Lecture 11 & 12

Reminder

$f_{X,Y}(x,y)$ Joint PDF

$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$ marginal PDF of X

$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$ marginal PDF of Y

$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \quad \forall x,y$ X,Y independent

$f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$ >0 Conditional PDF of Y given $X=x$

$$E(g(X,Y)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$$

$$g(y) = E(X|Y=y) = \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\,dx$$ conditional expectation

$$E(\underbrace{E(X|Y)}_{g(Y)}) = E(X)$$

Conditional Variance

$$Var(X|Y=y) = \begin{cases} \sum_i (x_i - E(X|Y=y))^2 \underbrace{f_{X|Y}(x|y)}_{P(X=x_i|Y=y)} & \text{discrete} \\ \int_{-\infty}^{+\infty} (x - E(X|Y=y))^2 f_{X|Y}(x|y)\,dx \end{cases}$$

$$Var(X) = E(\underbrace{Var(X|Y)}) + Var(E(X|Y))$$

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} (x - E(X))^2 f_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x - E(X))^2 f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x - E(X|Y=y) + E(X|Y=y) - E(X))^2 f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x - E(X|Y=y))^2 f_{X,Y}(x,y)\, dx\, dy \quad A$$

$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 2(x - E(X|Y=y))(E(X|Y=y) - E(X)) f_{X,Y}(x,y)\, dx\, dy \quad B$$

$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (E(X|Y=y) - \underbrace{E(X)}_{E(E(X|Y))})^2 f_{X,Y}(x,y)\, dx\, dy \quad C$$

$$A = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x - E(X|Y=y))^2 f_{X|Y}(x|y) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} (x - E(X|Y=y))^2 f_{X|Y}(x|y)\, dx}_{\mathrm{Var}(X|Y=y)}\, f_Y(y)\, dy$$

$$= E(\mathrm{Var}(X|Y))$$

$$B = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x - E(X|Y=y))(E(X|Y=y) - E(X)) f_{X|Y}(x|y) f_Y(y)\, dx\, dy$$

$$= 2\int_{-\infty}^{+\infty} \left[\underbrace{\int_{-\infty}^{+\infty} (x - E(X|Y=y)) f_{X|Y}(x|y)\, dx}\right] (E(X|Y=y) - E(X)) f_Y(y)\, dy$$

$$\underbrace{\int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\, dx}_{E(X|Y=y)} - E(X|Y=y) \underbrace{\int_{-\infty}^{+\infty} f_{X|Y}(x|y)\, dx}_{=1}$$

$$\rightarrow E(X|Y=y) - E(X|Y=y) = 0$$

$$= 0$$

$$C = \mathrm{Var}(E(X|Y))$$

X, Y

Covariance

$$\mathrm{Cov}(X,Y) = E\left((X-\mu_X)(Y-\mu_Y)\right) \qquad \mu_X = E(X), \quad \mu_Y = E(Y)$$

$$= \begin{cases} \sum_x \sum_y (x-\mu_X)(y-\mu_Y)\, f_{X,Y}(x,y) & \text{discrete} \\ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\mu_X)(y-\mu_Y)\, f_{X,Y}(x,y)\,dx\,dy & \text{continuous} \end{cases}$$

Properties

$$\underline{\mathrm{Cov}(X,Y) = E((X-\mu_X)(Y-\mu_Y))} = \mathrm{Cov}(Y,X)$$

$$\mathrm{Cov}(X,X) = E((X-\mu_X)(X-\mu_X))$$
$$= E((X-\mu_X)^2) = \mathrm{Var}(X)$$

$$\mathrm{Cov}(X,a) = E((X-\mu_X)(a-a)) = 0 \qquad a \text{ constant}$$

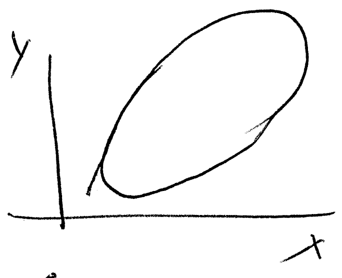
$$\mathrm{Cov}(aX+b,\ cY+d) =$$
$$E\Big((aX+b - \underbrace{E(aX+b)}\,)(cY+d - E(cY+d))\Big)$$
$$= E\left((aX+\cancel{b} - (a\mu_X+\cancel{b}))(cY+\cancel{d} - c\mu_Y - \cancel{d})\right)$$
$$= ac\ E((X-\mu_X)(Y-\mu_Y))$$
$$= ac \cdot \mathrm{Cov}(X,Y)$$

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

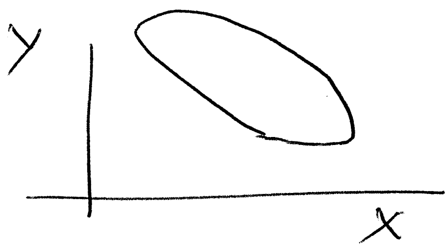
$$\mathrm{Cov}(X,Y) = E((X-\mu_X)(Y-\mu_Y))$$
$$= E(XY + \mu_X\mu_Y - \mu_X Y - X\mu_Y)$$
$$= E(XY) + \mu_X\mu_Y - \mu_X \underbrace{E(Y)}_{\mu_Y} - \underbrace{E(X)}_{\mu_X}\mu_Y$$
$$= E(XY) - \mu_X\mu_Y$$
$$= E(XY) - E(X)\,E(Y)$$

$Cov(X,Y) = E(XY) - E(X)\,E(Y)$

$Var(X) = Cov(X,X) = E(X^2) - E(X)^2$



$Cov(X,Y) > 0$



$Cov(X,Y) < 0$

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ex

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & x=3,\ y=4 \\ \frac{1}{3} & x=3,\ y=6 \\ \frac{1}{6} & x=5,\ y=6 \\ 0 & \text{oth} \end{cases}$$

$$Cov(x,y) = \underbrace{E(XY)}_{17} - \underbrace{E(X)}_{10/3}\ \underbrace{E(Y)}_{5} = 1/3 > 0$$

$$E(X) = 3\times\frac{1}{2} + 3\times\frac{1}{3} + 5\times\frac{1}{6} + 0 = \frac{10}{3}$$

$$E(Y) = 4\times\frac{1}{2} + 6\times\frac{1}{3} + 6\times\frac{1}{6} + 0 = 5$$

$$E(XY) = 3\times 4\times\frac{1}{2} + 3\times 6\times\frac{1}{3} + 5\times 6\times\frac{1}{6} = 17$$

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what if X,Y are independent?

$$Cov(X,Y) = E(XY) - E(X)\,E(Y)$$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\ f_{X,Y}(x,y)\,dx\,dy$$

independent $f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$ $\forall x,y$

$$\overset{\text{ind}}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\ f_X(x)\,f_Y(y)\,dx\,dy$$

$$= \underbrace{\int_{-\infty}^{+\infty} x\ f_X(x)\,dx}_{E(X)}\ \underbrace{\int_{-\infty}^{+\infty} y\ f_Y(y)\,dy}_{E(Y)}$$

X,Y independent $\Rightarrow$ $Cov(X,Y) = 0$ (uncorrelated)

$\not\Leftarrow$

ex $\Theta \sim \text{Unif}(0, 2\pi)$ Continuous

$$X = \cos(\theta) = h(\theta) \quad , \quad Y = \sin(\theta) = g(\theta) \qquad X^2 + Y^2 = 1$$

$$E(X) = E(h(\theta)) = \int_0^{2\pi} \cos(\theta) \frac{1}{2\pi} d\theta = 0$$

$$E(Y) = E(g(\theta)) = \int_0^{2\pi} \sin(\theta) \frac{1}{2\pi} d\theta = 0$$

$$E(XY) = E(h(\theta) g(\theta)) = \int_{-\infty}^{+\infty} h(\theta) g(\theta) f_\theta(\theta) d\theta$$

$$= \int_0^{2\pi} \sin\theta \cos\theta \frac{1}{2\pi}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \underbrace{2\sin\theta \cos\theta}_{\sin(2\theta)} d\theta = 0$$

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0$$

X, Y uncorrelated but dependent

X, Y independent

$$E(g(X) h(Y)) = E(g(X)) E(h(Y))$$

$$E(g(X) h(Y)) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x) h(y) f_{X,Y}(x,y) dx dy$$

$$\overset{\text{ind}}{=} \int_{-\infty}^{+\infty} g(x) f_X(x) dx \int_{-\infty}^{+\infty} h(y) f_Y(y) dy$$

$$= E(g(X)) \, E(h(Y))$$

ex discrete

X, Y

$$f_{X,Y}(x,y) = \begin{cases} 1/4, & x=3,\ y=5 \\ 1/4, & x=4,\ y=9 \\ 1/4, & x=6,\ y=9 \\ 1/4, & x=7,\ y=5 \\ 0 & \text{oth} \end{cases}$$

uncorrelated? $Cov(X,Y)$

$E(X) = 5 = 3 \times \frac{1}{4} + 4 \times \frac{1}{4} + 6 \times \frac{1}{4} + 7 \times \frac{1}{4}$

$E(Y) = 7$

$E(XY) = 3 \times 5 \times \frac{1}{4} + 4 \times 9 \times \frac{1}{4} \cdots = 35$

$Cov(X,Y) = 0$ $\quad$ X, Y uncorrelated

independent? $f_{X,Y}(6,5) = 0 \neq f_X(6)\, f_Y(5)$

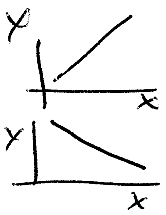
X, Y dependent

$Cov(kX, kY) = k^2\, Cov(X,Y)$

$$\rho = Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$ Correlation

$$\boxed{-1 \leq \rho \leq 1}$$

- $\rho = \pm 1$ $\quad$ $Y = aX + b$
  - $\rho = 1$ $\quad a > 0$
  - $\rho = -1$ $\quad a < 0$



- $\rho = 0$ $\quad$ no linear relationship ex $X \sim N(0,1)$, $Y = X^2$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= \underbrace{E(X^3)}_{=0} - \underbrace{E(X)}_{=0} E(X^2) = 0$$

Covariance Matrix

$$R = E\left[\underset{2\times 1}{\begin{bmatrix} (X-\mu_x) \\ (Y-\mu_y) \end{bmatrix}} \overset{1\times 2}{\begin{bmatrix} (X-\mu_x) & (Y-\mu_y) \end{bmatrix}} \right] \quad 2\times 2$$

$$= \begin{bmatrix} E((X-\mu_x)^2) = Var(X) & Cov(X,Y) \\ E((X-\mu_x)(Y-\mu_y)) = Cov(X,Y) & E((X-\mu_y)^2) = Var(Y) \end{bmatrix}$$

Joint Normal RVs

$$X,Y \sim N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$$

$\rho = 0$ $\quad f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall x, \forall y$

For Normal RVs, $\rho = 0$ (uncorrelated)

$=$ independent

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$$

Moments

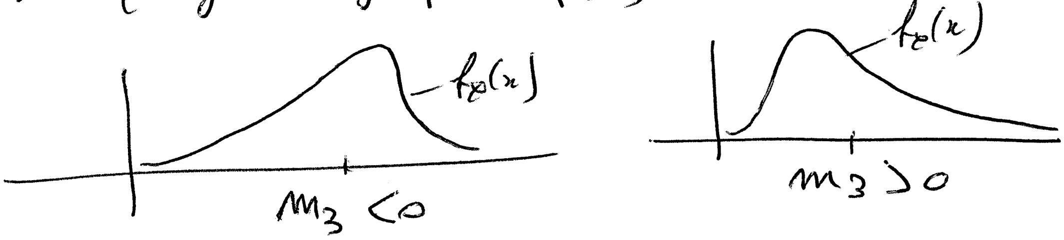
$m_1 = E(X)$ first moment

$m_2 = E(X^2)$ second moment

$$Var(X) = E(X^2) - E(X)^2 = m_2 - m_1^2$$

$m_3 = E(X^3)$ third moment

skewness (asymmetry of the PDF)



$m_r = E(X^r)$ r^{th} moment

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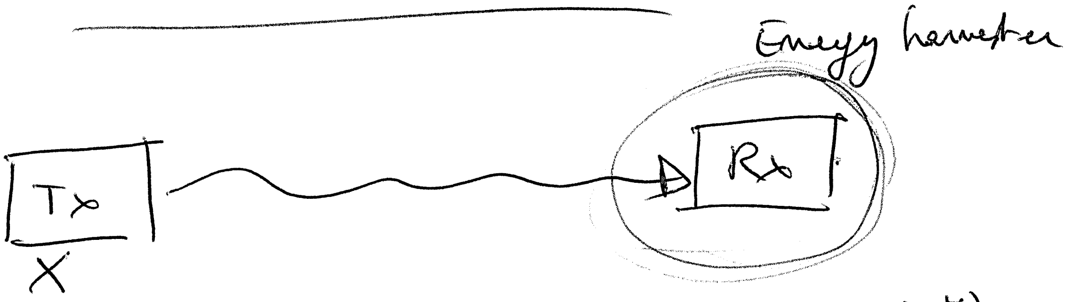
ex



$a\,E(X^2) + b\,E(X^4) + c\,E(X^6)$

Moment Generating Function (MGF)

$$m_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} f_X(x) & \text{discrete} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx & \text{cont.} \end{cases}$$

Purpose

1) Find $m_r = E(X^r)$
2) uniquely identifies a distribution (PDF)
3) $X_1 + \dots + X_n$ ($X_1, \dots X_n$ independent)

ex $X \sim N(\mu, \sigma^2)$ $\qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$m_X(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x^2 - 2\mu x + \mu^2 - 2tx\sigma^2)}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x^2 - 2x(\mu + t\sigma^2) + \mu^2)}{2\sigma^2}} dx$$

$$= e^{\frac{t^2\sigma^4 + 2\mu t\sigma^2}{2\sigma^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x^2 - 2x(\mu+t\sigma^2) + \overbrace{\mu^2 + t^2\sigma^4 + 2\mu t\sigma^2}^{(\mu+t\sigma^2)^2})}{2\sigma^2}} dx$$

$$e^{\frac{t^2\sigma^4}{2\sigma^2} + \mu t} \underbrace{\int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+t\sigma^2))^2}{2\sigma^2}}}_{\text{PDF of } N(\mu+\sigma^2 t,\ \sigma^2)} dx}_{=1}$$

$$= \boxed{e^{\mu t + \frac{t^2\sigma^2}{2}}}$$

$\underline{ex}$ $X \sim \text{Poisson}(\lambda)$

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2 \ldots$$

$$m_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\left[\sum_{x=0}^{\infty} \frac{e^{tx}\lambda^x}{x!}\right]$$

$$= e^{-\lambda}\left[\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}\right]$$

$\exp(a) = \sum_{x=0}^{\infty} \frac{a^x}{x!}$

$$= e^{-\lambda} \exp(\lambda e^t)$$

$$= \boxed{\exp((e^t - 1)\lambda)}$$

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① Find $E(X^r) = m_r$

$$m_X(t) = E(e^{tX}) = E\left(\sum_{r=0}^{\infty} \frac{(tX)^r}{r!}\right)$$

$$= \sum_{r=0}^{\infty} \frac{E(X^r)\, t^r}{r!}$$

$$= 1 + E(X)\, t + \frac{E(X^2)\, t^2}{2!} + \frac{E(X^3)\, t^3}{3!}$$

$$= 1 + m_1 t + \frac{m_2 t^2}{2!} + \frac{m_3 t^3}{3!} + \cdots$$

- Find m_1

$$\frac{d}{dt} m_X(t) = \underline{m_1} + \frac{2t\, m_2}{2!} + \frac{3t^2\, m_3}{3!} + \cdots$$

$$\left.\frac{d}{dt} m_X(t)\right|_{t=0} = m_1$$

- Find m_2

$$\frac{d^2}{dt^2} m_X(t) = m_2 + \frac{3\times 2\, t\, m_3}{3!} + \cdots$$

$$\left.\frac{d^2}{dt^2} m_X(t)\right|_{t=0} = m_2$$

$\vdots$

$$m_n = \left.\frac{d^n}{dt^n} m_X(t)\right|_{t=0}$$

ex $X \sim \text{Poisson}(\lambda)$

$$m_X(t) = \exp(\lambda(e^t - 1))$$

$$\frac{d\, m_X(t)}{dt} = \lambda e^t \exp(\lambda(e^t - 1))$$

$$\left.\frac{d\, m_X(t)}{dt}\right|_{t=0} = \lambda = m_1 = E(X)$$

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(2) MGF uniquely identifies PDF / PMF

$$m_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}} \longleftrightarrow X \sim N(\mu, \sigma^2)$$

$$m_X(t) = \exp(\lambda(e^t - 1)) \longleftrightarrow X \sim \text{Poisson}(\lambda)$$

③ $Z = X + Y$, X, Y independent

$$m_Z(t) = m_{X+Y}(t) = E\left(e^{t(X+Y)}\right)$$
$$= E\Big(\underbrace{e^{tX}}_{g(X)}\ \underbrace{e^{tY}}_{h(Y)}\Big)$$
$$\stackrel{ind}{=} E\left(e^{tX}\right) E\left(e^{tY}\right)$$
$$= m_X(t)\ m_Y(t)$$

ex $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ X, Y independent

$Z = X + Y$

$$m_Z(t) = m_X(t)\ m_Y(t)$$
$$= e^{t\underbrace{(\mu_1+\mu_2)}_{\mu} + \frac{t^2}{2}\underbrace{(\sigma_1^2+\sigma_2^2)}_{\sigma^2}}$$

$$\boxed{\begin{aligned} m_X(t) &= e^{\mu_1 t + \frac{t^2\sigma_1^2}{2}} \\ m_Y(t) &= e^{\mu_2 t + \frac{t^2\sigma_2^2}{2}} \end{aligned}}$$

↳ MGF of $N(\mu, \sigma^2)$
$= N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

ex $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ X, Y independent

$Z = aX + bY$

$$m_Z(t) = m_{aX+bY}(t) = E\left(e^{t(aX+bY)}\right)$$
$$= E\left(e^{atX}\ e^{btY}\right)$$
$$\stackrel{ind}{=} E\left(e^{atX}\right) E\left(e^{btY}\right)$$
$$= m_X(at)\ m_Y(bt)$$
$$= e^{a\mu_1 t + \frac{a^2t^2\sigma_1^2}{2}}\ e^{b\mu_2 t + \frac{b^2t^2\sigma_2^2}{2}}$$
$$= e^{t(a\mu_1 + b\mu_2) + \frac{t^2}{2}(a^2\sigma_1^2 + b^2\sigma_2^2)}$$

MGF of $N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

$$m_X(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\, dx$$

$$t = j\omega$$

$$\phi_X(\omega) = m_X(j\omega) = E(e^{j\omega X}) = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x)\, dx$$

characteristic function

$$\underline{F(\omega)} = \int_{-\infty}^{+\infty} e^{-j\omega t} \underline{f(t)}\, dt$$ Fourier TF of $f(t)$

direct	transformed
$f(t)$	$F(\omega)$
$f_X(x)$	$\phi_X(\omega)$
$f(t) \otimes g(t)$	$F(\omega)\ G(\omega)$
$f_X \otimes f_Y$	$\phi_{X+Y}(\omega) = \phi_X(\omega)\phi_Y(\omega)$

$$Z = aX + bY$$

$$E(Z) = E(aX + bY) = a\,E(X) + b\,E(Y)$$

$$Var(Z) = E\left((aX + bY - a\mu_X + b\mu_Y)^2\right)$$

$$= E\left((a(X - \mu_X) + b(Y - \mu_Y))^2\right)$$

$$= E\left(a^2(X - \mu_X)^2 + b^2(Y - \mu_Y)^2 + 2ab(X - \mu_X)(Y - \mu_Y)\right)$$

$$= a^2 \underbrace{E\left((X - \mu_X)^2\right)}_{Var(X)} + b^2 \underbrace{E\left((Y - \mu_Y)^2\right)}_{Var(Y)} + 2ab \underbrace{E\left((X - \mu_X)(Y - \mu_Y)\right)}_{Cov(X,Y)}$$

X, Y independent $\Rightarrow Cov(X,Y) = 0 \Rightarrow Var(Z) = a^2 Var(X) + b^2 Var(Y)$

Summary $\quad X, Y$ independent

$$\begin{cases} f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \quad \forall x,y \\ E(XY) = E(X)\,E(Y) \\ E(g(X)\,h(Y)) = E(g(X))\,E(h(Y)) \\ \mathrm{Cov}(X,Y) = 0 \\ \rho = 0 \\ \mathrm{Var}(aX+bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) \\ \quad \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \\ \quad \mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \\ m_{X+Y}(t) = m_X(t)\, m_Y(t) \\ f_{X|Y}(x|y) = f_X(x) \quad \forall x,y \\ E(X|Y) = E(X) \quad \forall y \end{cases}$$