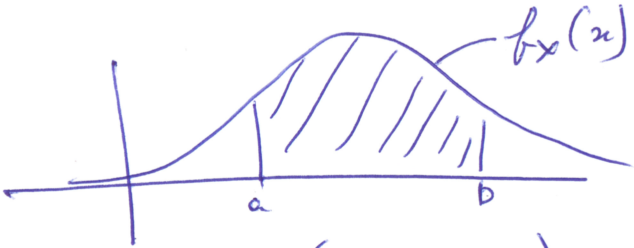
Lecture 7 and 8

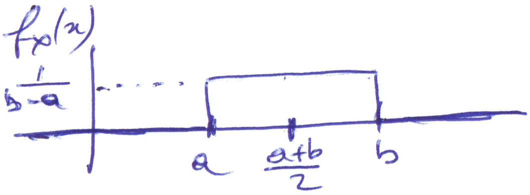
Continuous RV



$P(a \leq X \leq b)$

- PDF $f_X(x)$
- CDF $F_X(u) = P(X \leq u)$
- $\mu = E(X) = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx$
- $Var(X) = E\left((X-\mu)^2\right) = \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x) \, dx$

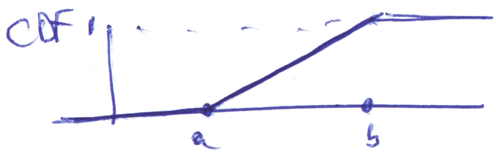
Uniform



$X \sim Unif(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & oth \end{cases}$$

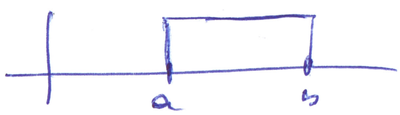
$$F_X(u) = P(X \leq u) = \begin{cases} 0 & u \leq a \\ \int_a^u \frac{1}{b-a} dx = \frac{u-a}{b-a} & a < u < b \\ 1 & u \geq b \end{cases}$$



$$E(X) = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx = \int_a^b \frac{x}{b-a} dx = \frac{b+a}{2}$$

$$Var(X) = E(X^2) - \underbrace{E(X)}_{\frac{b+a}{2}}{}^2 = \frac{(b-a)^2}{12}$$

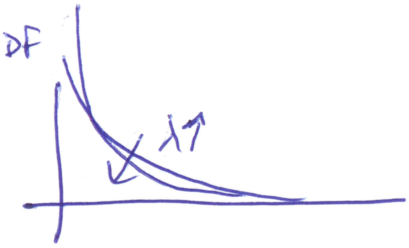
$$E(X^2) = \int_a^b \frac{x^2}{b-a} dx$$



Exponential

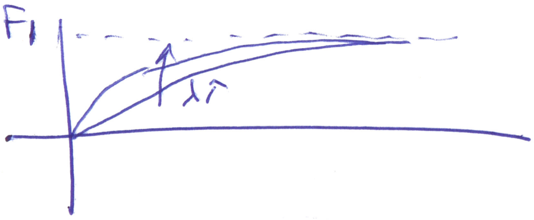
$X \sim EXPO(\lambda)$ $\qquad \lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{oth} \end{cases}$$

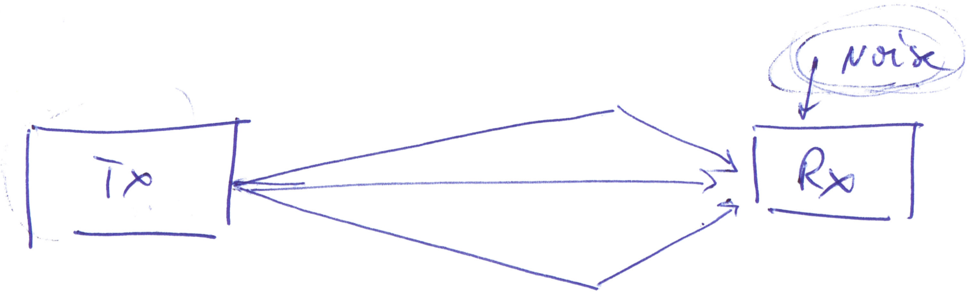


$$F_X(u) = P(X \leq u) = \begin{cases} 0 & u \leq 0 \\ \int_0^u \lambda e^{-\lambda x} dx & u > 0 \end{cases}$$

$$= \left[-e^{-\lambda x}\right]_0^u = \boxed{1 - e^{-\lambda u}}$$



ex



$$\int_0^{+\infty} \lambda e^{-\lambda x} dx = 1 \qquad \text{valid PDF}$$

memoryless property

$T \sim EXPO(\lambda)$

$$P(T > a+b \mid T > a) = \frac{P(T > a+b \wedge T > a)}{P(T > a)}$$

$$= \frac{P(T > a+b)}{P(T > a)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$

$$= e^{-\lambda b} = P(T > b)$$

$P(T > a) = 1 - P(T \leq a)$
$= 1 - F_T(a) = 1 - (1 - e^{-\lambda a})$
$= e^{-\lambda a}$

- Expectation

$$E(X) = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx$$

$$= \int_0^{+\infty} x \lambda e^{-\lambda x} dx$$

$u = x \qquad du = dx$

$dv = \lambda e^{-\lambda x} dx \qquad v = -e^{-\lambda x}$

$$= \underbrace{\left[-x e^{-\lambda x}\right]_0^{+\infty}}_{=0} + \underbrace{\int_0^{+\infty} e^{-\lambda x} dx}_{\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty} = \frac{1}{\lambda}}$$

$$= \frac{1}{\lambda}$$

- Variance

$$Var(X) = E(X^2) - \underbrace{E(X)}_{1/\lambda}^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$E(X^2) = \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx$$

$u = x^2 \qquad du = 2x\,dx$

$dv = \lambda e^{-\lambda x} dx \qquad v = -e^{-\lambda x}$

$$= \underbrace{\left[-x^2 e^{-\lambda x}\right]_0^{+\infty}}_{0} + \underbrace{\int_0^{+\infty} 2x e^{-\lambda x} dx}_{\frac{2}{\lambda} \underbrace{\int_0^{+\infty} \lambda x e^{-\lambda x} dx}_{E(X) = \frac{1}{\lambda}}} = \frac{2}{\lambda^2}$$

Change of Variable

$$X \longrightarrow Y = g(X)$$

$$f_X(x) \qquad\qquad f_Y(y) = ?$$

ex $X \sim EXPO(\lambda)$ $\qquad f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x>0 \\ 0 & oth \end{cases}$

(1) $Y = cX \qquad c>0$

$$F_Y(y) = P(Y \le y) = P(cX \le y) = P\left(X \le \frac{y}{c}\right) = F_X\left(\frac{y}{c}\right)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X\left(\frac{y}{c}\right) \frac{1}{c}$$

$$= \begin{cases} \lambda e^{-\lambda \frac{y}{c}} \frac{1}{c} & y>0 \\ 0 & oth \end{cases}$$

$$= \boxed{\begin{cases} \frac{\lambda}{c} e^{-\frac{\lambda}{c} y} & y>0 \\ 0 & oth \end{cases}}$$

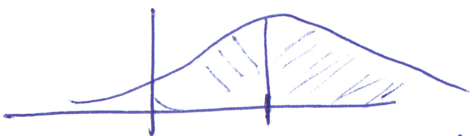
$$Y \sim EXPO\left(\frac{\lambda}{c}\right)$$

(2) $Y = cX \qquad c<0 \qquad d = -c > 0$

$$F_Y(y) = P(Y \le y) = P(cX \le y) = P(-dX \le y)$$

$$= P(dX \ge -y) = P\left(X \ge \frac{-y}{d}\right)$$

$$= 1 - P\left(X \le \frac{-y}{d}\right) = 1 - F_X\left(\frac{-y}{d}\right)$$



$$f_Y(y) = \frac{d F_Y(y)}{dy} = -f_X\left(\frac{-y}{d}\right)\left(\frac{-1}{d}\right)$$

$$f_X(x) = \begin{cases} \lambda \, e^{-\lambda x} & x > 0 \\ 0 & \text{oth} \end{cases}$$

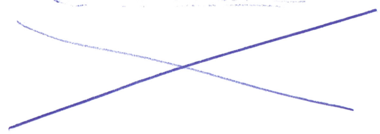
$$f_Y(y) = \begin{cases} \frac{1}{\alpha} \lambda \, e^{+\lambda \frac{y}{\alpha}} & y < 0 \\ 0 & y \geq 0 \end{cases}$$

$$= \begin{cases} \frac{-\lambda}{c} \, e^{-\lambda \frac{y}{c}} & y < 0 \\ 0 & y \geq 0 \end{cases}$$

$$f_Y(y) = \left| \frac{\lambda}{c} \right| e^{-\lambda \frac{y}{c}}$$

$$X \xrightarrow{g(x)} Y \qquad g(x)$$

monotonic



$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

$$y = cx$$

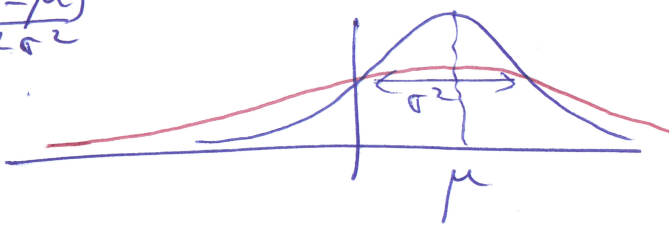
$$x = \frac{y}{c}$$

Normal

$$X \sim N(\mu, \sigma^2)$$

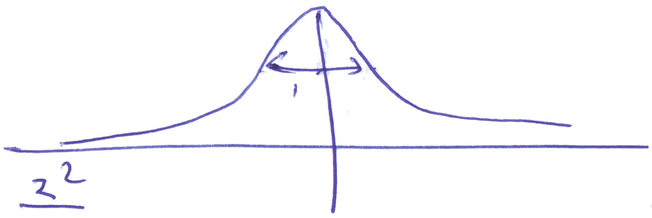
$\mu \to E(X)$, $\sigma^2 \to Var(X)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



$\mu = 0$, $\sigma^2 = 1$ standard Normal

$$Z \sim N(0, 1)$$

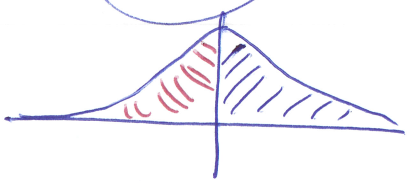


$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Valid PDF ?

$$\int_{-\infty}^{+\infty} f_Z(z)\, dz = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \stackrel{?}{=} 1$$

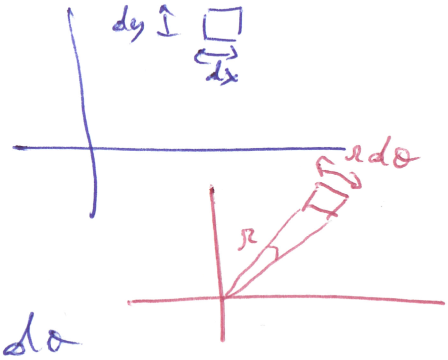
$$I = \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}$$



$$I = 2 \underbrace{\int_0^{+\infty} e^{-\frac{z^2}{2}} dz}_{J}$$

$$J^2 = \underbrace{\int_0^{+\infty} e^{-\frac{x^2}{2}} dx}_{J} \underbrace{\int_0^{+\infty} e^{-\frac{y^2}{2}} dy}_{J}$$

$$J^2 = \int_0^{+\infty}\int_0^{+\infty} e^{-\frac{x^2+y^2}{2}}\, dx\, dy$$



Cartesian to polar coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dx\,dy = \det(J)\, dr\, d\theta = r\, dr\, d\theta$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\det(J) = r$$

$$J^2 = \int_0^{+\infty}\int_0^{\pi/2} e^{-\frac{r^2}{2}}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^{+\infty} e^{-\frac{r^2}{2}}\, r\, dr\right] d\theta$$

$$\left[-e^{-\frac{r^2}{2}}\right]_0^{+\infty}$$

$$= 1$$

$$= \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

$$J^2 = \frac{\pi}{2}$$

$$I = 2J = 2\sqrt{\frac{\pi}{2}}$$

$$= \sqrt{2\pi}$$

$$E(Z) = \int_{-\infty}^{+\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 0$$

$$E(Z^2) = ? \qquad\qquad Z \sim N(0,1)$$

$$Var(Z) = \underbrace{E(Z^2)}_{?} - \underset{\overset{\shortparallel}{0}}{E(Z)^2} = 1$$

$$E(Z^2) = \int_{-\infty}^{+\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \underset{\text{int. by part}}{=} 1$$

$$X = \sigma Z + \mu \qquad\qquad f_X(x) = ?$$

$$P(X \leq x) = P(\sigma Z + \mu \leq x) = P\left(Z \leq \frac{x-\mu}{\sigma}\right)$$

$$= F_X(x) \qquad\qquad = F_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$F_Z(u) = \int_{-\infty}^{u} \boxed{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}} dz$$

$$f_X(x) = \frac{d F_X(x)}{dx} = f_Z\left(\frac{x-\mu}{\sigma}\right) \frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{\sigma^2}}$$

$$\rightarrow X \sim N(\mu, \sigma^2)$$

$$\underset{N(0,1)}{Z} \xrightarrow{X = \sigma Z + \mu} \underset{N(\mu, \sigma^2)}{X}$$

$$E(X) \overset{①}{=} \int_{-\infty}^{+\infty} x \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{pdf} dx = \mu \qquad X \sim N(\mu, \sigma^2)$$

$$\overset{②}{=} E(\sigma \underbrace{Z}_{N(0,1)} + \mu) = \sigma \underbrace{E(Z)}_{0} + \mu = \mu$$

$$E(aX+b) = aE(X)+b$$

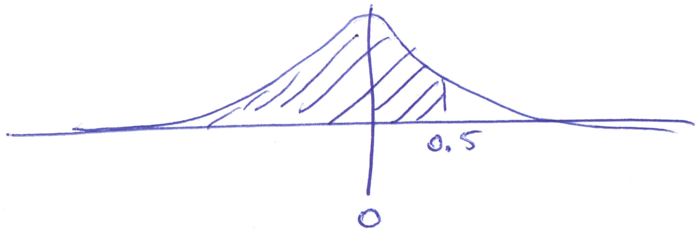
$$Var(X) = \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x) \, dx = \sigma^2$$

$$= Var(\underline{\sigma Z + \mu}) = \sigma^2 Var(Z) = \sigma^2$$

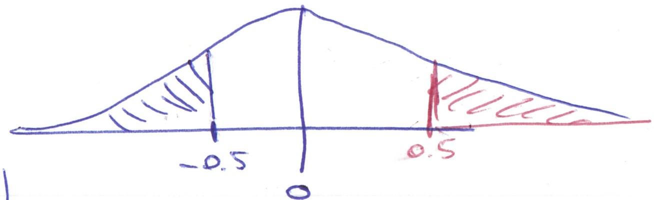
$$Var(aX+b) = a^2 Var(X)$$

$$Z \sim N(0,1)$$

$$P(Z \leq 0.5) = \int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \underline{0.6915}$$



$$P(Z \leq -0.5)$$

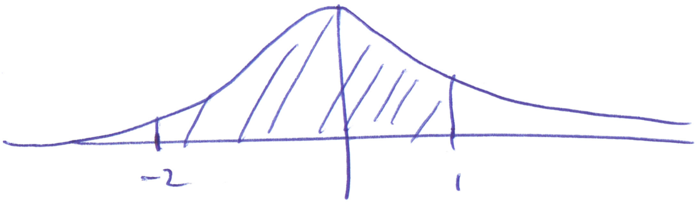


$$= 1 - \underbrace{P(Z \leq 0.5)}_{0.6915}$$

$$F_Z(-3) + F_Z(3) = 1$$

- $P(-2 < Z < 1)$ $\qquad Z \sim N(0,1)$

$= F_Z(1) - F_Z(-2)$



$= F_Z(1) - (1 - F_Z(2))$

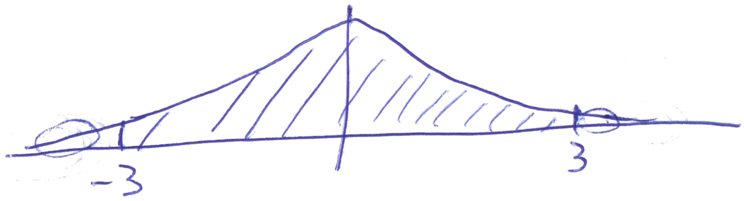
$= \underbrace{F_Z(1)}_{0.84} - 1 + \underbrace{F_Z(2)}_{0.97}$

- $P(-3 < Z < 3) = F_Z(3) - F_Z(-3)$

$= F_Z(3) - (1 - F_Z(3))$

$= 2\underbrace{F_Z(3)}_{0.9987} - 1 \simeq 0.99$

$Z \sim N(0,1)$



- $X \sim N(\mu, \sigma^2)$

$F_X(u) = P(X \le u)$? $\qquad X = \sigma Z + \mu$

$= P(\sigma Z + \mu \le u) = P\left(Z \le \frac{u-\mu}{\sigma}\right)$

$= F_Z\left(\frac{u-\mu}{\sigma}\right)$

$X \sim N(3, 5^2)$

$P(X<5) = P(\overset{5}{\sigma} Z + \overset{3}{\mu} < 5) = P\left(Z < \frac{5-3}{5}\right)$

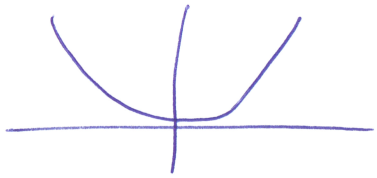
$= P(Z < 0.4) = 0.65$

$Z = \frac{X-\mu}{\sigma}$

$P(Z < 1.01)$

Chi-square

$Z \sim N(0,1) \qquad U = Z^2$



$f_U(u) = ?$

$$F_U(u) = P(U \le u) = P(Z^2 \le u)$$

$$= \begin{cases} 0 & u \le 0 \\ P(-\sqrt{u} \le Z \le \sqrt{u}) & u > 0 \end{cases}$$

$$P(-\sqrt{u} \le Z \le \sqrt{u}) = \int_{-\sqrt{u}}^{\sqrt{u}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du} \int_{\underbrace{-\sqrt{u}}_{a(u)}}^{\overbrace{\sqrt{u}}^{b(u)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

Leibnitz differentiation theorem

$$\frac{d}{du} \int_{a(u)}^{b(u)} f(z) \, dz = f(b(u)) \frac{d\, b(u)}{du} - f(a(u)) \frac{d\, a(u)}{du}$$

$$f_U(u) = \frac{1}{2} u^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}} - \left(-\frac{1}{2} u^{-1/2}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}}$$

$$= \begin{cases} \dfrac{1}{\sqrt{u}} \dfrac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}} & u > 0 \\ 0 & u \le 0 \end{cases}$$

Log-Normal

$X \sim N(\mu, \sigma^2) \qquad Y = e^X$

Y is log-normal because $\ln Y = X \sim N(\mu, \sigma^2)$

$f_Y(y) = ?$

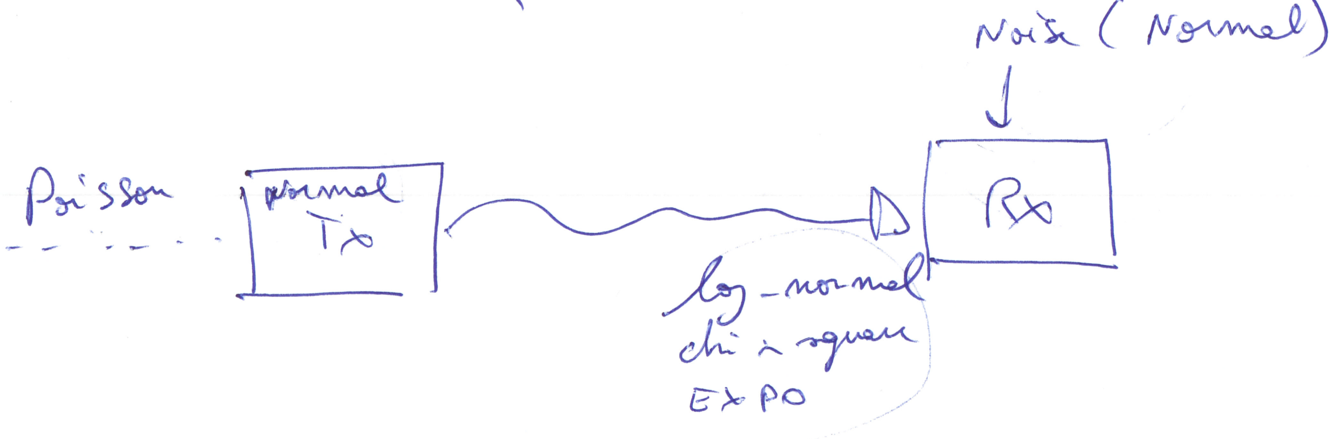
$$F_Y(y) = P(Y \leq y)$$

$$= \begin{cases} 0 & y \leq 0 \\ P(e^X \leq y) = P(X \leq \ln y) = F_X(\ln y) & y > 0 \end{cases}$$

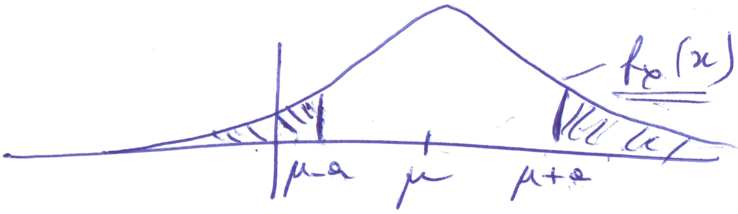
$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} f_X(\ln y)\,\frac{1}{y} & y > 0 \\ 0 & y \leq 0 \end{cases}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \begin{cases} \frac{1}{y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} & y > 0 \\ 0 & y \leq 0 \end{cases}$$



Chebyshev



$\mu = E(X)$

$\sigma^2 = Var(X)$

- $\forall a > 0, \quad P(|X-\mu| \geq a) \leq \dfrac{\sigma^2}{a^2}$

Valid for all $f_X(x)$

$\forall a > 0, \quad P(|X-\mu| < a) \geq 1 - \dfrac{\sigma^2}{a^2}$

$b = \mu$

- slightly more general version

$\forall b, \quad \forall a > 0, \quad P(|X-b| \geq a) \leq \dfrac{E((X-b)^2)}{a^2}$

- $E(X) = \mu, \quad Var(X) = 0$

$\forall a > 0, \ P(|X-\mu| \geq a) \leq 0 \implies P(X=\mu) = 1$

- $E(X^2) = 0$

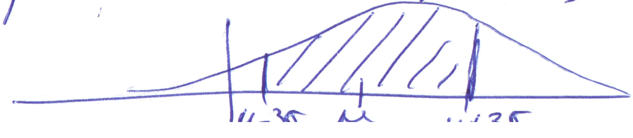
$\left.\begin{array}{l} E(X^2) = 0 \\ Var(X) = E(X^2) - E(X)^2 \geq 0 \end{array}\right\} \implies \begin{array}{l} E(X) = 0, \quad Var(X) = 0 \\ P(X=0) = 1 \end{array}$

- $a = k\sigma$

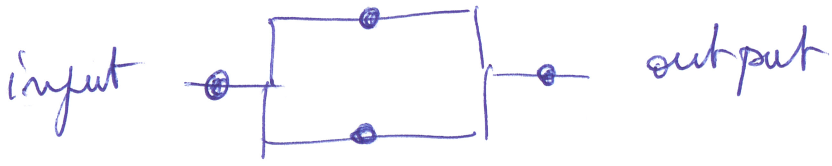
$\forall k > 0, \quad P(|X-\mu| < k\sigma) \geq 1 - \dfrac{\sigma^2}{k^2\sigma^2}$

$\forall k > 0, \quad P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \dfrac{1}{k^2}$

$k = 3 \qquad P(\mu - 3\sigma < X < \mu + 3\sigma) \geq 1 - \dfrac{1}{9} = \dfrac{8}{9}$

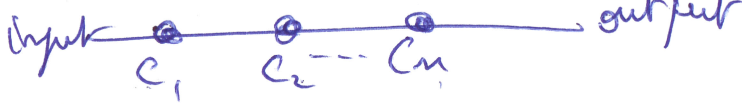


Reliability



system functions if there is a functioning path between input and output

Series



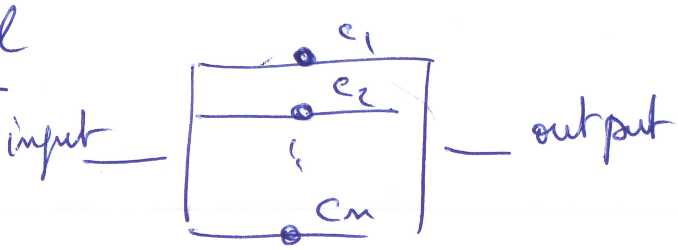
$c_1 \dots c_n$ identical, independently

"C_i" component C_i fails $P(C_i) = \theta$

"$\bar{C}_i$" " C_i functions $P(\bar{C}_i) = 1-\theta$

$$
\begin{aligned}
P(\text{system functions}) &= P(\bar{C}_1 \wedge \bar{C}_2 \wedge \dots \wedge \bar{C}_n) \quad \text{indep.} \\
&= P(\bar{C}_1)\, P(\bar{C}_2) \dots P(\bar{C}_n) \quad \text{identical} \\
&= (1-\theta)\ (1-\theta) \quad (1-\theta) \\
&= (1-\theta)^n
\end{aligned}
$$

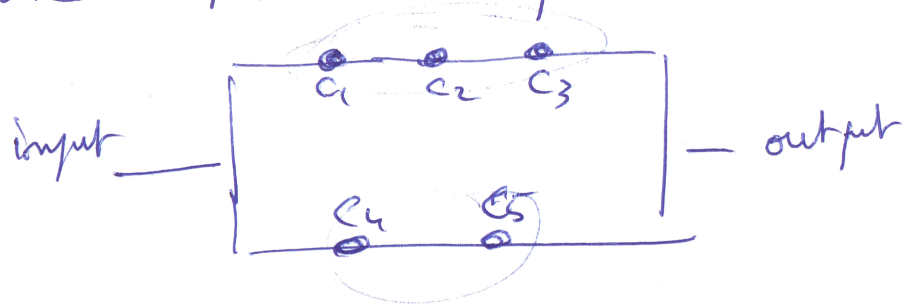
parallel

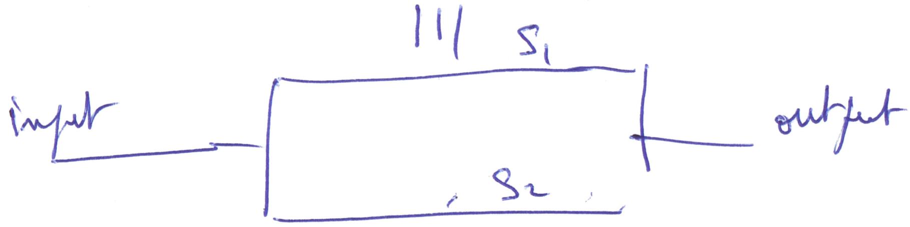


$$
\begin{aligned}
P(\text{system function}) &= P(\bar{C}_1 \cup \bar{C}_2 \cup \dots \bar{C}_n) \\
&= P(\overline{C_1 \wedge C_2 \wedge \dots \wedge C_n}) \\
&= 1 - P(C_1 \wedge C_2 \wedge \dots \wedge C_n) \\
&= 1 - P(C_1)\, P(C_2) \dots P(C_n) \quad \text{ind.} \\
&= 1 - \theta^n \quad \text{identical}
\end{aligned}
$$

$P(\bar{A}) = 1 - P(A)$

Combination of series and parallel





S_1 branch 1 fails

S_2 branch 2 fails

$$P(\text{system functions}) = P(\bar{S}_1 \cup \bar{S}_2)$$
$$= P(\overline{S_1 \cap S_2})$$
$$= 1 - P(S_1 \cap S_2)$$
$$= 1 - P(S_1)\, P(S_2)$$

$$P(S_1) = 1 - P(\bar{S}_1) = 1 - (1-\theta)^3$$
$$P(S_2) = 1 - P(\bar{S}_2) = 1 - (1-\theta)^2$$