Lecture 15

Random sample $X_1 \ldots X_n$ $f_X(x)$

independent, identically distributed

$N(\mu, \sigma^2)$

$EXPO(\lambda)$

Estimator $\hat{\theta}$ of parameter θ

RV

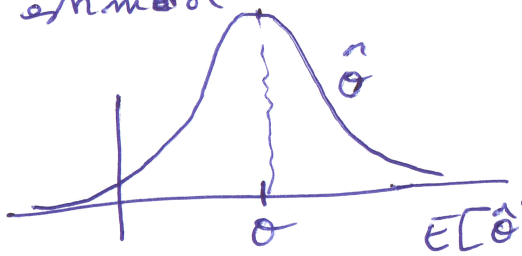
$\hat{\mu}$ of μ

$\hat{\sigma}^2$ of σ^2

$\hat{\lambda}$ of λ

Properties

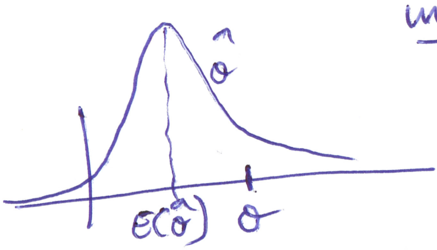
1) Biased or unbiased estimator



$E[\hat{\theta}] = \theta$

unbiased

$\text{bias} = E(\hat{\theta}) - \theta$

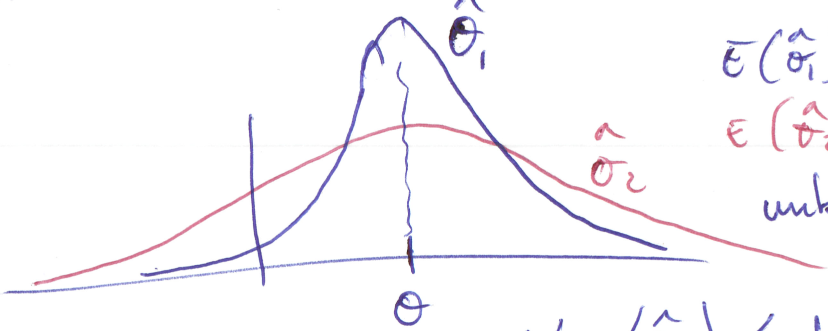


$E(\hat{\theta}) \neq \theta$

biased

$\bar{X}$ is an unbiased estimator of μ

2) Minimum variance unbiased estimator (MVUE)



$E(\hat{\theta}_1) = \theta$

$E(\hat{\theta}_2) = \theta$

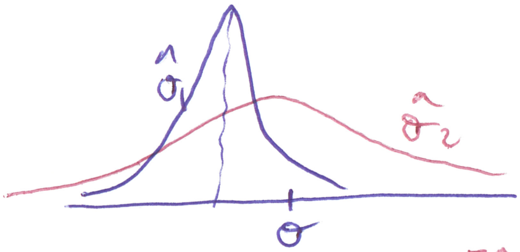
unbiased

$Var(\hat{\theta}_1) < Var(\hat{\theta}_2)$

3) MSE

(mean squared error)

$$MSE(\hat\theta) = E\left((\hat\theta - \theta)^2\right)$$



$E[\hat\theta_1] \neq \theta$ biased

$E[\hat\theta_2] = \theta$ unbiased

$Var(\hat\theta_1) < Var(\hat\theta_2)$

$$MSE(\hat\theta) = E\left(\left(\hat\theta - E(\hat\theta) + E(\hat\theta) - \theta\right)^2\right)$$

$$= E\left((\hat\theta - E(\hat\theta))^2 + (E(\hat\theta) - \theta)^2 + 2(\hat\theta - E(\hat\theta))(E(\hat\theta) - \theta)\right)$$

$$= \underbrace{E\left((\hat\theta - E(\hat\theta))^2\right)}_{Var(\hat\theta)} + \underbrace{E\left((E(\hat\theta) - \theta)^2\right)}_{bias^2} + \underbrace{2\, E(\hat\theta - E(\hat\theta))\,(E(\hat\theta) - \theta)}_{=0}$$

$E(\hat\theta - E(\hat\theta))$
$= E(\hat\theta) - E(\hat\theta)$

$$= Var(\hat\theta) + bias^2$$

estimators $\hat\theta_1$, $\hat\theta_2$

choose $\hat\theta_1$ if $MSE(\hat\theta_1) < MSE(\hat\theta_2)$

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how to construct estimators?

1) Method of Moments (MoM)

2) Maximum likelihood (ML)

1) Method of Moments (MoM)

theoretical moment

$m_k = E(X^k)$

random sample $X_1 \dots X_n$ f_X

sample moment

$\frac{1}{n}\sum_{i=1}^{n} X_i^k$

$m_1 = E(X)$ $\qquad \frac{1}{n}\sum_{i=1}^{n} X_i$

$m_2 = E(X^2)$ $\qquad \frac{1}{n}\sum_{i=1}^{n} X_i^2$

Construct the estimator by equating theoretical moment with sample moment

ex random sample $X_1, \dots X_n \sim EXPO(\lambda)$

Construct MoM estimator of λ, i.e. $\hat{\lambda}$

theoretical moment $\qquad$ sample moment

$m_1 = E(X) = \frac{1}{\lambda}$ $\qquad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Construct $\hat{\lambda}$ such that

$$\frac{1}{\hat{\lambda}} = \bar{X} \Rightarrow \hat{\lambda} = \frac{1}{\bar{X}}$$

[data sets $x_1, \dots x_n$

estimate of λ will $\frac{1}{\bar{x}}$ where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$]

is $\hat{\lambda}$ an biased or unbiased estimator of λ?

$$E[\hat{\lambda}] = E\left[\frac{1}{\bar{X}}\right] \neq \frac{1}{E[\bar{X}]} = \lambda$$

biased

$E[\bar{X}] = \frac{1}{\lambda}$

ex random sample $X_1 \dots X_n \sim N(\mu, \sigma^2)$

Construct MoM estimators of μ and σ^2

th. moment sample moment

$$m_1 = E(X) = \mu \qquad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$\sigma^2 = E(X^2) - E(X)^2 = E(X^2) - \mu^2$

$$m_2 = E(X^2) = \sigma^2 + \mu^2 \qquad \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Construct $\hat{\mu}$ and $\hat{\sigma}^2$ such that

$$\left.\begin{aligned} \hat{\mu} &= \bar{X} \\ \hat{\sigma}^2 + \hat{\mu}^2 &= \frac{1}{n}\sum_{i=1}^{n} X_i^2 \end{aligned}\right\} \qquad \boxed{\hat{\mu} = \bar{X}}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

$$\boxed{\hat{\sigma}^2 = \frac{1}{n}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]}$$

$\hat{\mu}$: $E[\hat{\mu}] = E[\bar{X}] = \mu$ unbiased

$\hat{\sigma}^2$: $E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum X_i^2\right] - E[\bar{X}^2]$

$$= \frac{n\,E[X^2]}{n} - E[\bar{X}^2]$$

$$= \sigma^2 + \mu^2 - E[\bar{X}^2] \neq \sigma^2 \quad \text{biased}$$

$\mu = E[\bar{X}]$

$\mu^2 \neq E[\bar{X}^2]$

$$E\left[\frac{1}{n}\sum X_i^2\right] = \frac{1}{n} E\left[\sum X_i^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} E(X_i^2)\right] = \frac{1}{n}\left[\sum_{i=1}^{n} (\sigma^2 + \mu^2)\right]$$

$$= \frac{n(\sigma^2 + \mu^2)}{n} = \sigma^2 + \mu^2$$

2) ML

(random) sample $X_1, \ldots X_n$ f_X

$$\underline{\underline{L(\theta)}} = f_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n | \theta)$$

Construct $\hat{\theta}$ that maximizes $L(\theta)$ MLE

ex random sample $X_1, \ldots X_n \sim \text{EXPO}(\lambda)$

MLE of λ ?

$$\underline{\underline{L(\lambda)}} = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$f_{X_1, X_2 \ldots X_n} = \prod_{i=1}^{n} f_{X_i}$$

$$= \lambda^n e^{-\lambda \left(\sum_{i=1}^{n} x_i\right)}$$

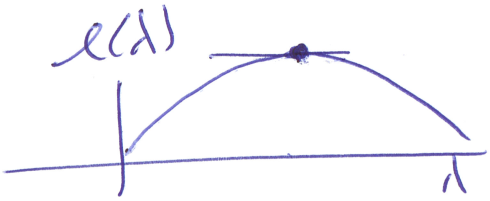
$$= \lambda^n e^{-n\lambda \bar{x}}$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\ell(\lambda) = \log(L(\lambda))$$

log-likelihood

$$= n \log \lambda - n \lambda \bar{x}$$



$$\frac{d}{d\lambda} \ell(\lambda) = \frac{n}{\lambda} - n\bar{x}$$

Construct $\hat{\lambda}$ such that $\frac{n}{\hat{\lambda}} = n\bar{x} \Rightarrow \hat{\lambda} = \frac{1}{\bar{x}}$

$$\boxed{\hat{\lambda} = \frac{1}{\bar{X}}}$$

biased

$$\frac{d^2 \ell(\lambda)}{d\lambda^2} < 0$$

ex random sample $X_1, \ldots X_n \sim N(\mu, \sigma^2)$

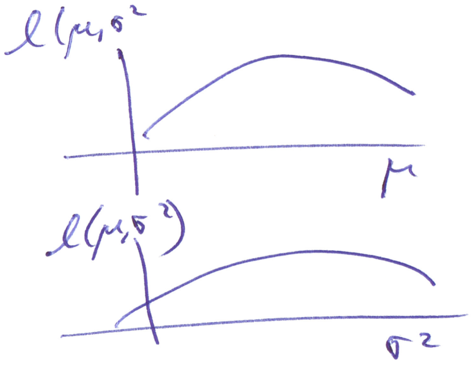
MLE of μ and σ^2

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$$

$$l(\mu, \sigma^2) = \log(L(\mu, \sigma^2))$$

$$= -\frac{n}{2}\log(2\pi\sigma^2) - \left(\sum_{i=1}^{n}(x_i-\mu)^2\right)\frac{1}{2\sigma^2}$$



$$\frac{\partial l}{\partial \mu} = \frac{+2\sum_{i=1}^{n}(x_i-\mu)}{2\sigma^2} = \frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma^2}$$

Choose $\hat{\mu}$ such that $\frac{\partial l}{\partial \mu} = 0$

$$\sum_{i=1}^{n} x_i = n\hat{\mu}$$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X}$$

$$\frac{\partial^2 l}{\partial \mu^2} < 0$$

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^4}$$

Choose $\hat{\sigma}^2$ such that $\frac{\partial l}{\partial \sigma^2} = 0$

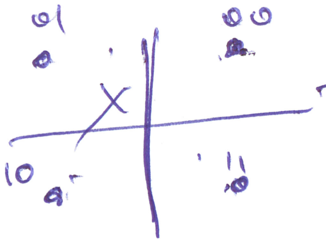
$$\left.\frac{\partial^2 l}{\partial \sigma^2}\right|_{\hat{\mu}, \hat{\sigma}^2} < 0$$

$$-\frac{n}{2}\frac{1}{\hat{\sigma}^2} + \frac{\sum(x_i-\hat{\mu})^2}{2\hat{\sigma}^4} = 0$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

biased

QPSK



$Y = X + N$

$N \sim N(0, \sigma^2)$

$Y \sim N(X, \sigma^2)$