Lecture 9/10

Continuous RVs : Uniform, Normal, Exponential, Chi-square, log normal

$$X \longrightarrow Y$$

$$f_X(x) \qquad\qquad f_Y(y)$$

$\boxed{Tx} \xrightarrow{\text{EXPO}} \boxed{Rx}$

Normal (comms 2nd Noise year)

Normal (comms 3/4 info theory)

EXPO, chi-square, lognormal (4th year wireless comms)

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ex
$$\int_{-\infty}^{2.35} \sqrt{\frac{2}{\pi}}\, e^{-2(u-2)^2}\, du$$

$$X \sim N(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \int_{-\infty}^{2.35} \underbrace{\sqrt{\frac{1}{2\pi\frac{1}{4}}}\, e^{-\frac{(u-2)^2}{2\frac{1}{4}}}}_{\text{PDF of } U \sim N(\mu=2,\ \sigma^2=\frac{1}{4})} du$$

$$U = \sigma \underset{N(0,1)}{Z} + \mu = \frac{1}{2}Z + 2$$

$$= F_U(2.35) = P(U \leq 2.35)$$

$$= P\left(\frac{1}{2}Z + 2 \leq 2.35\right)$$

$$= P(Z \leq 0.7) = 0.758$$

Reliability

- RV T — random time to failure (non-negative)
- CDF of T: $F_T(t) = P(T \le t) = \int_0^t f_T(t)\,dt$ — failure time distribution
- PDF of T: $f_T(t)$ — failure time density

$$f_T(t) = \frac{d}{dt} F_T(t) = \frac{d}{dt}(1 - R_T(t)) = -\frac{d}{dt} R_T(t)$$

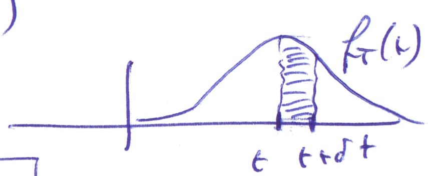
- $R_T(t) = 1 - F_T(t) = 1 - P(T \le t) = P(T > t)$

  reliability function

- $A = \{ t < T < t + \delta t \}$

$B = \{ t < T \}$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{\underset{P(T>t)}{\underset{\shortparallel}{P(B)}}} \approx \frac{f_T(t)\,\delta t}{R_T(t)}$$



$$\approx \underbrace{\boxed{\frac{f_T(t)}{R_T(t)}}}_{z_T(t)} \delta t$$

$z_T(t)$ hazard rate

(measure of proneness to failure at time t)

- cumulative hazard function

$$H_T(t) = \int_0^t z_T(u)\,du$$

$$= \int_0^t \frac{f_T(u)}{R_T(u)}\,du = -[\ln R_T]_0^t$$

$$= -\ln R_T(t) + \overbrace{\ln R_T(0)}^{0}$$

$R_T(0) = P(T > 0) = 1$

$H_T(t) = -\ln R_T(t)$

$R_T(t) = e^{-H_T(t)}$

ex hazard rate $z_T(t) = \lambda$

$$H_z(t) = \int_0^t z_T(u)\,du = \int_0^t \lambda\,du = \lambda t$$

$$R_T(t) = e^{-H_T(t)} = e^{-\lambda t}$$

$$f_T(t) = -\frac{d}{dt} R_T(t) = \lambda e^{-\lambda t} \qquad T \sim EXPO(\lambda)$$

$\rightarrow$ reminiscent of memoryless property

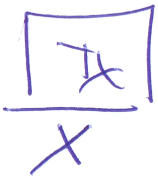
$$P(T > a+b \mid T > a) = P(T > b)$$

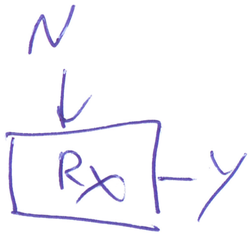
- $E[T]$ mean time to failure (MTTF)

$$E(T) = \int_0^\infty t\, f_T(t)\,dt = \int_0^\infty R_T(t)\,dt$$

(MTTF $< \infty$)

Jointly Distributed RVs





$Y = X + N$

RV X CDF $F_X(x) = P(X \le x)$

PDF $f_X(x)$

PMF

RVs X, Y joint CDF

$$F_{X,Y}(a,b) = P(X \le a \;,\; Y \le b)$$

AND

Discrete RVs

X, Y

ex X throw a die $\{1,2,3,4,5,6\}$

Y toss a coin $\{H, T\}$

$$S = \{ (1,H), (2,H), \ldots, (6,H), \\ (1,T), (2,T), \ldots, (6,T) \}$$

Joint PMF

$$f_{X,Y}(x,y) = P(X = x, Y = y)$$

Valid PMF
- $f_{X,Y}(x,y) \ge 0 \quad \forall x, y$
- $\sum_x \sum_y f_{X,Y}(x,y) = 1$

$$P((X,Y) \in A) = \sum_{x} \sum_{\substack{y \\ (x,y) \in A}} f_{X,Y}(x,y)$$

ex

	X			
Y	1	2	3	$f_Y(y)$
5	0.2	0.1	0	0.3
6	0.2	0.1	0.1	0.4
7	0.1	0.1	0.1	0.3
$f_X(x)$	0.5	0.3	0.2	

$P(X=1, Y=5) = 0.2$

$A = \{(1,5), (2,6), (3,7)\}$

$P((X,Y) \in A) =$
$P(X=1, Y=5) +$
$P(X=2, Y=6) + P(X=3, Y=7)$

$= f_{X,Y}(1,5) + f_{X,Y}(2,6) + f_{X,Y}(3,7)$

$= 0.2 + 0.1 + 0.1 = 0.4$

$$f_{X,Y}(x,y) \to f_X(x) \quad \text{marginal PMF of } X$$
$$f_{X,Y}(x,y) \to f_Y(y) \quad \text{marginal PMF of } Y$$

Joint PMF

$$f_X(x) = \sum_y f_{X,Y}(x,y)$$

$$f_Y(y) = \sum_x f_{X,Y}(x,y)$$

ex

x	1	2	3
$f_X(x)$	0.5	0.3	0.2

y	5	6	7
$f_Y(y)$	0.3	0.4	0.3

1) $f_X(x) \geq 0$
$f_Y(y) \geq 0$

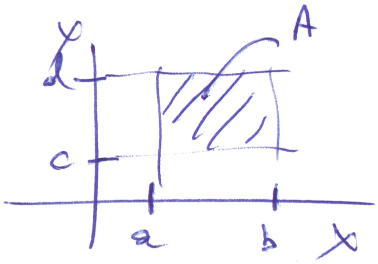
2) $\sum_x f_X(x) = 1$
$\sum_y f_Y(y) = 1$

Continuous RVs

$P(X \in A) = \int_A f_X(x)\, dx$

Joint PDF

$$P((X,Y) \in A) = \iint_A \underbrace{f_{X,Y}(x,y)}_{\text{joint PDF}}\, dx\, dy$$



$$P((X,Y) \in A) = \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

Valid PDF

- $f_{X,Y}(x,y) \geq 0 \quad \forall x,y$
- $\iint_{-\infty\,-\infty}^{+\infty\,+\infty} f_{X,Y}(x,y)\, dx\, dy = 1$

Joint CDF $\quad F_{X,Y}(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x,y)\, dx\, dy$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$

$f_X(x) = \frac{d}{dx} F_X(x)$

ex

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \leq x < 1,\ 0 \leq y \leq 1 \\ 0 & \text{oth} \end{cases}$$

Valid PDF?

- $f_{X,Y}(x,y) \geq 0 \quad \forall x,y$
- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx\, dy = \int_0^1 \int_0^1 \frac{6}{5}(x+y^2)\, dx\, dy = 1$

$$f_{X,Y}(x,y) \;<\; \begin{array}{ll} \to f_X(x) & \text{marginal PDF of } X \\ \to f_Y(y) & \text{marginal PDF of } Y \end{array}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$$

$$f_{X,Y}(x,y) \longrightarrow \begin{array}{l} f_X(x) \\ f_Y(y) \end{array}$$

$$\not\longleftarrow$$

<u>ex</u> <u>rectangular domain</u>

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{oth} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}\,dy = \int_0^1 \frac{6}{5}(x+y^2)\,dy$$

$$= \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_0^1 = \begin{cases} \frac{6}{5}\left[x + \frac{1}{3}\right] & 0 \le x \le 1 \\ 0 & \text{oth} \end{cases}$$

$f_{X,Y}(x,y)$
$\neq f_X(x)\, f_Y(y)$
X, Y
dependent

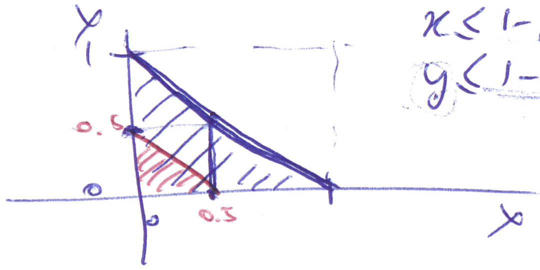
$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}\,dx = \int_0^1 \frac{6}{5}(x+y^2)\,dx$$

$$= \frac{6}{5}\left[\frac{x^2}{2} + y^2 x\right]_0^1 = \begin{cases} \frac{6}{5}\left(\frac{1}{2}+y^2\right) & 0 \le y \le 1 \\ 0 & \text{oth} \end{cases}$$

$$P(X \le 0.5) = \int_0^{0.5} f_X(x)\,dx$$

ex non rectangular domain

$$f_{X,Y}(x,y) = \begin{cases} 24xy & 0 \le x \le 1, \ 0 \le y \le 1, \ x+y \le 1 \\ 0 & \text{oth} \end{cases}$$



$x \le 1-y$

$y \le 1-x$

Valid Joint PDF?

- $f_{X,Y}(x,y) \ge 0 \quad \forall x, y$
- $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y} \, dx \, dy = \int_0^1 \int_0^{1-x} 24xy \, dy \, dx$

$$= \int_0^1 \int_0^{1-y} 24xy \, dx \, dy$$

$$= 1$$

. $A = \{ (x,y) : 0 \le x \le 1, \ 0 \le y \le 1, \ x+y \le 0.5 \}$

$$P((X,Y) \in A) = \int_0^{0.5} \int_0^{0.5-x} 24xy \, dy \, dx$$

$$= \int_0^{0.5} \int_0^{0.5-y} 24xy \, dx \, dy$$

. marginal of X

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y} \, dy$$

$$= \int_0^{1-x} f_{X,Y}(x,y) \, dy$$

$x+y \le 1$

$y \le 1-x$

independent RVs

A, B events

$$P(A|B) = P(A) \qquad A, B \text{ independent}$$

$$P(A \cap B) = P(A)P(B)$$

X, Y RVs independent

- $P(X \leq x, Y \leq y) = P(X \leq x) \, P(Y \leq y) \quad \forall x, y$

  $F_{X,Y}(x,y) = F_X(x) \, F_Y(y)$

- $f_{X,Y}(x,y) = f_X(x) \, f_Y(y)$ continuous $\forall x, y$

  $P(X = x, Y = y) = P(X = x) \, P(Y = y)$ discrete

More than 2 RVs

$$X_1, \ldots, X_n$$

Joint CDF $\quad F_{X_1, X_2, \ldots X_n}(x_1, \ldots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots X_n \leq x_n)$

Joint PDF $\quad f_{X_1, X_2, \ldots, X_n}(x_1, \ldots, x_n) = \dfrac{\partial^n F_{X_1, X_2, \ldots X_n}}{\partial x_1 \ldots \partial x_n}$

independent RVs

$$F_{X_1, X_2, \ldots X_n}(x_1, \ldots x_n) = F_{X_1}(x_1) \ldots F_{X_n}(x_n) \quad \forall x_1, x_2, \ldots x_n$$

$$f_{X_1, X_2 \ldots X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1) \ldots f_{X_n}(x_n) \quad \forall x_1 \ldots x_n$$

Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional PMF / PDF

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \quad > 0$$

Valid PDF/PMF

- $f_{Y|X}(y|x) \geq 0$
- $\int_{-\infty}^{+\infty} f_{Y|X}(y|x)\, dy = 1$

$$\int_{-\infty}^{+\infty} \frac{f_{X,Y}(x,y)}{f_X(x)}\, dy = \frac{1}{f_X(x)} \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy}_{f_X(x)}$$

$$= 1$$

- $P(Y \leq y \mid X = x) = \int_{-\infty}^{y} f_{Y|X}(y|x)\, dy$

- X, Y independent

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \overset{ind}{=} \frac{\cancel{f_X(x)}\, f_Y(y)}{\cancel{f_X(x)}} = f_Y(y)$$

$\forall x, \forall y$

$$P(A|B) = P(A)$$

ex

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}\left(x+y^2\right) & 0\le x\le 1,\ 0\le y\le 1 \\ 0 & \text{oth} \end{cases}$$

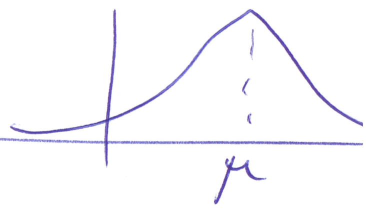
$$f_Y(y) = \begin{cases} \frac{6}{5}y^2 + \frac{3}{5} & 0\le y\le 1 \\ 0 & \text{oth} \end{cases}$$

$\underline{Y=0.3}$

$$f_{X|Y}(x\,|\,0.3) = \frac{f_{X,Y}(x,0.3)}{f_Y(0.3)} = \begin{cases} \dfrac{\frac{6}{5}\left(x+0.3^2\right)}{\frac{6}{5}\,0.3^2+\frac{3}{5}} & 0\le x\le 1 \\ 0 & \text{oth} \end{cases}$$

Expectation

$X \qquad \mu = E(X) = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx$



$Y = g(X) \qquad E(Y)$ — 1) first calculate $f_Y(y)$, then calculate $E(Y) = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy$

2) $E(g(X)) = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx$

$$E\Big(\underbrace{g(X,Y)}_{=Z}\Big) = \begin{cases} \sum_x \sum_y g(x,y)\, f_{X,Y}(x,y) & \text{discrete} \\ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy & \text{continuous} \end{cases}$$

- $\varphi(X,Y) = X$

$$E(\varphi(X,Y)) = E(X) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x \; f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{+\infty} x \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy}_{f_X(x)}\, dx$$

$$= \int_{-\infty}^{+\infty} x \; f_X(x)\, dx = \underline{E(X)}$$

- $\varphi(X,Y) = (X - \mu_X)^2 \qquad \mu_X = E(X)$

$$E(\varphi(X,Y)) = E((X-\mu_X)^2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (\cancel{X}x - \mu_X)^2 \, f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{+\infty} (x-\mu_X)^2 \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy}_{f_X(x)}\, dx$$

$$= \underline{\underline{\mathrm{Var}(X)}}$$

- $\varphi(X,Y) = X+Y$

$$E(\varphi(X,Y)) = E(X+Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x+y)\; f_{X,Y}(x,y)\, dy\, dx$$

$$= \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x \; f_{X,Y}(x,y)\, dy\, dx}_{E(X)} + \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y \; f_{X,Y}(x,y)\, dy\, dx}_{E(Y)}$$

$$E(X+Y) = E(X) + E(Y)$$

- $\varphi(X,Y) = XY$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\, y \; f_{X,Y}(x,y)\, dx\, dy$$

$$\neq E(X)\, E(Y)$$

ex

	X: 1	2	3	$f_Y(y)$
Y: 5	0.2	0.1	0	0.3
6	0.2	0.1	0.1	0.4
7	0.1	0.1	0.1	0.3
$f_X(x)$	0.5	0.3	0.2	

$$g(x,y) = x+y$$

$$E(X+Y) = \underbrace{E(X)}_{\sum_x x f_X(x)} + \underbrace{E(Y)}_{\sum_y y f_Y(y)}$$

$$E(X+Y) = \sum_x \sum_y (x+y)\, f_{X,Y}(x,y)$$

Conditional expectation

$E[X|Y]$ $\begin{cases} \text{is a random variable} \\ \text{is a function of } Y \quad g(Y) \end{cases}$

$Y=y$

$$g(y) = E[X \mid Y=y] = \begin{cases} \sum_i x_i\, P(X=x_i \mid Y=y) & \text{discrete} \\ \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\, dx & \text{continuous} \end{cases}$$

$E[X|Y=y]$ is a function of y

y	0	1	2
$E[X\mid Y=y]$	$E[X\mid Y=0]$	$E[X\mid Y=1]$	$E[X\mid Y=2]$

$g(y)$

$E(X)$ is a number

• X, Y independent

$$\forall y \quad E(X \mid Y=y) = \int_{-\infty}^{+\infty} x \, f_{X|Y}(x|y)\, dx$$

$$\underset{\text{ind}}{=} \int_{-\infty}^{+\infty} x \, f_X(x)\, dx = \underline{\underline{E(X)}}$$

$f_{X|Y} = f_X$

• $E(X \mid Y=y) = \rho(y)$ $\qquad E(g(Y)) = \int_{-\infty}^{+\infty} g(y)\, f_Y(y)\, dy$

$$E_Y\big(\underbrace{E(X \mid Y)}_{\rho(Y)}\big) \underset{(a)}{=} \int_{-\infty}^{+\infty} \rho(y)\, f_Y(y)\, dy$$

$\rho(Y) = E(X \mid Y)$

$$\rho(y) = E(X \mid Y=y) = \boxed{\int_{-\infty}^{+\infty} x \, f_{X|Y}(x|y)\, dx}$$

$$(a) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x \, \underbrace{f_{X|Y}(x|y)\, f_Y(y)}_{f_{X,Y}(x,y)}\, dx\, dy$$

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$$

$$= \int_{-\infty}^{+\infty} x \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy}_{f_X(x)}\, dx$$

$$= \int_{-\infty}^{+\infty} x \, f_X(x)\, dx = E(X)$$

$$E\big(E(X \mid Y)\big) = E(X)$$