

EE401: Advanced Communication Theory

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Principles of Decision and Detection Theory

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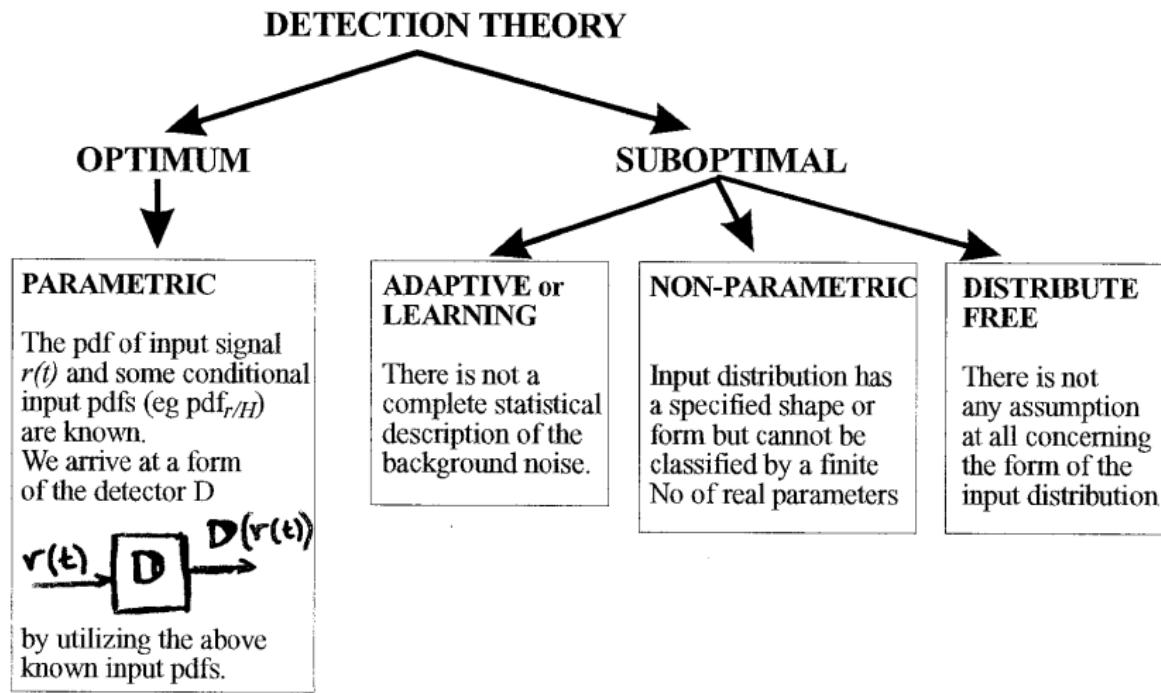
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Basic Decision Theory

- Detection Theory is concerned with determining **the existence of a signal in the presence of noise** - and can be classified as follows:



- N.B.:

① Adaptive or Learning :

- ★ system parameters change as a function of input;
- ★ difficult to analyze;
- ★ mathematically complex

② Non-Parametric :

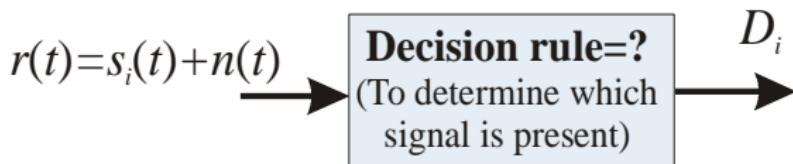
- ★ fairly constant level of performance because these are based on general assumptions on the input pdf;
- ★ easier to implement.

③ Consider a parametric detector which has been designed for Gaussian pdf input.

- ★ **if input is actually Gaussian** : then parametric's performance is significantly better;
- ★ **if input \neq Gaussian** but still symmetric: then non-parametric detector performance may be much better than the performance of the parametric detector

Hypothesis Testing in an M-ary Comm System

- A Hypothesis \triangleq a statement of a possible condition
- In an M -ary Comm. System we have M hypotheses:



where $s_i(t) =$ one of M signal (channel symbols)

- hypotheses:

$$\left\{ \begin{array}{l} H_1 : s_1(t) \text{ is } \overbrace{\text{present}}^{\text{is being sent}} \text{ with probability } \Pr(H_1) \\ H_2 : s_2(t) \text{ is present with probability } \Pr(H_2) \\ \dots & \dots \\ H_M : s_M(t) \text{ is present with probability } \Pr(H_M) \end{array} \right.$$

- The **statistics** of the observed signal

$$r(t) = s_i(t) + n(t) \quad (1)$$

are affected by **the presence of** $s_1(t)$, or $s_2(t)$, ..., or $s_M(t)$

- If $s_i(t), \forall i$ are **known**

then their distributions are **known**

and the problem is **translated** to make a decision about one of the M distributions after having observed $r(t)$

This is called 'Hypothesis Testing'

Terminology

- **A priori probabilities :**

$$\Pr(H_1), \Pr(H_2), \dots, \Pr(H_M)$$

(these are calculated **BEFORE** the experiment is performed)

- **A posterior probabilities**

$$\Pr(H_1/r), \Pr(H_2/r), \dots, \Pr(H_M/r)$$

That is, if r = observation variable
then we have M Conditional Probabilities

$$\Pr(H_i/r), \forall i \in [1, \dots, M]$$

known as a POSTERIOR PROBABILITIES
(since these are calculated **AFTER** the experiment is performed).

- $\Pr(H_i/r)$ $\forall i$: difficult to find.

A more natural approach is to find

$$\Pr(r/H_i), \forall i \quad (2)$$

since in general $\text{pdf}_{r/H_i}, \forall i$

- ▶ are known or
- ▶ can be found

- **Likelihood Functions (LF) :**

$$\text{pdf}_{r/H_1}(r), \text{pdf}_{r/H_2}(r), \dots, \text{pdf}_{r/H_M}(r) \quad (3)$$

i.e. the M conditional probability density functions $\text{pdf}_{r/H_i}(r), \forall i$, are known as "Likelihood Functions"

- **Likelihood Ratio (LR) :** The ratio

$$\frac{\text{pdf}_{r/H_i}(r)}{\text{pdf}_{r/H_j}(r)} \text{ for } i \neq j \quad (4)$$

is known as "Likelihood Ratio"

• MAP Criterion

DECISION—choose hypothesis H_i i.e. D_i :

$$\text{iff } \Pr(H_i/r) > \Pr(H_j/r), \forall j : j \neq i \quad (5)$$

$$\left\{ \begin{array}{ll} D_1 : & \text{iff } \Pr(H_1/r) > \Pr(H_j/r), \forall j \neq 1 \\ D_2 : & \text{iff } \Pr(H_2/r) > \Pr(H_j/r), \forall j \neq 2 \\ \dots & \dots \\ D_M : & \text{iff } \Pr(H_M/r) > \Pr(H_j/r), \forall j \neq M \end{array} \right.$$

This detection is known as max a posterior probability (MAP) criterion

- Equivalent MAP expressions:

$$\begin{aligned}
 D_i : & \text{ iff } \Pr(H_i) \times \text{pdf}_{r/H_i}(r) > \Pr(H_j) \times \text{pdf}_{r/H_j}(r); \forall j : j \neq i \\
 & \Updownarrow \\
 D_i = & \max_{j, \forall j} (\underbrace{\Pr(H_j) \times \text{pdf}_{r/H_j}(r)}_{\triangleq G_j})
 \end{aligned}$$

- Note:

- The above can be easily proven using the Bayes rule

$$\Pr(H_i/r) = \frac{\text{pdf}_{r/H_i}(r) \cdot \Pr(H_i)}{\text{pdf}_r(r)} \quad (6)$$

- G_j is known as "decision variable"
- In this topic the symbol G will be used to represent a "decision variable".

- Correlation between two **analogue energy signals** $r(t)$ and $s(t)$ of duration T_{cs} :

$$\text{corr} \triangleq \int_0^{T_{cs}} r(t).s(t).dt \quad (7)$$

- Correlation between the **discretised versions** \underline{r} and \underline{s} of the **signals** $r(t)$ and $s(t)$:

$$\text{corr} \triangleq \underline{r}^T \underline{s} \quad (8)$$

where

$$\underline{r} = [r_1, r_2, \dots, r_L]^T \quad (9)$$

and

$$\underline{s} = [s_1, s_2, \dots, s_L]^T \quad (10)$$

M-ary Decision Criteria

- Consider the sets of parameters \mathcal{P}_1 and \mathcal{P}_2 where:
 - \mathcal{P}_1 denotes the set of parameters $\Pr(H_1), \Pr(H_2), \dots, \Pr(H_M)$
 - \mathcal{P}_2 represents the set of costs/weights $C_{ij}, \forall i, j$ associated with the transition probabilities $\Pr(D_i|H_j)$ (i.e. one weight for every element of the channel transition matrix \mathbb{F} - see EE303, Topic on "Comm Channels")
- Estimate/identify the likelihood functions

$$\text{pdf}_{r|H_1}(r), \text{pdf}_{r|H_2}(r), \dots, \text{pdf}_{r|H_M}(r)$$

- Decision: choose the hypothesis H_i (i.e. D_i) with the maximum $G_i(r)$ where $G_i(r)$ depends on the chosen criterion as follows:
 - ★ **BAYES** Criterion
 - ★ **Minimum Probability of Error** ($\min(p_e)$) Criterion
 - ★ **MAP** Criterion
 - ★ **MINIMAX** Criterion
 - ★ Newman-Pearson (**N-P**) Criterion
 - ★ Maximum Likelihood (**ML**) Criterion

	\mathcal{P}_1	\mathcal{P}_2	choose Hypothesis with $\max(G_j(r))$
Bayes	known	known	$G_j(r) \triangleq \text{weight}_j \times \Pr(H_j) \times \text{pdf}_{r H_j}$
$\min(p_e)$ or MAP	known	unknown	$G_j(r) \triangleq \Pr(H_j) \times \text{pdf}_{r H_j}$
Minimax	unknown	known	$G_j(r) \triangleq \text{weight}_j \times \text{pdf}_{r H_j}$
N-P	unknown	unknown	by solving a constraint optim. problem
ML	don't care	don't care	$G_j(r) \triangleq \text{pdf}_{r H_j}$

- Notes:

- Note-1:

if an approximate/initial solution is required then any information about the sets of parameters \mathcal{P}_1 and/or \mathcal{P}_2 can be ignored. In this case the Maximum Likelihood (ML) Criterion should be used.

- Note-2:

$$\text{weight}_j \triangleq \sum_{\substack{i=1 \\ i \neq j}}^M (C_{ij} - C_{jj}) \quad (11)$$

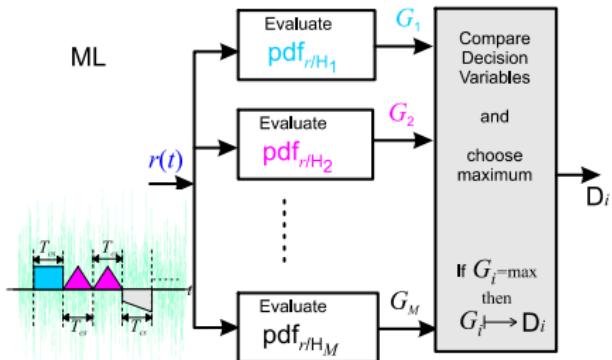
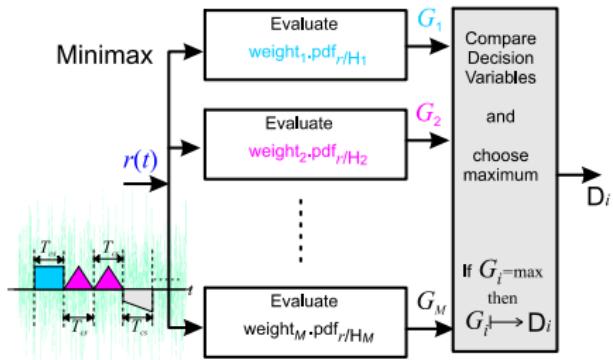
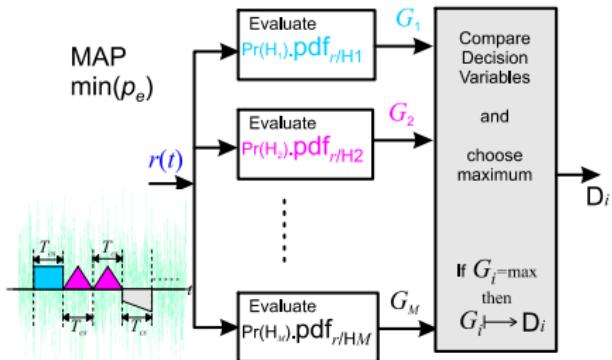
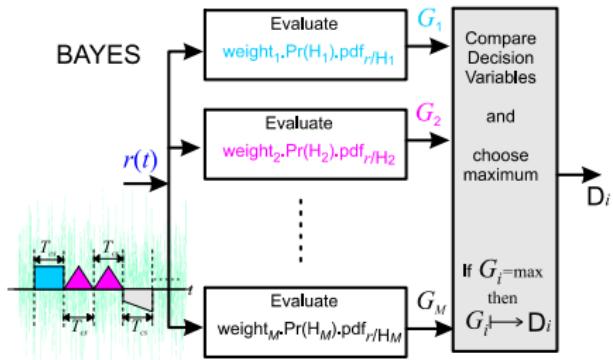
or (since the term for $i = j$ is equal to zero) simply,

$$\text{weight}_j = \sum_{i=1}^M (C_{ij} - C_{jj}) \quad (12)$$

- Note-3:

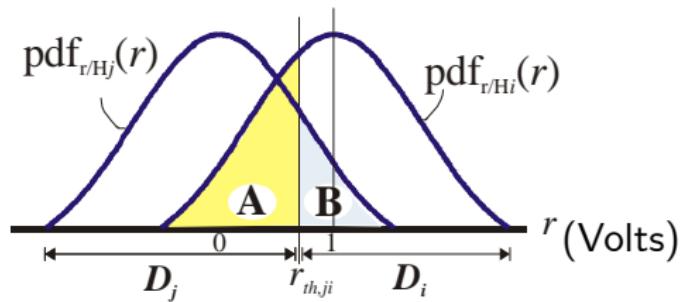
Sometimes, for convenience, G_j will be used (i.e. $G_j \triangleq G_j(r)$) - i.e. the argument will be ignored.

Decision Criteria: Mathematical Architecture



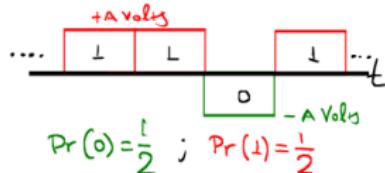
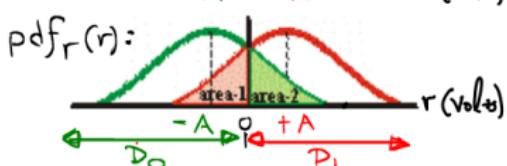
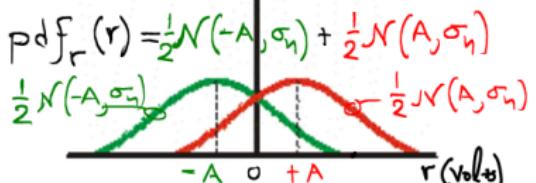
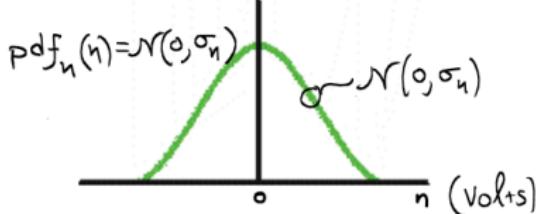
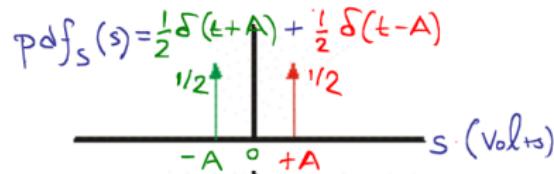
Examples

Example: Gaussian LFs

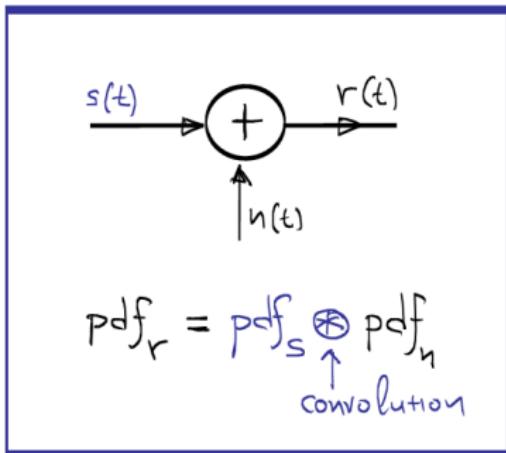


- By solving $G_j(r) = G_i(r)$ the decision threshold $r_{th,ji}$ can be estimated
- $\boxed{\text{area-A}} = \Pr(D_j|H_i)$ and $\boxed{\text{area-B}} = \Pr(D_i|H_j)$
- $p_{e,cs} = \sum_{j=1}^M \sum_{\substack{i=1 \\ i \neq j}}^M \Pr(D_j, H_i) = \text{symbol error probability/rate (SER)}$

Example: Signal + Gaussian Noise



$$\Pr(0) = \frac{1}{2}; \quad \Pr(1) = \frac{1}{2}$$

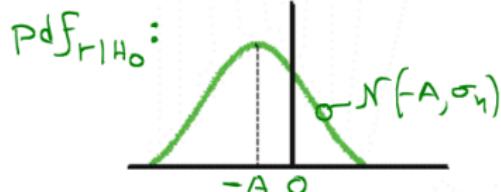
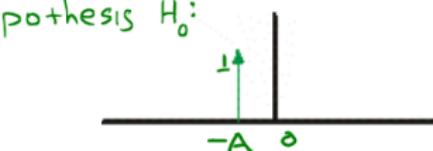
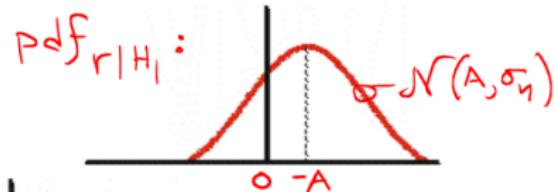
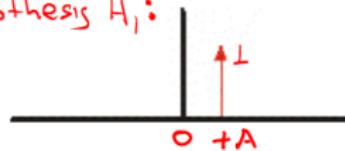


$$p\text{df}_r = p\text{df}_s \oplus p\text{df}_n$$

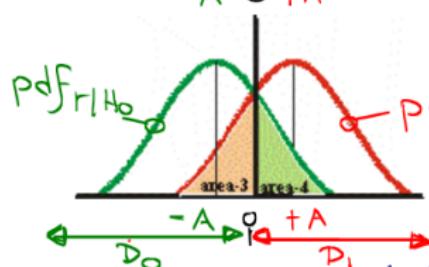
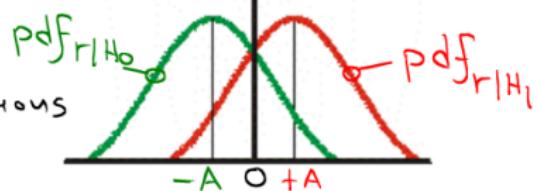
convolution

where

$\left\{ \begin{array}{l} \text{area-1} = \Pr(D_0, H_1) \\ \text{area-2} = \Pr(D_1, H_0) \end{array} \right.$

Hypothesis H_0 :Hypothesis H_1 :

Likelihood functions
put together



where
 $\text{area-3} = \Pr(D_0 | H_1)$
 $\text{area-4} = \Pr(D_1 | H_0)$

Note: $P_e = \frac{\Pr(D_0, H_1)}{\Pr(D_0, H_1) \cdot \Pr(H_1)} + \frac{\Pr(D_1, H_0)}{\Pr(D_1, H_0) \cdot \Pr(H_0)}$

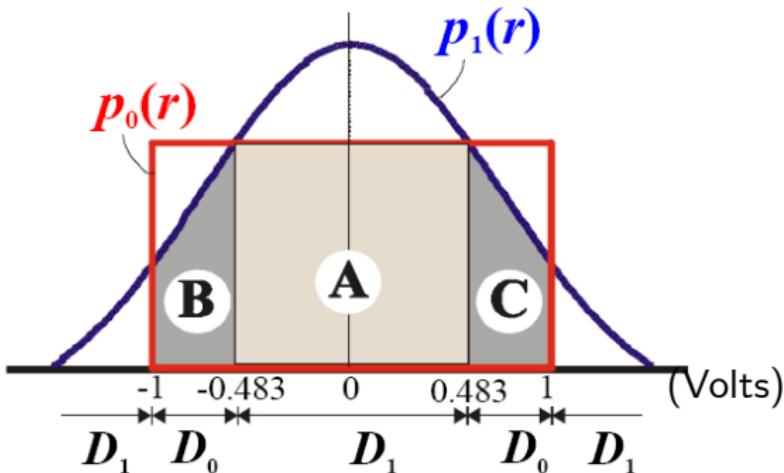
Example: Rectangular and Gaussian LFs

- A discrete channel employs two equally probable symbols and the likelihood functions are given by the following expressions

$$\text{pdf}_{r|H_0} = \frac{1}{2} \text{rect}\left\{\frac{r}{2}\right\}$$

$$\text{pdf}_{r|H_1} = N\left(0, \sigma^2 = \frac{1}{2}\right)$$

- If the detector employed has been designed in an optimum way, find the decision rule and model the above discrete channel.



Note:

$$\boxed{\text{area B}} = \boxed{\text{area C}} = \mathbf{T}\left\{\frac{0.483}{1/2}\right\} - \mathbf{T}\left\{\frac{1}{1/2}\right\} = 0.1490:$$

$$\Pr(D_0/H_1) = \boxed{\text{area B} + \text{area C}} = 0.298$$

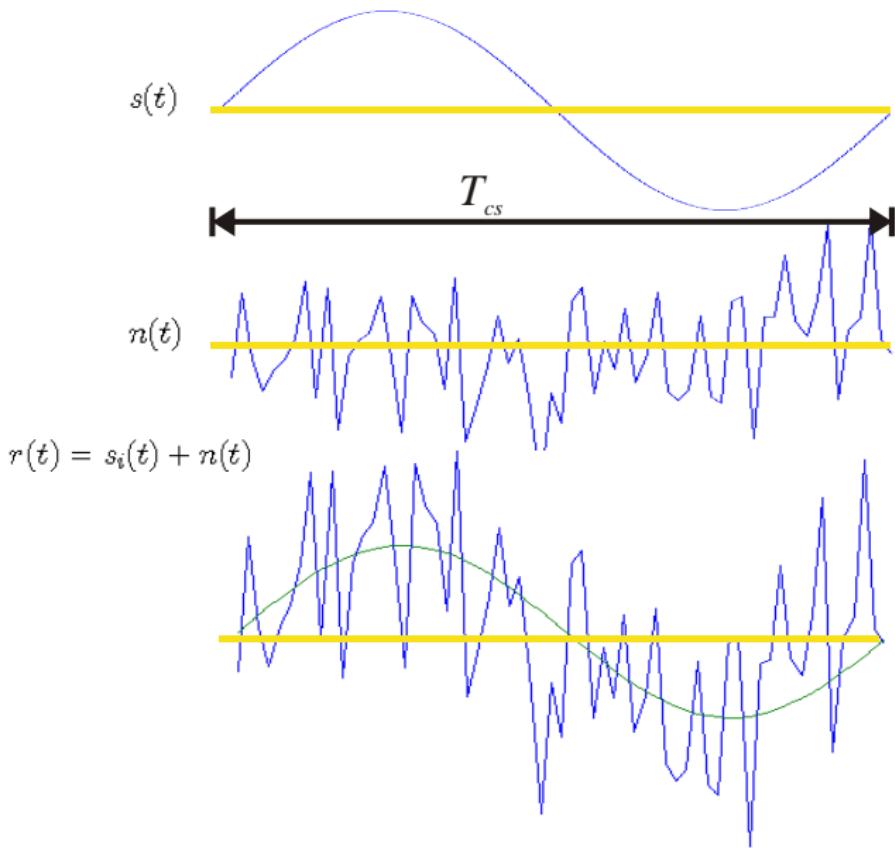
and $\Pr(D_1/H_0) = \boxed{\text{area A}} = 0.483$

Basic Concepts on Optimum Receivers

- Receiver = Detector + Decision Device
- Consider a M -ary communication model in which one of M signals $s_i(t)$, for $i = 1, 2, \dots, M$, is received in the time interval $(0, T_{cs})$ in the presence of white noise
i.e.

$$r(t) = \left\{ \begin{array}{l} s_1(t) \\ \text{or } s_2(t) \\ \dots \\ \text{or } s_M(t) \end{array} \right\} + n(t), \quad 0 \leq t \leq T_{cs} \quad (13)$$

where $r(t)$ denotes the received (**observable**) signal.



The Concept of "Continuous Sampling"

Continuous Sampling helps to get a probabilistic description of a continuous signal:

- assume that L amplitude samples of the received signal

$$r(t), \quad 0 \leq t \leq T_{cs}$$

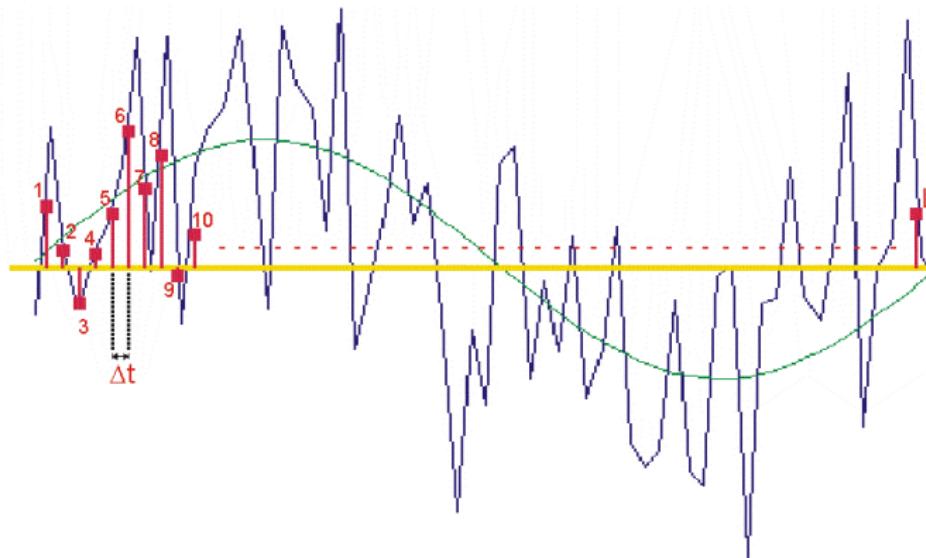
are available, i.e. r_1, r_2, \dots, r_L , (with sampling frequency $\frac{1}{\Delta t}$)

$$\text{where } r_k = \left\{ \begin{array}{l} s_{1k} \\ \text{or } s_{2k} \\ \dots \\ \text{or } s_{Mk} \end{array} \right\} + n_k; \quad k = 1, \dots, L$$

or, in more compact form,

$$\underline{r} = \left\{ \begin{array}{l} \underline{s}_1 \\ \text{or } \underline{s}_2 \\ \dots \\ \text{or } \underline{s}_M \end{array} \right\} + \underline{n} \quad \text{where} \quad \begin{cases} \underline{r} = [r_1, r_2, \dots, r_L]^T \\ \underline{n} = [n_1, n_2, \dots, n_L]^T \\ \underline{s}_i = [s_{i1}, s_{i2}, \dots, s_{iL}]^T \end{cases}$$

i.e. $\underline{r} = \underline{s}_i + \underline{n}$ (L samples)



2. take the limit as $L \rightarrow \infty$, $\Delta t \rightarrow 0$, $L \times \Delta t \rightarrow T_{cs}$

Optimum M-ary Receivers

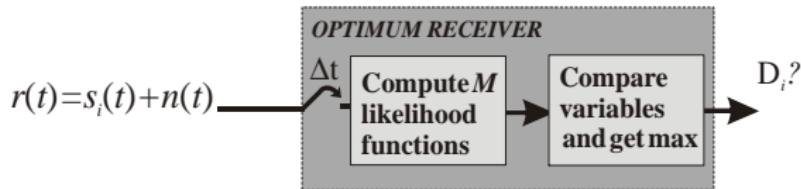
- **Objective** : to design a receiver which operates on $r(t)$ and chooses one of the following M hypotheses:

$$\left\{ \begin{array}{l} H_1 : r(t) = s_1(t) + n(t) \\ H_2 : r(t) = s_2(t) + n(t) \\ \dots \\ H_M : r(t) = s_M(t) + n(t) \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} H_1 : \underline{r} = \underline{s}_1 + \underline{n} \\ H_2 : \underline{r} = \underline{s}_2 + \underline{n} \\ \dots \\ H_M : \underline{r} = \underline{s}_M + \underline{n} \end{array} \right.$$

- **Optimum Decision Rule:**

depends on the likelihood functions $\text{pdf}_{r|H_i}(r)$

- **Optimum Receiver - Initial Conceptual Structure:**



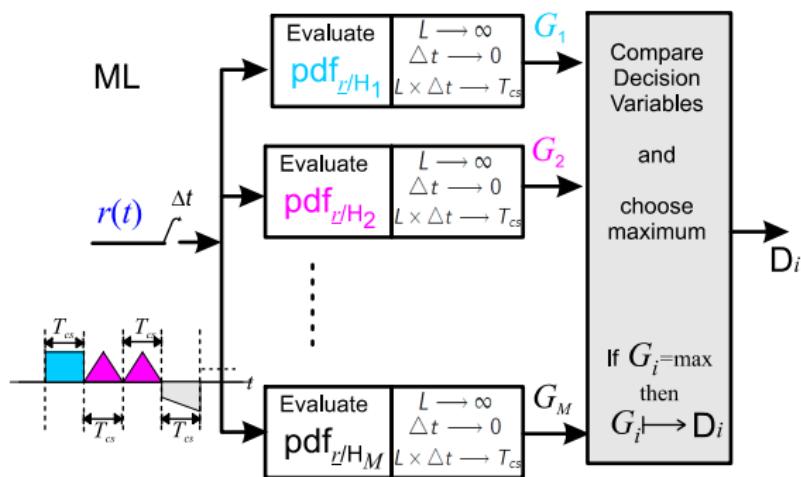
• Assumptions :

- Noise: $\text{PSD}_{n_i}(f) = \frac{N_0}{2} \Rightarrow \text{PSD}_n(f) = \frac{N_0}{2} \text{rect}\left\{\frac{f}{2B}\right\}$
Therefore, the received signal is sampled at intervals

$$\Delta t = \frac{1}{2B} \quad (14)$$

- Sampling:** the samples are uncorrelated and statistically independent

• Example - ML Receiver with "Cont. Sampling" :



Gaussian Multivariable Distribution

- Consider the Gaussian random vector: $\underline{r} = [r_1, r_2, \dots, r_L]^T$

Then

$$\text{pdf}(\underline{r}) = \frac{1}{\sqrt{(2\pi)^L \det(\mathbb{R}_{rr})}} \exp \left\{ -\frac{(\underline{r} - \underline{\mu}_r)^T \mathbb{R}_{rr}^{-1} (\underline{r} - \underline{\mu}_r)}{2} \right\} \quad (15)$$

where

$$\underline{\text{mean}} = \underline{\mu}_r = \mathcal{E}\{\underline{r}\} = [\mathcal{E}\{r_1\}, \mathcal{E}\{r_2\}, \dots, \mathcal{E}\{r_L\}]^T \quad (16)$$

$$\text{cov}(\underline{r}) = \mathbb{R}_{rr} = \mathcal{E} \left\{ \left(\underline{r} - \underline{\mu}_r \right) \left(\underline{r} - \underline{\mu}_r \right)^T \right\} \quad (17)$$

- If $\underline{r} = \underline{s} + \underline{n}$ then the mean $\underline{\mu}_r$ and covariance \mathbb{R}_{rr} are given as follows:

$$\underline{\mu}_r = \underline{s} \text{ and } \mathbb{R}_{rr} = \mathbb{R}_{nn} \quad (18)$$

proof :

$$\mathcal{E}\{\underline{r}\} = \underline{\mu}_r \triangleq \mathcal{E}\{\underline{s} + \underline{n}\} = \mathcal{E}\{\underline{s}\} + \mathcal{E}\{\underline{n}\} = \mathcal{E}\{\underline{s}\} = \underline{s} \quad (19)$$

$$\begin{aligned} \text{cov}\{\underline{r}\} &\triangleq \mathbb{R}_{rr} = \mathcal{E}\left\{(\underline{r} - \mathcal{E}\{\underline{r}\})(\underline{r} - \mathcal{E}\{\underline{r}\})^T\right\} \\ &= \mathcal{E}\left\{(\underline{r} - \underline{s})(\underline{r} - \underline{s})^T\right\} \\ &= \mathcal{E}\{\underline{n}\underline{n}^T\} \\ \Rightarrow \text{cov}\{\underline{r}\} &= \mathbb{R}_{nn} = \sigma_n^2 \mathbb{I}_L = N_0 B \mathbb{I}_L \end{aligned} \quad (20)$$

- Note that:

$$\text{If } \mathbb{R}_{nn} = \sigma_n^2 \mathbb{I}_L \text{ then } \det(\mathbb{R}_{nn}) = (\sigma_n^2)^L \quad (21)$$

Likelihood Functions (Continuous Sampling)

- $\text{pdf}_{r/H_i}(r)$: Based on the above (for AWGN) and for “Continuous Sampling” we have

$$\text{pdf}_{r/H_i}(r) = \left(\frac{1}{\sqrt{2\pi\sigma_n^2}} \right)^L \cdot \exp \left\{ -\frac{(\underline{r} - \underline{s}_i)^T (\underline{r} - \underline{s}_i)}{2\sigma_n^2} \right\} \quad (22)$$

- However $\begin{cases} \sigma_n^2 = N_0 B \\ \Delta t = \frac{1}{2B} \end{cases} \Rightarrow \sigma_n^2 = \frac{N_0}{2 \cdot \Delta t}$
- Therefore

$$\text{pdf}_{r/H_i}(\underline{r}) = \left(\sqrt{\frac{\Delta t}{\pi N_0}} \right)^L \cdot \exp \left\{ -\frac{(\underline{r} - \underline{s}_i)^T (\underline{r} - \underline{s}_i) \cdot \Delta t}{N_0} \right\}$$

and for “Continuous Sampling”, i.e. $L \rightarrow \infty, \Delta t \rightarrow 0, L \times \Delta t \rightarrow T_{cs}$
we have

$$\text{pdf}_{r/H_i}(r(t)) = \text{const} \cdot \exp \left\{ -\frac{1}{N_0} \cdot \int_0^{T_{cs}} (r(t) - s_i(t))^2 dt \right\} \quad (23)$$

- **ML APPROACH** using "Continuous Sampling":

Rule : choose the hypothesis H_i with the maximum likelihood

$\text{pdf}_{r/H_i}(r(t))$ where

$$\text{pdf}_{r/H_i}(r(t)) = \text{const} \cdot \exp \left\{ -\frac{1}{N_0} \cdot \int_0^{T_{cs}} (r(t) - s_i(t))^2 dt \right\} \quad (24)$$

for $i = 1, \dots, M$

- **MAP APPROACH** using "Continuous Sampling":

This is a more general approach than the Maximum Likelihood

Rule : choose the hypothesis H_i with the maximum $\Pr(H_i) \times \text{pdf}_{r/H_i}(r(t))$ where

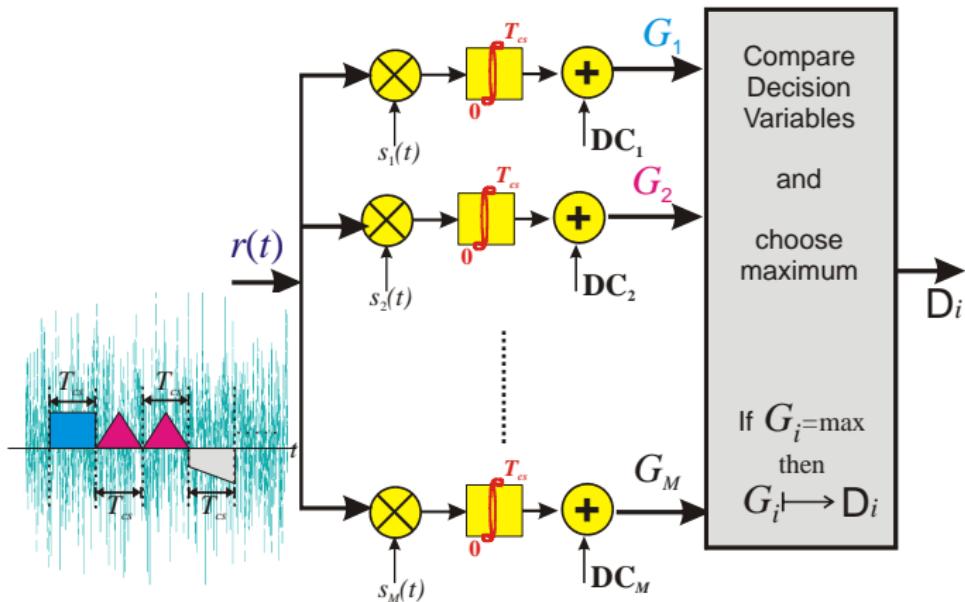
$$\text{pdf}_{r/H_i}(r(t)) = \text{const. exp} \left\{ -\frac{1}{N_0} \cdot \int_0^{T_{cs}} (r(t) - s_i(t))^2 dt \right\} \quad (25)$$

for $i = 1, \dots, M$

i.e. $\max \{ \Pr(H_i) \times \text{pdf}_{r/H_i}(r(t)) \}$

$$= \max \left\{ \int_0^{T_{cs}} r(t) s_i^*(t) dt + \frac{N_0}{2} \ln (\Pr(H_i)) - \frac{1}{2} E_i \right\} \quad (26)$$

- Equation-26 (for proof see Appendix 2) suggests that an M -ary receiver will have the following form:

OPTIMUM M -ary RECEIVER: CORREL. RECEIVER

where $DC_i \equiv \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2}E_i$ with $i = 1, 2, \dots, M$

- Note that if the a priori probabilities are equal,
i.e.

$$\Pr(H_1) = \Pr(H_2) = \dots = \Pr(H_i) \quad (27)$$

and the signals have the same energy,
i.e.

$$E_1 = E_2 = \dots = E_M$$

then Equation-26 is simplified as follows:

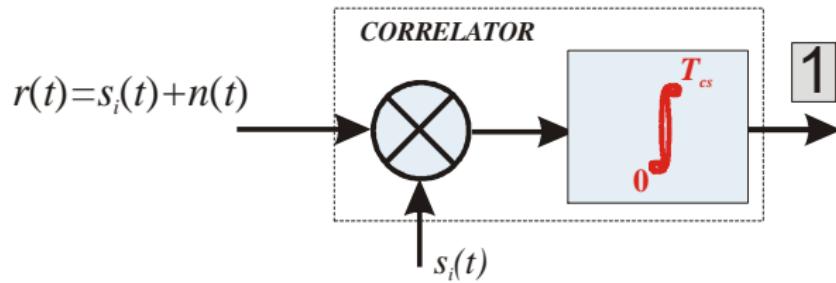
$$\begin{aligned} \max \left\{ \Pr(H_i) \times \text{pdf}_{r/H_i}(r(t)) \right\} &= \max \{ \text{pdf}_{r/H_i}(r(t)) \} \\ &= \max \left\{ \int_0^{T_{cs}} r(t) s_i^*(t) \cdot dt \right\} \end{aligned} \quad (28)$$

- N.B. - ML receiver: similar structure to MAP with $\text{DC}_i = \frac{1}{2} E_i$

Optimum M-ary Receivers using Matched Filters

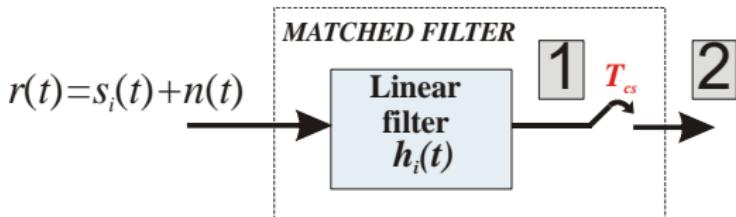
Known Signals in White Noise

- Consider the output of one branch of the correlation receiver:



$$\boxed{\text{at point-1}} = \text{output} = \int_0^{T_{cs}} r(t) \cdot s_i(t) \cdot dt$$

- In this section we will try to replace the **correlator** of a correlation receiver with a **linear filter** (**known as Matched Filter**).



$$\boxed{\text{at point-1}} \quad = \quad \int_0^t r(u) \cdot h_i(t-u) \cdot du$$

$$\boxed{\text{at point-2}} \quad = \quad \text{output} = \int_0^{T_{cs}} r(u) \cdot h_i(T_{cs}-u) \cdot du$$

- N.B.:

compare correlator's o/p (point-1) with matched filter's o/p (point-2).

- If we choose the impulse response of the linear filter as

$$h_i(t) = s_i(T_{cs} - t) \quad 0 \leq t \leq T_{cs} \quad (29)$$

then that linear filter, defined by the above equation, is called Matched Filter.

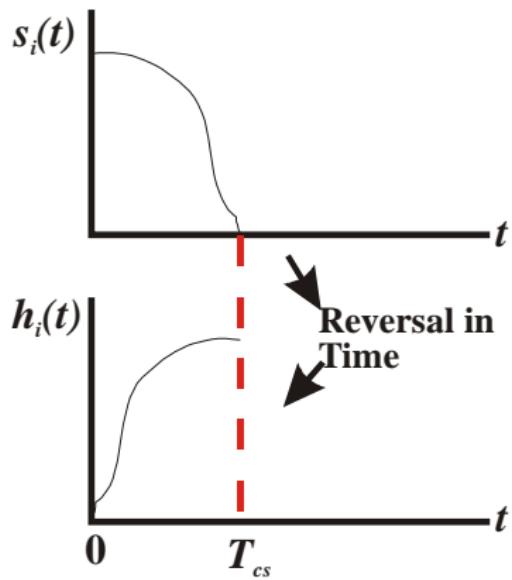
$$\begin{aligned} \boxed{\text{at point-2}} &= \int_0^{T_{cs}} r(u) \cdot s_i(u) \cdot du \\ \implies \boxed{\text{at point-2}} &= \int_0^{T_{cs}} r(t) \cdot s_i(t) \cdot dt \end{aligned}$$

- N.B.:

$$\boxed{\text{Output Of Correlator}} = \boxed{\text{Output Of Matched Filter}}$$

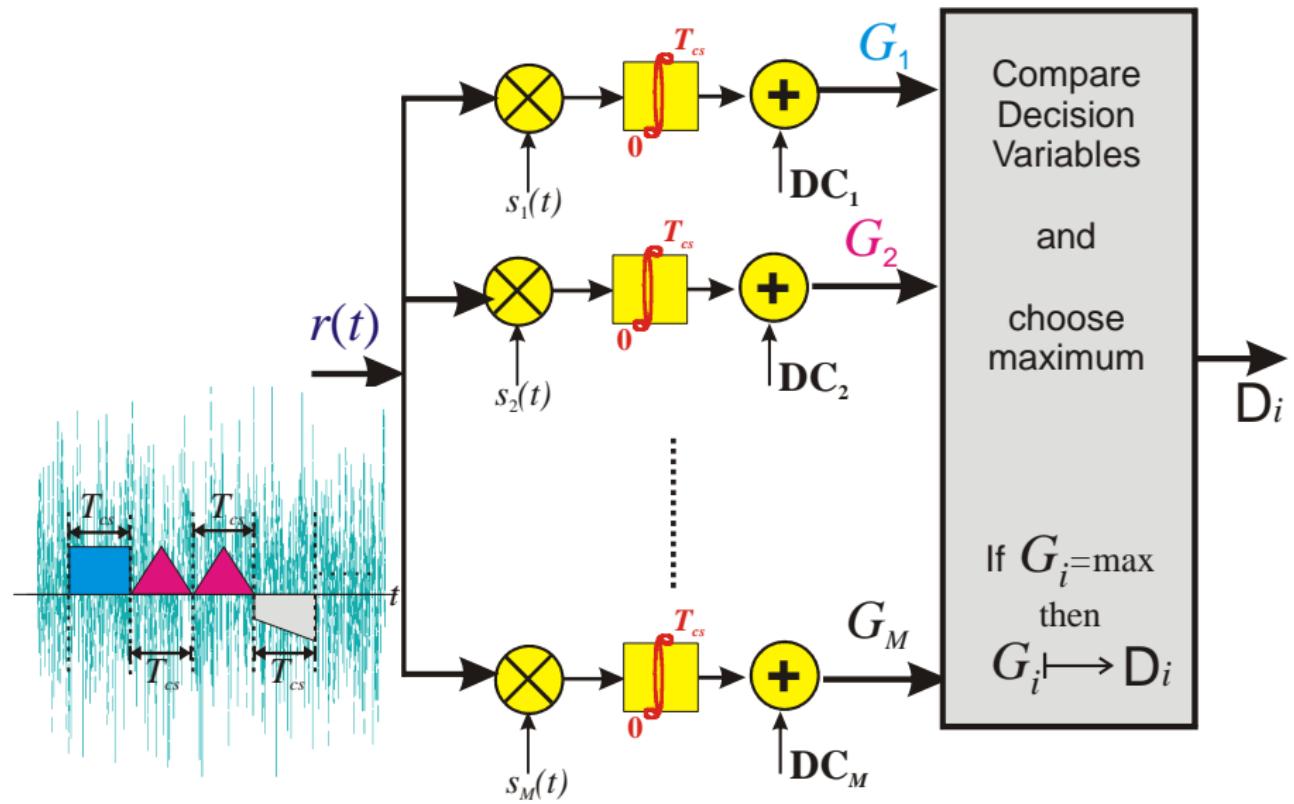
↑
only at time $t = T_{cs}$

- example:



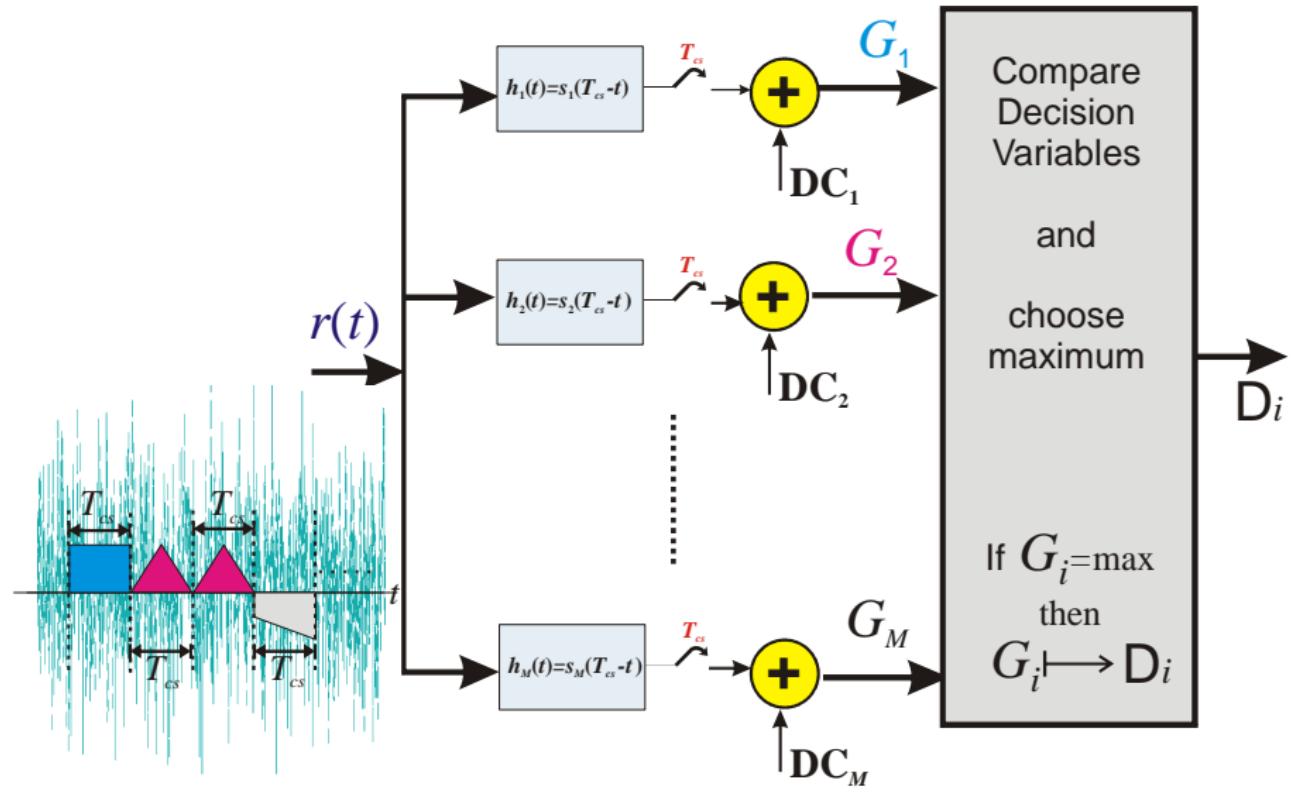
CORRELATION RECEIVER:

OPTIMUM M-ary RECEIVER: CORREL. RECEIVER



or, MATCHED FILTER RECEIVER:

OPTIMUM M -ary RECEIVER: MATCHED FILTER RECEIVER



Signals and Matched Filters in Freq. Domain

- Let $h(t) = \begin{cases} s(T_{cs} - t) & 0 \leq t \leq T_{cs} \\ 0 & \text{elsewhere} \end{cases}$



$$\begin{aligned}
 s(t) &\xrightarrow{\text{FT}} S(f) = \int_0^{T_{cs}} s(t) e^{-j2\pi ft} dt \\
 h(t) &\xrightarrow{\text{FT}} H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt \\
 &= \int_0^{T_{cs}} s(T_{cs} - t) e^{-j2\pi ft} dt \\
 &= \int_0^{T_{cs}} s(u) e^{-j2\pi f(T_{cs} - u)} du \\
 &= e^{-j2\pi fT_{cs}} \int_0^{T_{cs}} s(u) e^{+j2\pi fu} du \\
 &= e^{-j2\pi fT_{cs}} \cdot S^*(f)
 \end{aligned}$$

therefore

$$H(f) = e^{-j2\pi fT_{cs}} \cdot S^*(f)$$

(30)

Known Signals in Non-White Noise

- Now let us assume that the noise $n(t)$ is described as follows:

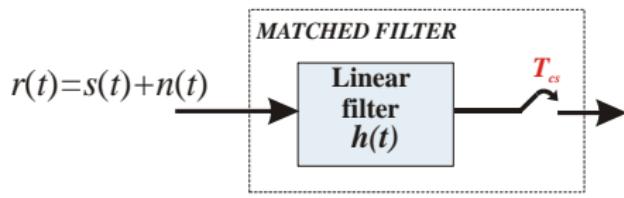
$$n(t) = \begin{cases} \text{zero mean} \\ R_{nn}(\tau) = \text{not necessarily white or Gaussian} \end{cases} \quad (31)$$

- Let

$$r(t) = \underbrace{s(t)}_{\text{completely known}} + n(t) \quad (32)$$

- objective: design a linear filter $h(t)$ such that

$$\text{SNR}_{out} = \max_{at \ t=T_{cs}} \quad (33)$$



$$\begin{aligned}
 & \text{output} = \int_0^{T_{cs}} h(\tau) \cdot \underbrace{r(T_{cs} - \tau)}_{\downarrow s(T_{cs} - \tau) + n(T_{cs} - \tau)} dt \\
 &= \underbrace{\int_0^{T_{cs}} h(\tau) \cdot s(T_{cs} - \tau) dt}_{=\tilde{s}(T_{cs})} + \underbrace{\int_0^{T_{cs}} h(\tau) \cdot n(T_{cs} - \tau) d\tau}_{=\tilde{n}(T_{cs})}
 \end{aligned}$$

$$\therefore \text{SNR}_{out} = \frac{\mathcal{E}\{\tilde{s}(T_{cs})^2\}}{\mathcal{E}\{\tilde{n}(T_{cs})^2\}} = \frac{\tilde{s}(T_{cs})^2}{\mathcal{E}\{\tilde{n}(T_{cs})^2\}} \quad (34)$$

$$\therefore \max_{h(t)}(\text{SNR}_{out}) = \left\{ \min_{h(t)} \mathcal{E}\{\tilde{n}(T_{cs})^2\} \text{ with } \tilde{s}(T_{cs}) = \text{constant} \right\} \quad (35)$$

$$\therefore \min_{h(t)} \xi \text{ where } \xi = \mathcal{E}\{\tilde{n}(T_{cs})^2\} - \lambda \cdot \tilde{s}(T_{cs}) \quad (36)$$

- i.e

$$\text{optimum impulse response} \triangleq h_{opt} = \arg \min_{h(t)} \xi \quad (37)$$

$$\text{where } \xi = \mathcal{E}\{\tilde{n}(T_{cs})^2\} - \lambda \cdot \tilde{s}(T_{cs})$$

- The solution of Equation 37 is the solution of the following

FREDHOLM INTEGRAL EQUATION of 1st KIND:

$$\int_0^{T_{cs}} h_{opt}(z).R_{nn}(\tau - z).dz = s(T_{cs} - \tau); \quad 0 \leq \tau \leq T_{cs} \quad (38)$$

GENERAL EXPRESSION FOR MATCHED FILTERS

- Proof of the above result (Equation 38) is not important. What is really important is that the proof does not involve any assumption about the pdf or PSD of the noise (i.e. that the noise is white Gaussian).

A Special Case

- In the case of white noise the above Equation (Equation-38) can be solved easily as follows:

$$\int_0^{T_{cs}} h_{opt}(z) \cdot \underbrace{\frac{N_0}{2} \cdot \delta(\tau - z)}_{=R_{nn}(\tau-z)} dz = s(T_{cs} - \tau) \quad (39)$$

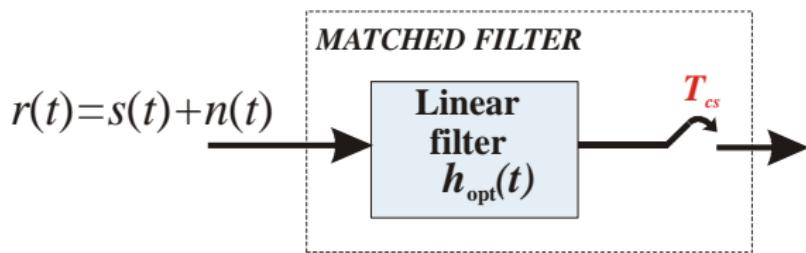
$$\implies h_{opt}(\tau) \cdot \frac{N_0}{2} = s(T_{cs} - \tau) \implies$$

$$h_{opt} = \frac{2}{N_0} \cdot s(T_{cs} - \tau) \quad (40)$$

- Be careful - SNR is not influenced by the factor $\frac{2}{N_0}$

Output SNR

- Consider next you have got the optimum impulse response $h_0(t)$ (by using Equation-38). This impulse response provides the maximum output-SNR which can be estimated as follows:



- Important Question: $\text{SNR}_{\text{out}} = ?$
we can answer this question as follows: 

$$\begin{aligned}
 \text{o/p} &= \int_0^{T_{cs}} h_{opt}(\tau) \cdot \underbrace{\frac{r(T_{cs} - \tau)}{s(T_{cs} - \tau) + n(T_{cs} - \tau)}}_{\uparrow} d\tau \\
 &= \underbrace{\int_0^{T_{cs}} h_{opt}(\tau) \cdot s(T_{cs} - \tau) d\tau}_{=\tilde{s}(T_{cs})} + \underbrace{\int_0^{T_{cs}} h_{opt}(\tau) \cdot n(T_{cs} - \tau) d\tau}_{=\tilde{n}(T_{cs})} \quad (41)
 \end{aligned}$$

therefore,

$$\begin{aligned}
 \text{SNR}_{\text{out}}_{\max} &= \frac{\mathcal{E}\{\tilde{s}(T_{cs})^2\}}{\mathcal{E}\{\tilde{n}(T_{cs})^2\}} = \frac{\tilde{s}(T_{cs})^2}{\mathcal{E}\{\tilde{n}(T_{cs})^2\}} = \\
 &= \dots \dots \quad (42)
 \end{aligned}$$

... (for you) ...

... (for you) ...

$$\text{i.e. } \underset{\max}{\text{SNR}_{out}} = \frac{\bar{s}(T_{cs})^2}{\mathcal{E}\{\tilde{n}(T_{cs})^2\}} = \int_0^{T_{cs}} h_{opt}(z) \cdot s(T_{cs} - z) dz \quad (43)$$

A Special Case: Output SNR (Signal plus white Noise)

- For white noise we have seen that: $h_{opt}(z) = \frac{2}{N_0} \cdot s(T_{cs} - z)$
- Then Equation-43 becomes:

$$\begin{aligned}
 \text{SNR}_{\max}^{\text{out}} &= \int_0^{T_{cs}} \underbrace{\frac{2}{N_0} \cdot s(T_{cs} - z) \cdot s(T_{cs} - z)}_{=h_{opt}} \cdot dz \\
 &= \frac{2}{N_0} \int_0^{T_{cs}} s^2(T_{cs} - z) \cdot dz = \\
 \implies \text{SNR}_{\max}^{\text{out}} &= 2 \frac{E}{N_0} \text{ for white noise} \tag{44}
 \end{aligned}$$

- Provided that the filter is matched to the signal, it is obvious from Equation-44 that

$$\begin{aligned}
 \text{SNIR}_{\max}^{\text{out}} &\neq f\{\text{signal waveform}\} \\
 &\neq f\{\text{signal bandwidth}\} \\
 &\neq f\{\text{peak power}\} \\
 &\neq f\{\text{time duration}\}
 \end{aligned}$$

Approximation To Matched Filter Solution

- We have seen that the matched filter can be found by solving the following integral equation:

$$\int_0^{T_{cs}} h_{opt}(z) \cdot R_{nn}(\tau - z) \cdot dz = s(T_{cs} - \tau); \quad 0 \leq \tau \leq T_{cs}$$

MATCHED FILTER GENERAL EQUATION

(a Fredholm Integral of the 1st kind.)

- However, it is very difficult to solve the above equation in a general case.
- N.B.: solution=easy when noise=white

- In order to find an approximation to matched filter solution, i.e.

$$w_0(z) \simeq h_{opt}(z)$$

we need to relax the condition $0 \leq \tau \leq T_{cs}$ to $-\infty \leq \tau \leq \infty$

$$0 \leq \tau \leq T_{cs} \text{ to } -\infty \leq \tau \leq \infty$$

.

Then

$$\int_{-\infty}^{\infty} w_0(z) R_{nn}(\tau - z) \cdot dz = s \left(\frac{T_0}{\nearrow} - \tau \right) \quad (45)$$

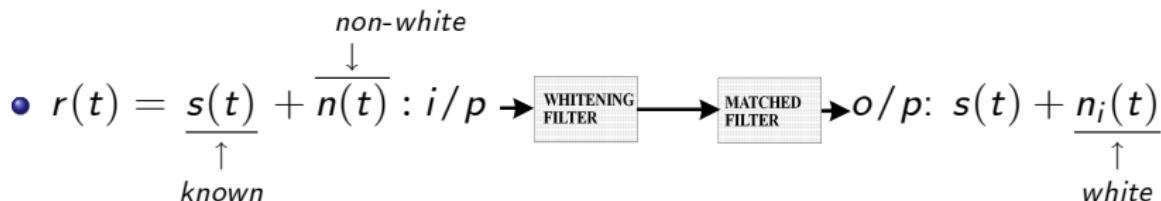
Maximizes SNR at some time

However the above integral is a convolution integral

$$\therefore W_0(f) \cdot PSD_n(f) = S^*(f) \cdot e^{-j2\pi f T_0} \quad (46)$$

$$\Rightarrow W_0(f) = \frac{S^*(f) \cdot e^{-j2\pi f T_0}}{PSD_n(f)} \quad (47)$$

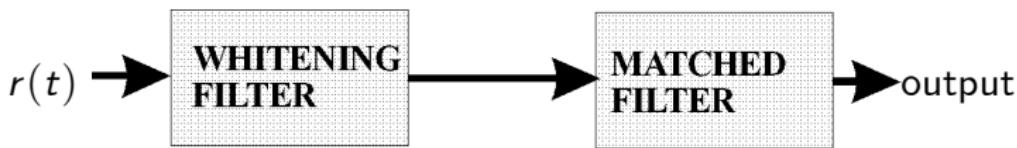
Whitening Filter



- whitening filter:

- ▶ attenuates → regions in Frequency Domain in which NOISE=LARGE
- ▶ accentuates → regions in Frequency Domain in which NOISE=LOW

- combined Whitening Filter-Matched Filter transfer function:



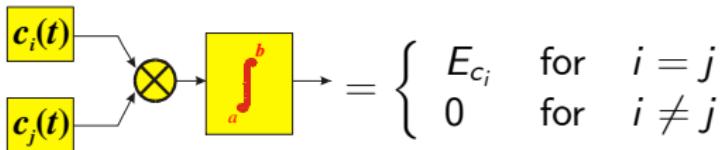
$$\text{combined transfer function} \equiv \mathbb{L}(f) = \frac{S^*(f) \cdot e^{-j2\pi f T_0}}{\text{PSD}_n(f)}$$

Optimum M-ary Rx based on Signal Constellation

Introduction - Orthogonal Signals

- Two signals $c_i(t)$ and $c_j(t)$ are called orthogonal signals (i.e. $c_i(t) \perp c_j(t)$) in the interval $[a, b]$ iff:

$$\int_a^b c_i(t) \cdot c_j^*(t) dt = \begin{cases} E_{c_i} & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad (48)$$



- Note:

- if $E_{c_i} = 1$ then the signals are called orthonormal
- $\int_a^b c_i(t) \cdot c_i^*(t) dt = E_{c_i} =$ the energy of $c_i(t)$ in the interval $[a, b]$.

The “Approximation Theorem” of an Energy Signal

- **Theorem** : Consider an energy signal $s(t)$. Furthermore consider a set of D orthogonal signals

$$\{c_1(t), c_2(t), \dots, c_D(t)\}$$

Then

- ▶ The signal $s(t)$ can be approximated by a linear combination of the orthogonal signals $c_i(t)$ as follows:

$$\hat{s}(t) = \sum_{i=1}^D w_i^* c_i(t) = \underline{w}_s^H \underline{c}(t) \quad (49)$$

where $\begin{cases} \underline{w}_s = [w_1, w_2, \dots, w_D]^T \\ \underline{c}(t) = [c_1(t), c_2(t), \dots, c_D(t)]^T \end{cases}$

- ▶ \underline{w}_s is a vector of coefficients or weights

- ▶ The best approximation is the one which uses the following coefficients

$$\underline{w}_s = \frac{1}{E_c} \odot \int_a^b s(t) \cdot \underline{c}^*(t) dt = \text{optimum weights} \quad (50)$$

where

$$\underline{E}_c = [E_{c_1}, E_{c_2}, \dots, E_{c_D}]^T \quad (51)$$

and \odot denotes Hadamard multiplication

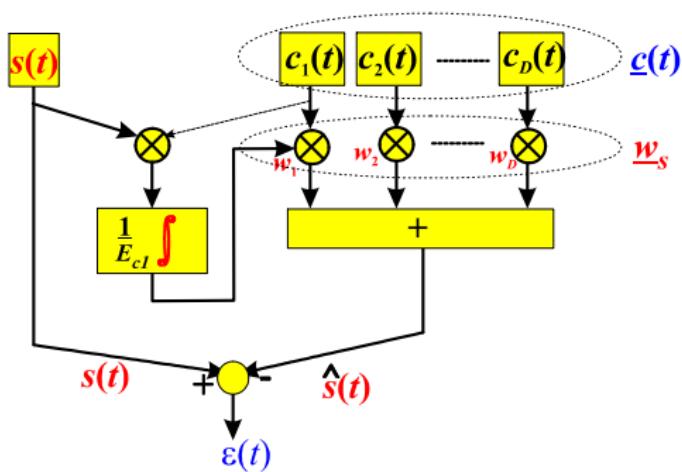
- ▶ The above coefficients are optimum in the sense that they minimize the energy E_ϵ of the error signal

$$\epsilon(t) = s(t) - \hat{s}(t) \quad (52)$$



Comments on the "Approximation" Theorem

- The above theorem states that any energy signal can be approximated by a linear combination of a set of orthogonal signals and can be equivalent described by the following diagram:



- The energy of the error signal $\epsilon(t)$ is minimum when the set of signals

$\{\epsilon(t), c_1(t), c_2(t), \dots, c_D(t)\}$ = is an orthogonal set

$$\text{i.e. if } \left\{ \begin{array}{l} \epsilon(t) \perp c_i(t), \forall i \\ \text{or} \\ \int_a^b (s(t) - \underline{w}_s^H \underline{c}(t)) \cdot \underline{c}^*(t) dt = 0 \end{array} \right\} \text{ then } E_\epsilon = \min \quad (53)$$

- In general

$$s(t) \approx \tilde{s}(t) \quad (54)$$

- However, if

$$s(t) = \hat{s}(t) \quad (55)$$

then $\{c_1(t), c_2(t), \dots, c_D(t)\} \triangleq$ a **complete set** of orthog. signals.

Gram-Schmidt Orthogonalization

- Suppose that we have a set of energy signals

$$\{s_1(t), s_2(t), \dots, s_M(t)\} = \text{non-orthogonal}$$

and we wish to construct a set of orthogonal signals

$$\{c_1(t), c_2(t), \dots, c_D(t)\} = \text{orthogonal}$$

- One popular approach to construct this orthogonal set is the **GRAM-SCHMIDT orthogonalisation**, described as follows:

1st orthogonal signal : $c_1(t) = \frac{s_1(t)}{\sqrt{\text{energy of numerator } E_{1,\text{num}}}}$

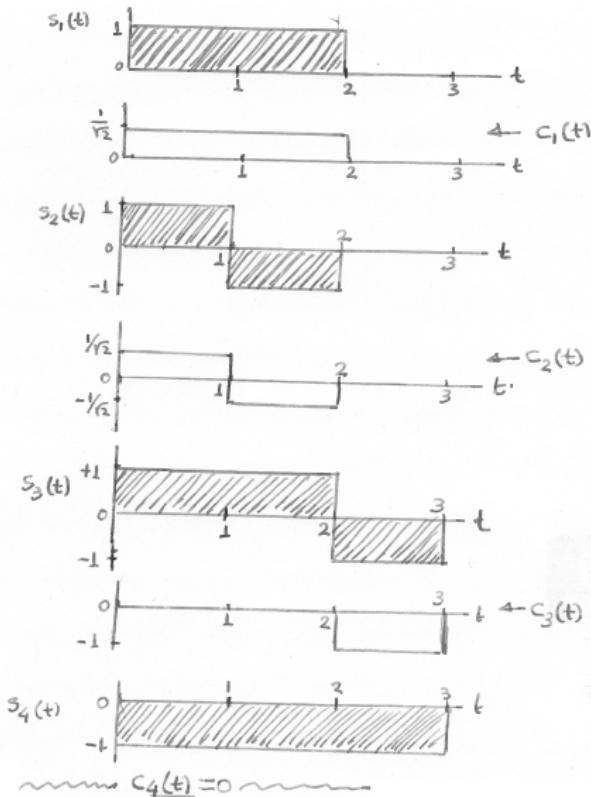
2nd orthogonal signal : $c_2(t) = \frac{s_2(t) - \alpha_{21} \cdot c_1(t)}{\sqrt{\text{energy of numerator } E_{2,\text{num}}}}$

⋮

or, in general,

k^{th} orthogonal signal : $c_k(t) = \frac{s_k(t) - \sum_{l=1}^{k-1} \alpha_{kl} \cdot c_l(t)}{\sqrt{\text{energy of numerator } E_{k,\text{num}}}}$

Example



M-ary Signals: Energy and Cross-Correlation

- **Binary Com. Systems** : use 2 possible waveforms $\{s_0(t), s_1(t)\}$
or $\{s_1(t), s_2(t)\}$
- **M-ary Com. Systems** : use M possible
waveforms $\{s_1(t), \dots, s_M(t)\}$
Note: these waveforms are energy signals of duration T_{cs}
- The M signals (or channel symbols) are characterized by their energy E_i

$$E_i = \int_0^{T_{cs}} s_i^2(t) dt \quad \forall i \quad (56)$$

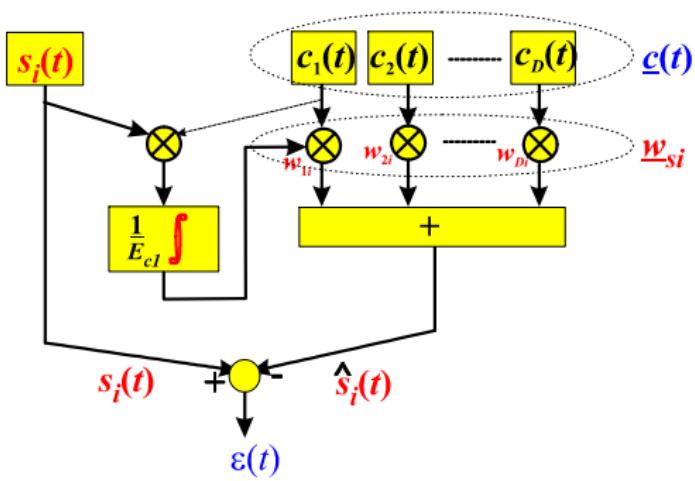
- Furthermore their similarity (or dissimilarity) is characterized by their cross-correlation

$$\rho_{ij} = \frac{1}{\sqrt{E_i E_j}} \int_0^{T_{cs}} s_i(t) \cdot s_j^*(t) dt \quad (57)$$

- The M signals may be also expressed, by using the Approx. Theorem described in the previous section, as a linear combination of a set of D orthogonal signals $\{c_k(t)\}$

$$s_i(t) = w_{s_i}^H c(t); \forall i \quad (58)$$

$$\text{where } \underline{c}(t) = [c_1(t), c_2(t), \dots, c_D(t)]^T \quad (59)$$



- Let us define the following $(D \times D)$ matrix

$$\begin{aligned}\mathbb{R}_{cc} &= \int_0^{T_{cs}} \underline{c}(t) \cdot \underline{c}(t)^H dt \\ &= \begin{bmatrix} \int_0^{T_{cs}} c_1^2(t) dt, & \int_0^{T_{cs}} c_1(t)c_2^*(t)dt, & \dots, & \int_0^{T_{cs}} c_1(t)c_D^*(t).dt \\ \int_0^{T_{cs}} c_2(t).c_1^*(t).dt, & \int_0^{T_{cs}} c_2^2(t).dt & \dots, & \int_0^{T_{cs}} c_2(t)c_D^*(t).dt \\ \dots, & \dots, & \dots, & \dots \\ \int_0^{T_{cs}} c_D(t).c_1^*(t).dt, & \int_0^{T_{cs}} c_D(t).c_2^*(t).dt, & \dots, & \int_0^{T_{cs}} c_D^2(t).dt \end{bmatrix}\end{aligned}$$

i.e.

$$\mathbb{R}_{cc} = \int_0^{T_{cs}} \underline{c}(t) \cdot \underline{c}(t)^H dt = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \mathbb{I}_D \quad (60)$$

- Then, by using Equation-56 in conjunction with Equation-58, the energy E_i of the signal $s_i(t), \forall i$, can be expressed as follows:

$$\begin{aligned}
 E_i &= \int_0^{T_{cs}} \underbrace{\underline{w}_{s_i}^H \underline{c}(t)}_{=s_i(t)} \cdot \underbrace{\underline{c}(t)^H \underline{w}_{s_i}}_{=s_i^*(t)} \cdot dt \\
 &= \underline{w}_{s_i}^H \left(\int_0^{T_{cs}} \underline{c}(t) \underline{c}(t)^H dt \right) \underline{w}_{s_i} \\
 &= \underline{w}_{s_i}^H \cdot \mathbb{R}_{cc} \cdot \underline{w}_{s_i} \\
 &= \underline{w}_{s_i}^H \cdot \mathbb{I}_D \cdot \underline{w}_{s_i} \\
 &= \underline{w}_{s_i}^H \underline{w}_{s_i} \\
 &= \left\| \underline{w}_{s_i} \right\|^2
 \end{aligned} \tag{61}$$

- Similarly, by using Equation-57 in conjunction with Equation-58 the cross-correlation coef. $\rho_{ij} \quad \forall ij$ becomes

$$\begin{aligned}
 \rho_{ij} &= \frac{1}{\sqrt{E_i E_j}} \int_0^{T_{cs}} \underline{w}_{s_i}^H \cdot \underline{c}(t) \cdot \underline{c}(t)^H \cdot \underline{w}_{s_j} \cdot dt \\
 &= \frac{1}{\sqrt{E_i E_j}} \cdot \underline{w}_{s_i}^T \cdot \left(\int_0^{T_{cs}} \underline{c}(t) \cdot \underline{c}(t)^H \cdot dt \right) \cdot \underline{w}_{s_j} \\
 &= \frac{1}{\sqrt{E_i E_j}} \underline{w}_{s_i}^H \cdot \mathbb{R}_{cc} \cdot \underline{w}_{s_j} \\
 &= \frac{1}{\sqrt{E_i E_j}} \underline{w}_{s_i}^H \cdot \mathbb{I}_D \cdot \underline{w}_{s_j} \\
 &= \frac{1}{\sqrt{E_i E_j}} \underline{w}_{s_i}^H \underline{w}_{s_j} = \frac{\underline{w}_{s_i}^H \underline{w}_{s_j}}{\|\underline{w}_{s_i}\| \|\underline{w}_{s_j}\|}
 \end{aligned}$$

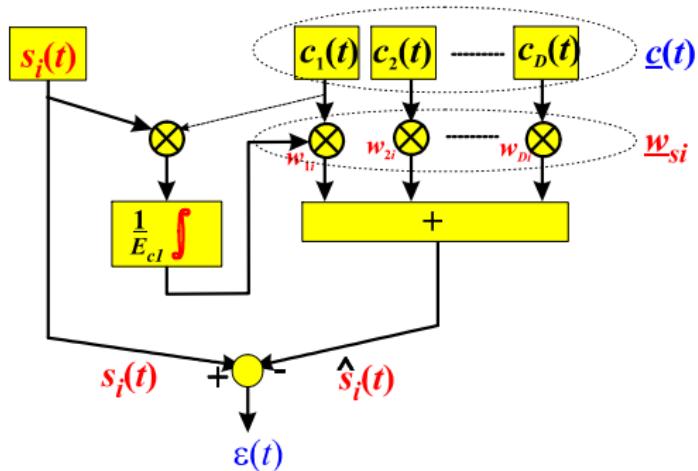
i.e.

$$E_i = \underline{w}_{s_i}^H \underline{w}_{s_i} = \left\| \underline{w}_{s_i} \right\|^2 \quad (62)$$

$$\rho_{ij} = \frac{1}{\sqrt{E_i E_j}} \underline{w}_{s_i}^H \cdot \underline{w}_{s_j} = \frac{\underline{w}_{s_i}^H \underline{w}_{s_j}}{\left\| \underline{w}_{s_i} \right\| \left\| \underline{w}_{s_j} \right\|} \quad (63)$$

Signal Constellation

- With reference to the figure below



- If the error signal $e(t) = 0$

then the knowledge of the vector w_s is as good as knowing the transmitted signal $s(t)$
 or, in the case of M signals (M -ary system)
 $\{\text{knowledge of } s_i(t)\} = \{\text{knowledge of } w_{s_i}\}, \forall i$

- Therefore, we may represent the signal $s_i(t)$ by a point (\underline{w}_{s_i}) in a D -dimensional Euclidean space with

$$D \leq M \quad (64)$$

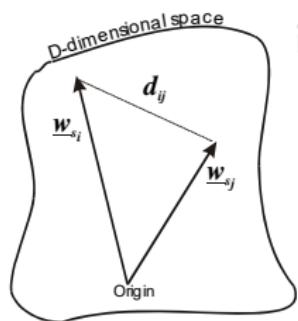
- The **set of points (vectors)** specified by the columns of the matrix

$$\mathbb{W} = [\underline{w}_{s_1}, \underline{w}_{s_2}, \dots, \underline{w}_{s_M}] \quad (65)$$

is known as "**signal constellation**" .

Distance between two M-ary signals

- The distance between two signals $s_i(t)$ and $s_j(t)$ is the Euclidean distance between their associate vectors \underline{w}_{s_i} and \underline{w}_{s_j}



$$\begin{aligned}
 \text{i.e. } d_{ij} &= \left\| \underline{w}_{s_i} - \underline{w}_{s_j} \right\| \\
 &= \sqrt{(\underline{w}_{s_i} - \underline{w}_{s_j})^H (\underline{w}_{s_i} - \underline{w}_{s_j})} \\
 &= \sqrt{\underline{w}_{s_i}^H \underline{w}_{s_i} + \underline{w}_{s_j}^H \underline{w}_{s_j} - 2 \operatorname{Re}(\underline{w}_{s_i}^H \underline{w}_{s_j})} \\
 &= \sqrt{E_i + E_j - 2\rho_{ij}\sqrt{E_i E_j}}
 \end{aligned}$$

$$d_{ij}^2 = E_i + E_j - 2\rho_{ij}\sqrt{E_i E_j} \quad (66)$$

It is clear from the above that the Euclidean distance d_{ij} of two signals indicates, like the cross-correlation coefficient, the similarity or dissimilarity of the signals.

- An **Important Bound** involving d_{ij} :

- ▶ If $p_{e,cs(s_i(t))}$ denotes the prob. of error associated with the channel symbol $s_i(t)$, it can be found that (see S. Haykin p.498)

$$p_{e,cs(s_i(t))} \leq \sum_{\substack{j=1 \\ j \neq i}}^M T \left\{ \frac{d_{ij}}{\sqrt{2N_0}} \right\} \quad (67)$$

Then, the symbol error probability $p_{e,sc}$ is bounded as follows:

$$p_{e,sc} \leq \frac{1}{M} \sum_{i=1}^M \sum_{\substack{j=1 \\ j \neq i}}^M T \left\{ \frac{d_{ij}}{\sqrt{2N_0}} \right\} \quad (68)$$

- ▶ N.B.: the following expression was used to go from Equ.67 to Equ.68

$$p_{e,sc} = \frac{1}{M} \sum_{i=1}^M p_{e,cs(s_i(t))} \quad (\text{see S.Haykin p.498}) \quad (69)$$

- “minimum distance”: $d_{\min} \triangleq \min_{\forall i,j} \{ d_{ij} \}$

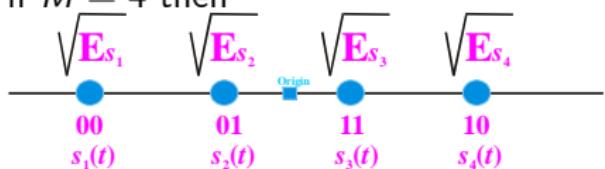
Examples of Signal Constellation Diagram

Consider an M-ary System having the following signals

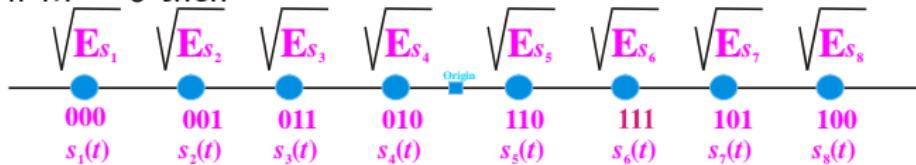
$$\{s_1(t), s_2(t), \dots, s_M(t)\} \text{ with } 0 \leq t \leq T_{cs}$$

- M-ary ASK

- ▶ channel symbols: $s_i(t) = A_i \cdot \cos(2\pi F_c t)$ where $A_i = \text{given} = 2i - 1 - M$ (say)
- ▶ dimensionality of signal space = $D = 1$
- ▶ if $M = 4$ then

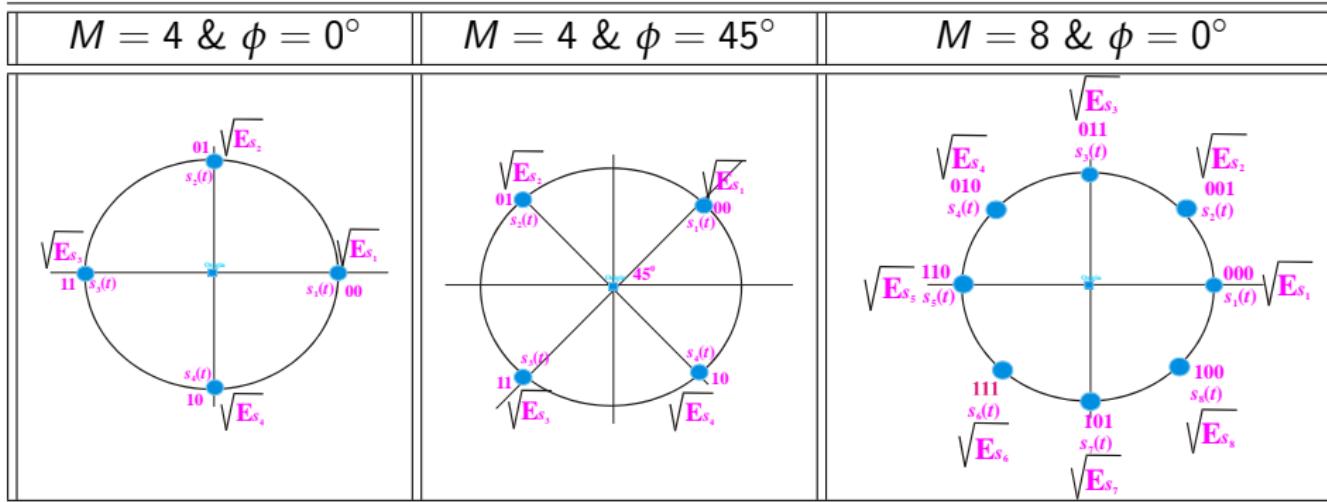


- ▶ if $M = 8$ then

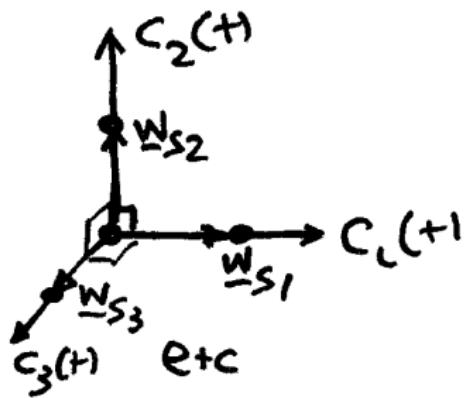


- M-ary PSK

- channel symbols: $s_i(t) = A \cos \left(2\pi F_c t + \frac{2\pi}{M} \cdot (i-1) + \phi \right)$
for $i = 1, 2, \dots$
- dimensionality of signal-space = $D = 2$



- M-ary FSK:
very difficult to be represented using constellation diagram

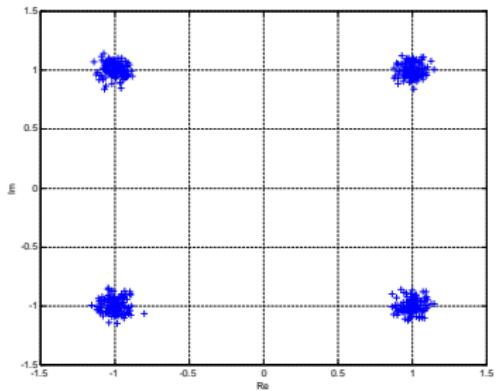


$$f_L - f_J = n \frac{1}{2T_{c_J}}$$

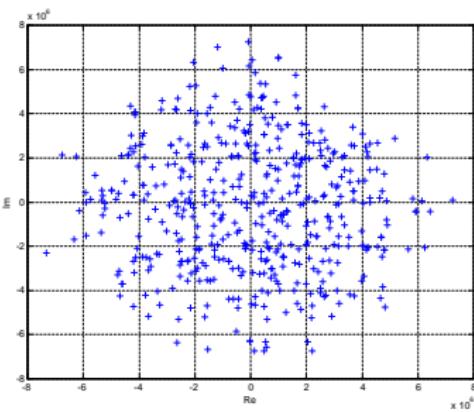
$$f_L + f_J = m \frac{1}{2T_{c_J}}$$

$$n, m = \text{integer}$$

Examples of Plots of Decision Variables (QPSK- Receiver)



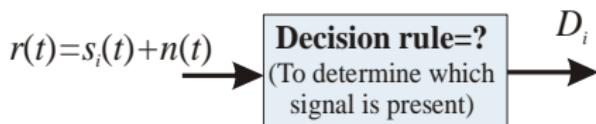
"good"



"bad"

Optimum M-ary Receiver using the Signal-Vectors

- Let us consider again the general M-ary problem:



where $s_i(t)$ = one of M signals (channel symbols)

- Hypotheses:

$$\left\{ \begin{array}{ll} H_1 : & s_1(t) \text{ is present with probability } \Pr(H_1) \\ H_2 : & s_2(t) \text{ is present with probability } \Pr(H_2) \\ \dots & \dots \dots \\ H_M : & s_M(t) \text{ is present with probability } \Pr(H_M) \end{array} \right.$$

- we have seen that the MAP criterion, using the "CONTINUOUS SAMPLING" concept, is described by the following rule:
Rule: choose the hypothesis H_i with the maximum

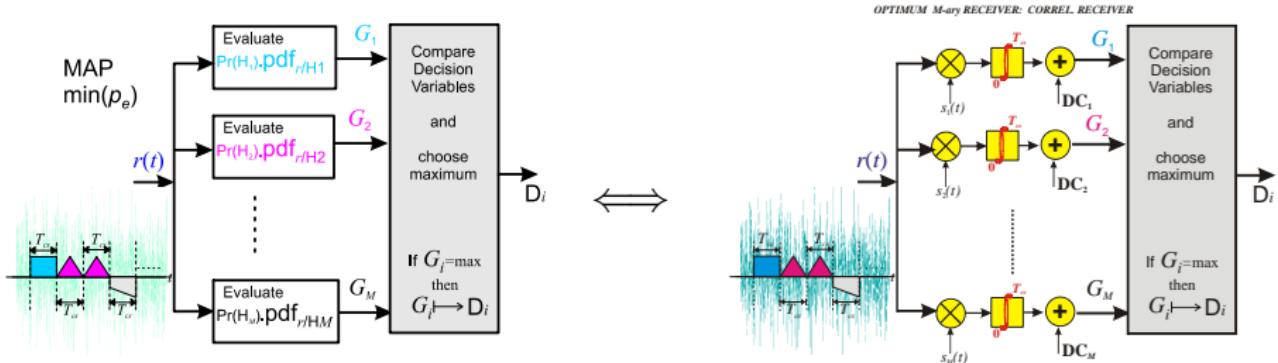
$$\Pr(H_i) \times \text{pdf}_{r/H_i}(r(t))$$

where

$$\begin{aligned}
 & \max \left\{ \Pr(H_i) \times \text{pdf}_{r/H_i}(r(t)) \right\} \\
 = & \max \left\{ \int_0^{T_{cs}} r(t) s_i^*(t) dt + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\} \quad (70)
 \end{aligned}$$

- Remember that this rule can also be described by the following two equivalent receivers:





where

$$DC_i = \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \quad (71)$$

- Now let us use the Approx. Theorem of energy signals to represent the received signals $r(t)$ as well as the M -ary signal $s_i(t), \forall i$,

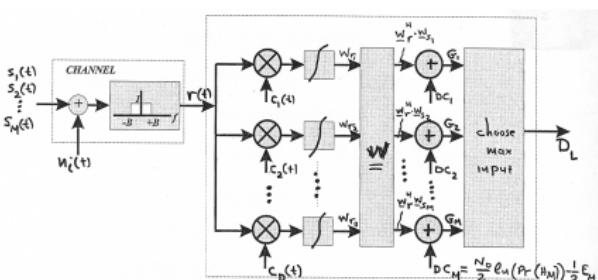
$$\text{i.e. } \begin{cases} r(t) = \underline{w}_r^H \underline{c}(t) \\ s_i(t) = \underline{w}_{s_i}^H \underline{c}(t) \end{cases} \implies s_i^*(t) = (\underline{w}_{s_i}^H \underline{c}(t))^* = \underline{c}(t)^H \underline{w}_{s_i} \quad \forall i$$

In this case the MAP receiver (Equ-70) becomes

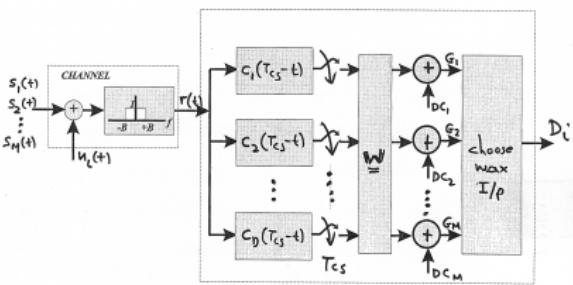
$$\begin{aligned} & \max \left\{ \Pr(H_i) \times \text{pdf}_{r/H_i}(r(t)) \right\} \\ = & \max \left\{ \int_0^{T_{cs}} r(t) s_i^*(t) dt + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\} \\ = & \max \left\{ \int_0^{T_{cs}} \underline{w}_r^H \underline{c}(t) \cdot \underline{c}(t)^H \underline{w}_{s_i} dt + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\} \\ = & \max \left\{ \underline{w}_r^H \left(\int_0^{T_{cs}} \underline{c}(t) \cdot \underline{c}(t)^H dt \right) \underline{w}_{s_i} + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\} \\ = & \max \left\{ \underline{w}_r^H \underline{w}_{s_i} + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\} \end{aligned}$$

$$\text{i.e. } \max \{ \Pr(H_i) \times \text{pdf}_{r/H_i}(r(t)) \} = \max \left\{ \underline{w}_r^H \underline{w}_{S_i} + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\} \quad (72)$$

The above equation may be implemented as follows:



or



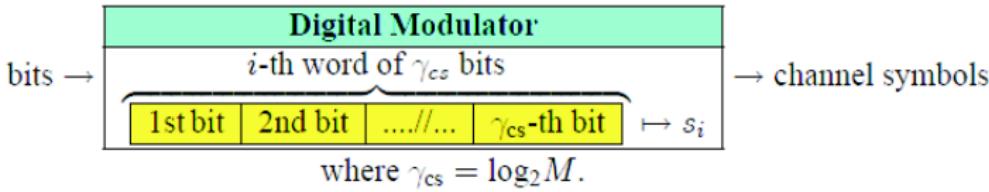
Encoding M-ary Signals

- The performance of M -ary systems is evaluated by means of the average probability of symbol error $p_{e,cs}$, which, for $M > 2$, is different than the average probability of bit error (or Bit-Error-Rate BER), p_e .

That is
$$\begin{cases} p_{e,cs} \neq p_e & \text{for } M > 2 \\ p_{e,cs} = p_e & \text{for } M = 2 \end{cases} \quad (73)$$

However, because we transmit binary data, the probability of bit error p_e is a more natural parameter for performance evaluation than $p_{e,cs}$.

- Although, these two probabilities are related, i.e. $p_e = f\{p_{e,cs}\}$ their relationship depends on the encoding approach which is employed by the digital modulator for mapping binary digits to M -ary signals (channel symbols)



Important Relationships Between BER and SER

- If the encoder provides a mapping where adjacent symbols differ in one binary digit then it can be proved (see Haykin p500) that

$$\frac{1}{\gamma_{cs}} \cdot p_{e,cs} \leq p_e \leq p_{e,cs} \quad (74)$$

e.g. M -ary PSK (for Gray code, otherwise difficult):

$$BER = p_e = \frac{1}{\gamma_{cs}} \cdot p_{e,cs} \quad (75)$$

- ▶ N.B.: Note that Gray encoder provides a mapping where adjacent symbols differ in one binary digit. This property is very important because the most likely errors (due to channel noise effects) are related with an erroneous selection (wrong decision) of an adjacent symbol.

- If all M -ary signals are equally likely, then it can be proved (see Haykin p.500) that

$$p_e = \frac{2^{\gamma_{cs}-1}}{2^{\gamma_{cs}} - 1} \cdot p_{e,cs} \quad (76)$$

which, for large γ_{cs} is simplified to

$$p_e = \frac{1}{2} \cdot p_{e,cs} \quad (77)$$

- Example:

M -ary FSK: $BER = p_e = \frac{2^{\gamma_{cs}-1}}{2^{\gamma_{cs}} - 1} \cdot p_{e,cs}$ (78)

Appendix-1: Proof of Equation-26

The above rule can be rewritten as follows:

$$\begin{aligned}
 & \max \left\{ \Pr(H_i) \times \text{pdf}_{r/H_i}(r(t)) \right\} \\
 = & \max \left\{ \ln(\Pr(H_i)) + \ln(\text{const.}) - \frac{1}{N_0} \int_0^{T_{cs}} (r(t) - s_i(t))^2 dt \right\} \\
 = & \max \left\{ \ln(\Pr(H_i)) - \frac{1}{N_0} \int_0^{T_{cs}} (r(t) - s_i(t))^2 dt \right\} \\
 = & \max \left\{ \ln(\Pr(H_i)) - \frac{1}{N_0} \int_0^{T_{cs}} r^2(t) dt - \frac{1}{N_0} \int_0^{T_{cs}} s_i^2(t) dt + \frac{2}{N_0} \int_0^{T_{cs}} r(t)s_i^*(t) dt \right\} \\
 = & \max \left\{ \ln(\Pr(H_i)) - \frac{1}{N_0} E_r - \frac{1}{N_0} E_i + \frac{2}{N_0} \int_0^{T_{cs}} r(t)s_i^*(t) dt \right\} \\
 = & \max \left\{ \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i + \int_0^{T_{cs}} r(t)s_i^*(t) dt \right\} \\
 = & \max \left\{ \int_0^{T_{cs}} r(t)s_i^*(t) dt + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\}
 \end{aligned}$$

Appendix-2: Walsh-Hadamard Orthogonal Sets and Signals

- $\mathbb{H}_1 = [1]$

$$\mathbb{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (2 \times 2) \text{ matrix}$$

$$\mathbb{H}_{64} = \underbrace{\mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2}_{6\text{-times}}$$

where \otimes denotes the Kronecker product of two matrices

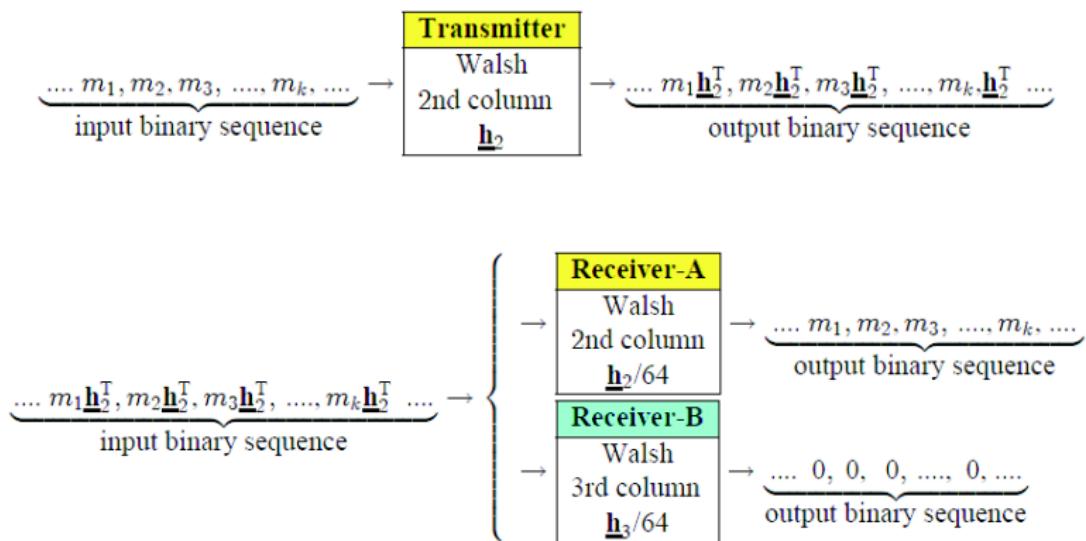
e.g. for two matrices \mathbb{A} and \mathbb{B}

$$\text{if } \mathbb{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ then } \mathbb{A} \otimes \mathbb{B} = \begin{bmatrix} A_{11}\mathbb{B} & A_{12}\mathbb{B} \\ A_{21}\mathbb{B} & A_{22}\mathbb{B} \end{bmatrix}$$

- Properties:

- ▶ $\mathbb{H}_{64}^T \mathbb{H}_{64} = 64 \mathbb{I}_{64}$
- ▶ Let \underline{h}_i denote the i -th column of \mathbb{H}_{64}
i.e. $\mathbb{H}_{64} = [\underline{h}_1, \underline{h}_2, \dots, \underline{h}_{64}]$

- The following example illustrates an application of “Walsh Hadamard” in a mobile communications where (each user in the same cell and same frequency channel is assigned one column of Walsh matrix,



Appendix-3: IS95 - Walsh Function of order-64

Walsh functions of order 64, as indexed in IS-95 (W_i is the Walsh notation, and H_i is the Hadamard notation).

