SPEECH PROCESSING
Properties of Digital Filters

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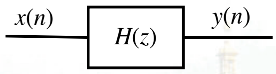
Aims of this module

- This lecture reviews some well known facts about filters and introduces some less known ones that will be needed later on.
  - Derive the power response of first order FIR and IIR filters and relate this to the geometry of the pole-zero diagram.
  - Relate the bandwidth of a 2nd-order resonance to the geometry of the pole-zero diagram.
  - Describe the bandwidth expansion transformation of a filter.
  - Describe the effect of reversing the coefficients of a filter.
  - Derive expressions for the log frequency response and its average value.

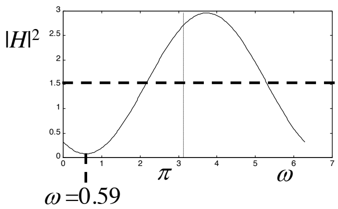
1^{st} order FIR filter



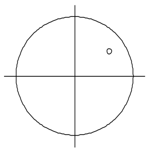
$$H(z) = 1 - az^{-1} \quad \Leftrightarrow \quad y(n) = x(n) - ax(n-1)$$

- Filter has a single zero at $z = a = re^{j\theta}$
- Frequency response of filter is given by: $H(e^{j\omega}) = 1 - ae^{-j\omega}$
- Power response of filter is given by:
$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega})$$
$$= \left(1 - ae^{-j\omega}\right)\left(1 - a^*e^{+j\omega}\right)$$
$$= 1 + r^2 - 2r\cos(\omega - \theta)$$
- Example: $a = 0.6 + 0.4j = 0.72e^{0.59j}$

$$\left|H(e^{j\omega})\right|^2 = 1.52 - 1.44\cos(\omega - 0.59)$$



Mean power gain = $1+r^2$



Log Frequency Response for |a|<1

- Calculate the log response of the filter $\log\left(H(e^{j\omega})\right) = \log\left(1 - ae^{-j\omega}\right)$
- If $|a|<1$ then $|ae^{-jw}|<1$ and we can expand the log as a power series using

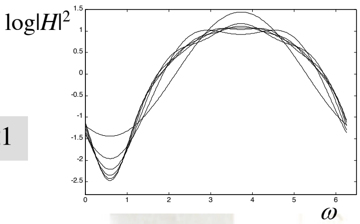
$$\log(1-d) = -\left(d + \frac{d^2}{2} + \frac{d^3}{3} + \ldots\right) \quad \text{for} \quad |d| < 1$$

- Hence $\log\left(H\left(e^{j\omega}\right)\right) = -\sum_{n=1}^{\infty}\frac{a^n}{n}e^{-jn\omega}$
- Note that $\log\left(re^{j\phi}\right) = \log(r) + j\phi \quad \Rightarrow \quad \log\left(|x|\right) = \Re\left(\log(x)\right)$
- Hence $\log\left(\left|H\left(e^{j\omega}\right)\right|^2\right) = -2\sum_{n=1}^{\infty}\frac{r^n}{n}\cos\left(n(\omega-\theta)\right)$ where $a = re^{j\theta}$

Log power spectrum

First six terms in the summation for:
$a = 0.6 + 0.4j$

The average of $\log|H|^2$ is always zero if $|a|<1$



Log Frequency Response for |a|>1

- If $|a|>1$, we can rearrange the formula in terms of a^{-1}

$$\log\left(H(e^{j\omega})\right)=\log\left(-ae^{-j\omega}\left(1-a^{-1}e^{j\omega}\right)\right)=\log\left(-ae^{-j\omega}\right)+\log\left(1-a^{-1}e^{j\omega}\right)$$

- Since $|a^{-1}|<1$, we can expand the log as before to obtain

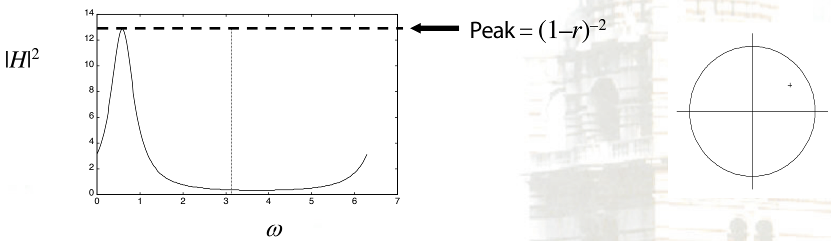
$$\log\left(\left|H\left(e^{j\omega}\right)\right|^2\right)=2\log|a|-2\sum_{n=1}^{\infty}\frac{r^{-n}}{n}\cos\left(n(\omega-\theta)\right)\quad \text{where}\quad a=re^{j\theta}$$

- The average of $\log|H|^2$ is $2\log|a|$ if $|a|>1$
- The log response of an arbitrary filter is just the sum of the log responses of each pole or zero. For a stable filter, all the poles must be within the unit circle. Hence ...
- Given a stable filter $H(z)=\dfrac{b_0+b_1z^{-1}+b_2z^{-2}+\ldots}{a_0+a_1z^{-1}+a_2z^{-2}+\ldots}$
- Then the average value of $\log|H|^2$ is given by $2\log\left(\left|\dfrac{b_0}{a_0}\right|\right)+2\sum_{\text{zeros with }|z_i|>1}\log\left(|z_i|\right)$

Single Pole Filter

$$H(z)=\frac{1}{1-az^{-1}}\quad\Leftrightarrow\quad y(n)=x(n)+ay(n-1)$$

- Filter has a single pole at $z=a=re^{j\theta}$
- Power response of filter is given by $\left|H(e^{j\omega})\right|^2=\dfrac{1}{1+r^2-2r\cos(\omega-\theta)}$



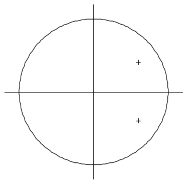
- Note: response is no longer a simple cosine wave

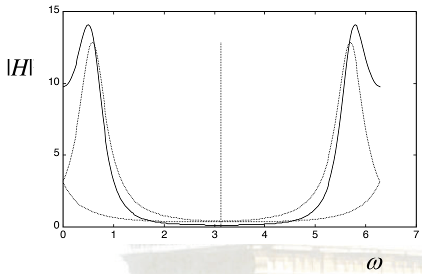
Pole Pairs

- If the filter coefficients are real, any complex zeros or poles will always occur in conjugate pairs
- The response of the filter is the product of the responses of the individual poles
  - Conjugate pole/zero pairs ensure a symmetric response
- Example: Poles at $0.6 \pm 0.4j \ = 0.72 e^{\pm 0.59 j} = r e^{\pm j\theta}$

$$H(z) = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

$$= \frac{1}{1 - 1.2z^{-1} + 0.52z^{-2}}$$

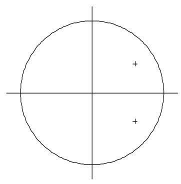




Geometrical Interpretation

$$H(z) = \frac{1}{(1 - az^{-1})(1 - a^* z^{-1})}$$

$$|H(z)| = \frac{1}{|1 - az^{-1}|\,|1 - a^* z^{-1}|}$$



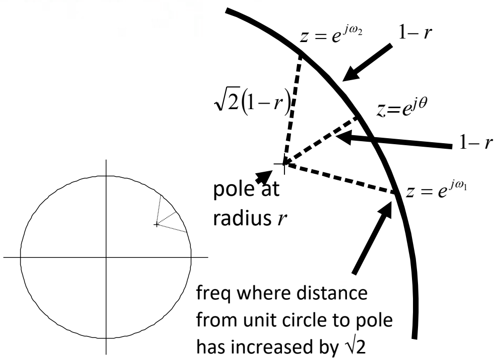
- But since $|z|=1$, we have $|1 - az^{-1}| = |z^{-1}|\,|z - a| = |z - a|$
  - This is just the distance between z and a
- The magnitude response of the filter

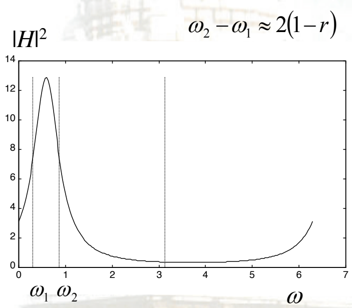
$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots} = \frac{b_0}{a_0} \times \frac{\prod\left(1 - x_i z^{-1}\right)}{\prod\left(1 - y_i z^{-1}\right)}$$

at a frequency ω is proportional to the product of the distance from the point $e^{j\omega}$ to all the zeros divided by the product of the distance to all the poles .
  - The constant of proportionality is b_0/a_0.

Bandwidth of a Resonance Peak

- The bandwidth of a resonance peak is the frequency range at which the magnitude response has decreased by √2 compared to the peak value.
- For poles near the unit circle this is approximately
  $2(1-r)$ rad/s = $(1-r)/\pi$ Hz (normalized).





Bandwidth Expansion

- Given a filter $H(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots}$

  form a new filter by multiplying coefficients a_i and b_i by k^i for some $k < 1$

$$G(z) = H(z/k) = \frac{b_0 + b_1 k z^{-1} + b_2 k^2 z^{-2} + \ldots}{a_0 + a_1 k z^{-1} + a_2 k^2 z^{-2} + \ldots}$$

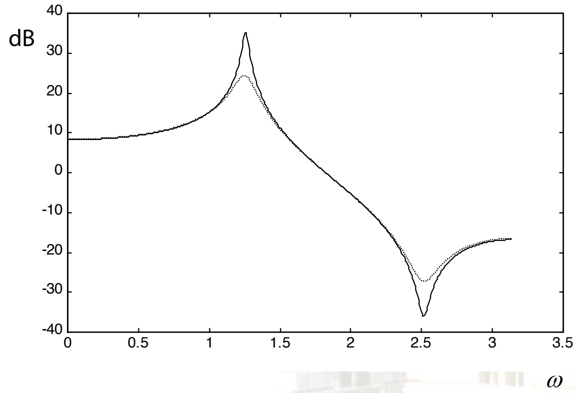
- If $H(z)$ has a pole/zero at z_0, then $G(z)$ will have one at kz_0
  - All poles and zero will be moved towards the origin by a factor k
- If the bandwidth of a pole of $H(z)$ is b=$2(1-r)$, then the bandwidth of the corresponding pole in $G(z)$ will be expanded to:

$$2(1-kr) = b + 2r(1-k)$$

Example

$$H(z) = \frac{1 + 1.57z^{-1} + 0.94z^{-2}}{1 - 0.67z^{-1} + 0.96z^{-2}}$$

$k = 0.95$



Coefficient Reversal

- Given a filter $H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_p z^{-p}$
form a new filter by conjugating the coefficients and putting them in reverse order:
$$G(z) = b_p^* + b_{p-1}^* z^{-1} + b_{p-2}^* z^{-2} + \ldots + b_0^* z^{-p} = z^{-p} H^*\left(z^{*-1}\right)$$
- If z_0 is a zero of $H(z)$ then $z_0{}^{*-1}$ is a zero of $G(z)$
  - This is called a *reflection* in the unit circle.
- The frequency response of $G(z)$ is given by:
$$G(e^{j\omega}) = e^{-jp\omega} H^*(e^{j\omega})$$

- Hence $G(z)$ has the same magnitude response as $H(z)$ but a different phase response:

$$\left|G(e^{j\omega})\right| = \left|H(e^{j\omega})\right| \qquad \mathrm{Arg}\big(G(e^{j\omega})\big) = -\mathrm{Arg}\big(H(e^{j\omega})\big) - p\omega$$