Wavelets, Sparsity and their Applications
Session 1: Introduction

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Note: many figures in the slides are taken from a course offered by Prof. M. Vetterli and Prof. V. Goyal

Overview

- **Material Provided**
  - Slides
  - Lecture notes
  - Matlab Mini-project
- **Assessment**
  - Final exam consisting of four compulsory questions each carrying equal mark
  - Matlab mini-project due on Monday 18th of December at 5:00pm (25% of final mark)
- **Material and recommended textbooks available on** Blackboard

Motivation

- **Data Data Data**
- Where is all this data coming from?
- 90% of the data exchanged over Internet is non-textual

- Understanding the origin of the data and its formation is key (biomedical data, multimedia data, physical measurements from sensor networks)
- Proper Signal/Data Representation is fundamental to restructure the original data in a more revealing form (*sparse signal representation*)

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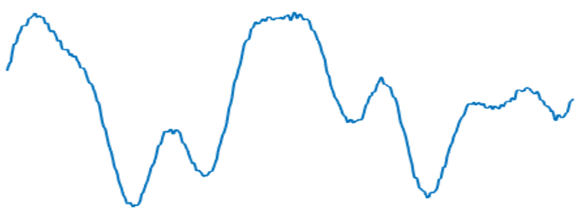
The Signal Representation Problem

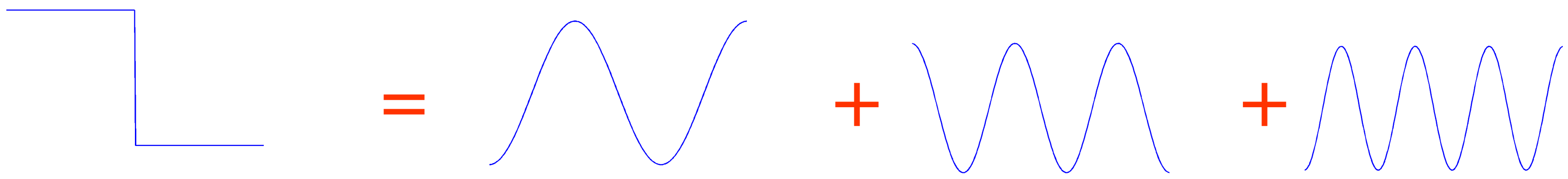
Signal processing aims to decompose complex signals using elementary functions which are then easier to manipulate



$$x(t)=\sum_{i=-\infty}^{\infty}\alpha_i\varphi_i(t)$$

- Wavelets and the wavelet transform - in much the same way as the Fourier transform - is an advanced signal processing tool that helps us to make things easy.
- This is achieved by decomposing a signal in elementary constituents and analysing or processing these constituents independently before reconstruction (re-synthesising the signal).

Signal Representation: Fourier Series

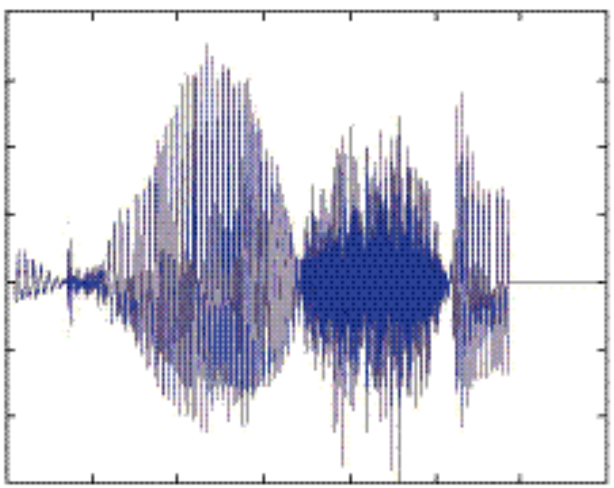


Any signal defined on a finite interval is represented by a sum of sinusoids at different frequencies.

What has Fourier ever done for us?

- Fourier Transform and Fourier series make things easy, e.g.,
  - Fast computation of convolution
  - Essential in LTI filter design and in filtering in general
  - Essential in communication
  - Valid signal analysis tool

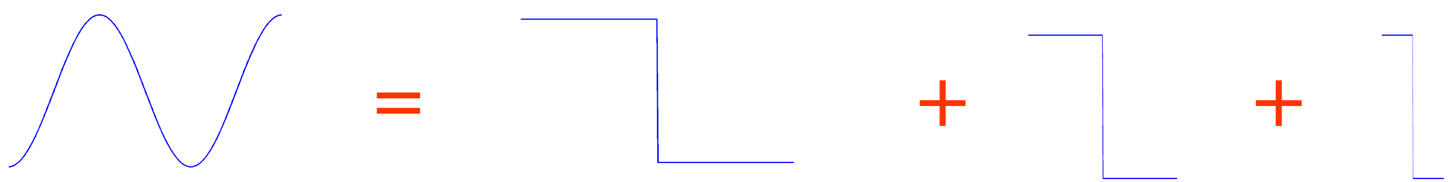
- Can we do better? Is FT good at analysing the signal below?



Signal Representation: Haar Wavelet

finite interval → Fourier series

finite energy → Haar function



Any function of finite energy is given by the sum of the Haar function and its translated and scaled versions.

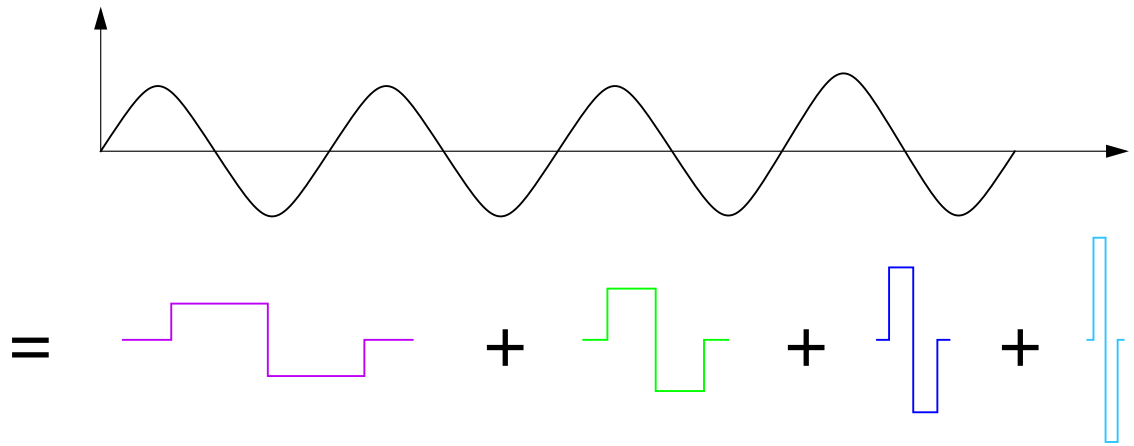
Haar function is the first example of a wavelet!

L^p space: f must be p-interable for f to be in L_p.

Signal Representation: Haar Wavelet

L^p norm: $\|f\|_p = \left(\sum_i |f_i|^p\right)^{\frac{1}{p}}$

1910: Alfred Haar discovers the Haar wavelet
dual to the Fourier construction



Why do this? What makes it work?

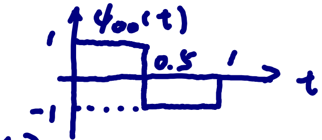
- basic atoms form an orthonormal set

Note

- sines/cosines and Haar functions are ON bases for $L_2(\Re)$
- both are structured orthonormal bases
- they have different time and frequency behavior

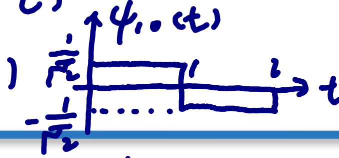
The Haar Decomposition

$\psi_{00}(t) = \psi(t)$



$\psi_{10}(t) = 2^{-\frac{1}{2}}\psi(2^{-1}t)$

$= \frac{1}{\sqrt{2}}\psi(\frac{t}{2})$

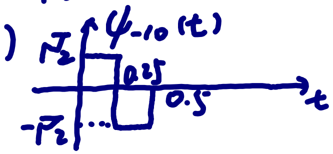


$\frac{t}{2} = 0.5$

$t = 1$

$\psi_{-10}(t) = 2^{\frac{1}{2}}\psi(2^{1}t)$

$= \sqrt{2}\,\psi(2t)$



$2t = 0.5$

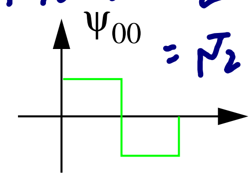
$t = 0.25$

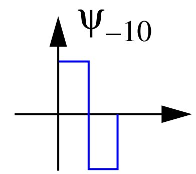
Basis functions $\psi(0) = 1$

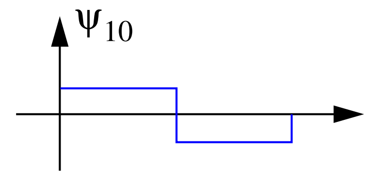
$\rightarrow \psi(0.5) = -1$

$\psi(1) = 0$

$$\psi(t) = \begin{cases} 1 & 0 \le t < 0.5 \\ -1 & 0.5 \le t < 1 \\ 0 & \text{else} \end{cases}$$

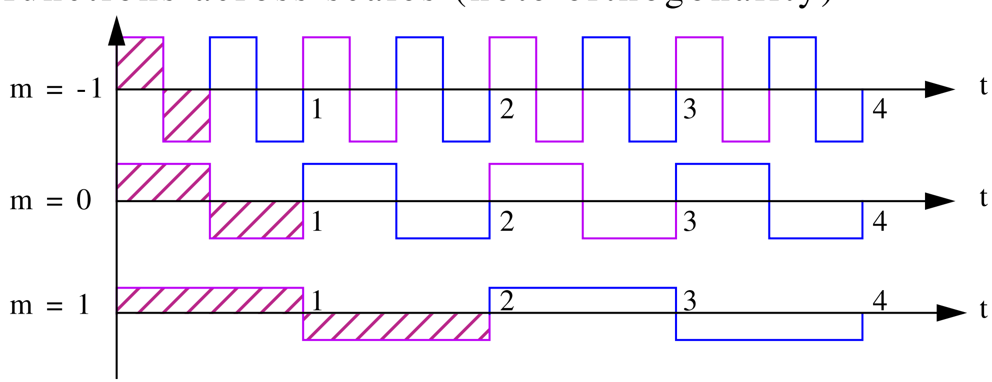






$$\psi_{mn}(t) = 2^{-m/2}\psi(2^{-m}t - n)$$

Basis functions across scales (note orthogonality)



Signal Representations

$$x(t)=\sum_{i=-\infty}^{\infty}\alpha_i\varphi_i(t)$$

Key Ingredients:

- a set of 'atoms': $\{\varphi_i\}$
- a inner product: $\langle x,\varphi_i\rangle=\int x(t)\varphi_i(t)dt$
- a synthesis formula: $x(t)=\sum_{i=-\infty}^{\infty}\alpha_i\varphi_i(t)$

Many choices of $\{\varphi_i\}$

- orthonormal bases (e.g., Fourier series, Haar wavelet, Daubechies wavelets)
- biorthogonal bases (e.g., splines)
- overcomplete expansions or frames

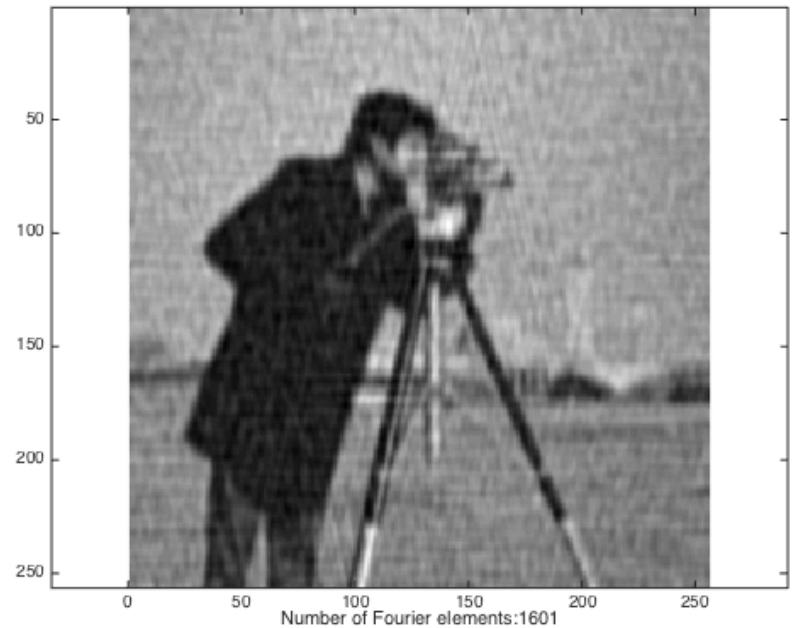
Key Questions: How can I find new sets of atoms? Which set is better?

Sparse Signal Representations

- Given two competing signal representations, which one is better?
- Signal Processing uses **Occam's razor** to answer this question:
  "among competing *representations* that predict equally well,
  the one with the *fewest number of components* should be selected."
- Wavelets are *better* because they provide sparse representations of most natural signals
- This is why wavelets are used successfully in many signal processing applications (e.g., image compression)
- The Sparse Representation Principle is now the cornerstone of 'Data Analysis'

Wavelets vs Fourier

Coefficients used: 2%



Wavelets

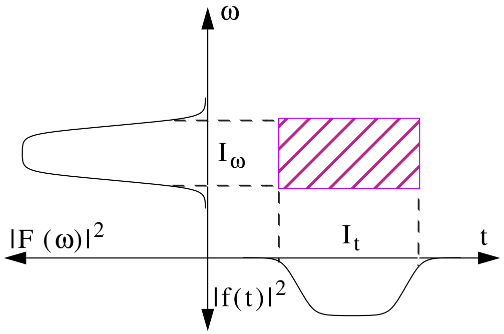
Fourier

Wavelets vs Fourier: Time-Frequency Tiling

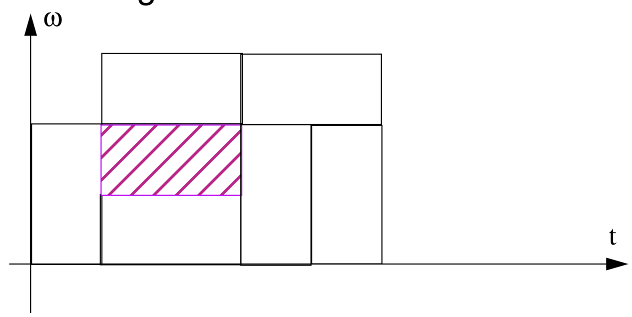
Time-frequency tiling

Basis functions have some spread in time and frequency

- leads to time-frequency tile or atom
- the area in which most of the signal's energy resides



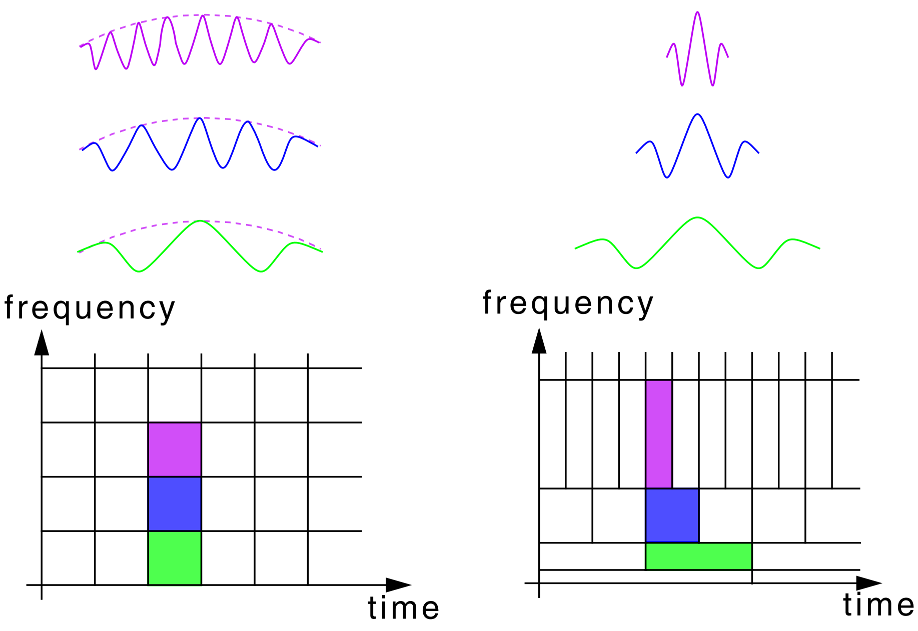
- to represent signals, the tiling cannot have "holes"



Wavelets vs Fourier: Time-Frequency Tiling

1945: Gabor localizes the Fourier transform ⇒ STFT

1980: Morlet proposes the continuous wavelet transform



short-time Fourier transform **wavelet transform**

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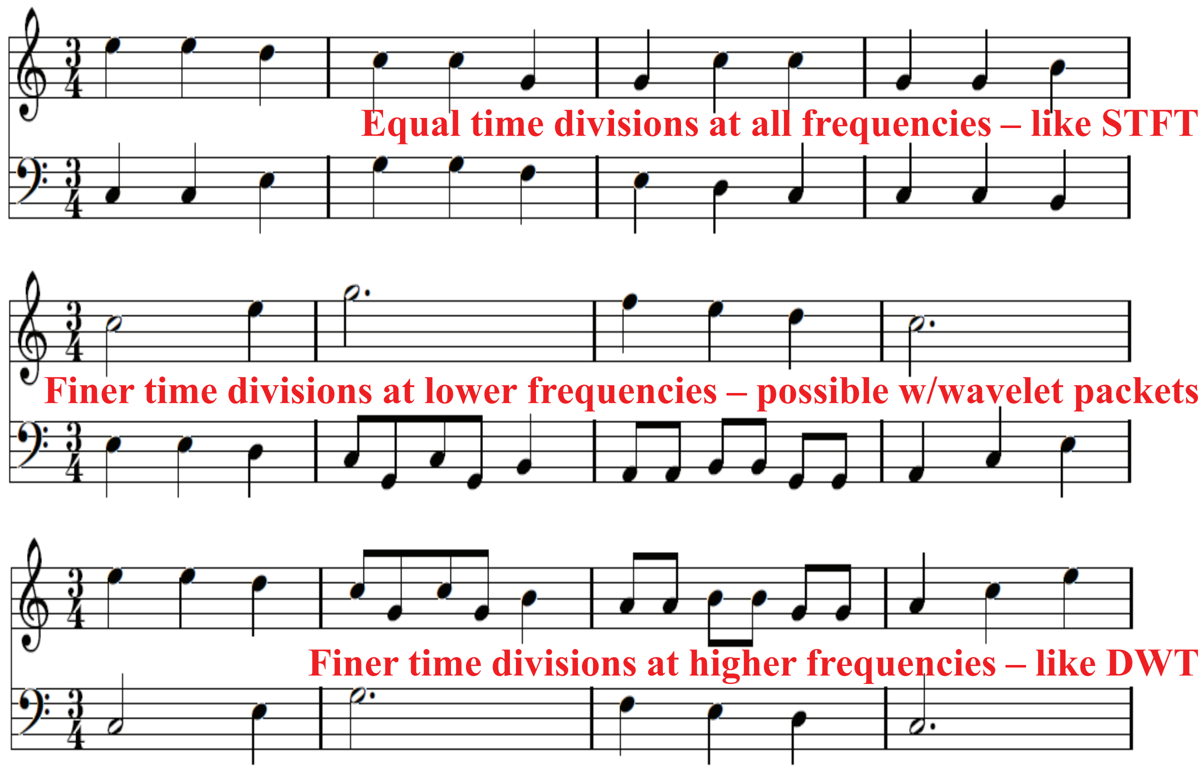
Wavelets vs Fourier

Which one looks like music?

(a) (b) (c)



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Wavelets vs Fourier

Preview 1: Approximation

Consider signals in the vector space $\mathbb{R}^N$

There are many orthonormal expansions

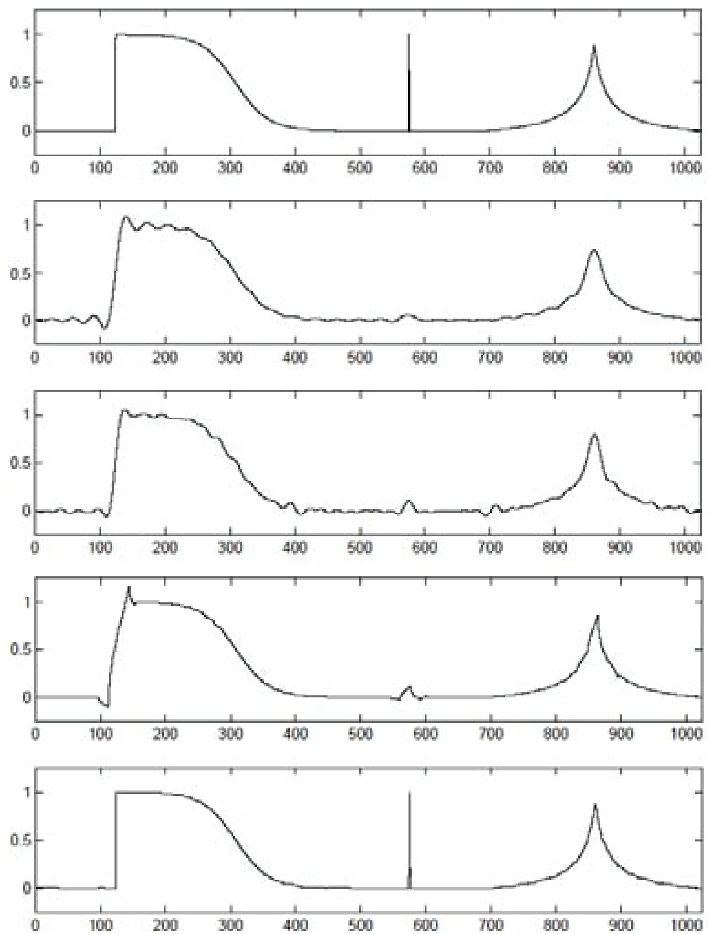
$$x = \sum_{i=1}^{N} \alpha_i \varphi_i$$

How do they differ for *approximation*?

$$x \;\approx\; \hat{x}_K \;=\; \sum_{i=1}^{K} \beta_i \varphi_{j_i}, \qquad K < N$$

- Different bases $\{\varphi_i\}_{i=1}^N$
- Different orders $j_1, j_2, j_3, \ldots$

Wavelets vs Fourier



Original signal, N=1024

D= 2.7 M=64, linear Fourier

D= 2.4 M=64, nonlinear Fourier

D=3.5 M=64, linear Daubechies wavelet

D=0.01 M=64, nonlinear Daubechies wavelet

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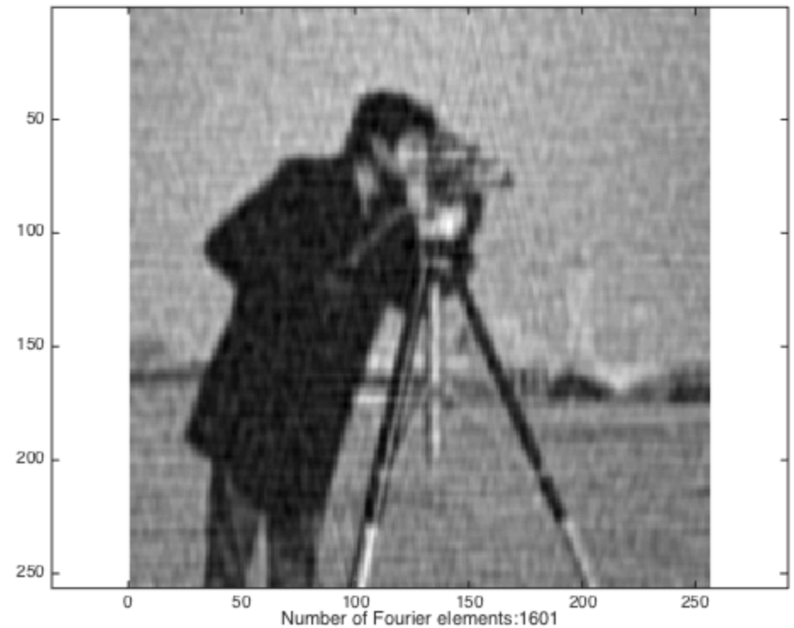
Why Sparsity?

Coefficients used: 2%

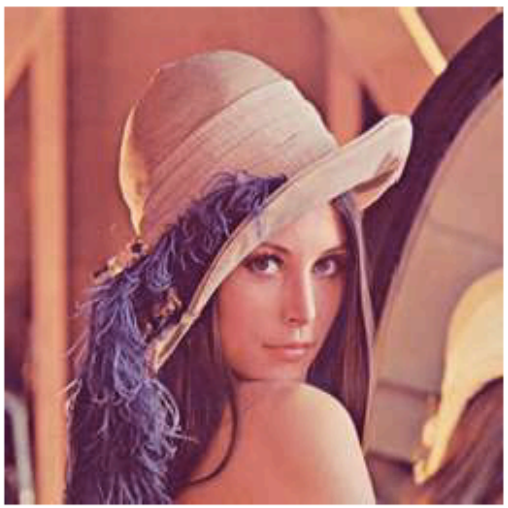
Wavelets



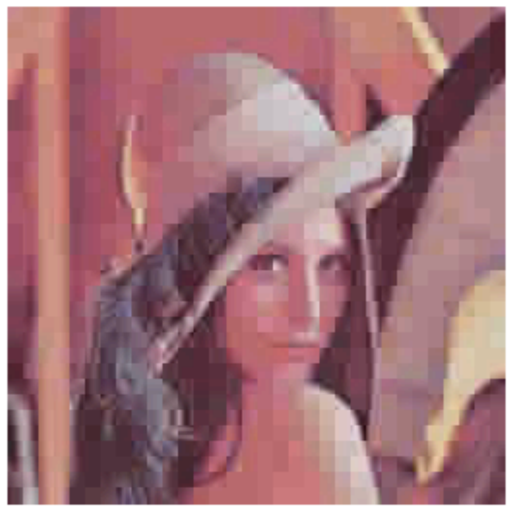
Fourier

Why Sparsity?

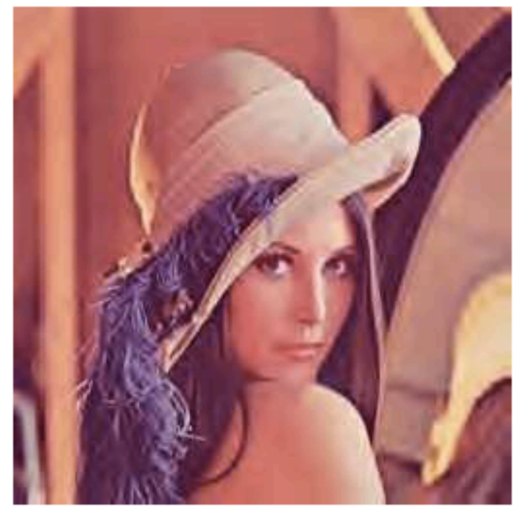
Image Compression



Original Lena Image (256×256 pixels)



JPEG (Compression Ratio 43:1)



JPEG2000 (Compression Ratio 43:1)

Note: images courtesy of dspworx.com

Why Sparsity?

- In signal processing we often have to solve ill-conditioned inverse problems
- Approach: given partial and noisy knowledge of your signal, amongst all possible valid solutions, pick the sparsest one
- This sparsity-driven principle has lead to state-of-the-art algorithms in denoising, inpainting, deconvolution, sampling etc.

Why Sparsity? Inpainting

The usual suspect

Why Sparsity? Inpainting

Inpainting based on Scholefield-Dragotti. IEEE Trans. Image Processing 2014

Wavelets vs Fourier

Preview 2: Estimation

Consider signal x observed through noisy observation

$$y = x + d \qquad d \sim \mathcal{N}(0, \sigma^2 I)$$

minimum mean square error

MMSE estimate $\hat{x}_{\text{MMSE}} = E[x \mid y]$ can be complicated (if even known)

For simplicity, we often insist on a diagonal estimator in a transform domain:

$$\hat{x} = T^{-1} D(Ty), \qquad D(\,\cdot\,) \text{ diagonal}$$

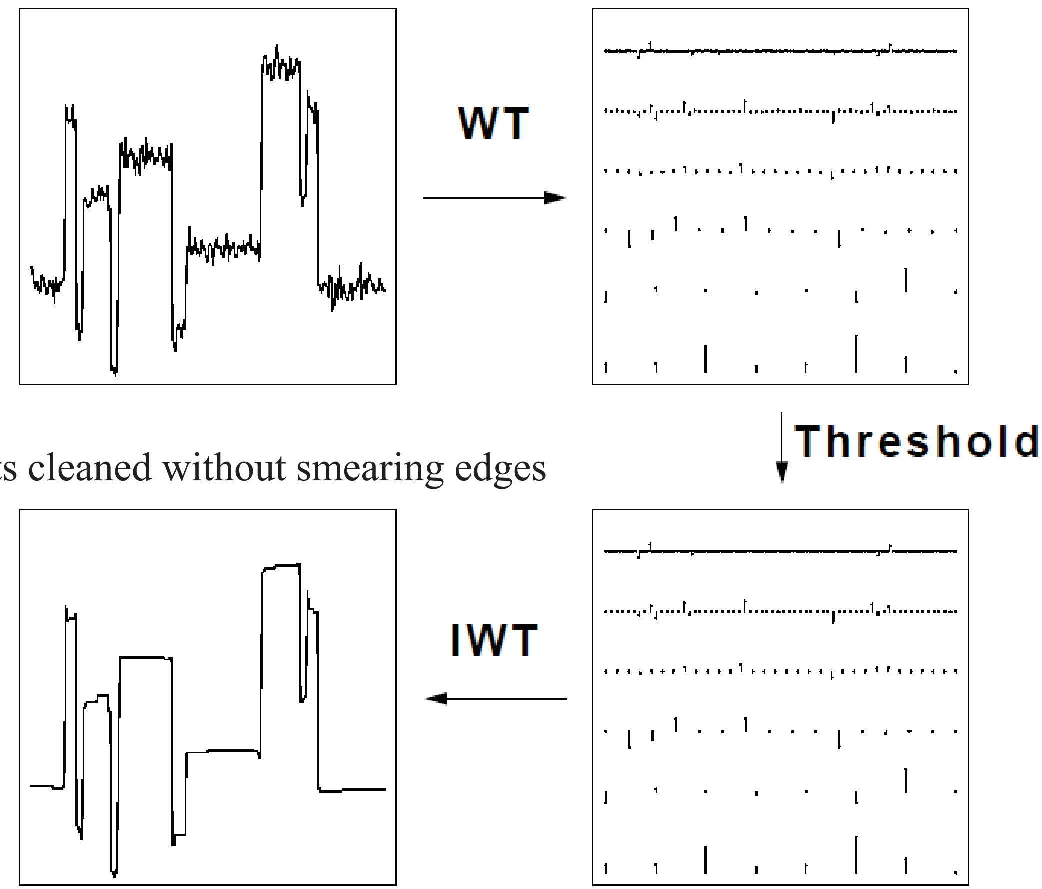
What are good choices for T and D?

Wavelets vs Fourier

Original image | Original image + i.i.d. Gaussian noise | Denoised in Fourier domain | Denoised in Wavelet domain

Wavelets vs Fourier

1-D Example



Smooth parts cleaned without smearing edges

Wavelets vs Fourier

Fourier	Wavelet
linear, time invariant	not linear, time invariant
Gaussian	non-Gaussian
stationary	non-stationary
continuous	discontinuous
smooth	containing edges
known structure	(partly) unknown structure
global	locally "adaptive"
single resolution	multiple resolutions
modeled by sinusoids	modeled by piecewise polynomials

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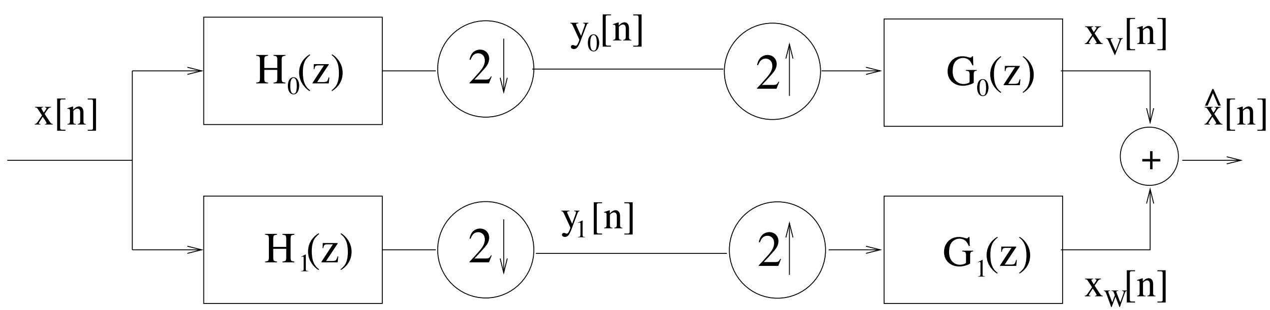
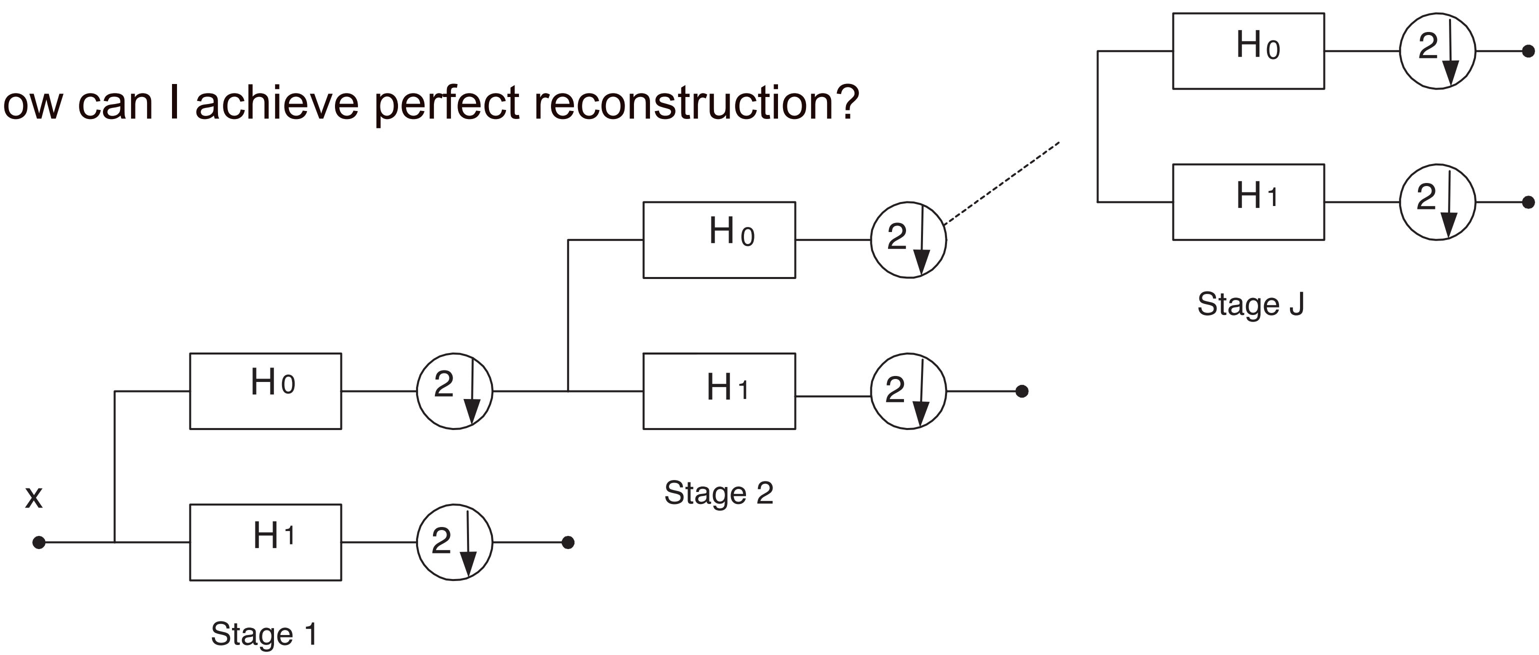
Implementation of the Wavelet Transform

- Two-channel filter-bank

- 

- How can I achieve perfect reconstruction?



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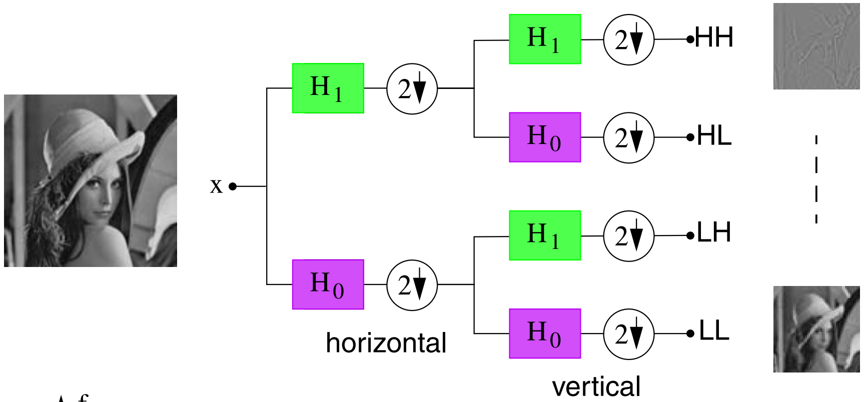
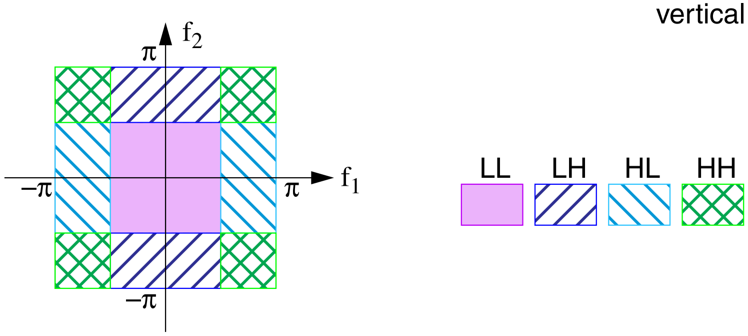
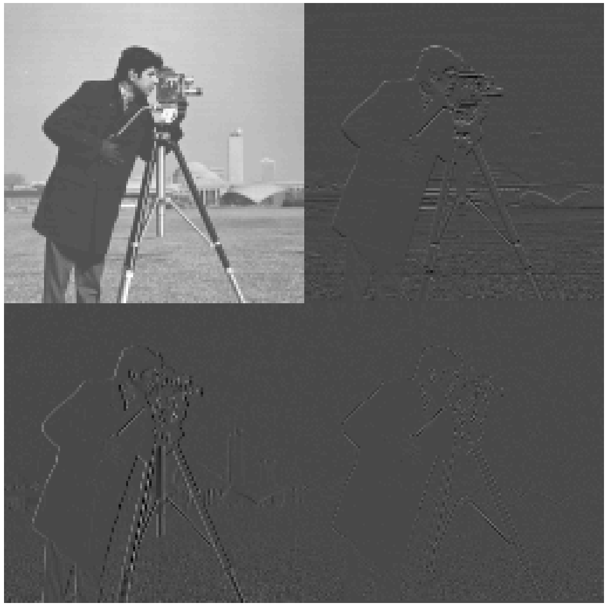
Implementation of the Wavelet Transform

1984: Lena gets critical (subband coding)

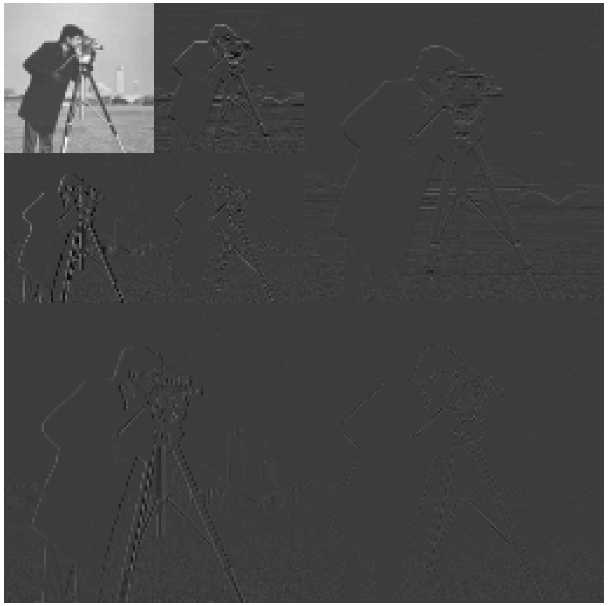




Implementation of the 2-D Wavelet Transform



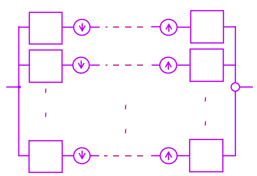
Implementation of the 2-D Wavelet Transform



Wavelet Theory: A Unifying Framework

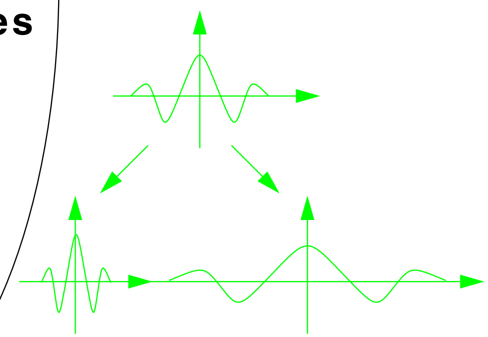
Wavelets
(applied mathematics)

Filter banks
(DSP)

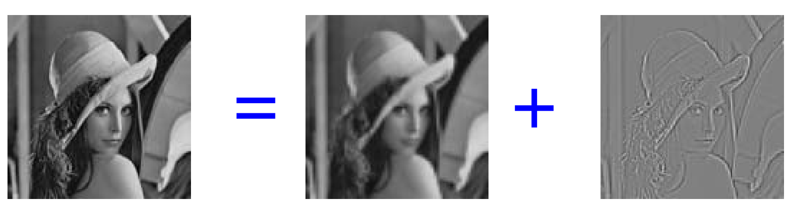


Construction of bases for signal expansions

$$x = \sum \langle \psi_i, x \rangle \psi_i$$



Multiresolution signal analysis
(computer vision)



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Course Aims and Course Outline

- **Course aim:** find systematic ways to design new signal representations (e.g., new wavelets beyond the Haar wavelet). This is achieved
  - using the geometry of Hilbert spaces
  - learning about filter banks (for fast implementation of new bases)
  - connecting ideas in approximation theory and spline theory with filter banks

- Wavelets had impact in many **applications**. We will touch upon**:**
  - Image/Video Compression
  - Super-Resolution Imaging
  - Neuroscience

- **Conceptually** wavelets:
  - provide a bridge between discrete-time and continuous-time processing
  - give us an understanding of the importance of compact representation (**sparsity**) in both signal processing and data analysis (e.g., sparse regression)
  - provide the tools to build new representations to tackle the applications at hand

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Wavelets and Beyond: Image Super-Resolution

(a)Original (512 x 512)

(b) Low-res. (64 x 64)

(c) Super-res (PSNR=24.2dB)

- *Forty low-resolution and shifted versions of the original.*
- *Accurate registration is achieved by retrieving the continuous moments of the 'Tiger' from the samples.*
- *The registered images are interpolated to achieve super-resolution.*