}}$ $\left|\frac{dx}{dz}\right| = \frac{1}{4} z^{-\frac{3}{4}}$

Suppose $f_X(x) = 0.5$ for $x \in (0,2) \Rightarrow F_X(x) = 0.5x$

$f_Z(z) = f_X(x) \cdot \left|\frac{dx}{dz}\right| = 0.5 \times \frac{1}{4} z^{-\frac{3}{4}} = \frac{1}{8} z^{-\frac{3}{4}}$ $z \in (0,16)$

(a) $y = 4x \Rightarrow x = 0.25y \Rightarrow f_Y(y) = 0.5 \times 0.25 = 0.125$ for $y \in (0,8)$

(b) $z = x^4 \Rightarrow x = z^{1/4} \Rightarrow f_Z(z) = 0.5 \times 1/4\, z^{-3/4} = 0.125\, z^{-3/4}$ for $z \in (0,16)$

# Joint Distributions

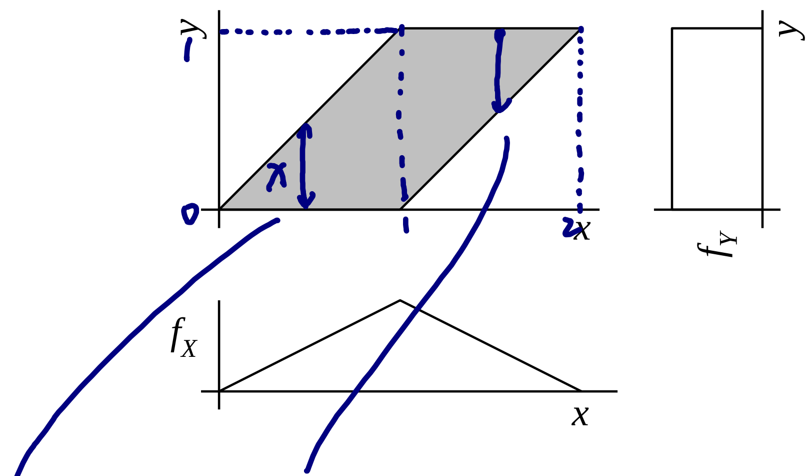
Joint pdf: $f_{x,y}(x,y)$

Marginal pdf: $f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)dy$

Independence: $\Leftrightarrow \quad f_{x,y}(x,y) = f_x(x) f_y(y)$

Conditional pdf: $f_{x|y}(x) = \dfrac{f_{x,y}(x,y)}{f_y(y)}$

## Example:

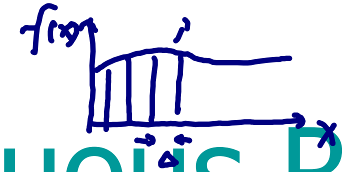



$f_{x,y} = 1$ for $y \in (0,1), x \in (y, y+1)$

$f_{x|y} = 1$ for $x \in (y, y+1)$

$f_{y|x} = \dfrac{1}{\min(x, 1-x)}$ for $y \in \left(\max(0, x-1), \min(x,1)\right)$

$\int f(x)dx = \lim_{\Delta\to 0}\sum_i f(x_i)\Delta = 1$ $\quad P_i = f(x_i)\Delta$ the prob. of $x$ in slot $i$.




# Entropy of Continuous R.V.

$H = -\sum_i P_i \log P_i = -\sum_i f(x_i)\Delta \log[f(x_i)\Delta] = -\sum_i f(x_i)\Delta[\log f(x_i) + \log\Delta]$

$= -\sum_i f(x_i)\Delta \log f(x_i) - \underbrace{\sum_i f(x_i)\Delta}_{1} \log\Delta \xrightarrow{\Delta\to 0} -\log\Delta - \underbrace{\int_{-\infty}^{\infty} f(x)\log f(x)dx}_{h(x)}$

- Given a continuous pdf $f(x)$, we divide the range of $x$ into bins of width $\Delta$
  - For each $i$, $\exists x_i$ with $f(x_i)\Delta = \int_{i\Delta}^{(i+1)\Delta} f(x)dx$ mean value theorem
- Define a discrete random variable Y
  - $Y = \{x_i\}$ and $p_Y = \{f(x_i)\Delta\}$
  - Scaled, quantised version of $f(x)$ with slightly unevenly spaced $x_i$
- $$H(y) = -\sum f(x_i)\Delta \log\left(f(x_i)\Delta\right)$$
  $$= -\log\Delta - \sum f(x_i)\log\left(f(x_i)\right)\Delta$$
  $$\underset{\Delta\to 0}{\to} -\log\Delta - \int_{-\infty}^{\infty} f(x)\log f(x)dx = -\log\Delta + h(x)$$
- Differential entropy: $\boxed{h(x) = -\int_{-\infty}^{\infty} f_x(x)\log f_x(x)dx}$

  - Similar to entropy of discrete r.v. but there are differences

# Differential Entropy

**Differential Entropy:** $h(x) \stackrel{\Delta}{=} -\int_{-\infty}^{\infty} f_x(x) \log f_x(x) dx = E -\log f_x(x)$

## Bad News:

- $h(x)$ does not give the amount of information in $x$
- $h(x)$ is not necessarily positive
- $h(x)$ changes with a change of coordinate system

## Good News:

- $h_1(x) - h_2(x)$ does compare the uncertainty of two continuous random variables provided they are quantised to the same precision
- Relative Entropy and Mutual Information still work fine
- If the range of $x$ is normalized to 1 and then $x$ is quantised to $n$ bits, the entropy of the resultant discrete random variable is approximately $h(x)+n$

# Differential Entropy Examples

$$h(X) = -\int_{-\infty}^{+\infty} f_X(x)\log f_X(x)\,dx = -\int_a^b \frac{1}{b-a}\log\frac{1}{b-a}\,dx = \log(b-a)$$

- Uniform Distribution: $X \sim U(a,b)$
  - $f(x) = (b-a)^{-1}$ for $x \in (a,b)$ and $f(x) = 0$ elsewhere
  - $h(X) = -\int_a^b (b-a)^{-1}\log\,(b-a)^{-1}\,dx = \log(b-a)$
  - Note that $h(X) < 0$ if $(b-a) < 1$

$$h(X) = -\int_{-\infty}^{+\infty} f_X(x)\log f_X(x)\,dx = -\int_{-\infty}^{+\infty} f(x)\log_e f(x)\,dx\cdot\log e = -\log e\int f(x)\left(\ln\frac{1}{\sqrt{2\pi}\sigma} - \frac{(x-\mu)^2}{2\sigma^2}\right)dx$$

- Gaussian Distribution: $X \sim N(\mu,\sigma^2)$
  - $f(x) = \left(2\pi\sigma^2\right)^{-½}\exp\left(-\frac{1}{2}(x-\mu)^2\sigma^{-2}\right)$
  - $h(X) = -(\log e)\int_{-\infty}^{\infty} f(x)\ln f(x)\,dx$

$$= -\frac{1}{2}\log e\int f(x)\left[-\ln 2\pi\sigma^2 - (x-\mu)^2\sigma^{-2}\right]dx$$

$$= \frac{1}{2}\log e\Big[\underbrace{\int f(x)\,dx}_{1}\cdot\ln 2\pi\sigma^2 + \underbrace{\int f(x)(x-\mu)^2\sigma^{-2}\,dx}_{\sigma^{-2}\underbrace{E[(x-\mu)^2]}_{\sigma^2}}\Big]$$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{2}\log e\left(\ln(2\pi\sigma^2)+1\right)$$

$$= -\frac{1}{2}(\log e)\int_{-\infty}^{\infty} f(x)\left(-\ln(2\pi\sigma^2) - (x-\mu)^2\sigma^{-2}\right)$$

$$= \frac{1}{2}\left[\log e\cdot\ln(2\pi\sigma^2) + \log e\right]$$

$$= \frac{1}{2}(\log e)\left(\ln(2\pi\sigma^2) + \sigma^{-2}E\left((x-\mu)^2\right)\right)$$

$$= \frac{1}{2}\left[\log e\cdot\frac{\log(2\pi\sigma^2)}{\log e} + \log e\right] = \frac{1}{2}\log(2\pi e\sigma^2)$$

$$\log_x y = \frac{\log_e y}{\log_e x}$$

$$= \frac{1}{2}(\log e)\left(\ln(2\pi\sigma^2)+1\right) = ½\log\left(2\pi e\sigma^2\right) \cong \log(4.1\sigma)\text{ bits}$$

$(x-m)^T K^{-1} (x-m) = tr((x-m)^T K^{-1}(x-m)) = tr((x-m)(x-m)^T K^{-1})$

$1\times n \quad n\times n \quad n\times 1$

# Multivariate Gaussian

Const → tr (matrix)

Given mean, $\mathbf{m}$, and symmetric positive definite covariance matrix $\mathbf{K}$,

$$X_{1:n} \sim \mathbf{N}(\mathbf{m},\mathbf{K}) \quad \Leftrightarrow \quad f(\mathbf{x}) = \overbrace{\left|2\pi\mathbf{K}\right|}^{\text{det}}{}^{-1/2} \exp\left(-\frac{1}{2}\underbrace{(\mathbf{x}-\mathbf{m})^T}_{1\times n}\underbrace{\mathbf{K}^{-1}}_{n\times n}\underbrace{(\mathbf{x}-\mathbf{m})}_{n\times 1}\right)$$

$$h(f) = -(\log e)\int f(\mathbf{x})\times\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\mathbf{K}^{-1}(\mathbf{x}-\mathbf{m}) - \tfrac{1}{2}\ln\left|2\pi\mathbf{K}\right|\right)d\mathbf{x}$$

$$= \tfrac{1}{2}\log(e)\times\left(\ln\left|2\pi\mathbf{K}\right| + E\left((\mathbf{x}-\mathbf{m})^T\mathbf{K}^{-1}\underline{(\mathbf{x}-\mathbf{m})}\right)\right)$$

$$= \tfrac{1}{2}\log(e)\times\left(\ln\left|2\pi\mathbf{K}\right| + E\,\mathrm{tr}\left(\underline{(\mathbf{x}-\mathbf{m})}(\mathbf{x}-\mathbf{m})^T\mathbf{K}^{-1}\right)\right) \qquad \mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$$

⇓ trace is a sum

$$= \tfrac{1}{2}\log(e)\times\left(\ln\left|2\pi\mathbf{K}\right| + \mathrm{tr}\left(\underbrace{E(\mathbf{x}-\mathbf{m})(\mathbf{x}-\mathbf{m})^T}_{\triangleq \mathbf{K}}\mathbf{K}^{-1}\right)\right) \qquad E_f\mathbf{x}\mathbf{x}^T = \mathbf{K}$$

$$= \tfrac{1}{2}\log(e)\times\left(\ln\left|2\pi\mathbf{K}\right| + \mathrm{tr}\left(\mathbf{K}\mathbf{K}^{-1}\right)\right) \quad = \tfrac{1}{2}\log(e)\times\left(\ln\left|2\pi\mathbf{K}\right| + n\right)$$

$$= \tfrac{1}{2}\log\left(e^n\right) + \tfrac{1}{2}\log\left(\underbrace{\left|2\pi\mathbf{K}\right|}_{n\times n}\right) \qquad \mathrm{tr}(\mathbf{I}) = n = \ln(e^n)$$

can be negative if $|K|$ is small

$$= \tfrac{1}{2}\log\left(\left|2\pi e\mathbf{K}\right|\right) = \tfrac{1}{2}\log\left((2\pi e)^n\left|\mathbf{K}\right|\right) \text{ bits}$$

$\det(aA) = a^n \det(A)$

# Other Differential Quantities

## Joint Differential Entropy

$$h(X,Y) = -\iint_{x,y} f_{X,Y}(x,y)\log f_{X,Y}(x,y)\,dxdy \;= E - \log f_{X,Y}(x,y)$$

## Conditional Differential Entropy

$$h(X \mid Y) = -\iint_{x,y} f_{X,Y}(x,y)\log f_{X,Y}(x \mid y)\,dxdy \;= h(X,Y) - h(Y)$$

## Mutual Information

$$I(X;Y) = \iint_{x,y} f_{X,Y}(x,y)\log \frac{f_{X,Y}(x,y)}{f_X(x)\,f_Y(y)}\,dxdy \;= h(X) + h(Y) - h(X,Y)$$

## Relative Differential Entropy of two pdf's:

$$D(f \,\|\, g) = \int f(x)\log\frac{f(x)}{g(x)}\,dx$$

$$= -h_f(X) - E_f \log g(X)$$

(a) must have $f(x)=0 \Rightarrow g(x)=0$

(b) continuity $\Rightarrow 0\log(0/0) = 0$

# Differential Entropy Properties

Chain Rules $$h(x, y) = h(x) + h(y \mid x) = h(y) + h(x \mid y)$$

$$I(x, y; z) = I(x; z) + I(y; z \mid x)$$

Information Inequality: $D(f \parallel g) \geq 0$

Proof: Define $S = \{\mathbf{x} : f(\mathbf{x}) > 0\}$ $D(f \| g) = -\int f(x) \log g(x) dx - h(x)$

$$-D(f \parallel g) = \int_{\mathbf{x} \in S} f(\mathbf{x}) \log \frac{g(\mathbf{x})}{f(\mathbf{x})} d\mathbf{x} = E_f \left( \log \frac{g(\mathbf{x})}{f(\mathbf{x})} \right) = -\int f(x) \log \frac{g(x)}{f(x)} dx$$

$$\leq \log \left( E \frac{g(\mathbf{x})}{f(\mathbf{x})} \right) = \log \left( \int_S f(\mathbf{x}) \frac{g(\mathbf{x})}{f(\mathbf{x})} d\mathbf{x} \right)$$

Jensen + $\log()$ is concave

$$= \log \left( \int_S g(\mathbf{x}) d\mathbf{x} \right) \leq \log 1 = 0$$

all the same as for discrete r.v. $H()$

# Information Inequality Corollaries

Mutual Information ≥ 0

$$I(x;y) = D(f_{x,y} \| f_x f_y) \geq 0$$

Conditioning reduces Entropy

$$h(x) - h(x \mid y) = I(x;y) \geq 0$$

Independence Bound

$$h(x_{1:n}) = \sum_{i=1}^{n} h(x_i \mid x_{1:i-1}) \leq \sum_{i=1}^{n} h(x_i)$$

all the same as for $H()$

# Change of Variable

## Change Variable: $y = g(x)$

from earlier
$$f_y(y) = f_x\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right|$$

$$f_y(y) = f_x(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$$
$$h(y) = h(x) + E\log\left|\frac{dy}{dx}\right|$$

$$h(y) = -E\log(f_y(y)) = -E\log\left(f_x(g^{-1}(y))\right) - E\log\left|\frac{dx}{dy}\right|$$

$$= -E\log\left(f_x(x)\right) - E\log\left|\frac{dx}{dy}\right| = h(x) + E\log\left|\frac{dy}{dx}\right|$$

## Examples:

- Translation: $y = x + a \Rightarrow dy/dx = 1 \Rightarrow h(y) = h(x)$
- Scaling: $y = cx \Rightarrow dy/dx = c \Rightarrow h(y) = h(x) + \log|c|$
- Vector version: $y_{1:n} = \mathbf{A}x_{1:n} \Rightarrow h(\mathbf{y}) = h(\mathbf{x}) + \log|\det(\mathbf{A})|$

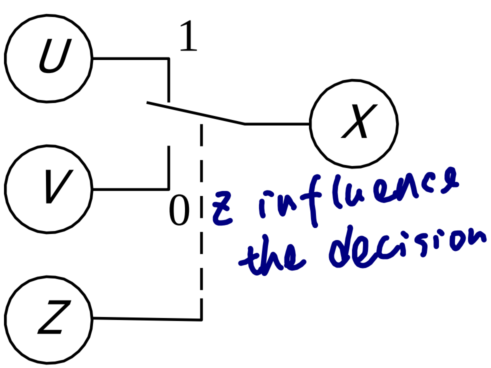
not the same as for $H()$

# Concavity & Convexity

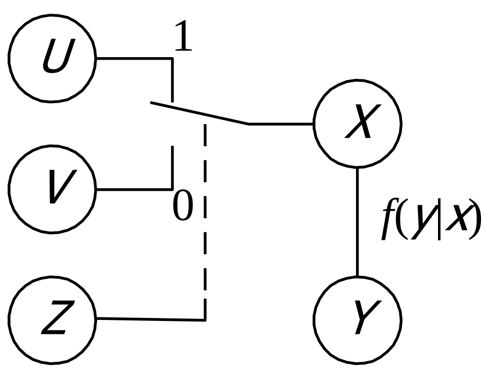
- Differential Entropy:
  - $h(x)$ is a concave function of $f_x(x)$ $\Rightarrow \exists$ a maximum
- Mutual Information:
  - $I(x\,;\,y)$ is a concave function of $f_x(x)$ for fixed $f_{y|x}(y)$
  - $I(x\,;\,y)$ is a convex function of $f_{y|x}(y)$ for fixed $f_x(x)$

## Proofs:

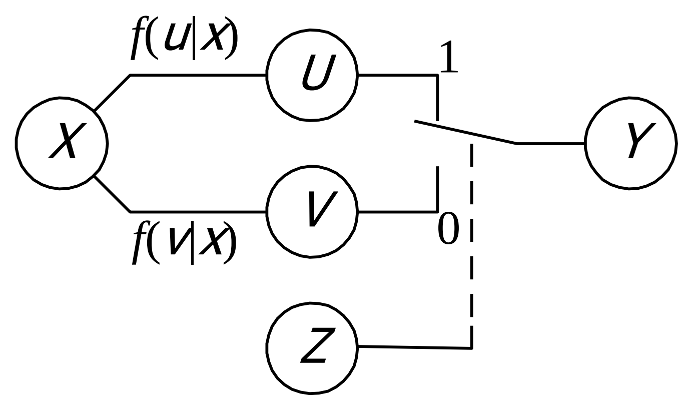
Exactly the same as for the discrete case: $\mathbf{p}_z = [1-\lambda,\ \lambda]^T$




$H(x) \geq H(x \mid z)$




$I(x\,;y) \geq I(x\,;y\mid z)$




$I(x\,;y) \leq I(x\,;y\mid z)$

# Uniform Distribution Entropy

What distribution over the finite range $(a,b)$ maximizes the entropy ?

Answer: A uniform distribution $u(x)=(b-a)^{-1}$

Proof:

Suppose $f(x)$ is a distribution for $x \in (a,b)$

$$0 \le D(f \| u) = -h_f(x) - E_f \log u(x)$$

$$= -h_f(x) + \log(b-a)$$

$$\Rightarrow \boxed{h_f(x) \le \log(b-a)}$$

$D(f\|u) = \int_{-\infty}^{+\infty} f(x) \log \frac{f(x)}{u(x)} dx$

$= -h(x) - \int_{-\infty}^{+\infty} f(x) \log u(x) dx$

$= -h(x) + \log(b-a) \ge 0$

$\therefore h(x) \le \log(b-a)$

$E_f \log(\phi(x)) = \log e \int f(x)$



# Maximum Entropy Distribution

What zero-mean distribution maximizes the entropy on $(-\infty, \infty)^n$ for a given covariance matrix $\mathbf{K}$ ? (known covariance)

Answer: A multivariate Gaussian $\phi(\mathbf{x}) = |2\pi\mathbf{K}|^{-½} \exp\left(-½\,\mathbf{x}^T\mathbf{K}^{-1}\mathbf{x}\right)$

Proof: $0 \le D(f \,\|\, \phi) = -h_f(\mathbf{x}) - E_f \log \phi(\mathbf{x})$

$$\Rightarrow h_f(\mathbf{x}) \le -(\log e) E_f\left(-½\ln\left(|2\pi\mathbf{K}|\right) - ½\,\mathbf{x}^T\mathbf{K}^{-1}\mathbf{x}\right)$$

$h_f(\underline{x}) \le \frac{1}{2}\log|2\pi e\underline{K}|$

$$= ½(\log e)\left(\ln\left(|2\pi\mathbf{K}|\right) + \mathrm{tr}\left(E_f\,\mathbf{x}\mathbf{x}^T\mathbf{K}^{-1}\right)\right)$$

$$= ½(\log e)\left(\ln\left(|2\pi\mathbf{K}|\right) + \mathrm{tr}(\mathbf{I})\right)$$

$E_f\mathbf{x}\mathbf{x}^T = \mathbf{K}$

$$= ½\log\left(|2\pi e\mathbf{K}|\right) = h_\phi(\mathbf{x})$$

$\mathrm{tr}(\mathbf{I}) = n = \ln(e^n)$

Since translation doesn't affect $h(X)$, we can assume zero-mean w.l.o.g.

# Summary

- Differential Entropy: $h(x) = -\int_{-\infty}^{\infty} f_x(x) \log f_x(x) dx$
  - Not necessarily positive
  - $h(x+a) = h(x)$, $h(ax) = h(x) + \log|a|$
- Many properties are formally the same
  - $h(x|y) \le h(x)$
  - $I(x; y) = h(x) + h(y) - h(x, y) \ge 0$, $D(f||g)=E \log(f/g) \ge 0$
  - $h(x)$ concave in $f_x(x)$; $I(x; y)$ concave in $f_x(x)$
- Bounds:
  - Finite range: Uniform distribution has max: $h(x) = \log(b-a)$

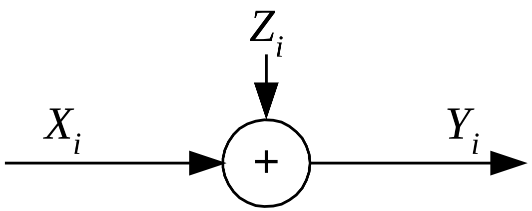
    (translation ~ mean)
  - Fixed Covariance: Gaussian has max: $h(x) = \frac{1}{2}\log((2\pi e)^n|\mathbf{K}|)$

# Lecture 14

- Discrete-Time Gaussian Channel Capacity
- Continuous Typical Set and AEP
- Gaussian Channel Coding Theorem
- Bandlimited Gaussian Channel
  - Shannon Capacity

# Capacity of Gaussian Channel

## Discrete-time channel: $y_i = x_i + z_i$




- Zero-mean Gaussian i.i.d. $z_i \sim N(0,N)$
- Average power constraint $n^{-1}\sum_{i=1}^{n} x_i^2 \le P$

$$Ey^2 = E(x+z)^2 = Ex^2 + 2E(x)E(z) + Ez^2 \le P + N$$

$X,Z$ indep and $EZ=0$

## Information Capacity

$h(x) = \frac{1}{2}\log((2\pi e)^n |K|)$

- Define information capacity: $C = \max_{Ex^2 \le P} I(x;y)$

$$I(x;y) = h(y) - h(y \mid x) = h(y) - h(x+z \mid x)$$

$E(y^2) \le P+N$

$$\overset{(a)}{=} h(y) - h(z \mid x) = h(y) - h(z)$$

(a) Translation independence

Power (var.) given, entropy is bounded by Gaussian distribution.

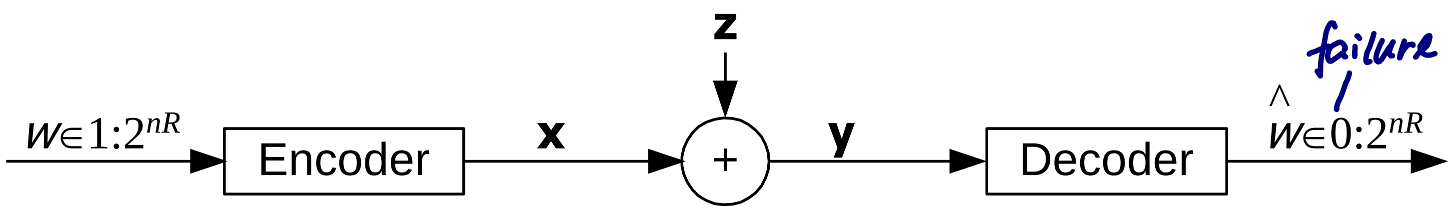
$$\le \tfrac{1}{2}\log 2\pi e(P+N) - \tfrac{1}{2}\log 2\pi eN$$

Gaussian Limit with equality when $x \sim N(0,P)$

$$= \frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

The optimal input is Gaussian

# Achievability




- An $(M,n)$ code for a Gaussian Channel with power constraint is
  - A set of $M$ codewords $\mathbf{x}(w) \in X^n$ for $w=1:M$ with $\mathbf{x}(w)^T\mathbf{x}(w) \le nP \quad \forall w$
  - A deterministic decoder $g(\mathbf{y}) \in 0:M$ where $0$ denotes failure
  - Errors: codeword : $\lambda_i$ max$_i$ : $\lambda^{(n)}$ average$_i$ : $P_e^{(n)}$
- Rate $R$ is achievable if $\exists$ seq of $(2^{nR},n)$ codes with $\lambda^{(n)} \underset{n\to\infty}{\to} 0$
- Theorem: $R$ achievable iff $R < C = ½ \log\left(1 + PN^{-1}\right)$ ♦

♦ = proved on next pages

$\#: \frac{\sqrt{n(P+N)}^{\,n}}{\sqrt{nN}^{\,n}} = \left(1+\frac{P}{N}\right)^{\frac{n}{2}}$

# Argument by Sphere Packing

$R = \frac{1}{n}\log \# = \frac{1}{n}\cdot\frac{n}{2}\log\left(1+\frac{P}{N}\right) = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$

- Each transmitted $\mathbf{x}_i$ is received as a probabilistic cloud $\mathbf{y}_i$
  - cloud 'radius' = $\sqrt{\mathrm{Var}(\mathbf{y} \mid \mathbf{x})} = \sqrt{nN}$

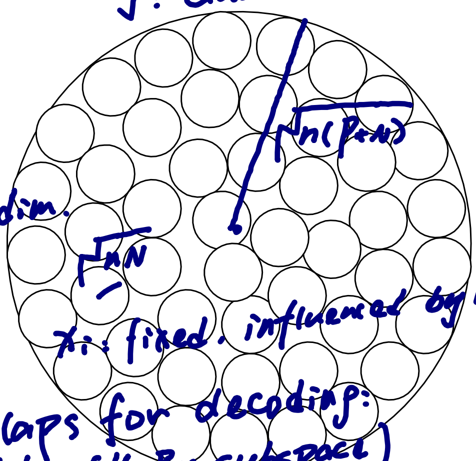
Law of large numbers

- Energy of $\mathbf{y}_i$ constrained to $n(P+N)$ so clouds must fit into a hypersphere of radius $\sqrt{n(P+N)}$
- Volume of hypersphere $\propto r^{n}$ hypersphere: sphere in n-dim.
- Max number of non-overlapping clouds:

$$\frac{(nP + nN)^{\frac{1}{2}n}}{(nN)^{\frac{1}{2}n}} = 2^{\frac{1}{2}n\log\left(1+PN^{-1}\right)}$$

- Max achievable rate is $\frac{1}{2}\log(1+P/N)$

y: Gaussian




no overlaps for decoding:

$\# = \frac{\mathrm{Vol}(y: \text{Rx subspace})}{\mathrm{Vol}(x: \text{under noise})} = \frac{(\sqrt{n(P+N)})^n}{(\sqrt{nN})^n} = \left(1+\frac{P}{N}\right)^{\frac{n}{2}}$

$R = \frac{1}{n}\log \# = \frac{1}{n}\cdot\frac{n}{2}\cdot\log\left(1+\frac{P}{N}\right)$

# Continuous AEP

## Typical Set: Continuous distribution, discrete time i.i.d.

For any $\varepsilon>0$ and any $n$, the typical set with respect to $f(\mathbf{x})$ is

$$T_\varepsilon^{(n)} = \left\{ \mathbf{x} \in S^n : \left| -n^{-1} \log f(\mathbf{x}) - h(X) \right| \le \varepsilon \right\}$$

where $S$ is the support of $f \Leftrightarrow \{\mathbf{x} : f(\mathbf{x})>0\}$

$$f(\mathbf{x}) = \prod_{i=1}^{n} f(x_i) \quad \text{since } x_i \text{ are independent}$$

$$h(X) = E - \log f(X) = -n^{-1} E \log f(\mathbf{x})$$

## Typical Set Properties

1. $p(\mathbf{x} \in T_\varepsilon^{(n)}) > 1 - \varepsilon$ for $n > N_\varepsilon$ typicality — Proof: LLN

2. $(1-\varepsilon) 2^{n(h(X)-\varepsilon)} \overset{n>N_\varepsilon}{\le} \text{Vol}(T_\varepsilon^{(n)}) \le 2^{n(h(X)+\varepsilon)}$ size $\approx 2^{nh(x)}$ — Proof: Integrate max/min prob

where $\text{Vol}(A) = \int_{\mathbf{x} \in A} d\mathbf{x}$

# Continuous AEP Proof

**Proof 1:** By law of large numbers

$$-n^{-1}\log f(x_{1:n}) = -n^{-1}\sum_{i=1}^{n}\log f(x_i) \xrightarrow{prob} E-\log f(x) = h(x)$$

Reminder: $x_n \xrightarrow{\text{prob}} y \quad \Rightarrow \quad \forall \varepsilon > 0, \exists N_\varepsilon$ such that $\forall n > N_\varepsilon, \quad P\left(|x_n - y| > \varepsilon\right) < \varepsilon$

**Proof 2a:** $1-\varepsilon \le \int_{T_\varepsilon^{(n)}} f(\mathbf{x})d\mathbf{x}$ for $n > N_\varepsilon$ Property 1

$\le 2^{-n(h(X)-\varepsilon)} \int_{T_\varepsilon^{(n)}} d\mathbf{x} = 2^{-n(h(X)-\varepsilon)} \mathrm{Vol}\left(T_\varepsilon^{(n)}\right)$ max $f(x)$ within $T$

**Proof 2b:** $1 = \int_{S^n} f(\mathbf{x})d\mathbf{x} \ge \int_{T_\varepsilon^{(n)}} f(\mathbf{x})d\mathbf{x}$

$\ge 2^{-n(h(X)+\varepsilon)} \int_{T_\varepsilon^{(n)}} d\mathbf{x} = 2^{-n(h(X)+\varepsilon)} \mathrm{Vol}\left(T_\varepsilon^{(n)}\right)$ min $f(x)$ within $T$

# Jointly Typical Set

## Jointly Typical: $x_i, y_i$ i.i.d from $\Re^2$ with $f_{x,y}(x_i,y_i)$

jointly typical:
$\left\{\begin{array}{l}x \text{ typical} \\ y \text{ typical} \\ (x,y) \text{ typical}\end{array}\right.$

$$J_\varepsilon^{(n)} = \Big\{ \mathbf{x}, \mathbf{y} \in \Re^{2n} : \left| -n^{-1} \log f_X(\mathbf{x}) - h(x) \right| < \varepsilon,$$
$$\left| -n^{-1} \log f_Y(\mathbf{y}) - h(y) \right| < \varepsilon,$$
$$\left| -n^{-1} \log f_{X,Y}(\mathbf{x}, \mathbf{y}) - h(x, y) \right| < \varepsilon \Big\}$$

## Properties:

1. Indiv p.d.: $\mathbf{x}, \mathbf{y} \in J_\varepsilon^{(n)} \Rightarrow \log f_{x,y}(\mathbf{x}, \mathbf{y}) = -nh(x, y) \pm n\varepsilon$
2. Total Prob: $p\left(\mathbf{x}, \mathbf{y} \in J_\varepsilon^{(n)}\right) > 1 - \varepsilon$ for $n > N_\varepsilon$
3. Size: $(1-\varepsilon)2^{n(h(x,y)-\varepsilon)} \overset{n>N_\varepsilon}{\leq} \mathrm{Vol}\left(J_\varepsilon^{(n)}\right) \leq 2^{n(h(x,y)+\varepsilon)}$
4. Indep $\mathbf{x}', \mathbf{y}'$: $(1-\varepsilon)2^{-n(I(x;y)+3\varepsilon)} \overset{n>N_\varepsilon}{\leq} p\left(\mathbf{x}', \mathbf{y}' \in J_\varepsilon^{(n)}\right) \leq 2^{-n(I(x;y)-3\varepsilon)}$

Proof of 4.: Integrate max/min $f(\mathbf{x}', \mathbf{y}') = f(\mathbf{x}')f(\mathbf{y}')$, then use known bounds on $\mathrm{Vol}(J)$

# Gaussian Channel Coding Theorem

$R$ is achievable iff $R < C = \frac{1}{2}\log\left(1 + PN^{-1}\right)$

## Proof ($\Leftarrow$):

Choose $\varepsilon > 0$

Random codebook: $\mathbf{x}_w \in \Re^n$ for $w = 1 : 2^{nR}$ where $x_w$ are i.i.d. $\sim N(0, P - \varepsilon)$

Use Joint typicality decoding

Errors: 1. Power too big $p\left(\mathbf{x}^T\mathbf{x} > nP\right) \to 0 \Rightarrow \le \varepsilon$ for $n > M_\varepsilon$

2. $\mathbf{y}$ not J.T. with $\mathbf{x}$ $p(\mathbf{x}, \mathbf{y} \notin J_\varepsilon^{(n)}) < \varepsilon$ for $n > N_\varepsilon$

3. another $\mathbf{x}$ J.T. with $\mathbf{y}$ $\sum_{j=2}^{2^{nR}} p(\mathbf{x}_j, \mathbf{y}_i \in J_\varepsilon^{(n)}) \le (2^{nR} - 1) \times 2^{-n(I(X;Y) - 3\varepsilon)}$

Total Err $P_\varepsilon^{(n)} \le \varepsilon + \varepsilon + 2^{-n\left(I(X;Y) - R - 3\varepsilon\right)} \le 3\varepsilon$ for large $n$ if $R < I(X;Y) - 3\varepsilon$

Expurgation: Remove half of codebook*: $\lambda^{(n)} < 6\varepsilon$ now max error

We have constructed a code achieving rate $R - n^{-1}$

*:Worst codebook half includes $\mathbf{x}_i$: $\mathbf{x}_i^T\mathbf{x}_i > nP \Rightarrow \lambda_i = 1$

# Gaussian Channel Coding Theorem

**Proof ($\Rightarrow$): Assume** $P_e^{(n)} \to 0$ and $n^{-1}\mathbf{x}^T\mathbf{x} < P$ for each $\mathbf{x}(w)$


$$nR = H(w) = I(w; y_{1:n}) + H(w \mid y_{1:n})$$

$$\le I(x_{1:n}; y_{1:n}) + H(w \mid y_{1:n})$$ Data Proc Inequal

$$= h(y_{1:n}) - h(y_{1:n} \mid x_{1:n}) + H(w \mid y_{1:n})$$

$$\le \sum_{i=1}^{n} h(y_i) - h(z_{1:n}) + H(w \mid y_{1:n})$$ Indep Bound + Translation

$$\le \sum_{i=1}^{n} I(x_i; y_i) + 1 + nRP_e^{(n)}$$ $Z$ i.i.d + Fano, $|W|=2^{nR}$

$$\le \sum_{i=1}^{n} ½ \log\left(1 + PN^{-1}\right) + 1 + nRP_e^{(n)}$$ max Information Capacity

$$R \le ½ \log\left(1 + PN^{-1}\right) + n^{-1} + RP_e^{(n)} \quad \to ½ \log\left(1 + PN^{-1}\right)$$

# Bandlimited Channel

- Channel bandlimited to $f \in (-W, W)$ and signal duration $T$
  - Not exactly
  - Most energy in the bandwidth, most energy in the interval
- Nyquist: Signal is defined by $2WT$ samples
  - white noise with double-sided p.s.d. $\frac{1}{2}N_0$ becomes i.i.d gaussian $N(0, \frac{1}{2}N_0)$ added to each coefficient
  - Signal power constraint = $P \Rightarrow$ Signal energy $\le PT$
    - Energy constraint per coefficient: $n^{-1}\mathbf{x}^T\mathbf{x} < PT/2WT = \frac{1}{2}W^{-1}P$
- Capacity: $C' = \frac{1}{2}\log\left(1+\frac{P'}{N'}\right) = \frac{1}{2}\log\left(1+\frac{\frac{P}{2W}}{\frac{N_0}{2}}\right) = \frac{1}{2}\log\left(1+\frac{P}{N_0 W}\right)$

$P' = \frac{P}{2W}$

$N' = \frac{N_0}{2}$

$$C = \frac{1}{2}\log\left(1+\frac{1/2 \cdot P/W}{N_0/2}\right) \times \frac{2WT}{T} = W\log\left(1+\frac{P}{WN_0}\right) \text{ bits/second}$$

$C' = \frac{\#\text{bits}}{T} = \frac{2WT \cdot \frac{1}{2}\log\left(1+\frac{P}{N_0 W}\right)}{T} = W\log\left(1+\frac{P}{WN_0}\right)$

- More precisely, it can be represented in a vector space of about $n=2WT$ dimensions with prolate spheroidal functions as an orthonormal basis

Compare discrete time version: $\frac{1}{2}\log(1+PN^{-1})$ bits per channel use

# Limit of Infinite Bandwidth

$$C = W \log\left(1 + \frac{P}{WN_0}\right) \text{ bits/second}$$

$$C \underset{W\to\infty}{\to} \frac{P}{N_0}\log e$$

$$\lim_{w\to\infty} C = W \log\left(1+\frac{P}{WN_0}\right)^{\frac{WN_0}{P}\cdot\frac{P}{WN_0}} = 1$$

$$= \log e \cdot \frac{P}{N_0}$$

Minimum signal to noise ratio (SNR)

$$\frac{E_b}{N_0} = \frac{PT_b}{N_0} = \frac{P/C}{N_0} \underset{W\to\infty}{\to} \ln 2 = -1.6 \text{ dB}$$

required for reliable transmission

Given capacity, trade-off between $P$ and $W$

$R \le W\log\left(1+\frac{P}{WN_0}\right)$

- Increase P, decrease W

$r \le \log\left(1+\frac{P}{WN_0}\right) = \log\left(1+\frac{E_b R}{N_0 W}\right) = \log\left(1+r\frac{E_b}{N_0}\right)$

- Increase W, decrease P

$2^r \le 1 + r\frac{E_b}{N_0}$

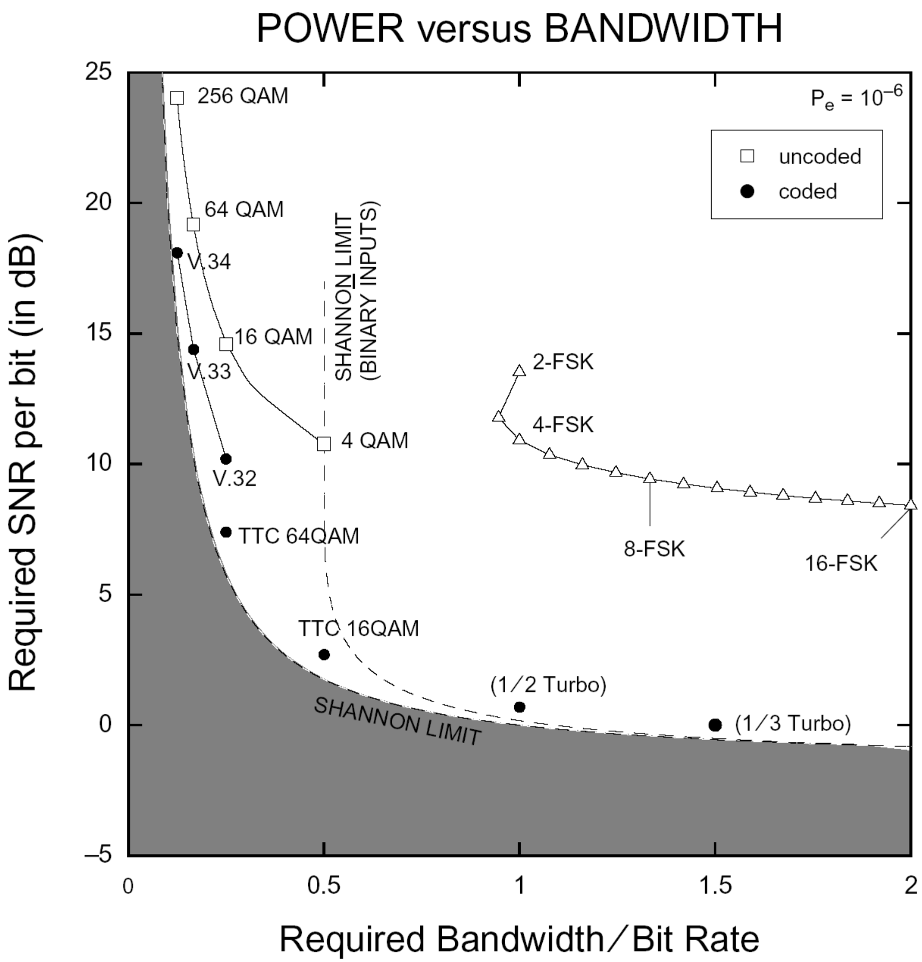
  - spread spectrum

$\frac{E_b}{N_0} \ge \frac{2^r-1}{r} \ge \lim_{r\to 0}\frac{2^r-1}{r} = \lim_{r\to 0}\frac{e^{\ln 2\cdot r}-1}{r}$

  - ultra wideband

$\underline{\underline{\text{der.}}}\ \frac{\ln 2\, e^{\ln 2\cdot r}}{1} = \ln 2 = -1.6\text{ dB}$

# Channel Code Performance




- Power Limited
  - High bandwidth
  - Spacecraft, Pagers
  - Use QPSK/4-QAM
  - Block/Convolution Codes
- Bandwidth Limited
  - Modems, DVB, Mobile phones
  - 16-QAM to 256-QAM
  - Convolution Codes
- Value of 1 dB for space
  - Better range, lifetime, weight, bit rate
  - $80 M (1999)

Diagram from “An overview of Communications” by John R Barry

# Summary

- Gaussian channel capacity

$$C = \frac{1}{2}\log\left(1+\frac{P}{N}\right) \quad \text{bits/transmission}$$

- Proved by using continuous AEP (or sphere packing)
- Bandlimited channel

$$C = W\log\left(1+\frac{P}{WN_0}\right) \quad \text{bits/second}$$

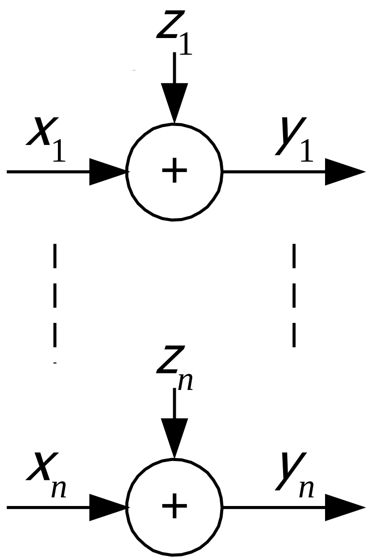
  – Minimum SNR = –1.6 dB as W $\to\infty$

# Lecture 15

- Parallel Gaussian Channels
  - Waterfilling
- Gaussian Channel with Feedback
  - Memoryless: no gain
  - Memory: at most ½ bits/transmission

# Parallel Gaussian Channels

- $n$ independent Gaussian channels
  - A model for nonwhite noise wideband channel where each component represents a different frequency
  - e.g. digital audio, digital TV, Broadband ADSL, WiFi (multicarrier/OFDM)
- Noise is independent $z_i \sim N(0,N_i)$
- Average Power constraint $E\mathbf{x}^T\mathbf{x} \le nP$
- Information Capacity: $C = \max_{f(\mathbf{x}):E_f \mathbf{x}^T\mathbf{x} \le nP} I(\mathbf{x};\mathbf{y})$
- $R<C \Leftrightarrow R$ achievable
  - proof as before
- What is the optimal $f(\mathbf{x})$ ?



# Parallel Gaussian: Max Capacity

Need to find $f(\mathbf{x})$: $C = \max_{f(\mathbf{x}):E_f \mathbf{x}^T\mathbf{x} \le nP} I(\mathbf{x};\mathbf{y})$

$$I(\mathbf{x};\mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y}\,|\,\mathbf{x}) = h(\mathbf{y}) - h(\mathbf{z}\,|\,\mathbf{x})$$

Translation invariance

$$= h(\mathbf{y}) - h(\mathbf{z}) = h(\mathbf{y}) - \sum_{i=1}^{n} h(z_i)$$

$\mathbf{x},\mathbf{z}$ indep; $Z_i$ indep

$$\overset{(a)}{\le} \sum_{i=1}^{n} \left(h(y_i) - h(z_i)\right) \overset{(b)}{\le} \sum_{i=1}^{n} ½ \log\left(1 + P_i N_i^{-1}\right)$$

(a) indep bound;
(b) capacity limit

Equality when: (a) $y_i$ indep $\Rightarrow$ $x_i$ indep; (b) $x_i \sim N(0, P_i)$

We need to find the $P_i$ that maximise $\sum_{i=1}^{n} ½ \log\left(1 + P_i N_i^{-1}\right)$

# Parallel Gaussian: Optimal Powers

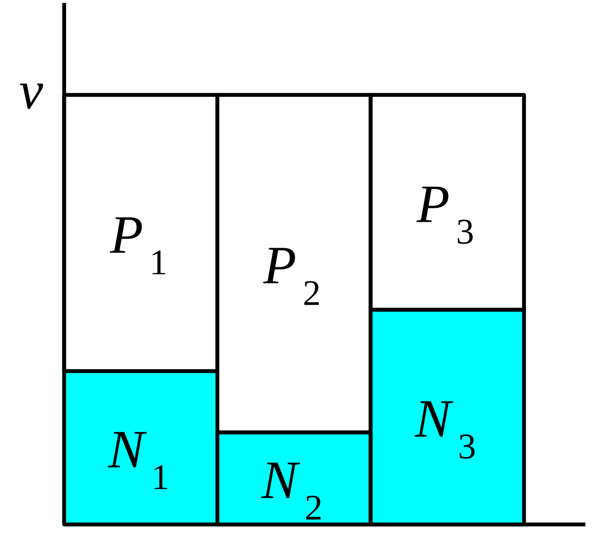
We need to find the $P_i$ that maximise $\log(e)\sum_{i=1}^{n} \tfrac{1}{2}\ln\left(1+P_i N_i^{-1}\right)$

- subject to power constraint $\sum_{i=1}^{n} P_i = nP$
- use Lagrange multiplier

$$J = \sum_{i=1}^{n} \tfrac{1}{2}\ln\left(1+P_i N_i^{-1}\right) - \lambda \sum_{i=1}^{n} P_i$$

$$\frac{\partial J}{\partial P_i} = \tfrac{1}{2}\left(P_i + N_i\right)^{-1} - \lambda = 0 \quad \Rightarrow \quad P_i + N_i = v$$

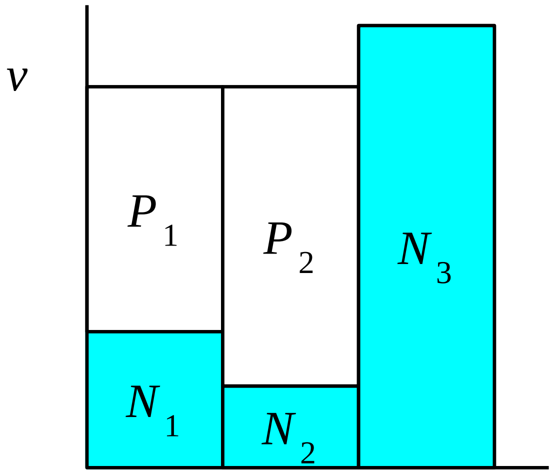
Also $\sum_{i=1}^{n} P_i = nP \quad \Rightarrow \quad v = P + n^{-1}\sum_{i=1}^{n} N_i$




Water Filling: put most power into least noisy channels to make equal power + noise in each channel

# Very Noisy Channels

- What if water is not enough?
- Must have $P_i \geq 0 \ \forall i$
- If $v < N_i$ then set $P_i$=0 and recalculate $v$ (i.e., $P_i = \max(v - N_i,0)$ )




## Kuhn-Tucker Conditions:

(not examinable)

- Max $f(\mathbf{x})$ subject to $\mathbf{Ax}+\mathbf{b}=\mathbf{0}$ and

  $g_i(\mathbf{x}) \geq 0$ for $i \in 1:M$ with $f, g_i$ concave

- set $J(\mathbf{x}) = f(\mathbf{x}) - \sum_{i=1}^{M} \mu_i g_i(\mathbf{x}) - \boldsymbol{\lambda}^T \mathbf{Ax}$
- Solution $\mathbf{x}_0$, λ, $\mu_i$ iff

$$\nabla J(\mathbf{x}_0) = 0, \quad \mathbf{Ax} + \mathbf{b} = \mathbf{0}, \quad g_i(\mathbf{x}_0) \geq 0, \quad \mu_i \geq 0, \quad \mu_i g_i(\mathbf{x}_0) = 0$$

# Colored Gaussian Noise

- Suppose $\mathbf{y} = \mathbf{x} + \mathbf{z}$ where $E\,\mathbf{z}\mathbf{z}^T = \mathbf{K_z}$ and $E\,\mathbf{x}\mathbf{x}^T = \mathbf{K_x}$
- We want to find $\mathbf{K_x}$ to maximize capacity subject to power constraint: $E\sum_{i=1}^{n} x_i^2 \le nP \quad \Leftrightarrow \quad \mathrm{tr}\left(\mathbf{K_x}\right) \le nP$
  - Find noise eigenvectors: $\mathbf{K}_Z = \mathbf{Q}\Lambda\mathbf{Q}^T$ with $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$
  - Now $\mathbf{Q}^T\mathbf{y} = \mathbf{Q}^T\mathbf{x} + \mathbf{Q}^T\mathbf{z} = \mathbf{Q}^T\mathbf{x} + \mathbf{w}$

    where $\mathrm{E}\;\mathbf{w}\mathbf{w}^T = \mathrm{E}\;\mathbf{Q}^T\mathbf{z}\mathbf{z}^T\mathbf{Q} = \mathrm{E}\;\mathbf{Q}^T\mathbf{K}_Z\mathbf{Q} = \mathbf{\Lambda}$ is diagonal
    - $\Rightarrow W_i$ are now independent (so previous result on P.G.C. applies)
  - Power constraint is unchanged $\mathrm{tr}\left(\mathbf{Q}^T\mathbf{K}_X\mathbf{Q}\right) = \mathrm{tr}\left(\mathbf{K}_X\mathbf{Q}\mathbf{Q}^T\right) = \mathrm{tr}\left(\mathbf{K}_X\right)$
  - Use water-filling and indep. messages $\mathbf{Q}^T\mathbf{K}_X\mathbf{Q} + \mathbf{\Lambda} = v\mathbf{I}$
  - Choose $\mathbf{Q}^T\mathbf{K}_X\mathbf{Q} = v\mathbf{I} - \mathbf{\Lambda}$ where $v = P + n^{-1}\,\mathrm{tr}\left(\mathbf{\Lambda}\right)$

$$\Rightarrow \quad \mathbf{K}_X = \mathbf{Q}\left(v\mathbf{I} - \mathbf{\Lambda}\right)\mathbf{Q}^T$$

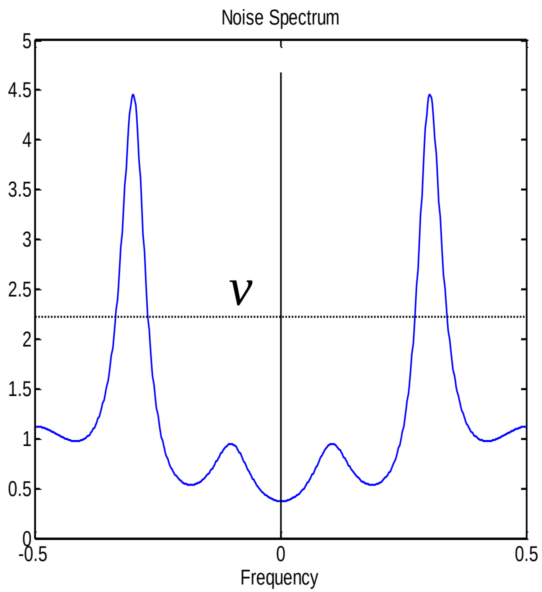
# Power Spectrum Water Filling

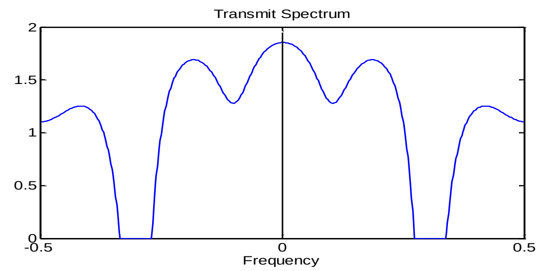
- If **z** is from a stationary process then $\mathrm{diag}(\Lambda) \underset{n\to\infty}{\rightarrow}$ power spectrum $N(f)$
  - To achieve capacity use waterfilling on noise power spectrum

$$P = \int_{-W}^{W} \max\left(v - N(f), 0\right) df$$

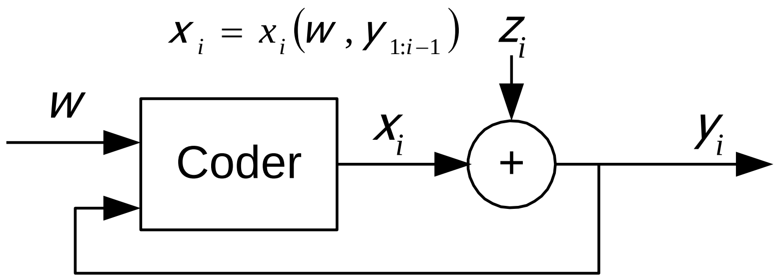
$$C = \int_{-W}^{W} \tfrac{1}{2} \log\left(1 + \frac{\max\left(v - N(f), 0\right)}{N(f)}\right) df$$

  - Waterfilling on spectral domain



# Gaussian Channel + Feedback




## Does Feedback add capacity ?

- White noise (& DMC) – No
- Coloured noise – Not much

$$I(w;\mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y}\,|\,w) = h(\mathbf{y}) - \sum_{i=1}^{n} h(y_i\,|\,w, y_{1:i-1})$$ Chain rule

$$= h(\mathbf{y}) - \sum_{i=1}^{n} h(y_i\,|\,w, y_{1:i-1}, x_{1:i}, z_{1:i-1})$$ $x_i = x_i(w, y_{1:i-1})$, $\mathbf{z} = \mathbf{y} - \mathbf{x}$

$$= h(\mathbf{y}) - \sum_{i=1}^{n} h(z_i\,|\,w, y_{1:i-1}, x_{1:i}, z_{1:i-1})$$ $\mathbf{z} = \mathbf{y} - \mathbf{x}$ and translation invariance

$$= h(\mathbf{y}) - \sum_{i=1}^{n} h(z_i\,|\,z_{1:i-1})$$ $\mathbf{z}$ may be colored; $z_i$ depends only on $z_{1:i-1}$

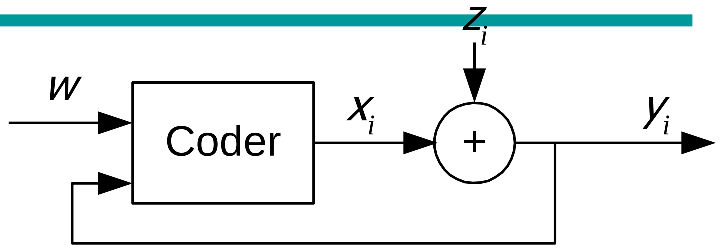
$$= h(\mathbf{y}) - h(\mathbf{z})$$ Chain rule, $h(\mathbf{z}) = ½\log\left(\left|2\pi e \mathbf{K}_{\mathbf{z}}\right|\right)$ bits

$$\le \frac{1}{2}\log\frac{|\mathbf{K}_{\mathbf{y}}|}{|\mathbf{K}_{\mathbf{z}}|}$$

$\Rightarrow$ maximize $I(w;\mathbf{y})$ by maximizing $h(\mathbf{y}) \Rightarrow \mathbf{y}$ gaussian

$\Rightarrow$ we can take $\mathbf{z}$ and $\mathbf{x} = \mathbf{y} - \mathbf{z}$ jointly gaussian

# Maximum Benefit of Feedback




$$C_{n,FB} = \max_{\mathrm{tr}(\mathbf{K_x}) \le nP} \tfrac{1}{2} n^{-1} \log \frac{|\mathbf{K_y}|}{|\mathbf{K_z}|}$$

$$\le \max_{\mathrm{tr}(\mathbf{K_x}) \le nP} \tfrac{1}{2} n^{-1} \log \frac{|2(\mathbf{K_x} + \mathbf{K_z})|}{|\mathbf{K_z}|}$$

Lemmas 1 & 2:

$|2(\mathbf{Kx}+\mathbf{Kz})| \ge |\mathbf{Ky}|$

$$= \max_{\mathrm{tr}(\mathbf{K_x}) \le nP} \tfrac{1}{2} n^{-1} \log \frac{2^n |\mathbf{K_x} + \mathbf{K_z}|}{|\mathbf{K_z}|}$$

$|k\mathbf{A}| = k^n |(\mathbf{A}|$

$$= \tfrac{1}{2} + \max_{\mathrm{tr}(\mathbf{K_x}) \le nP} \tfrac{1}{2} n^{-1} \log \frac{|\mathbf{K_x} + \mathbf{K_z}|}{|\mathbf{K_z}|} = \tfrac{1}{2} + C_n \quad \text{bits / transmission}$$

**Ky** = **Kx**+**Kz** if no feedback

$C_n$: capacity without feedback

Having feedback adds at most ½ bit per transmission for colored Gaussian noise channels

# Max Benefit of Feedback: Lemmas

Lemma 1: $\mathbf{K}_{\mathbf{x}+\mathbf{z}}+\mathbf{K}_{\mathbf{x}-\mathbf{z}} = 2(\mathbf{K}_{\mathbf{x}}+\mathbf{K}_{\mathbf{z}})$

$$\begin{aligned}\mathbf{K}_{\mathbf{x}+\mathbf{z}}+\mathbf{K}_{\mathbf{x}-\mathbf{z}} &= E(\mathbf{x}+\mathbf{z})(\mathbf{x}+\mathbf{z})^T + E(\mathbf{x}-\mathbf{z})(\mathbf{x}-\mathbf{z})^T \\ &= E\left(\mathbf{xx}^T+\mathbf{xz}^T+\mathbf{zx}^T+\mathbf{zz}^T + \mathbf{xx}^T-\mathbf{xz}^T-\mathbf{zx}^T+\mathbf{zz}^T\right) \\ &= E\left(2\mathbf{xx}^T+2\mathbf{zz}^T\right) = 2(\mathbf{K}_{\mathbf{x}}+\mathbf{K}_{\mathbf{z}})\end{aligned}$$

Lemma 2: If $\mathbf{F},\mathbf{G}$ are positive definite then $|\mathbf{F}+\mathbf{G}| \ge |\mathbf{F}|$

Consider two indep random vectors $\mathbf{f}\sim N(0,\mathbf{F})$, $\mathbf{g}\sim N(0,\mathbf{G})$

$$\begin{aligned}\tfrac{1}{2}\log\left((2\pi e)^n|\mathbf{F}+\mathbf{G}|\right) &= h(\mathbf{f}+\mathbf{g}) \\ &\ge h(\mathbf{f}+\mathbf{g}\,|\,\mathbf{g}) = h(\mathbf{f}\,|\,\mathbf{g}) \\ &= h(\mathbf{f}) = \tfrac{1}{2}\log\left((2\pi e)^n|\mathbf{F}|\right)\end{aligned}$$

Conditioning reduces $h()$

Translation invariance

$\mathbf{f},\mathbf{g}$ independent

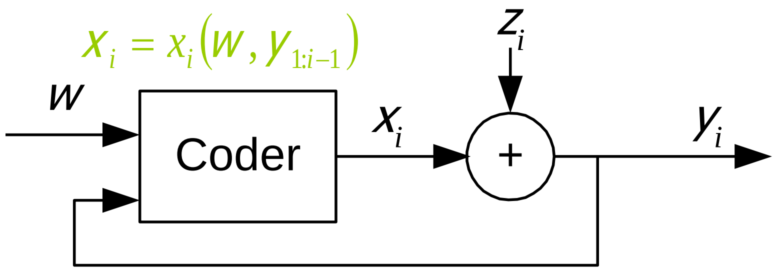
Hence: $|2(\mathbf{K}_{\mathbf{x}}+\mathbf{K}_{\mathbf{z}})| = |\mathbf{K}_{\mathbf{x}+\mathbf{z}}+\mathbf{K}_{\mathbf{x}-\mathbf{z}}| \ge |\mathbf{K}_{\mathbf{x}+\mathbf{z}}| = |\mathbf{K}_{\mathbf{y}}|$

# Gaussian Feedback Coder

**x** and **z** jointly gaussian $\Rightarrow$

$$\mathbf{x} = \mathbf{B}\mathbf{z} + \mathbf{v}(w)$$

where **v** is indep of **z** and
$\mathbf{B}$ is strictly lower triangular since $x_i$ indep of $z_j$ for $j>i$.




$$\mathbf{y} = \mathbf{x} + \mathbf{z} = (\mathbf{B} + \mathbf{I})\mathbf{z} + \mathbf{v}$$

$$\mathbf{K_y} = E\mathbf{y}\mathbf{y}^T = E\left((\mathbf{B} + \mathbf{I})\mathbf{z}\mathbf{z}^T(\mathbf{B} + \mathbf{I})^T + \mathbf{v}\mathbf{v}^T\right) = (\mathbf{B} + \mathbf{I})\mathbf{K_z}(\mathbf{B} + \mathbf{I})^T + \mathbf{K_v}$$

$$\mathbf{K_x} = E\mathbf{x}\mathbf{x}^T = E\left(\mathbf{B}\mathbf{z}\mathbf{z}^T\mathbf{B}^T + \mathbf{v}\mathbf{v}^T\right) = \mathbf{B}\mathbf{K_z}\mathbf{B}^T + \mathbf{K_v}$$

Capacity: $$C_{n,FB} = \max_{\mathbf{K_v},\mathbf{B}} \tfrac{1}{2} n^{-1} \frac{|\mathbf{K_y}|}{|\mathbf{K_z}|} = \max_{\mathbf{K_v},\mathbf{B}} \tfrac{1}{2} n^{-1} \log \frac{\left|(\mathbf{B} + \mathbf{I})\mathbf{K_z}(\mathbf{B} + \mathbf{I})^T + \mathbf{K_v}\right|}{|\mathbf{K_z}|}$$

subject to $$\mathbf{K_x} = \operatorname{tr}\left(\mathbf{B}\mathbf{K_z}\mathbf{B}^T + \mathbf{K_v}\right) \le nP$$

hard to solve ☹

**Optimization can be done numerically**

# Gaussian Feedback: Toy Example

$$n = 2, \quad P = 2, \quad \mathbf{K_z} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{Bz} + \mathbf{v} \Rightarrow x_1 = v_1, \; x_2 = bz_1 + v_2$$

Goal: Maximize (w.r.t. $\mathbf{K_v}$ and $b$)

$$|\mathbf{K_Y}| = |(\mathbf{B}+\mathbf{I})\mathbf{K_z}(\mathbf{B}+\mathbf{I})^T + \mathbf{K_v}|$$

Subject to:

$\mathbf{K_v}$ must be positive definite

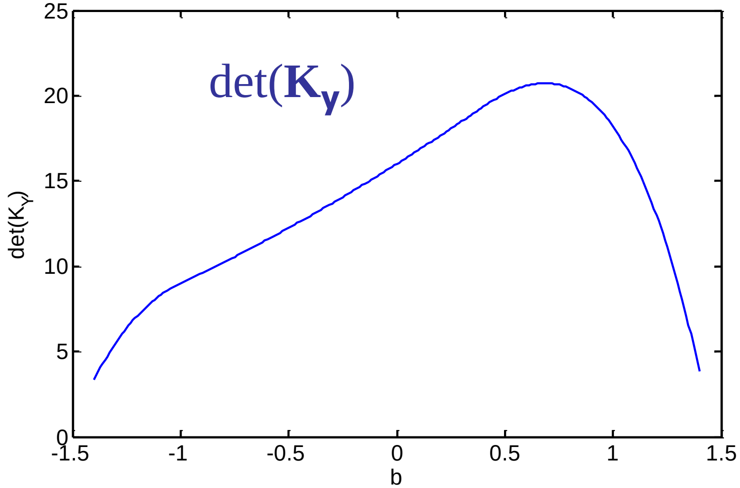
Power constraint : $\mathrm{tr}\left(\mathbf{BK_zB}^T + \mathbf{K_v}\right) \le 4$

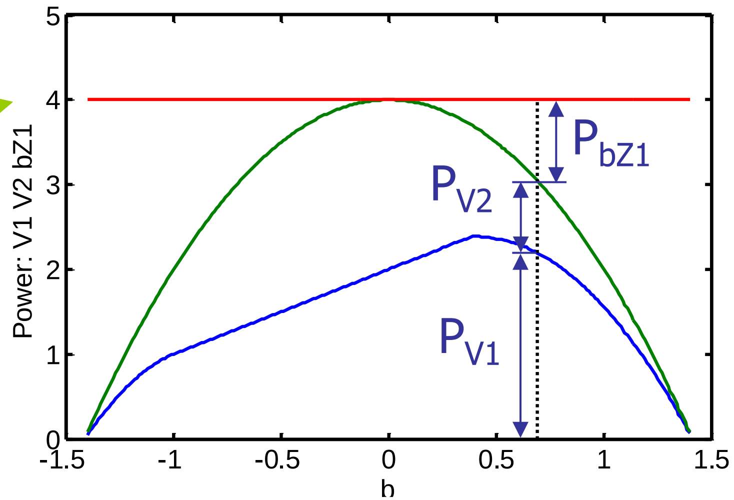
Solution (via numerically search):

b=0: $|\mathbf{K_y}|=16$ $C=0.604$ bits

b=0.69: $|\mathbf{K_y}|=20.7$ $C=0.697$ bits

Feedback increases C by 16%




$$P_{x1} = P_{v1} \qquad P_{x2} = P_{v2} + P_{bz1}$$

# Summary

- Water-filling for parallel Gaussian channel

$$C = \sum_{i=1}^{n} \frac{1}{2} \log\left(1 + \frac{(v - N_i)^+}{N_i}\right)$$

$$x^+ = \max(x, 0)$$

$$\sum (v - N_i)^+ = nP$$

- Colored Gaussian noise

$$C = \sum_{i=1}^{n} \frac{1}{2} \log\left(1 + \frac{(v - \lambda_i)^+}{\lambda_i}\right)$$

$\lambda_i$ eigenvealues of $\mathbf{K_z}$

$$\sum (v - \lambda_i)^+ = nP$$

- Continuous Gaussian channel

$$C = \int_{-W}^{W} ½ \log\left(1 + \frac{\left(v - N(f)\right)^+}{N(f)}\right) df$$

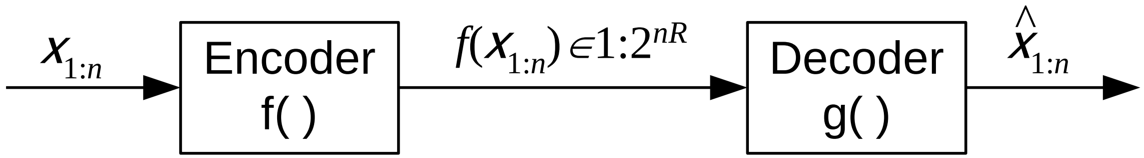
- Feedback bound

$$C_{n,FB} \leq C_n + \frac{1}{2}$$

# Lecture 16

- Lossy Source Coding
  - For both discrete and continuous sources
  - Bernoulli Source, Gaussian Source
- Rate Distortion Theory
  - What is the minimum distortion achievable at a particular rate?
  - What is the minimum rate to achieve a particular distortion?
- Channel/Source Coding Duality

# Lossy Source Coding




**Distortion function:** $d(x, \hat{x}) \geq 0$

- examples: (i) $d_S(x, \hat{x}) = (x - \hat{x})^2$ (ii) $d_H(x, \hat{x}) = \begin{cases} 0 & x = \hat{x} \\ 1 & x \neq \hat{x} \end{cases}$
- sequences: $d(\mathbf{x}, \hat{\mathbf{x}}) \stackrel{\Delta}{=} n^{-1} \sum_{i=1}^{n} d(x_i, \hat{x}_i)$

**Distortion of Code $f_n(\ )$, $g_n(\ )$:** $D = E_{\mathbf{x} \in \mathcal{X}^n} d\left(\mathbf{x}, \hat{\mathbf{x}}\right) = E\ d\left(\mathbf{x}, g(f(\mathbf{x}))\right)$

**Rate distortion pair (R,D) is achievable for source $X$ if**

$\exists$ a sequence $f_n(\ )$ and $g_n(\ )$ such that $\lim\limits_{n \to \infty} E_{\mathbf{x} \in \mathcal{X}^n} d(\mathbf{x}, g_n(f_n(\mathbf{x}))) \leq D$

# Rate Distortion Function

Rate Distortion function for $\{x_i\}$ with pdf $p(\mathbf{x})$ is defined as

$$R(D) = \min\{R\} \text{ such that } (R,D) \text{ is achievable}$$

Theorem: $R(D) = \min\ I(x;\hat{x})$ over all $p(x,\hat{x})$ such that :

(a) $p(x)$ is correct ♦

(b) $E_{x,\hat{x}}\, d(x,\hat{x}) \le D$

– this expression is the Rate Distortion function for $X$

Proof is not examinable

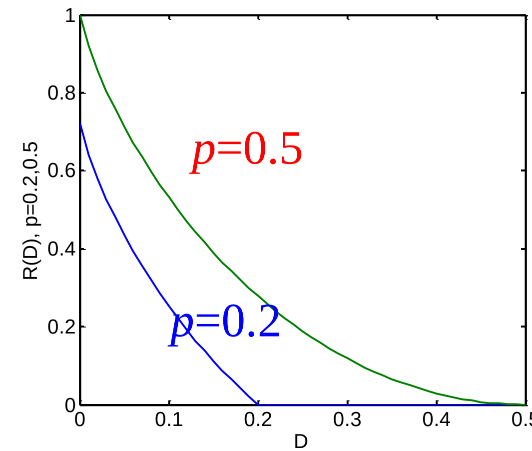
Lossless coding: If $D = 0$ then we have $R(D) = I(x;x) = H(x)$

♦ $p(x,\hat{x}) = p(x)q(\hat{x} \mid x)$

# $R(D)$ bound for Bernoulli Source

Bernoulli: X = $[0,1]$, $\mathbf{p}_X = [1-p\ ,\ p]$ assume $p \le ½$

- Hamming Distance: $d(x,\hat{x}) = x \oplus \hat{x}$
- If $D \ge p$, $R(D)=0$ since we can set $g(\ ) \equiv 0$
- For $D < p \le ½$, if $E\ d(x,\hat{x}) \le D$ then




$$I(x;\hat{x}) = H(x) - H(x \mid \hat{x})$$

$$= H(p) - H(x \oplus \hat{x} \mid \hat{x})$$ $\oplus$ is one-to-one

$$\ge H(p) - H(x \oplus \hat{x})$$ Conditioning reduces entropy

$$\ge H(p) - H(D)$$

$\text{Prob.}(x \oplus \hat{x} = 1) \le D$ for $D \le ½$

$H(x \oplus \hat{x}) \le H(D)$ as $H(p)$ monotonic

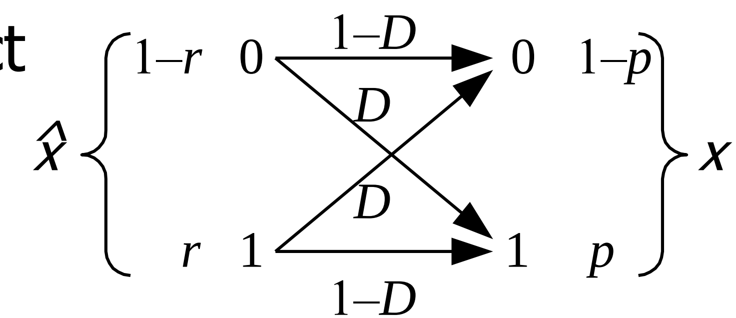
Hence $R(D) \ge H(p) - H(D)$

# $R(D)$ for Bernoulli source

We know optimum satisfies $R(D) \geq H(p) - H(D)$

- We show we can find a $p(\hat{x}, x)$ that attains this.
- Peculiarly, we consider a channel with $\hat{x}$ as the input and error probability $D$

Now choose $r$ to give $x$ the correct probabilities:




$$r(1-D) + (1-r)D = p$$

$$\Rightarrow r = (p - D)(1 - 2D)^{-1}, \qquad D \leq p$$

Now $I(x;\hat{x}) = H(x) - H(x \mid \hat{x}) = H(p) - H(D)$

and $p(x \neq \hat{x}) = D \quad \Rightarrow \quad \text{distortion} \leq D$

Hence $R(D) = H(p) - H(D)$

If $D \geq p$ or $D \geq$ 1- $p$, we can achieve $R(D)$=0 trivially.

# $R(D)$ bound for Gaussian Source

- Assume $X \sim N(0,\sigma^2)$ and $d(x,\hat{x}) = (x-\hat{x})^2$
- Want to minimize $I(x;\hat{x})$ subject to $E(x-\hat{x})^2 \le D$

$$I(x;\hat{x}) = h(x) - h(x\,|\,\hat{x})$$

$$= ½\log 2\pi e\sigma^2 - h(x-\hat{x}\,|\,\hat{x})$$ Translation Invariance

$$\ge ½\log 2\pi e\sigma^2 - h(x-\hat{x})$$ Conditioning reduces entropy

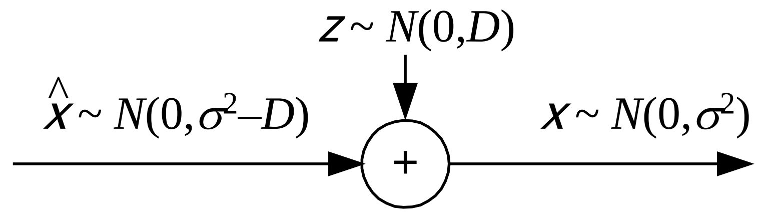
$$\ge ½\log 2\pi e\sigma^2 - ½\log\left(2\pi e \operatorname{Var}\left(x-\hat{x}\right)\right)$$ Gauss maximizes entropy for given covariance

$$\ge ½\log 2\pi e\sigma^2 - ½\log 2\pi e D$$ require $\operatorname{Var}\left(x-\hat{x}\right) \le E(x-\hat{x})^2 \le D$

$$I(x;\hat{x}) \ge \max\left(½\log\frac{\sigma^2}{D}, 0\right)$$ $I(x;y)$ always positive

# $R(D)$ for Gaussian Source

To show that we can find a $p(\hat{x}, x)$ that achieves the bound, we construct a test channel that introduces distortion $D < \sigma^2$



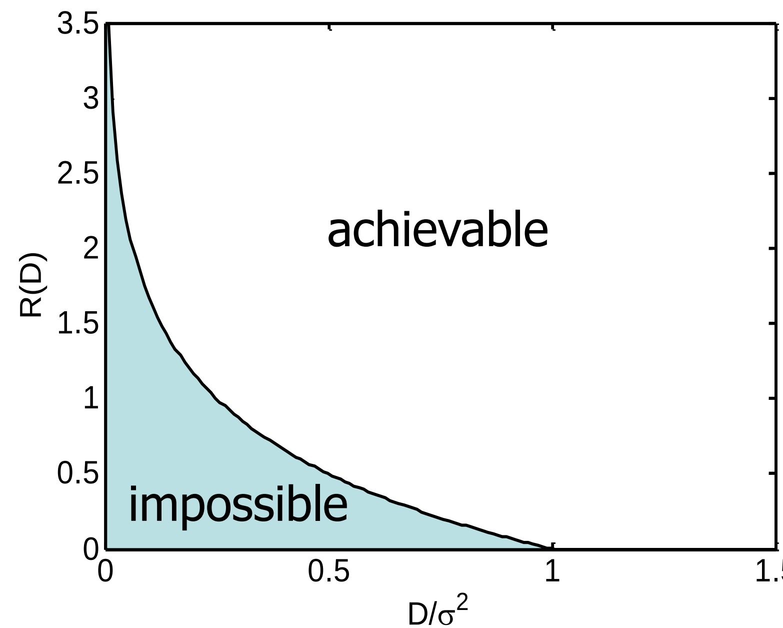

$$I(x;\hat{x}) = h(x) - h(x \mid \hat{x})$$

$$= ½ \log 2\pi e\sigma^2 - h(x - \hat{x} \mid \hat{x})$$

$$= ½ \log 2\pi e\sigma^2 - h(z \mid \hat{x})$$

$$= ½ \log \frac{\sigma^2}{D}$$

$$\Rightarrow R(D) = \max\{½ \log \frac{\sigma^2}{D}, 0\}$$




$$\Rightarrow \quad D(R) = \frac{\sigma^2}{2^{2R}} \qquad \text{cf. PCM} \quad D(R) = \frac{m_p^2/3}{2^{2R}} \overset{m_p = 4\sigma}{=} \frac{16/3 \cdot \sigma^2}{2^{2R}}$$

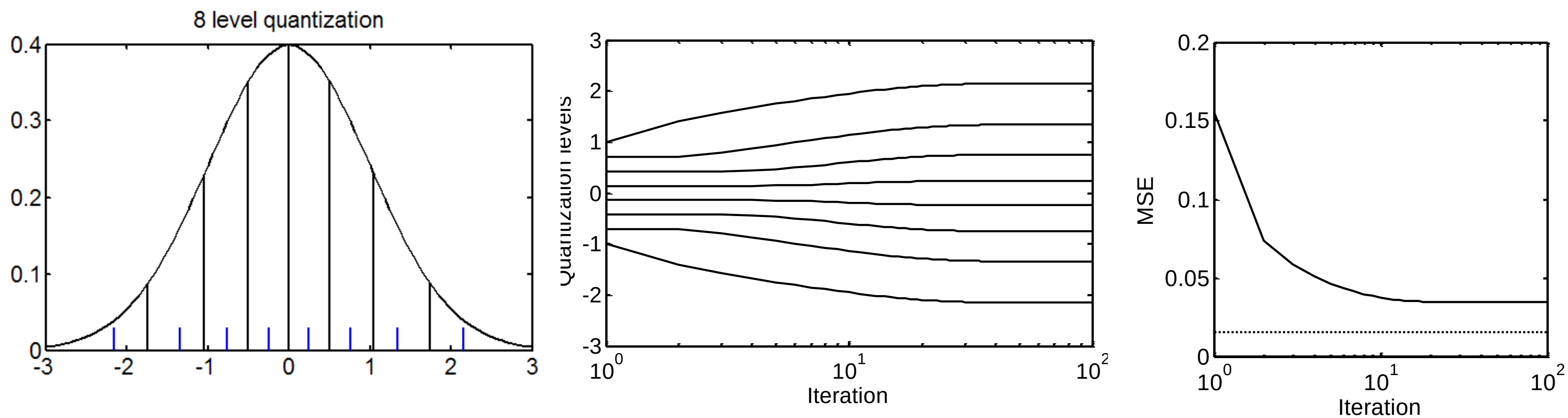
# Lloyd Algorithm

**Problem:** Find optimum quantization levels for Gaussian pdf

a. Bin boundaries are midway between quantization levels

b. Each quantization level equals the mean value of its own bin

**Lloyd algorithm:** Pick random quantization levels then apply conditions (a) and (b) in turn until convergence.



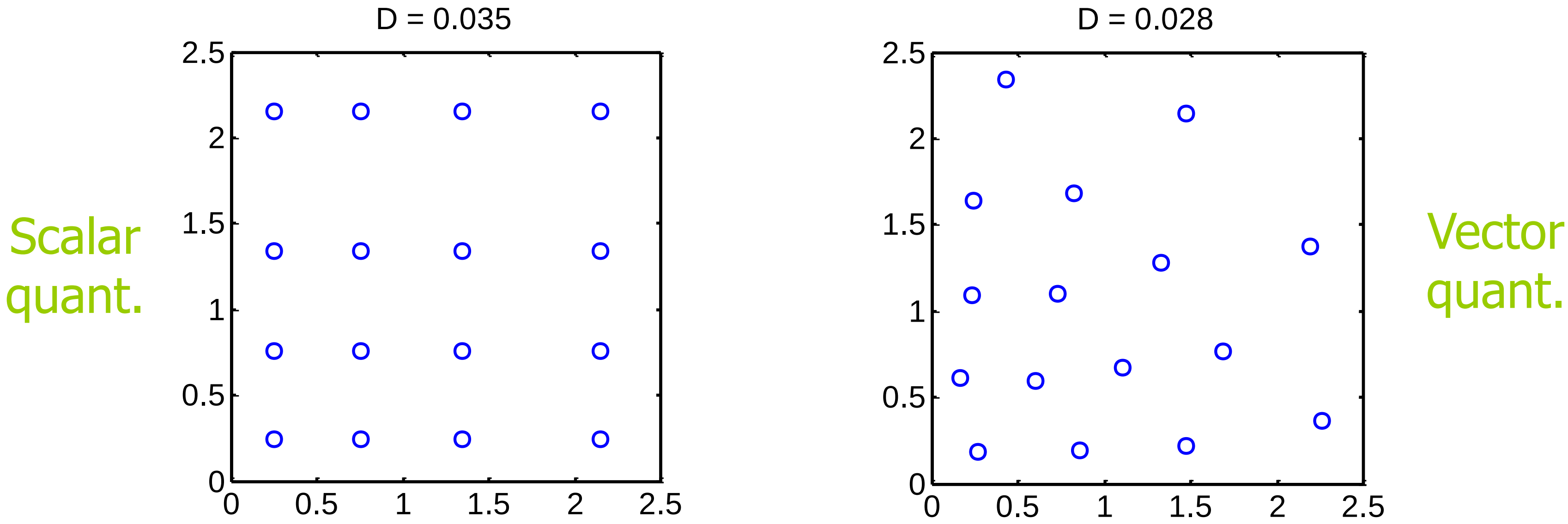

Solid lines are bin boundaries. Initial levels uniform in $[-1,+1]$ .

Best mean sq error for 8 levels = $0.0345\sigma^2$. Predicted $D(R) = (\sigma/8)^2 = 0.0156\sigma^2$

# Vector Quantization

To get $D(R)$, you have to quantize many values together

- True even if the values are independent




Two gaussian variables: one quadrant only shown

- Independent quantization puts dense levels in low prob areas
- Vector quantization is better (even more so if correlated)

# Multiple Gaussian Variables

- Assume $x_{1:n}$ are independent gaussian sources with different variances. How should we apportion the available total distortion between the sources?
- Assume $x_i \sim N(0,\sigma_i^2)$ and $d(\mathbf{x},\hat{\mathbf{x}}) = n^{-1}\left(\mathbf{x}-\hat{\mathbf{x}}\right)^T\left(\mathbf{x}-\hat{\mathbf{x}}\right) \le D$

$$I(x_{1:n};\hat{x}_{1:n}) \ge \sum_{i=1}^{n} I(x_i;\hat{x}_i)$$

Mut Info Independence Bound for independent $x_i$

$$\ge \sum_{i=1}^{n} R(D_i) = \sum_{i=1}^{n} \max\left(½\log\frac{\sigma_i^2}{D_i},0\right)$$

$R(D)$ for individual Gaussian

We must find the $D_i$ that minimize

$$\sum_{i=1}^{n} \max\left(½\log\frac{\sigma_i^2}{D_i},0\right)$$

$$\Rightarrow D_i = \begin{cases} D_0 & \text{if } D_0 < \sigma_i^2 \\ \sigma_i^2 & \text{otherwise} \end{cases}$$

$$\text{such that} \quad n^{-1}\sum_{i=1}^{n} D_i = D$$

# Reverse Water-filling

Minimize $\sum_{i=1}^{n} \max\left(\frac{1}{2}\log\frac{\sigma_i^2}{D_i}, 0\right)$ subject to $\sum_{i=1}^{n} D_i \le nD$

$R_i = \frac{1}{2}\log\frac{\sigma_i^2}{D}$

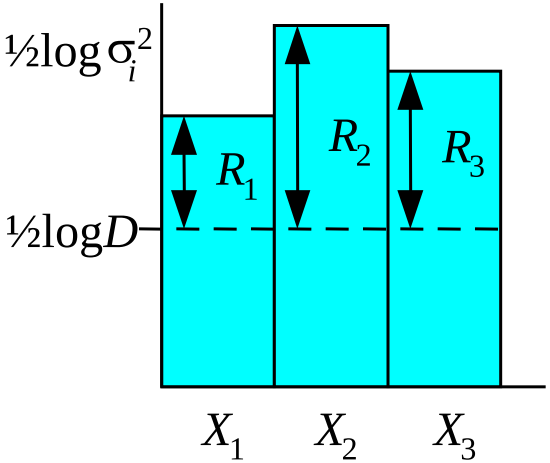
Use a Lagrange multiplier:

$$J = \sum_{i=1}^{n} \frac{1}{2}\log\frac{\sigma_i^2}{D_i} + \lambda\sum_{i=1}^{n} D_i$$

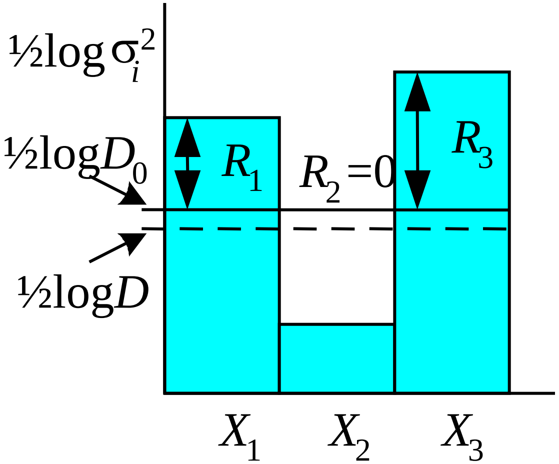
$$\frac{\partial J}{\partial D_i} = -\frac{1}{2}D_i^{-1} + \lambda = 0 \quad \Rightarrow \quad D_i = \frac{1}{2}\lambda^{-1} = D_0$$

$$\sum_{i=1}^{n} D_i = nD_0 = nD \qquad \Rightarrow \qquad D_0 = D$$

Choose $R_i$ for equal distortion

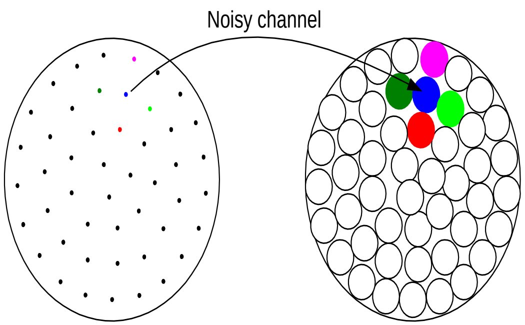



- If $\sigma_i^2 < D$ then set $R_i = 0$ (meaning $D_i = \sigma_i^2$) and increase $D_0$ to maintain the average distortion equal to $D$

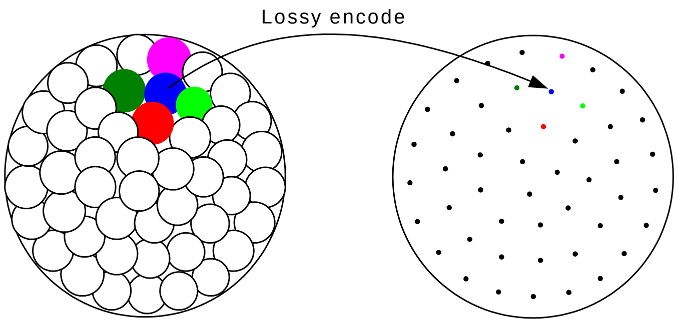


# Channel/Source Coding Duality

- Channel Coding
  - Find codes separated enough to give non-overlapping output images.
  - Image size = channel noise
  - The maximum number (highest rate) is when the images just don't overlap (some gap).
- Source Coding
  - Find regions that cover the sphere
  - Region size = allowed distortion
  - The minimum number (lowest rate) is when they just fill the sphere (with no gap).




Sphere Packing




Sphere Covering

# Gaussian Channel/Source

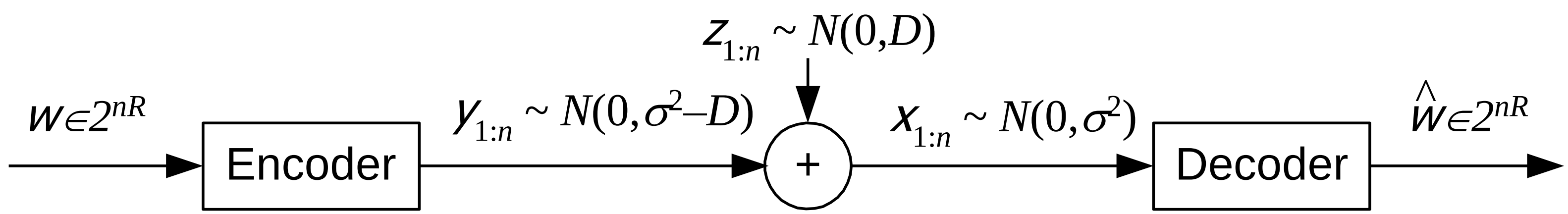
- Capacity of Gaussian channel ($n$: length)
  - Radius of big sphere $\sqrt{n(P+N)}$
  - Radius of small spheres $\sqrt{nN}$
  - Capacity $2^{nC} = \dfrac{\sqrt{n(P+N)}^{\,n}}{\sqrt{nN}^{\,n}} = \left(\dfrac{P+N}{N}\right)^{n/2}$

Maximum number of small spheres packed in the big sphere

- Rate distortion for Gaussian source
  - Variance $\sigma^2 \rightarrow$ radius of big sphere $\sqrt{n\sigma^2}$
  - Radius of small spheres $\sqrt{nD}$ for distortion $D$
  - Rate $2^{nR(D)} = \left(\dfrac{\sigma^2}{D}\right)^{n/2}$

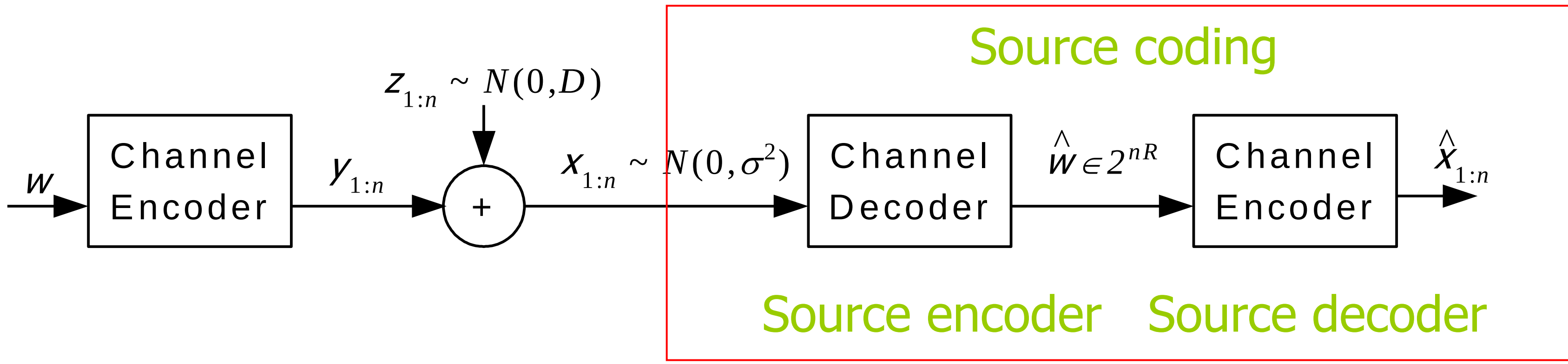
Minimum number of small spheres to cover the big sphere

# Channel Decoder as Source Encoder




- For $R \cong C = \frac{1}{2}\log\left(1 + \left(\sigma^2 - D\right)D^{-1}\right)$, we can find a channel encoder/decoder so that $p(\hat{w} \neq w) < \varepsilon$ and $E(x_i - y_i)^2 = D$
- Now reverse the roles of encoder and decoder. Since

$$p(\hat{x} \neq y) = p\left(w \neq \hat{w}\right) < \varepsilon \text{ and } E(x_i - \hat{x}_i)^2 \cong E(x_i - y_i)^2 = D$$




We have encoded $x$ at rate $R = \frac{1}{2}\log(\sigma^2 D^{-1})$ with distortion $D$!

# Summary

- Lossy source coding: tradeoff between rate and distortion
- Rate distortion function

$$R(D) = \min_{\mathbf{p}_{\hat{x}|x}\ s.t.\ Ed(x,\hat{x}) \le D} I(x;\hat{x})$$

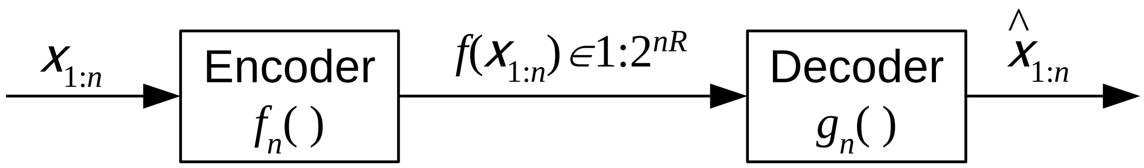
- Bernoulli source: $R(D) = (H(p) - H(D))^+$
- Gaussian source (reverse waterfilling): $R(D) = \left(\frac{1}{2}\log\frac{\sigma^2}{D}\right)^+$
- Duality: channel decoding (encoding) $\Leftrightarrow$ source encoding (decoding)

# Nothing But Proof

- Proof of Rate Distortion Theorem
  - Converse: if the rate is less than R(D), then distortion of any code is higher than D
  - Achievability: if the rate is higher than R(D), then there exists a rate-R code which achieves distortion D

Quite technical!

# Review




Rate Distortion function for $x$ whose $p_{\mathbf{x}}(\mathbf{x})$ is known is

$$R(D) = \inf R \text{ such that } \exists f_n, g_n \text{ with } \lim_{n\to\infty} E_{\mathbf{x}\in X^n} d(\mathbf{x}, \hat{\mathbf{x}}) \le D$$

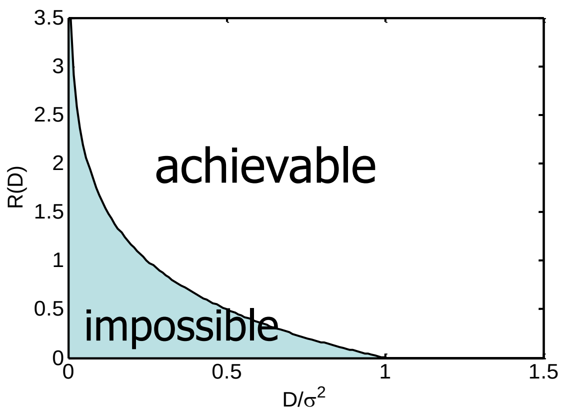
Rate Distortion Theorem:

$$R(D) = \min I(x;\hat{x}) \text{ over all } p(\hat{x}\,|\,x) \text{ such that } E_{x,\hat{x}} d(x,\hat{x}) \le D$$

We will prove this theorem for discrete $X$ and bounded $d(x,y) \le d_{\max}$

$R(D)$ curve depends on your choice of $d(,)$

Decreasing and convex



# Converse: Rate Distortion Bound

Suppose we have found an encoder and decoder at rate $R_0$ with expected distortion $D$ for independent $x_i$ (worst case)

We want to prove that $R_0 \geq R(D) = R\big(E\, d(\mathbf{x};\hat{\mathbf{x}})\big)$

– We show first that $R_0 \geq n^{-1}\sum_i I(x_i;\hat{x}_i)$

– We know that $I(x_i;\hat{x}_i) \geq R\big(E\ d(x_i;\hat{x}_i)\big)$ Def$^n$ of $R(D)$

– and use convexity of $R(D)$ to show

$$n^{-1}\sum_i R\left(E\ d(x_i;\hat{x}_i)\right) \geq R\left(n^{-1}\sum_i E\ d(x_i;\hat{x}_i)\right) = R\left(E\ d(\mathbf{x};\hat{\mathbf{x}})\right) = R(D)$$

We prove convexity first and then the rest

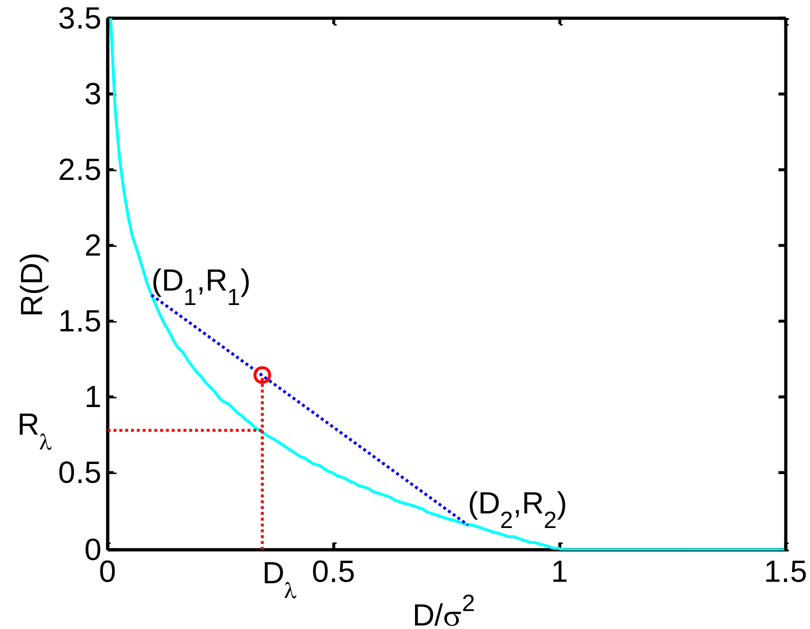
# Convexity of $R(D)$

If $p_1(\hat{x} \mid x)$ and $p_2(\hat{x} \mid x)$ are associated with $(D_1, R_1)$ and $(D_2, R_2)$ on the $R(D)$ curve we define

$$p_\lambda(\hat{x} \mid x) = \lambda p_1(\hat{x} \mid x) + (1-\lambda) p_2(\hat{x} \mid x)$$

Then

$$E_{p_\lambda} d(x, \hat{x}) = \lambda D_1 + (1-\lambda) D_2 = D_\lambda$$




$$R(D_\lambda) \le I_{p_\lambda}(X; \hat{X})$$

$$\le \lambda I_{p_1}(X; \hat{X}) + (1-\lambda) I_{p_2}(X; \hat{X})$$

$$= \lambda R(D_1) + (1-\lambda) R(D_2)$$

$$R(D) = \min_{p(\hat{x}|x)} I(X; \hat{X})$$

$I(X; \hat{X})$ convex w.r.t. $p(\hat{x} \mid x)$

$p_1$ and $p_2$ lie on the $R(D)$ curve

# Proof that $R \geq R(D)$

$$nR_0 \geq H(\hat{x}_{1:n}) \geq H(\hat{x}_{1:n}) - H(\hat{x}_{1:n} \mid x_{1:n})$$ Uniform bound; $H(\hat{x} \mid x) \geq 0$

$$= I(\hat{x}_{1:n}; x_{1:n})$$ Definition of $I(;)$

$$\geq \sum_{i=1}^{n} I(x_i; \hat{x}_i)$$ $x_i$ indep: Mut Inf Independence Bound

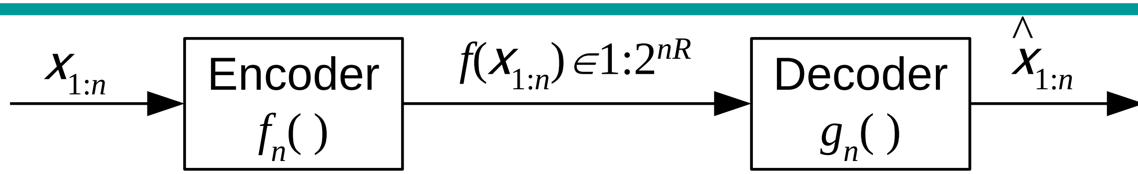
$$\geq \sum_{i=1}^{n} R\big(E\ d(x_i; \hat{x}_i)\big) = n \sum_{i=1}^{n} n^{-1} R\big(E\ d(x_i; \hat{x}_i)\big)$$ definition of $R$

$$\geq nR\left(n^{-1} \sum_{i=1}^{n} E\ d(x_i; \hat{x}_i)\right) = nR\big(E\ d(x_{1:n}; \hat{x}_{1:n})\big)$$ convexity

defn of vector $d(\ )$

$$\geq nR(D)$$ original assumption that $E(d) \leq D$ and $R(D)$ monotonically decreasing

# Rate Distortion Achievability




We want to show that for any $D$, we can find an encoder and decoder that compresses $x_{1:n}$ to $nR(D)$ bits.

- $\mathbf{p}_X$ is given
- Assume we know the $p(\hat{x} \mid x)$ that gives $I(x;\hat{x}) = R(D)$
- Random codebook: Choose $2^{nR}$ random $\hat{x}_i \sim \mathbf{p}_{\hat{x}}$
  - There must be at least one code that is as good as the average
- Encoder: Use joint typicality to design
  - We show that there is almost always a suitable codeword

First define the typical set we will use, then prove two preliminary results.

# Distortion Typical Set

## Distortion Typical: $(x_i, \hat{x}_i) \in \mathcal{X} \times \hat{\mathcal{X}}$ drawn i.i.d. $\sim p(x, \hat{x})$

$$
\begin{aligned}
J_{d,\varepsilon}^{(n)} = \Big\{ \mathbf{x}, \hat{\mathbf{x}} \in \mathcal{X}^n \times \hat{\mathcal{X}}^n : & \left| -n^{-1} \log p(\mathbf{x}) - H(x) \right| < \varepsilon, \\
& \left| -n^{-1} \log p(\hat{\mathbf{x}}) - H(\hat{x}) \right| < \varepsilon, \\
& \left| -n^{-1} \log p(\mathbf{x}, \hat{\mathbf{x}}) - H(x, \hat{x}) \right| < \varepsilon \\
& \left| d(\mathbf{x}, \hat{\mathbf{x}}) - E\, d(x, \hat{x}) \right| < \varepsilon \Big\}
\end{aligned}
$$

new condition

## Properties of Typical Set:

1. Indiv p.d.: $\mathbf{x}, \hat{\mathbf{x}} \in J_{d,\varepsilon}^{(n)} \Rightarrow \log p(\mathbf{x}, \hat{\mathbf{x}}) = -nH(x, \hat{x}) \pm n\varepsilon$
2. Total Prob: $p\left(\mathbf{x}, \hat{\mathbf{x}} \in J_{d,\varepsilon}^{(n)}\right) > 1 - \varepsilon$ for $n > N_\varepsilon$

weak law of large numbers; $d(x_i, \hat{x}_i)$ are i.i.d.

# Conditional Probability Bound

**Lemma:** $\mathbf{x}, \hat{\mathbf{x}} \in J_{d,\varepsilon}^{(n)} \Rightarrow p(\hat{\mathbf{x}}) \geq p(\hat{\mathbf{x}} \mid \mathbf{x}) 2^{-n\left(I(X;\hat{X})+3\varepsilon\right)}$

**Proof:**

$$p(\hat{\mathbf{x}} \mid \mathbf{x}) = \frac{p(\hat{\mathbf{x}}, \mathbf{x})}{p(\mathbf{x})}$$

$$= p(\hat{\mathbf{x}}) \frac{p(\hat{\mathbf{x}}, \mathbf{x})}{p(\hat{\mathbf{x}}) p(\mathbf{x})}$$

take max of top and min of bottom

$$\leq p(\hat{\mathbf{x}}) \frac{2^{-n\left(H(X,\hat{X})-\varepsilon\right)}}{2^{-n\left(H(X)+\varepsilon\right)} 2^{-n\left(H(\hat{X})+\varepsilon\right)}}$$

bounds from def$^n$ of $J$

$$= p(\hat{\mathbf{x}}) 2^{n\left(I(X;\hat{X})+3\varepsilon\right)}$$

def$^n$ of $I$

# Curious but Necessary Inequality

**Lemma:** $u,v \in [0,1], m > 0 \quad \Rightarrow \quad (1-uv)^m \le 1-u+e^{-vm}$

**Proof:** $u=0$: $\quad e^{-vm} \ge 0 \quad \Rightarrow \quad (1-0)^m \le 1-0+e^{-vm}$

$u=1$: Define $\quad f(v) = e^{-v} - 1 + v \quad \Rightarrow \quad f'(v) = 1 - e^{-v}$

$f(0) = 0$ and $f'(v) > 0$ for $v > 0 \quad \Rightarrow \quad f(v) \ge 0$ for $v \in [0,1]$

Hence for $v \in [0,1], \quad 0 \le 1 - v \le e^{-v} \quad \Rightarrow (1-v)^m \le e^{-vm}$

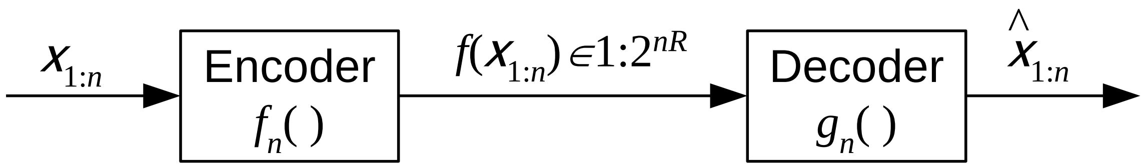
$0<u<1$: Define $\quad g_v(u) = (1-uv)^m$

$\Rightarrow g_v''(x) = m(m-1)v^2(1-uv)^{n-2} \ge 0 \Rightarrow g_v(u)$ convex $\qquad$ for $u,v \in [0,1]$

$(1-uv)^m = g_v(u) \le (1-u)g_v(0) + ug_v(1)$ $\qquad$ convexity for $u,v \in [0,1]$

$$= (1-u)1 + u(1-v)^m \le 1 - u + ue^{-vm} \le 1 - u + e^{-vm}$$

# Achievability of $R(D)$: preliminaries




- Choose $D$ and find a $p(\hat{x} \mid x)$ such that $I(x;\hat{x}) = R(D);\ E\, d(x,\hat{x}) \le D$

  Choose $\delta > 0$ and define $\mathbf{p}_{\hat{x}} = \{ \quad p(\hat{x}) = \sum_x p(x)\, p(\hat{x} \mid x) \quad \}$

- **Decoder:** For each $w \in 1:2^{nR}$ choose $g_n(w) = \hat{\mathbf{x}}_w$ drawn i.i.d. $\sim \mathbf{p}_{\hat{x}}^n$
- **Encoder:** $f_n(\mathbf{x}) = \min w$ such that $(\mathbf{x}, \hat{\mathbf{x}}_w) \in J_{d,\varepsilon}^{(n)}$ else $1$ if no such $w$
- **Expected Distortion:** $\overline{D} = E_{\mathbf{x},g}\, d(\mathbf{x}, \hat{\mathbf{x}})$
  - over all input vectors $\mathbf{x}$ and all random decoding functions, $g$
  - for large $n$ we show $\overline{D} = D + \delta$ so there must be one good code

# Expected Distortion

We can divide the input vectors $\mathbf{x}$ into two categories:

a) if $\exists w$ such that $(\mathbf{x},\hat{\mathbf{x}}_w) \in J_{d,\varepsilon}^{(n)}$ then $d(\mathbf{x},\hat{\mathbf{x}}_w) < D + \varepsilon$

since $E\ d(\mathbf{x},\hat{\mathbf{x}}) \leq D$

b) if no such $w$ exists we must have $d(\mathbf{x},\hat{\mathbf{x}}_w) < d_{\max}$ since we are assuming that $d(,)$ is bounded. Suppose the probability of this situation is $P_e$.

Hence $\overline{D} = E_{\mathbf{x},g}\ d(\mathbf{x},\hat{\mathbf{x}})$

$$\leq (1 - P_e)(D + \varepsilon) + P_e d_{\max}$$

$$\leq D + \varepsilon + P_e d_{\max}$$

We need to show that the expected value of $P_e$ is small

# Error Probability

Define the set of valid inputs for (random) code $g$

$$V(g) = \left\{ \mathbf{x} : \exists w \text{ with } (\mathbf{x}, g(w)) \in J_{d,\varepsilon}^{(n)} \right\}$$

We have $P_e = \sum_{g} p(g) \sum_{\mathbf{x} \notin V(g)} p(\mathbf{x}) = \sum_{\mathbf{x}} p(\mathbf{x}) \sum_{g : \mathbf{x} \notin V(g)} p(g)$ Change the order

Define $K(\mathbf{x}, \hat{\mathbf{x}}) = 1$ if $(\mathbf{x}, \hat{\mathbf{x}}) \in J_{d,\varepsilon}^{(n)}$ else $0$

Prob that a random $\hat{\mathbf{x}}$ does not match $\mathbf{x}$ is $1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}})$

Prob that an entire code does not match $\mathbf{x}$ is $\left(1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}})\right)^{2^{nR}}$

Codewords are i.i.d.

Hence $P_e = \sum_{\mathbf{x}} p(\mathbf{x}) \left(1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}})\right)^{2^{nR}}$

# Achievability for Average Code

Since $\mathbf{x}, \hat{\mathbf{x}} \in J_{d,\varepsilon}^{(n)} \Rightarrow p(\hat{\mathbf{x}}) \geq p(\hat{\mathbf{x}} \mid \mathbf{x}) 2^{-n\left(I(X;\hat{X})+3\varepsilon\right)}$

$$P_e = \sum_{\mathbf{x}} p(\mathbf{x}) \left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}}) \right)^{2^{nR}}$$

$$\leq \sum_{\mathbf{x}} p(\mathbf{x}) \left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}} \mid \mathbf{x}) K(\mathbf{x}, \hat{\mathbf{x}}) \cdot 2^{-n\left(I(X;\hat{X})+3\varepsilon\right)} \right)^{2^{nR}}$$

Using $(1-uv)^m \leq 1 - u + e^{-vm}$

with $u = \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}} \mid \mathbf{x}) K(\mathbf{x}, \hat{\mathbf{x}}); \quad v = 2^{-nI(X;\hat{X})-3n\varepsilon}; \quad m = 2^{nR}$

$$\leq \sum_{\mathbf{x}} p(\mathbf{x}) \left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}} \mid \mathbf{x}) K(\mathbf{x}, \hat{\mathbf{x}}) + \exp\left( -2^{-n\left(I(X;\hat{X})+3\varepsilon\right)} 2^{nR} \right) \right)$$

Note: $0 \leq u, v \leq 1$ as required

# Achievability for Average Code

$$P_e \le \sum_{\mathbf{x}} p(\mathbf{x}) \left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}} \mid \mathbf{x}) K(\mathbf{x}, \hat{\mathbf{x}}) + \exp\left( - 2^{-n\left(I(X;\hat{X})+3\varepsilon\right)} 2^{nR} \right) \right)$$

$$= 1 - \sum_{\mathbf{x},\hat{\mathbf{x}}} p(\mathbf{x}, \hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}}) + \exp\left( - 2^{n\left(R - I(X;\hat{X}) - 3\varepsilon\right)} \right)$$

Mutual information does not involve particular $\mathbf{x}$

$$= P\left\{ (\mathbf{x}, \hat{\mathbf{x}}) \notin J_{d,\varepsilon}^{(n)} \right\} + \exp\left( - 2^{n\left(R - I(X;\hat{X}) - 3\varepsilon\right)} \right)$$

$$\xrightarrow[n \to \infty]{} 0$$

since both terms $\to 0$ as $n \to \infty$ provided $nR > I(X, \hat{X}) + 3\varepsilon$

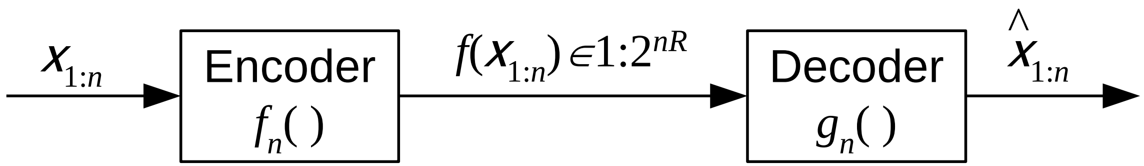
Hence $\forall \delta > 0$, $\overline{D} = E_{\mathbf{x},g}\, d(\mathbf{x}, \hat{\mathbf{x}})$ can be made $\le D + \delta$

# Achievability

Since $\forall\delta > 0$, $\overline{D} = E_{\mathbf{x},g}\, d(\mathbf{x},\hat{\mathbf{x}})$ can be made $\le D+\delta$

there must be at least one $g$ with $E_{\mathbf{x}}\, d(\mathbf{x},\hat{\mathbf{x}}) \le D+\delta$

Hence $(R,D)$ is achievable for any $R > R(D)$




that is $\lim\limits_{n\to\infty} E_{X_{1:n}}(\mathbf{x},\hat{\mathbf{x}}) \le D$

In fact a stronger result is true (proof in C&T 10.6):

$\forall\delta > 0, D$ and $R > R(D), \exists f_n, g_n$ with $p(d(\mathbf{x},\hat{\mathbf{x}}) \le D+\delta) \underset{n\to\infty}{\rightarrow} 1$

# Lecture 17

- Introduction to network information theory
- Multiple access
- Distributed source coding

# Network Information Theory

- System with many senders and receivers
- New elements: interference, cooperation, competition, relay, feedback…
- Problem: decide whether or not the sources can be transmitted over the channel
  - Distributed source coding
  - Distributed communication
  - The general problem has not yet been solved, so we consider various special cases
- Results are presented without proof (can be done using mutual information, joint AEP)

# Implications to Network Design

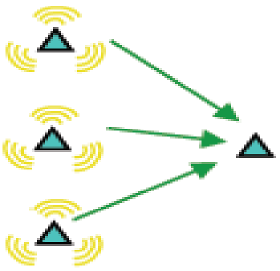
- Examples of large information networks
  - Computer networks
  - Satellite networks
  - Telephone networks
- A complete theory of network communications would have wide implications for the design of communication and computer networks
- Examples
  - CDMA (code-division multiple access): mobile phone network
  - Network coding: significant capacity gain compared to routing-based networks

# Network Models Considered

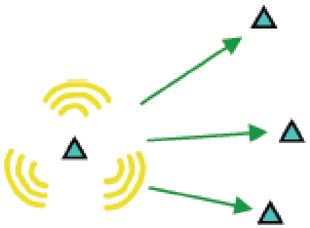
- Multi-access channel
- Broadcast channel
- Distributed source coding
- Relay channel
- Interference channel
- Two-way channel
- General communication network

# State of the Art

- Triumphs
  - Multi-access channel

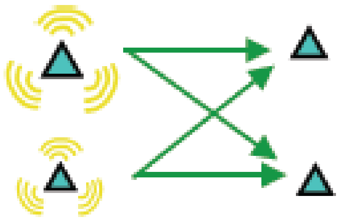


  - Gaussian broadcast channel



- Unknowns
  - The simplest relay channel

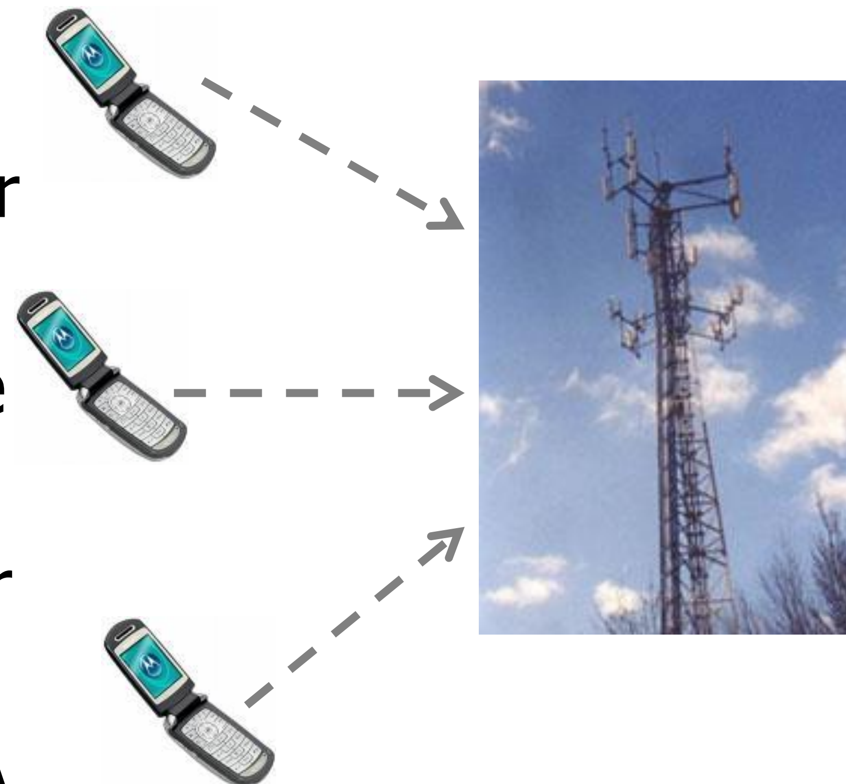
  - The simplest interference channel



Reminder: Networks being built (ad hoc networks, sensor networks) are much more complicated

# Multi-Access Channel

- Example: many users communicate with a common base station over a common channel
- What rates are achievable simultaneously?
- Best understood multiuser channel
- Very successful: 3G CDMA mobile phone networks



# Capacity Region

- Capacity of single-user Gaussian channel

$$C = \frac{1}{2}\log\left(1+\frac{P}{N}\right) = C\left(\frac{P}{N}\right)$$

- Gaussian multi-access channel with m users

$$Y = \sum_{i=1}^{m} X_i + Z$$

$X_i$ has equal power $P$
noise $Z$ has variance $N$

- Capacity region

$$R_i < C\left(\frac{P}{N}\right)$$

$$R_i + R_j < C\left(\frac{2P}{N}\right)$$

$$R_i + R_j + R_k < C\left(\frac{3P}{N}\right)$$

$$\vdots$$

$$\sum_{i=1}^{m} R_i < C\left(\frac{mP}{N}\right)$$

$R_i$: rate for user $i$

Transmission: independent and simultaneous (i.i.d. Gaussian codebooks)

Decoding: joint decoding, look for $m$ codewords whose sum is closest to $Y$

The last inequality dominates when all rates are the same

The sum rate goes to ∞ with m

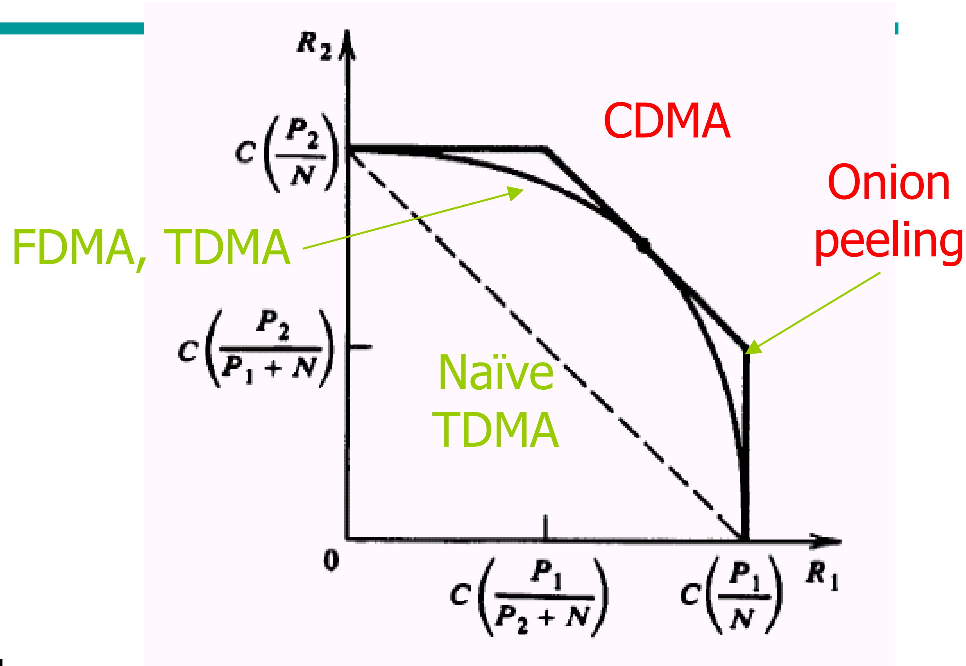
# Two-User Channel

- Capacity region

$$R_1 < C\left(\frac{P_1}{N}\right)$$

$$R_2 < C\left(\frac{P_2}{N}\right)$$

$$R_1 + R_1 < C\left(\frac{P_1 + P_2}{N}\right)$$




  - Corresponds to CDMA
  - Surprising fact: sum rate = rate achieved by a single sender with power $P_1+P_2$
- Achieves a higher sum rate than treating interference as noise, i.e.,

$$C\left(\frac{P_1}{P_2 + N}\right) + C\left(\frac{P_2}{P_1 + N}\right)$$

# Onion Peeling

- Interpretation of corner point: onion-peeling
  - First stage: decoder user 2, considering user 1 as noise
  - Second stage: subtract out user 2, decoder user 1
- In fact, it can achieve the entire capacity region
  - Any rate-pairs between two corner points achievable by time-sharing
- Its technical term is successive interference cancelation (SIC)
  - Removes the need for joint decoding
  - Uses a sequence of single-user decoders
- SIC is implemented in the uplink of CDMA 2000 EV-DO (evolution-data optimized)
  - Increases throughput by about 65%

# Comparison with TDMA and FDMA

- FDMA (frequency-division multiple access)

$$R_1 = W_1 \log\left(1 + \frac{P_1}{N_0 W_1}\right)$$

$$R_2 = W_2 \log\left(1 + \frac{P_2}{N_0 W_2}\right)$$

Total bandwidth $W = W_1 + W_2$

Varying $W_1$ and $W_2$ tracing out the curve in the figure

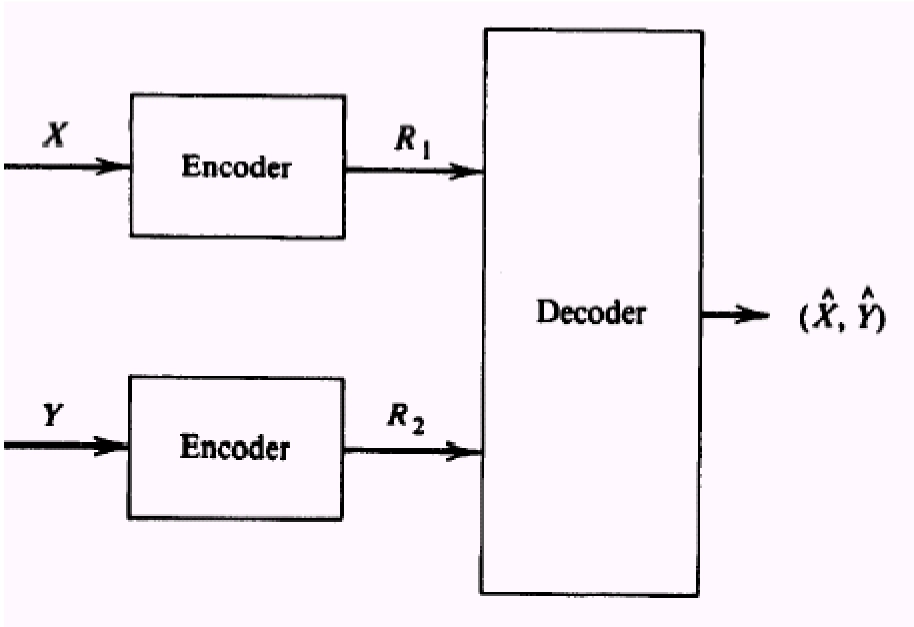
- TDMA (time-division multiple access)
  - Each user is allotted a time slot, transmits and other users remain silent
  - Naïve TDMA: dashed line
  - Can do better while still maintaining the same average power constraint; the same capacity region as FDMA
- CDMA capacity region is larger
  - But needs a more complex decoder

# Distributed Source Coding

- Associate with nodes are sources that are generally dependent
- How do we take advantage of the dependence to reduce the amount of information transmitted?
- Consider the special case where channels are noiseless and without interference
- Finding the set of rates associate with each source such that all required sources can be decoded at destination
- Data compression dual to multi-access channel

# Two-User Distributed Source Coding

- $X$ and $Y$ are correlated
- But the encoders cannot communicate; have to encode independently
- A single source: $R > H(X)$
- Two sources: $R > H(X,Y)$ if encoding together



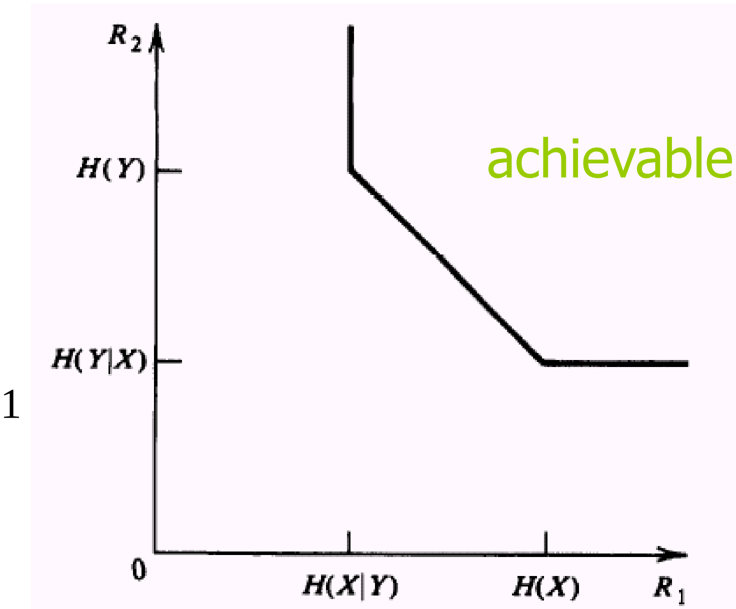

- What if encoding separately?
  - Of course one can do $R > H(X) + H(Y)$
  - Surprisingly, $R = H(X,Y)$ is sufficient (Slepian-Wolf coding, 1973)
  - Sadly, the coding scheme was not practical (again)

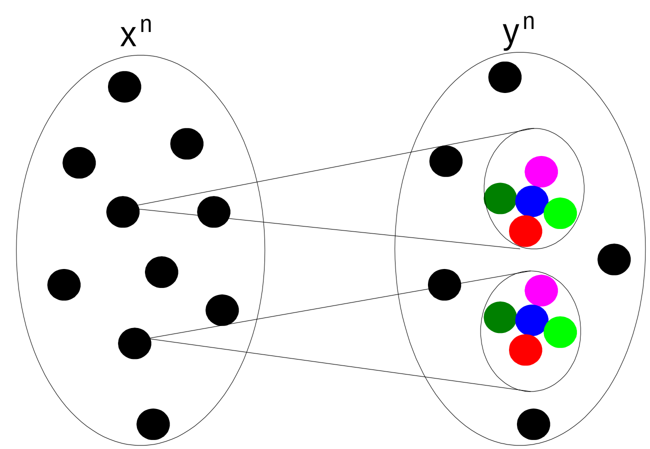
# Slepian-Wolf Coding

- Achievable rate region

$$R_1 \geq H(X \mid Y)$$
$$R_2 \geq H(Y \mid X)$$
$$R_1 + R_2 \geq H(X, Y)$$

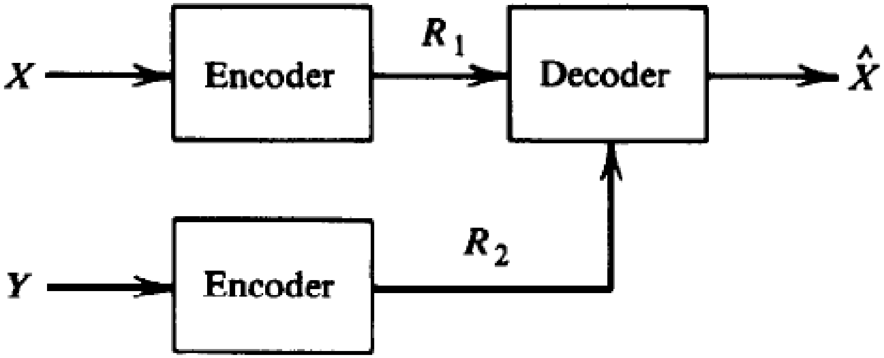
- Core idea: joint typicality
- Interpretation of corner point $R_1$ $H(X)$, $R_2 = H(Y|X)$
  - X can encode as usual
  - Associate with each $x^n$ is a jointly typical fan (however Y doesn't know)
  - Y sends the color (thus compression)
  - Decoder uses the color to determine the point in jointly typical fan associated with $x^n$
- Straight line: achieved by time-sharing



# Wyner-Ziv Coding

- Distributed source coding with side information
- Y is encoded at rate $R_2$
- Only X to be recovered
- How many bits $R_1$ are required?




- If $R_2 = H(Y)$, then $R_1 = H(X|Y)$ by Slepian-Wolf coding
- In general

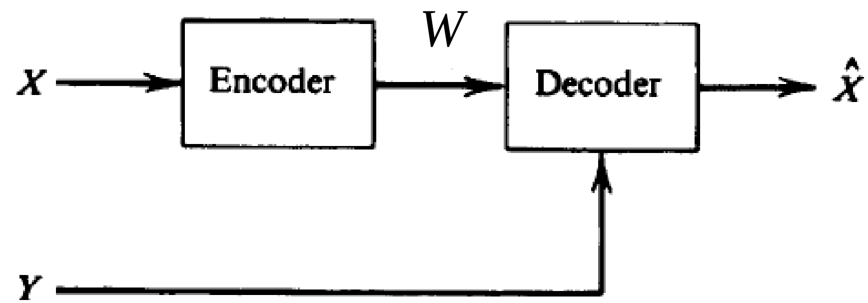
$$R_1 \geq H(X|U)$$

$$R_2 \geq I(Y;U)$$

where U is an auxiliary random variable (can be thought of as approximate version of Y)

# Rate-Distortion

- Given Y, what is the rate-distortion to describe X?

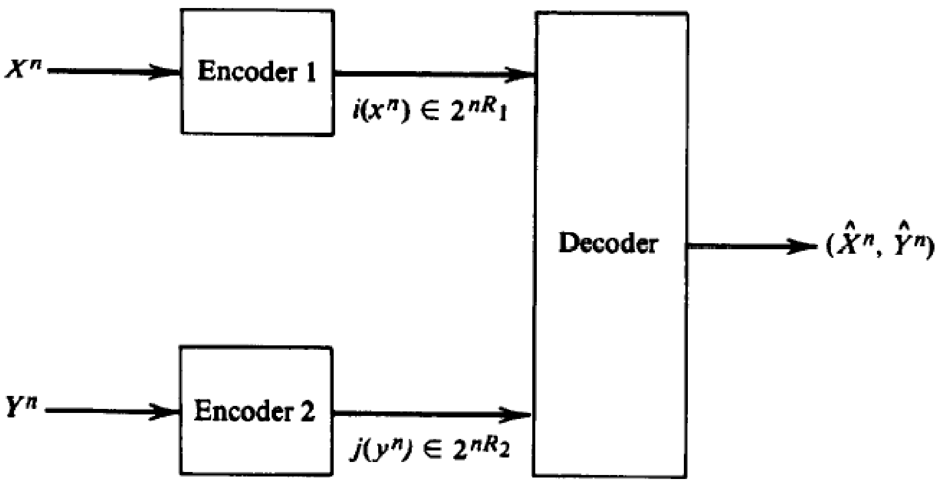



$$R_Y(D) = \min_{p(w|x)} \min_f \{ I(X;W) - I(Y;W) \}$$

over all decoding functions $f : Y \times W \to \hat{X}$

and all $p(w \mid x)$ such that $E_{x,w,y} d(x,\hat{x}) \le D$

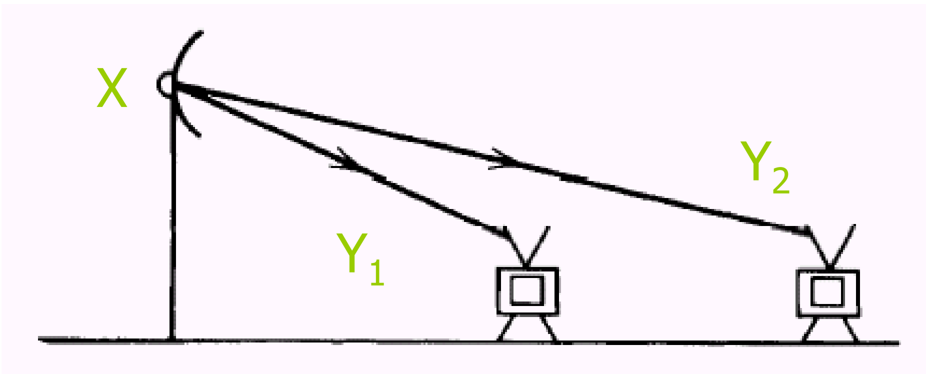
- The general problem of rate-distortion for correlated sources remains unsolved



# Lecture 18

- Network information theory – II
  - Broadcast
  - Relay
  - Interference channel
  - Two-way channel
  - Comments on general communication networks

# Broadcast Channel





- One-to-many: HDTV station sending different information simultaneously to many TV receivers over a common channel; lecturer in classroom
- What are the achievable rates for all different receivers?
- How does the sender encode information meant for different signals in a common signal?
- Only partial answers are known.

# Two-User Broadcast Channel

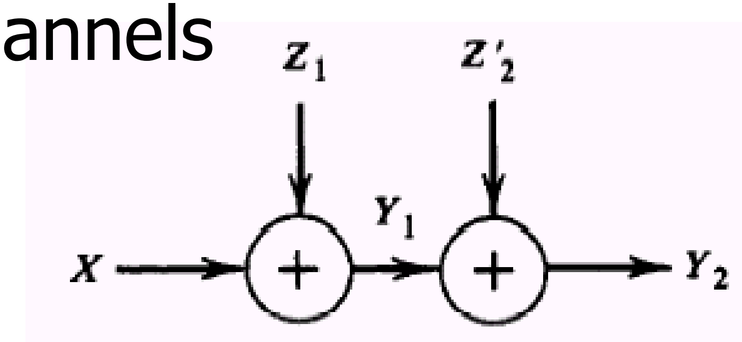
- Consider a memoryless broadcast channel with one encoder and two decoders
- Independent messages at rate $R_1$ and $R_2$
- Degraded broadcast channel: $p(y_1, y_2|x) = p(y_1|x)\, p(y_2|y_1)$
  - Meaning $X \to Y_1 \to Y_2$ (Markov chain)
  - $Y_2$ is a degraded version of $Y_1$ (receiver 1 is better)
- Capacity region of degraded broadcast channel

$$R_2 \le I(U;Y_2)$$
$$R_1 \le I(X;Y_1|U)$$

$U$ is an auxiliary random variable

# Scalar Gaussian Broadcast Channel

- All scalar Gaussian broadcast channels belong to the class of degraded channels



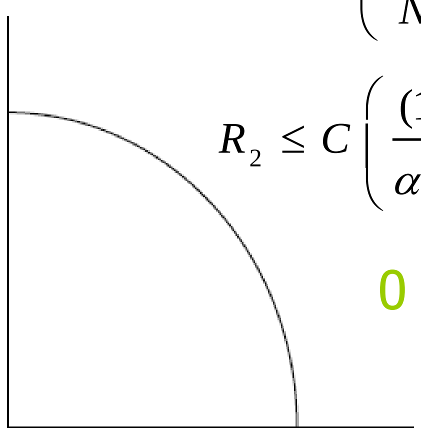

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$

Assume variance
$N_1 < N_2$

- Capacity region

$$R_1 \le C\left(\frac{\alpha P}{N_1}\right)$$
$$R_2 \le C\left(\frac{(1-\alpha)P}{\alpha P + N_2}\right)$$

$0 \le \alpha \le 1$



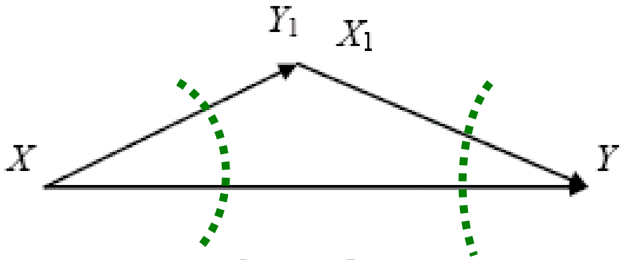
Coding Strategy

Encoding: one codebook with power $\alpha P$ at rate $R_1$, another with power $(1-\alpha)P$ at rate $R_2$, send the sum of two codewords

Decoding: Bad receiver $Y_2$ treats $Y_1$ as noise; good receiver $Y_1$ first decode $Y_2$, subtract it out, then decode his own message

# Relay Channel

- One source, one destination. one or more intermediate relays
- Example: one relay




  - A broadcast channel ($X$ to $Y$ and $Y_1$)
  - A multi-access channel ($X$ and $X_1$ to $Y$)
  - Capacity is unknown! Upper bound:

$$C \leq \sup_{p(x,x_1)} \min\{\, I(X, X_1; Y), I(X; Y, Y_1 \mid X_1)\}$$

  - Max-flow min-cut interpretation
    - First term: maximum rate from $X$ and $X_1$ to $Y$
    - Second term: maximum rate from $X$ to $Y$ and $Y_1$

# Degraded Relay Channel

- In general, the max-flow min-cut bound cannot be achieved
- Reason
  - Interference
  - What for the relay to forward?
  - How to forward?
- Capacity is known for degraded relay channel (i.e, $Y$ is a degradation of $Y_1$, or relay is better than receiver), i.e., the upper bound is achieved

$$C = \sup_{p(x,x_1)} \min\{I(X, X_1; Y), I(X; Y, Y_1 \mid X_1)\}$$

# Gaussian Relay Channel

- Channel model

$$Y_1 = X + Z_1 \qquad \text{Variance}(Z_1) = N_1$$

$$Y = X + Z_1 + X_1 + Z_2 \qquad \text{Variance}(Z_2) = N_2$$

- Encoding at relay: $X_{1i} = f_i(Y_{11}, Y_{12}, \cdots, Y_{1i-1})$
- Capacity

$$C = \max_{0 \le \alpha \le 1} \min \left\{ C\left( \frac{P + P_1 + 2\sqrt{(1-\alpha)PP_1}}{N_1 + N_2} \right), C\left( \frac{\alpha P}{N_1} \right) \right\}$$

X has power P

X1 has power P1

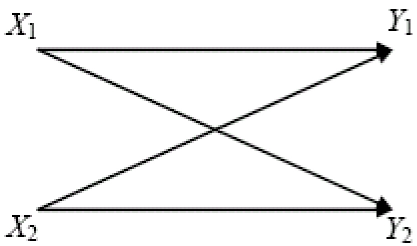
  - If

$$\text{relay} - \text{desitinati on SNR} \quad \frac{P_1}{N_2} \ge \frac{P}{N_1} \quad \text{source} - \text{relay SNR}$$

    then $C = C(P / N_1)$ (capacity from source to relay can be achieved; exercise)
  - Rate $C = C(P/(N_1 + N_2))$ without relay is increased by the relay to $C = C(P / N_1)$

# Interference Channel

- Two senders, two receivers, with crosstalk




  - $Y_1$ listens to $X_1$ and doesn't care what $X_2$ speaks or what $Y_2$ hears
  - Similarly with $X_2$ and $Y_2$
- Neither a broadcast channel nor a multiaccess channel
- This channel has not been solved
  - Capacity is known to within one bit (Etkin, Tse, Wang 2008)
  - A promising technique — interference alignment (Camdambe, Jafar 2008)

# Symmetric Interference Channel

- Model

$$Y_1 = X_1 + aX_2 + Z_1 \qquad \text{equal power } P$$

$$Y_2 = X_2 + aX_1 + Z_2 \qquad \mathrm{Var}(Z_1) = \mathrm{Var}(Z_2) = N$$

- Capacity has been derived in the strong interference case ($a \geq 1$) (Han, Kobayashi, 1981)
  - Very strong interference ($a^2 \geq 1 + P/N$) is equivalent to no interference whatsoever
- Symmetric capacity (for each user $R_1 = R_2$)

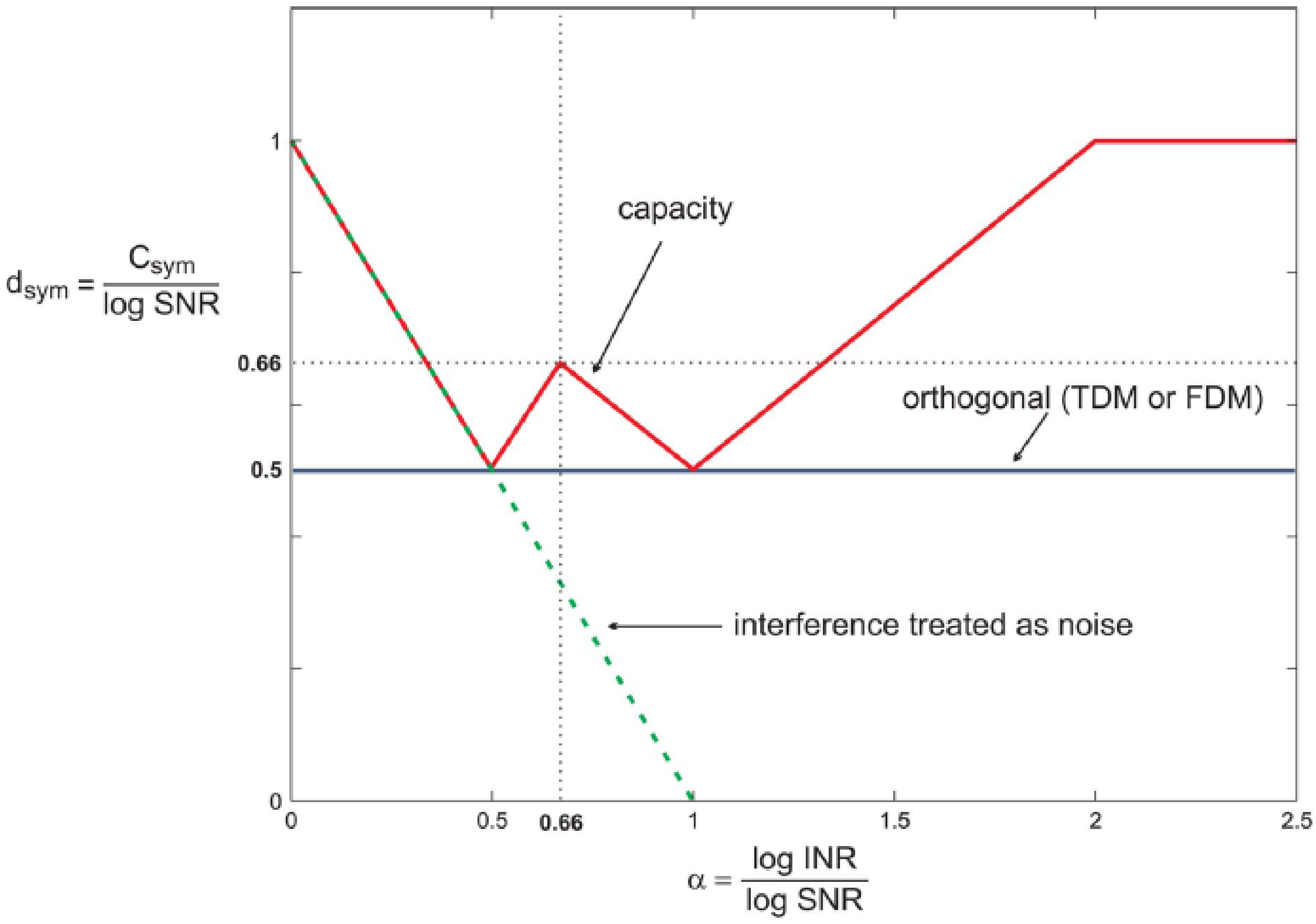
$$d_{\mathrm{sym}} = \begin{cases} 1-\alpha, & 0 \leq \alpha < \frac{1}{2} \\ \alpha, & \frac{1}{2} \leq \alpha < \frac{2}{3} \\ 1-\frac{\alpha}{2}, & \frac{2}{3} < \alpha \leq 1 \\ \frac{\alpha}{2}, & 1 \leq \alpha < 2 \\ 1, & \alpha \geq 2. \end{cases}$$

$$d_{\mathrm{sym}}(\alpha) := \lim_{\mathsf{SNR},\mathsf{INR}\to\infty;\ \frac{\log \mathsf{INR}}{\log \mathsf{SNR}}=\alpha} \frac{C_{\mathrm{sym}}(\mathsf{INR},\mathsf{SNR})}{C_{\mathrm{awgn}}(\mathsf{SNR})},$$

SNR = $P/N$

INR = $a^2P/N$

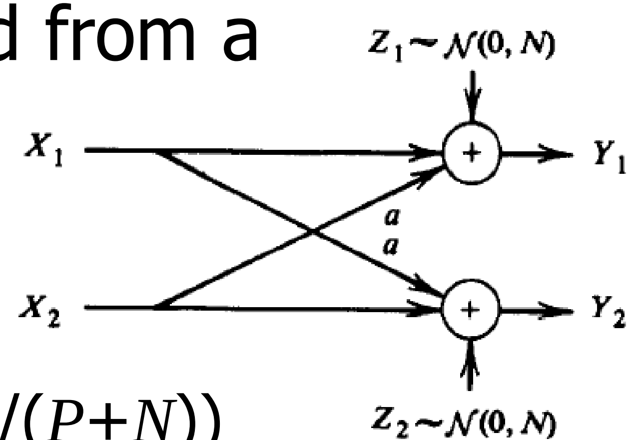
# Capacity



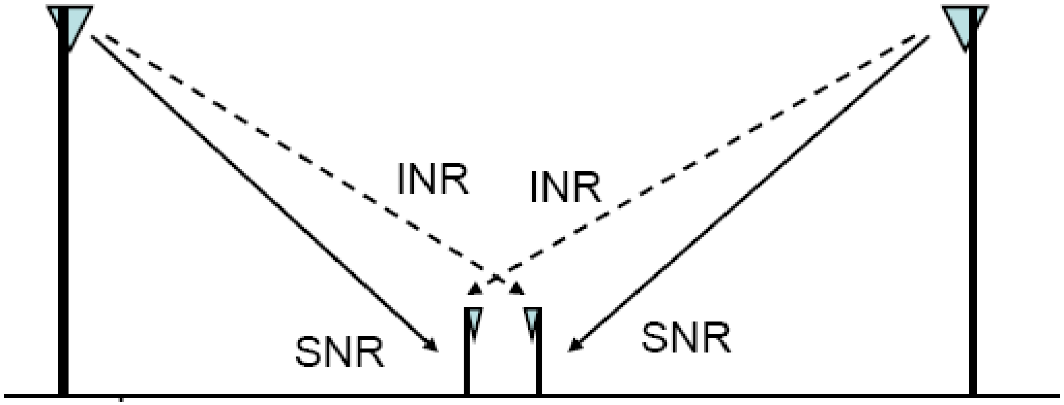
$d_{sym} = \frac{C_{sym}}{\log SNR}$
1
0.66
0.5
0
0
0.5
0.66
1
1.5
2
2.5
$\alpha = \frac{\log INR}{\log SNR}$
capacity
orthogonal (TDM or FDM)
interference treated as noise

# Very strong interference = no interference

- Each sender has power $P$ and rate $C(P/N)$
- Independently sends a codeword from a Gaussian codebook
- Consider receiver 1
  - Treats sender 1 as interference
  - Can decode sender 2 at rate $C(a^2P/(P+N))$
  - If $C(a^2P/(P+N)) > C(P/N)$, i.e.,
    rate 2 → 1 > rate 2 → 2 (crosslink is better)
    he can perfectly decode sender 2
  - Subtracting it from received signal, he sees a clean channel with capacity $C(P/N)$



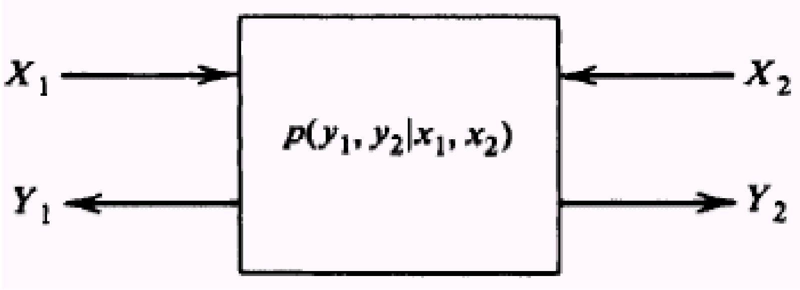
# An Example




- Two cell-edge users (bottleneck of the cellular network)
- No exchange of data between the base stations or between the mobiles
- Traditional approaches
  - Orthogonalizing the two links (reuse ½)
  - Universal frequency reuse and treating interference as noise
- Higher capacity can be achieved by advanced interference management

# Two-Way Channel

- Similar to interference channel, but in both directions (Shannon 1961)




- Feedback
  - Sender 1 can use previously received symbols from sender 2, and vice versa
  - They can cooperate with each other
- Gaussian channel:
  - Capacity region is known (not the case in general)
  - Decompose into two independent channels

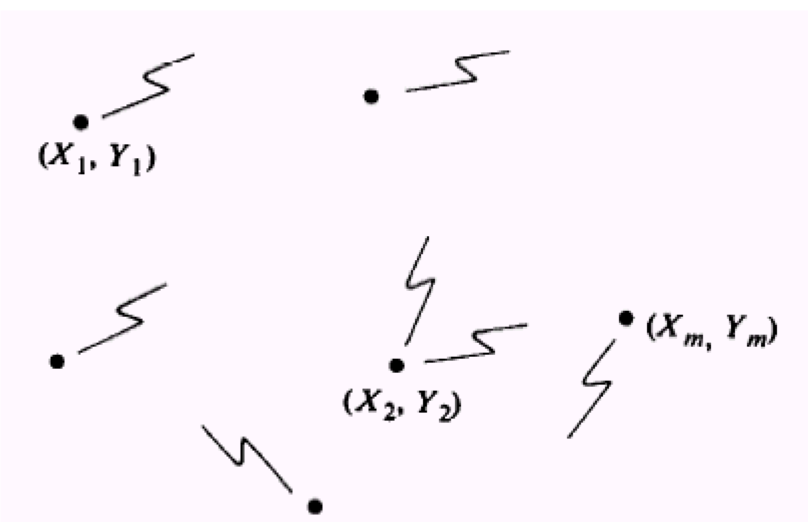
$$R_1 < C\left(\frac{P_1}{N_1}\right)$$

$$R_2 < C\left(\frac{P_2}{N_2}\right)$$

Coding strategy: Sender 1 sends a codeword; so does sender 2. Receiver 2 receives a sum but he can subtract out his own thus having an interference-free channel from sender 1.

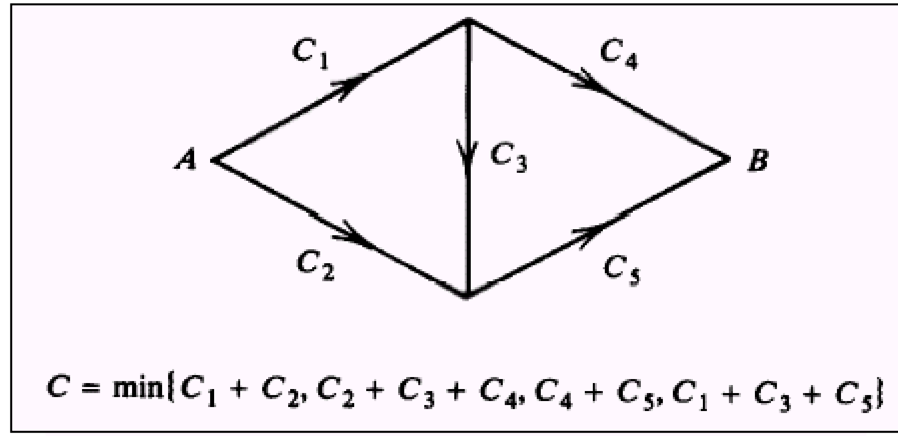
# General Communication Network

- Many nodes trying to communicate with each other
- Allows computation at each node using it own message and all past received symbols
- All the models we have considered are special cases
- A comprehensive theory of network information flow is yet to be found




# Capacity Bound for a Network

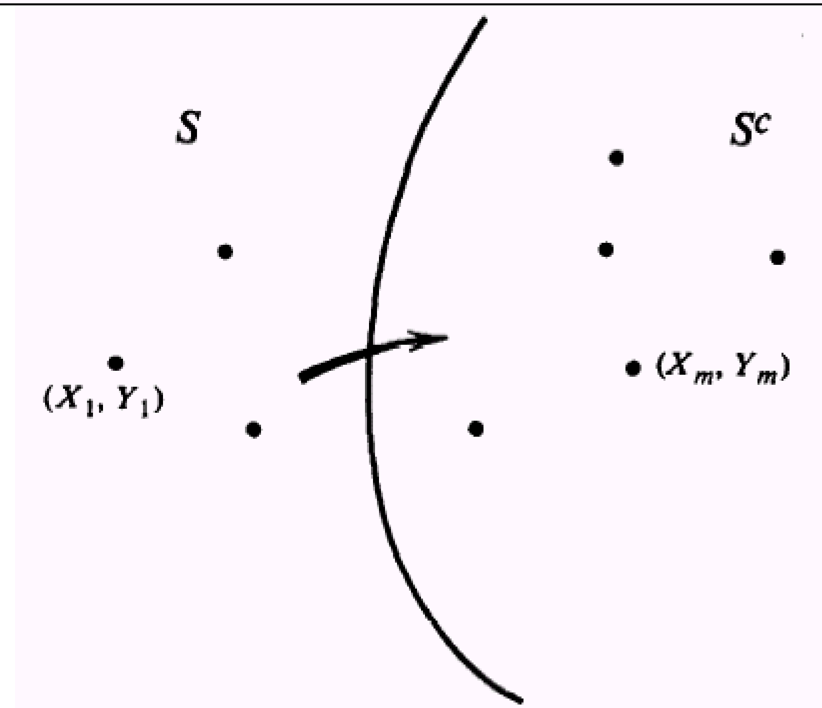
- Max-flow min-cut
  - Minimizing the maximum flow across cut sets yields an upper bound on the capacity of a network




- Outer bound on capacity region

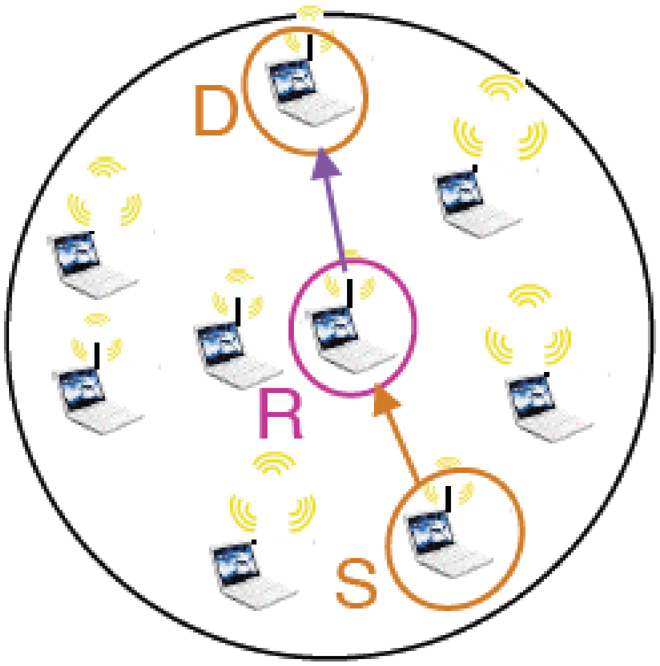
$$\sum_{i \in S,\, j \in S^c} R^{(i,j)} \le I(X^{(S)}; Y^{(S^c)} \mid X^{(S^c)})$$

  - Not achievable in general



# Questions to Answer

- Why multi-hop relay? Why decode and forward? Why treat interference as noise?
- Source-channel separation? Feedback?
- What is really the best way to operate wireless networks?
- What are the ultimate limits to information transfer over wireless networks?

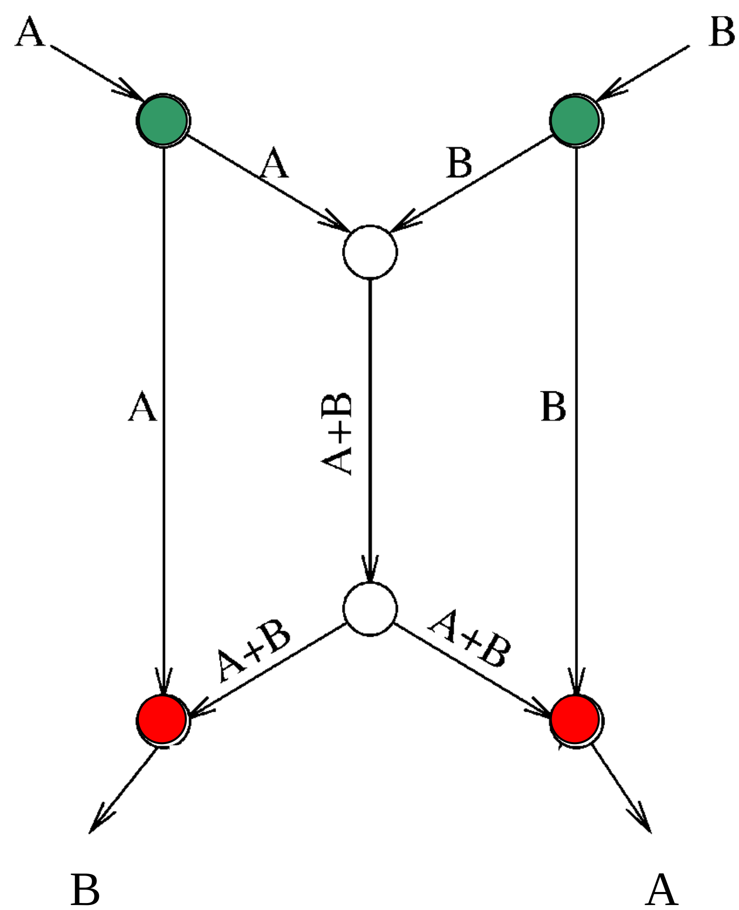


# Scaling Law for Wireless Networks

- High signal attenuation: (transport) capacity is O(n) bit-meter/sec for a planar network with n nodes (Xie-Kumar'04)
- Low attenuation: capacity can grow superlinearly
- Requires cooperation between nodes
- Multi-hop relay is suboptimal but order optimal

# Network Coding

- Routing: store and forward (as in Internet)
- Network coding: recompute and redistribute
- Given the network topology, coding can increase capacity (Ahlswede, Cai, Li, Yeung, 2000)
  - Doubled capacity for butterfly network
- Active area of research




Butterfly Network

# Lecture 19

- Revision Lecture

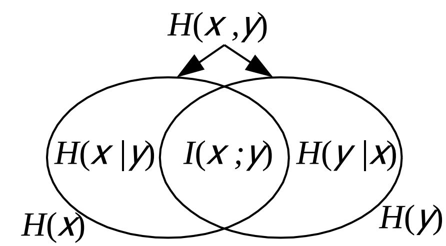
# Summary (1)

- Entropy: $$H(x) = \sum_{x \in \mathrm{X}} p(x) \times -\log_2 p(x) = E - \log_2 (p_X(x))$$
  - Bounds: $$0 \le H(x) \le \log |\mathrm{X}|$$
  - Conditioning reduces entropy: $$H(y \mid x) \le H(y)$$
  - Chain Rule: $$H(x_{1:n}) = \sum_{i=1}^{n} H(x_i \mid x_{1:i-1}) \le \sum_{i=1}^{n} H(x_i)$$
$$H(x_{1:n} \mid y_{1:n}) \le \sum_{i=1}^{n} H(x_i \mid y_i)$$
- Relative Entropy:

$$D(\mathbf{p} \,\|\, \mathbf{q}) = E_{\mathbf{p}} \log \left( p(x) / q(x) \right) \ge 0$$

# Summary (2)

- Mutual Information:




$$I(y;x) = H(y) - H(y|x)$$
$$= H(x) + H(y) - H(x,y) = D\left(\mathbf{p}_{x,y} \,\|\, \mathbf{p}_x \mathbf{p}_y\right)$$

- – Positive and Symmetrical: $I(x;y) = I(y;x) \geq 0$
- – $x$, $y$ indep $\Leftrightarrow$ $H(x,y) = H(y) + H(x) \Leftrightarrow I(x;y) = 0$
- – Chain Rule: $I(x_{1:n};y) = \sum_{i=1}^{n} I(x_i;y \mid x_{1:i-1})$

$$x_i \text{ independent} \quad \Rightarrow \quad I(x_{1:n};y_{1:n}) \geq \sum_{i=1}^{n} I(x_i;y_i)$$

$$p(y_i \mid x_{1:n};y_{1:i-1}) = p(y_i \mid x_i) \quad \Rightarrow \quad I(x_{1:n};y_{1:n}) \leq \sum_{i=1}^{n} I(x_i;y_i)$$

$n$-use DMC capacity

# Summary (3)

- **Convexity:** $f''(x) \geq 0 \Rightarrow f(x)$ convex $\Rightarrow Ef(\mathsf{x}) \geq f(E\mathsf{x})$
  - $H(\mathbf{p})$ concave in $\mathbf{p}$
  - $I(\mathsf{x}\,;\,\mathsf{y})$ concave in $\mathbf{p}_{\mathsf{x}}$ for fixed $\mathbf{p}_{\mathsf{y}|\mathsf{x}}$
  - $I(\mathsf{x}\,;\,\mathsf{y})$ convex in $\mathbf{p}_{\mathsf{y}|\mathsf{x}}$ for fixed $\mathbf{p}_{\mathsf{x}}$
- **Markov:** $\mathsf{x} \rightarrow \mathsf{y} \rightarrow \mathsf{z} \Leftrightarrow p(z \mid x, y) = p(z \mid y) \Leftrightarrow I(\mathsf{x}; \mathsf{z} \mid \mathsf{y}) = 0$

  $\Rightarrow I(\mathsf{x}\,;\,\mathsf{y}) \geq I(\mathsf{x}\,;\,\mathsf{z})$ and $I(\mathsf{x}\,;\,\mathsf{y}) \geq I(\mathsf{x};\,\mathsf{y} \mid \mathsf{z})$
- **Fano:** $\mathsf{x} \rightarrow \mathsf{y} \rightarrow \hat{\mathsf{x}} \Rightarrow p(\hat{\mathsf{x}} \neq \mathsf{x}) \geq \dfrac{H(\mathsf{x} \mid \mathsf{y}) - 1}{\log(|\mathcal{X}| - 1)}$
- **Entropy Rate:**
  - Stationary process

    $$H(\mathcal{X}) = \lim_{n \to \infty} n^{-1} H(\mathsf{x}_{1:n})$$

    $$H(\mathcal{X}) = \lim_{n \to \infty} H(\mathsf{x}_n \mid \mathsf{x}_{1:n-1})$$
  - Markov Process:

    $$H(\mathcal{X}) = \lim_{n \to \infty} H(\mathsf{x}_n \mid \mathsf{x}_{n-1})$$ if stationary

# Summary (4)

- Kraft: Uniquely Decodable $\Rightarrow \sum_{i=1}^{|X|} D^{-l_i} \le 1 \Rightarrow \exists$ instant code
- Average Length: Uniquely Decodable $\Rightarrow \quad L_C = E\, l(x) \ge H_D(x)$
- Shannon-Fano: Top-down 50% splits. $L_{SF} \le H_D(x)+1$

- Huffman: Bottom-up design. Optimal. $L_H \le H_D(x)+1$
  - Designing with wrong probabilities, $\mathbf{q} \Rightarrow$ penalty of $D(\mathbf{p}||\mathbf{q})$
  - Long blocks disperse the 1-bit overhead
- Lempel-Ziv Coding:
  - Does not depend on source distribution
  - Efficient algorithm widely used
  - Approaches entropy rate for stationary ergodic sources

# Summary (5)

- Typical Set
  - Individual Prob $\mathbf{x} \in T_{\varepsilon}^{(n)} \Rightarrow \log p(\mathbf{x}) = -nH(x) \pm n\varepsilon$
  - Total Prob $p(\mathbf{x} \in T_{\varepsilon}^{(n)}) > 1 - \varepsilon \text{ for } n > N_{\varepsilon}$
  - Size $(1-\varepsilon)2^{n(H(x)-\varepsilon)} \overset{n>N_{\varepsilon}}{<} \left|T_{\varepsilon}^{(n)}\right| \leq 2^{n(H(x)+\varepsilon)}$
  - No other high probability set can be much smaller
- Asymptotic Equipartition Principle
  - Almost all event sequences are equally surprising

# Summary (6)

- DMC Channel Capacity: $C = \max_{\mathbf{p}_x} I(x; y)$
- Coding Theorem
  - Can achieve capacity: random codewords, joint typical decoding
  - Cannot beat capacity: Fano inequality
- Feedback doesn't increase capacity of DMC but could simplify coding/decoding
- Joint Source-Channel Coding doesn't increase capacity of DMC

# Summary (7)

- Polar codes are low-complexity codes directly built from information theory.
- Their constructions are aided by the polarization phenomenon.
- For channel coding, polar codes achieve channel capacity.
- For source coding, polar codes achieve the entropy bound.
- And much more.

# Summary (8)

- Differential Entropy: $h(x) = E - \log f_x(x)$
  - Not necessarily positive
  - $h(x+a) = h(x)$, $h(ax) = h(x) + \log|a|$, $h(x|y) \le h(x)$
  - $I(x; y) = h(x) + h(y) - h(x, y) \ge 0$, $D(f||g) = E \log(f/g) \ge 0$
- Bounds:
  - Finite range: Uniform distribution has max: $h(x) = \log(b-a)$
  - Fixed Covariance: Gaussian has max: $h(x) = ½\log((2\pi e)^n|\mathrm{K}|)$
- Gaussian Channel
  - Discrete Time: $C=½\log(1+PN^{-1})$
  - Bandlimited: $C=W\log(1+PN_0^{-1}W^{-1})$
    - For constant C: $E_b N_0^{-1} = PC^{-1}N_0^{-1} = (W/C)\left(2^{(W/C)^{-1}} - 1\right) \underset{W\to\infty}{\rightarrow} \ln 2 = -1.6 \text{ dB}$
  - Feedback: Adds at most ½ bit for coloured noise

# Summary (9)

- Parallel Gaussian Channels: Total power constraint $\sum P_i = nP$
  - White noise: Waterfilling: $P_i = \max(v - N_i, 0)$
  - Correlated noise: Waterfill on noise eigenvectors
- Rate Distortion: $R(D) = \min_{\mathbf{p}_{\hat{x}|x} \, s.t.\, Ed(x,\hat{x}) \le D} I(x;\hat{x})$
  - Bernoulli Source with Hamming $d$: $R(D) = \max(H(\mathbf{p}_x) - H(D), 0)$
  - Gaussian Source with mean square $d$: $R(D) = \max(½\log(\sigma^2 D^{-1}), 0)$
  - Can encode at rate R: random decoder, joint typical encoder
  - Can't encode below rate R: independence bound

# Summary (10)

- Gaussian multiple access channel

$$R_1 < C\left(\frac{P_1}{N}\right), \qquad R_2 < C\left(\frac{P_2}{N}\right)$$

$$R_1 + R_1 < C\left(\frac{P_1 + P_2}{N}\right), \qquad C(x) = \frac{1}{2}\log(1+x)$$

- Distributed source coding
  - Slepian-Wolf coding

$$R_1 \geq H(X \mid Y), \quad R_2 \geq H(Y \mid X)$$

$$R_1 + R_2 \geq H(X, Y)$$

- Scalar Gaussian broadcast channel

$$R_1 \leq C\left(\frac{\alpha P}{N_1}\right), \qquad R_2 \leq C\left(\frac{(1-\alpha)P}{\alpha P + N_2}\right), \qquad 0 \leq \alpha \leq 1$$

- Gaussian Relay channel

$$C = \max_{0 \leq \alpha \leq 1} \min\left\{ C\left(\frac{P + P_1 + 2\sqrt{(1-\alpha)PP_1}}{N_1 + N_2}\right), C\left(\frac{\alpha P}{N_1}\right)\right\}$$

# Summary (11)

- Interference channel
  - Strong interference = no interference
- Gaussian two-way channel
  - Decompose into two independent channels
- General communication network
  - Max-flow min-cut theorem
  - Not achievable in general
  - But achievable for multiple access channel and Gaussian relay channel