EE9-SO27/ELEC70081
Wireless Communications and Optimisation

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Course Objectives

- Course on wireless communications and communication theory
  - Fundamentals from a 5G and 6G perspective
  - At the cross-road between information theory, coding theory, signal processing, microwave theory, and antenna/propagation theory
- Course on optimisation for wireless communications
  - Interplay between communications and optimisation to design intelligent wireless systems grounded in fundamental communication theory and optimized using modern optimization (and potentially machine learning) tools.
- Major focus of the course is on 1) fundamentals of wireless, 2) multi-antenna (MIMO) communication theory, 3) multi-user communication theory, 4) optimisation of wireless communications
  - Applications: everywhere in modern networks: 4G, 5G, 6G, WiMAX(IEEE 802.16e, IEEE 802.16m), WiFi(IEEE 802.11n), satellite, radar, sensing,...
- Valuable for those who want to either pursue a PhD in communications or a career in a high-tech telecom company (research centres, R&D branches of telecom manufacturers and operators,...).
- Skills
  - Mathematically model and analyse wireless communications systems
  - Design the fundamental building blocks (transmitters and receivers) of multi-user multi-antenna communication systems
  - Mathematically optimize wireless communication systems
  - Evaluate the performance of modern (5G and beyond) communications systems.

Content

- **Part 1: Basics of Wireless**
  - The wireless channel
  - Fading and diversity
  - Capacity of wireless channels
- **Part 2: MIMO systems**
  - Basics of MIMO
  - The MIMO channel
  - Capacity of MIMO wireless channels
  - Transmission and reception strategies
- **Part 3: Multiuser communications**
  - Capacity of multiuser downlink and uplink channels
  - Fairness, scheduling, and precoding/beamforming
  - Massive MIMO
  - Multiuser multicell communications
- **Part 4: Convex optimisation for wireless communications**
  - Convex optimisation problems (LP, QCQP, SOCP, SDP, SDR, GP, Lagrangian, KKT)
  - Applications in transmit/receive beamforming, MIMO detection, multicast beamforming, reconfigurable intelligent surfaces, power control, power allocation
- **Part 5: Future Topics**

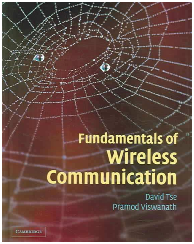
Important Information

- Course material on blackboard
- Prerequisite: For Meng (3rd year module) ELEC96008 - Communication Systems
- No Exam
- 3 courseworks (using Matlab): $20\%, 25\%, 25\%$ and 1 oral: 30%. 1) deadline: 5 February 2023, 2) deadline: 26 February 2023, 3) deadline: 21 March 2023. Oral with GTAs to be scheduled on 23-24 March 2023. Any questions on CW 1,2,3 can be asked at the oral.
  - CW1: 4G single-user link-level simulations
  - CW2: 5G multi-user Massive MIMO system-level simulations
  - CW3: 6G optimisation of multi-user reconfigurable intelligent surfaces
- Participate in the lab sessions with GTAs Yang Zhao and Hongyu Li. See the problem sheets on blackboard.
- More detailed teaching material on wireless communications available on
  - http://www.ee.ic.ac.uk/bruno.clerckx/Teaching.html
  - My youtube channel https://www.youtube.com/@prof.brunoclerckx1530/videos
  - wireless communications playlist https://www.youtube.com/playlist?list=PL3nE1Yo1b4CrAfN3lndrMImPFuS1hR5U-

References - Wireless Communications

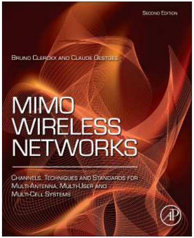
- Introductory

D. Tse and P. Viswanath, “Fundamentals of Wireless Communication,” Cambridge University Press, May 2005



- More advanced and MIMO-focused (primary reference)

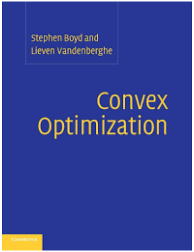
Bruno Clerckx and Claude Oestges, “MIMO Wireless Networks: Channels, Techniques and Standards for Multi-Antenna, Multi-User and Multi-Cell Systems,” Academic Press (Elsevier), Oxford, UK, Jan 2013.



References - Optimisation

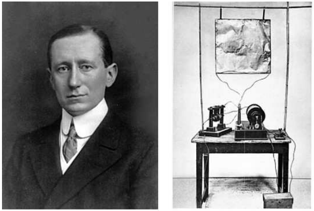
- Convex Optimization

  S. Boyd and L. Vandenberghe, “Convex Optimization,” Cambridge University Press, March 2004

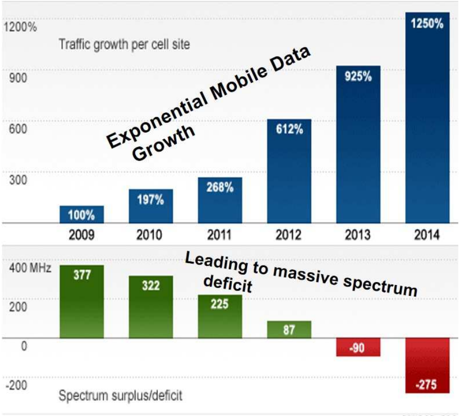


- Several papers on convex optimization for communications and signal processing

Wireless Revolution



Marconi 1896



Source: FCC

Cellular systems
1G – 1980's
2G (GSM) – 1990's
3G (UMTS) – 2000's
4G (LTE) – 2010's
5G – 2020's

Other systems
WiFi, Satellite, Bluetooth, Zigbee, ...

Challenges

Network/Radio Challenges

- Data rates
- Scarce/limited spectrum
- Reliability and coverage
- Mobility
- Energy efficiency
- Latency
- Explosion of the number of devices

Device/RF/Chip Challenges

- Performance
- Complexity
- Size, Power, Cost
- High frequencies/mmWave
- Multiple Antennas
- Multiradio Integration
- Coexistence

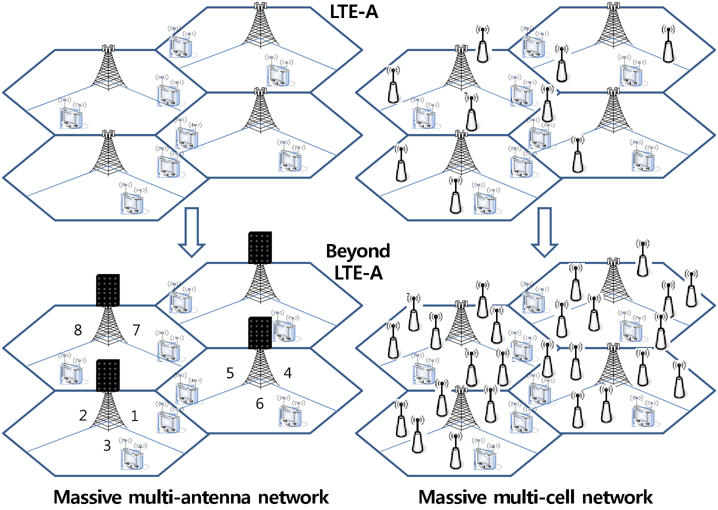
Spectrum Regulation

- Spectrum limited
- Worldwide spectrum controlled by ITU-R
- Regulation/spectrum allocation needed

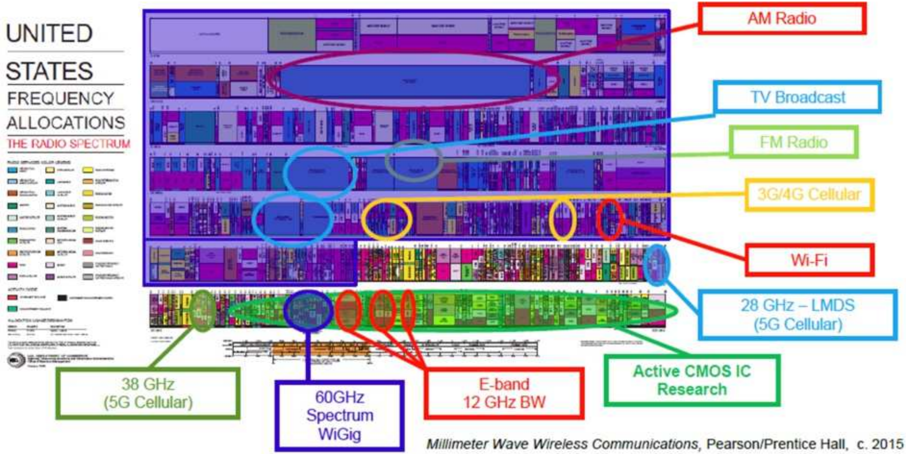
Standards

- Systems need to interact
- Companies want their systems adopted as standard
- Intellectual property (patents)
- IEEE, 3GPP

5G: more antennas, more cells



5G: higher frequencies and larger bandwidth



Millimeter Wave Wireless Communications, Pearson/Prentice Hall, c. 2015

Future Wireless Networks

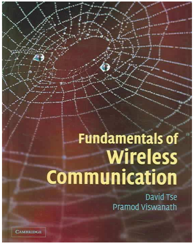
- Next-Gen Cellular/WiFi (5G, 6G, and beyond)
- Smart Homes/Spaces
- Smart Cities
- Internet of Things
- Autonomous Cars
- Body-Area Networks
- Non-terrestrial networks: satellite, drones, UAV
- Chemical/Molecular Communications
- Applications of Communications in Health, Bio-medicine, and Neuroscience
- Energy harvesting and transfer
- Integrated sensing and communications
- New/intelligent material and surfaces
- New communication and coding schemes
- New optimization and machine learning mechanisms
- ...

Part 1: Basics of Wireless

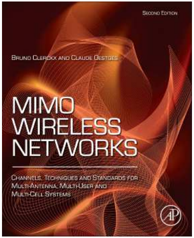
The Wireless Channel
Fading and Diversity
Capacity of Wireless Channels

Reference Book

Chapter 5 - Sections 5.2.1, 5.3.1, 5.3.2, 5.4.1, 5.4.2, 5.4.3, 5.4.5.



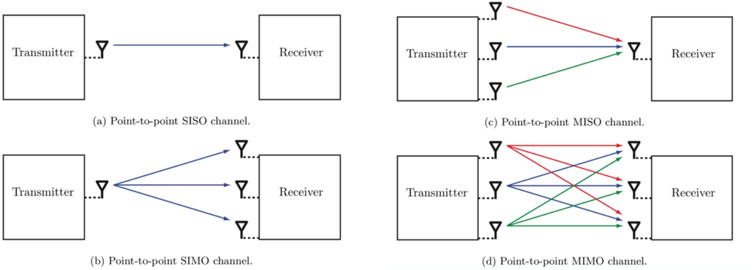
Chapter 1 - Sections 1.2, 1.3, 1.4, 1.5.



The Wireless Channel

Taxonomy

- Single-input single-output (SISO)
- Single-input multiple-output (SIMO)
- Multiple-input single-output (MISO)
- Multiple-input multiple-output (MIMO)



(a) Point-to-point SISO channel.
(c) Point-to-point MISO channel.
(b) Point-to-point SIMO channel.
(d) Point-to-point MIMO channel.

Discrete Time Representation

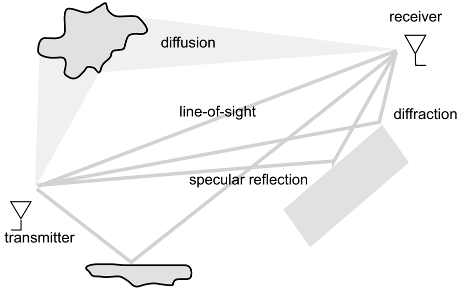
- *channel*: the impulse response of the linear time-varying communication system between one (or more) transmitter(s) and one (or more) receiver(s). Denote τ_{max} is the maximal length of the impulse response.
- Assume a SISO transmission where the digital signal is defined in discrete-time by the complex time series $\{c_l\}_{l \in \mathbb{Z}}$ and is transmitted at the symbol rate T_s.
- Transmit pulse-shaping filter, upconversion, channel, downconversion, receive pulse shaping filter (considering Nyquist criterion).
- After sampling the received signal at the symbol rate T_s and assuming $T_s >> \tau_{max}$ (frequency flat channel and narrowband transmission),

$$y_k = \sqrt{E_s}\mathsf{h}_k c_k + n_k$$

- More details https://youtu.be/SFIRXrvvXBQ

The Wireless Channel

- Multipath



- Wireless channel varies:
  - Long time scale: Large Scale Fading (Path-Loss and Shadowing)
  - Short time scale: Small Scale Fading

Large Scale Fading: Path-Loss and Shadowing

- Time constants associated with variations are very long as the mobile moves.
  - many seconds or minutes
- Important for cell site planning and rough estimate of network performance.
- Modeling:
  - Maxwell's equations: complex and impractical
  - Free space and 2-path models: (too) simple
  - Ray tracing models: requires site-specific information
  - Simplified power falloff models: good for high-level analysis
  - Empirical and Standards-based Models: not accurate, used to assess different designs

Free space model

- Path loss for unobstructed LOS path: distance R between Tx and Rx
- Friis equation: received power attenuates like $1/R^2$

$$P_r = P_t D_t D_r \left(\frac{\lambda}{4\pi R}\right)^2$$

$$P_r|_{\mathsf{dB}} = P_t|_{\mathsf{dB}} + D_t|_{\mathsf{dB}} + D_r|_{\mathsf{dB}} + 20\log_{10}\left(\frac{\lambda}{4\pi R}\right)$$

  - $|_{\mathsf{dB}}$ indicates the conversion to dB
  - D_t, D_r the transmit and receive antenna directivities
  - $R >> \lambda$ the wavelength

- Path Loss: $\Lambda_0|_{\mathsf{dB}} = 20\log_{10}\left(\frac{4\pi R}{\lambda}\right)$

Simplified Path Loss Model

- Used extensively in system design
- A real-valued deterministic attenuation term modeled as $\Lambda_0 \propto R^\eta$

$$P_r = P_t K \left(\frac{R_0}{R} \right)^\eta$$

- Important parameter: the path loss exponent η
  - Determined empirically: $2 \leq \eta \leq 8$.
- Path Loss: $\Lambda_0|_{\text{dB}} = L_0|_{\text{dB}} + 10\eta \log_{10} \left(\frac{R}{R_0} \right)$
  - L_0 is the deterministic path-loss at a reference distance R_0

- What about at higher frequencies? mmWave?
  - Less mature, on-going characterization
  - Path loss large due to frequency, rain, and oxygen
  - Limited to short range or would need massive MIMO

Shadowing

- Models attenuation from obstructions/obstacles
- Random due to random number and type of obstructions
- Modeled as a lognormal random variable S
  - $S|_{\mathsf{dB}} = 10\log_{10}(S)$ is a zero-mean normal variable of given standard deviation σ_S, i.e. $S|_{\mathsf{dB}} \sim \mathcal{N}(0, \sigma_S^2)$.
  - zero mean as mean captured in path loss
  - $4 < \sigma_S < 12$
  - Central Limit Theorem used to explain this model
- Path loss + shadowing

$$\Lambda|_{\mathsf{dB}} = \Lambda_0|_{\mathsf{dB}} + S|_{\mathsf{dB}} = L_0|_{\mathsf{dB}} + 10\eta\log_{10}\left(\frac{R}{R_0}\right) + S|_{\mathsf{dB}},$$

$$P_r|_{\mathsf{dB}} = P_t|_{\mathsf{dB}} + K|_{\mathsf{dB}} - 10\eta\log_{10}\left(\frac{R}{R_0}\right) - S|_{\mathsf{dB}}$$

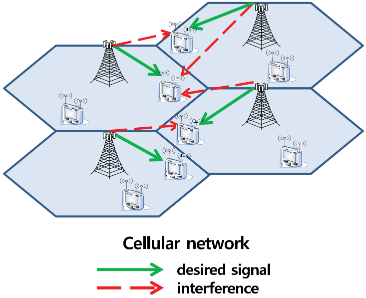
  - Λ is sometimes simply known as the path-loss.

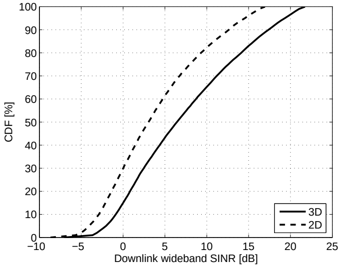
Large Scale Fading and Network Performance

- For user q in cell i, the *wideband/long-term SINR*

$$SINR_{w,q} = \frac{\Lambda_{q,i}^{-1} E_{s,i}}{\sigma_{n,q}^2 + \sum_{j \neq i} \Lambda_{q,j}^{-1} E_{s,j}}.$$

- Function of major propagation mechanisms (path loss, shadowing, antenna radiation patterns,...), base stations deployment and user distribution.
- CDF of $SINR_{w,q}$ in a frequency reuse 1 network (cells share the same frequency band) with 2D and 3D antenna patterns in urban macro deployment.





Integrating Large Scale and Small Scale Fading

- Assuming narrowband channels and given specific Tx and Rx locations, h_k is modeled as

$$\mathsf{h}_k = \frac{1}{\sqrt{\Lambda_0 \, S}} \, h_k,$$

  where

  - *path-loss* Λ_0: $\Lambda_0 \propto R^\eta$ where η.
  - *shadowing* S: Lognormal random variable, $S|_{\mathsf{dB}} \sim \mathcal{N}(0, \sigma_S^2)$.
  - *fading* h_k: caused by the combination of non coherent multipaths. By definition of Λ_0 and S, $\mathcal{E}\{ |h|^2 \} = 1$.

- Alternatively, $\mathsf{h}_k = \Lambda^{-1/2} \, h_k$ with Λ modeled on a logarithm scale

$$\Lambda|_{\mathsf{dB}} = \Lambda_0|_{\mathsf{dB}} + S|_{\mathsf{dB}} = L_0|_{\mathsf{dB}} + 10\eta \log_{10}\left(\frac{R}{R_0}\right) + S|_{\mathsf{dB}},$$

Reminder: Gaussian Random Variable

- *Real Gaussian random variable* x with mean $\mu = \mathcal{E}\left\{x\right\}$ and variance σ^2

$$p\left(x\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(x-\mu\right)^2}{2\sigma^2}\right).$$

  Standard Gaussian random variable: $\mu = 0$ and $\sigma^2 = 1$
- *Complex Gaussian random variable* $x = x_r + jx_i$: $\left[x_r, x_i\right]^T$ is a real Gaussian random vector.
- Important case: $x = x_r + jx_i$ is such that its real and imaginary parts are i.i.d. zero mean Gaussian variables of variance σ^2 (circularly symmetric complex Gaussian random variable).
- $s = |x| = \sqrt{x_r^2 + x_i^2}$ is Rayleigh distributed

$$p(s) = \frac{s}{\sigma^2} \exp\left(-\frac{s^2}{2\sigma^2}\right).$$

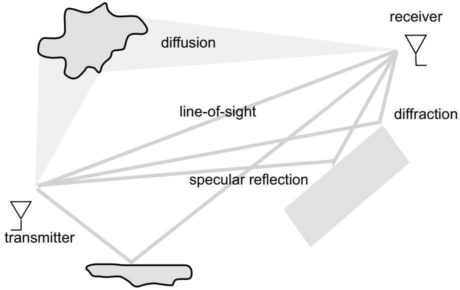
- $y = s^2 = |x|^2 = x_r^2 + x_i^2$ is χ_2^2 (i.e. exponentially) distributed (with two degrees of freedom)

$$p_y(y) = \frac{1}{2\sigma^2} \exp\left(-\frac{y}{2\sigma^2}\right).$$

  Hence, $\mu = \mathcal{E}\left\{y\right\} = 2\sigma^2$.

Small Scale Fading (or simply Fading)

- Multipaths



- Assuming that the signal reaches the receiver via a large number of paths of similar energy,
  - Central Limit Theorem used to explain this model
  - h is modeled such that its real and imaginary parts are i.i.d. zero mean Gaussian variables of variance σ^2 (circularly symmetric complex Gaussian variable).
  - Recall $\mathcal{E}\{\,|h|^2\,\} = 2\sigma^2 = 1$.

Small Scale Fading (or simply Fading)

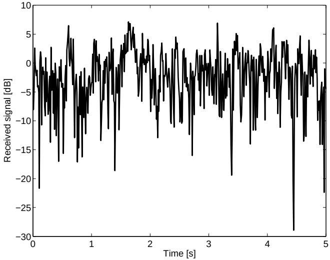
- The channel *amplitude* $s \triangleq |h|$ follows a *Rayleigh* distribution,

$$p_s(s) = \frac{s}{\sigma^2} \exp\left(-\frac{s^2}{2\sigma^2}\right),$$

whose first two moments are

$$\mathcal{E}\{s\} = \sigma\sqrt{\frac{\pi}{2}}, \qquad \mathcal{E}\{s^2\} = 2\sigma^2 = \mathcal{E}\{\,|h|^2\,\} = 1.$$

- The *phase* of h is uniformly distributed over $[0, 2\pi)$
- Typical received signal strength of a Rayleigh fading channel



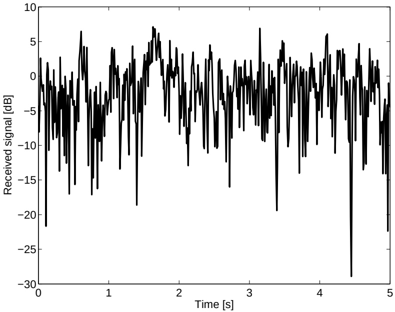
Fading and Diversity

System Model

- Path loss models are identical for both single- and multi-antenna systems.
- For point to point systems, it is common to discard the path loss and shadowing and only investigate the effect due to fading, i.e. the classical model for narrowband channels
$$y = \sqrt{E_s} hc + n,$$
where the time index is removed for better legibility and n is usually taken as white Gaussian distributed, $\mathcal{E}\{n_k n_l^*\} = \sigma_n^2 \delta(k-l)$.
- E_s can then be seen as an average received symbol energy.
- The *average SNR* is defined as $\rho \triangleq E_s/\sigma_n^2$.
- The *instantaneous SNR* is $E_s|h|^2/\sigma_n^2 = \rho|h|^2$.

Fading and Diversity

- The signal level randomly fluctuates, with some sharp declines of power and instantaneous received SNR known as *fades*.



- When the channel is in a deep fade, a reliable decoding of the transmitted signal may not be possible anymore, resulting in an error.
- How to recover the signal? Use of diversity techniques

Maximum Likelihood Detection

- Decision rule: choose the hypothesis that maximizes the conditional density

$$\arg\max_x p(y|x) = \arg\max_x \log p(y|x)$$

- If real AWGN $y = x + n$ with $n \sim N(0, \sigma_n^2)$,

$$p(y|x) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{(y-x)^2}{2\sigma_n^2}\right)$$

and

$$\arg\max_x p(y|x) = \arg\min_x (y-x)^2$$

- If $y = \sqrt{E_s}hc + n$, the ML decision rule becomes

$$\arg\min_c \left|y - \sqrt{E_s}hc\right|^2$$

Impact of Fading

- What is the impact of fading on system performance?
- Consider the simple case of BPSK transmission through an AWGN channel and a SISO Rayleigh fading channel:
  - In the absence of fading ($h = 1$), the symbol-error rate (SER) in an additive white Gaussian noise (AWGN) channel is given by
  $$\bar{P} = \mathcal{Q}\left(\sqrt{\frac{2E_s}{\sigma_n^2}}\right) = \mathcal{Q}\big(\sqrt{2\rho}\big),$$
  where $\mathcal{Q}\left(x\right)$ is the Gaussian $\mathcal{Q}$-function defined as
  $$\mathcal{Q}\left(x\right) \triangleq P\left(y \geq x\right) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy.$$
  - In the presence of (Rayleigh) fading, the received signal level fluctuates as $s\sqrt{E_s}$, and the SNR varies as ρs^2. As a result, the SER
  $$\begin{aligned} \bar{P} &= \int_0^\infty \mathcal{Q}\big(\sqrt{2\rho}s\big) p_s(s)\, ds \\ &= \frac{1}{2}\Big(1 - \sqrt{\frac{\rho}{1+\rho}}\Big) \\ &\overset{(\rho\nearrow)}{\cong} \frac{1}{4\rho} \end{aligned}$$
  although the average SNR $\bar{\rho} = \int_0^\infty \rho s^2\, p_s(s)\, ds$ remains equal to ρ.

Diversity in Multiple Antennas Wireless Systems

- How to combat the impact of fading? Use diversity techniques
- The principle of diversity is to provide the receiver with multiple versions (called diversity branch) of the same transmitted signal.
  - Independent fading conditions across branches needed.
  - Diversity stabilizes the link through channel hardening which leads to better error rate.
  - Multiple domains: time (coding and interleaving), frequency (equalization and multi-carrier modulations) and space (multiple antennas/polarizations).
- *Array Gain*: increase in average output SNR (i.e., at the input of the detector) relative to the single-branch average SNR ρ

$$g_a \triangleq \frac{\bar{\rho}_{out}}{\bar{\rho}} = \frac{\bar{\rho}_{out}}{\rho}$$

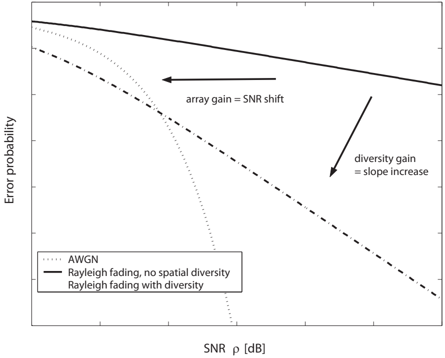
- *Diversity Gain*: increase in the error rate slope as a function of the SNR. Defined as the negative slope of the log-log plot of the average error probability $\bar{P}$ versus SNR

$$g_d^o(\rho) \triangleq -\frac{\log_2\left(\bar{P}\right)}{\log_2\left(\rho\right)}.$$

The diversity gain is commonly taken as the asymptotic slope, i.e., for $\rho \to \infty$.

Diversity in Multiple Antennas Wireless Systems

- Illustration of diversity and array gains



Careful that the curves have been plotted against the single-branch average SNR $\bar{\rho} = \rho$!
If plotted against the output average SNR $\bar{\rho}_{out}$, the array gain disappears.

SIMO Systems

- Receive diversity may be implemented via two rather different combining methods:
  - *selection combining*: the combiner selects the branch with the highest SNR among the n_r receive signals, which is then used for detection,
  - *gain combining*: the signal used for detection is a linear combination of all branches, $z = \mathbf{g}\mathbf{y}$, where $\mathbf{g} = [g_1, \ldots, g_{n_r}]$ is the combining vector.
    1. Equal Gain Combining
    2. Maximal Ratio Combining
    3. Minimum Mean Square Error Combining

- Space antennas sufficiently far apart from each other so as to experience independent fading on each branch.

- We assume that the receiver is able to acquire the perfect knowledge of the channel.

Selection Combining

- Assume that the n_r channels are independant and identically Rayleigh distributed (i.i.d.) with unit energy and that the noise levels are equal on each antenna.
- Choose the branch with the largest amplitude $s_{max} = \max\{s_1, \ldots, s_{n_r}\}$.
- The probability that s falls below a certain level S is given by its CDF

$$P[s < S] = 1 - e^{-S^2/2\sigma^2}.$$

- The probability that s_{max} falls below a certain level S is given by

$$P\left[s_{max} < S\right] = P[s_1, \ldots, s_{n_r} \leq S] = \left[1 - e^{-S^2}\right]^{n_r}.$$

- The PDF of s_{max} is then obtained by derivation of its CDF

$$p_{s_{max}}(s) = n_r\, 2s\, e^{-s^2} \left[1 - e^{-s^2}\right]^{n_r - 1}.$$

- The average SNR at the output of the combiner $\bar{\rho}_{out}$ is eventually given by

$$\bar{\rho}_{out} = \int_0^\infty \rho s^2 p_{s_{max}}(s)\, ds = \rho \sum_{n=1}^{n_r} \frac{1}{n} \overset{n_r \nearrow}{\approx} \rho \left[\gamma + \log(n_r) + \frac{1}{2n_r}\right].$$

where $\gamma \approx 0.57721566$ is Euler's constant. We observe that the array gain g_a is of the order of $\log(n_r)$.

Selection Combining

- For BPSK and a two-branch diversity, the SER as a function of the average SNR per channel ρ writes as

$$\begin{aligned}\bar{P} &= \int_0^\infty \mathcal{Q}\left(\sqrt{2\rho s}\right) p_{s_{max}}(s)\, ds \\ &= \frac{1}{2} - \sqrt{\frac{\rho}{1+\rho}} + \frac{1}{2}\sqrt{\frac{\rho}{2+\rho}} \\ &\overset{\rho \nearrow}{\cong} \frac{3}{8\rho^2}.\end{aligned}$$

  The slope of the bit error rate curve is equal to 2.

- In general, the diversity gain g_d^o of a n_r-branch selection diversity scheme is equal to n_r. Selection diversity extracts all the possible diversity out of the channel.

Equal Gain Combining

- In gain combining, the signal z used for detection is a linear combination of all branches,

$$z = \mathbf{gy} = \sum_{n=1}^{n_r} g_n y_n = \sqrt{E_s}\mathbf{gh}c + \mathbf{gn} = \sqrt{E_s}[\sum_{n=1}^{n_r} h_n g_n]c + \mathbf{gn}$$

  where
  - g_n's are the combining weights and $\mathbf{g} \triangleq [g_1, \ldots, g_{n_r}]$
  - the data symbol c is sent through the channel and received by n_r antennas
  - $\mathbf{h} \triangleq [h_1, \ldots, h_{n_r}]^T$
- Assume Rayleigh distributed channels $h_n = |h_n| e^{j\phi_n}$, $n = 1, \ldots, n_r$, with unit energy, all the channels being independent.
- *Equal Gain Combining*: fixes the weights as $g_n = e^{-j\phi_n}$.
  - Mean value of the output SNR $\bar{\rho}_{out}$ (averaged over the Rayleigh fading):

$$\bar{\rho}_{out} = \frac{E_s\mathcal{E}\left\{|\sum_{n=1}^{n_r} g_n h_n|^2\right\}}{n_r\sigma_n^2} = \frac{E_s\mathcal{E}\left\{\left[\sum_{n=1}^{n_r} |h_n|\right]^2\right\}}{n_r\sigma_n^2} = \rho\left[1 + (n_r - 1)\frac{\pi}{4}\right],$$

    where the expectation is taken over the channel statistics. The array gain grows linearly with n_r, and is therefore larger than the array gain of selection combining.
  - The diversity gain of equal gain combining is equal to n_r analogous to selection.

Reminder: Chi-Square Distribution and MGF

- χ_n^2 is the sum of the square of n i.i.d. zero-mean Gaussian random variables.
- Assume n i.i.d. zero mean complex Gaussian variables $h_1, \ldots, h_n$ (real and imaginary parts with variance σ^2). Defining $u = \sum_{k=1}^{n} |h_k|^2$, the MGF of u is given by

$$\mathcal{M}_u(\tau) = \mathcal{E}\{e^{\tau u}\} = \left[\frac{1}{1 - 2\sigma^2\tau}\right]^n,$$

Maximal Ratio Combining

- *Maximal Ratio Combining*: weights are chosen as $g_n = h_n^*$.
  - MRC maximizes the output SNR with white noise

$$\bar{\rho}_{out} = \frac{E_s \mathcal{E}\left\{\left|\sum_{n=1}^{n_r} h_n^* h_n\right|^2\right\}}{\|\mathbf{h}\|^2 \sigma_n^2} = \frac{E_s}{\sigma_n^2}\mathcal{E}\left\{\frac{\|\mathbf{h}\|^4}{\|\mathbf{h}\|^2}\right\} = \rho\mathcal{E}\left\{\|\mathbf{h}\|^2\right\} = \rho n_r.$$

    The array gain g_a is thus always equal to n_r, or equivalently, the output SNR is the sum of the SNR levels of all branches (holds true irrespective of the correlation between the branches).
  - For BPSK transmission, the symbol error rate reads as

$$\bar{P} = \int_0^\infty \mathcal{Q}\left(\sqrt{2\rho u}\right) p_u(u)\, du$$

    where $u = \|\mathbf{h}\|^2$ is χ^2 distribution with $2n_r$ degrees of freedom when the different channels are i.i.d. Rayleigh

$$p_u(u) = \frac{1}{(n_r - 1)!} u^{n_r - 1} e^{-u}.$$

    At high SNR, $\bar{P}$ becomes

$$\bar{P} = (4\rho)^{-n_r} \binom{2n_r - 1}{n_r}.$$

    The diversity gain is again equal to n_r.

Maximal Ratio Combining

– For alternative constellations, the error probability is given, assuming ML detection, by

$$\bar{P} \approx \int_0^\infty \bar{N}_e \mathcal{Q}\left(\sqrt{\frac{d_{min}^2 \rho u}{2}}\right) p_u(u)\, du,$$

$$\leq \bar{N}_e \mathcal{E}\left\{ e^{-\frac{d_{min}^2 \rho u}{4}} \right\} \qquad \text{(using Chernoff bound } \mathcal{Q}(x) \leq \exp\left(-\frac{x^2}{2}\right)\text{)}$$

where $\bar{N}_e$ and d_{min} are respectively the number of nearest neighbors and minimum distance of separation of the underlying constellation.

Since u is a χ^2 variable with $2n_r$ degrees of freedom, the above average upper-bound is given by

$$\bar{P} \leq \bar{N}_e \left(\frac{1}{1+\rho d_{min}^2/4}\right)^{n_r}$$

$$\overset{\rho\nearrow}{\leq} \bar{N}_e \left(\frac{\rho d_{min}^2}{4}\right)^{-n_r}.$$

The diversity gain g_d^o is equal to the number of receive branches in i.i.d. Rayleigh channels.

– Vector representation with normalization $\mathbf{g} = \mathbf{h}^H/||\mathbf{h}||$

$$z = \frac{\mathbf{h}^H}{||\mathbf{h}||}\mathbf{y} = \sqrt{E_s}\frac{\mathbf{h}^H}{||\mathbf{h}||}\mathbf{h}c + \frac{\mathbf{h}^H}{||\mathbf{h}||}\mathbf{n} = \sqrt{E_s}||\mathbf{h}||c + \frac{\mathbf{h}^H}{||\mathbf{h}||}\mathbf{n}$$

Minimum Mean Square Error Combining

- So far noise was white Gaussian. When the noise (and interference) is colored, MRC is not optimal anymore.
- Let us denote the combined noise plus interference signal as $\mathbf{n}_i$ such that $\mathbf{y} = \sqrt{E_s}\mathbf{h}c + \mathbf{n}_i$.
- An optimal gain combining technique is the minimum mean square error (MMSE) combining, where the weights are chosen in order to minimize the mean square error between the transmitted symbol c and the combiner output z, i.e.,

$$\mathbf{g}^{\star} = \arg\min_{\mathbf{g}} \mathcal{E}_{\mathbf{n}_i,c}\left\{ |\mathbf{g}\mathbf{y} - c|^2 \right\}.$$

- Two popular solutions:
  - The optimal weight vector $\mathbf{g}^{\star}$ is given by
  $$\mathbf{g}^{\star} = \alpha\mathbf{h}^H\mathbf{R}_{\mathbf{yy}}^{-1},$$
  where $R_{\mathbf{yy}} = \mathcal{E}\left\{\mathbf{y}\mathbf{y}^H\right\}$ is the covariance matrix of the received signal. α is a scalar.
  - The optimal weight vector $\mathbf{g}^{\star}$ is given by
  $$\mathbf{g}^{\star} = \beta\mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-1},$$
  where $\mathbf{R}_{\mathbf{n}_i} = \mathcal{E}\left\{\mathbf{n}_i\mathbf{n}_i^H\right\}$ is the covariance matrix of the combined noise plus interference signal $\mathbf{n}_i$. β is a scalar.

  Those two solutions have the same direction (more later).
- The Signal to Interference plus Noise Ratio (SINR) at the output of the MMSE combiner simply writes as $\rho_{out} = E_s\mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-1}\mathbf{h}$.

Minimum Mean Square Error Combining

- **Interpretation 1**: Solution comes from expanding the MSE and finding the minimum

$$\mathcal{E}_{n,c}\{\,|\mathbf{g}\mathbf{y}-c|^2\,\} \;=\; \mathbf{g}\underbrace{\mathcal{E}\{\mathbf{y}\mathbf{y}^H\}}_{\mathbf{R}_{yy}}\mathbf{g}^H \;-\; \mathbf{g}\underbrace{\mathcal{E}\{\mathbf{y}c^*\}}_{\mathbf{R}_{yc}}-\underbrace{\mathcal{E}\{c\mathbf{y}^H\}}_{\mathbf{R}_{cy}=\mathbf{R}_{yc}^H}\mathbf{g}^H \;+\; \underbrace{\mathcal{E}\{|c|^2\}}_{\mathbf{R}_{cc}=P_c}$$

Note that $\frac{d}{d\mathbf{g}}\mathbf{g}\mathbf{R}_{yy}\mathbf{g}^H=\mathbf{g}^*\mathbf{R}_{yy}^T$, $\frac{d}{d\mathbf{g}}\mathbf{g}\mathbf{R}_{yc}=\mathbf{R}_{yc}^T$, $\frac{d}{d\mathbf{g}}\mathbf{R}_{yc}^H\mathbf{g}^H=0$. Hence

$$\frac{d}{d\mathbf{g}}\mathcal{E}_{n,c}\{\,|\mathbf{g}\mathbf{y}-c|^2\,\}=\mathbf{g}^*\mathbf{R}_{yy}^T-\mathbf{R}_{yc}^T=0$$

leads to $\mathbf{g}=\mathbf{R}_{cy}\mathbf{R}_{yy}^{-1}=\alpha\mathbf{h}^H\mathbf{R}_{yy}^{-1}$ (with c and $\mathbf{n}_i$ independent and zero mean).

- **Interpretation 2**: *MMSE combiner can be thought of as first whitening the noise plus interference* by multiplying $\mathbf{y}$ by $\mathbf{R}_{\mathbf{n}_i}^{-1/2}$

$$\mathbf{R}_{\mathbf{n}_i}^{-1/2}\mathbf{y}=\sqrt{E_s}\;\underbrace{\mathbf{R}_{\mathbf{n}_i}^{-1/2}\mathbf{h}}_{\text{effective channel}}\;c+\;\underbrace{\mathbf{R}_{\mathbf{n}_i}^{-1/2}\mathbf{n}_i}_{\text{noise whitened}}$$

and then match filter the effective channel $\mathbf{R}_{\mathbf{n}_i}^{-1/2}\mathbf{h}$ using $\mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-H/2}$

$$\mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-H/2}\mathbf{R}_{\mathbf{n}_i}^{-1/2}\mathbf{y}=\sqrt{E_s}\,\underbrace{\mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-H/2}\mathbf{R}_{\mathbf{n}_i}^{-1/2}}_{\text{MMSE combiner}}\mathbf{h}c+\mathbf{h}^H\mathbf{R}_{\mathbf{n}_i}^{-H/2}\tilde{\mathbf{n}}_i$$

- In the absence of interference and the presence of white noise, MMSE combiner reduces to MRC filter up to a scaling factor.

Minimum Mean Square Error Combining

Example

Question: Assume a transmission of a signal c from a single antenna transmitter to a multi-antenna receiver through a SIMO channel $\mathbf{h}$. The transmission is subject to the interference from another transmitter sending signal x through the interfering SIMO channel $\mathbf{h}_i$.
The received signal model writes as

$$\mathbf{y} = \mathbf{h}c + \mathbf{h}_i x + \mathbf{n}$$

where $\mathbf{n}$ is the zero mean complex additive white Gaussian noise (AWGN) vector with $\mathcal{E}\{\mathbf{n}\mathbf{n}^H\} = \sigma_n^2 \mathbf{I}_{n_r}$.
We apply a combiner $\mathbf{g}$ at the receiver to obtain the observation $z = \mathbf{g}\mathbf{y}$.
Derive the expression of the MMSE combiner and the SINR at the output of the combiner.

Minimum Mean Square Error Combining

Example

Answer: The MMSE combiner g is given by

$$\mathbf{g} = \mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1}$$

where $\mathbf{R}_{\mathbf{n}_i} = \mathcal{E}\left\{\mathbf{n}_i \mathbf{n}_i^H\right\}$ with $\mathbf{n}_i = \mathbf{h}_i x + \mathbf{n}$.
Hence $\mathbf{R}_{\mathbf{n}_i} = \mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}$ with $P_x = \mathcal{E}\{\,|x|^2\,\}$, the power of the interfering signal.
Hence,

$$\mathbf{g} = \mathbf{h}^H \left(\mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}\right)^{-1}.$$

At the receiver, we obtain

$$z = \mathbf{g}\mathbf{y} = \mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h} c + \mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{n}_i.$$

Minimum Mean Square Error Combining

Example

Answer: The output SINR writes

$$
\begin{aligned}
\rho_{out} &= \frac{\left|\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}\right|^2 P_c}{\mathcal{E}\left\{\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{n}_i \left(\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{n}_i\right)^H\right\}} \\
&= \frac{\left|\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}\right|^2 P_c}{\mathcal{E}\left\{\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{n}_i \mathbf{n}_i^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}\right\}} \\
&= \frac{\left|\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}\right|^2 P_c}{\mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h}} \\
&= \mathbf{h}^H \mathbf{R}_{\mathbf{n}_i}^{-1} \mathbf{h} P_c \\
&= P_c \mathbf{h}^H \left(\mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}\right)^{-1} \mathbf{h} \\
&= \mathsf{SNR}\, \mathbf{h}^H \left(\mathsf{INR}\, \mathbf{h}_i \mathbf{h}_i^H + \mathbf{I}_{n_r}\right)^{-1} \mathbf{h}
\end{aligned}
$$

with $P_c = \mathcal{E}\{\, |c|^2 \,\}$, SNR $= P_c/\sigma_n^2$ (the average SNR), INR $= P_x/\sigma_n^2$ (the average INR - Interference to Noise Ratio). □

Minimum Mean Square Error Combining

Example

Answer: We could have used the other MMSE combiner expression

$$\mathbf{g} = \mathbf{h}^H \mathbf{R}_{yy}^{-1}$$

where $\mathbf{R}_{yy} = \mathcal{E}\left\{\mathbf{y}\mathbf{y}^H\right\}$.
Hence $\mathbf{R}_{yy} = \mathbf{h} P_c \mathbf{h}^H + \mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}$.
Hence,

$$\mathbf{g} = \mathbf{h}^H \left(\mathbf{h} P_c \mathbf{h}^H + \mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}\right)^{-1}.$$

Interestingly,

$$\frac{\mathbf{h}^H \left(\mathbf{h} P_c \mathbf{h}^H + \mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}\right)^{-1}}{||\mathbf{h}^H \left(\mathbf{h} P_c \mathbf{h}^H + \mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}\right)^{-1}||} = \frac{\mathbf{h}^H \left(\mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}\right)^{-1}}{||\mathbf{h}^H \left(\mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}\right)^{-1}||}$$

from matrix inversion lemma (see optimization part for a proof)

Minimum Mean Square Error Combining

- **Interpretation 3**: *MMSE combiner maximizes the SINR*
  - Assume again $\mathbf{y} = \mathbf{h}c + \mathbf{h}_i x + \mathbf{n}$ as in previous example, SINR after combining

$$\begin{aligned}\mathrm{SINR} &= \frac{|\mathbf{gh}|^2 P_c}{|\mathbf{gh}_i|^2 P_x + ||\mathbf{g}||^2 \sigma_n^2} \\ &= \frac{\mathbf{gh} P_c \mathbf{h}^H \mathbf{g}^H}{\mathbf{g}(\mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2)\mathbf{g}^H}\end{aligned}$$

  - This is a Rayleigh quotient

$$\arg\max_{\mathbf{g}} \mathrm{SINR} = \beta \mathbf{h}^H \left(\mathbf{h}_i P_x \mathbf{h}_i^H + \sigma_n^2 \mathbf{I}_{n_r}\right)^{-1}$$

  More details in the optimization part. Note that beta does not influence SINR.

MISO Systems

- MISO systems exploit diversity at the transmitter through the use of n_t transmit antennas in combination with pre-processing or precoding.

- A significant difference with receive diversity is that the transmitter might not have the knowledge of the MISO channel.
  - At the receiver, the channel is easily estimated.
  - At the transmit side, feedback from the receiver is required to inform the transmitter.

- There are basically two different ways of achieving *direct transmit diversity*:
  - when Tx has a *perfect channel knowledge*, beamforming can be performed to achieve both diversity and array gains,
  - when Tx has a *partial or no channel knowledge of the channel*, space-time coding is used to achieve a diversity gain (but no array gain in the absence of any channel knowledge).

- *Indirect transmit diversity* techniques convert spatial diversity to time or frequency diversity.

Transmit Diversity via Matched Beamforming

- The actual transmitted signal is a vector $\mathbf{x}$ that results from the multiplication of the signal c by a weight vector $\mathbf{w}$.
- At the receiver, the signal reads as

$$y = \sqrt{E_s}\mathbf{h}\mathbf{x} + n = \sqrt{E_s}\mathbf{h}\mathbf{w}c + n,$$

  where $\mathbf{h} \triangleq [h_1, \ldots, h_{n_t}]$ represents the MISO channel vector, and $\mathbf{w}$ is also known as the precoder.
- The choice that maximizes the receive SNR is given by

$$\mathbf{w} = \frac{\mathbf{h}^H}{\|\mathbf{h}\|}.$$

- Transmit along the direction of the matched channel. Hence also known as *matched beamforming* (MBF) or *transmit MRC* or *maximum ratio transmission (MRT)*.
- The array gain is equal to the number of transmit antennas, i.e. $\bar{\rho}_{out} = n_t \rho$.
- The diversity gain equal to n_t as the symbol error rate is upper-bounded at high SNR by

$$\bar{P} \leq \bar{N}_e \left(\frac{\rho d_{min}^2}{4} \right)^{-n_t}.$$

- Matched beamforming presents the same performance as receive MRC, but *requires a perfect transmit channel knowledge*.

Transmit Diversity via Space-Time Coding

- *Alamouti scheme* is an ingenious transmit diversity scheme for two transmit antennas which does not require transmit channel knowledge.
  - Assume that the flat fading channel remains constant over the two successive symbol periods, and is denoted by $\mathbf{h} = [h_1 \ h_2]$.
  - Two symbols c_1 and c_2 are transmitted simultaneously from antennas 1 and 2 during the first symbol period, followed by symbols $-c_2^*$ and c_1^*, transmitted from antennas 1 and 2 during the next symbol period:

$$y_1 = \sqrt{E_s} h_1 \frac{c_1}{\sqrt{2}} + \sqrt{E_s} h_2 \frac{c_2}{\sqrt{2}} + n_1, \qquad \text{(first symbol period)}$$

$$y_2 = -\sqrt{E_s} h_1 \frac{c_2^*}{\sqrt{2}} + \sqrt{E_s} h_2 \frac{c_1^*}{\sqrt{2}} + n_2. \qquad \text{(second symbol period)}$$

  The two symbols are spread over two antennas and over two symbol periods.
  - Equivalently

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \sqrt{E_s} \underbrace{\begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}}_{\mathbf{H}_{eff}} \underbrace{\begin{bmatrix} c_1/\sqrt{2} \\ c_2/\sqrt{2} \end{bmatrix}}_{\mathbf{c}} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}.$$

  - Applying the matched filter $\mathbf{H}_{eff}^H$ to the received vector $\mathbf{y}$ effectively decouples the transmitted symbols as shown below

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{H}_{eff}^H \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \sqrt{E_s} \left[|h_1|^2 + |h_2|^2 \right] \mathbf{I}_2 \begin{bmatrix} c_1/\sqrt{2} \\ c_2/\sqrt{2} \end{bmatrix} + \mathbf{H}_{eff}^H \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}$$

Transmit Diversity via Space-Time Coding

– The mean output SNR (averaged over the channel statistics) is thus equal to

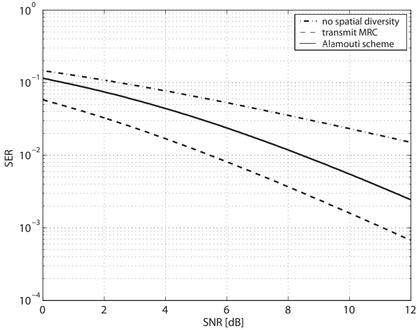
$$\bar{\rho}_{out} = \frac{E_s}{\sigma_n^2}\,\mathcal{E}\left\{\frac{\left[\,\|\mathbf{h}\|^2\,\right]^2}{2\,\|\mathbf{h}\|^2}\right\} = \rho.$$

No array gain owing to the lack of transmit channel knowledge.

– The average symbol error rate at high SNR can be upper-bounded according to

$$\bar{P} \le \bar{N}_e\left(\frac{\rho d_{min}^2}{8}\right)^{-2}.$$

The diversity gain is equal to $n_t = 2$ despite the lack of transmit channel knowledge.



Transmit MRC vs. Alamouti with 2 transmit antennas in i.i.d. Rayleigh fading channels (BPSK).

Observations:

– At high SNR, any increase in the SNR by 10dB leads to a decrease of SER by 10^{-n} for diversity order n.

- Alamouti, transmit MRC: 2
- No spatial diversity: 1

– Transmit MRC has 3 dB gain over Alamouti

Indirect Transmit Diversity

- It is also possible to convert spatial diversity to time or frequency diversity, which are then exploited using well-known SISO techniques.

- Assume that $n_t = 2$ and that the signal on the second transmit branch is
  - either delayed by one symbol period: the spatial diversity is converted into *frequency diversity* (delay diversity)
  - either phase-rotated: the spatial diversity is converted into *time diversity*
  - The effective SISO channel resulting from the addition of the two branches seen by the receiver now fades over frequency or time. This selective fading can be exploited by conventional diversity techniques, e.g. FEC/interleaving.

Capacity of Wireless Channels

AWGN Channel

- Real discrete-time AWGN channel

$$Y = X + N, \qquad N \sim N(0, \sigma^2)$$

where X is power-constrained input $\mathcal{E}\left\{X^2\right\} \leq E_s$

- The channel transition density is given by

$$p(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right)$$

Entropy

- Entropy is a measure of the average uncertainty of a random variable

Definition

For a continuous random variable X, the (differential) entropy $h(X)$ is defined as

$$h(X) = \mathcal{E}\left\{\log_2 \frac{1}{p(x)}\right\} = -\mathcal{E}\left\{\log_2 p(x)\right\} = -\int p(x) \log_2 p(x) dx,$$

where $p(x)$ is the probability density function of X.

Example

For $X \sim N(\mu, \sigma^2)$, $-\log_2 p(x) = \frac{(x-\mu)^2}{2\sigma^2} \log_2(e) + \frac{1}{2}\log_2(2\pi\sigma^2)$. Thus, $h(X) = -\mathcal{E}\left\{\log_2 p(x)\right\} = \frac{1}{2}\log_2(e) + \frac{1}{2}\log_2(2\pi\sigma^2) = \frac{1}{2}\log_2(2\pi e\sigma^2)$. The mean does not affect the differential entropy.

Theorem

Consider a RV with zero mean and variance σ^2. Then $h(X) \leq \frac{1}{2}\log_2(2\pi e\sigma^2)$, with equality iff $X \sim N(0, \sigma^2)$.

Mutual Information and Channel Capacity

- Mutual information $I(X;Y)$ is a measure of the amount of information that one RV contains about another RV. It is a measure of the dependence between the two RVs.
- $I(X;Y)$ is the reduction in the uncertainty of one random variable due to the knowledge of the other

$$I(X;Y) = h(Y) - h(Y|X)$$

  with $h(Y)$ the differential entropy and $h(Y|X)$ the conditional differential entropy
- Channel capacity is the highest information rate (in units of information per unit time) that can be achieved with arbitrarily small error probability.
  - Transmitting at a rate higher than the channel capacity leads to errors

Definition

Channel capacity

$$C = \max_{p(x)} I(X;Y)$$

is the maximum of the mutual information between the input and output of the channel, where the maximization is with respect to the input distribution.

AWGN Channel Capacity

Theorem

The capacity of the real AWGN channel is

$$C = \max_{p(x):\mathcal{E}\left\{X^2\right\}\leq E_s} I(X;Y) = \frac{1}{2}\log_2(1+\frac{E_s}{\sigma^2}).$$

Proof: Consider $Y = X + N$, with $N \sim N(0,\sigma^2)$ and $\mathcal{E}\left\{X^2\right\} \leq E_s$. Given $X = x$, $h(Y|X=x) = h(N)$, so that $h(Y|X) = h(N)$ and

$$I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(N).$$

Maximizing $I(X;Y)$ comes to maximize $h(Y)$. Since X and N are independent, $\mathcal{E}\left\{Y^2\right\} = \mathcal{E}\left\{X^2\right\} + \mathcal{E}\left\{N^2\right\} \leq E_s + \sigma^2$. We now know that

$$h(Y) \leq \frac{1}{2}\log_2(2\pi e(E_s+\sigma^2))$$

and equality is achieved iff $Y \sim N(0, E_s+\sigma^2)$. $Y \sim N(0,E_s+\sigma^2)$ is achieved if the input distribution is $X \sim N(0,E_s)$, independent of the noise. We then get

$$I(X;Y) = h(Y) - h(N) = \frac{1}{2}\log_2(2\pi e(E_s+\sigma^2)) - \frac{1}{2}\log_2(2\pi e\sigma^2) = \frac{1}{2}\log_2(1+\frac{E_s}{\sigma^2}).$$

SISO System Model

- A single-user SISO system with one transmit and one receive antennas over a frequency flat-fading channel.
- The transmit and received signals are related by

$$y_k = \sqrt{E_s} h_k c_k + n_k$$

  where
  - y_k is the received signal,
  - h_k is the complex channel,
  - n_k is a zero mean complex additive white Gaussian noise (AWGN) $\mathcal{CN}(0, \sigma_n^2)$, with $\mathcal{E}\{n_k n_l\} = \sigma_n^2$ if $k = l$ and 0 otherwise.
  - $\rho = E_s / \sigma_n^2$ represents the average SNR.
- Power constraint: $\mathcal{E}\left\{|c_k|^2\right\} \leq 1$.
- Channel time variation: T_{coh} coherence time
  - *slow fading*: T_{coh} is so long that coding is performed over a single channel realization.
  - *fast fading*: T_{coh} is so short that coding over multiple channel realizations is possible.

Capacity of Deterministic SISO Channel

Definition

The capacity of an AWGN channel is

$$C = \log_2 (1 + SNR) \quad [bits/s/H].$$

Definition

The capacity of a deterministic (time-invariant) SISO channel h is

$$C = \log_2 \left(1 + \rho |h|^2\right) \quad [bits/s/H].$$

Instantaneous SNR is $\rho |h|^2$.

Capacity of Deterministic SISO Channel

- Low SNR: $\log_2(1+x) \approx x \log_2(e)$ for x small

$$C \approx \rho |h|^2 \log_2(e).$$

  - C grows linearly with SNR at low SNR.
  - Double SNR and C is doubled.

- High SNR: $\log_2(1+x) \approx \log_2(x)$ for x large

$$C \approx \log_2\left(\rho |h|^2\right) = \log_2(\rho) + \log_2\left(|h|^2\right).$$

  - C grows logarithmically with SNR (or linearly with SNR in dB) at high SNR.
  - Double SNR (or 3dB increase) and C is increased by 1 bits/s/Hz.

Capacity of Deterministic SIMO Channel

- System model

$$\mathbf{y}_k = \sqrt{E_s}\mathbf{h}_k c_k + n_k$$

  - $\mathbf{y}_k$, $\mathbf{h}_k$ n_r-dimensional vectors

- MRC combining

$$\frac{\mathbf{h}_k^H}{\mathbf{h}_k}\mathbf{y}_k = \sqrt{E_s}\frac{\mathbf{h}_k^H}{\mathbf{h}_k}\mathbf{h}_k c_k + \frac{\mathbf{h}_k^H}{\mathbf{h}_k} n_k$$

Instantaneous SNR $\rho \|\mathbf{h}_k\|^2$.

Definition

The capacity of a deterministic (time-invariant) SIMO channel $\mathbf{h}$ is

$$C = \log_2\left(1 + \rho \|\mathbf{h}\|^2\right) \quad [bits/s/H].$$

Capacity of Deterministic MISO Channel

- System model using MRT $\mathbf{w}_k c_k$ with $\mathbf{w}_k = \frac{\mathbf{h}_k}{\|\mathbf{h}_k\|}$

$$y_k = \sqrt{E_s}\mathbf{h}_k\mathbf{w}_k c_k + n_k$$

Instantaneous SNR $\rho \|\mathbf{h}_k\|^2$.

Definition

The capacity of a deterministic (time-invariant) MISO channel $\mathbf{h}$ is

$$C = \log_2\left(1 + \rho\|\mathbf{h}\|^2\right) \quad [bits/s/H]$$

With perfect transmit channel knowledge!

Ergodic Capacity of Fast Fading Channels

- Fast fading:
  - Doppler frequency sufficiently high to allow for coding over many channel realizations/coherence time periods
  - The transmission capability is represented by a single quantity known as the ergodic capacity

- Rate of information flow between Tx and Rx at time instant k over channels h_k

$$\log_2\left(1+\rho|h_k|^2\right).$$

Such a rate varies over time according to the channel fluctuations. The average rate of information flow over a time duration $T >> T_{coh}$ is

$$\frac{1}{T}\sum_{k=0}^{T-1}\log_2\left(1+\rho|h_k|^2\right).$$

Definition

The ergodic capacity of fast-fading channel is given by

$$\bar{C}=\mathcal{E}\left\{\log_2\left(1+\rho|h|^2\right)\right\}.$$

Impact on Coding Strategy

- Perfect Transmit Channel Knowledge
  - Use a variable-rate code (family of codes of different rates) adapted as a function of the channel state.
  - Code for state h achieves the capacity $\log_2\left(1+\rho|h|^2\right)$.
  - Average throughput $\mathcal{E}\left\{\log_2\left(1+\rho|h|^2\right)\right\}$.
  - No need for the codeword to span many coherence time periods.
  - Code average out the effect of noise

- Partial Transmit Channel Knowledge (only channel distribution known)
  - Same average throughput $\mathcal{E}\left\{\log_2\left(1+\rho|h|^2\right)\right\}$ achievable.
  - Encoding requires a fixed-rate code (whose rate is given by the ergodic capacity) with encoding spanning many channel realizations.
  - Code length large enough $T >> T_c$ to average out both the noise and channel fluctuations.

Outage Capacity and Probability in Slow Fading Channels

- In slow fading, the encoding still averages out the randomness of the noise but cannot fully average out the randomness of the channel.
- For a given channel realization h and a target rate R, reliable transmission if

$$\log_2\left(1+\rho|h|^2\right) > R$$

  If not met, an outage occurs and the decoding error probability is strictly non-zero.
- Look at the tail probability of $\log_2\left(1+\rho|h|^2\right)$, not its average!

Definition

The outage probability $P_{out}\left(R\right)$ of a wireless channel with a target rate R is given by

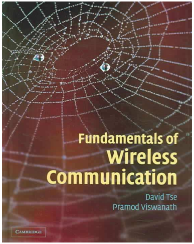
$$P_{out}\left(R\right) = P\left(\log_2\left(1+\rho|h|^2\right) < R\right).$$

- An outage occurs when channel is too weak: coding can average out the noise but cannot average out the channel fade.
- More meaningful in the absence of CSI knowledge at the transmitter: the transmitter cannot adjust its transmit strategy $\rightarrow$ hopes the channel is good enough
- More details on https://youtu.be/LS5EbFXSvfE

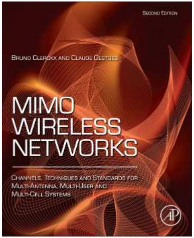
Part 2: MIMO Systems

Basics of MIMO
The MIMO Channel
Capacity of MIMO Channels
Tx and Rx Strategies

Reference Book



Chapter 1 - Sections 1.2.4, 1.3.2, 1.6
Chapter 2 - Sections 2.2.1, 2.3.1
Chapter 3 - Sections 3.2.1, 3.4.1
Chapter 5 - Sections 5.2, 5.3, 5.4.2, 5.5.1, 5.7, 5.8.1
Chapter 6 - Sections 6.1, 6.2, 6.3.1, 6.5.2, 6.5.4, 6.5.8



Basics of MIMO

Previous Lectures

- Discrete Time Representation
  - SISO: $y = \sqrt{E_s} hc + n$
  - SIMO: $\mathbf{y} = \sqrt{E_s}\mathbf{h}c + \mathbf{n}$
  - MISO (with perfect CSIT): $y = \sqrt{E_s}\mathbf{h}\mathbf{w}c + n$
- h is fading
  - amplitude Rayleigh distributed
  - phase uniformly distributed
- Diversity
  - Diversity gain: $g_d^o(\rho) \triangleq -\frac{\log_2(\bar{P})}{\log_2(\rho)}$
  - Array gain: $g_a \triangleq \frac{\bar{\rho}_{out}}{\bar{\rho}} = \frac{\bar{\rho}_{out}}{\rho}$
- SIMO
  - selection combining
  - gain combining
- MISO
  - with perfect channel knowledge at Tx: Matched Beamforming
  - without channel knowledge at Tx: Space-Time Coding (Alamouti Scheme), indirect (time, frequency) transmit diversity

MIMO Systems

- In MIMO systems, the fading channel between each transmit-receive antenna pair can be modeled as a SISO channel.
- For uni-polarized antennas and small inter-element spacings (of the order of the wavelength), path loss and shadowing of all SISO channels are identical.
- Stacking all inputs and outputs in vectors $\mathbf{c}_k = [c_{1,k}, \ldots, c_{n_t,k}]^T$ and $\mathbf{y}_k = [y_{1,k}, \ldots, y_{n_r,k}]^T$, the input-output relationship at any given time instant k reads as

$$\mathbf{y}_k = \sqrt{E_s}\mathbf{H}_k\mathbf{x}_k + \mathbf{n}_k,$$

  where
  - $\mathbf{x}_k$ is a precoded version of $\mathbf{c}_k$ that depends on the channel knowledge at the Tx.
  - $\mathbf{H}_k$ is defined as the $n_r \times n_t$ MIMO channel matrix, $\mathbf{H}_k(n, m) = h_{nm,k}$, with h_{nm} denoting the narrowband channel between transmit antenna m $(m = 1, \ldots, n_t)$ and receive antenna n $(n = 1, \ldots, n_r)$,
  - $\mathbf{n}_k = [n_{1,k}, \ldots, n_{n_r,k}]^T$ is the sampled noise vector, containing the noise contribution at each receive antenna, such that the noise is white in both time and spatial dimensions, $\mathcal{E}\{\mathbf{n}_k\mathbf{n}_l^H\} = \sigma_n^2\mathbf{I}_{n_r}\delta(k-l)$.
- Using the same channels normalization as for SISO channels, $\mathcal{E}\{\|\mathbf{H}\|_F^2\} = n_t n_r$.
- when Tx has a *perfect channel knowledge*: (dominant and multiple) eigenmode transmission
- when Tx has *no knowledge of the channel*: space-time coding (with $\mathbf{x}_k = \mathbf{c}_k$)

Reminder: Linear Algebra

- *Vector Orthogonality*: $\mathbf{a}^H\mathbf{b} = 0$ (^{H} stands for Hermitian, i.e. conjugate transpose)
- *Hermitian matrix*: $\mathbf{A} = \mathbf{A}^H$
- *Unitary matrix*: $\mathbf{A}^H\mathbf{A} = \mathbf{I}$
- *Singular Value Decomposition* (SVD) of a matrix $\mathbf{H}$ $[n_r \times n_t]$: $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$
  - $\mathbf{U}$ $[n_r \times r(\mathbf{H})]$: unitary matrix of left singular vectors
  - $\mathbf{\Sigma} = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{r(\mathbf{H})}\}$: diagonal matrix containing the singular values of $\mathbf{H}$
  - $\mathbf{V}$ $[n_t \times r(\mathbf{H})]$: unitary matrix of left singular vectors
  - $r(\mathbf{H})$: the rank of $\mathbf{H}$

  We will often look at Hermitian matrices of the form $\mathbf{A} = \mathbf{H}^H\mathbf{H}$ whose *Eigenvalue Value Decomposition* (EVD) writes as $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$ with $\mathbf{\Lambda} = \mathbf{\Sigma}^2$.
- $\mathbf{A} = \mathbf{H}^H\mathbf{H}$ is a positive-semidefinite matrix (≥ 0), i.e. all eigenvalues of $\mathbf{A}$ are nonnegative.
- *Trace* of a matrix $\mathbf{A}$: $\text{Tr}\{\mathbf{A}\} = \sum_i \mathbf{A}(i,i)$.
- *Frobenius norm* of a matrix $\mathbf{A}$: $\|\mathbf{A}\|_F^2 = \sum_i \sum_j |A(i,j)|^2$
- $\|\mathbf{A}\|_F^2 = \text{Tr}\left\{\mathbf{A}\mathbf{A}^H\right\} = \text{Tr}\left\{\mathbf{A}^H\mathbf{A}\right\}$
- $\text{Tr}\{\mathbf{A}\mathbf{B}\} = \text{Tr}\{\mathbf{B}\mathbf{A}\}$
- $\det(\mathbf{I} + \mathbf{A}\mathbf{B}) = \det(\mathbf{I} + \mathbf{B}\mathbf{A})$

Reminder: Linear Algebra and Matrix Properties

- *Kronecker product*: $\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} \mathbf{A}(1,1)\mathbf{B} & \dots & \mathbf{A}(1,n)\mathbf{B} \\ \vdots & \dots & \vdots, \\ \mathbf{A}(m,1)\mathbf{B} & \dots & \mathbf{A}(m,n)\mathbf{B} \end{bmatrix}$
- $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$
- $(\mathbf{A} \otimes \mathbf{B})^H = \mathbf{A}^H \otimes \mathbf{B}^H$
- $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC} \otimes \mathbf{BD})$
- $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$ if $\mathbf{A}, \mathbf{B}$ square and non singular.
- $\det(\mathbf{A}_{m\times m} \otimes \mathbf{B}_{n\times n}) = \det(\mathbf{A})^n \det(\mathbf{B})^m$
- $\mathrm{Tr}\{\mathbf{A} \otimes \mathbf{B}\} = \mathrm{Tr}\{\mathbf{A}\}\,\mathrm{Tr}\{\mathbf{B}\}$
- $\mathrm{vec}(\mathbf{A})$ converts $[m \times n]$ matrix into $mn \times 1$ vector by stacking the columns of $\mathbf{A}$ on top of one another.
  - $\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})\,\mathrm{vec}(\mathbf{B})$
- $\mathrm{Tr}\{\mathbf{ABB}^H\mathbf{A}^H\} = \mathrm{vec}(\mathbf{A}^H)^H (\mathbf{I} \otimes \mathbf{BB}^H)\,\mathrm{vec}(\mathbf{A}^H)$
- $\det(\mathbf{I} + \epsilon\mathbf{A}) = 1 + \epsilon\mathrm{Tr}\{\mathbf{A}\}$ if $\epsilon << 1$

Space-Time Coding

- MIMO without Transmit Channel Knowledge
- Array/diversity gains are exploitable in SIMO, MISO and ... MIMO
- *Alamouti scheme* can easily be applied to 2×2 MIMO channels

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

- Received signal vector (make sure the channel remains constant over two symbol periods!)

$$\mathbf{y}_1 = \sqrt{E_s}\mathbf{H} \begin{bmatrix} c_1/\sqrt{2} \\ c_2/\sqrt{2} \end{bmatrix} + \mathbf{n}_1, \qquad \text{(first symbol period)}$$

$$\mathbf{y}_2 = \sqrt{E_s}\mathbf{H} \begin{bmatrix} -c_2^*/\sqrt{2} \\ c_1^*/\sqrt{2} \end{bmatrix} + \mathbf{n}_2. \qquad \text{(second symbol period)}$$

- Equivalently

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2^* \end{bmatrix} = \sqrt{E_s} \underbrace{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{12}^* & -h_{11}^* \\ h_{22}^* & -h_{21}^* \end{bmatrix}}_{\mathbf{H}_{eff}} \underbrace{\begin{bmatrix} c_1/\sqrt{2} \\ c_2/\sqrt{2} \end{bmatrix}}_{\mathbf{c}} + \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2^* \end{bmatrix}.$$

Space-Time Coding

- Apply the matched filter $\mathbf{H}_{eff}^{H}$ to $\mathbf{y}$ ($\mathbf{H}_{eff}^{H}\mathbf{H}_{eff} = \|\mathbf{H}\|_F^2\, \mathbf{I}_2$)

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \sqrt{E_s}\mathbf{H}_{eff}^{H}\mathbf{y} = \sqrt{E_s}\, \|\mathbf{H}\|_F^2\, \mathbf{I}_2\, \mathbf{c} + \mathbf{n}'$$

where $\mathbf{n}'$ is such that $\mathcal{E}\{\mathbf{n}'\} = \mathbf{0}_{2\times 1}$ and $\mathcal{E}\{\mathbf{n}'\mathbf{n}'^{H}\} = \|\mathbf{H}\|_F^2\ \sigma_n^2\mathbf{I}_2$.

- Average output SNR

$$\bar{\rho}_{out} = \frac{E_s}{\sigma_n^2}\,\mathcal{E}\left\{\frac{\left[\,\|\mathbf{H}\|_F^2\,\right]^2}{2\,\|\mathbf{H}\|_F^2}\right\} = 2\rho,$$

Receive array gain ($g_a = n_r = 2$) but no transmit array gain!

- Average symbol error rate

$$\bar{P} \leq \bar{N}_e\left(\frac{\rho d_{min}^2}{8}\right)^{-4}.$$

Full diversity ($g_d^o = n_t n_r = 4$)

Dominant Eigenmode Transmission

- MIMO with Perfect Transmit Channel Knowledge
- Extension of Matched Beamforming to MIMO

$$\mathbf{y} = \sqrt{E_s}\mathbf{H}\mathbf{x} + \mathbf{n} = \sqrt{E_s}\mathbf{H}\mathbf{w}c + \mathbf{n},$$
$$z = \mathbf{g}\mathbf{y} = \sqrt{E_s}\mathbf{g}\mathbf{H}\mathbf{w}c + \mathbf{g}\mathbf{n}.$$

- Decompose

$$\mathbf{H} = \mathbf{U}_{\mathbf{H}}\mathbf{\Sigma}_{\mathbf{H}}\mathbf{V}_{\mathbf{H}}^H,$$
$$\mathbf{\Sigma}_{\mathbf{H}} = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{r(\mathbf{H})}\}.$$

- Received SNR is maximized by matched filtering, leading to

$$\mathbf{w} = \mathbf{v}_{max}$$
$$\mathbf{g} = \mathbf{u}_{max}^H$$

  where $\mathbf{v}_{max}$ and $\mathbf{u}_{max}$ are respectively the right and left singular vectors corresponding to the maximum singular value of $\mathbf{H}$, $\sigma_{max} = \max\{\sigma_1, \sigma_2, \ldots, \sigma_{r(\mathbf{H})}\}$. Note the generalization of matched beamforming (MISO) and MRC (SIMO)!
- Equivalent channel: $z = \sqrt{E_s}\sigma_{max}c + \tilde{n}$ where $\tilde{n} = \mathbf{g}\mathbf{n}$ has a variance equal to σ_n^2.

Dominant Eigenmode Transmission

- Array gain: $\mathcal{E}\{\sigma_{max}^2\} = \mathcal{E}\{\lambda_{max}\}$ where λ_{max} is the largest eigenvalue of $\mathbf{H}\mathbf{H}^H$. Commonly, $\max\{n_t, n_r\} \leq g_a \leq n_t n_r$.

- Diversity gain: the dominant eigenmode transmission extracts a full diversity gain of $n_t n_r$ in i.i.d. Rayleigh channels.

Dominant Eigenmode Transmission

Example

Question: Show that the optimum (in the sense of SNR maximization) transmit precoder and combiner in dominant eigenmode transmission is given by the dominant right and left singular vector of the channel matrix, respectively.
Answer: Let us write

$$\mathbf{y} = \sqrt{E_s}\mathbf{H}\mathbf{x} + \mathbf{n} = \sqrt{E_s}\mathbf{H}\mathbf{w}c + \mathbf{n},$$
$$z = \mathbf{g}\mathbf{y} = \sqrt{E_s}\mathbf{g}\mathbf{H}\mathbf{w}c + \mathbf{g}\mathbf{n}.$$

where $\|\mathbf{w}\|^2 = 1$ (power constraint). We decompose

$$\mathbf{H} = \mathbf{U}_\mathbf{H}\mathbf{\Sigma}_\mathbf{H}\mathbf{V}_\mathbf{H}^H, \qquad \mathbf{\Sigma}_\mathbf{H} = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{r(\mathbf{H})}\}.$$

In order to maximize the SNR, we choose $\mathbf{g}$ as a matched filter, i.e. $\mathbf{g} = (\mathbf{H}\mathbf{w})^H$ such that

$$\mathbf{g}\mathbf{H}\mathbf{w} = \mathbf{w}^H\mathbf{H}^H\mathbf{H}\mathbf{w} = \mathbf{w}^H\mathbf{V}_\mathbf{H}\mathbf{\Sigma}_\mathbf{H}^2\mathbf{V}_\mathbf{H}^H\mathbf{w} = \sum_{i=1}^{r(\mathbf{H})} \sigma_i^2 \left|\mathbf{v}_i^H\mathbf{w}\right|^2 \leq \sigma_{max}^2$$

where $\mathbf{v}_i$ is the i column of $\mathbf{V}_\mathbf{H}$ and $\sigma_{max} = \max_{i=1,\ldots,r(\mathbf{H})} \sigma_i$.

Dominant Eigenmode Transmission

Example

Answer: The inequality is replaced by an equality if $\mathbf{w} = \mathbf{v}_{max}$. By choosing $\mathbf{w} = \mathbf{v}_{max}$,

$$\begin{aligned} \mathbf{g} = \mathbf{w}^H \mathbf{H}^H &= \mathbf{v}_{max}^H \mathbf{V_H} \boldsymbol{\Sigma}_\mathbf{H} \mathbf{U_H}^H \\ &= \sigma_{max} \mathbf{u}_{max}^H \end{aligned}$$

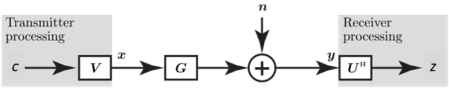
where $\mathbf{u}_{max}$ is the column of $\mathbf{U_H}$ corresponding to the dominant singular value σ_{max} of $\mathbf{H}$. If we normalize g such that $\|\mathbf{g}\|^2 = 1$, we can write $\mathbf{g} = \mathbf{u}_{max}$. □

Multiple Eigenmode Transmission

- Assume $n_r \geq n_t$ an that $r(\mathbf{H}) = n_t$, i.e. n_t singular values in $\mathbf{H}$. Hence, what about spreading symbols over all non-zero eigenmodes of the channel:
  - Tx side: multiply the input vector $\mathbf{c}$ $(n_t \times 1)$ using $\mathbf{V_H}$, i.e. $\mathbf{c}' = \mathbf{V_H c}$.
  - Rx side: multiply the received vector $\mathbf{y}$ by $\mathbf{G} = \mathbf{U_H^{\mathit{H}}}$.
  - Overall,

$$\mathbf{z} = \sqrt{E_s}\mathbf{GHx} + \mathbf{Gn} = \sqrt{E_s}\mathbf{U_H^{\mathit{H}} H V_H c} + \mathbf{U}^H\mathbf{n} = \sqrt{E_s}\mathbf{\Sigma_H c} + \tilde{\mathbf{n}}.$$

  Channel decomposed into n_t parallel SISO channels given by $\{\sigma_1, \ldots, \sigma_{n_t}\}$.



- The rate achievable in the MIMO channel is the sum of the SISO channel capacities

$$R = \sum_{k=1}^{n_t} \log_2(1 + \rho s_k \sigma_k^2),$$

  where $\{s_1, \ldots, s_{n_t}\}$ is the power allocation on each of the channel eigenmodes.
- The rate scales linearly in n_t.
- In general, the rate scales linearly with the rank of $\mathbf{H}$.

Multiple Eigenmode Transmission

Example

Question: Is the rate achievable in a MIMO channel with multiple eigenmode transmission and uniform power allocation across modes always larger than that achievable with dominant eigenmode transmission?
Answer: No! The achievable rate with multiple eigenmode transmission in the MIMO channel is the sum of the SISO channel achievable rates

$$R = \sum_{k=1}^{r(\mathbf{H})} \log_2(1 + \rho s_k \sigma_k^2),$$

where $\{s_1, \ldots, s_{r(\mathbf{H})}\}$ is the power allocation on each of the channel eigenmodes.
Two strategies (for a total power constraint $\sum_{k=1}^{r(\mathbf{H})} s_k = 1$):

- Uniform power allocation: $R_u = \sum_{k=1}^{r(\mathbf{H})} \log_2(1 + \rho 1/r(\mathbf{H})\sigma_k^2)$
- Dominant eigenmode transmission: $R_d = \log_2(1 + \rho\sigma_{max}^2)$

R_u could be either greater or smaller than R_d. For instance, if $\sigma_1 >> 0$ and $\sigma_k \approx \epsilon$ for $k > 1$, $R_u \approx \log_2(1 + \rho\sigma_1^2/r(\mathbf{H})) \leq R_d$ for small values of ρ. At very high SNR, despite the little contributions of $\sigma_k \approx \epsilon$, R_u will become higher than R_d. □

Multiplexing gain

- Array/diversity gains are exploitable in SIMO, MISO and MIMO but MIMO can offer much more than MISO and SIMO.

- MIMO channels offer *multiplexing gain*: measure of the number of independent streams that can be transmitted in parallel in the MIMO channel. Defined as

$$g_s \triangleq \lim_{\rho \to \infty} \frac{R(\rho)}{\log_2(\rho)}$$

  where $R(\rho)$ is the transmission rate.

- The multiplexing gain is the pre-log factor of the rate at high SNR, i.e.

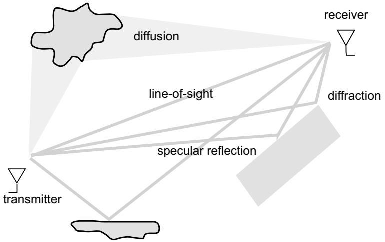
$$R \approx g_s \log_2(\rho)$$

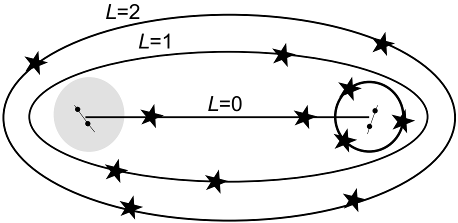
- Modeling only the individual SISO channels from one Tx antenna to one Rx antenna not enough:
  - MIMO performance depends on the channel matrix properties
  - characterize all statistical correlations between all matrix elements necessary!

The MIMO Channel

Double-Directional Channel Modeling

- Space comes as an additional dimension
  - *directional*: model the angular distribution of the energy at the antennas
  - *double*: there are multiple antennas at transmit and receive sides
- Spatial distribution of scatterers matters
  - delay-spread $\Longleftrightarrow$ channel frequency selectivity
  - angle-spread $\Longleftrightarrow$ channel spatial selectivity





The MIMO Channel Matrix

- $n_r \times n_t$ narrowband MIMO channel (the channel is not frequency selective)

$$\mathbf{H}(t) = \left[\begin{array}{cccc} h_{11}(t) & h_{12}(t) & \ldots & h_{1n_t}(t) \\ h_{21}(t) & h_{22}(t) & \ldots & h_{2n_t}(t) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n_r1}(t) & h_{n_r2}(t) & \ldots & h_{n_rn_t}(t) \end{array} \right],$$

Statistical Properties of the MIMO Channel Matrix

- Assume narrowband channels, the spatial correlation matrix of the MIMO channel

$$\mathbf{R} = \mathcal{E}\{\text{vec}(\mathbf{H}^H)\text{vec}(\mathbf{H}^H)^H\}$$

  This is a $n_t n_r \times n_t n_r$ positive semi-definite Hermitian matrix.
- It describes the correlation between all pairs of transmit-receive channels:
  - $\mathcal{E}\left\{\mathbf{H}(n,m)\mathbf{H}^*(n,m)\right\}$: the average energy of the channel between antenna m and antenna n,
  - $r_m^{(nq)} = \mathcal{E}\left\{\mathbf{H}(n,m)\mathbf{H}^*(q,m)\right\}$: the receive correlation between channels originating from transmit antenna m and impinging upon receive antennas n and q,
  - $t_n^{(mp)} = \mathcal{E}\left\{\mathbf{H}(n,m)\mathbf{H}^*(n,p)\right\}$: the transmit correlation between channels originating from transmit antennas m and p and arriving at receive antenna n,
  - $\mathcal{E}\left\{\mathbf{H}(n,m)\mathbf{H}^*(q,p)\right\}$: the cross-channel correlation between channels (m,n) and (q,p).

Example

2x2 MIMO

$$\mathbf{R} = \begin{bmatrix} 1 & t_1^* & r_1^* & s_1^* \\ t_1 & 1 & s_2^* & r_2^* \\ r_1 & s_2 & 1 & t_2^* \\ s_1 & r_2 & t_2 & 1 \end{bmatrix}$$

$$t_1 = \mathcal{E}\left\{\mathbf{H}(1,1)\mathbf{H}^*(1,2)\right\}$$

$$r_1 = \mathcal{E}\left\{\mathbf{H}(1,1)\mathbf{H}^*(2,1)\right\}$$

Spatial Correlation

- When the energy spreading is very large at both sides and d_t/d_r are sufficiently large, elements of $\mathbf{H}$ become uncorrelated, and $\mathbf{R}$ becomes diagonal.

Example

Consider two transmit antennas spaced by d_t.

- *isotropic scattering*: very rich scattering environment around the transmitter with a uniform distribution of the energy.

$$t = \mathrm{J}_0\left(2\pi\frac{d_t}{\lambda}\right).$$

  The transmit correlation only depends on the spacing between the two antennas.

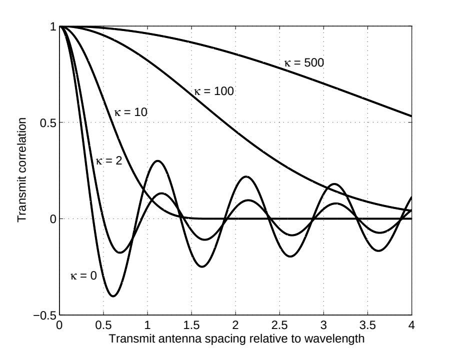
- *highly directional scattering*: scatterers around the transmit array are concentrated along a narrow direction $\theta_{t,0}$, i.e., $\mathcal{A}_t(\theta_t) \to \delta(\theta_t - \theta_{t,0})$

$$t \to e^{j\varphi_t(\theta_{t,0})} = e^{j2\pi(d_t/\lambda)\cos\theta_{t,0}}.$$

  Very high transmit correlation approaching one. The scattering direction is directly related to the phase of the transmit correlation.

Spatial Correlation

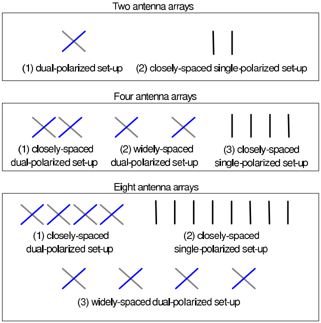
Example



- $\mathcal{A}_t(\theta_t)$ in real-world channels: neither uniform nor a delta.
- isotropic scattering ($\kappa = 0$): first minimum for $d_t = 0.38\lambda$
- directional scattering ($\kappa = \infty$): correlation never reaches 0
- in practice, decorrelation in rich scattering is reached for $d_t \approx 0.5\lambda$
- The more directional the azimuthal dispersion (i.e. for κ increasing), the larger the antenna spacing required to obtain a null correlation.

Analytical Representation of Rayleigh MIMO Channels

- Independent and Identically Distributed (I.I.D.) Rayleigh fading
  - $\mathbf{R} = \mathbf{I}_{n_t n_r}$
  - $\mathbf{H} = \mathbf{H}_w$ is a random fading matrix with unit variance and i.i.d. circularly symmetric complex Gaussian entries.
- Realistic in practice only if both conditions are satisfied:
  - the antenna spacings and/or the angle spreads at Tx and Rx are large enough,
  - all individual channels characterized by the same average power (i.e., balanced array).
- What about real-world channels? Sometimes significantly deviate from this ideal channel:
  - *limited angular spread and/or reduced array sizes* cause the channels to becom *correlated* (channels are not independen anymore)
  - a *coherent contribution* may induce the channel statistics to become *Ricean* (channels are not Rayleigh distributed anymore),
  - the use of multiple *polarizations* creates gain imbalances between the various elements of the channel matrix (channel are not identically distributed anymore).



Correlated Rayleigh Fading Channels

- For identically distributed Gaussian channels, $\mathbf{R}$ constitutes a sufficient description of the stochastic behavior of the MIMO channel.
- Any channel realization is obtained by

$$\text{vec}\left(\mathbf{H}^H\right) = \mathbf{R}^{1/2}\,\text{vec}(\mathbf{H}_w),$$

  where $\mathbf{H}_w$ is one realization of an i.i.d. channel matrix.
- Complicated to use because
  - cross-channel correlation not intuitive and not easily tractable
  - Too many parameters: dimensions of $\mathbf{R}$ rapidly become large as the array sizes increase
  - vec operation complicated for performance analysis
- Kronecker model: use a separability assumption

$$\mathbf{R} = \mathbf{R}_r \otimes \mathbf{R}_t,$$

$$\mathbf{H} = \mathbf{R}_r^{1/2}\mathbf{H}_w\mathbf{R}_t^{1/2}$$

  where $\mathbf{R}_t$ and $\mathbf{R}_r$ are respectively the transmit and receive correlation matrices.
- Strictly valid only if $r_1 = r_2 = r$ and $t_1 = t_2 = t$ and $s_1 = rt$ and $s_2 = rt^*$ (for 2×2)

$$\mathbf{R} = \begin{bmatrix} 1 & t_1^* & r_1^* & s_1^* \\ t_1 & 1 & s_2^* & r_2^* \\ r_1 & s_2 & 1 & t_2^* \\ s_1 & r_2 & t_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t^* & r^* & r^*t^* \\ t & 1 & r^*t & r^* \\ r & rt^* & 1 & t^* \\ rt & r & t & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & r^* \\ r & 1 \end{bmatrix}}_{\mathbf{R}_r} \otimes \underbrace{\begin{bmatrix} 1 & t^* \\ t & 1 \end{bmatrix}}_{\mathbf{R}_t}$$

Capacity of MIMO Channels

Previous Lectures

- Transmission strategies
  - Space-Time Coding when no Tx channel knowledge
  - Multiple (including dominant) eigenmode transmission when Tx channel knowledge

$$\begin{aligned}\mathbf{z} &= \sqrt{E_s}\mathbf{G}\mathbf{H}\mathbf{x} + \mathbf{G}\mathbf{n} \\ &= \sqrt{E_s}\mathbf{U}_{\mathbf{H}}^H\mathbf{H}\mathbf{V}_{\mathbf{H}}\mathbf{c} + \mathbf{U}^H\mathbf{n} \\ &= \sqrt{E_s}\mathbf{\Sigma}_{\mathbf{H}}\mathbf{c} + \tilde{\mathbf{n}}.\end{aligned}$$

  Multiple parallel data pipes $\rightarrow$ Spatial multiplexing gain!

- Performance highly depends on the channel matrix properties
  - Angle spread and inter-element spacing
  - Spatial Correlation: spread antennas far apart to decrease spatial correlation
  - Rayleigh and Ricean distribution

System Model

- A single-user MIMO system with n_t transmit and n_r receive antennas over a frequency flat-fading channel.
- The transmit and received signals in a MIMO channel are related by

$$\mathbf{y}_k = \sqrt{E_s}\mathbf{H}_k\mathbf{x}_k + \mathbf{n}_k$$

  where
  - $\mathbf{y}_k$ is the $n_r \times 1$ received signal vector,
  - $\mathbf{H}_k$ is the $n_r \times n_t$ channel matrix
  - $\mathbf{n}_k$ is a $n_r \times 1$ zero mean complex additive white Gaussian noise (AWGN) vector with $\mathcal{E}\{\mathbf{n}_k\mathbf{n}_l^H\} = \sigma_n^2 \mathbf{I}_{n_r}\delta(k-l)$.
  - $\rho = E_s/\sigma_n^2$ represents the SNR.
- The input covariance matrix is defined as the covariance matrix of the transmit signal $\mathbf{x}$ (we drop the time index) and writes as $\mathbf{Q} = \mathcal{E}\left\{\mathbf{x}\mathbf{x}^H\right\}$.
- Power constraint: Tr$\{\mathbf{Q}\} \leq 1$.
- Channel time variation: T_{coh} coherence time
  - *slow fading*: T_{coh} is so long that coding is performed over a single channel realization.
  - *fast fading*: T_{coh} is so short that coding over multiple channel realizations is possible.

Capacity of Deterministic MIMO Channels

Proposition

For a deterministic MIMO channel $\mathbf{H}$*, the mutual information* $\mathcal{I}$ *is written as*

$$\mathcal{I}(\mathbf{H}, \mathbf{Q}) = \log_2 \det \left[\mathbf{I}_{n_r} + \rho \mathbf{H} \mathbf{Q} \mathbf{H}^H \right]$$

where $\mathbf{Q}$ *is the input covariance matrix whose trace is normalized to unity.*

Definition

The capacity of a deterministic $n_r \times n_t$ MIMO channel with perfect channel state information at the transmitter is

$$C\left(\mathbf{H}\right) = \max_{\mathbf{Q} \geq 0 : \mathrm{Tr}\{\mathbf{Q}\} = 1} \log_2 \det \left[\mathbf{I}_{n_r} + \rho \mathbf{H} \mathbf{Q} \mathbf{H}^H \right].$$

Note the difference with SISO capacity.

If noise not white, whiten the noise (recall MMSE combiner).

Capacity and Water-Filling Algorithm

- What is the best transmission strategy, i.e. the optimum input covariance matrix $\mathbf{Q}$?
- *First*, create $n = \min\{n_t, n_r\}$ parallel data pipes (Multiple Eigenmode Transmission)
  - Decouple the channel along the individual channel modes (in the directions of the singular vectors of the channel matrix $\mathbf{H}$ at both the transmitter and the receiver)

$$\mathbf{H} = \mathbf{U_H}\mathbf{\Sigma_H}\mathbf{V_H}^H,$$

$$\mathbf{U_H}^H\mathbf{H}\mathbf{V_H} = \mathbf{U_H}^H\mathbf{U_H}\mathbf{\Sigma_H}\mathbf{V_H}^H\mathbf{V_H} = \mathbf{\Sigma_H}$$

  - Optimum input covariance matrix $\mathbf{Q}^\star$ writes as

$$\mathbf{Q}^\star = \mathbf{V_H}\text{diag}\left\{s_1^\star, \ldots, s_n^\star\right\}\mathbf{V_H}^H,$$

- *Second*, allocate power to data pipes
  - $\mathbf{\Sigma_H} = \text{diag}\left\{\sigma_1, \ldots, \sigma_n\right\}$, and $\sigma_k^2 \triangleq \lambda_k$
  - Capacity: $C(\mathbf{H}) = \max_{\{s_k\}_{k=1}^n} \sum_{k=1}^n \log_2\left[1 + \rho s_k \lambda_k\right] = \sum_{k=1}^n \log_2\left[1 + \rho s_k^\star \lambda_k\right]$

Proposition

The power allocation strategy $\{s_1, \ldots, s_n\} = \{s_1^\star, \ldots, s_n^\star\}$ *that maximizes* $\sum_{k=1}^n \log_2\left(1 + \rho\lambda_k s_k\right)$ *under the power constraint* $\sum_{k=1}^n s_k = 1$, *is given by the water-filling solution,*

$$s_k^\star = \left(\mu - \frac{1}{\rho\lambda_k}\right)^+, \; k = 1, \ldots, n$$

where μ *is chosen so as to satisfy the power constraint* $\sum_{k=1}^n s_k^\star = 1$.

Water-Filling Algorithm

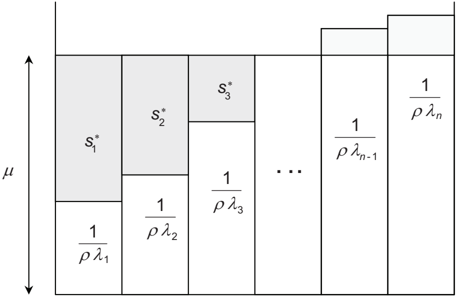
- Iterative power allocation
  - Order eigenvalues λ_k in decreasing order of magnitude
  - At iteration i, evaluate the constant μ from the power constraint

$$\mu(i) = \frac{1}{n-i+1}\left(1 + \sum_{k=1}^{n-i+1} \frac{1}{\rho\lambda_k}\right)$$

  - Calculate power

$$s_k(i) = \mu(i) - \frac{1}{\rho\lambda_k}, \qquad k = 1, \ldots, n-i+1.$$

    If $s_{n-i+1} < 0$, set to 0
  - Iterate till the power allocated on each mode is non negative.



Water-Filling Algorithm

Example

Question: Consider the transmission $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$ with perfect CSIT over a deterministic point to point MIMO channel whose matrix is given by

$$\mathbf{H} = \begin{bmatrix} a & 0 & a & 0 \\ 0 & b & 0 & b \end{bmatrix}$$

where a and b are complex scalars with $|a| \geq |b|$. The input covariance matrix is given by $\mathbf{Q} = \mathcal{E}\left\{\mathbf{x}\mathbf{x}^H\right\}$ and is subject to the transmit power constraint $\text{Tr}\left\{\mathbf{Q}\right\} \leq P$.

1. Compute the capacity with perfect CSIT of that deterministic channel. Particularize to the case $a = b$. Explain your reasoning.
2. Explain how to achieve that capacity.
3. In which deployment scenario, could such channel matrix structure be encountered?

Water-Filling Algorithm

Example

Answer:

1. Let us write $\mathbf{Q} = \mathbf{V}\mathbf{P}\mathbf{V}^H$ with the diagonal element of $\mathbf{P}$, denoted as P_k (satisfying $\sum_{k=1}^{n_t} P_k = P$), refers to the power allocated to stream k. The capacity with perfect CSIT over the deterministic channel $\mathbf{H}$ is given by

$$C\left(\mathbf{H}\right) = \max_{P_1,\ldots,P_k} \sum_{k=1}^{\min\{2,4\}} \log_2\left(1 + \frac{P_k}{\sigma_n^2}\lambda_k\right)$$

where λ_k refers the non-zero eigenvalue of $\mathbf{H}^H\mathbf{H}$, respectively equal to $2\left|a\right|^2$ and $2\left|b\right|^2$. Hence,

$$C\left(\mathbf{H}\right) = \max_{P_1,P_2}\left(\log_2\left(1 + \frac{P_1}{\sigma_n^2}2\left|a\right|^2\right) + \log_2\left(1 + \frac{P_2}{\sigma_n^2}2\left|b\right|^2\right)\right).$$

The optimal power allocation is given by the water-filling solution

$$P_1^\star = \left(\mu - \frac{\sigma_n^2}{2\left|a\right|^2}\right)^+, \qquad P_2^\star = \left(\mu - \frac{\sigma_n^2}{2\left|b\right|^2}\right)^+$$

with μ computed such that $P_1^\star + P_2^\star = P$.

Water-Filling Algorithm

Example

Answer:

Assuming $P_1^\star$ and $P_2^\star$ are positive, $\mu = \frac{P}{2} + \frac{\sigma_n^2}{4}\left(\frac{1}{|a|^2} + \frac{1}{|b|^2}\right)$. If $\mu - \frac{\sigma_n^2}{2|b|^2} \leq 0$, i.e. $\frac{P}{2} + \frac{\sigma_n^2}{4|a|^2} - \frac{\sigma_n^2}{4|b|^2} \leq 0$, $P_2^\star = 0$ and $P_1^\star = P$. The capacity writes as

$$C(\mathbf{H}) = \log_2\left(1 + \frac{P}{\sigma_n^2} 2\,|a|^2\right).$$

If $\frac{P}{2} + \frac{\sigma_n^2}{4|a|^2} - \frac{\sigma_n^2}{4|b|^2} > 0$, $P_1^\star = \frac{P}{2} - \frac{\sigma_n^2}{4|a|^2} + \frac{\sigma_n^2}{4|b|^2}$ and $P_2^\star = \frac{P}{2} + \frac{\sigma_n^2}{4|a|^2} - \frac{\sigma_n^2}{4|b|^2}$. The capacity writes as

$$C(\mathbf{H}) = \log_2\left(1 + \frac{P_1^\star}{\sigma_n^2} 2\,|a|^2\right) + \log_2\left(1 + \frac{P_2^\star}{\sigma_n^2} 2\,|b|^2\right).$$

In the particular case where $a = b$, uniform power allocation $P_1^\star = P_2^\star = \frac{P}{2}$ is optimal and

$$C(\mathbf{H}) = 2\log_2\left(1 + \frac{P}{\sigma_n^2}\,|a|^2\right).$$

Example

Answer:

2. Transmit along $\mathbf{V}$, given by the two dominant eigenvector of $\mathbf{H}^H\mathbf{H}$. They are easily computed given the orthogonality of the channel matrix $\mathbf{H}$ as

$$\mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   The power allocated to the two streams is given by $P_1^\star$ and $P_2^\star$. At the receiver, the precoded channel is already decoupled and no further combiner is necessary. Each stream can be decoded using the corresponding SISO decoder.

3. Dual-polarized antenna deployment (e.g. VHVH-VH) with LoS and good antenna XPD.

□

Capacity Bounds and Suboptimal Power Allocations

- Low SNR: power allocated to the dominant eigenmode

$$C\left(\mathbf{H}\right) \overset{\rho \to 0}{\to} \log_2\left(1+\rho\lambda_{max}\right).$$

- High SNR: power is uniformly allocated among the non-zero modes

$$C\left(\mathbf{H}\right) \overset{\rho \to \infty}{\to} \sum_{k=1}^{n} \log_2\left(1+\frac{\rho}{n}\lambda_k\right) \cong n\log_2\left(\frac{\rho}{n}\right) + \sum_{k=1}^{n}\log_2(\lambda_k).$$

Observations: $C\left(\mathbf{H}\right)$ scales linearly with n. The spatial multiplexing gain is $g_s = n$. MISO fading channels do not offer any multiplexing gain.

- At any SNR

$$C\left(\mathbf{H}\right) \geq \log_2\left(1+\rho\lambda_{max}\right),$$

$$C\left(\mathbf{H}\right) \geq \sum_{k=1}^{n} \log_2\left(1+\frac{\rho}{n}\lambda_k\right).$$

Impact on Coding Architecture

- Transmit independent streams in the directions of the eigenvectors of the channel matrix $\mathbf{H}$.

- For a total transmission rate R, each stream k can then be encoded using a capacity-achieving Gaussian code with rate R_k such that $\sum_{k=1}^{n} R_k = R$, ascribed a power λ_k and be decoded independently of the other streams.

- The optimal power allocation based on the water-filling allocation strategy.

Ergodic Capacity of Fast Fading Channels

- Fast fading:
  - Doppler frequency sufficiently high to allow for coding over many channel realizations/coherence time periods
  - The transmission capability is represented by a single quantity known as the ergodic capacity
- With Perfect Transmit Channel Knowledge, similar to deterministic channels
- Focus on Partial Transmit Channel Knowledge

MIMO Capacity with Partial Transmit Channel Knowledge

- $\mathbf{H}$ is not known to the transmitter $\rightarrow$ we cannot adapt $\mathbf{Q}$ at all time instants
- Rate of information flow between Tx and Rx at time instant k over channels $\mathbf{H}_k$

$$\log_2 \det \left[\mathbf{I}_{n_r} + \rho \mathbf{H}_k \mathbf{Q} \mathbf{H}_k^H\right].$$

Such a rate varies over time according to the channel fluctuations. The average rate of information flow over a time duration $T >> T_{coh}$ is

$$\frac{1}{T} \sum_{k=0}^{T-1} \log_2 \det \left[\mathbf{I}_{n_r} + \rho \mathbf{H}_k \mathbf{Q} \mathbf{H}_k^H\right].$$

Definition

The ergodic capacity of a $n_r \times n_t$ MIMO channel with channel distribution information at the transmitter (CDIT) is given by

$$\bar{C}_{CDIT} \triangleq \bar{C} = \max_{\mathbf{Q} \geq 0: \mathrm{Tr}\{\mathbf{Q}\}=1} \mathcal{E}\left\{\log_2 \det \left[\mathbf{I}_{n_r} + \rho \mathbf{H} \mathbf{Q} \mathbf{H}^H\right]\right\},$$

where $\mathbf{Q}$ is the input covariance matrix optimized as to maximize the ergodic mutual information.

- $T >> T_c$ to average out the noise and the channel fluctuations

I.I.D. Rayleigh Fast Fading Channels: Partial Transmit Channel Knowledge

- Optimal covariance matrix

Proposition

In i.i.d. Rayleigh fading channels, the ergodic capacity with CDIT is achieved under an equal power allocation scheme $\mathbf{Q} = \mathbf{I}_{n_t}/n_t$, i.e.,

$$\bar{C}_{CDIT} = \bar{\mathcal{I}}_e = \mathcal{E}\left\{ \log_2 \det \left[\mathbf{I}_{n_r} + \frac{\rho}{n_t} \mathbf{H}_w \mathbf{H}_w^H \right] \right\} = \mathcal{E}\left\{ \sum_{k=1}^{n} \log_2 \left[1 + \frac{\rho}{n_t} \lambda_k \right] \right\}.$$

Encoding requires a fixed-rate code (whose rate is given by the ergodic capacity) with encoding spanning many channel realizations.

- Low SNR:

$$\bar{C}_{CDIT} \geq \mathcal{E}\left\{ \log_2 \left[1 + \frac{\rho}{n_t} \|\mathbf{H}_w\|_F^2 \right] \right\} \approx \frac{\rho}{n_t} \mathcal{E}\left\{ \|\mathbf{H}_w\|_F^2 \right\} \log_2 (e) = n_r \rho \log_2 (e)$$

Observations:

- $\bar{C}_{CDIT}$ is only determined by the energy of the channel.
- A MIMO channel only yields a n_r gain over a SISO channel. Increasing the number of transmit antennas is not useful (contrary to perfect CSIT). SIMO and MIMO channels reach the same capacity for a given n_r.

I.I.D. Rayleigh Fast Fading Channels: Partial Transmit Channel Knowledge

- High SNR:

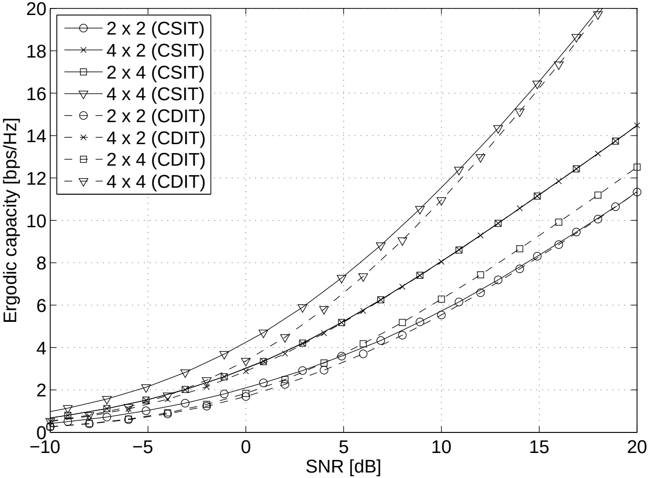
$$\bar{C}_{CDIT} \approx \mathcal{E}\left\{\sum_{k=1}^{n} \log_2\left[\frac{\rho}{n_t}\lambda_k\right]\right\} = n\log_2\left(\frac{\rho}{n_t}\right) + \mathcal{E}\left\{\sum_{k=1}^{n} \log_2(\lambda_k)\right\}$$

Observations:

- $\bar{C}_{CDIT}$ at high SNR scales linearly with n (by contrast to the low SNR regime).
- The multiplexing gain g_s is equal to n, similarly to the CSIT case.
- $\bar{C}_{CDIT}$ and $\bar{C}_{CSIT}$ are not equal: constant gap equal to $n \log_2(n_t/n)$ at high SNR.

I.I.D. Rayleigh Fast Fading Channels

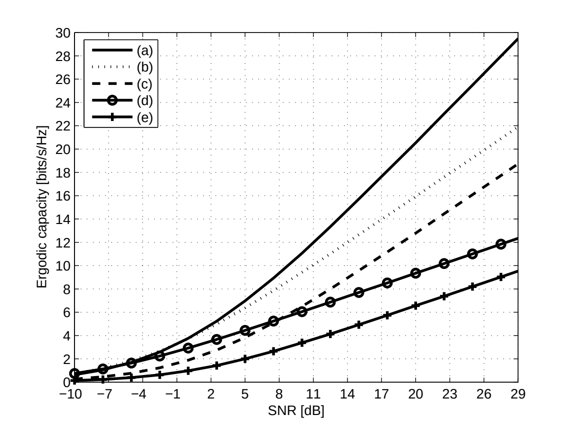
- Ergodic capacity of various $n_r \times n_t$ i.i.d. Rayleigh channels with full (CSIT) and partial (CDIT) channel knowledge at the transmitter.



I.I.D. Rayleigh Fast Fading Channels

Example

Question: Here is the ergodic capacity of point-to-point i.i.d. Rayleigh fast fading channels with Channel Distribution Information at the Transmitter (CDIT) for five antenna ($n_r \times n_t$) configurations (denoted as (a) to (e)) with $n_t + n_r = 8$.



I.I.D. Rayleigh Fast Fading Channels

Example

Question: What is the achievable (spatial) multiplexing gain (at high SNR) for cases (a), (b), (c), (d) and (e)? Provide your reasoning.

Answer: The multiplexing gain is the pre-log factor of the ergodic capacity at high SNR, i.e. $g_s = \lim_{\rho \to \infty} \frac{\bar{C}_{CDIT}}{\log_2(\rho)}$. Hence by increasing the SNR by 3dB (e.g. from 17dB to 20dB), the ergodic capacity increases by g_s bits/s/Hz.

- a $g_s = 3$.
- b $g_s = 2$.
- c $g_s = 2$.
- d $g_s = 1$.
- e $g_s = 1$.

I.I.D. Rayleigh Fast Fading Channels

Example

Question: For (a), (b), (c), (d) and (e), identify an antenna configuration, i.e. n_t and n_r, satisfying $n_t + n_r = 8$ that achieves such multiplexing gain. Provide your reasoning.

Answer: There are several possible configurations that satisfy to $n_r + n_t = 8$, namely 5×3, 3×5, 6×2, 2×6, 7×1 and 1×7, 4×4. The matching between curves and antenna configurations is easily identified by using the following two arguments: 1) The multiplexing gain with CDIT at high SNR is given by $\min \{n_t, n_r\}$. 2) With CDIT only, the input covariance matrix in i.i.d. channel is $\mathbf{Q} = 1/n_t \mathbf{I}_{n_t}$. This implies that 6×2 and 7×1 outperform 2×6 and 1×7, respectively.

- a) $n_r \times n_t = 5 \times 3$ or 3×5
- b) $n_r \times n_t = 6 \times 2$
- c) $n_r \times n_t = 2 \times 6$
- d) $n_r \times n_t = 7 \times 1$
- e) $n_r \times n_t = 1 \times 7$

Impact on Coding Architecture

- When the channel is i.i.d. Rayleigh fading, $\mathbf{Q} = (1/n_t)\,\mathbf{I}_{n_t}$.
- Transmission of independent information symbols may be performed in parallel over n virtual spatial channels.
- The transmitter is very similar to the CSIT case except that all eigenmodes now receive the same amount of power.
- Transmit with uniform power allocation over n_t independent streams, each stream using an AWGN capacity-achieving code and perform joint ML decoding (independent decoding of all streams is clearly suboptimal due to interference between streams).

Outage Capacity and Probability in Slow Fading Channels

- In slow fading, the encoding still averages out the randomness of the noise but cannot fully average out the randomness of the channel.
- For a given channel realization $\mathbf{H}$ and a target rate R, reliable transmission if

$$\log_2 \det\left(\mathbf{I}_{n_r} + \rho\mathbf{H}\mathbf{Q}\mathbf{H}^H\right) > R$$

  If not met with any $\mathbf{Q}$, an outage occurs and the decoding error probability is strictly non-zero.
- Look at the tail probability of $\log_2 \det\left(\mathbf{I}_{n_r} + \rho\mathbf{H}\mathbf{Q}\mathbf{H}^H\right)$, not its average!
- Leads to the fundamental Diversity Multiplexing tradeoff of MIMO channel https://youtu.be/p-ZQ9h4OJx4

Transmission and Reception Strategies

Previous Lectures

- Previous lecture
  - Capacity of deterministic MIMO channels

$$C(\mathbf{H}) = \max_{\mathbf{Q} \geq 0 : \mathsf{Tr}\{\mathbf{Q}\}=1} \log_2 \det \left[\mathbf{I}_{n_r} + \rho \mathbf{H}\mathbf{Q}\mathbf{H}^H \right].$$

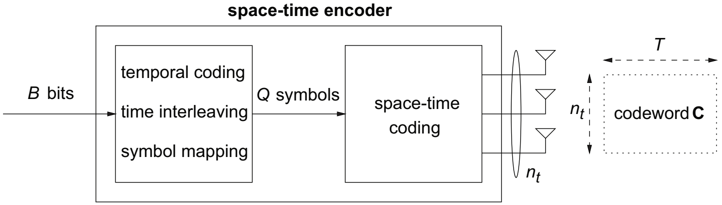
  - Ergodic capacity of fast fading channels
  - Outage capacity and probability of slow fading channels

- MIMO provides huge gains in terms of reliability and transmission rate
  - diversity gain, array gain, coding gain, spatial multiplexing gain, interference management

- What we further need
  - practical methodologies to achieve these gains?
  - how to code across space and time?
  - Some preliminary answers: multimode eigenmode transmission when channel knowledge available at the Tx, Alamouti scheme when no channel knowledge available at the Tx

Overview of a Space-Time Encoder

- Space-time encoder: sequence of two black boxes



- First black box: combat the randomness created by the noise at the receiver.
- Second black box: spatial interleaver which spreads symbols over several antennas in order to mitigate the spatial selective fading.
- The ratio B/T is the signaling rate of the transmission.
- The ratio Q/T is defined as the spatial multiplexing rate (representative of how many symbols are packed within a codeword per unit of time).

System Model

- MIMO system with n_t transmit and n_r receive antennas over a frequency flat-fading channel
- Transmit a codeword $\mathbf{C} = [\mathbf{c}_0 \ldots \mathbf{c}_{T-1}]$ $[n_t \times T]$ contained in the codebook $\mathcal{C}$
- At the k^{th} time instant, the transmitted and received signals are related by

$$\mathbf{y}_k = \sqrt{E_s}\mathbf{H}_k\mathbf{c}_k + \mathbf{n}_k$$

  where
  - $\mathbf{y}_k$ is the $n_r \times 1$ received signal vector,
  - $\mathbf{H}_k$ is the $n_r \times n_t$ channel matrix,
  - $\mathbf{n}_k$ is a $n_r \times 1$ zero mean complex AWGN vector with $\mathcal{E}\{\mathbf{n}_k\mathbf{n}_l^H\} = \sigma_n^2\mathbf{I}_{n_r}\delta(k-l)$,
  - The parameter E_s is the energy normalization factor. SNR $\rho = E_s/\sigma_n^2$.
- No transmit channel knowledge but we know it is i.i.d. Rayleigh fading.
- Codeword average transmit power $\mathcal{E}\left\{\text{Tr}\left\{\mathbf{C}\mathbf{C}^H\right\}\right\} = T$. Assume $\mathcal{E}\left\{\|\mathbf{H}\|_F^2\right\} = n_t n_r$.
- Many possible designs of the codebook $\mathcal{C}$. We focus on Spatial Multiplexing (most popular) but Alamouti is also particular instance of a general space-time coding framework. Details on design of space-time code in Chapters 6,8,9,10,11 of MIMO Wireless Networks book. See also https://youtu.be/Zoa-lpT6Y7M

Spatial Multiplexing/V-BLAST/D-BLAST

- Spatial Multiplexing (SM), also called V-BLAST, is a full rate code $(r_s = n_t)$ that consists in transmitting independent data streams on each transmit antenna.
- In uncoded transmissions, we assume one symbol duration $(T = 1)$ and codeword $\mathbf{C}$ is a symbol vector of size $n_t \times 1$.

Example

$$\mathbf{C} = \frac{1}{\sqrt{n_t}} \begin{bmatrix} c_1 & \ldots & c_{n_t} \end{bmatrix}^T.$$

Each element c_q is a symbol chosen from a given constellation.

ML decoding

- With instantaneous channel realizations perfectly known at the receive side, the ML decoder computes an estimate of the transmitted codeword according to

$$\hat{\mathbf{C}} = \arg\min_{\mathbf{C}\in\mathcal{C}} \left\| \mathbf{y} - \sqrt{E_s}\mathbf{H}\mathbf{C} \right\|^2$$

  where the minimization is performed over all possible codeword vectors $\mathbf{C} \in \mathcal{C}$.
- What is $\mathcal{C}$ for SM?

$$\mathbf{C} = \frac{1}{\sqrt{n_t}} \left[\begin{array}{ccc} c_1 & \ldots & c_{n_t} \end{array} \right]^T$$

  Assume all streams c_q use the same M-ary constellation of points $\mathcal{X}$, e.g. QPSK, 64-QAM, etc. Hence a search over M^{n_t} possible transmitted vectors is needed.
- Complexity gets prohibitive if M or number of streams increases.

ML decoding

- Pairwise Error Probability (PEP): probability that the ML decoder decodes the codeword $\mathbf{E}$ instead of the transmitted codeword $\mathbf{C}$. System performance dominated at high SNR by the couples of codewords that lead to the worst PEP.
- Error probability

$$P\left(\mathbf{C} \rightarrow \mathbf{E}\right) \leq \left(\frac{\rho}{4n_t}\right)^{-n_r} \left(\sum_{q=1}^{n_t} \left|c_q - e_q\right|^2\right)^{-n_r}$$

  The SNR exponent is equal to n_r. Due to the lack of coding across transmit antennas, no transmit diversity is achieved and only receive diversity is exploited.
- Over fast fading channels, we know that it is not necessary to code across antennas to achieve the ergodic capacity.

Proposition

Spatial Multiplexing with ML decoding and equal power allocation achieves the ergodic capacity of i.i.d. Rayleigh fast fading channels.

Zero-Forcing (ZF) Linear Receiver

- MIMO ZF receiver acts similarly to a ZF equalizer in frequency selective channels.
- ZF filtering effectively decouples the channel into n_t parallel channels
  - interference from other transmitted symbols is suppressed
  - scalar decoding may be performed on each of these channels
- The complexity of ZF decoding similar to SISO ML decoding, but the inversion step is responsible for the noise enhancement (especially at low SNR).
- Assuming that a symbol vector $\mathbf{C} = 1/\sqrt{n_t}\begin{bmatrix} c_1 & \dots & c_{n_t} \end{bmatrix}^T$ is transmitted, the output of the ZF filter $\mathbf{G}_{ZF}$ is given by

$$\mathbf{z} = \mathbf{G}_{ZF}\mathbf{y} = \begin{bmatrix} c_1 & \dots & c_{n_t} \end{bmatrix}^T + \mathbf{G}_{ZF}\mathbf{n}$$

  where $\mathbf{G}_{ZF}$ inverts the channel,

$$\mathbf{G}_{ZF} = \sqrt{\frac{n_t}{E_s}}\mathbf{H}^{\dagger}$$

  with $\mathbf{H}^{\dagger} = \left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H$ denoting the Moore-Penrose pseudo inverse.
- Then each stream can be decoded $\hat{c}_q = \arg\min_{c_q \in \mathcal{X}} |z_q - c_q|$, $\forall q$.

Zero-Forcing (ZF) Linear Receiver

- ZF is the solution to problem

$$\mathbf{z} = \arg\min_{\mathbf{x}} \left\| \mathbf{y} - \sqrt{\frac{E_s}{n_t}} \mathbf{H}\mathbf{x} \right\|^2$$

$$= \arg\min_{\mathbf{x}} \mathbf{y}^H \mathbf{y} - \sqrt{\frac{E_s}{n_t}} \mathbf{y}^H \mathbf{H}\mathbf{x} - \sqrt{\frac{E_s}{n_t}} \mathbf{x}^H \mathbf{H}^H \mathbf{y} + \frac{E_s}{n_t} \mathbf{x}^H \mathbf{H}^H \mathbf{H}\mathbf{x}$$

  where the minimization is performed over all possible vectors $\mathbf{x}$, i.e. we remove the constellation constraints (in contrast to ML).
- $\mathbf{z}$ is the solution of a least square problem

$$\frac{d}{d\mathbf{z}} \left\| \mathbf{y} - \sqrt{\frac{E_s}{n_t}} \mathbf{H}\mathbf{z} \right\|^2 = 0 - \sqrt{\frac{E_s}{n_t}} \mathbf{y}^H \mathbf{H} + \frac{E_s}{n_t} \mathbf{z}^H \mathbf{H}^H \mathbf{H} = 0$$

$$\mathbf{y}^H \mathbf{H} (\mathbf{H}^H \mathbf{H})^{-1} = \sqrt{\frac{E_s}{n_t}} \mathbf{z}^H$$

$$\mathbf{z} = \sqrt{\frac{n_t}{E_s}} (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{y}$$

Zero-Forcing (ZF) Linear Receiver

- Covariance matrix of the noise at the output of the ZF filter

$$\mathcal{E}\left\{\mathbf{G}_{ZF}\mathbf{n}\left(\mathbf{G}_{ZF}\mathbf{n}\right)^{H}\right\} = \frac{n_t}{\rho}\mathbf{H}^{\dagger}\left(\mathbf{H}^{\dagger}\right)^{H} = \frac{n_t}{\rho}\left(\mathbf{H}^{H}\mathbf{H}\right)^{-1}.$$

- The output SNR on the q^{th} subchannel is thus given by

$$\rho_q = \frac{\rho}{n_t}\frac{1}{\left(\mathbf{H}^{H}\mathbf{H}\right)^{-1}(q,q)}, \qquad q = 1,\ldots,n_t.$$

- Inversion leads to noise enhancement. Severe degradation at low SNR.
- Assuming that the channel is i.i.d. Rayleigh distributed, ρ_q is a χ^2 random variable with $2(n_r - n_t + 1)$ degrees of freedom, denoted as $\chi^2_{2(n_r-n_t+1)}$. The average PEP on the q^{th} subchannel is thus upper-bounded by

$$P\left(c_q \to e_q\right) \leq \left(\frac{\rho}{4n_t}\right)^{-(n_r-n_t+1)} \left|c_q - e_q\right|^{-2(n_r-n_t+1)}.$$

  The lower complexity of the ZF receiver comes at the price of a diversity gain limited to $n_r - n_t + 1$. Clearly, the system is undetermined if $n_t > n_r$.

Zero-Forcing (ZF) Linear Receiver

- In fast fading channels, the average maximum achievable rate $\bar{C}_{ZF}$ is equal to the sum of the maximum rates achievable by all layers

$$\begin{aligned}\bar{C}_{ZF} &= \sum_{q=1}^{\min\{n_t,n_r\}} \mathcal{E}\left\{\log_2\left(1+\rho_q\right)\right\} \\ &\overset{(\rho\nearrow)}{\approx} \min\left\{n_t,n_r\right\}\log_2\left(\frac{\rho}{n_t}\right)+\min\left\{n_t,n_r\right\}\mathcal{E}\left\{\log_2\left(\chi^2_{2(n_r-n_t+1)}\right)\right\}.\end{aligned}$$

Spatial Multiplexing in combination with a ZF decoder allows for transmitting over $n=\min\left\{n_t,n_r\right\}$ independent data pipes.

Zero-Forcing (ZF) Linear Receiver

- ZF receiver maximizes the SNR under the constraint that the interferences from all other layers are nulled out.
  - For a given layer q, the ZF combiner $\mathbf{g}_q$ is such that this layer is detected through a projection of the output vector $\mathbf{y}$ onto the direction closest to $\mathbf{H}(:,q)$ within the subspace orthogonal to the one spanned by the set of vectors $\mathbf{H}(:,p)$, $p \neq q$.
- Assume the following system model with $n_r \geq n_t$

$$\begin{aligned} \mathbf{y} &= \mathbf{Hc} + \mathbf{n}, \\ &= \mathbf{h}_q c_q + \sum_{p \neq q} \mathbf{h}_p c_p + \mathbf{n} \end{aligned}$$

  where $\mathbf{h}_q$ is the q^{th} column of $\mathbf{H}$.
- Let us build the following $n_r \times (n_t - 1)$ matrix by collecting all h_p with $p \neq q$:

$$\begin{aligned} \mathbf{H}_{-q} &= \begin{bmatrix} \ldots & \mathbf{h}_p & \ldots \end{bmatrix}_{p \neq q}, \\ &= \begin{bmatrix} \mathbf{U}' & \tilde{\mathbf{U}} \end{bmatrix} \mathbf{\Lambda} \mathbf{V}^H \end{aligned}$$

  where $\tilde{\mathbf{U}}$ is the matrix containing the left singular vectors corresponding to the null singular values. Similarly we define

$$\mathbf{c}_{-q} = \begin{bmatrix} \ldots & c_p & \ldots \end{bmatrix}_{p \neq q}^T.$$

Zero-Forcing (ZF) Linear Receiver

- By multiplying by $\tilde{\mathbf{U}}^H$, we project the output vector onto the subspace orthogonal to the one spanned by the columns of $\mathbf{H}_{-q}$

$$\begin{aligned}\tilde{\mathbf{U}}^H\mathbf{y} &= \tilde{\mathbf{U}}^H\mathbf{h}_q c_q + \tilde{\mathbf{U}}^H\mathbf{H}_{-q}\mathbf{c}_{-q} + \tilde{\mathbf{U}}^H\mathbf{n} \\ &= \tilde{\mathbf{U}}^H\mathbf{h}_q c_q + \tilde{\mathbf{U}}^H\mathbf{n}.\end{aligned}$$

- To maximize the SNR, noting the noise is still white, we match to the effective channel $\tilde{\mathbf{U}}^H\mathbf{h}_q$ such that

$$z = \left(\tilde{\mathbf{U}}^H\mathbf{h}_q\right)^H \tilde{\mathbf{U}}^H\mathbf{h}_q c_q + \left(\tilde{\mathbf{U}}^H\mathbf{h}_q\right)^H \tilde{\mathbf{U}}^H\mathbf{n}$$

and the ZF combiner is $g_q = \left(\tilde{\mathbf{U}}^H\mathbf{h}_q\right)^H \tilde{\mathbf{U}}^H = \mathbf{h}_q^H\tilde{\mathbf{U}}\tilde{\mathbf{U}}^H$.

Minimum Mean Squared Error (MMSE) Linear Receiver

- Filter maximizing the SINR. Minimize the total resulting noise: find $\mathbf{G}$ such that $\mathcal{E}\{\left\|\mathbf{Gy} - \begin{bmatrix} c_1 & \ldots & c_{n_t} \end{bmatrix}^T\right\|^2\}$ is minimum.
- The combined noise plus interference signal $\mathbf{n}_{i,q}$ when estimating symbol c_q writes as

$$\mathbf{n}_{i,q} = \sum_{p \neq q} \sqrt{\frac{E_s}{n_t}} \mathbf{h}_p c_p + \mathbf{n}.$$

The covariance matrix of $\mathbf{n}_{i,q}$ reads as

$$\mathbf{R}_{\mathbf{n}_{i,q}} = \mathcal{E}\left\{\mathbf{n}_{i,q}\mathbf{n}_{i,q}^H\right\} = \sigma_n^2 \mathbf{I}_{n_r} + \sum_{p \neq q} \frac{E_s}{n_t} \mathbf{h}_p \mathbf{h}_p^H$$

and the MMSE combiner for stream q is given by

$$\mathbf{g}_{MMSE,q} = \sqrt{\frac{E_s}{n_t}} \mathbf{h}_q^H \left(\sigma_n^2 \mathbf{I}_{n_r} + \sum_{p \neq q} \frac{E_s}{n_t} \mathbf{h}_p \mathbf{h}_p^H\right)^{-1}.$$

Minimum Mean Squared Error (MMSE) Linear Receiver

- Recall the other MMSE solution based on

$$\mathbf{R}_{yy} = \sigma_n^2 \mathbf{I}_{n_r} + \sum_p \frac{E_s}{n_t} \mathbf{h}_p \mathbf{h}_p^H$$

and the MMSE combiner for stream q is given by

$$\begin{aligned} \mathbf{g}_{MMSE,q} &= \sqrt{\frac{E_s}{n_t}} \mathbf{h}_q^H \left(\sigma_n^2 \mathbf{I}_{n_r} + \sum_p \frac{E_s}{n_t} \mathbf{h}_p \mathbf{h}_p^H \right)^{-1}. \\ &= \sqrt{\frac{E_s}{n_t}} \mathbf{h}_q^H \left(\sigma_n^2 \mathbf{I}_{n_r} + \frac{E_s}{n_t} \mathbf{H}\mathbf{H}^H \right)^{-1} \\ &= \sqrt{\frac{n_t}{E_s}} \mathbf{h}_q^H \underbrace{\left(\frac{n_t}{\rho} \mathbf{I}_{n_r} + \mathbf{H}\mathbf{H}^H \right)^{-1}}_{\text{common term for all streams}} \end{aligned}$$

- We can stack up combiners of all streams into one matrix

$$\mathbf{G}_{MMSE} = \sqrt{\frac{n_t}{E_s}} \left(\mathbf{H}^H \mathbf{H} + \frac{n_t}{\rho} \mathbf{I}_{n_t} \right)^{-1} \mathbf{H}^H = \sqrt{\frac{n_t}{E_s}} \mathbf{H}^H \left(\mathbf{H}\mathbf{H}^H + \frac{n_t}{\rho} \mathbf{I}_{n_r} \right)^{-1}$$

which is a popular representation of the MMSE filter
- Bridge between matched filtering at low SNR and ZF at high SNR.

Minimum Mean Squared Error (MMSE) Linear Receiver

- Similarly to ZF, after MMSE filtering, each stream can be decoded $\hat{c}_q = \arg\min_{c_q \in \mathcal{X}} |z_q - c_q|$, $\forall q$.
- The output SINR on the q^{th} subchannel (stream) is given by

$$\rho_q = \frac{E_s}{n_t} \mathbf{h}_q^H \left(\sigma_n^2 \mathbf{I}_{n_r} + \sum_{p \neq q} \frac{E_s}{n_t} \mathbf{h}_p \mathbf{h}_p^H \right)^{-1} \mathbf{h}_q .$$

- At high SNR, the MMSE filter is practically equivalent to ZF and the diversity achievable is thus limited to $n_r - n_t + 1$.

Minimum Mean Squared Error (MMSE) Linear Receiver

- **Interpretation 4**: MMSE is the solution to problem

$$\mathbf{z} = \arg\min_{\mathbf{x}} \left\| \mathbf{y} - \sqrt{\frac{E_s}{n_t}}\mathbf{H}\mathbf{x} \right\|^2 + \mu \left\|\mathbf{x}\right\|^2$$

$$= \arg\min_{\mathbf{x}} \mathbf{y}^H\mathbf{y} - \sqrt{\frac{E_s}{n_t}}\mathbf{y}^H\mathbf{H}\mathbf{x} - \sqrt{\frac{E_s}{n_t}}\mathbf{x}^H\mathbf{H}^H\mathbf{y} + \mathbf{x}^H\left(\frac{E_s}{n_t}\mathbf{H}^H\mathbf{H} + \mu\mathbf{I}\right)\mathbf{x}$$

  where the minimization is performed over all possible vectors $\mathbf{x}$, i.e. we remove the constellation constraints (in contrast to ML).
- $\mathbf{z}$ is the solution of a regularized least square problem with proper regularization μ

$$\frac{d}{d\mathbf{z}} \left\| \mathbf{y} - \sqrt{\frac{E_s}{n_t}}\mathbf{H}\mathbf{z} \right\|^2 + \mu \left\|\mathbf{z}\right\|^2 = 0 - \sqrt{\frac{E_s}{n_t}}\mathbf{y}^H\mathbf{H} + \mathbf{z}^H\left(\frac{E_s}{n_t}\mathbf{H}^H\mathbf{H} + \mu\mathbf{I}\right) = 0$$

$$\sqrt{\frac{E_s}{n_t}}\mathbf{y}^H\mathbf{H}\left(\frac{E_s}{n_t}\mathbf{H}^H\mathbf{H} + \mu\mathbf{I}\right)^{-1} = \mathbf{z}^H$$

$$\mathbf{z} = \sqrt{\frac{E_s}{n_t}}\left(\frac{E_s}{n_t}\mathbf{H}^H\mathbf{H} + \mu\mathbf{I}\right)^{-1}\mathbf{H}^H\mathbf{y}$$

Successive Interference Canceler

- Successively decode one symbol (or more generally one layer/stream) and cancel the effect of this symbol from the received signal.
- Decoding order based on the SINR of each symbol/layer: the symbol/layer with the highest SINR is decoded first at each iteration.
- SM with (ordered) SIC is generally known as V-BLAST, and ZF and MMSE V-BLAST refer to SM with respectively ZF-SIC and MMSE-SIC receivers.

- The diversity order experienced by the decoded layer is increased by one at each iteration. Therefore, the symbol/layer detected at iteration i will achieve a diversity of $n_r - n_t + i$.
- Major issue: error propagation
  - The error performance is mostly dominated by the weakest stream.
  - Non-ordered SIC: diversity order approximately $n_r - n_t + 1$.
  - Ordered SIC: performance improved by reducing the error propagation caused by the first decoded stream. The diversity order remains lower than n_r.

Successive Interference Canceler

1. *Initialization:* $i \longleftarrow 1$, $\mathbf{y}^{(1)} = \mathbf{y}$, $\mathbf{G}^{(1)} = \mathbf{G}_{ZF}(\mathbf{H})$, $q_1 \overset{(*)}{=} \arg\min_j \left\|\mathbf{G}^{(1)}(j,:)\right\|^2$ where $\mathbf{G}_{ZF}(\mathbf{H})$ is defined as the ZF filter of the matrix $\mathbf{H}$.
2. *Recursion:*
   1. *step 1:* extract the q_i^{th} transmitted symbol from the received signal $\mathbf{y}^{(i)}$

      $$\tilde{c}_{q_i} = \mathbf{G}^{(i)}(q_i,:)\,\mathbf{y}^{(i)}$$

      where $\mathbf{G}^{(i)}(q_i,:)$ is the q_i^{th} row of $\mathbf{G}^{(i)}$;
   2. *step 2:* slice $\tilde{c}_{q_i}$ to obtain the estimated transmitted symbol $\hat{c}_{q_i}$;
   3. *step 3:* assume that $\hat{c}_{q_i} = c_{q_i}$ and construct the received signal

      $$\begin{aligned} \mathbf{y}^{(i+1)} &= \mathbf{y}^{(i)} - \sqrt{\frac{E_s}{n_t}}\mathbf{H}(:,q_i)\,\hat{c}_{q_i} \\ \mathbf{G}^{(i+1)} &= \mathbf{G}_{ZF}\left(\mathbf{H}_{\overline{q_i}}\right) \\ i &\longleftarrow i+1 \\ q_{i+1} &\overset{(*)}{=} \arg\min_{j\notin\{q_1,\ldots,q_i\}} \left\|\mathbf{G}^{(i+1)}(j,:)\right\|^2 \end{aligned}$$

      where $\mathbf{H}_{\overline{q_i}}$ is the matrix obtained by zeroing columns $q_1,\ldots,q_i$ of $\mathbf{H}$. Here $\mathbf{G}_{ZF}\left(\mathbf{H}_{\overline{q_i}}\right)$ denotes the ZF filter applied to $\mathbf{H}_{\overline{q_i}}$.

Successive Interference Canceler

- In fast fading channels, the maximum rate achievable with ZF-SIC

$$\bar{C}_{ZF-SIC} = \sum_{q=1}^{\min\{n_t,n_r\}} \mathcal{E}\left\{\log_2\left(1+\rho_q\right)\right\}$$

$$\overset{(\rho\nearrow)}{\approx} \min\left\{n_t, n_r\right\}\log_2\left(\frac{\rho}{n_t}\right) + \sum_{q=1}^{\min\{n_t,n_r\}} \mathcal{E}\left\{\log_2\left(\chi^2_{2(n_r-n_t+q)}\right)\right\} = \bar{C}_{CDIT}$$

The loss that was observed with ZF filtering is now compensated because the successive interference cancellation improves the SNR of each decoded layer.

Proposition

Spatial Multiplexing with ZF-SIC (ZF V-BLAST) and equal power allocation achieves the ergodic capacity of i.i.d. Rayleigh fast fading MIMO channels at asymptotically high SNR.

This only holds true when error propagation is neglected.

Successive Interference Canceler

- MMSE-SIC does better for any SNR

$$\bar{C}_{MMSE-SIC} = \sum_{q=1}^{\min\{n_t,n_r\}} \mathcal{E}\left\{\log_2\left(1+\rho_q\right)\right\} = \mathcal{E}\left\{\log_2 \det\left(\mathbf{I}_{n_r} + \frac{\rho}{n_t}\mathbf{H}\mathbf{H}^H\right)\right\} = \bar{\mathcal{I}}_e,$$

Proposition

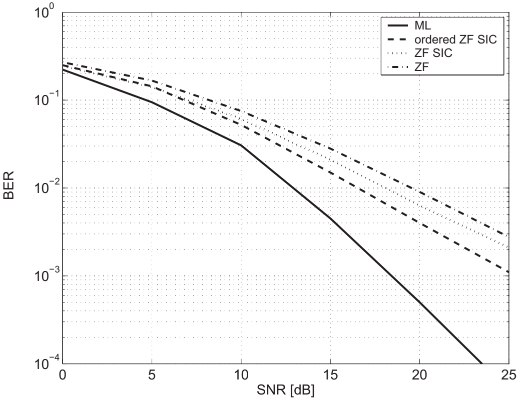
Spatial Multiplexing with MMSE-SIC (MMSE V-BLAST) and equal power allocation achieves the ergodic capacity for all SNR in i.i.d. Rayleigh fast fading MIMO channels.

Result also valid for a deterministic channel.

- Diversity achieved similar to that of ZF/MMSE receiver. This comes from the fact that the first layer dominates the error probability since its error exponent is the smallest.

Impact of Decoding Strategy on Error Probability

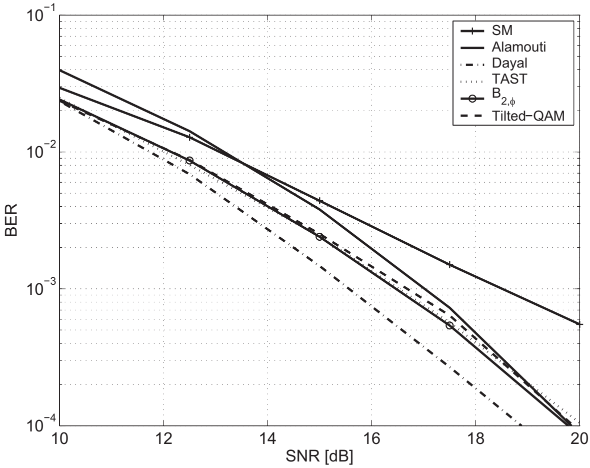
- SM with ML, ordered and non ordered ZF SIC (with perfect cancellation) and simple ZF decoding in 2×2 i.i.d. Rayleigh fading channels for 4 bits/s/Hz.



The slope of the ML curve approaches 2. ZF only achieves a diversity order of $n_r - n_t + 1 = 1$.

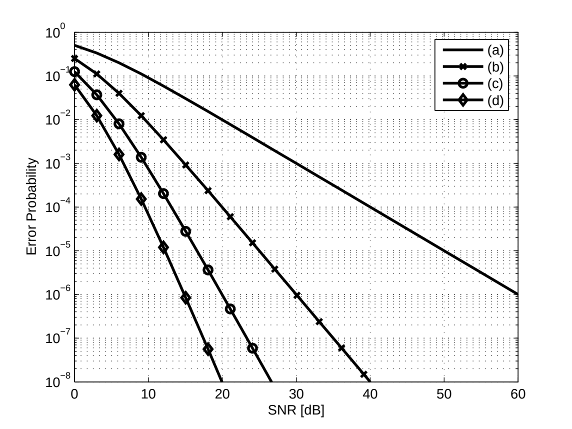
Global Performance Comparison

- Bit error rate (BER) of several space-time block codes in i.i.d. slow Rayleigh fading channels with $n_t = 2$ and $n_r = 2$ in a 4-bit/s/Hz transmission. ML decoding is used.



Example

Question: Here is the average Error Probability of one scheme (i.e., one transmission and reception strategy) vs. SNR for point-to-point channels with i.i.d. Rayleigh slow fading and four different antenna configurations (a) to (d). The CSI is unknown to the transmitter.



Example

Question: What is the diversity gain (at high SNR) achieved by that scheme in each antenna configuration? Provide your reasoning.

Answer: The diversity gain is the slope at high SNR of the error curve vs. the SNR on a log-log scale, i.e. $-\lim_{\rho\to\infty}\frac{\log(P_e(\rho))}{\log(\rho)}$ with ρ being the SNR.
For (a), the diversity gain is 1 as the error rate decreases by 10^{-1} when the SNR is increased from 50dB to 60dB.
For (b), the diversity gain is 2 as the error rate decreases by 10^{-2} when the SNR is increased from 30dB to 40dB.
For (c), the diversity gain is 3 as the error rate decreases by 10^{-3} when the SNR is increased from roughly 26dB to 36dB.
For (d), the diversity gain is 4 as the error rate decreases by 10^{-4} when the SNR is increased from 10dB to 20dB.

Example

Question: For each scenario (a) to (d), identify an antenna configuration (i.e., n_t and n_r) and the corresponding transmission/reception strategy that can achieve such diversity gain. Provide your reasoning.

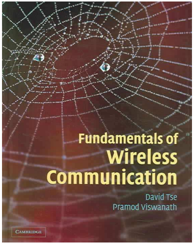
Answer: The simplest approach is to perform
for (a), receive matched filter with $n_r = 1$, $n_t = 1$
for (b), receive matched filter with $n_r = 2$, $n_t = 1$
for (c), receive matched filter with $n_r = 3$, $n_t = 1$
for (d), receive matched filter with $n_r = 4$, $n_t = 1$

Alternative strategies are possible, for instance selection combining at the receiver for all 4 cases. We could also perform transmit diversity based on space-time coding and use O-STBC for (b),(c),(d) to achieve diversity order of 2 with $n_t = 2$ and $n_r = 1$, 3 with $n_t = 3$ and $n_r = 1$, 4 with $n_t = 4$ and $n_r = 1$, respectively. Alternatively, we could as well use Spatial Multiplexing with ZF receiver and transmit two streams over two transmit antennas and use 2,3,4,5 receive antennas for a,b,c,d respectively.

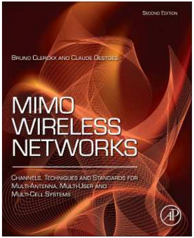
Part 3: Multiuser Communications

Capacity of Multiuser (SISO and MIMO) Channels
Fairness, Scheduling and Precoding
Massive MIMO
Multiuser Multicell Communications

Reference Book



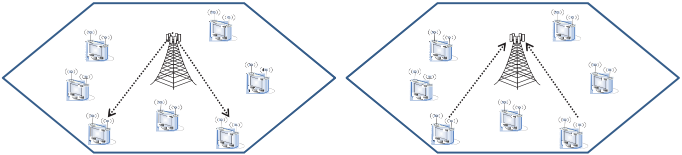
Chapter 12 - Sections 12.1, 12.2.1, 12.3.1, 12.5.1, 12.6.2,12.8.1,12.8.2,12.8.10 Chapter 13 - Section 13.2



Capacity of Multiuser (SISO and MIMO) Channels

BC vs MAC

- So far, we looked at a single link/user. Most systems are multi-user!
- How to deal with multiple users?
- Broadcast Channel (BC) and Multiple Access Channel(MAC)



(a) Broadcast Channel - Downlink **(b) Multiple Access Channel - Uplink**

Differences between BC and MAC:

- there are multiple independent receivers (and therefore multiple independent additive noises) in BC while there is a single receiver (and therefore a single noise term) in MAC.
- there is a single transmitter (and therefore a single transmit power constraint) in BC while there are multiple transmitters (and therefore multiple transmit power constraints) in MAC.
- the desired signal and the interference (originating from the co-scheduled signals) propagate through the same channel in the BC while they propagate through different channels in the MAC.

Capacity Region

- In a multi-user setup, given that all users share the same spectrum, the rate achievable by a given user q, denoted as R_q, will depend on the rate of the other users R_p, $p \neq q \rightarrow$ Trade-off between rates achievable by different users!
- The capacity region $\mathcal{C}$ formulates this trade-off by expressing the set of all user rates $(R_1, \ldots, R_K)$ that are simultaneously achievable.

Definition

The capacity region $\mathcal{C}$ of a channel $\mathbf{H}_{ul}$ is the set of all rate vectors $(R_1, \ldots, R_K)$ such that simultaneously user 1 to user K can reliably communicate at rate R_1 to rate R_K, respectively.

Any rate vector not in the capacity region is not achievable (i.e. transmission at those rates will lead to errors).

Definition

The sum-rate capacity C of a capacity region $\mathcal{C}$ is the maximum achievable sum of rates

$$C = \max_{(R_1,\ldots,R_K)\in\mathcal{C}} \sum_{q=1}^{K} R_q.$$

SISO MAC System Model

- Uplink multi-user SISO transmission
  - total number of K users ($\mathcal{K} = \{1, \ldots, K\}$) distributed in a cell,
  - 1 transmit antennas at each mobile terminal
  - 1 receive antenna at the base station
  - each user encodes its message into a stream $c_{ul,q}$
- Received signal (we drop the time dimension)

$$y_{ul} = \sum_{q=1}^{K} \Lambda_q^{-1/2} h_{ul,q} c_{ul,q} + n_{ul}$$

  where
  - $h_{ul,q}$ models the small scale time-varying fading process and Λ_q^{-1} refers to the large-scale fading accounting for path loss and shadowing
  - n_{ul} is a complex Gaussian noise $\mathcal{CN}\left(0, \sigma_n^2\right)$.
- Power constraint: $\mathcal{E}\left\{\left|c_{ul,q}\right|^2\right\} \leq E_{s,q}$.

SISO MAC System Model

- By stacking up the transmit signal vectors and the channel matrices of all K users,

$$\mathbf{c}_{ul} = \left[c_{ul,1}, \ldots, c_{ul,K}\right]^T,$$
$$\mathbf{h}_{ul} = \left[\Lambda_1^{-1/2} h_{ul,1}, \ldots, \Lambda_K^{-1/2} h_{ul,K}\right],$$

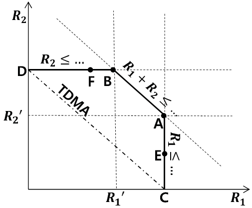
  the system model also writes as

$$y_{ul} = \mathbf{h}_{ul}\mathbf{c}_{ul} + n_{ul}.$$

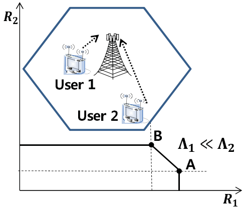
- Long term SNR of user q defined as $\eta_q = E_{s,q}\Lambda_q^{-1}/\sigma_n^2$.
- Note on the notations: the dependence on the path loss and shadowing is made explicit in order to stress that the co-scheduled users experience different path losses and shadowings and therefore receive power.
- The receiver (i.e. the BS in a UL scenario) has always perfect knowledge of the CSI.

Capacity Region of a Two-User SISO MAC

- The capacity region is a pentagon with two corner points A and B.



(a) Two-user MIMO MAC rate region for fixed $Q_{ul,1}$ and $Q_{ul,2}$



(b) Rate regions with various path losses

- $\mathcal{C}_{MAC}$ is the set of all rates pair (R_1, R_2) satisfying to

$$R_q \leq \log_2\left(1 + \eta_q \left|h_{ul,q}\right|^2\right), q = 1, 2$$

$$R_1 + R_2 \leq \log_2\left(1 + \eta_1 \left|h_{ul,1}\right|^2 + \eta_2 \left|h_{ul,2}\right|^2\right).$$

- Remarkably, at point A, user 1 can transmit at a rate equal to its single-link SISO rate and user 2 can simultaneously transmit at a rate $R_2' > 0$ equal to

$$R_2' = \log_2\left(1 + \eta_1 \left|h_{ul,1}\right|^2 + \eta_2 \left|h_{ul,2}\right|^2\right) - \log_2\left(1 + \eta_1 \left|h_{ul,1}\right|^2\right)$$

$$= \log_2\left(1 + \frac{\eta_2 \left|h_{ul,2}\right|^2}{1 + \eta_1 \left|h_{ul,1}\right|^2}\right) = \log_2\left(1 + \frac{\Lambda_2^{-1} \left|h_{ul,2}\right|^2 E_{s,2}}{\sigma_n^2 + \Lambda_1^{-1} \left|h_{ul,1}\right|^2 E_{s,1}}\right).$$

Capacity Region of SISO MAC

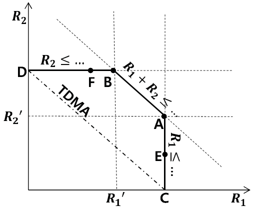
Proposition

$$\mathcal{C}_{MAC} = \left\{ (R_1, \ldots, R_K) : \sum_{q \in S} R_q \leq \log_2 \left(1 + \sum_{q \in S} \eta_q \left|h_{ul,q}\right|^2 \right), \forall S \subseteq \mathcal{K} \right\}$$

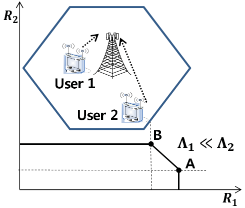
where $\eta_q = \Lambda_q^{-1} E_{s,q} / \sigma_n^2$.

Achievability of the Capacity Region

- SIC is optimal for achieving the corner points of the SISO MAC rate region.
- The exact corner point that is achieved on the capacity region depends on the stream cancellation ordering:
  - Point A, user 2 is canceled first (i.e. all streams from user 2) such that user 1 is left with the Gaussian noise and can achieve a rate equal to the single-link bound.
  - Assuming $n_t = 1$, $R_2' = \log_2(1 + \rho_q)$ where ρ_q is the SINR of the receiver for user 2's stream treating user 1's stream as Gaussian interference.



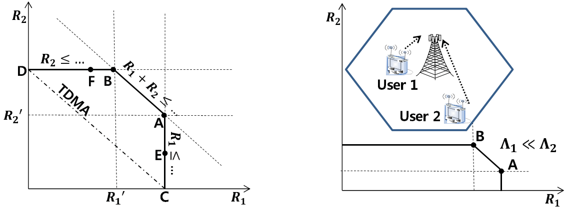
(a) Two-user MIMO MAC rate region for fixed $Q_{ul,1}$ and $Q_{ul,2}$



(b) Rate regions with various path losses

Comparisons with TDMA

- TDMA allocates the time resources in an orthogonal manner such that users are never transmitting at the same time (line D-C in the rate region).
- TDMA rate region is strictly smaller than the one achievable with SIC.



(a) Two-user MIMO MAC rate region for fixed $Q_{ul,1}$ and $Q_{ul,2}$

(b) Rate regions with various path losses

SISO BC System Model

- Downlink multi-user SISO transmission
  - total number of K users ($\mathcal{K} = \{1, \ldots, K\}$) distributed in a cell,
  - 1 receive antenna at each mobile terminal
  - 1 transmit antenna at the base station
- Received signal (we drop the time dimension)

$$y_q = \Lambda_q^{-1/2} h_q c + n_q$$

where
  - h_q models the small scale time-varying fading process and Λ_q^{-1} refers to the large-scale fading accounting for path loss and shadowing
  - n_q is a complex Gaussian noise $\mathcal{CN}\left(0, \sigma_{n,q}^2\right)$.
- Transmit power constraint: $\mathcal{E}\left\{|c|^2\right\} = E_s$.
- Multicast vs unicast
  - Multicast (one-to-many): c carries one message that is intended to all users), e.g. used in radio/tv broadcasting
  - Unicast (one-to-one): c carries a superposition of streams, each carrying one message intended to a given user, e.g. most common multi-user transmission in modern cellular systems and focus of this section
  - non-orthogonal unicast and multicast: multicast superimposed on unicast

SISO BC System Model

- By stacking up the received signal vectors, the noise vectors and the channel matrices of all K users,

$$\mathbf{y} = [y_1, \ldots, y_K]^T,$$
$$\mathbf{n} = [n_1, \ldots, n_K]^T,$$
$$\mathbf{h} = \left[\Lambda_1^{-1/2} h_1, \ldots, \Lambda_K^{-1/2} h_K\right]^T,$$

the system model also writes as

$$\mathbf{y} = \mathbf{h}c + \mathbf{n}.$$

- SNR of user q defined as $\eta_q = E_s \Lambda_q^{-1} / \sigma_{n,q}^2$.
- Perfect instantaneous channel state information (CSI) at the Tx and all Rx.
- Generally speaking, c is written as the superposition of statistically independent signals c_q

$$c = \sum_{q=1}^{K} c_q.$$

Stream c_q carries the message of user-q.

Capacity Region of two-user Deterministic SISO BC

- In *two-user* SISO MAC, point A was obtained by canceling user 2's signal first such that user 1 is left with Gaussian noise.
- Let us apply the same philosophy to the SISO BC:
  - transmit $c = c_1 + c_2$, with power of c_q denoted as s_q
  - user 1 cancels user 2's signal c_2 so as to be left with its own Gaussian noise
  - user 2 decodes its signal by treating user 1's signal c_1 as Gaussian noise.
- Achievable rates of such strategy (with sum-power constraint $s_1 + s_2 = E_s$)

$$R_1 = \log_2\left(1 + \frac{\Lambda_1^{-1} s_1}{\sigma_{n,1}^2} |h_1|^2\right)$$

$$R_2 = \log_2\left(1 + \frac{\Lambda_2^{-1} |h_2|^2 s_2}{\sigma_{n,2}^2 + \Lambda_2^{-1} |h_2|^2 s_1}\right).$$

- *Careful!* For user 1 to be able to correctly cancel user 2's signal, user 1's channel has to be good enough to support R_2, i.e.

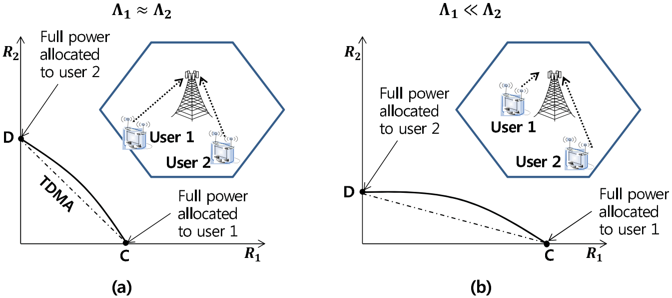
$$R_2 \leq \log_2\left(1 + \frac{\Lambda_1^{-1} |h_1|^2 s_2}{\sigma_{n,1}^2 + \Lambda_1^{-1} |h_1|^2 s_1}\right).$$

- The channel gains normalized w.r.t. their respective noise power should be ordered

$$\frac{\Lambda_2^{-1} |h_2|^2}{\sigma_{n,2}^2} \leq \frac{\Lambda_1^{-1} |h_1|^2}{\sigma_{n,1}^2}.$$

Capacity Region of two-user SISO Deterministic BC

- If the ordering condition is satisfied, the above strategy achieves the boundary of the capacity region of the two-user SISO BC for any power allocation s_1 and s_2 satisfying $s_1 + s_2 = E_s$.
- The capacity region is given by the union of all rate pairs (R_1, R_2) over all power allocations s_1 and s_2 satisfying $s_1 + s_2 = E_s$.



Capacity Region of K-user Deterministic SISO BC

- Define $\mathsf{h}_q = \Lambda_q^{-1/2} h_q / \sigma_{n,q}$. Assume $|\mathsf{h}_1|^2 \geq |\mathsf{h}_2|^2 \geq \ldots \geq |\mathsf{h}_K|^2$.

Proposition

With the ordering $|\mathsf{h}_1|^2 \geq |\mathsf{h}_2|^2 \geq \ldots \geq |\mathsf{h}_K|^2$, the capacity region $\mathcal{C}_{BC}$ of the Gaussian SISO BC is the set of all achievable rate vectors $(R_1, \ldots, R_K)$ given by

$$\bigcup_{s_q : \sum_{q=1}^{K} s_q = E_s} \left\{ (R_1, \ldots, R_K) : R_q \leq \log_2 \left(1 + \frac{|\mathsf{h}_q|^2 s_q}{1 + |\mathsf{h}_q|^2 \left[\sum_{p=1}^{q-1} s_p \right]} \right), \forall q \right\}.$$

Proposition

The sum-rate capacity of the SISO BC is achieved by allocating the transmit power to the strongest user

$$C_{BC} = \log_2 \left(1 + E_s \max_{q=1,\ldots,K} |\mathsf{h}_q|^2 \right) = \log_2 \left(1 + \max_{q=1,\ldots,K} \eta_q |h_q|^2 \right).$$

Recall that the MAC sum-rate capacity is obtained with all users simultaneously transmitting at their respective full power.

Achievability of the SISO BC Capacity Region

- Receiver cancellation - *Superposition coding with SIC and appropriate ordering*:
  - User ordering: decode and cancel out weaker users signals before decoding their own signal.
  - The weakest user decodes only the coarsest constellation. The strongest user decodes and subtracts all constellation points in order to decode the finest constellation.
- Transmitter cancellation - *Dirty-Paper Coding* (DPC)
  - Assume a system model $y = hc + i + n$ with i, n Gaussian interference and noise. Simply subtracting i for transmit signal is not a good idea!

Proposition

If Tx has full (non-causal) knowledge of the interference, the capacity of the dirty paper channel is equal to the capacity of the channel with the interference completely absent.

- By encoding users in the increasing order of their normalized channel gains, DPC achieves the capacity region of the SISO BC.

Example

Assume $|\mathrm{h}_1|^2 \geq |\mathrm{h}_2|^2$. By treating user 2's signal c_2 as known Gaussian interference at Tx and encoding user 1's signal c_1 using DPC, user 1 can achieve a rate as high as if user 2's signal was absent. User 2 treats user 1's signal as Gaussian noise.

Achievability of the SISO BC Capacity Region

Proposition

With the appropriate cancellation/encoding ordering, Superposition Coding with SIC and DPC are both optimal for achieving the SISO BC capacity region.

Proposition

The SISO BC sum-rate capacity is achievable with dynamic TDMA (to the strongest user) and Superposition Coding with SIC (with the appropriate cancellation ordering).

Comparisons with TDMA

- Similarly to MAC, TDMA rate region is contained in the BC capacity region.
- The gap between the BC capacity region and the TDMA rate region increases proportionally with the asymmetry between users normalized channel gains.
- TDMA achieves the sum-rate capacity of SISO BC.

SIMO MAC System Model

- Uplink multi-user MIMO (MU-MIMO) transmission
  - total number of K users ($\mathcal{K} = \{1, \ldots, K\}$) distributed in a cell,
  - single transmit antennas at mobile terminal q
  - n_r receive antenna at the base station
- Received signal (we drop the time dimension)

$$\mathbf{y}_{ul} = \sum_{q=1}^{K} \Lambda_q^{-1/2} \mathbf{h}_{ul,q} c_{ul,q} + \mathbf{n}_{ul}$$

  where
  - $\mathbf{y}_{ul}$ $[n_r \times 1]$
  - $\mathbf{h}_{ul,q}$ $[n_r \times 1]$ models the small scale time-varying fading process and Λ_q^{-1} refers to the large-scale fading accounting for path loss and shadowing
  - $\mathbf{n}_{ul}$ is a complex Gaussian noise $\mathcal{CN}\left(0, \sigma_n^2 \mathbf{I}_{n_r}\right)$.
- Power constraint: $\text{Tr}\{|c_{ul,q}|^2\} \leq E_{s,q}$.

SIMO MAC System Model

- By stacking up the transmit signal vectors and the channel matrices of all K users,

$$\mathbf{c}_{ul} = \left[c_{ul,1}, \ldots, c_{ul,K}\right]^T,$$
$$\mathbf{H}_{ul} = \left[\Lambda_1^{-1/2}\mathbf{h}_{ul,1}, \ldots, \Lambda_K^{-1/2}\mathbf{h}_{ul,K}\right],$$
$$\mathbf{y}_{ul} = \mathbf{H}_{ul}\mathbf{c}_{ul} + \mathbf{n}_{ul},$$

  $\mathbf{H}_{ul}$ is assumed to be full-rank as it would be the case in a typical user deployment.
- Long term SNR of user q defined as $\eta_q = E_{s,q}\Lambda_q^{-1}/\sigma_n^2$.
- Note on the notations: the dependence on the path loss and shadowing is made explicit in order to stress that the co-scheduled users experience different path losses and shadowings and therefore receive power.
- We assume that the receiver (i.e. the BS in a UL scenario) has always perfect knowledge of the CSI, but we will consider strategies where the transmitters have perfect or partial knowledge of the CSI.
- Capacity region of SIMO MAC can be characterized.

Capacity Region of SIMO MAC

Proposition

$$\mathcal{C}_{MAC} = \left\{ \begin{array}{l} (R_1, \ldots, R_K) : \sum_{q \in S} R_q \leq \\ \log_2 \det \left[\mathbf{I}_{n_r} + \sum_{q \in S} \eta_q \mathbf{h}_{ul,q} \mathbf{h}_{ul,q}^H \right], \forall S \subseteq \mathcal{K} \end{array} \right\}$$

where $\eta_q = \Lambda_q^{-1} E_{s,q} / \sigma_n^2$.

- Note

$$\log_2 \det \left[\mathbf{I}_{n_r} + \sum_{q \in S} \eta_q \mathbf{h}_{ul,q} \mathbf{h}_{ul,q}^H \right] = \log_2 \det \left[\mathbf{I}_{n_r} + \mathbf{H}_{ul,q} \mathrm{diag}\{\eta_q\}_{\forall q \in S} \mathbf{H}_{ul,q}^H \right]$$

with $\mathbf{H}_{ul,q} = [\mathbf{h}_{ul,q}]_{\forall q \in S}$

Capacity Region of SIMO MAC

Example

Two-user SIMO: $\mathcal{C}_{MAC}$ is the set of all rates pair (R_1, R_2) satisfying to

$$R_q \leq \log_2\left(1+\eta_q \left\|\mathbf{h}_{ul,q}\right\|^2\right) = \log_2 \det\left(\mathbf{I}_{n_r} + \eta_q \mathbf{h}_{ul,q}\mathbf{h}_{ul,q}^H\right), q = 1, 2$$

$$R_1 + R_2 \leq \log_2 \det\left(\mathbf{I}_{n_r} + \eta_1 \mathbf{h}_{ul,1}\mathbf{h}_{ul,1}^H + \eta_2 \mathbf{h}_{ul,2}\mathbf{h}_{ul,2}^H\right).$$

$$\begin{aligned}
R_2' &= \log_2 \det\left(\mathbf{I}_{n_r} + \eta_1 \mathbf{h}_{ul,1}\mathbf{h}_{ul,1}^H + \eta_2 \mathbf{h}_{ul,2}\mathbf{h}_{ul,2}^H\right) - \log_2 \det\left(\mathbf{I}_{n_r} + \eta_1 \mathbf{h}_{ul,1}\mathbf{h}_{ul,1}^H\right) \\
&= \log_2 \det\left(\mathbf{I}_{n_r} + \eta_2 \mathbf{h}_{ul,2}\mathbf{h}_{ul,2}^H \left(\mathbf{I}_{n_r} + \eta_1 \mathbf{h}_{ul,1}\mathbf{h}_{ul,1}^H\right)^{-1}\right) \\
&= \log_2\left(1 + \underbrace{\eta_2 \mathbf{h}_{ul,2}^H \left(\mathbf{I}_{n_r} + \eta_1 \mathbf{h}_{ul,1}\mathbf{h}_{ul,1}^H\right)^{-1} \mathbf{h}_{ul,2}}_{\text{SINR of MMSE receiver}}\right)
\end{aligned}$$

Achievability of the SIMO MAC Capacity Region

- For $n_t = 1$, the SIMO MAC architecture is reminiscent of the Spatial Multiplexing architecture discussed for a single-link MIMO channel.
- We can therefore fully reuse the various receiver architectures derived for single-link MIMO.
- Recall the optimality of the MMSE V-BLAST (also called Spatial Multiplexing with MMSE-SIC receiver)

Proposition

MMSE-SIC is optimal for achieving the corner points of the MIMO MAC rate region.

- The exact corner point that is achieved on the rate region depends on the stream cancellation ordering:
  - Point A, user 2 is canceled first (i.e. all streams from user 2) such that user 1 is left with the Gaussian noise and can achieve a rate equal to the single-link bound.
  - Assuming $n_t = 1$, $R_2' = \log_2(1 + \rho_q)$ where ρ_q is the SINR of the MMSE receiver for user 2's stream treating user 1's stream as colored Gaussian interference.
- Extension to MIMO MAC also available in the book.

Comparisons with TDMA

- TDMA allocates the time resources in an orthogonal manner such that users are never transmitting at the same time (line D-C in the rate region).
- SISO: both TDMA and SIC exploit a single degree of freedom but TDMA rate region is strictly smaller than the one achievable with SIC.
- SIMO: TDMA incurs a big loss compared to SIMO MAC (with MMSE-SIC) as it only exploits a single degree of freedom despite the presence of $\min\{n_r, K\}$ degrees of freedom achievable with SIMO MAC at high SNR.
- MIMO: As n_t increases, the gap between the TDMA and MIMO MAC rate regions decreases.

MISO BC System Model

- Downlink multi-user MIMO (MU-MIMO) transmission
  - total number of K users ($\mathcal{K} = \{1, \ldots, K\}$) distributed in a cell,
  - single receive antennas at mobile terminal q
  - n_t transmit antenna at the base station
- Received signal (we drop the time dimension)

$$y_q = \Lambda_q^{-1/2} \mathbf{h}_q \mathbf{x} + \mathbf{n}_q$$

  where
  - y_q scalar
  - $\mathbf{h}_q$ $[1 \times n_t]$ models the small scale time-varying fading process and Λ_q^{-1} refers to the large-scale fading accounting for path loss and shadowing
  - n_q is a complex Gaussian noise $\mathcal{CN}\left(0, \sigma_{n,q}^2\right)$.
- The input covariance matrix is defined as the covariance matrix of the transmit signal as $\mathbf{Q} = \mathcal{E}\left\{\mathbf{x}\mathbf{x}^H\right\}$.
- Power constraint: Tr$\{\mathbf{Q}\} \leq E_s$.

MISO BC System Model

- By stacking up the received signal vectors, the noise vectors and the channel matrices of all K users,

$$\mathbf{y} = [y_1, \ldots, y_K]^T,$$
$$\mathbf{n} = [n_1, \ldots, n_K]^T,$$
$$\mathbf{H} = \left[\Lambda_1^{-1/2}\mathbf{h}_1^T, \ldots, \Lambda_K^{-1/2}\mathbf{h}_K^T\right]^T,$$
$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n},$$

  $\mathbf{H}$ is assumed to be full-rank as it would be the case in a typical user deployment.
- SNR of user q defined as $\eta_q = E_s \Lambda_q^{-1}/\sigma_{n,q}^2$.
- Perfect instantaneous channel state information (CSI) at the Tx and all Rx.
- Generally speaking, $\mathbf{x}$ is written as the superposition of statistically independent signals $\mathbf{x}_q$

$$\mathbf{x} = \sum_{q=1}^{K} \mathbf{x}_q.$$

  The input covariance matrix of user q is defined as $\mathbf{Q}_q = \mathcal{E}\left\{\mathbf{x}_q \mathbf{x}_q^H\right\}$.

Capacity Region of MISO BC and its Achievability

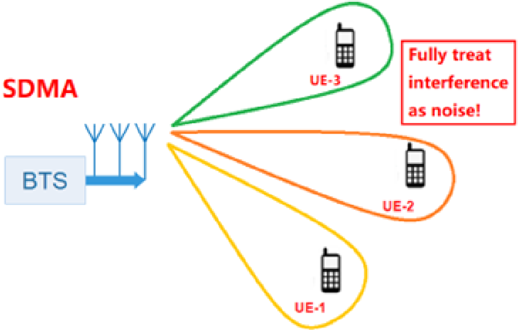
- MAC with multiple Rx antennas provides a tremendous capacity increase compared to suboptimal TDMA. So does BC with multiple Tx antennas!
- MISO BC difficult problem: users' channels cannot be ranked anymore.
- Assume an increasing encoding order from user 1 to K:
  1. Encode user 1's signal into $\mathbf{x}_1$.
  2. With full knowledge of $\mathbf{x}_1$, encode user 2's signal into $\mathbf{x}_2$ using DPC: $\mathbf{x}_1$ appears invisible to user 2 but $\mathbf{x}_2$ appears like a Gaussian interference to user 1.
  3. With full knowledge of user 1 and user 2's signals, encode user 3's signal into $\mathbf{x}_3$ using DPC.
  4. ... till K users are encoded.
- A given user q sees signals from users $p > q$ as a Gaussian interference but does not see any interference signals from users $p < q$:
  - Noise plus Interference at user q: $\sigma_{n,q}^2 + \Lambda_q^{-1}\mathbf{h}_q\left[\sum_{p>q}\mathbf{Q}_p\right]\mathbf{h}_q^H$.
  - Rate of user-q

$$R_q = \log_2\left[1 + \Lambda_q^{-1}\mathbf{h}_q\mathbf{Q}_q\mathbf{h}_q^H\left(\sigma_{n,q}^2 + \Lambda_q^{-1}\mathbf{h}_q\left[\sum_{p>q}\mathbf{Q}_p\right]\mathbf{h}_q^H\right)^{-1}\right]$$

- Capacity region: Repeat for all covariance matrices $\mathbf{Q}_1, \ldots, \mathbf{Q}_K$ satisfying the sum-power constraint $\sum_q \mathrm{Tr}\left\{\mathbf{Q}_q\right\} = E_s$ and all user ordering.
- Only DPC can achieve the MISO/MIMO BC sum-rate capacity.
- Careful! CSI assumed perfectly known to transmitter and receivers. If imperfect CSI, capacity unknown.

Capacity Region of MISO BC and its Achievability

- DPC not used in practice because too complex encoding.
- MU-MIMO/Space-Division Multiple Access (SDMA) using multi-user linear precoding used instead.



Fairness, Scheduling and Precoding

Multi-User Diversity

- In single-link systems, channel fading is viewed as a source of unreliability mitigated through diversity techniques (e.g. space-time coding).
- In multi-user communications, fading is viewed as a source of randomization that can be exploited!
- Multi-User (MU) diversity is a form of selection diversity among users provided by independent time-varying channels across the different users.
- Provided that the BS is able to track the user channel fluctuations (based on feedback), it can schedule transmissions to the users with favorable channel fading conditions, i.e. near their peaks, to improve the total cell throughput.
- Recall that MU diversity was already identified as part of the sum-rate maximization in SISO BC.

Multi-User Diversity Gain

- Assume that the fading distribution of the K users are independent and identically ($\Lambda_q^{-1} = \Lambda^{-1}$ and channel gains h_q are drawn from the same) Rayleigh distributed and that users experience the same average SNR $\eta_q = \eta$ ($\sigma_{n,q}^2 = \sigma_n^2$) $\forall q$:

$$y_q = \Lambda^{-1/2} h_q c + \mathbf{n}_q.$$

- Assume MU-SISO where one user is scheduled at a time in a TDMA manner: select the user with the largest channel gain.
- Mathematically same as antenna selection diversity.
- Average SNR gain
  - Average SNR after user selection $\bar{\rho}_{out}$

$$\bar{\rho}_{out} = \mathcal{E}\left\{\eta \max_{q=1,\ldots,K} |h_q|^2\right\} = \eta \sum_{q=1}^{K} \frac{1}{q}.$$

  - SNR gain provided by MU diversity g_m

$$g_m = \frac{\bar{\rho}_{out}}{\eta} = \sum_{q=1}^{K} \frac{1}{q} \overset{K\to\infty}{\cong} \log(K).$$

  g_m is of the order of $\log(K)$ and hence the gain of the strongest user grows as $\log(K)$!
- Heavily relies on CSIT (partial or imperfect feedback impacts the performance) and independent user fading distributions (correlated fading or LOS are not good for MU diversity)

Multi-User Diversity Gain

- Sum-rate capacity

$$\bar{C}_{TDMA} = \mathcal{E}\left\{C_{TDMA}\right\} = \mathcal{E}\left\{\log_2\left(1+\eta \max_{q=1,\ldots,K}|h_q|^2\right)\right\}.$$

  - low SNR

$$\bar{C}_{TDMA} \approx \mathcal{E}\left\{\max_{q=1,\ldots,K}|h_q|^2\right\}\eta\log_2(e) \approx g_m C_{awgn}.$$

    Observations: capacity of the fading channel $\log(K)$ times larger than the AWGN capacity.
  - high SNR (Use Jensen's inequality: $\mathcal{E}_x\left\{\mathcal{F}(x)\right\} \leq \mathcal{F}\left(\mathcal{E}_x\left\{x\right\}\right)$ if $\mathcal{F}$ concave)

$$\begin{aligned}
\bar{C}_{TDMA} &\approx \log_2(\eta) + \mathcal{E}\left\{\log_2\left(\max_{q=1,\ldots,K}|h_q|^2\right)\right\},\\
&\approx C_{awgn} + \mathcal{E}\left\{\log_2\left(\max_{q=1,\ldots,K}|h_q|^2\right)\right\},\\
&\overset{(a)}{\leq} C_{awgn} + \log_2\left(\mathcal{E}\left\{\max_{q=1,\ldots,K}|h_q|^2\right\}\right),\\
&= C_{awgn} + \log_2(g_m).
\end{aligned}$$

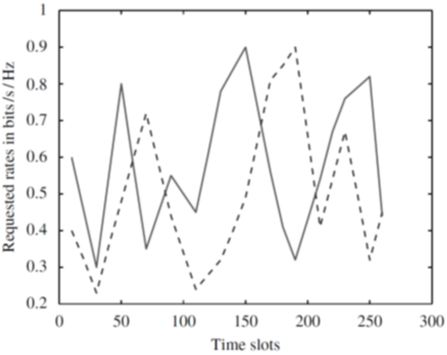
    Observations: capacity of a fading channel is larger than the AWGN capacity by a factor roughly equal to $\log_2(g_m) \approx \log\log(K)$.
- Fading channels are significantly more useful in a multi-user setting than in a single-user setting

Multi-User Diversity

- Few fundamental differences with classical spatial/time/frequency diversity:
  - Diversity techniques, like space-time coding, mainly focus on improving reliability by decreasing the outage probability in slow fading channels. MU diversity on the other hand increases the data rate over time-varying channels.
  - Classical diversity techniques mitigate fading while MU diversity exploits fading.
  - MU diversity takes a system-level view while classical diversity approaches focus on a single-link. This system-level view becomes increasingly important as we shift from single-cell to multi-cell scenarios.

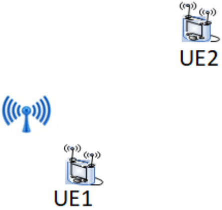
Fairness and Scheduling

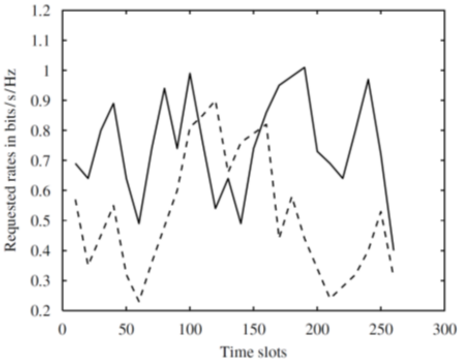
Figure 6.14 For symmetric channel statistics of users, the scheduling algorithm reduces to serving each user with the largest requested rate.



Fairness and Scheduling

Figure 6.15 In general, with asymmetric user channel statistics, the scheduling algorithm serves each user when it is near its peak within the latency time-scale t_c.





Fairness and Scheduling

- Sum-rate maximization in SISO BC: pick the strongest user. Is that fair?
- An appropriate scheduler should allocate resources (time, frequency, spatial, power) to the users in a fair manner while exploiting the MU diversity gain.
- Goal of the resource allocation strategy at the scheduler: maximize the utility metric $\mathcal{U}$.

$$q^\star = \arg\max_{q \in \mathcal{K}} \mathcal{U}$$

  where $q^\star$ refers to the optimum user (or more generally subset of users) to be scheduled.
- Two major kinds of resource allocation strategies:
  - *rate-maximization policy*: maximizes the sum-rate - no fairness among users
  - *fairness oriented policy*, commonly relying on a *proportional fair* (PF) metric: maximizes a weighted sum-rate and guarantees fairness among users.
- Those two strategies can be addressed by using two different utility metrics:

$$q^\star = \arg\max_{q \in \mathcal{K}} w_q R_q$$

  where
  - rate-maximization approach: $w_q = 1$
  - proportional fair approach: $w_q = \frac{\gamma_q}{\bar{R}_q}$ ($\bar{R}_q$ is the long-term average rate of user q and γ_q is the Quality of Service (QoS) of each user).

Practical Proportional Fair Scheduling

- The long-term average rate $\bar{R}_q$ of user q is updated using an exponentially weighted low-pass filter such that the estimate of $\bar{R}_q$ at time $k+1$, denoted as $\bar{R}_q\left(k+1\right)$, is function of the long-term average rate $\bar{R}_q\left(k\right)$ and of the current rate $R_q(k)$ at current time instant k as outlined by

$$\bar{R}_q\left(k+1\right) = \begin{cases} \left(1 - 1/t_c\right)\bar{R}_q\left(k\right) + 1/t_c R_q(k), & q \text{ scheduled at time } k \\ \left(1 - 1/t_c\right)\bar{R}_q\left(k\right), & q \text{ not scheduled at time } k \end{cases}$$

  where t_c is the scheduling time scale. User to be scheduled at time instant k is

$$q^{\star} = \arg\max_{q} \gamma_q \frac{R_q(k)}{\bar{R}_q(k)}.$$

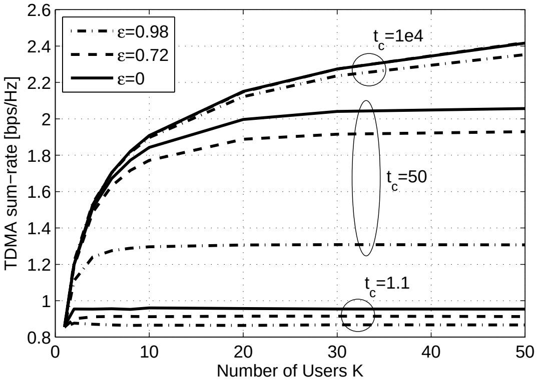
- The scheduling time scale t_c is a design parameter of the system that highly influences the user fairness and the performance
  - Very large t_c: assuming all users experience identical fading statistics and have the same QoS, the PF scheduler is equivalent to the rate-maximization scheduler, i.e. users contributing to the highest sum-rate are selected.
  - Small t_c: assuming all users have the same QoS, the scheduler divides the available resources equally among users (*Round-Robin* scheduling). No MU diversity is exploited.

Proportional Fair Scheduling

- Sum-rate of SISO TDMA with PF scheduling at SNR=0 dB as a function of the number of users K, the scheduling time scale t_c and the channel model

$$h_k = \epsilon h_{k-1} + \sqrt{1-\epsilon^2} n_k$$

with ϵ the channel time correlation coefficient.



User Grouping

- Given the presence of K users in the cell, the scheduler for MU-MIMO aims at finding the best scheduled user set among all possible candidates within $\mathcal{K}$.
- The *exhaustive search* is computationally intensive. Assuming a single stream transmission per user and $n_e \leq \min\{n_t, K\}$, a search like

$$\mathbf{K}^{\star} = \arg \max_{\substack{\mathbf{K} \subset \mathcal{K} \\ n_e \leq \min\{n_t, K\}}} \sum_{q \in \mathbf{K}} w_q R_q$$

  requires to consider a large number of different sets and has a complexity that quickly becomes cumbersome as K increases.

Precoding

- Downlink multi-user MIMO (MU-MIMO) transmission
  - total number of K users ($\mathcal{K} = \{1, \ldots, K\}$) distributed in a cell,
  - $n_{r,q}$ receive antennas at mobile terminal q (we simply drop the index q and write n_r if $n_{r,q} = n_r \ \forall q$)
  - n_t transmit antenna at the base station
- Received signal (we drop the time dimension)

$$\mathbf{y}_q = \Lambda_q^{-1/2} \mathbf{H}_q \mathbf{x} + \mathbf{n}_q$$

where

  - $\mathbf{y}_q \in \mathbb{C}^{n_{r,q}}$
  - $\mathbf{H}_q \in \mathbb{C}^{n_{r,q} \times n_t}$ models the small scale time-varying fading process and Λ_q^{-1} refers to the large-scale fading accounting for path loss and shadowing
  - $\mathbf{n}_q$ is a complex Gaussian noise $\mathcal{CN}\left(0, \sigma_{n,q}^2 \mathbf{I}_{n_{r,q}}\right)$.
- Long term SNR of user q defined as $\eta_q = E_s \Lambda_q^{-1} / \sigma_{n,q}^2$.
- Generally speaking, $\mathbf{c}'$ is written as the superposition of statistically independent signals $\mathbf{x}_q$

$$\mathbf{x} = \sum_{q=1}^{K} \mathbf{x}_q.$$

- Power constraint: $\text{Tr}\{\mathbf{Q}\} \leq E_s$ with $\mathbf{Q} = \mathcal{E}\left\{\mathbf{x}\mathbf{x}^H\right\}$.

Precoding

- *scheduled user set*, denoted as $\mathbf{K} \subset \mathcal{K}$, is the set of users who are actually scheduled (with a non-zero transmit power) by the transmitter at the time instant of interest.
- The transmitter serves users belonging to $\mathbf{K}$ with n_e data streams and user $q \in \mathbf{K}$ is served with $n_{u,q}$ data streams $(n_{u,q} \leq n_e)$. Hence, $n_e = \sum_{q \in \mathbf{K}} n_{u,q}$.
- Linear Precoding

$$\mathbf{x} = \mathbf{P}\mathbf{c} = \mathbf{W}\mathbf{S}^{1/2}\mathbf{c} = \sum_{q \in \mathbf{K}} \mathbf{P}_q \mathbf{c}_q = \sum_{q \in \mathbf{K}} \mathbf{W}_q \mathbf{S}_q^{1/2} \mathbf{c}_q$$

  where
  - $\mathbf{c}$ is the symbol vector made of n_e unit-energy independent symbols.
  - $\mathbf{P} \in \mathbb{C}^{n_t \times n_e}$ is the precoder subject to Tr$\{\mathbf{P}\mathbf{P}^H\} \leq E_s$, made of two matrices: a power control diagonal matrix denoted as $\mathbf{S} \in \mathbb{R}^{n_e \times n_e}$ and a transmit beamforming matrix $\mathbf{W} \in \mathbb{C}^{n_t \times n_e}$.
  - $\mathbf{P}_q \in \mathbb{C}^{n_t \times n_{u,q}}$, $\mathbf{W}_q \in \mathbb{C}^{n_t \times n_{u,q}}$, $\mathbf{S}_q \in \mathbb{R}^{n_{u,q} \times n_{u,q}}$, and $\mathbf{c}_q \in \mathbb{C}^{n_{u,q}}$ are user q's sub-matrices and sub-vector of $\mathbf{P}$, $\mathbf{W}$, $\mathbf{S}$, and $\mathbf{c}$, respectively.
- Received signal $\mathbf{y}_q \in \mathbb{C}^{n_{r,q}}$

$$\mathbf{y}_q = \Lambda_q^{-1/2} \mathbf{H}_q \mathbf{W}_q \mathbf{S}_q^{1/2} \mathbf{c}_q + \sum_{p \in \mathbf{K},\, p \neq q} \Lambda_q^{-1/2} \mathbf{H}_q \mathbf{W}_p \mathbf{S}_p^{1/2} \mathbf{c}_p + \mathbf{n}_q.$$

- Filtered received signal $\mathbf{z}_q = \mathbf{G}_q \mathbf{y}_q \in \mathbb{C}^{n_{u,q}}$ at user q with $\mathbf{G}_q \in \mathbb{C}^{n_{u,q} \times n_{r,q}}$

$$\mathbf{z}_q = \Lambda_q^{-1/2} \mathbf{G}_q \mathbf{H}_q \mathbf{W}_q \mathbf{S}_q^{1/2} \mathbf{c}_q + \sum_{p \in \mathbf{K},\, p \neq q} \Lambda_q^{-1/2} \mathbf{G}_q \mathbf{H}_q \mathbf{W}_p \mathbf{S}_p^{1/2} \mathbf{c}_p + \mathbf{G}_q \mathbf{n}_q.$$

Achievable Rate

- Maximum rate achievable by user q with linear precoding is

$$R_q = \sum_{l=1}^{n_{u,q}} \log_2 \left(1 + \rho_{q,l}\right)$$

where $\rho_{q,l}$ denotes the SINR experienced by stream l of user q

$$\rho_{q,l} = \frac{\Lambda_q^{-1} \left|\mathbf{g}_{q,l}\mathbf{H}_q\mathbf{p}_{q,l}\right|^2}{I_l + I_c + \left\|\mathbf{g}_{q,l}\right\|^2 \sigma_{n,q}^2} = \frac{\Lambda_q^{-1} \left|\mathbf{g}_{q,l}\mathbf{H}_q\mathbf{w}_{q,l}\right|^2 s_{q,l}}{I_l + I_c + \left\|\mathbf{g}_{q,l}\right\|^2 \sigma_{n,q}^2}$$

with $\mathbf{p}_{q,l} = \mathbf{w}_{q,l}s_{q,l}$ (resp. $\mathbf{g}_{q,l}$) the precoder (resp. combiner) attached to stream l of user q, I_l the inter-stream interference and I_c the intra-cell interference (also called multi-user interference)

$$I_l = \sum_{m \neq l} \Lambda_q^{-1} \left|\mathbf{g}_{q,l}\mathbf{H}_q\mathbf{p}_{q,m}\right|^2 = \sum_{m \neq l} \Lambda_q^{-1} \left|\mathbf{g}_{q,l}\mathbf{H}_q\mathbf{w}_{q,m}\right|^2 s_{q,m},$$

$$I_c = \sum_{\substack{p \in \mathbf{K} \\ p \neq q}} \sum_{m=1}^{n_{u,p}} \Lambda_q^{-1} \left|\mathbf{g}_{q,l}\mathbf{H}_q\mathbf{p}_{p,m}\right|^2 = \sum_{\substack{p \in \mathbf{K} \\ p \neq q}} \sum_{m=1}^{n_{u,p}} \Lambda_q^{-1} \left|\mathbf{g}_{q,l}\mathbf{H}_q\mathbf{w}_{p,m}\right|^2 s_{p,m}.$$

- If $n_r = 1$, the SINR of user q simply reads as $\rho_q = \frac{\Lambda_q^{-1}\left|\mathbf{h}_q\mathbf{w}_q\right|^2 s_q}{\sum_{\substack{p \in \mathbf{K} \\ p \neq q}} \Lambda_q^{-1}\left|\mathbf{h}_q\mathbf{w}_p\right|^2 s_p + \sigma_{n,q}^2}$.
- Various designs: MBF/MRT, ZFBF, R-ZFBF, BD, ...

Zero-Forcing Beamforming (ZFBF)

- Most popular MU-MIMO precoder. Assume single receive antenna per user ($n_{r,q} = 1$).
- Channel Direction Information (CDI) of user q: $\bar{\mathbf{h}}_q = \mathbf{h}_q / \|\mathbf{h}_q\|$.
- Idea is to force the intra-cell interference I_c to zero: the precoder of a user q, $\mathbf{w}_q$, is chosen such that $\mathbf{h}_p \mathbf{w}_q = 0 \ \forall p \in \mathbf{K} \setminus q$. Only possible if $n_e \leq n_t$!

$$y_q = \Lambda_q^{-1/2} \mathbf{h}_q \mathbf{w}_q s_q^{1/2} c_q + \sum_{p \in \mathbf{K},\, p \neq q} \Lambda_q^{-1/2} \underbrace{\mathbf{h}_q \mathbf{w}_p}_{\text{to force to zero}} s_p^{1/2} c_p + n_q.$$

- Define

$$\mathbf{H} = \left[\Lambda_i^{-1/2} \mathbf{h}_i^T, \ldots, \Lambda_j^{-1/2} \mathbf{h}_j^T \right]_{i,j \in \mathbf{K}}^T = \mathbf{D} \bar{\mathbf{H}}$$

with $\mathbf{D} = \text{diag}\left\{ \Lambda_i^{-1/2} \|\mathbf{h}_i\|, \ldots, \Lambda_j^{-1/2} \|\mathbf{h}_j\| \right\}_{i,j \in \mathbf{K}}$, $\bar{\mathbf{H}} = \left[\bar{\mathbf{h}}_i^T, \ldots, \bar{\mathbf{h}}_j^T \right]_{i,j \in \mathbf{K}}^T$. The ZFBF aims at designing $\mathbf{W} = [\mathbf{w}_i, \ldots, \mathbf{w}_j]_{i,j \in \mathbf{K}}$ such that $\mathbf{HW}$ is diagonal.

- Assuming $n_e \leq n_t$ and $\bar{\mathbf{H}}$ is full rank, the precoders can be chosen as the normalized columns of the right pseudo inverse of $\mathbf{H}$

$$\mathbf{F} = \mathbf{H}^H \left(\mathbf{H}\mathbf{H}^H \right)^{-1} = \bar{\mathbf{F}} \mathbf{D}^{-1} = \bar{\mathbf{H}}^H \left(\bar{\mathbf{H}} \bar{\mathbf{H}}^H \right)^{-1} \mathbf{D}^{-1}.$$

Transmit precoder $\mathbf{w}_q$ for user $q \in \mathbf{K}$: $\mathbf{w}_q = \mathbf{F}(:,q) / \|\mathbf{F}(:,q)\| = \bar{\mathbf{F}}(:,q) / \|\bar{\mathbf{F}}(:,q)\|$ where $\mathbf{F}(:,q)$ is to be viewed as the column of $\mathbf{F}$ corresponding to user q.

Zero-Forcing Beamforming (ZFBF)

- Assuming that $\mathbf{c} = \begin{bmatrix} c_i, \ldots, c_j \end{bmatrix}^T_{i,j \in \mathbf{K}}$, the received signal of user $q \in \mathbf{K}$ is

$$\mathbf{y}_q = \Lambda_q^{-1/2} \mathbf{h}_q \mathbf{w}_q s_q^{1/2} c_q + n_q = d_q c_q + n_q,$$

  with $d_q = \Lambda_q^{-1/2} \mathbf{h}_q \mathbf{w}_q s_q^{1/2} = \Lambda_q^{-1/2} \frac{\left\| \mathbf{h}_q \right\|}{\left\| \bar{\mathbf{F}}(:,q) \right\|} s_q^{1/2}$.
  Observations: MU-MIMO channel with ZFBF is split into n_e parallel (non-interfering) channels.
- The rate achievable by user q is given by

$$R_q = \log_2 \left(1 + d_q^2 / \sigma_{n,q}^2 \right).$$

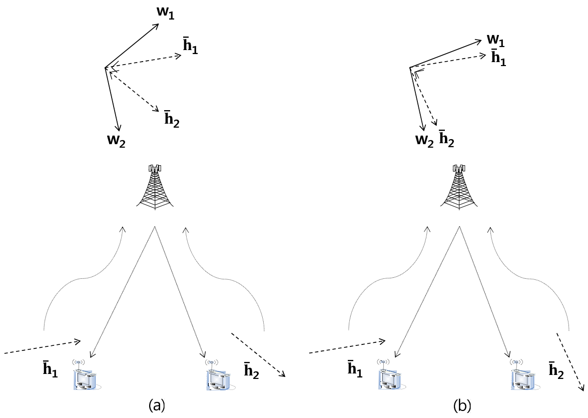
  d_q^2 is low if $\mathbf{H}$ is badly conditioned but would get larger if users' CDI are orthogonal or quasi-orthogonal.
  - reminiscent of the loss caused by noise enhancement incurred by the linear ZF
- For large K, better conditioning of matrix $\mathbf{H}$ through the use of user grouping.
- By uniformly allocating the power across user streams $s_q = E_s / n_e$ and by choosing $n_e = \tilde{n} = \min \{ n_t, K \}$, $d_q^2 / \sigma_{n,q}^2 = \alpha_q^2 \eta_q / n_e$ with $\alpha_q^2 = \left| \mathbf{h}_q \mathbf{w}_q \right|^2 = \left\| \mathbf{h}_q \right\|^2 / \left\| \bar{\mathbf{F}}(:,q) \right\|^2$

$$C_{BF}(\mathbf{H}) = \sum_{q=1}^{\min\{n_t, K\}} \log_2 \left(1 + \alpha_q^2 \frac{\eta_q}{\tilde{n}} \right).$$

  At high SNR with $\eta_q = \eta$, $C_{BF}(\mathbf{H}) \approx \min \{ n_t, K \} \log_2 (\eta_q)$. The multiplexing gain $\min \{ n_t, K \}$ is achieved (same as with DPC).

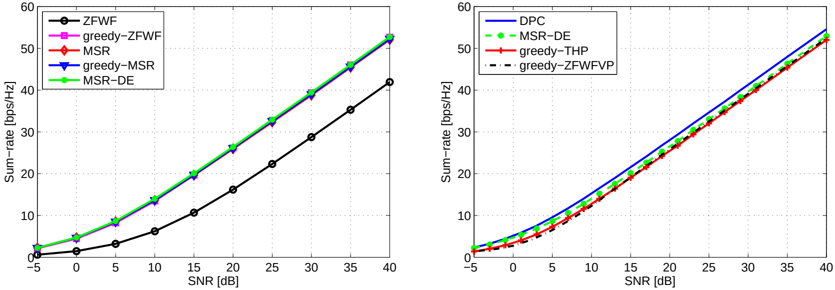
Zero-Forcing Beamforming (ZFBF)

- Illustration of ZFBF precoding for a two-user scenario: (a) non-orthogonal user set, (b) quasi-orthogonal user set.



Global Performance Comparison

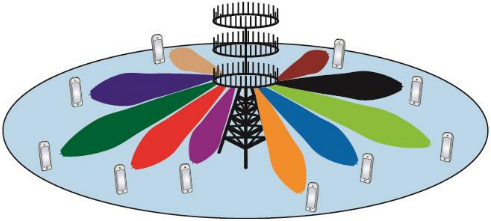
- Sum-rate of linear (left) and non-linear (right) MU-MIMO precoders vs SNR in $n_t = 4, K = 20$ i.i.d. Rayleigh fading channels



Observations: ZFBF without user selection (ZFWF) performs poorly. ZFBF with user selection (greedy-ZFWF) is a competitive strategy for MU-MIMO broadcast channels, in terms of both performance and complexity.
Keep in mind the assumptions: perfect CSIT, the same average SNR for all users and a max-rate scheduler (i.e. there is no fairness issue involved here).

Massive MIMO

Massive MIMO



- Very large number of transmit antennas n_t in the downlink (MISO/MIMO BC) or receive antennas n_r in the uplink (SIMO/MIMO MAC).
- Narrow beams: large beamforming gain and low multi-user interference.
- Simple precoder design sufficient.
- Simultaneously serve many users in the same resource block, simplified scheduling.
- Massive demand for CSIT.

Massive MIMO: Intuition

- Assume independent and identically Rayleigh fading (across antennas and independent across users). By the law of large numbers, channels become mutually orthogonal as n_t becomes large (decorrelation effect),

$$\lim_{n_t \to \infty} \frac{1}{n_t} \mathbf{H}_l \mathbf{H}_p^H = \mathbf{I}_{n_r} \delta_{lp}, \qquad \forall l, p = 1, \ldots, K,$$

  for MIMO,

$$\lim_{n_t \to \infty} \frac{1}{n_t} \mathbf{h}_l \mathbf{h}_p^H = \delta_{lp}, \qquad \forall l, p = 1, \ldots, K,$$

  for MISO ($\delta_{lp} = 0$ if $l \neq p$, 1 if $l = p$).

Massive MIMO: Matched Beamforming (MBF)

- As n_t increases, with perfect CSIT, the channel of user q becomes orthogonal to co-scheduled users $(p \neq q)$ channels. Hence, the multi-user interference is naturally eliminated by matching user q precoder to user q channel.
- Massive MIMO is spectrally and energy efficient:
  - Assume a single receive antenna for simplicity, and transmit with a MBF/MRT precoder $\mathbf{w}_p = \bar{\mathbf{h}}_p^H = \mathbf{h}_p^H / \|\mathbf{h}_p\|$ and a transmit power $s_p = E_s/n_t \ \forall p = 1, \ldots, K$
  - For large n_t, the SINR ρ_q of user q simplifies as

$$\rho_q = \frac{\Lambda_q^{-1} \left|\mathbf{h}_q \mathbf{w}_q\right|^2 E_s/n_t}{\sum_{\substack{p \in \mathbf{K} \\ p \neq q}} \Lambda_q^{-1} \left|\mathbf{h}_q \mathbf{w}_p\right|^2 E_s/n_t + \sigma_{n,q}^2} \overset{n_t \nearrow}{\approx} \frac{\Lambda_q^{-1} \left\|\mathbf{h}_q\right\|^2 E_s/n_t}{\sigma_{n,q}^2} \approx \frac{\Lambda_q^{-1} E_s}{\sigma_{n,q}^2} = \eta_q$$

  and the sum-rate is equal to

$$C_{BF}(\mathbf{H}) \approx \bar{C}_{BF} \approx \bar{C}_{BF} \approx \sum_{q=1}^{K} \log_2 (1 + \eta_q)$$

  for a total transmit power $K E_s/n_t$.
  - By MBF with a power E_s/n_t per user in a large MISO system (i.e. the transmit power is scaled down proportionally to $1/n_t$), each of the K users gets the same rate as if it were scheduled on a SISO AWGN channel with a transmit power E_s (and received SNR η_q) without any intra-cell interference and without any fading.
  - Assuming $\eta_q = \eta \ \forall q$, the total achievable sum-rate writes as K times the SISO AWGN rate.
  - The transmit power is scaled down proportionally to $1/n_t$ and the multiplexing gain increased proportional to K.

Massive MIMO: Zero-Forcing Beamforming

- ZFBF precoding: $\mathbf{w}_q$ for user $q \in \mathbf{K}$ writes as

$$\mathbf{w}_q = \mathbf{F}(:,q)/\|\mathbf{F}(:,q)\| = \bar{\mathbf{F}}(:,q)/\|\bar{\mathbf{F}}(:,q)\|$$

with

$$\begin{aligned} \mathbf{F} &= \mathbf{H}^H \left(\mathbf{H}\mathbf{H}^H\right)^{-1}, \\ &= \underbrace{\bar{\mathbf{H}}^H \left(\bar{\mathbf{H}}\bar{\mathbf{H}}^H\right)^{-1}}_{\bar{\mathbf{F}}} \mathbf{D}^{-1}. \end{aligned}$$

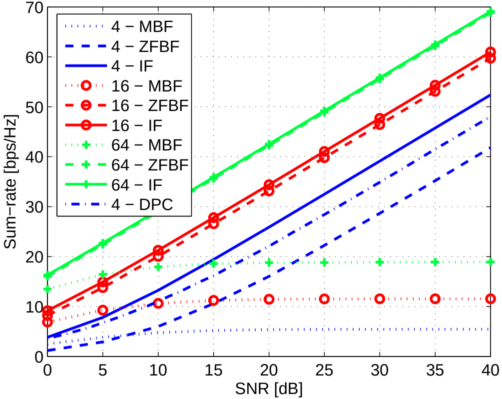
- Massive MIMO effect also benefits ZFBF:
  - As n_t grows, $\mathbf{H}\mathbf{H}^H$ and $\bar{\mathbf{H}}\bar{\mathbf{H}}^H$ become better conditioned, thereby simplifying the computation of the matrix inverse.
  - In the limit where user channels are orthogonal, $\mathbf{H}\mathbf{H}^H$ and $\bar{\mathbf{H}}\bar{\mathbf{H}}^H$ are diagonal and ZFBF boils down to MBF.

Massive MIMO: Channel Hardening and Scheduling

- The transmit (and also receive) beamforming gain approximates as n_t for large n_t. As n_t increases, the value of $\|\mathbf{h}_q\|^2$, being a $\chi^2_{2n_t}$ distributed random variable, concentrates indeed more and more around its mean (channel hardening).
- The SINR and the sum-rates become exclusively a function of Λ_q and not of the fading (that is so useful to benefit from MU diversity).
- Transmit/Receive beamforming and MU diversity are somehow not complementary. A large n_t benefits the array gain and multiplexing gain but restricts the MU diversity gain.
- In general, any transmission scheme that exploits spatial diversity reduces the multi-user diversity gain because of the channel hardening effect.
- Scheduling in Massive MIMO becomes much simpler.

Sum-Rate Evaluations

- Performance of MBF, ZFBF, DPC, IF in $n_t = 4, 16, 64$ and $K = 4$ i.i.d. Rayleigh fading channels.
  - IF stands for interference free and is the upper bound on the performance obtained assuming perfect matched beamforming, no intra-cell interference and an uniform power allocation across the four users, leading to a sum-rate of $\sum_{q=1}^{K} \log_2 \left(1 + \eta_q / K \left\| \mathbf{h}_q \right\|^2 \right)$.



Sum-Rate Evaluations

- As n_t grows, the gap between IF and ZFBF shrinks significantly:
  - Severe gap exists in the four transmit antenna case between IF, DPC and ZFBF,
  - The gap completely vanishes with 64 transmit antennas with ZFBF performing as well as an IF system.
  - Hence the performance gain of advanced precoding techniques does not justify the complexity increase.

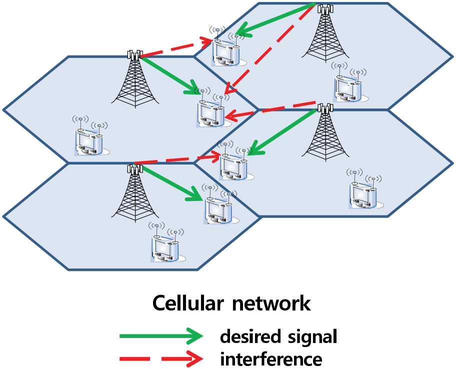
- MBF on the other hand performs relatively poorly (except at low SNR)
  - sum-rate performance fundamentally limited by intra-cell interference and his SINR is limited at high SNR by the ratio $\alpha = n_t/K$.
  - MBF requires a much larger number of antennas to reach the same performance as ZFBF.

- In summary, for $K >> n_t$ (with scheduling) and $n_t >> K$, simple linear precoding schemes provide very competitive alternatives to more complex (non-linear) strategies.

Multiuser Multicell Communications

Interference

- Current wireless networks primarily operate using a frequency reuse 1 (or close to 1), i.e. all cells share the same frequency band
- Interference is not only made of intra-cell (i.e. multi-user interference), but also of inter-cell (i.e. multi-cell) interference.
- Cell edge performance is primarily affected by the inter-cell interference.



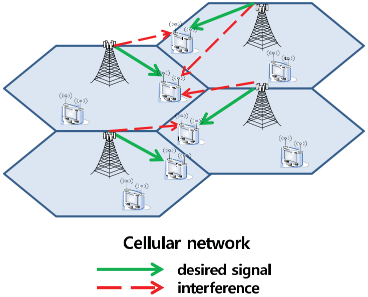
Cellular network

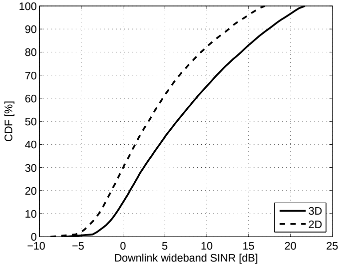
Wideband/long-term SINR

- For user q in cell i, the wideband/long-term SINR is commonly evaluated by ignoring the effect of fading but only account for path loss and shadowing

$$SINR_{w,q} = \frac{\Lambda_{q,i}^{-1} E_{s,i}}{\sigma_{n,q}^2 + \sum_{j \neq i} \Lambda_{q,j}^{-1} E_{s,j}}.$$

- Provides a rough estimate of the network performance. Function of major propagation mechanisms (path loss, shadowing, antenna radiation patterns,...), base stations deployment and user distribution.
- CDF of $SINR_{w,q}$ in a frequency reuse 1 network (cells share the same frequency band) with 2D and 3D antenna patterns in urban macro deployment.





Multi-User Multi-Cell Network

- General downlink multi-cell multi-user MIMO network with a total number of K_T users distributed in n_c cells.
- K_i users in every cell i, $n_{t,i}$ transmit antennas at BS i, $n_{r,q}$ receive antennas at mobile terminal q.
- The received signal of a given user q in cell i is

$$\mathbf{y}_q = \Lambda_{q,i}^{-1/2}\mathbf{H}_{q,i}\mathbf{c}_i' + \underbrace{\sum_{j\neq i} \Lambda_{q,j}^{-1/2}\mathbf{H}_{q,j}\mathbf{c}_j'}_{\text{inter-cell interference}} + \mathbf{n}_q$$

where

- $\mathbf{y}_q \in \mathbb{C}^{n_{r,q}}$,
- $\mathbf{n}_q$ is a complex Gaussian noise $\mathcal{CN}\left(0, \sigma_{n,q}^2 \mathbf{I}_{n_{r,q}}\right)$,
- $\Lambda_{q,i}^{-1}$ refers to the path-loss and shadowing between transmitter i and user q,
- $\mathbf{H}_{q,i} \in \mathbb{C}^{n_{r,q} \times n_{t,i}}$ models the MIMO fading channel between transmitter i and user q.

Linear Precoding

- *scheduled user set* of cell i, denoted as $\mathbf{K}_i$, as the set of users who are actually scheduled by BS i at the time instant of interest

- Transmit $n_{e,i}$ streams in each cell i using MU-MIMO linear precoding

$$\mathbf{c}_i' = \mathbf{P}_i \mathbf{c}_i = \mathbf{W}_i \mathbf{S}_i^{1/2} \mathbf{c}_i = \sum_{q \in \mathbf{K}_i} \mathbf{P}_{q,i} \mathbf{c}_{q,i} = \sum_{q \in \mathbf{K}_i} \mathbf{W}_{q,i} \mathbf{S}_{q,i}^{1/2} \mathbf{c}_{q,i}$$

  where

  - $\mathbf{c}_i$ is the symbol vector made of $n_{e,i}$ unit-energy independent symbols
  - $\mathbf{P}_i \in \mathbb{C}^{n_{t,i} \times n_{e,i}}$ is the precoder made of two matrices, namely a power control diagonal matrix denoted as $\mathbf{S}_i \in \mathbb{R}^{n_{e,i} \times n_{e,i}}$ and a transmit beamforming matrix $\mathbf{W}_i \in \mathbb{C}^{n_{t,i} \times n_{e,i}}$.
  - $\mathbf{P}_{q,i} \in \mathbb{C}^{n_{t,i} \times n_{u,q}}$, $\mathbf{W}_{q,i} \in \mathbb{C}^{n_{t,i} \times n_{u,q}}$, $\mathbf{S}_{q,i} \in \mathbb{R}^{n_{u,q} \times n_{u,q}}$, and $\mathbf{c}_{q,i} \in \mathbb{C}^{n_{u,q}}$ are user q's sub-matrices and sub-vector of $\mathbf{P}_i$, $\mathbf{W}_i$, $\mathbf{S}_i$, and $\mathbf{c}_i$, respectively.
  - The input covariance matrix at cell i is $\mathbf{Q}_i = \mathcal{E}\left\{\mathbf{c}_i' \mathbf{c}_i'^H\right\}$ subject to the transmit power constraint Tr$\{\mathbf{Q}_i\} \leq E_{s,i}$.

Linear Precoding

- The received signal $\mathbf{y}_q \in \mathbb{C}^{n_{r,q}}$ of user $q \in \mathbf{K}_i$

$$\mathbf{y}_q = \Lambda_{q,i}^{-1/2}\mathbf{H}_{q,i}\mathbf{P}_{q,i}\mathbf{c}_{q,i} + \underbrace{\sum_{p\in\mathbf{K}_i,\, p\neq q} \Lambda_{q,i}^{-1/2}\mathbf{H}_{q,i}\mathbf{P}_{p,i}\mathbf{c}_{p,i}}_{\text{intra-cell (multi-user) interference}}$$

$$+ \underbrace{\sum_{j\neq i}\sum_{l\in\mathbf{K}_j} \Lambda_{q,j}^{-1/2}\mathbf{H}_{q,j}\mathbf{P}_{l,j}\mathbf{c}_{l,j}}_{\text{inter-cell interference}} + \mathbf{n}_q.$$

- Apply a receive combiner to stream l of user q in cell i

$$z_{q,l} = \mathbf{g}_{q,l}\mathbf{y}_q = \Lambda_{q,i}^{-1/2}\mathbf{g}_{q,l}\mathbf{H}_{q,i}\mathbf{p}_{q,i,l}c_{q,i,l} + \underbrace{\sum_{m\neq l}\Lambda_{q,i}^{-1/2}\mathbf{g}_{q,l}\mathbf{H}_{q,i}\mathbf{p}_{q,i,m}c_{q,i,m}}_{\text{inter-stream interference}}$$

$$+ \underbrace{\sum_{p\in\mathbf{K}_i,\, p\neq q} \Lambda_{q,i}^{-1/2}\mathbf{g}_{q,l}\mathbf{H}_{q,i}\mathbf{P}_{p,i}\mathbf{c}_{p,i}}_{\text{intra-cell (multi-user) interference}} + \underbrace{\sum_{j\neq i}\sum_{l\in\mathbf{K}_j} \Lambda_{q,j}^{-1/2}\mathbf{g}_{q,l}\mathbf{H}_{q,j}\mathbf{P}_{l,j}\mathbf{c}_{l,j}}_{\text{inter-cell interference}} + \mathbf{g}_{q,l}\mathbf{n}_q.$$

Achievable Rate

- By treating all interference as noise, the maximum rate achievable by user q in cell i with linear precoding is

$$R_{q,i} = \sum_{l=1}^{n_{u,q}} \log_2 \left(1 + \rho_{q,l}\right).$$

- The quantity $\rho_{q,l}$ denotes the SINR experienced by stream l of user-q and writes as

$$\rho_{q,l} = \frac{S}{I_l + I_c + I_o + \left\|\mathbf{g}_{q,l}\right\|^2 \sigma_{n,q}^2}.$$

where S refers to the received signal power of the intended stream, I_l the inter-stream interference, I_c the intra-cell interference (i.e. interference from co-scheduled users) and I_o the inter-cell interference and they write as

$$S = \Lambda_{q,i}^{-1} \left|\mathbf{g}_{q,l}\mathbf{H}_{q,i}\mathbf{p}_{q,i,l}\right|^2,$$

$$I_l = \sum_{m \neq l} \Lambda_{q,i}^{-1} \left|\mathbf{g}_{q,l}\mathbf{H}_{q,i}\mathbf{p}_{q,i,m}\right|^2,$$

$$I_c = \sum_{p \in \mathbf{K}_i, p \neq q} \sum_{m=1}^{n_{u,p}} \Lambda_{q,i}^{-1} \left|\mathbf{g}_{q,l}\mathbf{H}_{q,i}\mathbf{p}_{p,i,m}\right|^2,$$

$$I_o = \sum_{j \neq i} \Lambda_{q,j}^{-1} \left\|\mathbf{g}_{q,l}\mathbf{H}_{q,j}\mathbf{P}_j\right\|^2.$$

Achievable Rate

Example

Given the precoders in all cells, what is the SINR of stream l of user-q in cell i?

- Noise plus interference: $I_l + I_c + I_o + \|\mathbf{g}_{q,l}\|^2 \sigma_{n,q}^2 = \mathbf{g}_{q,l}\mathbf{R}_{\mathbf{n_i}}\mathbf{g}_{q,l}^H$ where

$$\mathbf{R}_{\mathbf{n_i}} = \sum_{m \neq l} \Lambda_{q,i}^{-1}\mathbf{H}_{q,i}\mathbf{p}_{q,i,m}\left(\mathbf{H}_{q,i}\mathbf{p}_{q,i,m}\right)^H + \sum_{p \in \mathbf{K}_i, p \neq q} \sum_{m=1}^{n_{u,p}} \Lambda_{q,i}^{-1}\mathbf{H}_{q,i}\mathbf{p}_{p,i,m}\left(\mathbf{H}_{q,i}\mathbf{p}_{p,i,m}\right)^H + \sum_{j \neq i} \Lambda_{q,j}^{-1}\mathbf{H}_{q,j}\mathbf{P}_j\left(\mathbf{H}_{q,j}\mathbf{P}_j\right)^H + \sigma_{n,q}^2\mathbf{I}_{n_{r,q}}$$

  is the covariance matrix of the noise plus interference.

- MMSE combiner for stream l: $\mathbf{g}_{q,l} = \Lambda_{q,i}^{-1/2}\left(\mathbf{H}_{q,i}\mathbf{p}_{q,i,l}\right)^H \mathbf{R}_{\mathbf{n_i}}{}^{-1}$
- SINR $\rho_{q,l}$ experienced by stream l of user-q

$$\rho_{q,l} = \frac{\Lambda_{q,i}^{-1}\left|\mathbf{g}_{q,l}\mathbf{H}_{q,i}\mathbf{p}_{q,i,l}\right|^2}{\mathbf{g}_{q,l}\mathbf{R}_{\mathbf{n_i}}\mathbf{g}_{q,l}^H} = \Lambda_{q,i}^{-1}\left(\mathbf{H}_{q,i}\mathbf{p}_{q,i,l}\right)^H \mathbf{R}_{\mathbf{n_i}}{}^{-1}\mathbf{H}_{q,i}\mathbf{p}_{q,i,l}.$$

Part 4: Convex Optimization for Wireless Communications

Introduction and Fundamentals

Optimization Methods and Applications in Wireless Communications

Convex Optimization for Wireless Communications

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January 2023

Outline

Table 1: List of Notations and Descriptions

Notation	Description
$\mathbf{A}$, $\mathbf{a}$, and a	Matrix, vector, and scalar
$\mathbb{C}$ and R	Set of complex numbers and real numbers
$(\cdot)^*$ and $(\cdot)^{-1}$	Conjugate and inversion
$(\cdot)^H$ and $(\cdot)^T$	Conjugate-transpose and transpose
$\Re\{\cdot\}$ and $\Im\{\cdot\}$	Real and imaginary part of a complex number
$\lvert \cdot \rvert$	Absolute value of a scalar
$\lVert \cdot \rVert_2, \lVert \cdot \rVert$	ℓ-2 (Euclidean) norm of a vector
$\mathrm{diag}(\cdot)$	Diagonal matrix
$\mathbf{Tr}(\cdot)$	Summation of diagonal elements of a matrix
$\mathrm{rank}(\cdot)$	The rank of a matrix
$[\mathbf{A}]_{i,j}$	The (i,j)-th element of matrix $\mathbf{A}$
$[\mathbf{a}]_i$	The i-th element of vector $\mathbf{a}$

General Optimization Problem

- A mathematical optimization problem has the form[1]

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0, \forall i = 1, \ldots, m \\ & h_j(\mathbf{x}) = 0, \forall j = 1, \ldots, p \end{aligned} \tag{1}$$

  - $\mathbf{x} = [x_1, x_2, \ldots, x_n]$: optimization variable
  - $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$: objective function
  - $f_i : \mathbb{R}^n \rightarrow \mathbb{R}, \forall i$: inequality constraint functions
  - $h_j : \mathbb{R}^n \rightarrow \mathbb{R}, \forall j$: equality constraint functions
- $\mathbf{x}^\star$ is the globally optimal solution if $f_0(\mathbf{x}^\star) \leq f_0(\mathbf{x})$ for all feasible $\mathbf{x}$ satisfying the constraints.
- $\mathbf{x}'$ is the locally optimal solution if $f_0(\mathbf{x}') \leq f_0(\mathbf{x})$ for all feasible $\mathbf{x}$ satisfying the constraints and $||\mathbf{x} - \mathbf{x}'|| < \epsilon$ for some $\epsilon > 0$.

[1] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Solving Optimization Problems

- General optimization problem is difficult to solve.
- Certain problems, e.g., convex optimization, can be solved efficiently and reliably. We can efficiently find the global solution of convex optimization problems in many cases.
- A *convex optimization* is one in which the objective and constraint functions are all convex[2], i.e.,

$$f_i(\alpha \mathbf{x} + \beta \mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y}), \forall i = 0, \ldots, m \tag{2}$$

  - $\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\forall\ \alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$

- When the equality of (2) is achieved, the convex optimization boils down to *linear programming*, i.e., linear $\subset$ convex.

[2] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Research Steps

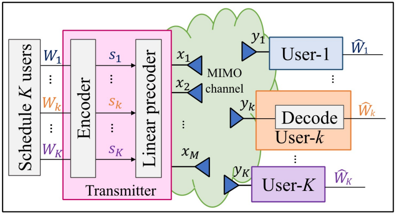
- Find a research topic and learn the background knowledge (thorough literature review).
- Grasp the research gap and fill the blanks.
- Establish the mathematical model based on communication theory principles and formulate the corresponding optimization problem.
- **Design algorithms to solve the optimization problem based on convex/non-convex optimization theories.**
- Verify the effectiveness of the proposed design based on numerical simulations.

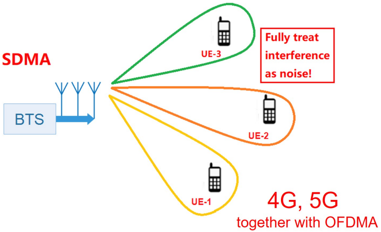
Wireless Communications and Optimization

- **Design algorithms to solve the optimization problem based on convex/non-convex optimization theories.**
  - The optimization problems in wireless communications are usually non-convex problems and difficult to solve.
  - One common solution is to transform non-convex problems into convex ones.
- **Q:** How to transform non-convex problem into convex ones (or decouple the original problem into several convex sub-problems)?
- **Key Point:** Identify different kinds of mathematical problems and map them into problems in wireless communications

Applications in Wireless Communications

- MU-MISO/SDMA with linear precoding: $y_k = \mathbf{h}_k \left(\sum_{i=1}^{K} \mathbf{w}_i s_i \right) + n_k$





- Common problems: Find precoding/beamforming vectors to
  - minimize the total transmit power
  - maximize (weighted) sum-rate, minimum rate (max-min fair), etc

Applications in Wireless Communications

- Transmit Beamforming - Power Minimization
- MIMO Detection
- Multicast Beamforming - Power Minimization
- Multicast Beamforming - Max-Min Fair
- Reconfigurable Intelligent Surfaces
- Power Control in Multi-Cell
- Power Allocation by Water-Filling
- Waveform Design for Wireless Power Transfer
- Transmit Beamforming - Rate Maximization
- Receive Beamforming

Outline

Convex Sets

- A set C is convex if the line segment between any two points in C lies in C, i.e., $\theta\mathbf{x} + (1-\theta)\mathbf{y} \in C$, $\forall\theta \in [0,1]$ and $\mathbf{x}, \mathbf{y} \in C$.
- Examples:
  - unit ball $C = \{\mathbf{x} \mid ||\mathbf{x}|| \leq 1\}$ is convex
  - unit sphere $C = \{\mathbf{x} \mid ||\mathbf{x}|| = 1\}$ is not convex
  - convex and nonconvex sets[3]

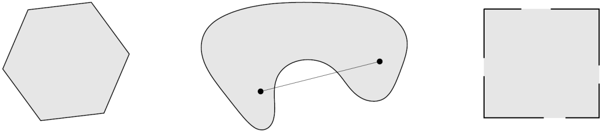


Figure 2.2 Some simple convex and nonconvex sets. *Left.* The hexagon, which includes its boundary (shown darker), is convex. *Middle.* The kidney shaped set is not convex, since the line segment between the two points in the set shown as dots is not contained in the set. *Right.* The square contains some boundary points but not others, and is not convex.

[3] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Convex Sets

- Property: intersection of any number of convex sets is convex.
  - A polyhedron is the intersection of halfspaces $\{\mathbf{x} \mid \mathbf{a}^T\mathbf{x} \leq b\}$ and hyperplanes $\{\mathbf{x} \mid \mathbf{a}^T\mathbf{x} = b\}$ (which are convex), and therefore is convex[4].

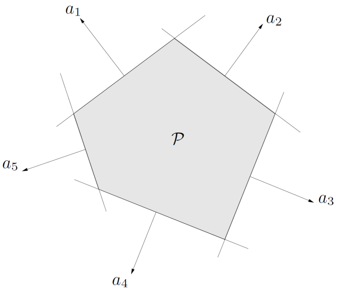


Figure 2.11 The polyhedron $\mathcal{P}$ (shown shaded) is the intersection of five halfspaces, with outward normal vectors $a_1, \ldots, a_5$.

[4] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Convex Cones

- A convex cone K is a special type of convex set which is closed under positive scaling: $\forall \mathbf{x}, \mathbf{y} \in K$ and $\theta_1, \theta_2 \geq 0$, $\theta_1 \mathbf{x} + \theta_2 \mathbf{y} \in K$.
- Pie slice[5]

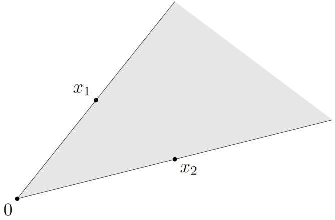


Figure 2.4 The pie slice shows all points of the form $\theta_1 x_1 + \theta_2 x_2$, where $\theta_1,\ \theta_2 \geq 0$. The apex of the slice (which corresponds to $\theta_1 = \theta_2 = 0$) is at 0; its edges (which correspond to $\theta_1 = 0$ or $\theta_2 = 0$) pass through the points x_1 and x_2.

- Nonnegative orthant $\mathbb{R}^n_+$ ($n = 2$: $\mathbf{x} = [x_1; x_2]$ with $x_1 \geq 0$, $x_2 \geq 0$)

[5] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Convex Cones

- Second-order cone[6] (ℓ-2 norm) $K = \{(t, \mathbf{x}) \in \mathbb{R}^{n+1} \mid ||\mathbf{x}|| \leq t\}$

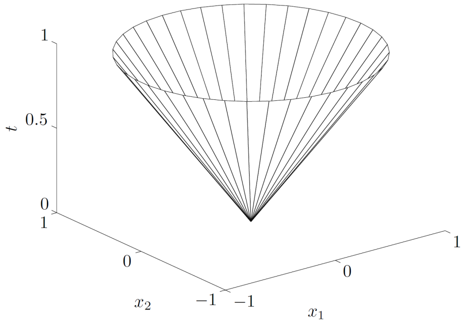


Figure 2.10 Boundary of second-order cone in $\mathbf{R}^3$, $\{(x_1, x_2, t) \mid (x_1^2 + x_2^2)^{1/2} \leq t\}$.

This set is the intersection of an infinite number of halfspaces, hence is convex

[6] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Convex Cones

- Positive semidefinite cone[7]
  - Positive semidefinite (PSD) matrix: $\mathbf{M} \succeq 0 \leftrightarrow \mathbf{x}^T\mathbf{M}\mathbf{x} \geq 0$, $\forall \mathbf{x} \in \mathbb{R}^n$. A matrix is positive semidefinite if and only if it is symmetric and all its eigenvalues are non-negative.
  - The set of PSD matrices $\mathbf{S}_+^n = \{X \in \mathbf{S}^n | X \succeq 0\}$
  - $\mathbf{S}_+^n$ is a convex cone: for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^T(\theta_1\mathbf{A} + \theta_2\mathbf{B})\mathbf{x} = \theta_1\mathbf{x}^T\mathbf{A}\mathbf{x} + \theta_2\mathbf{x}^T\mathbf{B}\mathbf{x} \geq 0, \tag{3}$$

if $\mathbf{A} \succeq 0$, $\mathbf{B} \succeq 0$, and $\theta_1, \theta_2 \geq 0$

[7] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Convex Functions

- A function $f(\mathbf{x}) : \mathbb{R}_+^n \to \mathbb{R}$ is convex[8] if for any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$,

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}), \forall \theta \in [0,1]. \qquad (4)$$

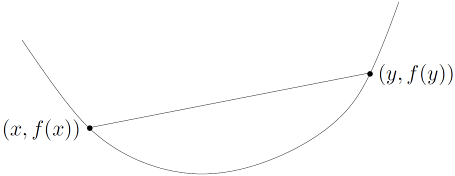


Figure 3.1 Graph of a convex function. The chord (*i.e.*, line segment) between any two points on the graph lies above the graph.

- f is concave is $-f$ is convex

[8]Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Convex Functions

- Examples:
  - convex: $|x|$, e^{ax} for any $a \in \mathbb{R}$, x^2, $x \log x$ over $\mathbb{R}_+$,
  - concave: $x^{1/2}$ over $\mathbb{R}_+$, $\log x$ over $\mathbb{R}_+$
  - affine (linear plus constant) are convex and concave: $\mathbf{a}^T\mathbf{x} + b$
  - x^3 convex over $[0, \infty)$ and concave over $(-\infty, 0]$, but neither convex nor concave over $\mathbb{R}$
  - Norm $||\mathbf{x}||_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ (real $p \geq 1$) on $\mathbb{R}^n$ is convex

$$
\begin{aligned}
||\theta\mathbf{x} + (1-\theta)\mathbf{y}||_p &\leq ||\theta\mathbf{x}||_p + ||(1-\theta)\mathbf{y}||_p \text{ (triangle ineq.)} \quad (5) \\
&= \theta||\mathbf{x}||_p + (1-\theta)||\mathbf{y}||_p \text{ (homogeneity)} \quad (6)
\end{aligned}
$$

Convex Functions

- First-order conditions: If f differentiable, convexity of f equivalent to[9]

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}), \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \tag{7}$$

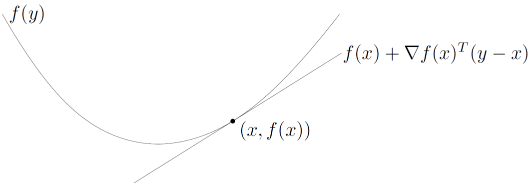


Figure 3.2 If f is convex and differentiable, then $f(x)+\nabla f(x)^T(y-x) \leq f(y)$ for all x, $y \in \mathbf{dom}\, f$.

  - convex function $\rightarrow$ first-order Taylor series expansion is a global underestimator of f.
  - if the first order Taylor approximation of f is always a global underestimator of f $\rightarrow$ f is convex.

[9] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Convex Functions

- Second-order conditions: If f is twice differentiable, then convexity of f $\leftrightarrow$ its Hessian is positive semidefinite $\nabla^2 f(\mathbf{x}) \succeq 0, \ \ \forall x \in \mathbb{R}^n$, i.e. positive curvature ($n = 1$: $f''(x) \geq 0$)
  - Quadratic $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} + \mathbf{q}^T\mathbf{x} + r \tag{8}$$

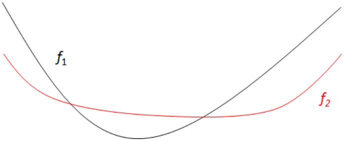
    with $\mathbf{P} \in \mathbf{S}^n$ (symmetric $n \times n$ matrix), $\mathbf{q} \in \mathbb{R}^n$, $r \in \mathbb{R}$.
    $\nabla f = \mathbf{P}\mathbf{x} + \mathbf{q}$. $\nabla^2 f = \mathbf{P}$. Hence, f convex if and only if $\mathbf{P} \succeq 0$.
  - Quadratic-over-linear x^2/y ($y > 0$) is convex

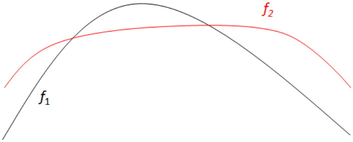
$$\nabla^2 f = \frac{2}{y^3}\begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} = \frac{2}{y^3}\begin{bmatrix} y \\ -x \end{bmatrix}\begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0 \tag{9}$$

    NB: convex fct $\rightarrow$ convex in each variable (converse not true)
  - Log-sum-exp $f(\mathbf{x}) = \log(e^{x_1} + \ldots + e^{x_n})$ convex

Convex Functions

- Properties:
  - Non-negative weighted sum of convex functions, $f = \sum_n w_n f_n$, is convex
    - Example: $\sum_i x_i \log x_i$ over $\mathbb{R}^n_+$ is convex
  - Pointwise maximum of m convex functions $f_1,\ldots,f_m$, i.e., $f(\mathbf{x}) = \max\{f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})\}$, is convex.
    - Pointwise minimum of m concave functions is concave.





Convex Optimization Problems

- Optimization problem

$$
\begin{aligned}
\min \quad & f_0(\mathbf{x}) \\
\text{s.t.} \quad & f_i(\mathbf{x}) \leq 0, \forall i = 1, \ldots, m \\
& h_j(\mathbf{x}) = 0, \forall j = 1, \ldots, p
\end{aligned} \tag{10}
$$

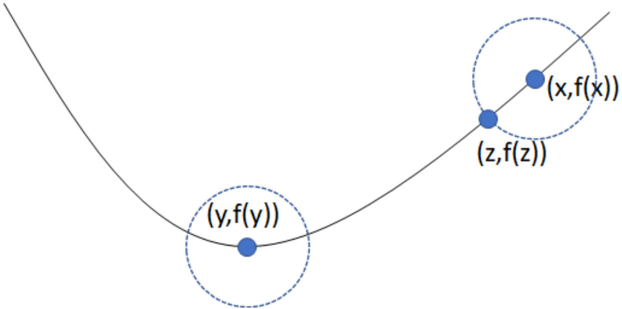
  is convex if

  - objective function f_0 is convex
  - inequality constraint functions f_i $(i = 1, \ldots, m)$ are convex
  - equality constraint functions $h_j(\mathbf{x}) = \mathbf{a}_j^T \mathbf{x} - b_j$ are affine

- Feasible set of a convex optimization problem is convex: minimize a convex objection function over a convex set.

Convex Optimization Problems

- Every locally optimal solution is also globally optimal.



For a convex problem, if you have a candidate point y and restrict the search locally (arbitrarily small) and find that there is no better point locally, you can conclude that even if you were searching everywhere, you would not find anything better.

Mathematical Formulation

- A set of problems with linear objective function and linear constraints[10] - not so common in wireless communications.
- General LP problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x} + d \tag{11a}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{G}\mathbf{x} \preceq \mathbf{h} \tag{11b}$$

where $\mathbf{G} \in \mathbb{R}^{m\times n}$, $\mathbf{A} \in \mathbb{R}^{p\times n}$, $\preceq$ refers to element-wise inequality.

- Standard LP problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x} \tag{12a}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \succeq \mathbf{0} \tag{12b}$$

- Inequality form LP problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x} \tag{13a}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{x} \preceq \mathbf{b} \tag{13b}$$

[10] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Mathematical Formulation

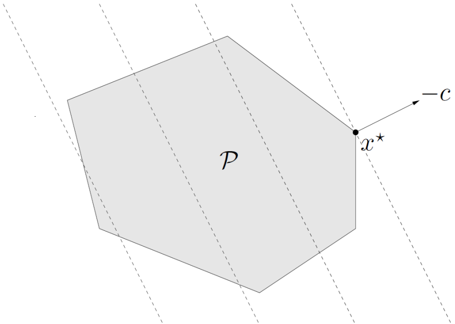


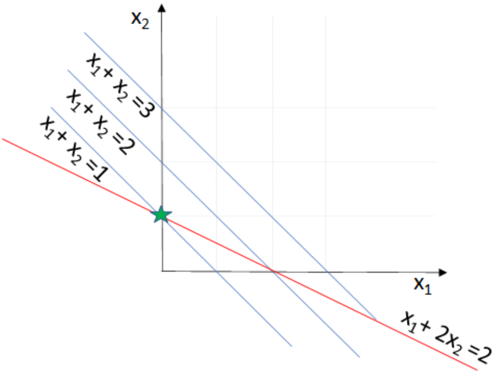
Figure 4.4 Geometric interpretation of an LP. The feasible set $\mathcal{P}$, which is a polyhedron, is shaded. The objective $c^T x$ is linear, so its level curves are hyperplanes orthogonal to c (shown as dashed lines). The point $x^\star$ is optimal; it is the point in $\mathcal{P}$ as far as possible in the direction $-c$.

Mathematical Formulation

- Example:

$$\min_{x_1, x_2} \quad x_1 + x_2 \tag{14a}$$

$$\text{s.t.} \quad x_1 + 2x_2 = 2, (x_1, x_2) \in \mathbb{R}_+^2 \tag{14b}$$



Mathematical Formulation

- A set of convex optimization problems with quadratic objective function and quadratic constraints[11]
- Quadratic programming (QP) problem

$$\min_{\mathbf{x}} \quad \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} + \mathbf{q}^T\mathbf{x} + r \tag{15a}$$

$$\text{s.t.} \quad \mathbf{G}\mathbf{x} \preceq \mathbf{h}, \mathbf{A}\mathbf{x} = \mathbf{b} \tag{15b}$$

- Convex if $\mathbf{P} \in \mathbf{S}_+^n$ ($\succeq 0$)
  - Minimize a convex quadratic function over a polyhedron

[11] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Mathematical Formulation

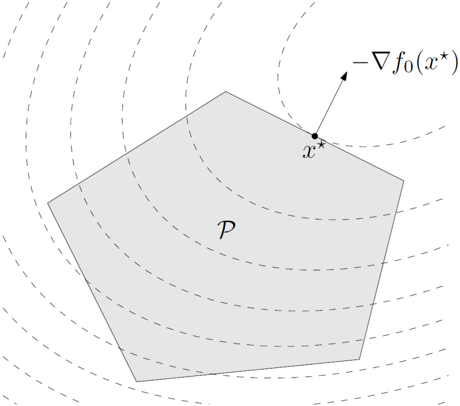


Figure 4.5 Geometric illustration of QP. The feasible set $\mathcal{P}$, which is a polyhedron, is shown shaded. The contour lines of the objective function, which is convex quadratic, are shown as dashed curves. The point $x^\star$ is optimal.

Mathematical Formulation

- Quadratically constrained quadratic program (QCQP) problem

$$\min_{\mathbf{x}} \quad \frac{1}{2}\mathbf{x}^T\mathbf{P}_0\mathbf{x} + \mathbf{q}_0^T\mathbf{x} + r_0 \tag{16a}$$

$$\text{s.t.} \quad \frac{1}{2}\mathbf{x}^T\mathbf{P}_i\mathbf{x} + \mathbf{q}_i^T\mathbf{x} + r_i \leq 0, \forall i = 1, \ldots, m \tag{16b}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{16c}$$

- Convex if $\mathbf{P}_0, \mathbf{P}_i \in \mathbf{S}_+^n$ $(\succeq 0)$
  - Minimize a convex quadratic function over a feasible region that is the intersection of ellipsoids
- LP $(\mathbf{P} = 0) \subset$ QP $(\mathbf{P}_i = 0) \subset$ QCQP

Example 1: Transmit (Tx) Beamforming - Power Min.[12]

- Find a set of beamforming vectors to minimize the total transmit power for MU-MISO subject to SINR constraint γ_i for user i, $i = 1, \ldots, K$

$$\min_{\mathbf{w}_1,\ldots,\mathbf{w}_K} \sum_{j=1}^{K} \|\mathbf{w}_j\|^2 \quad \text{s.t.} \quad \frac{|\mathbf{h}_i^H \mathbf{w}_i|^2}{\sum_{j\neq i} |\mathbf{h}_i^H \mathbf{w}_j|^2 + \sigma^2} \geq \gamma_i, \forall i \tag{17}$$

or equivalently

$$\min_{\mathbf{w}_1,\ldots,\mathbf{w}_K} \sum_{j=1}^{K} \mathbf{w}_j^H \mathbf{w}_j \tag{18}$$

$$\text{s.t.} \quad \underbrace{\mathbf{w}_i^H \mathbf{h}_i \mathbf{h}_i^H \mathbf{w}_i - \gamma_i \sum_{j\neq i} \mathbf{w}_j^H \mathbf{h}_i \mathbf{h}_i^H \mathbf{w}_j}_{\text{difference of convex functions is non-convex}} - \gamma_i \sigma^2 \geq 0, \forall i \tag{19}$$

- Non-convex QCQP: QP with non-convex QC

[12] Zhi-Quan Luo and Wei Yu. "An introduction to convex optimization for communications and signal processing". In: *IEEE Journal on selected areas in communications* 24.8 (2006), pp. 1426–1438.

Mathematical Formulation

- SOCP problem[13]

$$\min_{\mathbf{x}} \quad \mathbf{f}^T\mathbf{x} \tag{20a}$$

$$\text{s.t.} \quad \|\mathbf{A}_i\mathbf{x}+\mathbf{b}_i\| \leq \mathbf{c}_i^T\mathbf{x}+d_i, \forall i=1,\ldots,m \tag{20b}$$

$$\mathbf{F}\mathbf{x}=\mathbf{g} \tag{20c}$$

  - The inequality constraints $\|\mathbf{A}_i\mathbf{x}+\mathbf{b}_i\| \leq \mathbf{c}_i^T\mathbf{x}+d_i$, $\forall i=1,\ldots,m$ are called second-order cone (SOC) constraints
  - Note constraint with $\|.\|$ and not $\|.\|^2$ (QCQP)
  - Note $\|\mathbf{A}_i\mathbf{x}+\mathbf{b}_i\|-\mathbf{c}_i^T\mathbf{x}-d_i$ is not differentiable at 0

Tips:

- When $\mathbf{c}_i=\mathbf{0}$, $\forall i$, SOCP boils down to QCQP (after squaring)
- When $\mathbf{A}_i=\mathbf{0}$, $\forall i$, SOCP boils down to LP
- SOCP is more general than QCQP and LP

[13] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Mathematical Formulation

Why is $\|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\| \leq \mathbf{c}_i^T\mathbf{x} + d_i$ a convex constraint?

- Approach 1: $$\underbrace{\underbrace{\|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|}_{\text{Convex because norm}} \quad \underbrace{-\mathbf{c}_i^T\mathbf{x} - d_i}_{\text{Affine, hence convex}}}_{\text{sum of convex is convex}} \leq 0$$
- Approach 2:

$$\begin{aligned}
&\|\mathbf{A}_i(\theta\mathbf{x} + (1-\theta)\mathbf{y}) + \mathbf{b}_i\| - \mathbf{c}_i^T(\theta\mathbf{x} + (1-\theta)\mathbf{y}) - d_i \\
&= \|\theta(\mathbf{A}_i\mathbf{x}+\mathbf{b}_i)+(1-\theta)(\mathbf{A}_i\mathbf{y}+\mathbf{b}_i)\| - \theta(\mathbf{c}_i^T\mathbf{x}+d_i) - (1-\theta)(\mathbf{c}_i^T\mathbf{y}+d_i) \\
&\leq \theta(\|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\| - \mathbf{c}_i^T\mathbf{x} - d_i) + (1-\theta)(\|\mathbf{A}_i\mathbf{y} + \mathbf{b}_i\| - \mathbf{c}_i^T\mathbf{y} - d_i)
\end{aligned} \tag{21}$$

  using triangle inequality

- Approach 3: Use Hessian, but not here because non differentiable
- Approach 4: $\mathbf{A}_i\mathbf{x} + \mathbf{b}_i$ vector, $\mathbf{c}_i^T\mathbf{x} + d_i$ scalar and $(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \mathbf{c}_i^T\mathbf{x} + d_i) \in$ second-order cone in $\mathbb{R}^{n+1}$

Example 1: Tx Beamforming - Power Min. - SOCP[14]

- Recall

$$\min_{\mathbf{w}_1,\ldots,\mathbf{w}_K} \quad \sum_{j=1}^{K} \|\mathbf{w}_j\|^2 \tag{22a}$$

$$\text{s.t.} \quad \frac{|\mathbf{h}_i^H \mathbf{w}_i|^2}{\sum_{j\neq i} |\mathbf{h}_i^H \mathbf{w}_j|^2 + \sigma^2} \geq \gamma_i, \forall i \tag{22b}$$

  - γ_i, $\forall i$: SINR constraint for user i, $\forall i$
  - The SINR constraint is non-convex

- Three solutions:
  - SOCP transformation
  - SDP relaxation
  - KKT

[14] Zhi-Quan Luo and Wei Yu. "An introduction to convex optimization for communications and signal processing". In: *IEEE Journal on selected areas in communications* 24.8 (2006), pp. 1426–1438.

Example 1:Tx Beamforming - Power Min. - SOCP[15]

- Transform SINR constraints of problem (17) into SOC constraints

$$|\mathbf{h}_i^H\mathbf{w}_i|^2 + \frac{1}{\gamma_i}|\mathbf{h}_i^H\mathbf{w}_i|^2 \geq |\mathbf{h}_i^H\mathbf{w}_i|^2 + \sum_{j\neq i}|\mathbf{h}_i^H\mathbf{w}_j|^2 + \sigma^2 \tag{23}$$

$$(1+\frac{1}{\gamma_i})|\mathbf{h}_i^H\mathbf{w}_i|^2 \geq \left\| \; \mathbf{h}_i^H\mathbf{W} \quad \sigma^2 \; \right\|^2, \forall i \tag{24}$$

  - $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_K]$

- Transform objective function of problem (17) as

$$\sum_{j=1}^{K}\|\mathbf{w}_j\|^2 = \mathrm{trace}(\mathbf{W}^H\mathbf{W}) = \mathrm{vec}(\mathbf{W})^H\mathrm{vec}(\mathbf{W}) = \|\mathrm{vec}(\mathbf{W})\|^2 \tag{25}$$

[15] Ami Wiesel, Yonina C Eldar, and Shlomo Shamai. "Linear precoding via conic optimization for fixed MIMO receivers". In: *IEEE transactions on signal processing* 54.1 (2005), pp. 161–176.

Example 1: Tx Beamforming - Power Min. - SOCP

- Problem (17) becomes

$$\min_{\mathbf{w}_1,\ldots,\mathbf{w}_K} \quad \tau \tag{26a}$$

$$\text{s.t.} \quad (1+\frac{1}{\gamma_i})|\mathbf{h}_i^H\mathbf{w}_i|^2 \geq \left\| \quad \mathbf{h}_i^H\mathbf{W} \quad \sigma^2 \quad \right\|^2, \forall i \tag{26b}$$

$$\|\mathrm{vec}(\mathbf{W})\| \leq \sqrt{\tau} \tag{26c}$$

- Note that $\mathbf{h}_i^H\mathbf{w}_i$ can be chosen (arbitrary phase rotation can be added to beamforming vectors without affecting SINR)
- Problem (17) becomes a convex SOCP

$$\min_{\mathbf{w}_1,\ldots,\mathbf{w}_K} \quad \tau \tag{27a}$$

$$\text{s.t.} \quad \left\| \quad \mathbf{h}_i^H\mathbf{W} \quad \sigma^2 \quad \right\| \leq \sqrt{1+\frac{1}{\gamma_i}} \ \mathbf{h}_i^H\mathbf{w}_i, \forall i \tag{27b}$$

$$\|\mathrm{vec}(\mathbf{W})\| \leq \sqrt{\tau} \tag{27c}$$

Example 1: Tx Beamforming - Power Min. - SOCP

Example of code for this problem using CVX software[16]

```
cvx_begin
variable tau nonnegative
variable W(M,K) complex

minimize(tau)
subject to
    for i=1:K
        norm([H(i,1:M)*W 1])<=sqrt(1+1/gamma)*real(H(i,1:M)*W(1:M,i)) %note how H(i,1:M)*W(1:M,i) has been replaced by real(H(i,1:M)*W(1:M,i))
        imag(H(i,1:M)*W(1:M,i))==0 % we impose imaginary part to be 0
        %real(H(i,1:M)*W(1:M,i))>=0 not needed
    end
    norm(vec(W))<=sqrt(tau)

cvx_end
```

[16] Michael Grant and Stephen Boyd. *CVX: Matlab Software for Disciplined Convex Programming, version 2.1*. http://cvxr.com/cvx. Mar. 2014.

Mathematical Formulation

- A set of problems having linear objective function, linear equality constraints, and semidefiniteness constraint on variable $\mathbf{X}$[17]
- Standard SDP problem

$$\min_{\mathbf{X}} \quad \mathrm{Tr}(\mathbf{CX}) \tag{28a}$$
$$\text{s.t.} \quad \mathrm{Tr}(\mathbf{A}_i\mathbf{X}) = b_i, \forall i = 1, \ldots, m \tag{28b}$$
$$\mathbf{X} \succeq \mathbf{0} \tag{28c}$$

- Useful property of PSD and Schur complement $\mathbf{S}$

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \succeq \mathbf{0} \Longleftrightarrow \mathbf{A} \succeq \mathbf{0}, \mathbf{S} = \mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \succeq \mathbf{0} \tag{29}$$

$$\begin{bmatrix} (\mathbf{c}^T\mathbf{x} + d)\mathbf{I} & \mathbf{Ax} + \mathbf{b} \\ (\mathbf{Ax} + \mathbf{b})^T & \mathbf{c}^T\mathbf{x} + d \end{bmatrix} \succeq \mathbf{0} \Longleftrightarrow \mathbf{c}^T\mathbf{x} + d \geq 0, (\mathbf{c}^T\mathbf{x} + d)^2 \geq \|\mathbf{Ax} + \mathbf{b}\|^2 \tag{30}$$

[17] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

LP as SDP

- LP and equivalent SDP

$$\text{LP:} \quad \min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x} \tag{31a}$$
$$\text{s.t.} \quad \mathbf{A}\mathbf{x} \preceq \mathbf{b} \tag{31b}$$
$$\Updownarrow$$
$$\text{SDP:} \quad \min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x} \tag{31c}$$
$$\text{s.t.} \quad \mathrm{diag}(\mathbf{A}\mathbf{x} - \mathbf{b}) \preceq \mathbf{0} \tag{31d}$$

Tips:

- Model Generality: LP $\prec$ (convex) QCQP $\prec$ SOCP $\prec$ SDP
- Solution efficiency: LP $\succ$ (convex) QCQP $\succ$ SOCP $\succ$ SDP

SOCP as SDP

- SOCP and equivalent SDP

$$\text{SOCP:} \quad \min_{\mathbf{x}} \quad \mathbf{f}^T\mathbf{x} \tag{32a}$$

$$\text{s.t.} \quad \|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\| \leq \mathbf{c}_i^T\mathbf{x} + d_i, \forall i = 1, \ldots, m \tag{32b}$$

$$\Updownarrow$$

$$\text{SDP:} \quad \min_{\mathbf{x}} \quad \mathbf{f}^T\mathbf{x} \tag{32c}$$

$$\text{s.t.} \quad \begin{bmatrix} (\mathbf{c}_i^T\mathbf{x} + d_i)\mathbf{I} & \mathbf{A}_i\mathbf{x} + \mathbf{b}_i \\ (\mathbf{A}_i\mathbf{x} + \mathbf{b}_i)^T & \mathbf{c}_i^T\mathbf{x} + d_i \end{bmatrix} \succeq \mathbf{0}, \forall i = 1, \ldots, m \tag{32d}$$

Tips:

- Model Generality: LP $\prec$ (convex) QCQP $\prec$ SOCP $\prec$ SDP
- Solution efficiency: LP $\succ$ (convex) QCQP $\succ$ SOCP $\succ$ SDP

Example 1: Tx Beamforming - Power Min. - SDP

- Transform objective function of problem (17)

$$\sum_{j=1}^{K} \|\mathbf{w}_j\|^2 = \sum_{j=1}^{K} \mathrm{Tr}(\mathbf{w}_j \mathbf{w}_j^H) \tag{33}$$

- Transform SINR constraints of problem (17)

$$\frac{|\mathbf{h}_i^H \mathbf{w}_i|^2}{\sum_{j \neq i} |\mathbf{h}_i^H \mathbf{w}_j|^2 + \sigma^2} \geq \gamma_i$$

$$\rightarrow \underbrace{\mathrm{Tr}(\mathbf{h}_i^H \mathbf{w}_i \mathbf{w}_i^H \mathbf{h}_i)}_{=\mathrm{Tr}(\mathbf{h}_i \mathbf{h}_i^H \mathbf{w}_i \mathbf{w}_i^H)} - \gamma_i \sum_{j \neq i} \underbrace{\mathrm{Tr}(\mathbf{h}_i^H \mathbf{w}_j \mathbf{w}_j^H \mathbf{h}_i)}_{\mathrm{Tr}(\mathbf{h}_i \mathbf{h}_i^H \mathbf{w}_j \mathbf{w}_j^H)} \geq \gamma_i \sigma^2 \tag{34}$$

Example 1: Tx Beamforming - Power Min. - SDP[19]

- Reformulate problem (17) into an SDP

$$\min_{\mathbf{B}_1,\dots,\mathbf{B}_K} \quad \sum_{j=1}^{K} \mathrm{Tr}(\mathbf{B}_j) \tag{35a}$$

$$\text{s.t.} \quad \mathrm{Tr}(\mathbf{H}_i\mathbf{B}_i) - \gamma_i \sum_{j\neq i} \mathrm{Tr}(\mathbf{H}_i\mathbf{B}_j) \geq \gamma_i\sigma^2, \forall i \tag{35b}$$

$$\mathbf{B}_i \succeq \mathbf{0}, \forall i \tag{35c}$$

  - $\mathbf{B}_i = \mathbf{w}_i\mathbf{w}_i^H$, $\mathbf{H}_i = \mathbf{h}_i\mathbf{h}_i^H$, $\forall i$
  - $\mathbf{B}_i$ complex Hermitian, $\forall i$
- Relax the rank-1 constraint → SDP relaxation (SDR)
- Guaranteed to have at least one optimal solution which is rank one[18]

[18] Mats Bengtsson and Björn Ottersten. "Optimum and suboptimum transmit beamforming". In: *Handbook of antennas in wireless communications*. CRC press, 2018, pp. 18–1.

[19] Zhi-Quan Luo and Wei Yu. "An introduction to convex optimization for communications and signal processing". In: *IEEE Journal on selected areas in communications* 24.8 (2006), pp. 1426–1438.

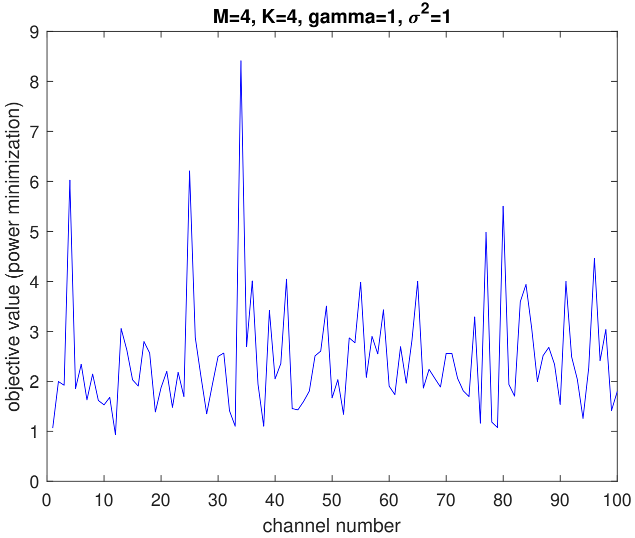
Example 1: Tx Beamforming - Power Min. - SDP

Example of code for this problem using CVX software

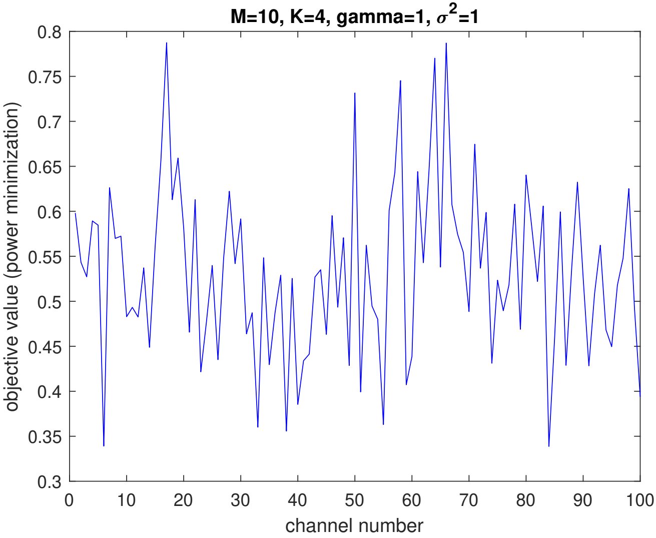
```
cvx_begin quiet
variable X(M,M,K) hermitian
variable s(K,1)
obj=0;
for i=1:K
    obj=obj+trace(X(:,:,i));
end
minimize(obj)
subject to
    for i=1:K
        cstr=0;
        for j=1:K
            if j~=i
                cstr=cstr+trace(Q(:,:,i)*X(:,:,j));
            end
        end
        trace(Q(:,:,i)*X(:,:,i))-gamma*cstr-s(i,1)==gamma %assume noise variance equal to 1
        s(i,1)>=0
        X(:,:,i) == hermitian_semidefinite(M) % careful to use hermitian_semidefinite for complex
    end
cvx_end
% solutions indeed look like rank 1
```

Note the use of slack variables to replace inequality constraints with equality constraints and nonnegativity constraints

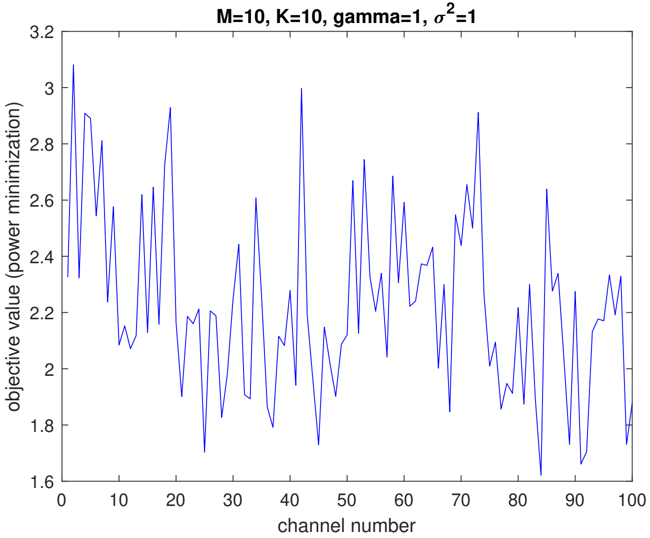
Example 1: Tx Beamforming - Power Min. - SDP



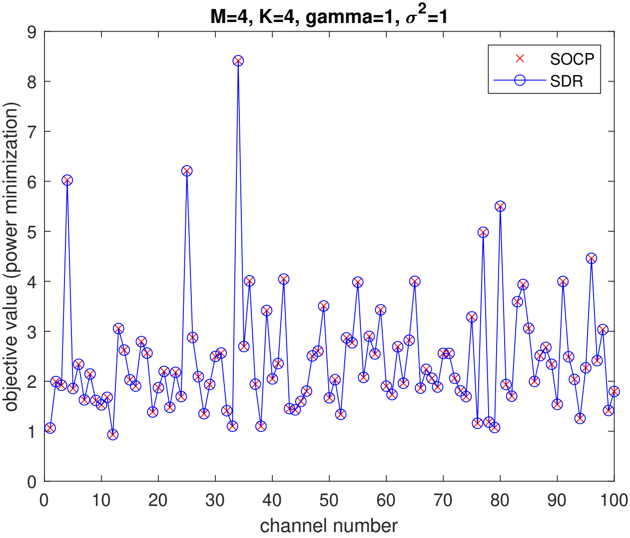
Example 1: Tx Beamforming - Power Min. - SDP



Example 1: Tx Beamforming - Power Min. - SDP

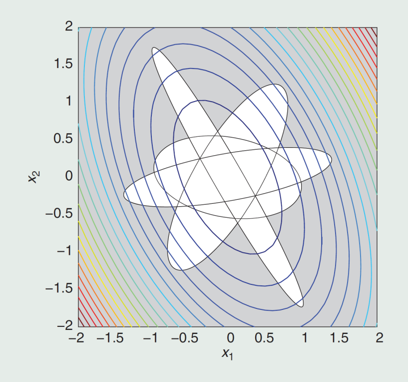


Example 1: Tx BF - Power Min. - SDP vs SOCP



Mathematical Formulation

- SDR is mainly utilized to solve non-convex QCQP problem[20]



[FIG1] A nonconvex QCQP in $\mathbb{R}^2$: Colored lines represent the contour of the objective function, the gray area represents the feasible set, and the black lines represent the boundary of each constraint.

[20] Zhi-Quan Luo et al. "Semidefinite relaxation of quadratic optimization problems". In: *IEEE Signal Processing Magazine* 27.3 (2010), pp. 20–34.

Mathematical Formulation

- A homogeneous (i.e. no linear terms) QCQP ($\mathbf{C} \succeq \mathbf{0}, \mathbf{A}_i \succeq \mathbf{0}\ \forall i$)

$$\min_{\mathbf{x}} \quad \mathbf{x}^H \mathbf{C} \mathbf{x} \tag{36a}$$

$$\text{s.t.} \quad \mathbf{x}_i^H \mathbf{A}_i \mathbf{x} \geq b_i, \forall i \tag{36b}$$

- Introduce $\mathbf{X} = \mathbf{x}\mathbf{x}^H$. Problem (36) is equivalently transformed into

$$\min_{\mathbf{X}} \quad \text{Tr}(\mathbf{C}\mathbf{X}) \tag{37a}$$

$$\text{s.t.} \quad \text{Tr}(\mathbf{A}_i\mathbf{X}) \geq b_i, \forall i \tag{37b}$$

$$\mathbf{X} \succeq \mathbf{0}, \text{rank}(\mathbf{X}) = 1 \tag{37c}$$

  - Objective function is linear in $\mathbf{X}$
  - Trace constraints are linear inequalities and set of PSD matrices is convex
  - rank-1 constraint non-convex (sum of two rank-one matrices is likely rank 2)

- Ignore the rank-1 constraint and solve the convex problem by CVX

Mathematical Formulation

- Check the rank of the solution $\mathbf{X}^\star$:
  - If $\text{rank}(\mathbf{X}^\star) = 1$, the optimal solution of problem (37), $\mathbf{x}^\star$ is obtained by performing eigenvalue decomposition (EVD) for $\mathbf{X}^\star$ ($\mathbf{x}^\star$ is essentially the eigenvector of $\mathbf{X}^\star$)
  - If $\text{rank}(\mathbf{X}^\star) \neq 1$, using Gaussian randomization[21] to find an approximate solution

Process of Randomization

- Generate L random samples with zero mean and covariance $\mathbf{X}^\star$
- Manipulate/scale samples so that the quadratic constraints are satisfied (turn samples into feasible solutions)
- Choose one from samples so that the objective is minimized
- Randomization procedure is problem dependent
- Larger L: higher accuracy, longer computational time

[21] Zhi-Quan Luo et al. "Semidefinite relaxation of quadratic optimization problems". In: *IEEE Signal Processing*

Example 2: MIMO Detection[22]

- Maximum-likelihood (ML) MIMO detection problem

$$\min_{\mathbf{s}_c \in \{\pm 1 \pm j\}^N} \quad \|\mathbf{y}_c - \mathbf{H}_c \mathbf{s}_c\|^2 \tag{38}$$

  - Discrete least square problem
  - Transmitted symbols follow QPSK constellation, i.e., $[\mathbf{s}_c]_i \in \{\pm 1 \pm j\}, \forall i$

- Transform problem (38) into real-valued QP

$$\min_{\mathbf{s} \in \{\pm 1\}^{2N}} \quad \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2 \tag{39}$$

  - $\mathbf{y}_c = \begin{bmatrix} \Re\{\mathbf{y}_c\} \\ \Im\{\mathbf{y}_c\} \end{bmatrix}, \mathbf{s}_c = \begin{bmatrix} \Re\{\mathbf{s}_c\} \\ \Im\{\mathbf{s}_c\} \end{bmatrix},$
    $\mathbf{H}_c = \begin{bmatrix} \Re\{\mathbf{H}_c\} & -\Im\{\mathbf{H}_c\} \\ \Im\{\mathbf{H}_c\} & \Re\{\mathbf{H}_c\} \end{bmatrix}$

[22] Zhi-Quan Luo et al. "Semidefinite relaxation of quadratic optimization problems". In: *IEEE Signal Processing Magazine* 27.3 (2010), pp. 20–34.

Example 2: MIMO Detection[23]

- Rewrite problem (39) as

$$\min_{\mathbf{s}\in\mathbb{R}^{2N},t\in\mathbb{R}} \quad \|t\mathbf{y}-\mathbf{H}\mathbf{s}\|^2 = t^2\|\mathbf{y}\|^2 + \mathbf{s}^T\mathbf{H}^T\mathbf{H}\mathbf{s} - t\mathbf{y}^T\mathbf{H}\mathbf{s} - \mathbf{s}^T\mathbf{H}^T\mathbf{y}t \tag{40a}$$

$$\text{s.t.} \quad t^2 = 1, s_i^2 = 1, \forall i = 1,\ldots,2N \tag{40b}$$

- Express problem (40) as homogeneous QCQP

$$\min_{\mathbf{s}\in\mathbb{R}^{2N},t\in\mathbb{R}} \quad \underbrace{[\mathbf{s}^T \quad t]}_{=\mathbf{x}^T} \underbrace{\begin{bmatrix} \mathbf{H}^T\mathbf{H} & -\mathbf{H}^T\mathbf{y} \\ -\mathbf{y}^T\mathbf{H} & \|\mathbf{y}\|^2 \end{bmatrix}}_{=\mathbf{Q}} \underbrace{\begin{bmatrix} \mathbf{s} \\ t \end{bmatrix}}_{=\mathbf{x}} \tag{41a}$$

$$\text{s.t.} \quad t^2 = 1, s_i^2 = 1, \forall i = 1,\ldots,2N \tag{41b}$$

$$\Rightarrow \min_{\mathbf{X}} \quad \mathrm{Tr}(\mathbf{QX}) \tag{41c}$$

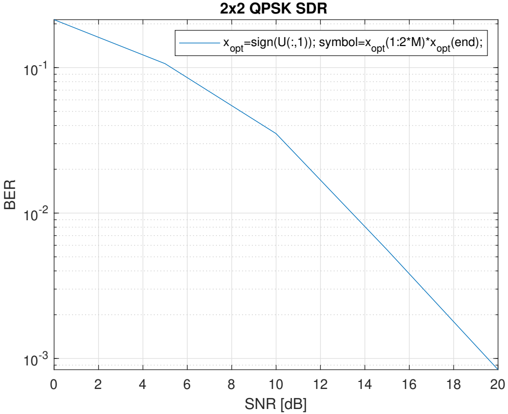
$$\text{s.t.} \quad \mathbf{X} \succeq \mathbf{0}, [\mathbf{X}]_{i,i} = 1, \forall i \tag{41d}$$

$$\mathrm{rank}(\mathbf{X}) = 1 \tag{41e}$$

[23] Zhi-Quan Luo et al. "Semidefinite relaxation of quadratic optimization problems". In: *IEEE Signal Processing Magazine* 27.3 (2010), pp. 20–34.

Example 2: MIMO Detection

- Ignore the rank-1 constraint and solve the relaxed SDP problem; then perform randomization to generate a feasible rank-1 solution
- Compare with prior ML, ZF, SIC performance!



Example 3: Multicast Beamforming - Power Min.[24]

- A transmitter simultaneously broadcasts common information to m receivers using a single beamforming vector $\mathbf{w}$
- Receiver i is equipped with $|\mathcal{I}_i|$ antennas and performs MRC
- Power minimization problem

$$\begin{aligned} \min_{\mathbf{w}} \quad & \|\mathbf{w}\|^2 && \text{(42a)} \\ \text{s.t.} \quad & \sum_{l \in \mathcal{I}_i} |\mathbf{h}_l^H \mathbf{w}|^2 \geq 1, \forall i = 1, \ldots, m && \text{(42b)} \end{aligned}$$

- Introduce $\mathbf{B} = \mathbf{w}\mathbf{w}^H$, $\mathbf{H}_i = \sum_{l \in \mathcal{I}_i} \mathbf{h}_l \mathbf{h}_l^H$. (42) is transformed into

$$\begin{aligned} \min_{\mathbf{B}} \quad & \mathrm{Tr}(\mathbf{B}) && \text{(43a)} \\ \text{s.t.} \quad & \mathrm{Tr}(\mathbf{H}_i \mathbf{B}) \geq 1, \forall i = 1, \ldots, m && \text{(43b)} \\ & \mathbf{B} \succeq \mathbf{0}, \mathrm{rank}(\mathbf{B}) = 1 && \text{(43c)} \end{aligned}$$

- Relax the rank-1 constraint and use randomization to construct a feasible solution to the original problem

[24] Zhi-Quan Luo and Wei Yu. "An introduction to convex optimization for communications and signal processing". In: *IEEE Journal on selected areas in communications* 24.8 (2006), pp. 1426–1438.

Example 3: Multicast Beamforming - Power Min.

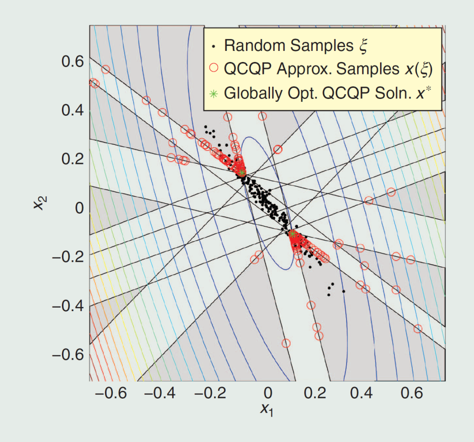
Example of code for this problem using CVX software

```
cvx_begin
variable X(M,M) hermitian  % optimization variable
variable s(K,1)  % slack variable to replace inequality constraint with an equality constraint, and a nonnegativity constraint
minimize(trace(X))
subject to
    for i=1:K
        %trace(X*Q(:,:,i))>=1 % Disciplined convex programming error:   Invalid constraint: {complex affine} >= {real constant}
        trace(X*Q(:,:,i))-s(i,1)==1
        s(i,1)>=0
    end
    X == hermitian_semidefinite(M)  % careful to use hermitian_semidefinite(M) and not semidefinite(M) for complex matrices
cvx_end
```

Here again we use slack variables

Then we need to perform rank-1 approximation or randomization

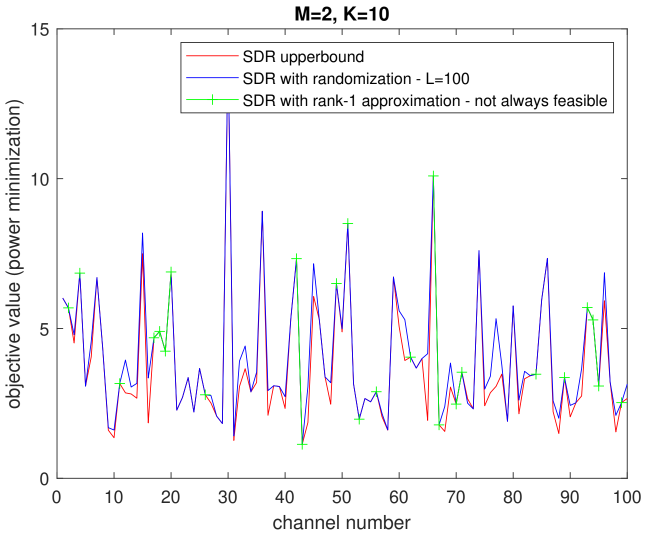
Example 3: Multicast Beamforming - Power Min.[25]



[FIG6] Illustration of randomizations in $\mathbb{R}^2$ for (16). The gray area is the feasible set and colored lines the contour of the objective.

[25] Zhi-Quan Luo et al. "Semidefinite relaxation of quadratic optimization problems". In: *IEEE Signal Processing Magazine* 27.3 (2010), pp. 20–34.

Example 3: Multicast Beamforming - Power Min.



Example 4: Multicast Beamforming - Max-Min Fair[26]

- Max-min fair beamforming problem (assume single antenna Rx)

$$\max_{\mathbf{w}} \min_{i} \quad |\mathbf{h}_i^H \mathbf{w}|^2 \tag{44a}$$
$$\text{s.t.} \quad \|\mathbf{w}\|^2 \leq 1 \tag{44b}$$

- Introduce $\mathbf{X} = \mathbf{w}\mathbf{w}^H$ and rewrite as

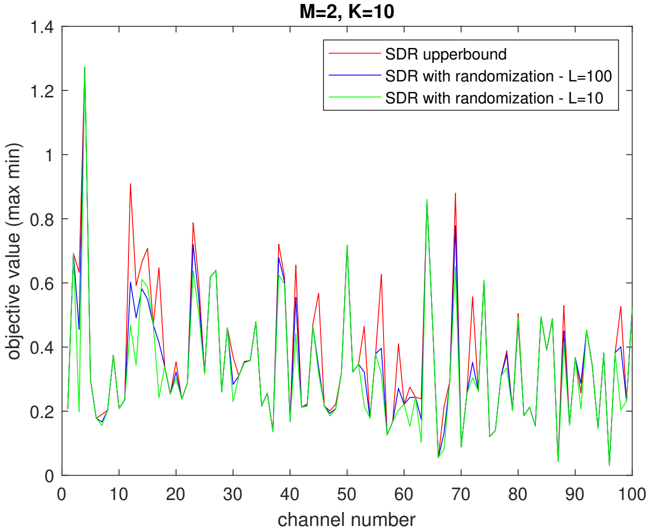
$$\min_{\mathbf{X},t} -t \tag{45a}$$
$$\text{s.t.} \quad \text{Tr}(\mathbf{X}\mathbf{H}_i) \geq t, \forall i \tag{45b}$$
$$\text{Tr}(\mathbf{X}) = 1 \tag{45c}$$
$$\mathbf{X} \succeq \mathbf{0} \tag{45d}$$
$$\text{rank}(\mathbf{X}) = 1 \tag{45e}$$

where $\mathbf{H}_i = \mathbf{h}_i\mathbf{h}_i^H$ and t is an auxiliary variable.

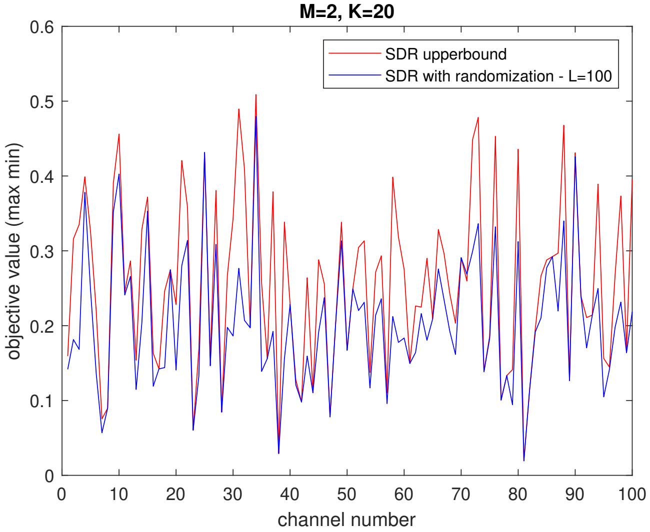
- Relax the rank-1 constraint and use randomization

[26] N.D. Sidiropoulos, T.N. Davidson, and Zhi-Quan Luo. "Transmit beamforming for physical-layer multicasting". In: *IEEE Transactions on Signal Processing* 54.6 (2006), pp. 2239–2251.

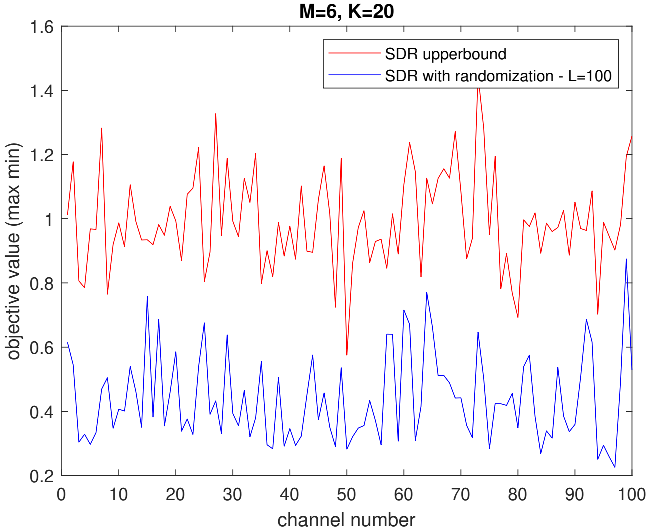
Example 4: Multicast Beamforming - Max-Min Fair



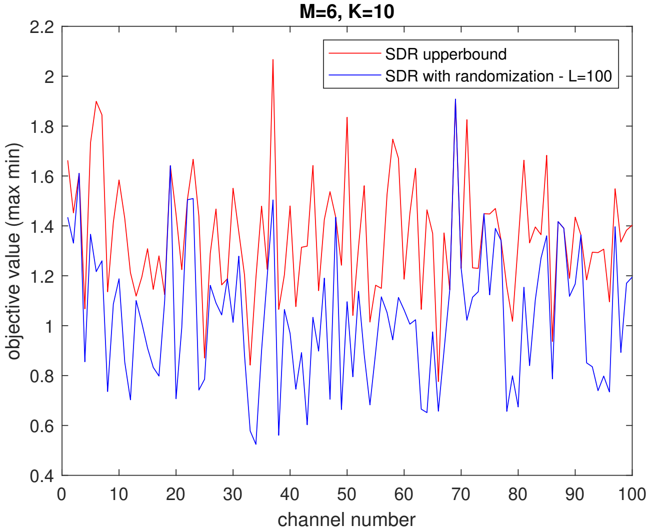
Example 4: Multicast Beamforming - Max-Min Fair



Example 4: Multicast Beamforming - Max-Min Fair

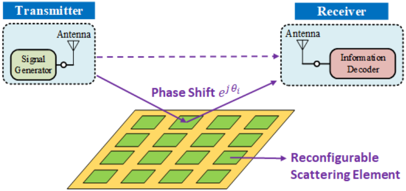


Example 4: Multicast Beamforming - Max-Min Fair



Example 5: Reconfigurable Intelligent Surfaces

- RIS consists of large number of reconfigurable scattering elements
- Each element adjusts the phase shift to "engineer the channel"
- Collaboratively adjust the phase shifts (constructively/destructively)
- Advantages of RIS: nearly passive, low cost low power consumption, no active thermal noise, no self-interference
- Enable spectrum efficient and energy efficient wireless



$$y = (\mathbf{h}_r^H \mathbf{\Theta} \mathbf{G} + \mathbf{h}_d^H)\mathbf{w}x + n \tag{46}$$

Various RIS architectures https://youtu.be/x3VMQ-ZU0ek

Example 5: Reconfigurable Intelligent Surfaces[27]

- Choose $\mathbf{w}$ as maximum-ratio transmission (MRT)
- Channel gain maximization problem for RIS-assisted SU-MISO

$$\max_{\boldsymbol{\Theta}} \quad \|\mathbf{h}_r^H \boldsymbol{\Theta} \mathbf{G} + \mathbf{h}_d^H\|^2 \tag{47a}$$

$$\text{s.t.} \quad \boldsymbol{\Theta} = \text{diag}(\theta_1, \dots, \theta_M), |\theta_m| = 1, \forall m \tag{47b}$$

  - $\boldsymbol{\Theta}$: RIS matrix with unit modulus constraint

- Write $\boldsymbol{\theta} = [\theta_1, \dots, \theta_M]$ and $\mathbf{h}_r^H \boldsymbol{\Theta} \mathbf{G} = \boldsymbol{\theta} \underbrace{\text{diag}(\mathbf{h}_r^H)\mathbf{G}}_{\boldsymbol{\Phi}}$ so that

$$\|\mathbf{h}_r^H \boldsymbol{\Theta} \mathbf{G} + \mathbf{h}_d^H\|^2 = \boldsymbol{\theta}\boldsymbol{\Phi}\boldsymbol{\Phi}^H\boldsymbol{\theta}^H + \boldsymbol{\theta}\boldsymbol{\Phi}\mathbf{h}_d + \mathbf{h}_d^H\boldsymbol{\Phi}^H\boldsymbol{\theta}^H + \|\mathbf{h}_d\|^2 \tag{48}$$

- Problem (47) is a non-convex QCQP and can be reformulated as a homogeneous QCQP.

[27] Qingqing Wu and Rui Zhang. "Intelligent Reflecting Surface Enhanced Wireless Network via Joint Active and Passive Beamforming". In: *IEEE Transactions on Wireless Communications* 18.11 (2019), pp. 5394–5409.

Example 5: Reconfigurable Intelligent Surfaces

- Define $\mathbf{v} = [\boldsymbol{\theta}, t]$, t an auxiliary variable, $\mathbf{R} = \begin{bmatrix} \boldsymbol{\Phi}\boldsymbol{\Phi}^H & \boldsymbol{\Phi}\mathbf{h}_d \\ \mathbf{h}_d^H\boldsymbol{\Phi}^H & 0 \end{bmatrix}$, problem (47) can be equivalently transformed into

$$\max_{\mathbf{v}} \quad \mathbf{v}\mathbf{R}\mathbf{v}^H \tag{49a}$$
$$\text{s.t.} \quad |[\mathbf{v}]_m| = 1, \forall m = 1, \ldots, M+1 \tag{49b}$$

- With $\mathbf{V} = \mathbf{v}^H\mathbf{v}$, transform problem (49) into

$$\max_{\mathbf{V}} \quad \mathrm{Tr}(\mathbf{RV}) \tag{50a}$$
$$\text{s.t.} \quad [\mathbf{V}]_{m,m} = 1, \forall m = 1, \ldots, M+1 \tag{50b}$$
$$\mathbf{V} \succeq \mathbf{0}, \;\; \mathrm{rank}(\mathbf{V}) = 1 \tag{50c}$$

Example 5: Reconfigurable Intelligent Surfaces

- Relaxing the rank-1 constraint, problem (50) can be simplified as

$$\max_{\mathbf{V}} \quad \mathrm{Tr}(\mathbf{RV}) \tag{51a}$$

$$\text{s.t.} \quad [\mathbf{V}]_{m,m} = 1, \forall m = 1, \ldots, M+1 \tag{51b}$$

$$\mathbf{V} \succeq \mathbf{0} \tag{51c}$$

- Problem (51) can be solved by CVX, and the solution to the original problem is obtained by Gaussian randomization

Mathematical Formulation

- *Monomial* function with $c > 0$ and $a_i \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n_{++}$

$$f(\mathbf{x}) = c x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \tag{52}$$

- *Posynomial* function with K monomial terms and $c_k > 0$

$$f(\mathbf{x}) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}} \tag{53}$$

Tips:

- If a posynomial is multiplied by a monomial, the result is a posynomial
- A posynomial can be divided by a monomial, with the result a posynomial[a]

[a] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Mathematical Formulation

- GP problem

$$\min_{\mathbf{x}} \quad f_0(\mathbf{x}) \tag{54a}$$
$$\text{s.t.} \quad f_i(\mathbf{x}) \leq 1, \forall i = 1, \ldots, m \tag{54b}$$
$$h_i(\mathbf{x}) = 1, \forall i = 1, \ldots, p \tag{54c}$$

  - $f_0, \ldots, f_m$: posynomial functions
  - $h_1, \ldots, h_p$: monomial functions

Tips:

- Non-convex in the natural form
- Can be transformed to convex optimization problem[a]

[a] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

GP in Convex Form[28]

- Monomial function $f(\mathbf{x}) = cx_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$ transforms to

$$\log f(e^{y_1},\ldots,e^{y_n}) = \mathbf{a}^T\mathbf{y} + b \tag{55}$$

  - $y_i = \log x_i,\ \forall i = 1,\ldots,n,\ b = \log c$

- Posynomial function $f(\mathbf{x}) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \ldots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1},\ldots,e^{y_n}) = \log\left(\sum_{k=1}^{K} e^{\mathbf{a}_k^T\mathbf{y}+b_k}\right) \tag{56}$$

  - $y_i = \log x_i,\ \forall i = 1,\ldots,n,\ b_k = \log c_k,\ \forall k = 1,\ldots,K$
  - $\mathbf{a}_k = [a_{1k},\ldots,a_{nk}]^T,\ \forall k = 1,\ldots,K$

[28] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

GP in Convex Form

- GP transforms to convex problem

$$\min_{\mathbf{y}} \quad \tilde{f}_0(\mathbf{y}) = \log\left(\sum_{k=1}^{K_0} e^{\mathbf{a}_{0k}^T\mathbf{y}+b_{0k}}\right) \tag{57a}$$

$$\text{s.t.} \quad \tilde{f}_i(\mathbf{y}) = \log\left(\sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}^T\mathbf{y}+b_{ik}}\right) \leq 0, \forall i = 1, \ldots, m \tag{57b}$$

$$\tilde{h}_i(\mathbf{y}) = \mathbf{g}_i^T\mathbf{y} + h_i = 0, \forall i = 1, \ldots, p \tag{57c}$$

- $y_i = \log x_i, \ \forall i = 1, \ldots, n$
- $b_{ik} = \log c_{ik}, \ \forall k = 1, \ldots, K, \ \forall i = 0, \ldots, m$
- $\mathbf{a}_{ik} = [a_{i1k}, \ldots, a_{ink}]^T, \ \forall k = 1, \ldots, K, \ \forall i = 0, \ldots, m$

Tips:

- Convex optimization
- Reduce to linear programming when $K_0 = K_1 = \ldots = K_m = 1$

Example 6: Power Control in Multi-Cell[29]

- Consider a multi-cell network in which there are L cells each with a single-antenna transmitter at the (BS), and a user with a single-antenna receiver.
- The power received from BS l at user i is given by $G_{il}P_l$ where P_l is the transmit power available at BS l, and $G_{il} = |h_{il}|^2$ models the squared channel gain between the BS and user.
- The SINR for the user in cell l can therefore be computed as

$$\mathrm{SINR}_l = \frac{G_{ll}P_l}{\sum_{p \neq l}^{L} G_{lp}P_p + \sigma_n^2}, \tag{58}$$

  where σ_n^2 is the noise power at the users.
- For $\mathrm{SINR}_l >> 1, \forall l$, the data rate of the user in cell l can be computed as $R_l = \log_2(1 + \mathrm{SINR}_l) \approx \log_2(\mathrm{SINR}_l)$.

[29] Mung Chiang et al. "Power Control By Geometric Programming". In: *IEEE Transactions on Wireless Communications* 6.7 (2007), pp. 2640–2651.

Example 6: Power Control in Multi-Cell

- The aggregate data rate of the network can be written as

$$R_{\mathrm{network}} = \sum_{l=1}^{L} R_l = \log_2 \left(\Pi_{l=1}^{L} \mathrm{SINR}_l \right). \tag{59}$$

- With P_{max} as the power budget of each cell, maximizing the network aggregate data rate problem can be formulated as

$$\begin{aligned} \max_{P_1,\ldots,P_L \geq 0} \quad & \left(\Pi_{l=1}^{L} \frac{G_{ll} P_l}{\sum_{p \neq l}^{L} G_{lp} P_p + \sigma_n^2} \right) \\ s.t. \quad & P_l \leq P_{max}, \; l = 1, \ldots, L. \end{aligned} \tag{60}$$

- Introducing auxiliary variables t_l, $\forall l$ such that

$$t_l \left(\sum_{p \neq l}^{L} G_{lp} P_p + \sigma_n^2 \right) \leq G_{ll} P_l, \tag{61}$$

Example 6: Power Control in Multi-Cell

- Problem (60) can be transformed into

$$\min_{\substack{P_1,\ldots,P_L \geq 0,\\ t_1,\ldots,t_L \geq 0}} \quad \Pi_{l=1}^{L} \frac{1}{t_l}$$

$$s.t. \quad \sum_{p \neq l}^{L} \frac{t_l G_{lp} P_p}{G_{ll} P_l} + \frac{t_l \sigma_n^2}{G_{ll} P_l} \leq 1, \quad l = 1, \ldots, L$$

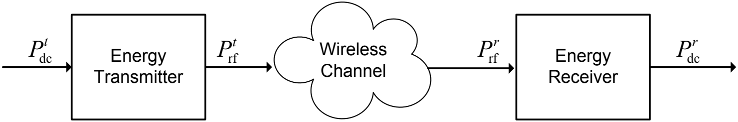
$$P_l \leq P_{max}, \quad l = 1, \ldots, L. \tag{62}$$

- We notice that the objective function and the constraint functions in (62) are posynomials w.r.t variables P and t, and thus it is a GP.
- Also useful in Massive MIMO and cell-free Massive MIMO[30]

[30] Anup Mishra et al. "Mitigating Intra-Cell Pilot Contamination in Massive MIMO: A Rate Splitting Approach". In: *IEEE Transactions on Wireless Communications* (2022), pp. 1–1.

Other Wireless Examples

- Wireless power transfer (WPT) https://youtu.be/Tkj9JSZsFg8



WPT: GP[31], SDR[32][33]; Wireless information and power transfer (WIPT): GP[34]; RIS-aided WPT: SDR[35], RIS-aided SWIPT: GP[36]

[31] Bruno Clerckx and Ekaterina Bayguzina. "Waveform design for wireless power transfer". In: *IEEE Transactions on Signal Processing* 64.23 (2016), pp. 6313–6328.

[32] Yang Huang and Bruno Clerckx. "Large-Scale Multiantenna Multisine Wireless Power Transfer". In: *IEEE Transactions on Signal Processing* 65.21 (2017), pp. 5812–5827. DOI: 10.1109/TSP.2017.2739112.

[33] Shanpu Shen and Bruno Clerckx. "Beamforming optimization for MIMO wireless power transfer with nonlinear energy harvesting: RF combining versus DC combining". In: *IEEE Transactions on Wireless Communications* 20.1 (2020), pp. 199–213.

[34] Bruno Clerckx. "Wireless Information and Power Transfer: Nonlinearity, Waveform Design, and Rate-Energy Tradeoff". In: *IEEE Transactions on Signal Processing* 66.4 (2018), pp. 847–862.

[35] Zhenyuan Feng, Bruno Clerckx, and Yang Zhao. "Waveform and beamforming design for intelligent reflecting surface aided wireless power transfer: Single-user and multi-user solutions". In: *IEEE Transactions on Wireless Communications* (2022).

[36] Yang Zhao, Bruno Clerckx, and Zhenyuan Feng. "IRS-aided SWIPT: Joint waveform, active and passive beamforming design under nonlinear harvester model". In: *IEEE Transactions on Communications* 70.2 (2021),

Lagrangian[37]

- For a general (not necessarily convex) optimization problem (primal problem)

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0, \forall i = 1, \ldots, m \\ & h_j(\mathbf{x}) = 0, \forall j = 1, \ldots, l \end{aligned} \tag{63}$$

  - $f_i(\mathbf{x}), \forall i = 1, \ldots, m$: inequality constraint functions
  - $h_j(\mathbf{x}), \forall j = 1, \ldots, l$: equality constraint functions
- Let $p^\star$ denote the global minimum value of (63)
- Introduce dual variables (also called Lagrange multipliers) $\boldsymbol{\lambda} \in \mathbb{R}^m$ and $\boldsymbol{\mu} \in \mathbb{R}^l$, and formulate the Lagrangian function

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{j=1}^{l} \mu_j h_j(\mathbf{x}) \tag{64}$$

[37] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

(Lagrange) Dual Function

- Dual function associated with (63)

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \min_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{65}$$

  - $g(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is always concave (even if (63) is non convex): pointwise minimum of a family of linear functions in $(\boldsymbol{\lambda}, \boldsymbol{\mu})$
  - $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is called dual feasible if $\boldsymbol{\lambda} \geq 0$ and $g(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is finite
  - For any (primal) feasible vector $\mathbf{x}$ and any dual feasible vector $(\boldsymbol{\lambda}, \boldsymbol{\mu})$,

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) \leq L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \leq f_0(\mathbf{x}). \tag{66}$$

  Thus

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) \leq p^\star \tag{67}$$

  i.e. the dual function value yields lower bounds on the optimal value $p^\star$.

(Lagrange) Dual Function[38]

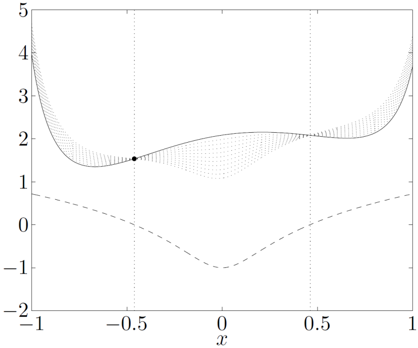


Figure 5.1 *Lower bound from a dual feasible point.* The solid curve shows the objective function f_0, and the dashed curve shows the constraint function f_1. The feasible set is the interval $[-0.46, 0.46]$, which is indicated by the two dotted vertical lines. The optimal point and value are $x^\star = -0.46$, $p^\star = 1.54$ (shown as a circle). The dotted curves show $L(x,\lambda)$ for $\lambda = 0.1,\ 0.2, \ldots, 1.0$. Each of these has a minimum value smaller than $p^\star$, since on the feasible set (and for $\lambda \geq 0$) we have $L(x,\lambda) \leq f_0(x)$.

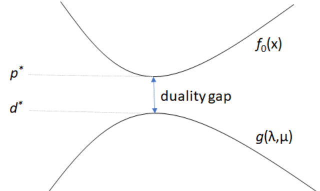
[38] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

(Lagrange) Dual Problem

- What is the best lower bound for $p^\star$?

$$\begin{aligned} \max_{\boldsymbol{\lambda},\boldsymbol{\mu}} \quad & g(\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \text{s.t.} \quad & \boldsymbol{\lambda} \geq 0, \boldsymbol{\mu} \in \mathbb{R}^l \end{aligned} \tag{68}$$

  - Dual problem (68) always convex regardless of convexity of primal problem (63), since $g(\boldsymbol{\lambda},\boldsymbol{\mu})$ is concave
  - Weak duality $d^\star \leq p^\star$ ($d^\star$ is the maximum value of (68)).
  - Strong duality $d^\star = p^\star$ holds for convex problems (satisfying some mild constraint qualification conditions, such as existence of a strict interior point - Slater's condition)
  - In general, $g(\boldsymbol{\lambda},\boldsymbol{\mu})$ difficult to compute



(Lagrange) Dual Problem

- Example (14) for which the optimal solution is $(x_1^\star, x_2^\star) = (0, 1)$
  - $L(x, \lambda_1, \lambda_2, \mu) = x_1 + x_2 + \lambda_1(-x_1) + \lambda_2(-x_2) + \mu(2 - x_1 - 2x_2)$
  - Dual function

$$\begin{aligned} g(\lambda_1, \lambda_2, \mu) &= \min_{(x_1,x_2)\in\mathbb{R}^2} x_1 + x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \mu(2 - x_1 - 2x_2) \\ &= 2\mu + \min_{(x_1,x_2)\in\mathbb{R}^2} (1 - \mu - \lambda_1)x_1 + (1 - 2\mu - \lambda_2)x_2 \\ &= \begin{cases} 2\mu, & \text{if } 1 = \mu + \lambda_1, 1 = 2\mu + \lambda_2 \\ -\infty, & \text{otherwise} \end{cases} \end{aligned} \tag{69}$$

  - Dual problem

$$\begin{aligned} \max_{\lambda_1,\lambda_2,\mu} \quad & 2\mu \\ \text{s.t.} \quad & \lambda_1 = 1 - \mu \geq 0, \lambda_2 = 1 - 2\mu \geq 0 \end{aligned} \tag{70}$$

  - $\mu^\star = 1/2$, $d^\star = 1$, $d^\star = p^\star$
  - Since $p^\star \geq g(\mu)$, the dual feasible optimal solution $\mu^\star$ serves as a certificate/proof for the primal optimality of $(x_1^\star, x_2^\star)$.

(Lagrange) Dual Problem

- Alternative way of computing Lagrangian and dual[39]
  - Lagrangian $L(x, \mu) = x_1 + x_2 + \mu(2 - x_1 - 2x_2)$
  - Dual function

$$g(\mu) = \min_{(x_1,x_2)\in\mathbb{R}_+^2} x_1 + x_2 + \mu(2 - x_1 - 2x_2) \tag{71}$$

$$= 2\mu + \min_{(x_1,x_2)\in\mathbb{R}_+^2} (1-\mu)x_1 + (1-2\mu)x_2 \tag{72}$$

$$= \begin{cases} 2\mu, & \text{if } \mu \le 1/2 \\ -\infty, & \text{otherwise} \end{cases} \tag{73}$$

  - Dual problem

$$\begin{aligned} \max_{\mu} \quad & 2\mu \\ \text{s.t.} \quad & \mu \le 1/2 \end{aligned} \tag{74}$$

[39] Zhi-Quan Luo and Wei Yu. "An introduction to convex optimization for communications and signal processing". In: *IEEE Journal on selected areas in communications* 24.8 (2006), pp. 1426–1438.

Karush-Kuhn-Tucker (KKT) Conditions

- A necessary condition for $\mathbf{x}^\star$ to be a local optimal solution of (63) is that there exists some $(\boldsymbol{\lambda}^\star, \boldsymbol{\mu}^\star)$ such that[40]

$$f_i(\mathbf{x}^\star) \leq 0, \forall i = 1, \ldots, m \;\; \text{(primal feasibility of } \mathbf{x}^\star) \tag{75a}$$
$$h_j(\mathbf{x}^\star) = 0, \forall j = 1, \ldots, l \;\; \text{(primal feasibility of } \mathbf{x}^\star) \tag{75b}$$
$$\lambda_i^\star \geq 0, \forall i = 1, \ldots, m \;\; \text{(dual feasibility)} \tag{75c}$$
$$\lambda_i^\star f_i(\mathbf{x}^\star) = 0, \forall i = 1, \ldots, m \;\; \text{(complementary slackness)} \tag{75d}$$
$$\nabla_{\mathbf{x}} L(\mathbf{x}^\star, \boldsymbol{\lambda}^\star, \boldsymbol{\mu}^\star) = 0 \;\; \text{(stationarity)} \tag{75e}$$

  - KKT necessary but not sufficient for global optimality
  - For convex problems, KKT necessary and sufficient
  - Without constraints, KKT boil down to conventional stationarity condition $\nabla_{\mathbf{x}} f_0(\mathbf{x}^\star) = 0$, i.e. unconstrained local minimum must be attained at a stationary point of f_0 (gradient of f_0 vanishes). With constraints, local optimum attained at a KKT point $\mathbf{x}^\star$; not longer at a stationary point of f_0.

[40] Zhi-Quan Luo and Wei Yu. "An introduction to convex optimization for communications and signal processing". In: *IEEE Journal on selected areas in communications* 24.8 (2006), pp. 1426–1438.

Interpretation[41]

- Rewrite (63) as an unconstrained problem $\min_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$
  - The hard constraints f_i and h_j have been replaced by soft versions
  - If we violate a constraint, say $f_i(\mathbf{x}) > 0$, our displeasure grows linearly with $\lambda_i \geq 0$
  - λ_i has the interpretation of the price for violating $f_i(\mathbf{x}) \leq 0$
  - Lagrangian can be viewed as a form of scalarization in a multicriterion problem $(f_0, f_1, \ldots, f_m)$
- If $f_i(\mathbf{x}^\star) < 0$ (inactive constraint), then the constraint can be tightened/loosened a small amount without affecting $p^\star$.
  - By complementary slackness, $\lambda_i^\star$ must be zero
- If $f_i(\mathbf{x}^\star) = 0$ (active constraint), then $\lambda_i^\star$ tells how active this constraint is
  - $\lambda_i^\star$ small: little effect on $p^\star$ if constraint is tightened/loosened
  - $\lambda_i^\star$ large: great effect on $p^\star$ if constraint is tightened/loosened

[41] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Karush-Kuhn-Tucker (KKT) Conditions

- Example (14)
  - $L(x, \lambda_1, \lambda_2, \mu) = x_1 + x_2 + \lambda_1(-x_1) + \lambda_2(-x_2) + \mu(2 - x_1 - 2x_2)$
  - KKT

$$-x_1 \leq 0, -x_2 \leq 0, \quad \text{(primal feasibility)} \tag{76a}$$

$$x_1 + 2x_2 - 2 = 0, \quad \text{(primal feasibility of)} \tag{76b}$$

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \quad \text{(dual feasibility)} \tag{76c}$$

$$\lambda_1 x_1 = 0, \lambda_2 x_2 = 0, \quad \text{(complementary slackness)} \tag{76d}$$

$$\frac{\partial L}{\partial x_1} = 1 - \lambda_1 - \mu = 0, \frac{\partial L}{\partial x_2} = 1 - \lambda_2 - 2\mu = 0 \tag{76e}$$

  - $(x_1^\star, x_2^\star) = (0, 1)$, $(\lambda_1^\star, \lambda_2^\star, \mu^\star) = (1/2, 0, 1/2)$
  - $\lambda_1^\star > 0 \rightarrow f_1(x_1^\star) = -x_1^\star = 0$; $f_2(x_2^\star) = -x_2^\star < 0 \rightarrow \lambda_2^\star = 0$

Example 7: Power Allocation by Water-Filling[42]

- The maximization problem may be rewritten as

$$\min_{\{s_k\}_{k=1}^n} \quad -\sum_{k=1}^{n} \log_2\left(1+\rho\lambda_k s_k\right)$$

$$\text{subject to} \quad s_k \geq 0,\ k=1,\ldots,n \text{ and } \sum_{k=1}^{n} s_k = 1.$$

- Lagrangian

$$L\left(s_k,\xi_k,\nu\right) = -\sum_{k=1}^{n} \log_2\left(1+\rho\lambda_k s_k\right) - \xi_k s_k + \nu\left(\sum_{k=1}^{n} s_k - 1\right) \quad (77)$$

with ξ_k the Lagrange multiplier associated to the inequality constraint $s_k \geq 0$ and ν the Lagrange multiplier for the power constraint $\sum_{k=1}^{n} s_k = 1$.

[42] Bruno Clerckx and Claude Oestges. *MIMO Wireless Networks: Channels, Techniques and Standards for Multi-Antenna, Multi-User and Multi-Cell Systems.* Academic Press (Elsevier), Oxford, UK, 2013.

Example 7: Power Allocation by Water-Filling

- KKT conditions for each value of $k = 1, \ldots, n$

$$s_k^\star \geq 0 \tag{78}$$

$$\sum_{k=1}^{n} s_k^\star = 1 \tag{79}$$

$$\xi_k^\star \geq 0 \tag{80}$$

$$\xi_k^\star s_k^\star = 0 \tag{81}$$

$$\left.\frac{\partial L\left(s_k, \xi_k, \nu\right)}{\partial s_k}\right|_{s_k^\star, \xi_k^\star, \nu^\star} = -\frac{\rho\lambda_k}{1+\rho\lambda_k s_k^\star} - \xi_k^\star + \nu^\star = 0 \tag{82}$$

- Based on the above equations, we have

$$\nu^\star \geq \frac{\rho\lambda_k}{1+\rho\lambda_k s_k^\star} \qquad k = 1, \ldots, n \tag{83}$$

$$\left(\nu^\star - \frac{\rho\lambda_k}{1+\rho\lambda_k s_k^\star}\right) s_k^\star = 0 \qquad k = 1, \ldots, n \tag{84}$$

Example 7: Power Allocation by Water-Filling

- If $\nu^\star < \rho\lambda_k$, (83) holds only if $s_k^\star > 0$, which implies by (84) that $\nu^\star = \rho\lambda_k/(1+\rho\lambda_k s_k^\star)$ or equivalently $s_k^\star = \frac{1}{\nu^\star} - \frac{1}{\rho\lambda_k}$.
- If $\nu^\star \geq \rho\lambda_k$, we cannot have $s_k^\star > 0$ since this would imply $\nu^\star \geq \rho\lambda_k > \rho\lambda_k/(1+\rho\lambda_k s_k^\star)$, thereby violating (84). Hence, $s_k^\star$ must be equal to 0 in this case.
- As a result, we may write that

$$s_k^\star = \begin{cases} \frac{1}{\nu^\star} - \frac{1}{\rho\lambda_k}, & \text{if } \nu^\star < \rho\lambda_k \\ 0, & \text{if } \nu^\star \geq \rho\lambda_k \end{cases} \tag{85}$$

or equivalently

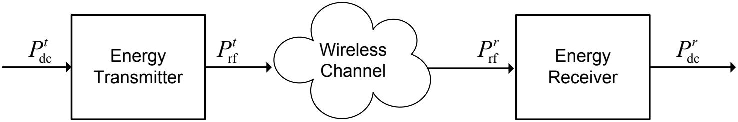
$$s_k^\star = \left(\frac{1}{\nu^\star} - \frac{1}{\rho\lambda_k}\right)^+ . \tag{86}$$

The level $\nu^\star$ is determined from the power constraint $\sum_{k=1}^n s_k^\star = 1$.

Observation:

Allocate more power to stronger channels and less power to weaker ones

Example 8: Waveform Design for Wireless Power[43]



- Transmit a multicarrier waveform $\sum_{n=0}^{N-1} s_n \cos(2\pi f_n t + \phi_n)$ at energy transmitter, receive $y(t)$ after the channel, harvest energy from $y(t)$ at energy receiver
- Allocate power to a multicarrier waveform to maximize P^r_{DC}

$$\max_{s_k} \quad P^r_{DC} = \underbrace{k_2 \mathcal{E}\left\{y(t)^2\right\}}_{\text{Linear term}} + \underbrace{k_4 \mathcal{E}\left\{y(t)^4\right\}}_{\text{Nonlinear terms}} \quad \text{s. t.} \quad \sum_{k=1}^{N} s_k^2 \leq P$$

- $N = 2$: $P_{DC} = k_2 \left(s_0^2 A_0^2 + s_1^2 A_1^2\right) + k_4 (s_0^4 A_0^4 + s_1^4 A_1^4 + 4 s_0^2 s_1^2 A_0^2 A_1^2)$
  - s_k^2 power allocated to frequency carrier k
  - A_k magnitude of the channel on carrier k

[43] Bruno Clerckx and Ekaterina Bayguzina. "Waveform design for wireless power transfer". In: *IEEE Transactions on Signal Processing* 64.23 (2016), pp. 6313–6328.

Example 8: Waveform Design for Wireless Power

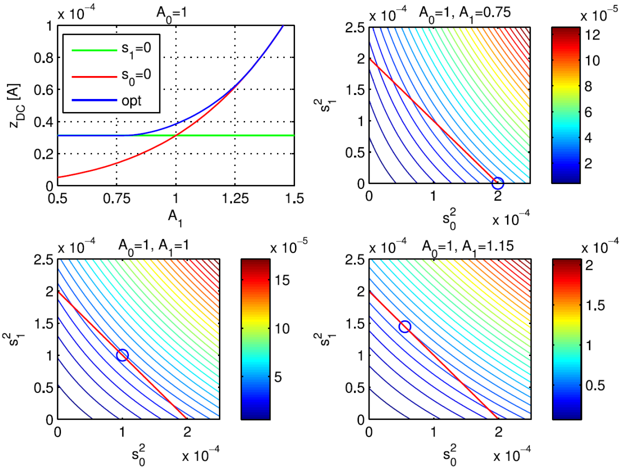
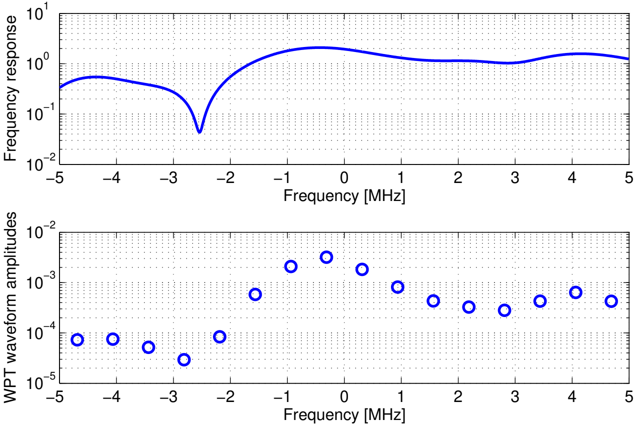


Fig. 2. z_{DC} as a function of A_1 and contours of z_{DC} as a function of s_0^2 and s_1^2. The straight line refers to the power constraint and the circle to the optimal power allocation strategy. $P = -40$ dBW.

Example 8: Waveform Design for Wireless Power



Observation reminiscent of Water-Filling:

Allocate more power to stronger channels and less power to weaker ones

Example 1: Tx Beamforming - Power Min. - KKT[44]

- Recall problem (17)

$$\min_{\mathbf{w}_1,\ldots,\mathbf{w}_K} \sum_{k=1}^{K} \|\mathbf{w}_k\|^2 \ \text{s.t.} \ \frac{|\mathbf{h}_k^H \mathbf{w}_k|^2}{\sum_{i\neq k} |\mathbf{h}_k^H \mathbf{w}_i|^2 + \sigma^2} \geq \gamma_k, \forall k \tag{87}$$

- Compute Lagrangian and KKT stationarity condition

$$L(\mathbf{w}_1,\ldots,\mathbf{w}_K,\lambda_1,\ldots,\lambda_K) = \sum_{k=1}^{K} \|\mathbf{w}_k\|^2 + \sum_{k=1}^{K} \lambda_k \left(\sum_{i\neq k} \frac{1}{\sigma^2} |\mathbf{h}_k^H \mathbf{w}_i|^2 + 1 - \frac{1}{\gamma_k \sigma^2} |\mathbf{h}_k^H \mathbf{w}_k|^2 \right), \tag{88}$$

$$\frac{\partial L}{\partial \mathbf{w}_k} = 2\mathbf{w}_k + 2\sum_{i\neq k} \frac{\lambda_i}{\sigma^2} \mathbf{h}_i \mathbf{h}_i^H \mathbf{w}_k - 2\frac{\lambda_k}{\gamma_k \sigma^2} \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}_k = 0, \ \ \forall k \tag{89}$$

- $\frac{\partial \|\mathbf{w}_k\|^2}{\partial \mathbf{w}_k} = 2\mathbf{w}_k, \ \frac{\partial \mathbf{w}_k^H \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}_k}{\partial \mathbf{w}_k} = 2\mathbf{h}_k \mathbf{h}_k^H \mathbf{w}_k$

[44] Emil Björnson, Mats Bengtsson, and Björn Ottersten. "Optimal Multiuser Transmit Beamforming: A Difficult

Example 1: Tx Beamforming - Power Min. - KKT

$$\mathbf{w}_k + \sum_{i \neq k} \frac{\lambda_i}{\sigma^2} \mathbf{h}_i \mathbf{h}_i^H \mathbf{w}_k = \frac{\lambda_k}{\gamma_k \sigma^2} \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}_k \tag{90a}$$

$$\mathbf{w}_k + \sum_{i \neq k} \frac{\lambda_i}{\sigma^2} \mathbf{h}_i \mathbf{h}_i^H \mathbf{w}_k + \frac{\lambda_k}{\sigma^2} \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}_k = \frac{\lambda_k}{\gamma_k \sigma^2} \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}_k + \frac{\lambda_k}{\sigma^2} \mathbf{h}_k \mathbf{h}_k^H \mathbf{w}_k \tag{90b}$$

$$\mathbf{w}_k = \left(\mathbf{I}_M + \sum_{i=1}^{K} \frac{\lambda_i}{\sigma^2} \mathbf{h}_i \mathbf{h}_i^H \right)^{-1} \mathbf{h}_k \underbrace{\frac{\lambda_k}{\sigma^2} \left(1 + \frac{1}{\gamma_k} \right) \mathbf{h}_k^H \mathbf{w}_k}_{\text{scalar}} \tag{91a}$$

$$\mathbf{w}_k = \left(\mathbf{I}_M + \sum_{i \neq k} \frac{\lambda_i}{\sigma^2} \mathbf{h}_i \mathbf{h}_i^H \right)^{-1} \mathbf{h}_k \underbrace{\frac{\lambda_k}{\sigma^2 \gamma_k} \mathbf{h}_k^H \mathbf{w}_k}_{\text{scalar}} \tag{91b}$$

Example 1: Tx Beamforming - Power Min. - KKT

- Derive optimal precoder structure (KKT 1 and KKT 2)

$$\mathbf{w}_k^\star = \underbrace{\sqrt{p_k}}_{p_k \text{ beamforming power}} \underbrace{\frac{\left(\mathbf{I}_M + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2}\mathbf{h}_i\mathbf{h}_i^H\right)^{-1}\mathbf{h}_k}{\left\|\left(\mathbf{I}_M + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2}\mathbf{h}_i\mathbf{h}_i^H\right)^{-1}\mathbf{h}_k\right\|}}_{\bar{\mathbf{w}}_k^\star \text{ optimal beamforming direction}} \tag{92a}$$

$$\mathbf{w}_k^\star = \sqrt{p_k}\frac{\left(\mathbf{I}_M + \sum_{i\neq k} \frac{\lambda_i}{\sigma^2}\mathbf{h}_i\mathbf{h}_i^H\right)^{-1}\mathbf{h}_k}{\left\|\left(\mathbf{I}_M + \sum_{i\neq k} \frac{\lambda_i}{\sigma^2}\mathbf{h}_i\mathbf{h}_i^H\right)^{-1}\mathbf{h}_k\right\|} \tag{92b}$$

with, respectively,

$$\frac{\lambda_k}{\sigma^2}\left(1+\frac{1}{\gamma_k}\right)\mathbf{h}_k^H\mathbf{w}_k \left\|\left(\mathbf{I}_M + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2}\mathbf{h}_i\mathbf{h}_i^H\right)^{-1}\mathbf{h}_k\right\| = \sqrt{p_k} \tag{93a}$$

$$\frac{\lambda_k}{\sigma^2\gamma_k}\mathbf{h}_k^H\mathbf{w}_k \left\|\left(\mathbf{I}_M + \sum_{i\neq k} \frac{\lambda_i}{\sigma^2}\mathbf{h}_i\mathbf{h}_i^H\right)^{-1}\mathbf{h}_k\right\| = \sqrt{p_k} \tag{93b}$$

Example 1: Tx Beamforming - Power Min. - KKT

- Derive beamforming power from SINR constraints:
  - SINR constraints hold with equality at the optimal solution[45]

$$\frac{|\mathbf{h}_k^H \mathbf{w}_k^\star|^2}{\sum_{i\neq k} |\mathbf{h}_k^H \mathbf{w}_i^\star|^2 + \sigma^2} = \gamma_k, \; k = 1, \ldots, K \tag{94}$$

$$\frac{1}{\gamma_k} p_k |\mathbf{h}_k^H \bar{\mathbf{w}}_k^\star|^2 - \sum_{i\neq k} p_i |\mathbf{h}_k^H \bar{\mathbf{w}}_i^\star|^2 = \sigma^2, \; k = 1, \ldots, K \tag{95}$$

  - Hence, K equations with K unknowns p_k

$$\begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} \sigma^2 \\ \vdots \\ \sigma^2 \end{bmatrix} \tag{96}$$

with

$$[\mathbf{M}]_{ij} = \begin{cases} \frac{1}{\gamma_i} |\mathbf{h}_i^H \bar{\mathbf{w}}_i^\star|^2, & i = j \\ -|\mathbf{h}_i^H \bar{\mathbf{w}}_j^\star|^2, & i \neq j \end{cases} \tag{97}$$

[45] If the 1st inequality is strict, $\mathbf{w}_1$ can be scaled down to give an equality in the 1st constraint, hence a smaller value of the objective function while still satisfying other constraints; hence contradicts that the solution is optimal

Example 1: Tx Beamforming - Power Min. - KKT

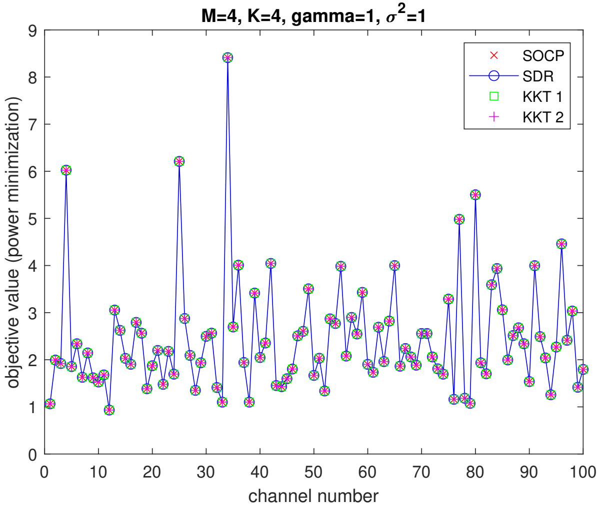
- Compute the Lagrange multipliers by plugging (92) into (93) (solve using fixed-point iterations)

$$\lambda_k = \frac{\sigma^2}{\left(1 + \frac{1}{\gamma_k}\right) \mathbf{h}_k^H \left(\mathbf{I}_M + \sum_{i=1}^{K} \frac{\lambda_i}{\sigma^2} \mathbf{h}_i \mathbf{h}_i^H\right)^{-1} \mathbf{h}_k} \tag{98a}$$

$$\lambda_k = \frac{\sigma^2 \gamma_k}{\mathbf{h}_k^H \left(\mathbf{I}_M + \sum_{i \neq k} \frac{\lambda_i}{\sigma^2} \mathbf{h}_i \mathbf{h}_i^H\right)^{-1} \mathbf{h}_k} \tag{98b}$$

- λ_k, p_k, $\mathbf{w}_k$ $\forall k$ same for solutions KKT1 and KKT2

Example 1: Tx BF - Power Min. - SDP vs SOCP vs KKT



Example 9: Transmit Beamforming - Rate Max.[46]

- Consider an objective function $f(\mathrm{SINR}_1, \ldots, \mathrm{SINR}_K)$ strictly increasing in the SINR of each user

$$\min_{\mathbf{w}_1,\ldots,\mathbf{w}_K} f(\mathrm{SINR}_1, \ldots, \mathrm{SINR}_K) \text{ s.t. } \sum_{k=1}^{K} \|\mathbf{w}_k\|^2 \leq P \tag{99}$$

- Sum-Rate

$$f = \sum_{k=1}^{K} R_k = \sum_{k=1}^{K} \log_2 \left(1 + \frac{|\mathbf{h}_k^H \mathbf{w}_k|^2}{\sum_{i\neq k} |\mathbf{h}_k^H \mathbf{w}_i|^2 + \sigma^2} \right) \tag{100}$$

- Weighted ($w_k \geq 0$) Sum-Rate

$$f = \sum_{k=1}^{K} w_k R_k = \sum_{k=1}^{K} w_k \log_2 \left(1 + \frac{|\mathbf{h}_k^H \mathbf{w}_k|^2}{\sum_{i\neq k} |\mathbf{h}_k^H \mathbf{w}_i|^2 + \sigma^2} \right) \tag{101}$$

[46] Emil Björnson, Mats Bengtsson, and Björn Ottersten. "Optimal Multiuser Transmit Beamforming: A Difficult Problem with a Simple Solution Structure [Lecture Notes]". In: *IEEE Signal Processing Magazine* 31.4 (2014), pp. 142–148.

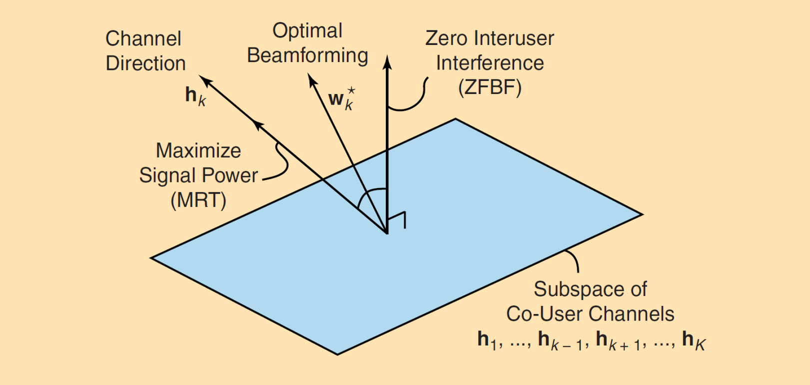
Example 9: Transmit Beamforming - Rate Max.

- Problem (99) more difficult than (17)
  - SINR predefined in (17) but optimal SINR to be found in (99)
  - If optimal SINR of (99) were known, optimal solution of (17) also solves (99)
- Optimal structure

$$\mathbf{w}_k^\star = \sqrt{p_k}\frac{\overbrace{\left(\mathbf{I}_M + \sum_{i=1}^{K}\frac{\lambda_i}{\sigma^2}\mathbf{h}_i\mathbf{h}_i^H\right)^{-1}}^{\text{rotation}}\overbrace{\mathbf{h}_k}^{\text{MRT}}}{\left\|\left(\mathbf{I}_M + \sum_{i=1}^{K}\frac{\lambda_i}{\sigma^2}\mathbf{h}_i\mathbf{h}_i^H\right)^{-1}\mathbf{h}_k\right\|} \tag{102}$$

  - MRT (optimal for $K = 1$) and rotation (fct of λ_i): rotates MRT to reduce interference caused in co-user directions $\mathbf{h}_{\forall i \neq k}$
  - Optimal choice of $\lambda_1, \ldots, \lambda_K \geq 0$ depends on f
  - λ_i: priority of user i where a larger value means that other users' beamforming vectors will be more orthogonal to $\mathbf{h}_i$

Example 9: Transmit Beamforming - Rate Max.[47]



[FIG2] The optimal beamforming $\mathbf{w}_k^\star$ is based on the channel direction $\mathbf{h}_k$ but rotated to balance between high signal power and being orthogonal to the co-user channels.

[47] Emil Björnson, Mats Bengtsson, and Björn Ottersten. "Optimal Multiuser Transmit Beamforming: A Difficult Problem with a Simple Solution Structure [Lecture Notes]". In: *IEEE Signal Processing Magazine* 31.4 (2014), pp. 142–148.

Example 10: Receive Beamforming

- Receive beamforming in multiuser setting: $\mathbf{y} = \sum_{i=1}^{K} \mathbf{h}_i s_i + \mathbf{n}$ with user k transmitting signal s_k with uplink transmit power q_k.
- Uplink SINR

$$SINR_k^{uplink} = \frac{q_k |\mathbf{g}_k \mathbf{h}_k|^2}{\sum_{i \neq k} q_i |\mathbf{g}_k \mathbf{h}_i|^2 + \sigma^2 \mathbf{g}_k \mathbf{g}_k^H} \tag{103}$$

- Note the difference with downlink SINR
- Uplink SINR of user k only a function of its own receive beamfoming vector $\mathbf{g}_k$
- Optimize the beamforming separately for each k

$$\arg\max_{\mathbf{g}_k} \frac{q_k |\mathbf{g}_k \mathbf{h}_k|^2}{\sum_{i \neq k} q_i |\mathbf{g}_k \mathbf{h}_i|^2 + \sigma^2 \mathbf{g}_k \mathbf{g}_k^H} \tag{104}$$

Example 10: Receive Beamforming - Rayleigh quotient

- Maximization of a generalized Rayleigh quotient

$$\frac{q_k|\mathbf{g}_k\mathbf{h}_k|^2}{\sum_{i\neq k} q_i|\mathbf{g}_k\mathbf{h}_i|^2 + \sigma^2\mathbf{g}_k\mathbf{g}_k^H} = \frac{\mathbf{g}_k\mathbf{A}_k\mathbf{g}_k^H}{\mathbf{g}_k\mathbf{B}_k\mathbf{g}_k^H} \tag{105}$$

with

$$\mathbf{A}_k = q_k\mathbf{h}_k\mathbf{h}_k^H, \mathbf{B}_k = \sum_{i\neq k} q_i\mathbf{h}_i\mathbf{h}_i^H + \sigma^2\mathbf{I} \tag{106}$$

- Take $\mathbf{z}_k = \mathbf{g}_k\mathbf{B}_k^{1/2}$, $\mathbf{g}_k = \mathbf{z}_k\mathbf{B}_k^{-1/2}$

$$\frac{q_k|\mathbf{g}_k\mathbf{h}_k|^2}{\sum_{i\neq k} q_i|\mathbf{g}_k\mathbf{h}_i|^2 + \sigma^2\mathbf{g}_k\mathbf{g}_k^H} = \frac{\mathbf{z}_k\mathbf{B}_k^{-1/2}\mathbf{A}_k\mathbf{B}_k^{-H/2}\mathbf{z}_k^H}{\mathbf{z}_k\mathbf{z}_k^H} \tag{107}$$

- Choose optimal $\mathbf{z}_k$ as dominant eigenvector of $\mathbf{B}_k^{-1/2}\mathbf{A}_k\mathbf{B}_k^{1/2}$, i.e., $\mathbf{z}_k = \mathbf{h}_k^H\mathbf{B}_k^{-H/2}/\|\mathbf{h}_k^H\mathbf{B}_k^{-H/2}\|$ and $\mathbf{g}_k = \mathbf{h}_k^H\mathbf{B}_k^{-H/2}\mathbf{B}_k^{-1/2}/\|\mathbf{h}_k^H\mathbf{B}_k^{-H/2}\mathbf{B}_k^{-1/2}\| = \mathbf{h}_k^H\mathbf{B}_k^{-1}/\|\mathbf{h}_k^H\mathbf{B}_k^{-1}\|$.

Example 10: Receive Beamforming - Rayleigh quotient

- Choose optimal beamforming vector as

$$\mathbf{g}_k = \frac{\mathbf{h}_k^H \mathbf{B}_k^{-1}}{\|\mathbf{h}_k^H \mathbf{B}_k^{-1}\|} \tag{108}$$

$$= \frac{\mathbf{h}_k^H \left(\sum_{i \neq k} q_i \mathbf{h}_i \mathbf{h}_i^H + \sigma^2 \mathbf{I}\right)^{-1}}{\left\|\mathbf{h}_k^H \left(\sum_{i \neq k} q_i \mathbf{h}_i \mathbf{h}_i^H + \sigma^2 \mathbf{I}\right)^{-1}\right\|} \tag{109}$$

$$= \frac{\mathbf{h}_k^H \left(\sum_{i=1}^{K} q_i \mathbf{h}_i \mathbf{h}_i^H + \sigma^2 \mathbf{I}\right)^{-1}}{\left\|\mathbf{h}_k^H \left(\sum_{i=1}^{K} q_i \mathbf{h}_i \mathbf{h}_i^H + \sigma^2 \mathbf{I}\right)^{-1}\right\|} \tag{110}$$

  i.e. the MMSE combiner/filter. Last equality coming from matrix inversion lemma.

- Optimal transmit and receive beamforming have the same structure
- Rayleigh quotient can also be solved using Lagrangian of the problem max $\mathbf{g}_k \mathbf{A}_k \mathbf{g}_k^H$ s.t. $\mathbf{g}_k \mathbf{B}_k \mathbf{g}_k^H = 1$

Appendix: Matrix Inversion Lemma and MMSE

- Show $\frac{\mathbf{h}_k^H\left(\sum_{i\neq k} q_i\mathbf{h}_i\mathbf{h}_i^H+\sigma^2\mathbf{I}\right)^{-1}}{\left\|\mathbf{h}_k^H\left(\sum_{i\neq k} q_i\mathbf{h}_i\mathbf{h}_i^H+\sigma^2\mathbf{I}\right)^{-1}\right\|} = \frac{\mathbf{h}_k^H\left(\sum_{i=1}^K q_i\mathbf{h}_i\mathbf{h}_i^H+\sigma^2\mathbf{I}\right)^{-1}}{\left\|\mathbf{h}_k^H\left(\sum_{i=1}^K q_i\mathbf{h}_i\mathbf{h}_i^H+\sigma^2\mathbf{I}\right)^{-1}\right\|}$
- Take $\mathbf{C} = \sum_{i\neq k}\frac{q_i}{\sigma^2}\mathbf{h}_i\mathbf{h}_i^H + \mathbf{I}$ so we need to show

$$\frac{\mathbf{h}_k^H\mathbf{C}^{-1}}{\left\|\mathbf{h}_k^H\mathbf{C}^{-1}\right\|} = \frac{\mathbf{h}_k^H\left(\frac{q_k}{\sigma^2}\mathbf{h}_k\mathbf{h}_k^H+\mathbf{C}\right)^{-1}}{\left\|\mathbf{h}_k^H\left(\frac{q_k}{\sigma^2}\mathbf{h}_k\mathbf{h}_k^H+\mathbf{C}\right)^{-1}\right\|} \tag{111}$$

- Matrix inversion lemma: $\left(\mathbf{x}\mathbf{x}^H+\mathbf{C}\right)^{-1} = \mathbf{C}^{-1} - \frac{\mathbf{C}^{-1}\mathbf{x}\mathbf{x}^H\mathbf{C}^{-1}}{1+\mathbf{x}^H\mathbf{C}^{-1}\mathbf{x}}$
- $\mathbf{h}_k^H\left(\frac{q_k}{\sigma^2}\mathbf{h}_k\mathbf{h}_k^H+\mathbf{C}\right)^{-1} = \mathbf{h}_k^H\mathbf{C}^{-1}\underbrace{\frac{1}{1+\frac{q_k}{\sigma^2}\mathbf{h}_k^H\mathbf{C}^{-1}\mathbf{h}_k}}_{=scalar}$

Advanced Optimization Methods

- Alternating Optimization, Block Coordinate Descent
- Majorization Minimization (MM)
- Successive Convex Approximation (SCA)
- Manifold
- Fractional Programming (FP)
- Weighted Minimum Mean Square Error (WMMSE)
- Alternating Direction Method of Multipliers (ADMM)
- Penalty Dual Decomposition (PDD)

Advanced Communication Systems

... used together with advanced communication schemes/systems

- Rate-Splitting Multiple Access (RSMA)
  https://www.youtube.com/playlist?list=PL3nE1Yo1b4CqJzE1Xn0ft91KtgU0Rmznq
  https://://sites.google.com/view/ieee-comsoc-wtc-sig-rsma/home
- Multi-user/Massive MIMO
- Reconfigurable Intelligent Surfaces
- Wireless Information and Power Transfer
- Integrated Sensing and Communications
- Non-terrestrial Networks
- Resource allocation problems (time, frequency, power, space, etc)
- ...

Outline

Other tips about CVX

Tips:

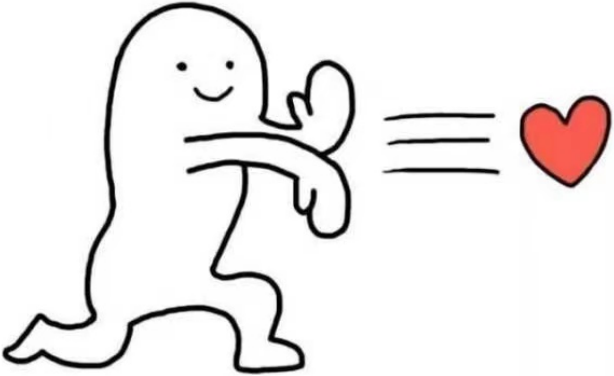
- There is specific threshold of the iterative methods in CVX. When the value of the matrix is too small/large, the iterative method cannot get correct results. Therefore, sometimes the input matrix needs to be scaled without changing the original problem

Useful Links

- Matrix cookbook[48] (including matrix derivatives, inverse, solution, decomposition, etc.)
- Coding grammar of CVX[49]
- Matrix calculus I[50]
- Matrix calculus II[51]
- Computational complexity of mathematical operations[52]
- ...

[48] https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf
[49] http://web.cvxr.com/cvx/doc/
[50] http://www.matrixcalculus.org/
[51] http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/calculus.html
[52] https://en.wikipedia.org/wiki/Computational_complexity_of_mathematical_operations

Thank You!

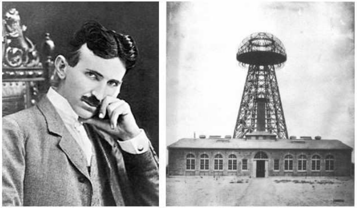


Part 5: Future Topics

Wireless is More than just Communications

Radio waves carry both energy and information

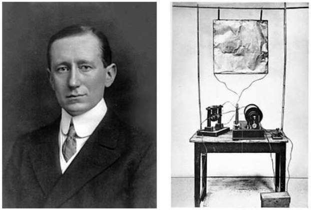
Wireless Power Transmission (WPT)



Tesla 1901

0G

Wireless Information Transmission (WIT)

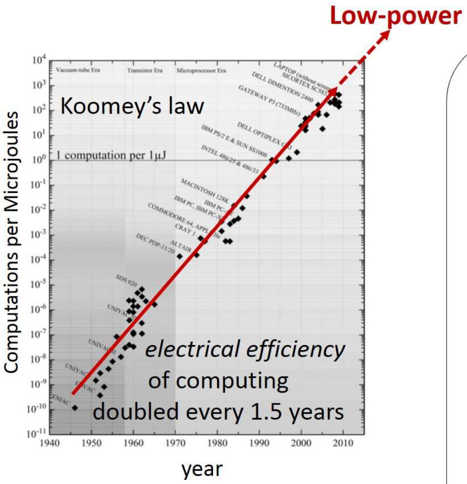


Marconi 1896

5G

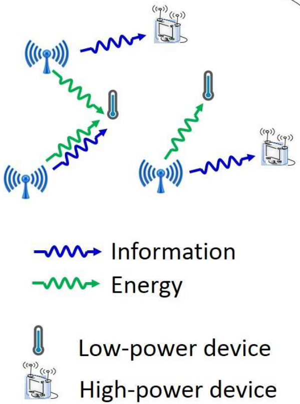
Unified Wireless Information and Power Transmission (WIPT)

In 20 Years from Now ... **Trillions** of **Low-Power** Devices



2017: 5 Billions phones

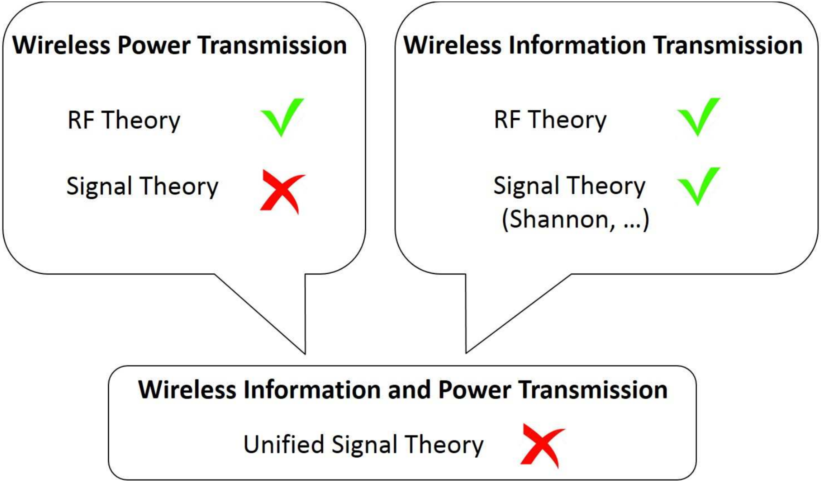
2035: **Trillions** IoT devices



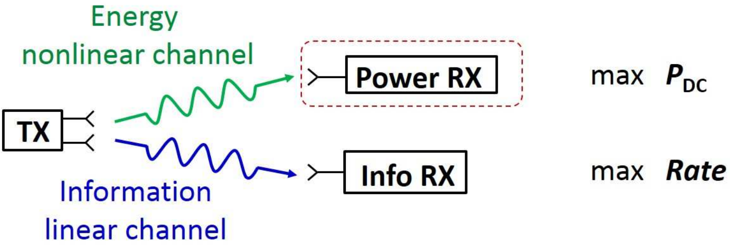
Future of Wireless:

Communicate and Energize

A **Missing** Signal Theory of Wireless Transmission



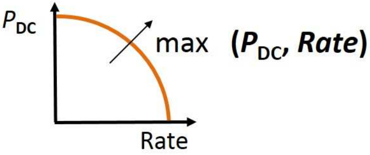
Signal Theory and Design



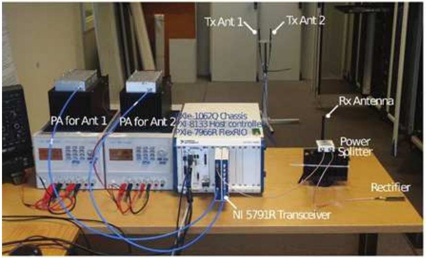
→ **New communications and signal design for WPT**

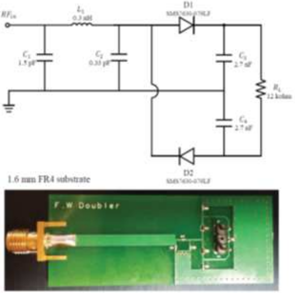
→ **Novel and unified signal theory and design for WIPT**

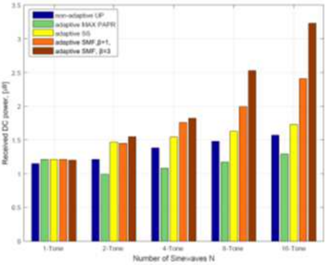
→ **Fundamental tradeoff**



System Design, Prototyping and Experimentation

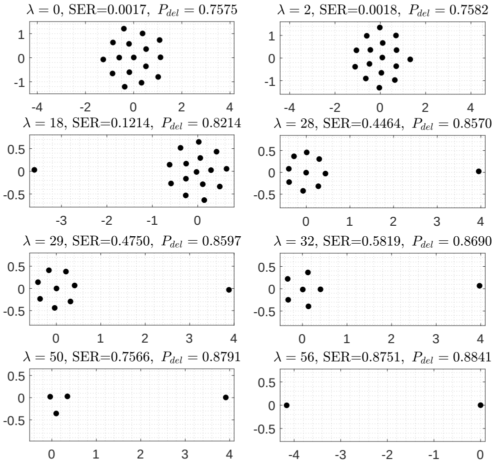






Machine learning for Wireless Communications and Power Transfer

16-symbols modulation for different values of energy



6G

Many on-going research activities:

- enhanced eMBB, enhanced URLLC, enhanced mMTC
- new services: integrated sensing (radar, localization, ...) and communications
- new surfaces: reconfigurable intelligent surfaces https://youtu.be/x3VMQ-ZU0ek
- new multi-antenna techniques (beyond massive MIMO)
- new multi-user/multiple access schemes and interference management techniques: Rate-Splitting Multiple Access https://www.youtube.com/playlist?list=PL3nE1Yo1b4CqJzE1Xn0ft91KtgU0Rmznq https://sites.google.com/view/ieee-comsoc-wtc-sig-rsma/home
- machine learning for wireless communications
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