

# Lectures

---

## **Introduction**

[1](#) Introduction - 1

## **Entropy Properties**

[2](#) Entropy I - 19

[3](#) Entropy II – 32

## **Lossless Source Coding**

[4](#) Theorem - 47

[5](#) Algorithms - 60

## **Channel Capacity**

[6](#) Data Processing Theorem - 76

[7](#) Typical Sets - 86

[8](#) Channel Capacity - 98

[9](#) Joint Typicality - 112

[10](#) Coding Theorem - 123

[11](#) Separation Theorem – 131

[12](#) Polar codes - 143

## **Continuous Variables**

[13](#) Differential Entropy - 170

[14](#) Gaussian Channel - 185

[15](#) Parallel Channels – 198

## **Lossy Source Coding**

[16](#) Rate Distortion Theory - 211

## **Network Information Theory**

[17](#) NIT I - 241

[18](#) NIT II - 256

## **Revision**

[19](#) Revision - 274

[20](#)

# Claude Shannon

---

- C. E. Shannon, "A mathematical theory of communication," Bell System Technical Journal, 1948.
- Two fundamental questions in communication theory:
- Ultimate limit on data compression
  - entropy
- Ultimate transmission rate of communication
  - channel capacity
- Almost all important topics in information theory were initiated by Shannon



1916 - 2001

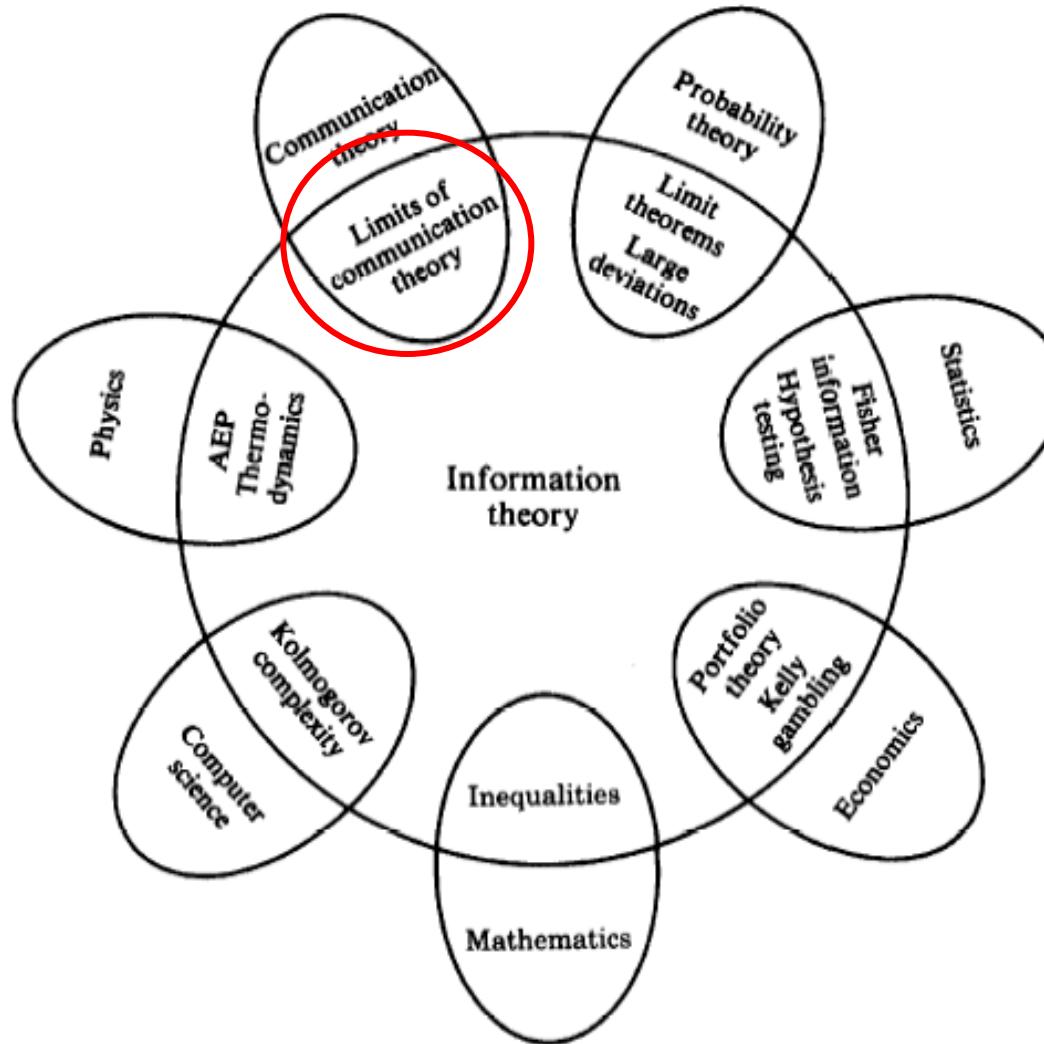
# Origin of Information Theory

---

- Common wisdom in 1940s:
  - It is impossible to send information error-free at a positive rate
  - Error control by using retransmission: rate  $\rightarrow 0$  if error-free
- Still in use today
  - ARQ (automatic repeat request) in TCP/IP computer networking
- Shannon showed reliable communication is possible for all rates below channel capacity
- As long as source entropy is less than channel capacity, asymptotically error-free communication can be achieved
- And anything can be represented in bits
  - Rise of digital information technology

# Relationship to Other Fields

---



# Course Objectives

---

- In this course we will (focus on communication theory):
  - Define what we mean by information.
  - Show how we can compress the information in a source to its theoretically minimum value and show the tradeoff between data compression and distortion.
  - Prove the channel coding theorem and derive the information capacity of different channels.
  - Generalize from point-to-point to network information theory.

# Relevance to Practice

---

- Information theory suggests means of achieving ultimate limits of communication
  - Unfortunately, these theoretically optimum schemes are computationally impractical
  - So some say “little info, much theory” (wrong)
- Today, information theory offers useful guidelines to design of communication systems
  - Polar code (achieves channel capacity)
  - CDMA (has a higher capacity than FDMA/TDMA)
  - Channel-coding approach to source coding (duality)
  - Network coding (goes beyond routing)

# Books/Reading

---

Book of the course:

- *Elements of Information Theory* by T M Cover & J A Thomas, Wiley, £39 for 2<sup>nd</sup> ed. 2006, or £14 for 1<sup>st</sup> ed. 1991 (Amazon)

Free references

- *Information Theory and Network Coding* by R. W. Yeung, Springer  
<http://iest2.ie.cuhk.edu.hk/~whyeung/book2/>
- *Information Theory, Inference, and Learning Algorithms* by D MacKay, Cambridge University Press  
<http://www.inference.phy.cam.ac.uk/mackay/itila/>
- *Lecture Notes on Network Information Theory* by A. E. Gamal and Y.-H. Kim, (Book is published by Cambridge University Press)  
<http://arxiv.org/abs/1001.3404>
- C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, Vol. 27, pp. 379–423, 623–656, July, October, 1948.

# Other Information

---

- Course webpage:  
<http://www.commsp.ee.ic.ac.uk/~ cling>
- Assessment: Exam only – no coursework.
- Students are encouraged to do the problems in problem sheets.
- Background knowledge
  - Mathematics
  - Elementary probability
- Needs intellectual maturity
  - Doing problems is not enough; spend some time thinking

# Notation

---

- Vectors and matrices
  - $\mathbf{v}$ =vector,  $\mathbf{V}$ =matrix
- Scalar random variables
  - $x = R.V$ ,  $x$  = specific value,  $X$  = alphabet
- Random column vector of length  $N$ 
  - $\mathbf{x} = R.V$ ,  $\mathbf{x}$  = specific value,  $X^N$  = alphabet
  - $x_i$  and  $x_i$  are particular vector elements
- Ranges
  - $a:b$  denotes the range  $a, a+1, \dots, b$
- Cardinality
  - $|X|$  = the number of elements in set  $X$

# Discrete Random Variables

---

- A random variable  $x$  takes a value  $x$  from the alphabet  $X$  with probability  $p_x(x)$ . The vector of probabilities is  $\mathbf{p}_x$ .

**Examples:**



$$X = [1; 2; 3; 4; 5; 6], \mathbf{p}_x = [\frac{1}{6}; \frac{1}{6}; \frac{1}{6}; \frac{1}{6}; \frac{1}{6}; \frac{1}{6}]$$

$\mathbf{p}_x$  is a “probability mass vector”

“english text”

$$X = [a; b; \dots, y; z; \text{<space>}]$$

$$\mathbf{p}_x = [0.058; 0.013; \dots; 0.016; 0.0007; 0.193]$$

Note: we normally drop the subscript from  $p_x$  if unambiguous

# Expected Values

---

- If  $g(x)$  is a function defined on  $X$  then

$$E_x g(X) = \sum_{x \in X} p(x)g(x) \quad \text{often write } E \text{ for } E_X$$

Examples:



$$X = [1; 2; 3; 4; 5; 6], \mathbf{p}_X = [1/6; 1/6; 1/6; 1/6; 1/6; 1/6]$$

$$E X = 3.5 = \mu \quad \sigma^2 = E(X^2) - (E(X))^2 = 15.17 - 12.25 = 2.92$$

$$E X^2 = 15.17 = \sigma^2 + \mu^2$$

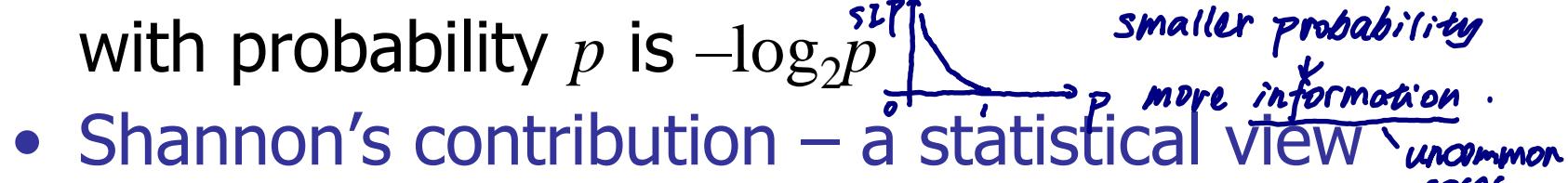
$$E \sin(0.1X) = 0.338$$

$$E - \log_2(p(X)) = 2.58 \quad \text{This is the "entropy" of } X$$

# Shannon Information Content

SIP: the amount of info. associated with an event with probability  $P$ .

- The Shannon Information Content of an outcome with probability  $p$  is  $-\log_2 p$
- Shannon's contribution – a statistical view
  - Messages, noisy channels are random
  - Pre-Shannon era: deterministic approach (Fourier...)
- Example 1:** Coin tossing
  - $X = [\text{Head}; \text{Tail}], p = [1/2; 1/2], \text{SIC} = [1; 1] \text{ bits}$
- Example 2:** Is it my birthday ?
  - $X = [\text{No}; \text{Yes}], p = [364/365; 1/365], \text{SIC} = [0.004; 8.512] \text{ bits}$



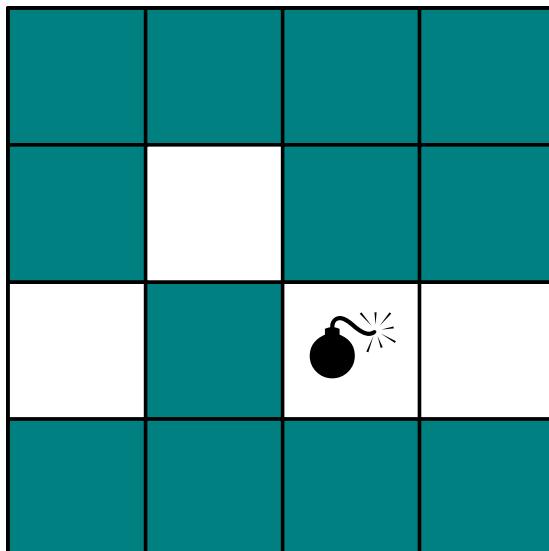
Unlikely outcomes give more information

# Minesweeper

---

- Where is the bomb ?
- 16 possibilities – needs 4 bits to specify

$$4 \geq \log_2 16$$



Guess	Prob	SIC
1. No	$15/16$	0.093 bits
2. No	$14/15$	0.100 bits
3. No	$13/14$	0.107 bits
4. Yes	$1/13$	3.700 bits
	Total	4.000 bits

$SIC = -\log_2 p$

$$H'(X) = \log_2 |X| \geq H(X)$$

$\therefore$  apply at uniform distribution.

# Entropy

entropy

a measurement of uncertainty	16
bits numbers required to	
describe a r.v..	
<u>expectation of information</u>	

$$H(X) = E - \log_2(p_x(x)) = - \sum_{x \in X} p_x(x) \log_2 p_x(x)$$

"transmission cost  
or verification"

- $H(x)$  = the average Shannon Information Content of  $x$
- $H(x)$  = the average information gained by knowing its value
- the average number of "yes-no" questions needed to find  $x$  is in the range  $[H(x), H(x)+1]$
- $H(x)$  = the amount of uncertainty before we know its value

We use  $\log(x) \equiv \log_2(x)$  and measure  $H(x)$  in bits

- if you use  $\log_e$  it is measured in nats
- 1 nat =  $\log_2(e)$  bits = 1.44 bits

- $\log_2(x) = \frac{\ln(x)}{\ln(2)}$

$$\frac{d \log_2 x}{dx} = \frac{\log_2 e}{x}$$

$H(X)$  depends only on the probability vector  $p_x$  not on the alphabet  $X$ , so we can write  $H(p_x)$

$$H(x) = -(1-p)\log(1-p) - p\log p$$

$$H'(x) = \log(1-p) + 1 - \log p - 1 = \log(1-p) - \log p$$

$$H'(x) = 0 \Rightarrow p = \frac{1}{2}$$

$$H(x) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1$$

# Entropy Examples

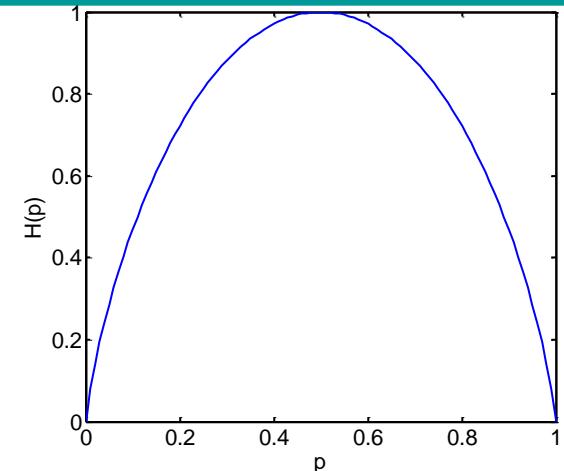
## (1) Bernoulli Random Variable

$$X = [0;1], \mathbf{p}_x = [1-p; p]$$

$$H(x) = -(1-p)\log(1-p) - p\log p$$

Very common – we write  $H(p)$  to mean  $H([1-p; p]).$

Maximum is when  $p=1/2$



$$H(p) = -(1-p)\log(1-p) - p\log p$$

$$H'(p) = \log(1-p) - \log p$$

$$H''(p) = -p^{-1}(1-p)^{-1}\log e$$

## (2) Four Coloured Shapes

$$X = [\text{Red circle}; \text{Green square}; \text{Blue diamond}; \text{Black asterisk}], \mathbf{p}_x = [\frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \frac{1}{8}]$$

$$\begin{aligned} H(x) &= H(\mathbf{p}_x) = \sum -\log(p(x))p(x) \\ &= 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 3 \times \frac{1}{8} = 1.75 \text{ bits} \end{aligned}$$

entropy {  
    monotonic non-decreasing  
    additive  
    non-negative  
     $\Rightarrow H(x) := -c \sum_{x \in X} P(x) \log P(x) \quad (c > 0 \rightarrow c = 1)$

# Comments on Entropy

---

- Entropy plays a central role in information theory
- Origin in thermodynamics
  - $S = k \ln \Omega$ ,  $k$ : Boltzmann's constant,  $\Omega$ : number of microstates
  - The second law: entropy of an isolated system is non-decreasing
- Shannon entropy
  - Agrees with intuition: additive, monotonic, continuous
  - Logarithmic measure could be derived from an axiomatic approach (Shannon 1948)

# Lecture 2

---

- Joint and Conditional Entropy
  - Chain rule
- Mutual Information
  - If  $x$  and  $y$  are correlated, their mutual information is the average information that  $y$  gives about  $x$ 
    - E.g. Communication Channel:  $x$  transmitted but  $y$  received
    - It is the amount of information transmitted through the channel
- Jensen's Inequality

# Joint and Conditional Entropy

**Joint Entropy:**  $H(X, Y)$

$$H(X, Y) = E - \log P(X, Y)$$

$$= -\sum_{x,y} P(x, y) \log P(x, y)$$

$$= \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = 1.5 \text{ bits}$$

$$\underline{H(X, Y) = E - \log p(X, Y)}$$

$$= -\frac{1}{2} \log \frac{1}{2} - \frac{1}{4} \log \frac{1}{4} - 0 \log 0 - \frac{1}{4} \log \frac{1}{4} = 1.5 \text{ bits}$$

$p(X, Y)$	$y=0$	$y=1$
$x=0$	$\frac{1}{2}$	$\frac{1}{4}$
$x=1$	0	$\frac{1}{4}$

Note:  $0 \log 0 = 0$

$$0 \log 0 = \lim_{x \rightarrow 0} x \log x$$

$$= \lim_{x \rightarrow 0} \frac{\log x}{\frac{1}{x}} = \lim_{x \rightarrow 0} \frac{-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0} 1 = 0$$

**Conditional Entropy:**  $H(Y|X)$

$$H(Y|X) = E - \log P(Y|X)$$

$$\underline{H(Y|X) = E - \log p(Y|X)}$$

$$= -\sum_{x,y} P(x, y) \log P(Y|X)$$

$$= -\sum_{x,y} p(x, y) \log p(y|x)$$

$$= -\frac{1}{2} \times \log \frac{2}{3} - \frac{1}{4} \times \log \frac{1}{3} - \frac{1}{4} \times \log 1$$

$$= 0.689 \text{ bits}$$

$$= -\frac{1}{2} \log \frac{2}{3} - \frac{1}{4} \log \frac{1}{3} - 0 \log 0 - \frac{1}{4} \log 1 = 0.689 \text{ bits}$$

$p(Y X)$	$y=0$	$y=1$
$x=0$	$\frac{2}{3}$	$\frac{1}{3}$
$x=1$	0	1

Note: rows sum to 1

# Conditional Entropy – View 1

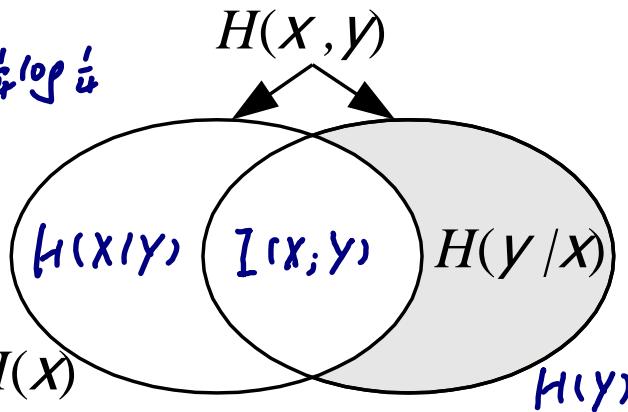
$$\begin{aligned}
 P(Y|X) &= \frac{P(X,Y)}{P(X)} & H(Y|X) &= E - \log P(Y|X) & = \sum_{x,y} P(x,y) \log P(x) \\
 \text{Additional Entropy:} & & &= E - \log \frac{P(X,Y)}{P(X)} & = \sum_x P(x) \log P(x) \\
 p(Y|X) &= p(X,Y) \div p(X) & &= E - \log P(x,y) - E - \log P(x) & & p(X) \\
 H(Y|X) &= E - \log p(Y|X) & &= \sum_{x,y} P(x,y) \log P(x,y) + \sum_{x,y} P(x,y) \log P(x) & & \\
 &= \sum_{x,y} P(x,y) \log \frac{P(x,y)}{P(x)} & & & & \\
 &= E \{-\log p(X,Y)\} - E \{-\log p(X)\} & & & & \\
 &= H(X,Y) - H(X) & & & & \\
 &= H(X,Y) - H(X) = H(\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}) - H(\frac{3}{4}, \frac{1}{4}) = 0.689 \text{ bits}
 \end{aligned}$$

$H(Y|X)$  is the average <sup>remaining uncertainty</sup> additional information in  $Y$  when you know  $X$

$$\begin{aligned}
 H(X,Y) &= - \sum_{x,y} P(x,y) \log P(x,y) \\
 &= - \frac{1}{2} \log \frac{1}{2} - \frac{1}{4} \log \frac{1}{4} - \frac{1}{4} \log \frac{1}{4} \\
 &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1.5 \text{ bits}
 \end{aligned}$$

$$\begin{aligned}
 H(X) &= - \sum_x P(x) \log P(x) \\
 &= - \frac{3}{4} \log \frac{3}{4} - \frac{1}{4} \log \frac{1}{4} \\
 &= 0.311 + 0.5 = 0.811 \text{ bits}
 \end{aligned}$$

$$H(Y|X) = H(X,Y) - H(X) = 0.689 \text{ bits}$$



# Conditional Entropy – View 2

## Average Row Entropy:

$$H(y|x) = - \sum_{x,y} p(x,y) \log p(y|x)$$

$$= - \sum_{x,y} p(y|x)p(x) \log p(y|x)$$

$$= - \sum_x p(x) \sum_y p(y|x) \log p(y|x) = \sum_x p(x) H(y|x=x)$$

$$H(y|x) = E - \log p(y|x) = \sum_{x,y} -p(x,y) \log p(y|x) \stackrel{\text{conditional entropy } H(y|x)}{\text{remaining uncertainty of } y \text{ known } x} \stackrel{\text{weighted average row entropy}}{=} \sum_{x,y} -p(x,y) \log p(y|x) = \sum_{x \in X} p(x) \sum_{y \in Y} -p(y|x) \log p(y|x)$$

$$= \sum_{x \in X} p(x) H(y|x=x) = \frac{3}{4} \times H\left(\frac{1}{3}\right) + \frac{1}{4} \times H(0) = 0.689 \text{ bits}$$

$$H(y|x=0) = - \sum_y p(y|x=0) \log p(y|x=0) = -\frac{2}{3} \log \frac{2}{3} - \frac{1}{3} \log \frac{1}{3} = 0.918 \text{ bits}$$

$$H(y|x=1) = - \sum_y p(y|x=1) \log p(y|x=1) = -1 \log 1 = 0 \text{ bits}$$

Take a weighted average of the entropy of each row using  $p(x)$  as weight

$$\therefore H(y|x) = \sum_x p(x) H(y|x=x) = \frac{3}{4} \times 0.918 + \frac{1}{4} \times 0 = 0.689 \text{ bits}$$

$p(x, y)$	$y=0$	$y=1$	$H(y x=x)$	$p(x)$
$x=0$	$\frac{1}{2}$	$\frac{1}{4}$	$H(1/3)$	$\frac{3}{4}$
$x=1$	0	$\frac{1}{4}$	$H(1)$	$\frac{1}{4}$

Conditional entropy  $H(y|x)$   
 remaining uncertainty of  $y$  known  $x$   
 $H(y|x) = H(x,y) - H(x)$   
 weighted average row entropy  
 $H(y|x) = \sum_x p(x) H(y|x=x)$

$$\begin{aligned}
 H(x, y, z) &= -\sum_{x,y,z} P(x, y, z) \log P(x, y, z) = -\sum_{x,y,z} P(x, y, z) (\log P(z|x, y) P(y|x) P(x)) \\
 &= -\sum_{x,y,z} P(x, y, z) (\log P(z|x, y) - \sum_{x,y,z} P(x, y, z) (\log P(y|x) - \sum_{x,y,z} P(x, y, z) (\log P(x))) \\
 &= -\sum_z P(z) (\log P(z|x, y) - \sum_y P(y) \log P(y|x) - \sum_x P(x) \log P(x)) \\
 &= H(z|x, y) + H(y|x) + H(x)
 \end{aligned}$$

## Chain Rules

- Probabilities

$$p(x, y, z) = p(z | x, y) p(y | x) p(x)$$

- Entropy

$$H(x, y, z) = H(z | x, y) + H(y | x) + H(x)$$

$$H(x_{1:n}) = \sum_{i=1}^n H(x_i | x_{1:i-1})$$

The log in the definition of entropy converts products of probability into sums of entropy

Mutual information: the reduction of uncertainty in  $x$  given  $y$ .



Conditional entropy: the remaining uncertainty of  $x$  given  $y$ .

# Mutual Information

Mutual information is the average amount of information that you get about  $x$  from observing the value of  $y$

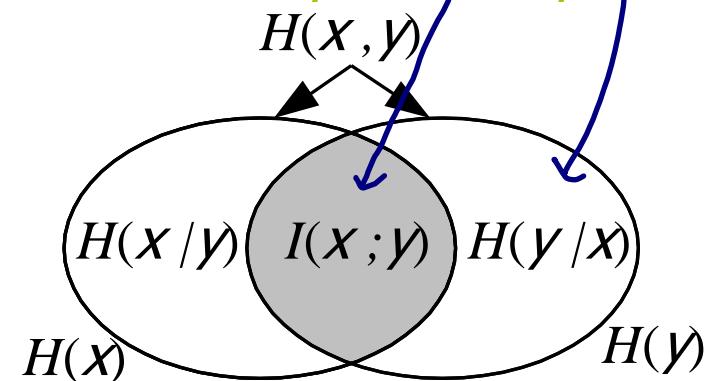
- Or the reduction in the uncertainty of  $x$  due to knowledge of  $y$

$$I(x; y) = H(x) - H(x | y) = H(x) + H(y) - H(x, y)$$

Information in  $x$                       Information in  $x$  when you already know  $y$

Mutual information is symmetrical

$$I(x; y) = I(y; x)$$



Use ";" to avoid ambiguities between  $I(x;y,z)$  and  $I(x,y;z)$

$$H(x) = - \sum_x P(x) \log P(x) = - \frac{3}{4} \log \frac{3}{4} - \frac{1}{4} \log \frac{1}{4} = 0.811 \text{ bits}$$

$$H(y) = - \sum_y P(y) \log P(y) = - \frac{1}{2} \log \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} = 1 \text{ bit}$$

$$H(x,y) = - \sum_{x,y} P(x,y) \log P(x,y)$$

$$= - \frac{1}{2} \log \frac{1}{2} - \frac{1}{4} \log \frac{1}{4} - \frac{1}{4} \log \frac{1}{4} = 1.5 \text{ bits}$$

# Mutual Information Example

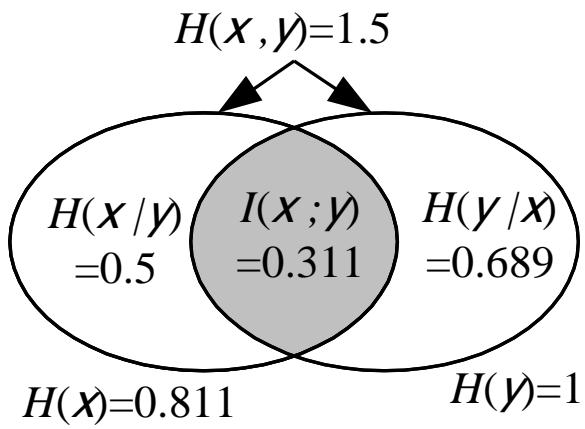
		$H(X Y)$	$H(Y X)$
		$H(X)$	$H(Y)$
		$y=0$	$y=1$
$x=0$		$\frac{1}{2}$	$\frac{1}{4}$
$x=1$		0	$\frac{1}{4}$

$$H(X|Y) = H(X.Y) - H(Y) = 0.5 \text{ bits}$$

$$H(Y|X) = H(X.Y) - H(X) = 0.689 \text{ bits}$$

$$I(X;Y) = H(X) + H(Y) - H(X,Y) = 0.311 \text{ bits}$$

- If you try to guess  $y$  you have a 50% chance of being correct.
- However, what if you know  $x$ ?
  - Best guess: choose  $y = x$
  - If  $x=0$  ( $p=0.75$ ) then 66% correct prob
  - If  $x=1$  ( $p=0.25$ ) then 100% correct prob
  - Overall 75% correct probability



$$I(X;Y) = H(X) - H(X|Y)$$

$$= H(X) + H(Y) - H(X,Y)$$

$$H(X) = 0.811, \quad H(Y) = 1, \quad H(X,Y) = 1.5$$

$$I(X;Y) = 0.311$$

$$I(X;Y) = H(X) - H(X|Y)$$

$$I(X_1, X_2, \dots, X_n; Y) = H(X_1, X_2, \dots, X_n) - H(X_1, X_2, \dots, X_n|Y)$$

# Conditional Mutual Information

$$= \sum_{i=1}^n H(X_i | X_{1:i-1}) - \sum_{i=1}^n H(X_i | X_{1:i-1}, Y)$$

$$= \sum_{i=1}^n I(X_i; Y | X_{1:i-1})$$

$$\begin{aligned} I(X_1, X_2; Y) &= I(X_1; Y) + I(X_2; Y | X_1) \\ &= H(X_1) - H(X_1|Y) + \\ &\quad H(X_2|X_1) - H(X_2|X_1, Y) \\ &= H(X_1, X_2) - H(X_1, X_2|Y) \end{aligned}$$

## Conditional Mutual Information

$$\begin{aligned} I(X;Y | Z) &= H(X | Z) - H(X | Y, Z) \\ &= H(X | Z) + H(Y | Z) - H(X, Y | Z) \end{aligned}$$

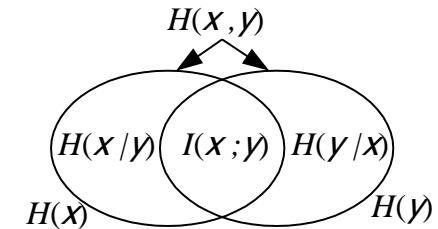
**Note:**  $Z$  conditioning applies to both  $X$  and  $Y$

## Chain Rule for Mutual Information

►  $I(X_1, X_2, X_3; Y) = I(X_1; Y) + I(X_2; Y | X_1) + I(X_3; Y | X_1, X_2)$

$$I(X_{1:n}; Y) = \sum_{i=1}^n I(X_i; Y | X_{1:i-1})$$

# Review/Preview



- **Entropy:**  $H(x) = \sum_{x \in X} -\log_2(p(x)) p(x) = E - \log_2(p_x(x))$ 
  - Positive and bounded  $0 \leq H(x) \leq \log |X|$   
*fixed pattern*    *uniform distribution*
- **Chain Rule:**  $H(x, y) = H(x) + H(y | x) \leq H(x) + H(y)$ 
  - Conditioning reduces entropy  $H(y | x) \leq H(y)$
- **Mutual Information:**

$$I(y; x) = H(y) - H(y | x) = H(x) + H(y) - H(x, y)$$

- Positive and Symmetrical  $I(x; y) = I(y; x) \geq 0$
- $x$  and  $y$  independent  $\Leftrightarrow H(x, y) = H(y) + H(x)$ 
  - $\Leftrightarrow I(x; y) = 0$

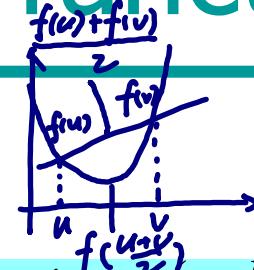
◆ = inequalities not yet proved

# Convex & Concave functions

$f(x)$  is strictly convex over  $(a, b)$  if

$$f(\lambda u + (1 - \lambda)v) < \lambda f(u) + (1 - \lambda)f(v) \quad \forall u \neq v \in (a, b), 0 < \lambda < 1$$

definition



- every chord of  $f(x)$  lies above  $f(x)$
- $f(x)$  is **concave**  $\Leftrightarrow -f(x)$  is **convex**

- Examples

- Strictly Convex:  $x^2, x^4, e^x, x \log x [x \geq 0]$
- Strictly Concave:  $\log x, \sqrt{x} [x \geq 0]$
- Convex and Concave:  $x$

Concave is like this



- **Test:**  $\frac{d^2 f}{dx^2} > 0 \quad \forall x \in (a, b) \Rightarrow f(x) \text{ is strictly convex}$

“convex” (not strictly) uses “ $\leq$ ” in definition and “ $\geq$ ” in test

# Jensen's Inequality

**Jensen's Inequality:** (a)  $f(x)$  convex  $\Rightarrow Ef(x) \geq f(Ex)$

(b)  $f(x)$  strictly convex  $\Rightarrow Ef(x) > f(Ex)$  unless  $x$  constant

Proof by induction on  $|X|$

–  $|X|=1$ :  $E f(x) = f(E x) = f(x_1)$

–  $|X|=k$ :  $E f(x) = \sum_{i=1}^k p_i f(x_i) = p_k f(x_k) + (1 - p_k) \sum_{i=1}^{k-1} \frac{p_i}{1 - p_k} f(x_i)$

convex:  $E f(x) \geq f(Ex)$

$$\geq p_k f(x_k) + (1 - p_k) f\left( \sum_{i=1}^{k-1} \frac{p_i}{1 - p_k} x_i \right)$$

definition:  $f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda)f(v)$

$$\geq f\left( p_k x_k + (1 - p_k) \sum_{i=1}^{k-1} \frac{p_i}{1 - p_k} x_i \right) = f(E x)$$

These sum to 1

Assume JI is true  
for  $|X|=k-1$

Follows from the definition of convexity for two-mass-point distribution

# Jensen's Inequality Example

Mnemonic example:

$f(x) = x^2$  : strictly convex

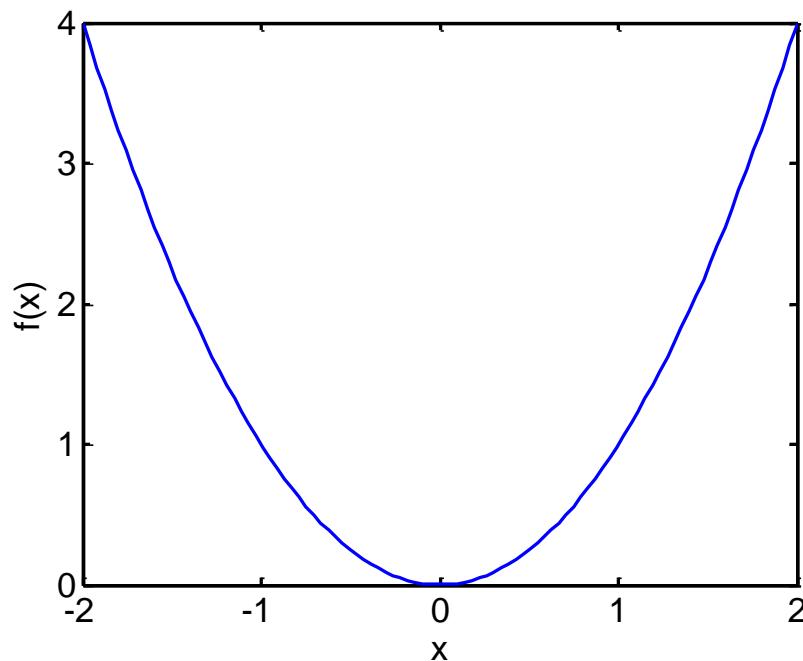
$X = [-1; +1]$

$p = [1/2; 1/2]$

$E X = 0$

$f(E X) = 0$

$E f(X) = 1 > f(E X)$



# Summary

- Chain Rule:

$$H(x, y) = H(y | x) + H(x)$$

- Conditional Entropy:

$$H(y | x) = H(x, y) - H(x) = \sum_{x \in X} p(x)H(y | x)$$

- Conditioning reduces entropy

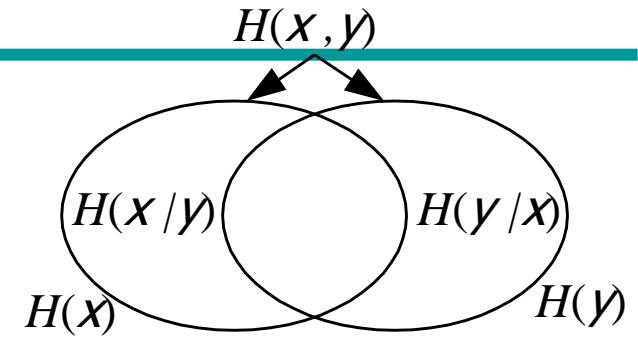
$$H(y | x) \leq H(y)$$

- Mutual Information  $I(x; y) = H(x) - H(x | y) \leq H(x)$

- In communications, mutual information is the amount of information transmitted through a noisy channel

- Jensen's Inequality  $f(x)$  convex  $\Rightarrow E f(x) \geq f(E x)$

◆ = inequalities not yet proved



# Lecture 3

---

- Relative Entropy
  - A measure of how different two probability mass vectors are
- Information Inequality and its consequences
  - Relative Entropy is always positive
    - Mutual information is positive
    - Uniform bound
    - Conditioning and correlation reduce entropy
- Stochastic Processes
  - Entropy Rate
  - Markov Processes

entropy  $H(X) = - \sum_x P(x) \log_2 P(x) = \sum_x P(x) \log_2 \frac{1}{P(x)}$ : a measurement of uncertainty

cross entropy  $H(P, Q) = - \sum_x P(x) \log_2 Q(x) = \sum_x P(x) \log_2 \frac{1}{Q(x)}$ : the cost of selected scheme to eliminate the system uncertainty.

relative entropy  $D(P||Q) = \sum_x P(x) \log_2 \frac{P(x)}{Q(x)}$ : the cost difference of schemes corresponding to different distributions.

**Relative Entropy or Kullback-Leibler Divergence** between two probability mass vectors  $p$  and  $q$

$$D(p \parallel q) = \sum_{x \in X} p(x) \log \frac{p(x)}{q(x)} = E_p \log \frac{p(x)}{q(x)} = E_p (-\log q(x)) - H(x)$$

where  $E_p$  denotes an expectation performed using probabilities  $p$   
 $D(P||Q) \geq 0$ : the assumed distribution  $q(x)$  cannot be more accurate than the real case.  
 $D(p \parallel q)$  measures the "distance" between the probability mass functions  $p$  and  $q$ .

We must have  $p_i=0$  whenever  $q_i=0$  else  $D(p \parallel q)=\infty$

Beware:  $D(p \parallel q)$  is not a true distance because:

- (1) it is asymmetric between  $p$ ,  $q$  and
- (2) it does not satisfy the triangle inequality.

# Relative Entropy Example

$$D(P \parallel Q) = E_p \left( \log_2 \frac{P(x)}{Q(x)} \right) = E_p \left( \log_2 \frac{1}{q(x)} - H(P(x)) \right)$$



$$X = [1 \ 2 \ 3 \ 4 \ 5 \ 6]^T$$

$$H(P) = - \sum_x P(x) \log_2 P(x) = 2.585 \quad H(Q) = - \sum_x Q(x) \log_2 Q(x) = 2.161$$

$$D(P \parallel Q) = E_p \left( \log_2 \frac{1}{q(x)} - H(P(x)) \right) = 2.935 - 2.585 = 0.35$$

$$p = \left[ \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \cancel{6} & \cancel{6} & \cancel{6} & \cancel{6} & \cancel{6} & \cancel{6} \end{array} \right] \Rightarrow H(p) = 2.585$$

$$D(Q \parallel P) = E_q \left( \log_2 \frac{1}{p(x)} - H(Q(x)) \right) = 2.585 - 2.161 = 0.424$$

$$q = \left[ \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \cancel{10} & \cancel{10} & \cancel{10} & \cancel{10} & \cancel{10} & \cancel{2} \end{array} \right] \Rightarrow H(q) = 2.161$$

$$D(p \parallel q) = E_p \left( -\log q_x \right) - H(p) = 2.935 - 2.585 = 0.35$$

$$D(q \parallel p) = E_q \left( -\log p_x \right) - H(q) = 2.585 - 2.161 = 0.424$$

# Information Inequality

*The assumed distribution  $q(x)$  cannot be more accurate than the real case.*

Information (Gibbs') Inequality:  $\underline{D(p \parallel q) \geq 0}$

- Define  $A = \{x : p(x) > 0\} \subseteq X$

• Proof

*(log: concave function  $E f(x) \leq f(E x)$ )*

$$D(p \parallel q) = - \sum_{x \in A} p(x) \log \frac{p(x)}{q(x)} = \sum_{x \in A} p(x) \log \frac{q(x)}{p(x)}$$

Jensen's inequality  $\leq \log \left( \sum_{x \in A} p(x) \frac{q(x)}{p(x)} \right) = \log \left( \sum_{x \in A} q(x) \right) \leq \log \left( \sum_{x \in X} q(x) \right) = \log 1 = 0$

If  $D(p \parallel q) = 0$ : Since  $\log(\cdot)$  is strictly concave we have equality in the proof only if  $q(x)/p(x)$ , the argument of  $\log$ , equals a constant.

But  $\sum_{x \in X} p(x) = \sum_{x \in X} q(x) = 1$  so the constant must be 1 and  $p \equiv q$

# Information Inequality Corollaries

---

- Uniform distribution has highest entropy
  - Set  $\mathbf{q} = [|\mathcal{X}|^{-1}, \dots, |\mathcal{X}|^{-1}]^T$  giving  $H(\mathbf{q}) = \log |\mathcal{X}|$  bits
$$D(\mathbf{p} \parallel \mathbf{q}) = E_{\mathbf{p}} \left\{ -\log q(x) \right\} - H(\mathbf{p}) = \log |\mathcal{X}| - H(\mathbf{p}) \geq 0$$

- Mutual Information is non-negative

$$\begin{aligned}
 D(p \parallel q) &= E_p \log_2 \frac{p(x)}{q(x)} \\
 I(X;Y) &= E \log_2 \frac{p(x,y)}{p(x)p(y)} = D(p(x,y) \parallel p(x)p(y)) = \\
 &\quad \boxed{E \log \frac{p(x,y)}{p(x)p(y)}} \\
 &= D(p(x,y) \parallel p(x)p(y)) \geq 0
 \end{aligned}$$

with equality only if  $p(x,y) \equiv p(x)p(y) \Leftrightarrow x$  and  $y$  are independent.

# More Corollaries

---

- Conditioning reduces entropy

$$0 \leq I(x; y) = H(y) - H(y | x) \Rightarrow H(y | x) \leq H(y)$$

with equality only if  $x$  and  $y$  are independent.

- Independence Bound

$$H(x_{1:n}) = \sum_{i=1}^n H(x_i | x_{1:i-1}) \leq \sum_{i=1}^n H(x_i)$$

with equality only if all  $x_i$  are independent.

E.g.: If all  $x_i$  are identical  $H(x_{1:n}) = H(x_1)$

independence bound  $H(X_{1:n}) = \sum_{i=1}^n H(X_i | X_{1:i-1}) \leq \sum_{i=1}^n H(X_i)$   
 conditional independence bound  $H(X_{1:n}|Y_{1:n}) = \sum_{i=1}^n H(X_i | X_{1:i-1}, Y_{1:n}) \leq \sum_{i=1}^n H(X_i | Y_i)$   
 mutual information independence bound  $I(X_{1:n}; Y_{1:n}) = H(X_{1:n}) - H(X_{1:n}|Y_{1:n}) \geq \sum_{i=1}^n H(X_i) - \sum_{i=1}^n H(X_i | Y_i) = \sum_{i=1}^n I(X_i; Y_i)$

## Conditional Independence Bound

- Conditional Independence Bound

$$H(X_{1:n} | Y_{1:n}) = \underbrace{\sum_{i=1}^n H(X_i | X_{1:i-1}, Y_{1:n})}_{\text{Conditional Independence Bound}} \leq \sum_{i=1}^n H(X_i | Y_i)$$

- Mutual Information Independence Bound

If all  $x_i$  are independent or, by symmetry, if all  $y_i$  are independent:

$$\begin{aligned} I(X_{1:n}; Y_{1:n}) &= H(X_{1:n}) - H(X_{1:n} | Y_{1:n}) \\ &\geq \sum_{i=1}^n H(X_i) - \sum_{i=1}^n H(X_i | Y_i) = \sum_{i=1}^n I(X_i; Y_i) \end{aligned}$$

E.g.: If  $n=2$  with  $x_i$  i.i.d. Bernoulli ( $p=0.5$ ) and  $y_1=x_2$  and  $y_2=x_1$ , then  $I(X_i; Y_i)=0$  but  $I(X_{1:2}; Y_{1:2}) = 2$  bits.

# Stochastic Process

---

Stochastic Process  $\{X_i\} = X_1, X_2, \dots$

Entropy:  $H(\{X_i\}) = H(X_1) + H(X_2 | X_1) + \dots = \infty$  often

Entropy Rate:  $H(X) = \lim_{n \rightarrow \infty} \frac{1}{n} H(X_{1:n})$  if limit exists

*the increasing rate of entropy  
w.r.t. n.*

- Entropy rate estimates the additional entropy per new sample.
- Gives a lower bound on number of code bits per sample.

Examples:

- Typewriter with  $m$  equally likely letters each time:  $H(X) = \log m$
- $X_i$  i.i.d. random variables:  $H(X) = H(X_i)$

# Stationary Process

---

Stochastic Process  $\{X_i\}$  is **stationary** iff

$$p(X_{1:n} = a_{1:n}) = p(X_{k+(1:n)} = a_{1:n}) \quad \forall k, n, a_i \in X$$

If  $\{X_i\}$  is stationary then  $H(X)$  exists and

$$\left. \begin{aligned} H(X) &= \lim_{n \rightarrow \infty} \frac{1}{n} H(X_{1:n}) = \lim_{n \rightarrow \infty} H(X_n | X_{1:n-1}) \\ &\text{(a) } \qquad \qquad \qquad \text{(b)} \end{aligned} \right\}$$

**Proof:**  $0 \leq H(X_n | X_{1:n-1}) \leq H(X_n | X_{2:n-1}) = H(X_{n-1} | X_{1:n-2})$

(a) conditioning reduces entropy, (b) stationarity

Hence  $H(X_n | X_{1:n-1})$  is positive, decreasing  $\Rightarrow$  tends to a limit, say  $b$

Hence

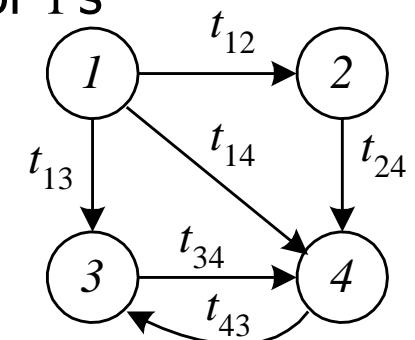
$$H(X_k | X_{1:k-1}) \rightarrow b \quad \Rightarrow \quad \frac{1}{n} H(X_{1:n}) = \frac{1}{n} \sum_{k=1}^n H(X_k | X_{1:k-1}) \rightarrow b = H(X)$$

# Markov Process (Chain)

---

Discrete-valued stochastic process  $\{x_i\}$  is

- Independent iff  $p(x_n|x_{0:n-1})=p(x_n)$
- Markov iff  $p(x_n|x_{0:n-1})=p(x_n|x_{n-1})$ 
  - time-invariant iff  $p(x_n=b|x_{n-1}=a) = p_{ab}$  indep of  $n$
  - States
  - Transition matrix:  $\mathbf{T} = \{t_{ab}\}$ 
    - Rows sum to 1:  $\mathbf{T}\mathbf{1} = \mathbf{1}$  where  $\mathbf{1}$  is a vector of 1's
    - $\mathbf{p}_n = \mathbf{T}^T \mathbf{p}_{n-1}$
    - Stationary distribution:  $\mathbf{p}_\$ = \mathbf{T}^T \mathbf{p}_\$$



Independent Stochastic Process is easiest to deal with, Markov is next easiest

# Stationary Markov Process

---

If a Markov process is

- a) **irreducible**: you can go from any state  $a$  to any  $b$  in a finite number of steps
- b) **aperiodic**:  $\forall$  state  $a$ , the possible times to go from  $a$  to  $a$  have highest common factor = 1

then it has exactly one stationary distribution,  $\mathbf{p}_\$$ .

- $\mathbf{p}_\$$  is the eigenvector of  $\mathbf{T}^T$  with  $\lambda = 1$ :  $\mathbf{T}^T \mathbf{p}_\$ = \mathbf{p}_\$$   
 $\mathbf{T}^n \xrightarrow[n \rightarrow \infty]{} \mathbf{1} \mathbf{p}_\$^T$  where  $\mathbf{1} = [1 \quad 1 \quad \dots \quad 1]^T$
- Initial distribution becomes irrelevant (**asymptotically stationary**)  $(\mathbf{T}^T)^n \mathbf{p}_0 = \mathbf{p}_\$ \mathbf{1}^T \mathbf{p}_0 = \mathbf{p}_\$, \quad \forall \mathbf{p}_0$

# Chess Board

$$H(X) = \lim_{n \rightarrow \infty} H(X_n | X_{n-1})$$

$$= \lim_{n \rightarrow \infty} - \sum P(X_n, X_{n-1}) \log_2 P(X_n | X_{n-1})$$

$$\text{Random Walk} = - \sum P(X_{n-1}) P(X_n | X_{n-1}) \log_2 P(X_n | X_{n-1})$$

- Move  $\leftrightarrow \uparrow \downarrow \leftarrow \rightarrow$  equal prob
- $$P(X_n | X_{n-1}) = \frac{P(X_n, X_{n-1})}{P(X_{n-1})}$$

- $p_1 = [1 \ 0 \dots \ 0]^T$

- $- H(p_1) = 0$

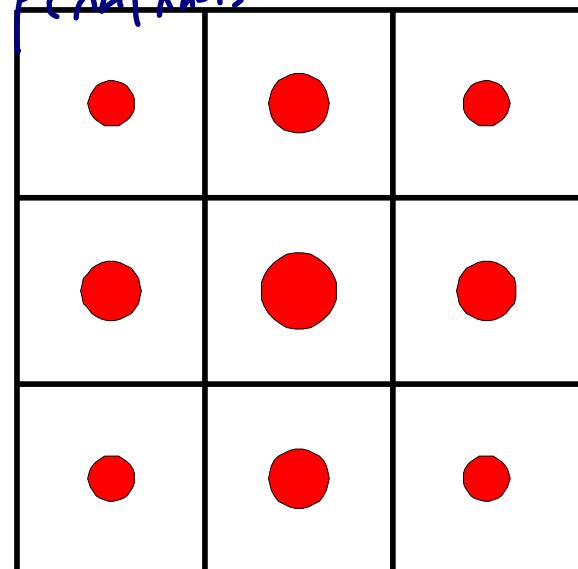
- $p_{\$} = \frac{1}{40} \times [3 \ 5 \ 3 \ 5 \ 8 \ 5 \ 3 \ 5 \ 3]^T$

- $- H(p_{\$}) = 3.0855$

- $H(X) = \lim_{n \rightarrow \infty} H(X_n | X_{n-1})$

$$= \lim_{n \rightarrow \infty} - \sum p(x_n, x_{n-1}) \log p(x_n | x_{n-1}) = \sum_{i,j} - p_{\$,i} t_{i,j} \log(t_{i,j}) = 2.2365$$

$H(p_8) = 3.0827, \quad H(p_8 | p_7) = 2.23038$



# Summary

---

- **Relative Entropy:**  $D(\mathbf{p} \parallel \mathbf{q}) = E_{\mathbf{p}} \log \frac{p(x)}{q(x)} \geq 0$ 
  - $D(\mathbf{p} \parallel \mathbf{q}) = 0$  iff  $\mathbf{p} \equiv \mathbf{q}$
- **Corollaries**
  - Uniform Bound: Uniform  $\mathbf{p}$  maximizes  $H(\mathbf{p})$
  - $I(X; Y) \geq 0 \Rightarrow$  Conditioning reduces entropy
  - Indep bounds:  $H(X_{1:n}) \leq \sum_{i=1}^n H(X_i)$        $H(X_{1:n} | Y_{1:n}) \leq \sum_{i=1}^n H(X_i | Y_i)$   
 $I(X_{1:n}; Y_{1:n}) \geq \sum_{i=1}^n I(X_i; Y_i)$     if  $X_i$  or  $Y_i$  are indep
- **Entropy Rate of stochastic process:**
  - $\{X_i\}$  stationary:  $H(X) = \lim_{n \rightarrow \infty} H(X_n | X_{1:n-1})$
  - $\{X_i\}$  stationary Markov:  

$$H(X) = H(X_n | X_{n-1}) = \sum_{i,j} -p_{\$,i} t_{i,j} \log(t_{i,j})$$


# Lecture 4

---

- Source Coding Theorem
  - $n$  i.i.d. random variables each with entropy  $H(X)$  can be compressed into more than  $nH(X)$  bits as  $n$  tends to infinity
- Instantaneous Codes
  - Symbol-by-symbol coding
  - Uniquely decodable
- Kraft Inequality
  - Constraint on the code length
- Optimal Symbol Code lengths
  - Entropy Bound

# Source Coding

---

- **Source Code:**  $C$  is a mapping  $X \rightarrow D^+$ 
  - $X$  a random variable of the message
  - $D^+ =$  set of all finite length strings from  $D$
  - $D$  is often binary
  - e.g.  $\{E, F, G\} \rightarrow \{0,1\}^+$ :  $C(E)=0, C(F)=10, C(G)=11$   
*an extension contains all codewords derived from the basis set.*
- **Extension:**  $C^+$  is mapping  $X^+ \rightarrow D^+$  formed by concatenating  $C(x_i)$  without punctuation
  - e.g.  $C^+(EFEEGE) = 01000110$

# Desired Properties

---

- **Non-singular:**  $x_1 \neq x_2 \Rightarrow C(x_1) \neq C(x_2)$ 
  - Unambiguous description of a single letter of X
- **Uniquely Decable:**  $C^+$  is non-singular
  - The sequence  $C^+(x^+)$  is unambiguous
  - A stronger condition
  - Any encoded string has only one possible source string producing it
  - However, one may have to examine the entire encoded string to determine even the first source symbol
  - One could use punctuation between two codewords but inefficient

# Instantaneous Codes

- Instantaneous (or Prefix) Code
  - No codeword is a prefix of another
  - Can be decoded **instantaneously** without reference to future codewords
- Instantaneous  $\Rightarrow$  Uniquely Decodable  $\Rightarrow$  Non-singular

Examples:

- $C(E, F, G, H) = (0, 1, 00, 11)$   $uv \ 1x \bar{IU}$
- $C(E, F) = (0, 101)$   $uv \ 1v \ IU$
- $C(E, F) = (1, 101)$   $uv \ 1x \bar{IU}$
- $C(E, F, G, H) = (00, 01, 10, 11)$   $uv \ 1v \ IU$
- $C(E, F, G, H) = (0, 01, 011, 111)$   $uv \ 1x \bar{IU}$



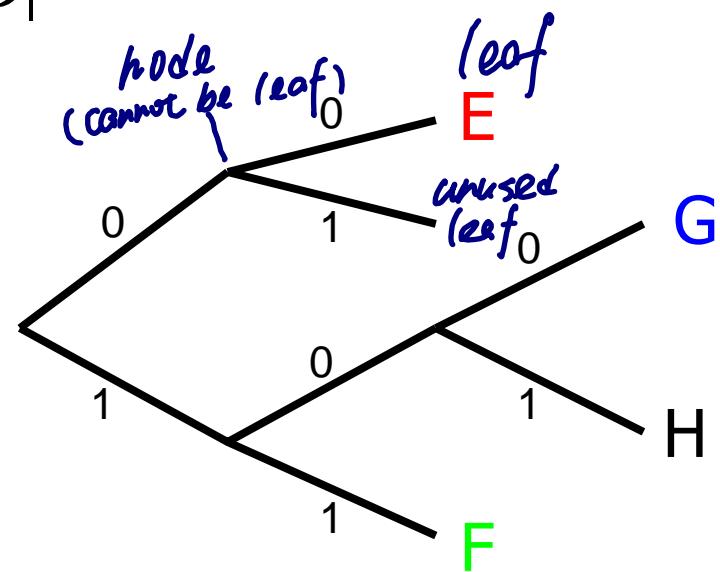
# Code Tree

---

Instantaneous code:  $C(E,F,G,H) = (00, 11, 100, 101)$

Form a  $D$ -ary tree where  $D = |D|$

- $D$  branches at each node
- Each codeword is a leaf
- Each node along the path to a leaf is a prefix of the leaf  
⇒ can't be a leaf itself
- Some leaves may be unused



$111011000000 \rightarrow F H G E E$

kraft inequality (for instantaneous codes)

$$\sum_{i=1}^{|X|} 2^{-l_i} \leq (\text{budget})$$

## Kraft Inequality (instantaneous codes)

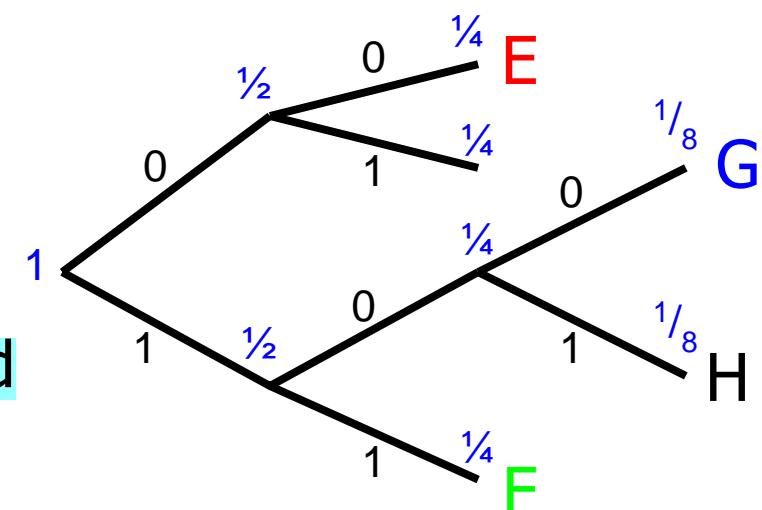
code cost  $\propto \frac{1}{\text{code length}}$

- Limit on codeword lengths of instantaneous codes
  - Not all codewords can be too short
- Codeword lengths  $l_1, l_2, \dots, l_{|X|} \Rightarrow$
- Label each node at depth  $l$  with  $2^{-l}$
- Each node equals the sum of all its leaves
- Equality iff all leaves are utilised
- Total code budget = 1

Code 00 uses up  $\frac{1}{4}$  of the budget

Code 100 uses up  $\frac{1}{8}$  of the budget

$$\boxed{\sum_{i=1}^{|X|} 2^{-l_i} \leq 1}$$



Same argument works with D-ary tree

McMillan inequality (for uniquely decodable codewords)

$$\sum_{i=1}^{|X|} D^{-l_i} \leq 1 \quad (\text{same with Kraft ineq.})$$

**McMillan Inequality (uniquely decodable codes)**

$$S^N = \left( \sum_{i=1}^{|X|} D^{-l_i} \right)^N = \sum_{i_1=1}^{|X|} \sum_{i_2=1}^{|X|} \dots \sum_{i_N=1}^{|X|} D^{-\sum_{i=1}^N l_i} = \sum_{x \in X^N} D^{-\text{length}\{C^+(x)\}}$$

If uniquely decodable  $C$  has codeword lengths

$$l_1, l_2, \dots, l_{|X|}, \text{ then } \sum_{i=1}^{|X|} D^{-l_i} \leq 1$$

every codebook length  $D_{\text{sum}}$

The same

**Proof:** Let  $S = \sum_{i=1}^{|X|} D^{-l_i}$  and  $M = \max l_i$  then for any  $N$ ,

$$S^N \leq NM \quad \text{for } N \geq 1 \Rightarrow S \leq 1.$$

exp linear  $\sum_{i=1}^{|X|} D^{-l_i}$

$$S^N = \left( \sum_{i=1}^{|X|} D^{-l_i} \right)^N = \sum_{i_1=1}^{|X|} \sum_{i_2=1}^{|X|} \dots \sum_{i_N=1}^{|X|} D^{-\sum_{i=1}^N l_i} = \sum_{\mathbf{x} \in X^N} D^{-\text{length}\{C^+(\mathbf{x})\}}$$

Sum over all possible codeword length

$$N=1 \quad \sum_{l=1}^{NM} D^{-l} \quad | \mathbf{x} : l = \text{length}\{C^+(\mathbf{x})\}| \leq \sum_{l=1}^{NM} D^{-l} \quad \text{re-order sum by total length}$$

$$NM \quad \sum_{l=1}^{NM} 1 = NM$$

Sum over all sequences of length  $N$

If  $S > 1$  then  $S^N > NM$  for some  $N$ . Hence  $S \leq 1$ .

max number of distinct sequences of length  $l$

Implication: uniquely decodable codes doesn't offer further reduction of codeword lengths than instantaneous codes

# McMillan Inequality (uniquely decodable codes)

If uniquely decodable  $C$  has codeword lengths

$$l_1, l_2, \dots, l_{|X|}, \text{ then } \sum_{i=1}^{|X|} D^{-l_i} \leq 1$$

The same

**Proof:** Let  $S = \sum_{i=1}^{|X|} D^{-l_i}$  and  $M = \max l_i$  then for any  $N$ ,

$$S^N = \left( \sum_{i=1}^{|X|} D^{-l_i} \right)^N = \sum_{i_1=1}^{|X|} \sum_{i_2=1}^{|X|} \dots \sum_{i_N=1}^{|X|} D^{-\left(l_{i_1} + l_{i_2} + \dots + l_{i_N}\right)} = \sum_{\mathbf{x} \in X^N} D^{-\underbrace{\text{length}\{C^+(\mathbf{x})\}}_{\text{length}}}$$

$$= \sum_{l=1}^{NM} D^{-l} |\{\mathbf{x} : l = \text{length}\{C^+(\mathbf{x})\}\}| \leq \sum_{l=1}^{NM} D^{-l} D^l = \sum_{l=1}^{NM} 1 = NM$$

If  $S > 1$  then  $S^N > NM$  for some  $N$ . Hence  $S \leq 1$ .

Implication: uniquely decodable codes doesn't offer further reduction of codeword lengths than instantaneous codes

# How Short are Optimal Codes?

---

If  $l(x) = \text{length}(C(x))$  then  $C$  is **optimal** if  
 $L=E$   $l(x)$  is as small as possible.

We want to minimize  $\sum_{x \in X} p(x)l(x)$  subject to

1.  $\sum_{x \in X} D^{-l(x)} \leq 1$
2. all the  $l(x)$  are integers

Simplified version:

Ignore condition 2 and assume condition 1 is satisfied with equality.

*optimistic mistakes*  
less restrictive so lengths may be shorter than actually possible  $\Rightarrow$  lower bound

# Optimal Codes (non-integer $l_i$ )

- Minimize  $\sum_{i=1}^{|X|} p(x_i)l_i$  subject to  $\sum_{i=1}^{|X|} D^{-l_i} = 1$  optimum at bound

Use Lagrange multiplier:

Define  $J = \sum_{i=1}^{|X|} p(x_i)l_i + \lambda \sum_{i=1}^{|X|} D^{-l_i}$  and set  $\frac{\partial J}{\partial l_i} = 0$

$$\frac{\partial J}{\partial l_i} = p(x_i) - \lambda \ln(D) D^{-l_i} = 0 \Rightarrow D^{-l_i} = p(x_i) / \lambda \ln(D)$$

also  $\sum_{i=1}^{|X|} D^{-l_i} = 1 \Rightarrow \lambda = 1 / \ln(D) \Rightarrow l_i = -\log_D(p(x_i))$

$$E l(x) = E - \log_D(p(x)) = \frac{E - \log_2(p(x))}{\log_2 D} = \frac{H(x)}{\log_2 D} = H_D(x)$$

no uniquely decodable code can do better than this  $D$ .

# Bounds on Optimal Code Length

Round up optimal code lengths:

- $l_i$  are bound to satisfy the Kraft Inequality (since the optimum lengths do)  $\sum D^{-l_i} \leq 1$  ?   
  $\begin{array}{ll} \text{True: } & \checkmark \\ \text{False: } & \times \end{array}$
- For this choice,  $-\log_D(p(x_i)) \leq l_i \leq -\log_D(p(x_i)) + 1$
- Average shortest length:



$$H_D(X) \leq L^* < H_D(X) + 1$$

(since we added  $<1$  to optimum values)

- We can do better by encoding blocks of  $n$  symbols

$$n^{-1} H_D(X_{1:n}) \leq n^{-1} E l(X_{1:n}) \stackrel{\text{normalise the length}}{\leq} n^{-1} H_D(X_{1:n}) + n^{-1}$$

- If entropy rate  $\overset{n \rightarrow \infty}{\Rightarrow}$  tighter bound  $\Rightarrow$  avg. symbol length  $\rightarrow$  entropy. of  $x_i$  exists ( $\Leftarrow x_i$  is stationary process)

$$n^{-1} H_D(X_{1:n}) \rightarrow H_D(X) \Rightarrow n^{-1} E l(X_{1:n}) \rightarrow H_D(X)$$

Also known as source coding theorem

# Block Coding Example

$$X = [A; B], p_x = [0.9; 0.1]$$

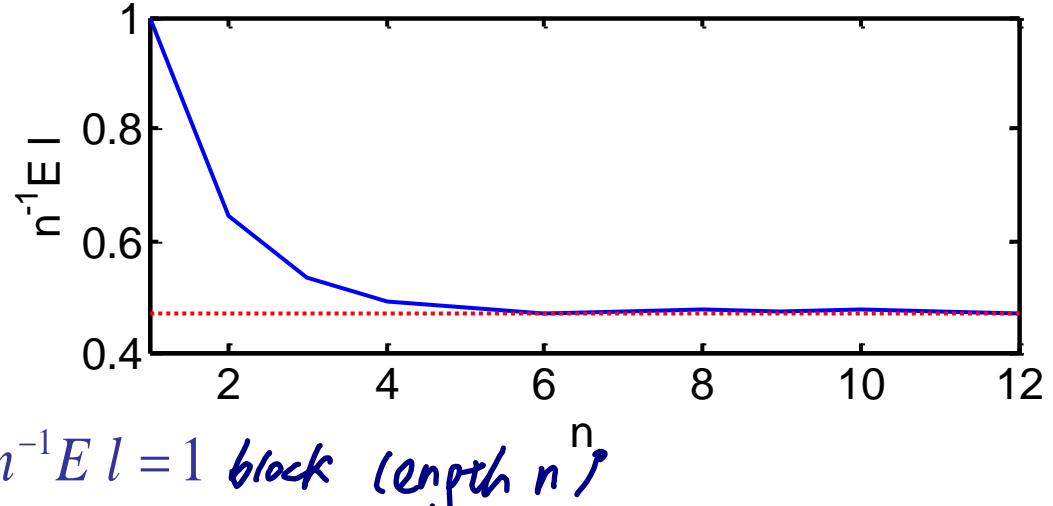
$$H(x_i) = 0.469$$

Huffman coding:

- $n=1$
- | sym  | A   | B   |
|------|-----|-----|
| prob | 0.9 | 0.1 |
| code | 0   | 1   |
- $$n^{-1}E_l = \frac{0.9 \times 1 + 0.1 \times 1}{1} = 1$$

- $n=2$
- | sym  | AA   | AB   | BA   | BB   |
|------|------|------|------|------|
| prob | 0.81 | 0.09 | 0.09 | 0.01 |
| code | 0    | 11   | 100  | 101  |
- $$n^{-1}E_l = \frac{0.81 \times 1 + 0.09 \times 2 + 0.09 \times 3}{2} = 0.645$$

- $n=3$
- | sym  | AAA   | AAB   | ... | BBA   | BBB   |
|------|-------|-------|-----|-------|-------|
| prob | 0.729 | 0.081 | ... | 0.009 | 0.001 |
| code | 0     | 101   | ... | 10010 | 10011 |



$\Rightarrow$  avg. symbol length  $\rightarrow$  entropy  
 $n^{-1}E_l$  (longer block, less uncertainty)

$$n^{-1}E_l = 0.583$$

The extra 1 bit inefficiency becomes insignificant for large blocks

# Summary

---

- McMillan Inequality for D-ary codes:
  - any uniquely decodable C has  $\sum_{i=1}^{|X|} D^{-l_i} \leq 1$
  - Any uniquely decodable code:

$$E l(x) \geq H_D(x)$$

- Source coding theorem
  - Symbol-by-symbol encoding

$$H_D(x) \leq E l(x) \leq H_D(x) + 1$$

- Block encoding  $n^{-1} E l(x_{1:n}) \rightarrow H_D(X)$

# Lecture 5

---

- Source Coding Algorithms
- Huffman Coding
- Lempel-Ziv Coding

# Huffman Code

---

An optimal binary instantaneous code must satisfy:

1.  $p(x_i) > p(x_j) \Rightarrow l_i \leq l_j$  (else swap codewords)
  2. The two longest codewords have the same length  
(else chop a bit off the longer codeword)
  3.  $\exists$  two longest codewords differing only in the last bit  
(else chop a bit off all of them)
- 

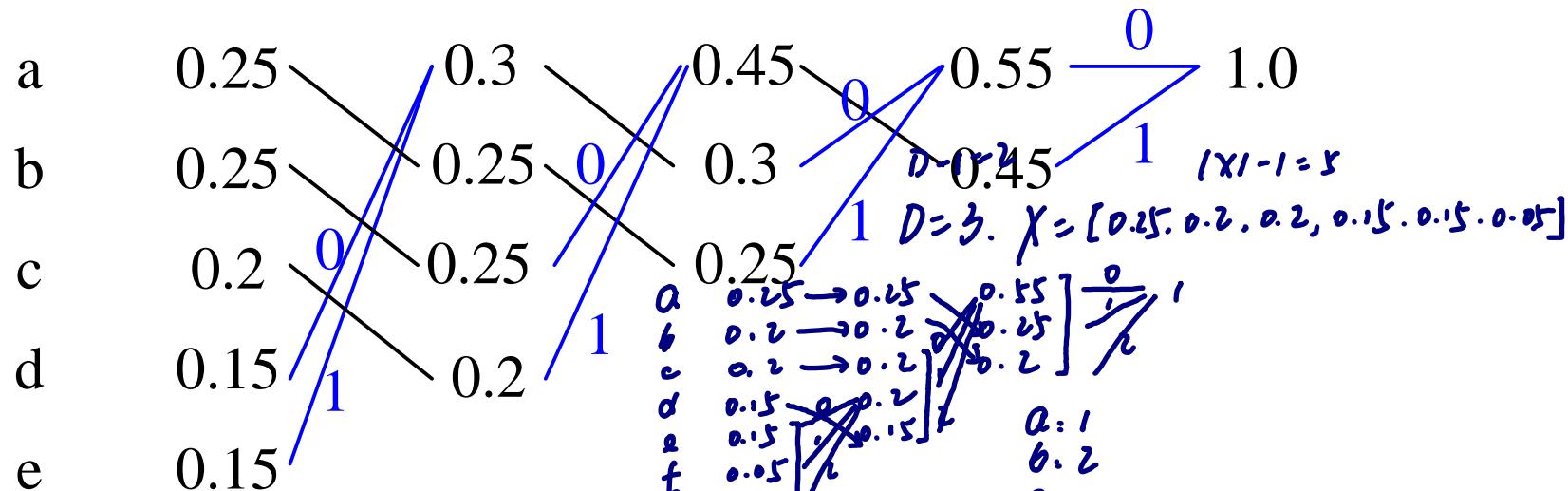
## Huffman Code construction

1. Take the two smallest  $p(x_i)$  and assign each a different last bit. Then merge into a single symbol.
2. Repeat step 1 until only one symbol remains

Used in JPEG, MP3...

# Huffman Code Example

$$X = [a, b, c, d, e], p_X = [0.25 \ 0.25 \ 0.2 \ 0.15 \ 0.15]$$



① go from symbol to the end    ② read backwards

Read diagram backwards for codewords:

$$C(X) = [01 \ 10 \ 11 \ 000 \ 001], L = 2.3, H(X) = 2.286$$

For D-ary code, first add extra zero-probability symbols until  $|X|-1$  is a multiple of  $D-1$  and then group D symbols at a time

# Huffman Code is Optimal Instantaneous Code

Huffman traceback gives codes for progressively larger alphabets:  $\begin{array}{c} a: 0.55 \\ b: 0.45 \end{array}$ ,  $L_2 = 0.55 + 0.45 = 1$

$$\mathbf{p}_2 = [0.55 \ 0.45],$$

$$\mathbf{c}_2 = [0 \ 1], L_2 = 1$$

<del>a: 0.45</del>	<del>b: 0.3</del>	<del>c: 0.25</del>	<del>0.55 0</del>	<del>0.45 1</del>
<del>a: 1</del>	<del>b: 0</del>	<del>c: 0</del>	<del>0.55 0</del>	<del>0.45 1</del>

$$L_3 = 0.45 \times 1 + 0.55 \times 2 = 1.55$$

$$\mathbf{p}_3 = [0.45 \ 0.3 \ 0.25],$$

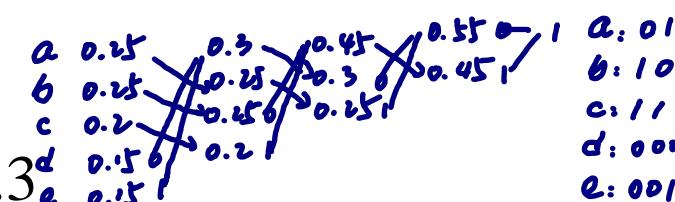
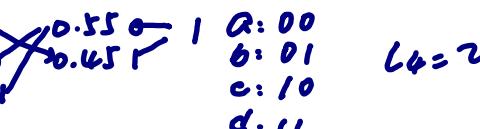
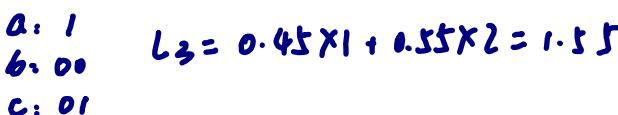
$$\mathbf{c}_3 = [1 \ 00 \ 01], L_3 = 1.55$$

$$\mathbf{p}_4 = [0.3 \ 0.25 \ 0.25 \ 0.2],$$

$$\mathbf{c}_4 = [00 \ 01 \ 10 \ 11], L_4 = 2$$

$$\mathbf{p}_5 = [0.25 \ 0.25 \ 0.2 \ 0.15 \ 0.15],$$

$$\mathbf{c}_5 = [01 \ 10 \ 11 \ 000 \ 001], L_5 = 2.3$$



$$L_5 = 2 \times 0.7 + 3 \times 0.3 = 2.3$$

We want to show that all these codes are optimal including  $C_5$

# Huffman Code is Optimal Instantaneous Code

---

Huffman traceback gives codes for progressively larger alphabets:

$$\mathbf{p}_2 = [0.55 \ 0.45],$$

$$\mathbf{c}_2 = [0 \ 1], L_2 = 1$$

$$\mathbf{p}_3 = [0.45 \ 0.3 \ 0.25],$$

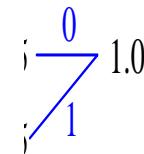
$$\mathbf{c}_3 = [1 \ 00 \ 01], L_3 = 1.55$$

$$\mathbf{p}_4 = [0.3 \ 0.25 \ 0.25 \ 0.2],$$

$$\mathbf{c}_4 = [00 \ 01 \ 10 \ 11], L_4 = 2$$

$$\mathbf{p}_5 = [0.25 \ 0.25 \ 0.2 \ 0.15 \ 0.15],$$

$$\mathbf{c}_5 = [01 \ 10 \ 11 \ 000 \ 001], L_5 = 2.3$$



We want to show that all these codes are optimal including  $C_5$

# Huffman Optimality Proof

---

Suppose one of these codes is sub-optimal:

- $\exists m > 2$  with  $c_m$  the first sub-optimal code (note  $c_2$  is definitely optimal)
- An optimal  $c'_m$  must have  $L_{C'm} < L_{Cm}$
- Rearrange the symbols with longest codes in  $c'_m$  so the two lowest probs  $p_i$  and  $p_j$  differ only in the last digit (doesn't change optimality)
- Merge  $x_i$  and  $x_j$  to create a new code  $c'_{m-1}$  as in Huffman procedure
- ~~$L_{C'm-1} = L_{Cm} - p_i - p_j$  since identical except 1 bit shorter with prob  $p_i + p_j$~~
- But also  $L_{C'm-1} = L_{Cm} - p_i - p_j$  hence  $L_{C'm-1} < L_{Cm-1}$  which contradicts assumption that  $c_m$  is the first sub-optimal code

Hence, Huffman coding satisfies  $H_D(x) \leq L < H_D(x) + 1$

Note: Huffman is just one out of many possible optimal codes

# Shannon-Fano Code

---

Fano code Fano: split

1. Put probabilities in decreasing order
2. Split as close to 50-50 as possible; repeat with each half

*source of information loss*

a	0.20	<u>largest probability: shortest. all zeros</u>	00	$I(X) = 2.81 \text{ bits}$
b	0.19	0	010	
c	0.17	1	011	$L_{SF} = 2.89 \text{ bits}$
d	0.15	1	100	
e	0.14	0	101	
f	0.06	1	110	Not necessarily optimal: the
g	0.05	1	1110	best code for this $p$ actually
h	0.04	0	1111	has $L = 2.85 \text{ bits}$
<i>smallest probability. longest. all ones.</i>				

# Shannon versus Huffman

Shannon

$$F_i = \sum_{k=1}^{i-1} p(x_k), \quad p(x_1) \geq p(x_2) \geq \dots \geq p(x_m)$$

Shannon:  $\lceil -\log_2 p(x_i) \rceil$

encoding: round the number  $F_i \in [0,1]$  to  $\lceil -\log p(x_i) \rceil$  bits

$$H_D(x) \leq L_{SF} \leq H_D(x) + 1 \quad (\text{excercise})$$

$$\mathbf{p}_x = [0.36 \quad 0.34 \quad 0.25 \quad 0.05] \Rightarrow H(x) = 1.78 \text{ bits}$$

$$-\log_2 \mathbf{p}_x = [1.47 \quad 1.56 \quad 2 \quad 4.32]$$

$$\mathbf{l}_S = \lceil -\log_2 \mathbf{p}_x \rceil = [2 \quad 2 \quad 2 \quad 5]$$

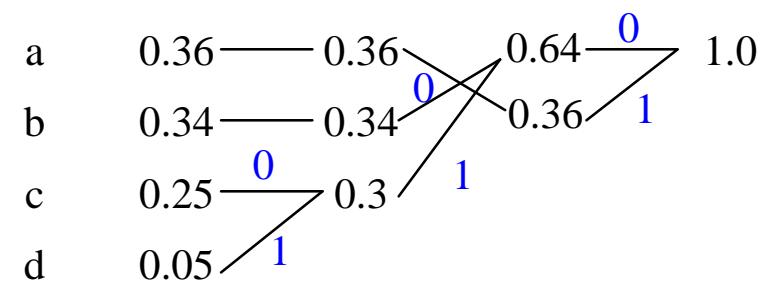
$$L_S = 2.15 \text{ bits}$$

Huffman

$$\mathbf{l}_H = [1 \quad 2 \quad 3 \quad 3]$$

$$L_H = 1.94 \text{ bits}$$

Individual codewords may be longer in Huffman than Shannon but not the average



# Issues with Huffman Coding

---

- Requires the probability distribution of the source
  - Must recompute entire code if any symbol probability changes
    - A block of  $N$  symbols needs  $|X|^N$  pre-calculated probabilities
- For many practical applications, however, the underlying probability distribution is unknown
  - Estimate the distribution
    - Arithmetic coding: extension of Shannon-Fano coding; can deal with large block lengths
  - Without the distribution
    - Universal coding: Lempel-Ziv coding

# Universal Coding

- Does not depend on the distribution of the source
- Compression of an individual sequence
- Run length coding
  - Runs of data are stored (e.g., in fax machines)  
Example: 

The sequence consists of 9 white 'W's followed by 2 black 'B's, then 7 white 'W's, 6 black 'B's, and 2 white 'W's. Below the sequence, the run lengths are listed as 9W2B7W6B2W.
- Lempel-Ziv coding *encode strings into phrases*
  - Generalization that takes advantage of runs of strings of characters (such as **WWWWWWWWB**)
  - Adaptive dictionary compression algorithms
  - Asymptotically optimum: achieves the entropy rate for any stationary ergodic source

# Lempel-Ziv Coding (LZ78)

Memorize previously occurring substrings in the input data

- parse input into the shortest possible distinct 'phrases', i.e., each phrase is the shortest phrase not seen earlier

phrases	#	codewords ( <u>head location + tail</u> )
A	1	0A
B	2	0B
AA	3	1A
BA	4	2A
BAB	5	4B
BB	6	2B
AB	7	HB

number the phrases starting from 1 (0 is the empty string)

ABAABABABBBAB... *new phrase = old phrase (head) + tail*

Look up a dictionary

- each phrase consists of a previously occurring phrase (head) followed by an additional A or B (tail)

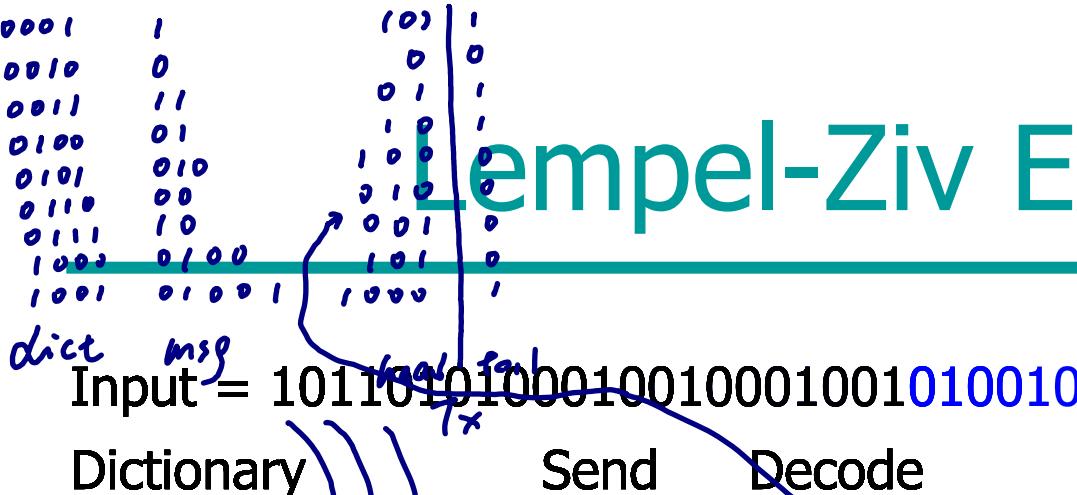
- encoding: give location of head followed by the additional symbol for tail

0A0B1A2A4B2B1B...

- decoder uses an identical dictionary

locations are underlined

# Lempel-Ziv Example



Dictionary		Send	Decode
0000	φ	(0)1	1
0001	1	00	0
0010	0	011	11
0011	11	101	01
0100	01	1000	010
0101	010	0100	00
0110	00	0010	10
0111	10	1010	0100
1000	0100	10001	01001
1001	01001	10010	010010

location

No need to always send 4 bits

Remark:

- No need to send the dictionary (imagine zip and unzip!)
- Can be reconstructed
- Need to send 0's in 01, 010 and 001 to avoid ambiguity (i.e., instantaneous code)

# Lempel-Ziv Comments

---

Dictionary  $D$  contains  $K$  entries  $D(0), \dots, D(K-1)$ . We need to send  $M=\text{ceil}(\log K)$  bits to specify a dictionary entry. Initially  $K=1$ ,  $D(0)=\phi$  = null string and  $M=\text{ceil}(\log K) = 0$  bits.

Input	Action
1	"1" $\notin D$ so send "1" and set $D(1)="1"$ . Now $K=2 \Rightarrow M=1$ .
0	"0" $\notin D$ so split it up as " $\phi$ "+"0" and send location "0" (since $D(0)=\phi$ ) followed by "0". Then set $D(2)="0"$ making $K=3 \Rightarrow M=2$ .
1	"1" $\in D$ so don't send anything yet – just read the next input bit.
1	"11" $\notin D$ so split it up as "1" + "1" and send location "01" (since $D(1)="1"$ and $M=2$ ) followed by "1". Then set $D(3)="11"$ making $K=4 \Rightarrow M=2$ .
0	"0" $\in D$ so don't send anything yet – just read the next input bit.
1	"01" $\notin D$ so split it up as "0" + "1" and send location "10" (since $D(2)="0"$ and $M=2$ ) followed by "1". Then set $D(4)="01"$ making $K=5 \Rightarrow M=3$ .
0	"0" $\in D$ so don't send anything yet – just read the next input bit.
1	"01" $\in D$ so don't send anything yet – just read the next input bit.
0	"010" $\notin D$ so split it up as "01" + "0" and send location "100" (since $D(4)="01"$ and $M=3$ ) followed by "0". Then set $D(5)="010"$ making $K=6 \Rightarrow M=3$ .

So far we have sent **1000111011000** where dictionary entry numbers are in **red**.

# Lempel-Ziv Properties

---

- Simple to implement
- Widely used because of its speed and efficiency
  - applications: compress, gzip, GIF, TIFF, modem ...
  - variations: LZW (considering last character of the current phrase as part of the next phrase, used in Adobe Acrobat), LZ77 (sliding window)
  - different dictionary handling, etc
- Excellent compression in practice
  - many files contain repetitive sequences
  - worse than arithmetic coding for text files

# Asymptotic Optimality

---

- Asymptotically optimum for stationary ergodic source (i.e. achieves entropy rate)
- Let  $c(n)$  denote the number of phrases for a sequence of length  $n$
- Compressed sequence consists of  $c(n)$  pairs (location, last bit)
- Needs  $\underline{c(n)[\log c(n)+1]}$  bits in total
- $\{X_i\}$  stationary ergodic  $\Rightarrow$

$$\limsup_{n \rightarrow \infty} n^{-1} l(X_{1:n}) = \limsup_{n \rightarrow \infty} \frac{c(n)[\log c(n)+1]}{n} \leq H(X) \text{ with probability } 1$$

- 
- Proof: C&T chapter 12.10
  - may only approach this for an enormous file

# Summary

---

- **Huffman Coding:**  $H_D(x) \leq E l(x) \leq H_D(x) + 1$ 
  - Bottom-up design
  - Optimal  $\Rightarrow$  shortest average length
- **Shannon-Fano Coding:**  $H_D(x) \leq E l(x) \leq H_D(x) + 1$ 
  - Intuitively natural top-down design
- **Lempel-Ziv Coding**
  - Does not require probability distribution
  - Asymptotically optimum for stationary ergodic source (i.e. achieves entropy rate)

# Lecture 6

---

- Markov Chains
  - Have a special meaning
  - Not to be confused with the standard definition of Markov chains (which are sequences of discrete random variables)
- Data Processing Theorem
  - You can't create information from nothing
- Fano's Inequality
  - Lower bound for error in estimating  $X$  from  $Y$

# Markov Chains

---

If we have three random variables:  $x, y, z$

$$p(x, y, z) = p(z | x, y) p(y | x) p(x)$$

they form a **Markov chain**  $x \rightarrow y \rightarrow z$  if

$$p(z | x, y) = p(z | y) \Leftrightarrow p(x, y, z) = p(z | y) p(y | x) p(x)$$

A Markov chain  $x \rightarrow y \rightarrow z$  means that

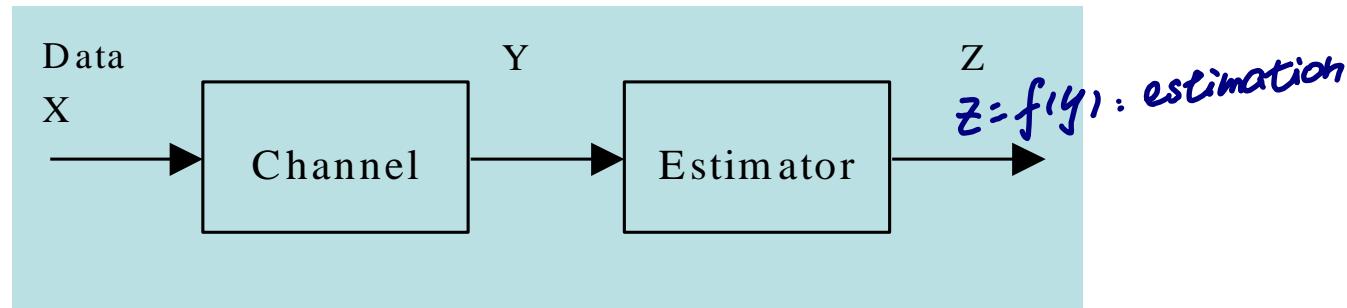
- the only way that  $x$  affects  $z$  is through the value of  $y$
- if you already know  $y$ , then observing  $x$  gives you no additional information about  $z$ , i.e.  $I(x; z | y) = 0 \Leftrightarrow H(z | y) = H(z | x, y)$
- if you know  $y$ , then observing  $z$  gives you no additional information about  $x$ .

$$I(x; z | y) = H(z | y) - H(z | x, y)$$

$$H(x | y) = H(x | y, z)$$

# Data Processing

- Estimate  $z = f(y)$ , where  $f$  is a function
- A special case of a Markov chain  $x \rightarrow y \rightarrow f(y)$



- Does processing of  $y$  increase the information that  $y$  contains about  $x$ ? *No. even less!*

# Markov Chain Symmetry

If  $x \rightarrow y \rightarrow z$        $\xrightarrow{MC} x \rightarrow y \rightarrow z$

$$p(x, z | y) = \frac{p(x, y, z)}{p(y)} \stackrel{(a)}{=} \frac{p(x, y)p(z | y)}{p(y)} = p(x | y)p(z | y)$$

*given y. x, z are independent.*

$$(a) \quad p(z | x, y) = p(z | y)$$

Hence  $x$  and  $z$  are conditionally independent given  $y$

Also  $x \rightarrow y \rightarrow z$  iff  $z \rightarrow y \rightarrow x$  since  $p(x|y)p(z|y) = p(x,z|y)$

$$p(x | y) = p(x | y) \frac{p(z | y)p(y)}{p(y, z)} \stackrel{(a)}{=} \frac{p(x, z | y)p(y)}{p(y, z)} = \frac{p(x, y, z)}{p(y, z)}$$

$$= p(x | y, z) \quad (a) \quad p(x, z | y) = p(x | y)p(z | y)$$

*given y, & does not provide additional information* Conditionally indep.  
Markov chain property is symmetrical

# Data Processing Theorem

---

If  $x \rightarrow y \rightarrow z$  then  $I(x; y) \geq I(x; z)$  *(even decrease)* *extract useful info  
do not lose mutual info*

- processing  $y$  cannot add new information about  $x$

If  $x \rightarrow y \rightarrow z$  then  $I(x; y) \geq I(x; y | z)$  *(even less)*

- Knowing  $z$  does not increase the amount  $y$  tells you about  $x$

**Proof:**

Apply chain rule in different ways

$$I(x; y, z) = I(x; y) + \underbrace{I(x; z | y)}_{(a)} = I(x; z) + I(x; y | z)$$

$x \rightarrow y$ : observation

$y \rightarrow z$ : data processing but  $I(x; z | y) = 0$

hence  $I(x; y) = I(x; z) + I(x; y | z)$

so  $I(x; y) \geq I(x; z)$  and  $I(x; y) \geq I(x; y | z)$

(a)  $I(x; z) = 0$  iff  $x$  and  $z$  are independent; Markov  $\Rightarrow p(x, z | y) = p(x | y)p(z | y)$

# So Why Processing?

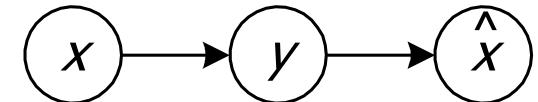
---

- One can not create information by manipulating the data
- But no information is lost if equality holds
- Sufficient statistic
  - $z$  contains all the information in  $y$  about  $x$
  - Preserves mutual information  $I(x; y) = I(x; z)$
- The estimator should be designed in a way such that it outputs sufficient statistics
- Can the estimation be arbitrarily accurate?

# Fano's Inequality

If we estimate  $x$  from  $y$ , what is  $p_e = p(\hat{x} \neq x)$  ?

$$H(x|y) \leq H(p_e) + p_e \log |X|$$



$$\Rightarrow p_e \geq \frac{(H(x|y) - H(p_e))}{\log |X|} \stackrel{(a)}{\geq} \frac{(H(x|y) - 1)}{\log |X|}$$

(a) the second form is weaker but easier to use

Proof: Define a random variable  $e = \begin{cases} 1 & \hat{x} \neq x \\ 0 & \hat{x} = x \end{cases}$

$$H(e, x | \hat{x}) = H(x | \hat{x}) + H(e | x, \hat{x}) = H(e | \hat{x}) + H(x | e, \hat{x}) \quad \text{chain rule}$$

$$\begin{aligned} \Rightarrow H(x | \hat{x}) + 0 &\leq H(e) + H(x | e, \hat{x}) && H \geq 0; H(e | y) \leq H(e) \\ &= H(e) + H(x | \hat{x}, e = 0)(1 - p_e) + H(x | \hat{x}, e = 1)p_e \\ &\leq H(p_e) + 0 \times (1 - p_e) + p_e \log |X| && H(e) = H(p_e) \end{aligned}$$

$$H(x | y) \leq H(x | \hat{x}) \text{ since } I(x; \hat{x}) \leq I(x; y) \quad \text{Markov chain}$$

# Implications

---

- Zero probability of error  $p_e = 0 \Rightarrow H(x|y) = 0$
- Low probability of error if  $H(x|y)$  is small
- If  $H(x|y)$  is large then the probability of error is high
- Could be slightly strengthened to

$$H(x|y) \leq H(p_e) + p_e \log(|X|-1)$$

- Fano's inequality is used whenever you need to show that errors are inevitable
  - E.g., Converse to channel coding theorem

# Fano Example

---

$$X = \{1:5\}, p_x = [0.35, 0.35, 0.1, 0.1, 0.1]^T$$

$Y = \{1:2\}$  if  $x \leq 2$  then  $y = x$  with probability  $6/7$   
while if  $x > 2$  then  $y = 1$  or  $2$  with equal prob.

Our best strategy is to guess  $\hat{x} = y$  ( $x \rightarrow y \rightarrow \hat{x}$ )

- $p_{x|y=1} = [0.6, 0.1, 0.1, 0.1, 0.1]^T$
- actual error prob:  $p_e = 0.4$

**Fano bound:**  $p_e \geq \frac{H(x|y) - 1}{\log(|X|-1)} = \frac{1.771 - 1}{\log(4)} = 0.3855$  **(exercise)**

Main use: to show when error free transmission is impossible since  $p_e > 0$

# Summary

---

- **Markov:**  $x \rightarrow y \rightarrow z \Leftrightarrow p(z | x, y) = p(z | y) \Leftrightarrow I(x; z | y) = 0$
- **Data Processing Theorem:** if  $x \rightarrow y \rightarrow z$  then
  - $I(x; y) \geq I(x; z), I(y; z) \geq I(x; z)$
  - $I(x; y) \geq I(x; y | z)$  can be false if not Markov
  - Long Markov chains: If  $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5 \rightarrow x_6$ , then Mutual Information increases as you get closer together:
    - e.g.  $I(x_3, x_4) \geq I(x_2, x_4) \geq I(x_1, x_5) \geq I(x_1, x_6)$
- **Fano's Inequality:** if  $x \rightarrow y \rightarrow \hat{x}$  then

$$p_e \geq \frac{H(x | y) - H(p_e)}{\log(|X| - 1)} \geq \frac{H(x | y) - 1}{\log(|X| - 1)} \geq \frac{H(x | y) - 1}{\log |X|}$$

weaker but easier to use since independent of  $p_e$

# Lecture 7

---

- Law of Large Numbers
  - Sample mean is close to expected value
- Asymptotic Equipartition Principle (AEP)
  - $-\log P(x_1, x_2, \dots, x_n)/n$  is close to entropy  $H$
- The Typical Set
  - Probability of each sequence close to  $2^{-nH}$
  - Size ( $\sim 2^{nH}$ ) and total probability ( $\sim 1$ )
- The Atypical Set
  - Unimportant and could be ignored

# Typicality: Example

---

$$X = \{a, b, c, d\}, p = [0.5 \ 0.25 \ 0.125 \ 0.125]$$

$$-\log p = [1 \ 2 \ 3 \ 3] \Rightarrow H(p) = 1.75 \text{ bits}$$

Sample eight i.i.d. values

- typical  $\Rightarrow$  correct proportions

$$\text{adbabaac} \quad -\log p(\mathbf{x}) = 14 = 8 \times 1.75 = nH(\mathbf{x})$$

- not typical  $\Rightarrow \log p(\mathbf{x}) \neq nH(\mathbf{x})$

$$\text{dddddddd} \quad -\log p(\mathbf{x}) = 24$$

# Convergence of Random Variables

---

- Convergence

$$x_n \xrightarrow[n \rightarrow \infty]{} y \Rightarrow \forall \varepsilon > 0, \exists m \text{ such that } \forall n > m, |x_n - y| < \varepsilon$$

Example:  $x_n = \pm 2^{-n}, \quad y = 0$

choose  $m = -\log \varepsilon$

- Convergence in probability (weaker than convergence)

$$x_n \xrightarrow{\text{prob}} y \Rightarrow \forall \varepsilon > 0, \quad P(|x_n - y| > \varepsilon) \rightarrow 0$$

Example:  $x_n \in \{0; 1\}, \quad p = [1 - n^{-1}; n^{-1}]$

for any small  $\varepsilon$ ,  $p(|x_n| > \varepsilon) = n^{-1} \xrightarrow{n \rightarrow \infty} 0$

so  $x_n \xrightarrow{\text{prob}} 0$  (but  $x_n \not\rightarrow 0$ )

Note:  $y$  can be a constant or another random variable

# Law of Large Numbers

---

Given i.i.d.  $\{x_i\}$ , sample mean  $s_n = \frac{1}{n} \sum_{i=1}^n x_i$

$$- E s_n = E x = \mu \quad \text{Var } s_n = n^{-1} \text{Var } x = n^{-1} \sigma^2$$

As  $n$  increases,  $\text{Var } s_n$  gets smaller and the values become clustered around the mean

LLN:  $s_n \xrightarrow{\text{prob}} \mu$

$$\Leftrightarrow \forall \varepsilon > 0, \quad P(|s_n - \mu| > \varepsilon) \xrightarrow{n \rightarrow \infty} 0$$

The expected value of a random variable is equal to the long-term average when sampling repeatedly.

# Asymptotic Equipartition Principle

---

- $\mathbf{x}$  is the i.i.d. sequence  $\{x_i\}$  for  $1 \leq i \leq n$ 
  - Prob of a particular sequence is  $p(\mathbf{x}) = \prod_{i=1}^n p(x_i)$
  - Average  $E - \log p(\mathbf{x}) = n E - \log p(x_i) = nH(X)$

- AEP:

$$-\frac{1}{n} \log p(\mathbf{x}) \xrightarrow{\text{prob}} H(X)$$

- Proof:

$$-\frac{1}{n} \log p(\mathbf{x}) = -\frac{1}{n} \sum_{i=1}^n \log p(x_i)$$

law of large numbers

$$\xrightarrow{\text{prob}} E - \log p(x_i) = H(X)$$

# Typical Set

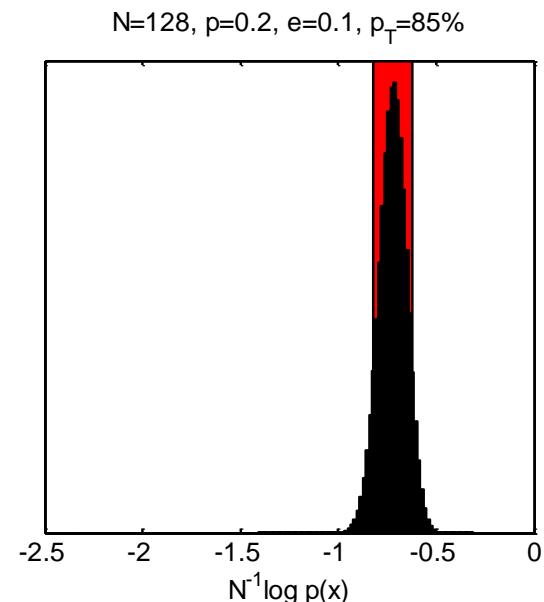
---

## Typical set (for finite n)

$$T_{\varepsilon}^{(n)} = \left\{ \mathbf{x} \in X^n : \left| -n^{-1} \log p(\mathbf{x}) - H(X) \right| < \varepsilon \right\}$$

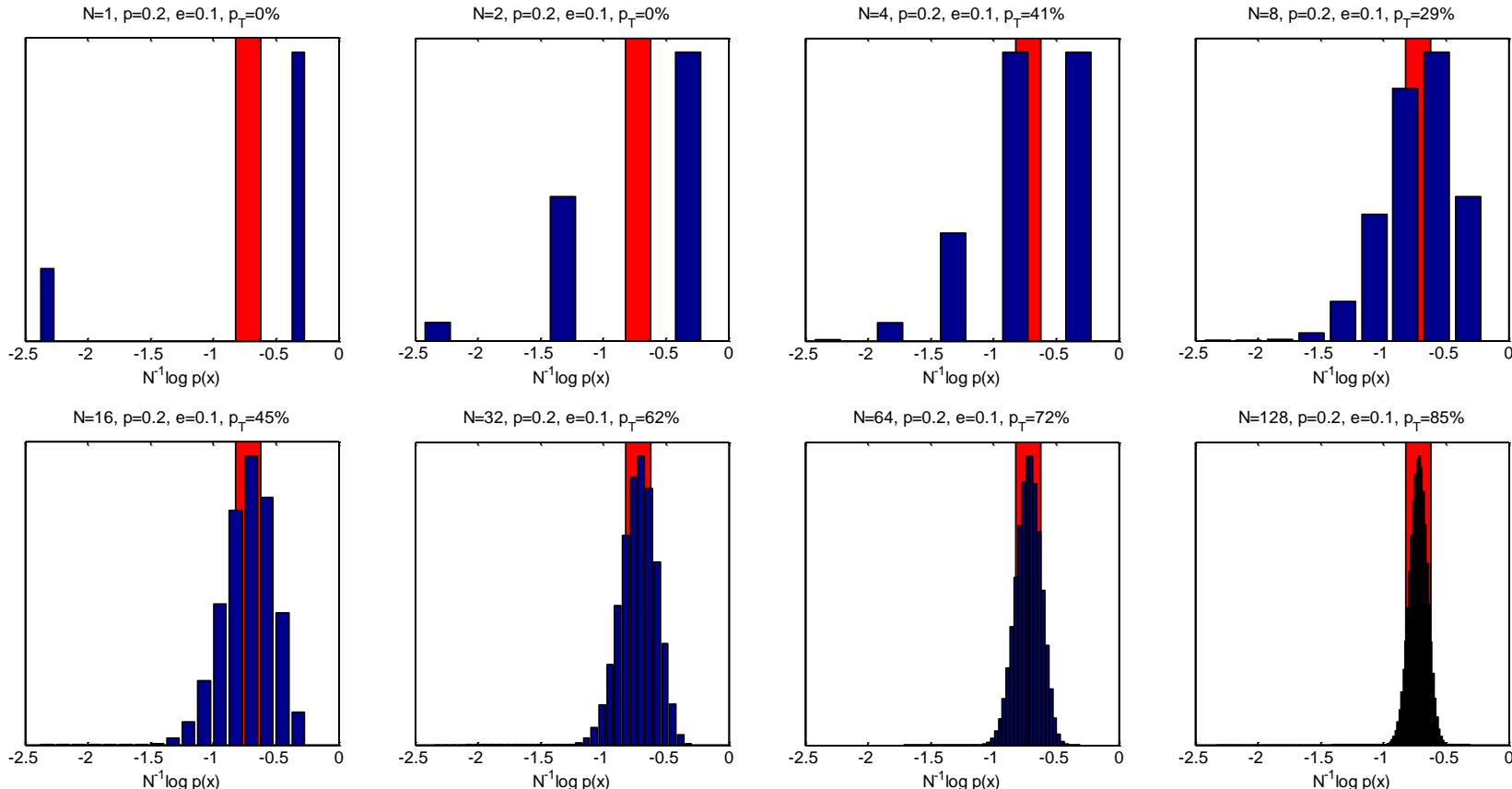
### Example:

- $x_i$  Bernoulli with  $p(x_i=1)=p$
- e.g.  $p([0 \ 1 \ 1 \ 0 \ 0 \ 0])=p^2(1-p)^4$
- For  $p=0.2$ ,  $H(X)=0.72$  bits
- Red bar shows  $T_{0.1}^{(n)}$



# Typical Set Frames

---



00,0000,00100,0100,00,0000,000010010000,0000100001000010000,  
0000100001000000, 0000000010000000

# Typical Set: Properties

---

$$\mathbf{x} \in T_{\varepsilon}^{(n)} \Rightarrow \log p(\mathbf{x}) = -nH(X) \pm n\varepsilon$$

$$p(\mathbf{x} \in T_{\varepsilon}^{(n)}) > 1 - \varepsilon \text{ for } n > N_{\varepsilon}$$

$$(1 - \varepsilon)2^{n(H(X) - \varepsilon)} < |T_{\varepsilon}^{(n)}| \leq 2^{n(H(X) + \varepsilon)}$$

**Proof 2:**

$$-n^{-1} \log p(\mathbf{x}) = n^{-1} \sum_{i=1}^n -\log p(x_i) \xrightarrow{\text{prob}} E - \log p(x_i) = H(X)$$

$$\text{Hence } \forall \varepsilon > 0 \exists N_{\varepsilon} \text{ s.t. } \forall n > N_{\varepsilon} \quad p(|-n^{-1} \log p(\mathbf{x}) - H(X)| > \varepsilon) < \varepsilon$$

**Proof 3a:**

$$\text{f.l.e. } n, \quad 1 - \varepsilon < p(\mathbf{x} \in T_{\varepsilon}^{(n)}) \leq \sum_{\mathbf{x} \in T_{\varepsilon}^{(n)}} 2^{-n(H(X) - \varepsilon)} = 2^{-n(H(X) - \varepsilon)} |T_{\varepsilon}^{(n)}|$$

**Proof 3b:**

$$1 = \sum_{\mathbf{x}} p(\mathbf{x}) \geq \sum_{\mathbf{x} \in T_{\varepsilon}^{(n)}} p(\mathbf{x}) \geq \sum_{\mathbf{x} \in T_{\varepsilon}^{(n)}} 2^{-n(H(X) + \varepsilon)} = 2^{-n(H(X) + \varepsilon)} |T_{\varepsilon}^{(n)}|$$

# Consequence

- for any  $\varepsilon$  and for  $n > N_\varepsilon$   
 “Almost all events are almost equally surprising”

- $p(\mathbf{x} \in T_\varepsilon^{(n)}) > 1 - \varepsilon \quad \text{and} \quad \log p(\mathbf{x}) = -nH(x) \pm n\varepsilon$

## Coding consequence

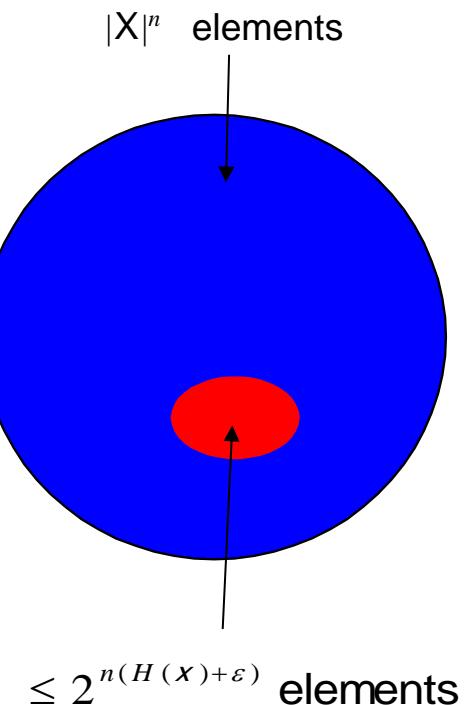
- $\mathbf{x} \in T_\varepsilon^{(n)}$  : ‘0’ + at most  $1+n(H+\varepsilon)$  bits
- $\mathbf{x} \notin T_\varepsilon^{(n)}$  : ‘1’ + at most  $1+n\log|X|$  bits
- $L = \text{Average code length}$

$$\leq p(\mathbf{x} \in T_\varepsilon^{(n)})[2 + n(H + \varepsilon)]$$

$$+ p(\mathbf{x} \notin T_\varepsilon^{(n)})[2 + n \log |X|]$$

$$\leq n(H + \varepsilon) + \varepsilon(n \log |X|) + 2\varepsilon + 2$$

$$= n(H + \varepsilon + \varepsilon \log |X| + 2(\varepsilon + 2)n^{-1}) = n(H + \varepsilon')$$



# Source Coding & Data Compression

---

For any choice of  $\varepsilon > 0$ , we can, by choosing block size,  $n$ , large enough, do the following:

- make a lossless code using only  $H(x) + \varepsilon$  bits per symbol on average:

$$\frac{L}{n} \leq H + \varepsilon$$

- The coding is one-to-one and decodable
  - However impractical due to exponential complexity
- Typical sequences have short descriptions of length  $\approx nH$ 
  - Another proof of source coding theorem (Shannon's original proof)
- However, encoding/decoding complexity is exponential in  $n$

# Smallest high-probability Set

---

$T_\varepsilon^{(n)}$  is a small subset of  $X^n$  containing most of the probability mass. Can you get even smaller ?

For any  $0 < \varepsilon < 1$ , choose  $N_0 = -\varepsilon^{-1} \log \varepsilon$ , then for any  $n > \max(N_0, N_\varepsilon)$  and any subset  $S^{(n)}$  satisfying  $|S^{(n)}| < 2^{n(H(x) - 2\varepsilon)}$

$$\begin{aligned}
 p(\mathbf{x} \in S^{(n)}) &= p(\mathbf{x} \in S^{(n)} \cap T_\varepsilon^{(n)}) + p(\mathbf{x} \in S^{(n)} \cap \overline{T_\varepsilon^{(n)}}) \\
 &< |S^{(n)}| \max_{\mathbf{x} \in T_\varepsilon^{(n)}} p(\mathbf{x}) + p(\mathbf{x} \in \overline{T_\varepsilon^{(n)}}) \\
 &< 2^{n(H - 2\varepsilon)} 2^{-n(H - \varepsilon)} + \varepsilon \quad \text{for } n > N_\varepsilon \\
 &= 2^{-n\varepsilon} + \varepsilon < 2\varepsilon \quad \text{for } n > N_0, \quad 2^{-n\varepsilon} < 2^{\log \varepsilon} = \varepsilon
 \end{aligned}$$

Answer: No

# Summary

---

- Typical Set
  - Individual Prob  $\mathbf{x} \in T_\varepsilon^{(n)} \Rightarrow \log p(\mathbf{x}) = -nH(x) \pm n\varepsilon$
  - Total Prob  $p(\mathbf{x} \in T_\varepsilon^{(n)}) > 1 - \varepsilon \text{ for } n > N_\varepsilon$
  - Size  $(1 - \varepsilon)2^{n(H(x) - \varepsilon)} < |T_\varepsilon^{(n)}| \leq 2^{n(H(x) + \varepsilon)}$
- No other high probability set can be much smaller than  $T_\varepsilon^{(n)}$
- Asymptotic Equipartition Principle
  - Almost all event sequences are equally surprising
- Can be used to prove source coding theorem

# Lecture 8

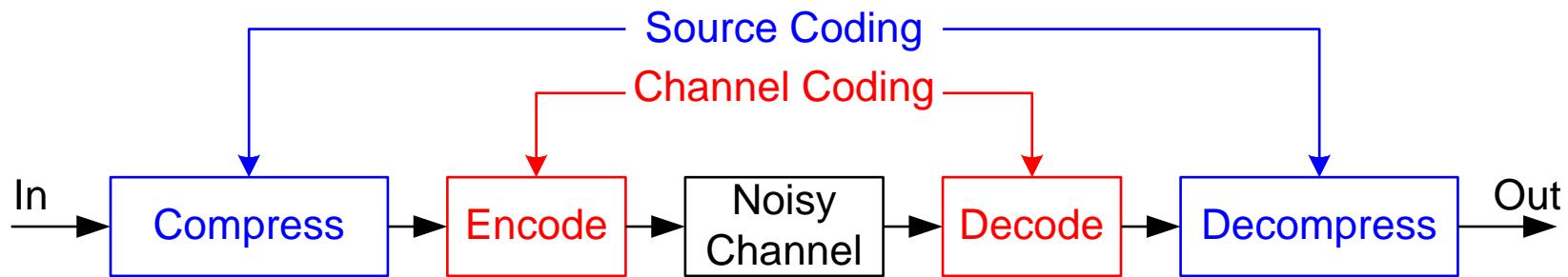
---

- Channel Coding
- Channel Capacity
  - The highest rate in bits per channel use that can be transmitted reliably
  - The maximum mutual information
- Discrete Memoryless Channels
  - Symmetric Channels
  - Channel capacity
    - Binary Symmetric Channel
    - Binary Erasure Channel
    - Asymmetric Channel



◆ = proved in channel coding theorem

# Model of Digital Communication



- **Source Coding**
  - Compresses the data to remove redundancy
- **Channel Coding**
  - Adds redundancy/structure to protect against channel errors

# Discrete Memoryless Channel

---

- Input:  $x \in X$ , Output  $y \in Y$



- Time-Invariant Transition-Probability Matrix

$$(\mathbf{Q}_{y|x})_{i,j} = p(y = y_j | x = x_i)$$

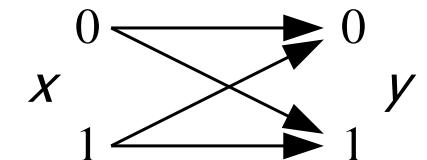
- Hence  $\mathbf{p}_y = \mathbf{Q}_{y|x}^T \mathbf{p}_x$
- $\mathbf{Q}$ : each row sum = 1, average column sum =  $|X||Y|^{-1}$
- **Memoryless**:  $p(y_n | x_{1:n}, y_{1:n-1}) = p(y_n | x_n)$
- **DMC** = Discrete Memoryless Channel

# Binary Channels

---

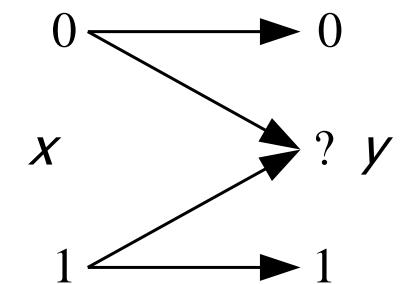
- Binary Symmetric Channel
  - $X = [0 \ 1]$ ,  $Y = [0 \ 1]$

$$\begin{pmatrix} 1-f & f \\ f & 1-f \end{pmatrix}$$



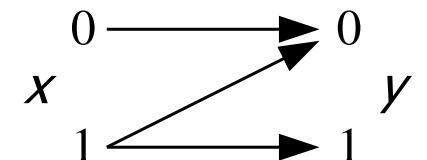
- Binary Erasure Channel
  - $X = [0 \ 1]$ ,  $Y = [0 \ ? \ 1]$

$$\begin{pmatrix} 1-f & f & 0 \\ 0 & f & 1-f \end{pmatrix}$$



- Z Channel
  - $X = [0 \ 1]$ ,  $Y = [0 \ 1]$

$$\begin{pmatrix} 1 & 0 \\ f & 1-f \end{pmatrix}$$



Symmetric: rows are permutations of each other; columns are permutations of each other  
 Weakly Symmetric: rows are permutations of each other; columns have the same sum

# Weakly Symmetric Channels

---

Weakly Symmetric:

1. All columns of  $\mathbf{Q}$  have the same sum =  $|X||Y|^{-1}$

- If  $x$  is uniform (i.e.  $p(x) = |X|^{-1}$ ) then  $y$  is uniform

$$p(y) = \sum_{x \in X} p(y|x)p(x) = |X|^{-1} \sum_{x \in X} p(y|x) = |X|^{-1} \times |X||Y|^{-1} = |Y|^{-1}$$

2. All rows are permutations of each other

- Each row of  $\mathbf{Q}$  has the same entropy so

$$H(Y|X) = \sum_{x \in X} p(x)H(Y|X=x) = H(\mathbf{Q}_{1,:})\sum_{x \in X} p(x) = H(\mathbf{Q}_{1,:})$$

where  $\mathbf{Q}_{1,:}$  is the entropy of the first (or any other) row of the  $\mathbf{Q}$  matrix

Symmetric:

1. All rows are permutations of each other
2. All columns are permutations of each other

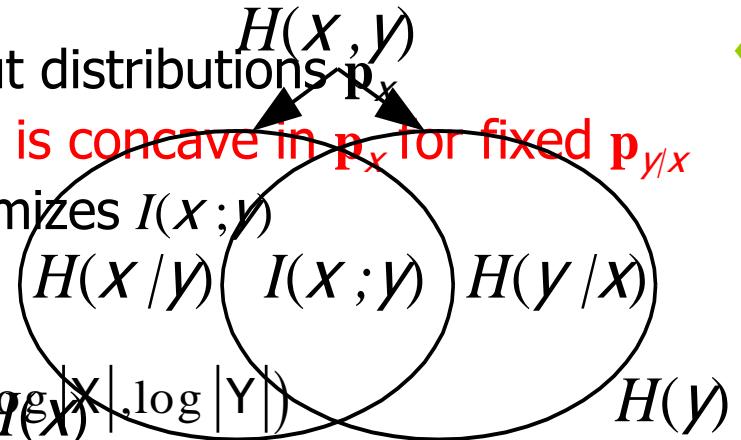
Symmetric  $\Rightarrow$  weakly symmetric

# Channel Capacity

- Capacity of a DMC channel:  $C = \max_{\mathbf{p}_x} I(x; y)$

- Mutual information (not entropy itself) is what could be transmitted through the channel
- Maximum is over all possible input distributions  $\mathbf{p}_x$
- $\exists$  only one maximum since  $I(x; y)$  is concave in  $\mathbf{p}_x$  for fixed  $\mathbf{p}_{y/x}$
- We want to find the  $\mathbf{p}_x$  that maximizes  $I(x; y)$
- Limits on  $C$ :

$$0 \leq C \leq \min(H(X), H(Y)) \leq \min(H(X) + \log |Y|)$$

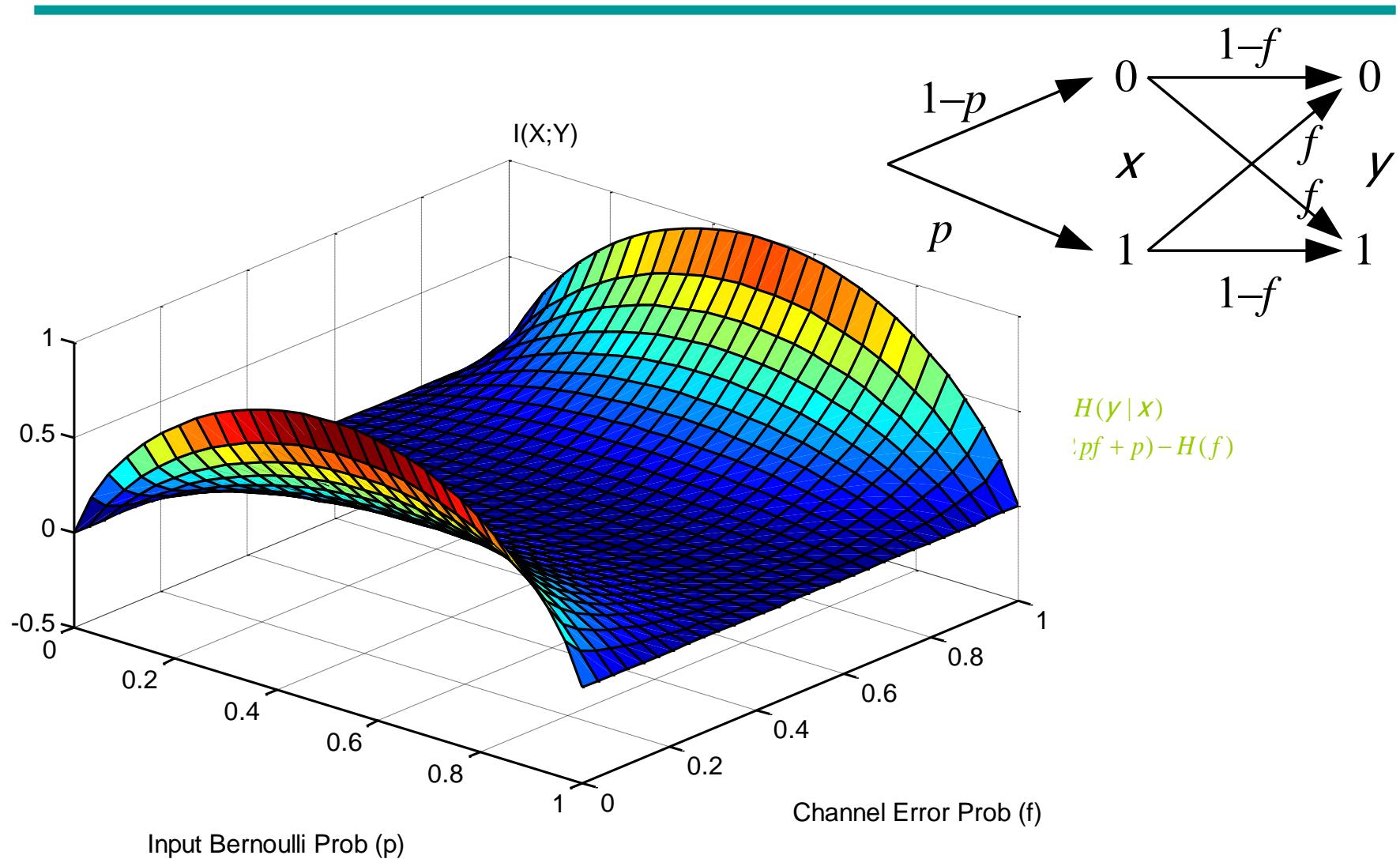


- Capacity for  $n$  uses of channel:

$$C^{(n)} = \frac{1}{n} \max_{\mathbf{p}_{x_{1:n}}} I(x_{1:n}; y_{1:n})$$

◆ = proved in two pages time

# Mutual Information Plot



# Mutual Information Concave in $p_X$

---

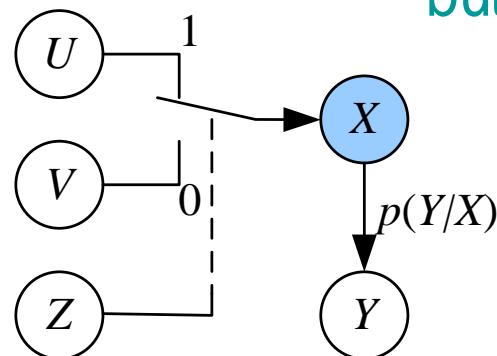
Mutual Information  $I(x; y)$  is **concave** in  $p_X$  for fixed  $p_{y/x}$

**Proof:** Let  $u$  and  $v$  have prob mass vectors  $p_u$  and  $p_v$

- Define  $z$ : bernoulli random variable with  $p(1) = \lambda$
- Let  $x = u$  if  $z=1$  and  $x=v$  if  $z=0 \Rightarrow p_x = \lambda p_u + (1-\lambda) p_v$

$$I(x, z; y) = I(x; y) + I(z; y | x) = I(z; y) + I(x; y | z)$$

but  $I(z; y | x) = H(y | x) - H(y | x, z) = 0$  so



$$\begin{aligned} I(x; y) &\geq I(x; y | z) \\ &= \lambda I(x; y | z=1) + (1-\lambda) I(x; y | z=0) \\ &= \lambda I(u; y) + (1-\lambda) I(v; y) \end{aligned}$$

Special Case:  $y=x \Rightarrow I(X; X)=H(X)$  is concave in  $p_X$

# Mutual Information Convex in $p_{Y|X}$

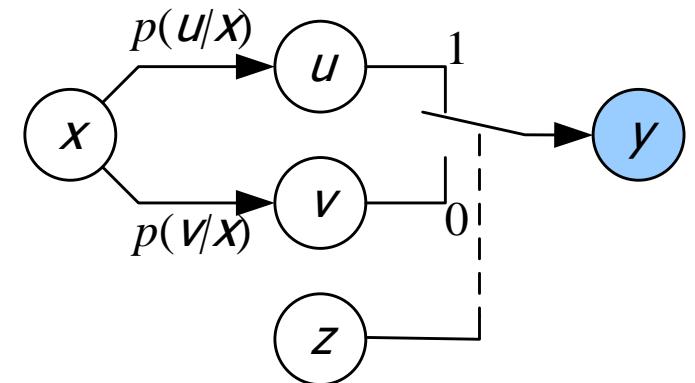
---

Mutual Information  $I(X;Y)$  is **convex** in  $p_{Y|X}$  for fixed  $p_X$

**Proof:** define  $u, v, x$  etc:

- $\mathbf{p}_{Y|X} = \lambda \mathbf{p}_{u|X} + (1 - \lambda) \mathbf{p}_{v|X}$

$$\begin{aligned} I(X;Y,Z) &= I(X;Y|Z) + I(X;Z) \\ &= I(X;Y) + I(X;Z|Y) \end{aligned}$$



but  $I(X;Z)=0$  and  $I(X;Z|Y)\geq 0$  so

$$\begin{aligned} I(X;Y) &\leq I(X;Y|Z) \\ &= \lambda I(X;Y|Z=1) + (1 - \lambda) I(X;Y|Z=0) \\ &= \lambda I(X;U) + (1 - \lambda) I(X;V) \end{aligned}$$

# $n$ -use Channel Capacity

---

For Discrete Memoryless Channel:

$$\begin{aligned}
 I(\mathbf{x}_{1:n}; \mathbf{y}_{1:n}) &= H(\mathbf{y}_{1:n}) - H(\mathbf{y}_{1:n} | \mathbf{x}_{1:n}) \\
 &= \sum_{i=1}^n H(\mathbf{y}_i | \mathbf{y}_{1:i-1}) - \sum_{i=1}^n H(\mathbf{y}_i | \mathbf{x}_i) && \text{Chain; Memoryless} \\
 &\leq \sum_{i=1}^n H(\mathbf{y}_i) - \sum_{i=1}^n H(\mathbf{y}_i | \mathbf{x}_i) = \sum_{i=1}^n I(\mathbf{x}_i; \mathbf{y}_i) && \text{Conditioning Reduces Entropy}
 \end{aligned}$$

with equality if  $y_i$  are independent  $\Rightarrow x_i$  are independent

We can maximize  $I(\mathbf{x}; \mathbf{y})$  by maximizing each  $I(\mathbf{x}_i; \mathbf{y}_i)$  independently and taking  $x_i$  to be i.i.d.

- We will concentrate on maximizing  $I(x; y)$  for a single channel use
- The elements of  $X_i$  are not necessarily i.i.d.

# Capacity of Symmetric Channel

If channel is **weakly symmetric**:

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Q_{1,:}) \leq \log |Y| - H(Q_{1,:})$$

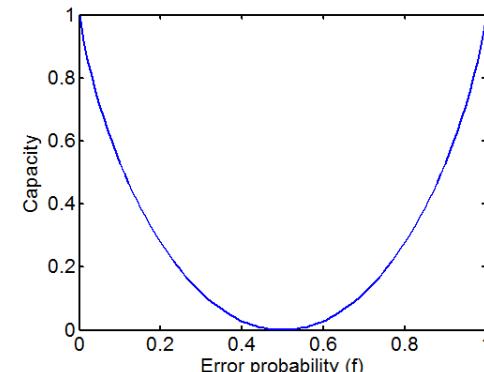
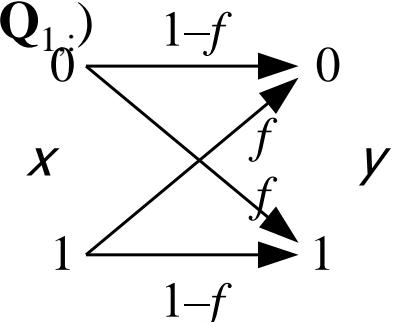
with equality iff input distribution is uniform

∴ Information Capacity of a WS channel is  $C = \log |Y| - H(Q_{1,:})$

For a **binary symmetric channel (BSC)**:

- $|Y| = 2$
- $H(Q_{1,:}) = H(f)$
- $I(X;Y) \leq 1 - H(f)$

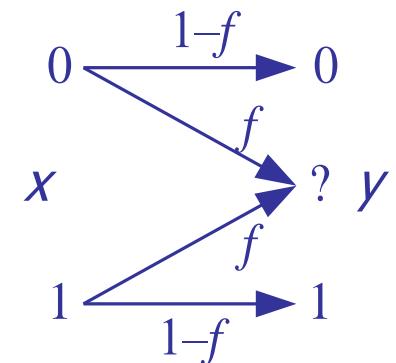
∴ Information Capacity of a BSC is  $1 - H(f)$



# Binary Erasure Channel (BEC)

---

$$\begin{pmatrix} 1-f & f & 0 \\ 0 & f & 1-f \end{pmatrix}$$



$$I(x; y) = H(x) - H(x | y)$$

$$= H(x) - p(y = 0) \times 0 - p(y = ?) H(x) - p(y = 1) \times 0$$

$$= H(x) - H(x)f$$

$H(x|y) = 0$  when  $y=0$  or  $y=1$

$$= (1 - f)H(x)$$

$$\leq 1 - f$$

since max value of  $H(x) = 1$

with equality when  $x$  is uniform

$$C = 1 - f$$

since a fraction  $f$  of the bits are lost, the capacity is only  $1-f$  and this is achieved when  $x$  is uniform

# Asymmetric Channel Capacity

Let  $\mathbf{p}_x = [a \ a \ 1-2a]^T \Rightarrow \mathbf{p}_y = \mathbf{Q}^T \mathbf{p}_x = \mathbf{p}_x$

$$H(y) = -2a \log a - (1-2a) \log (1-2a)$$

$$H(y|x) = 2aH(f) + (1-2a)H(1) = 2aH(f)$$

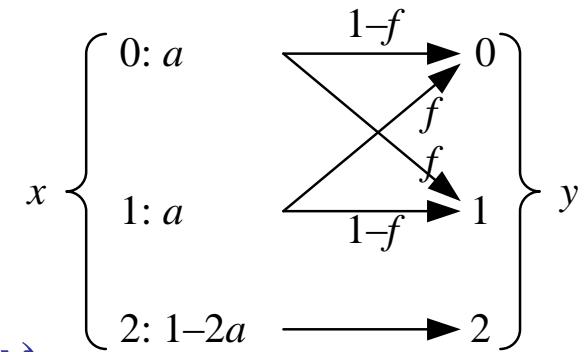
To find  $C$ , maximize  $I(x;y) = H(y) - H(y|x)$

$$I = -2a \log a - (1-2a) \log (1-2a) - 2aH(f)$$

$$\frac{dI}{da} = -2 \log e - 2 \log a + 2 \log e + 2 \log(1-2a) - 2H(f) = 0$$

$$\log \frac{1-2a}{a} = \log(a^{-1} - 2) = H(f) \Rightarrow a = (2 + 2^{H(f)})^{-1}$$

$$\Rightarrow C = -2a \log(a 2^{H(f)}) - (1-2a) \log(1-2a) = -\log(1-2a)$$



$$\mathbf{Q} = \begin{pmatrix} 1-f & f & 0 \\ f & 1-f & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note:

$$d(\log x) = x^{-1} \log e$$

Examples:  $f=0 \Rightarrow H(f)=0 \Rightarrow a=1/3 \Rightarrow C=\log 3 = 1.585$  bits/use  
 $f=1/2 \Rightarrow H(f)=1 \Rightarrow a=1/4 \Rightarrow C=\log 2 = 1$  bits/use

# Summary

---

- Given the channel, mutual information is concave in input distribution
- Channel capacity  $C = \max_{\mathbf{p}_x} I(x; y)$ 
  - The maximum exists and is unique
- DMC capacity
  - Weakly symmetric channel:  $\log|\mathbf{Y}| - H(\mathbf{Q}_{1,:})$
  - BSC:  $1 - H(f)$
  - BEC:  $1 - f$
  - In general it very hard to obtain closed-form;  
numerical method using convex optimization instead

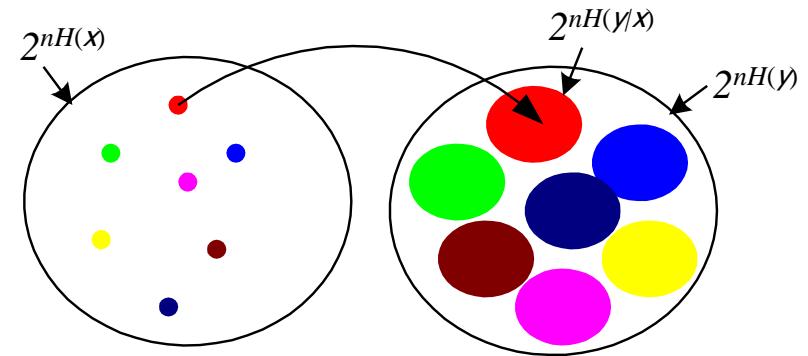
# Lecture 9

---

- Jointly Typical Sets
- Joint AEP
- Channel Coding Theorem
  - Ultimate limit on information transmission is channel capacity
  - The central and most successful story of information theory
  - Random Coding
  - Jointly typical decoding

# Intuition on the Ultimate Limit

- Consider blocks of  $n$  symbols:



- For large  $n$ , an average input sequence  $x_{1:n}$  corresponds to about  $2^{nH(y|x)}$  typical output sequences
- There are a total of  $2^{nH(y)}$  typical output sequences
- For nearly error free transmission, we select a number of input sequences whose corresponding sets of output sequences hardly overlap
- The maximum number of distinct **sets** of output sequences is  $2^{n(H(y)-H(y|x))} = 2^{nI(y;x)}$
- One can send  $I(y;x)$  bits per channel use  
for large  $n$  can transmit at any rate  $< C$  with negligible errors

# Jointly Typical Set

---

$\mathbf{x}, \mathbf{y}$  is the i.i.d. sequence  $\{x_i, y_i\}$  for  $1 \leq i \leq n$

- Prob of a particular sequence is  $p(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^N p(x_i, y_i)$
- $E - \log p(\mathbf{x}, \mathbf{y}) = n E - \log p(x_i, y_i) = nH(x, y)$
- Jointly Typical set:

$$\begin{aligned} J_{\varepsilon}^{(n)} = \left\{ \mathbf{x}, \mathbf{y} \in \mathcal{X} \mathcal{Y}^n : \right. & \left| -n^{-1} \log p(\mathbf{x}) - H(x) \right| < \varepsilon, \\ & \left| -n^{-1} \log p(\mathbf{y}) - H(y) \right| < \varepsilon, \\ & \left. \left| -n^{-1} \log p(\mathbf{x}, \mathbf{y}) - H(x, y) \right| < \varepsilon \right\} \end{aligned}$$

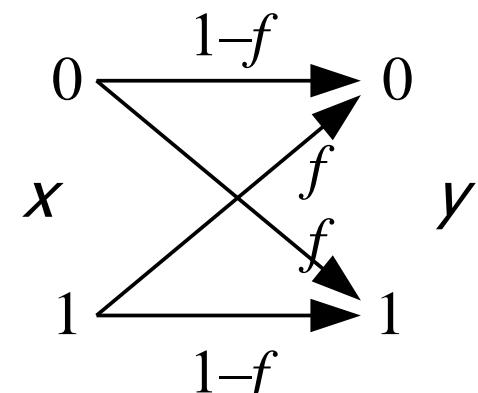
# Jointly Typical Example

---

## Binary Symmetric Channel

$$f = 0.2, \quad \mathbf{p}_x = (0.75 \quad 0.25)^T$$

$$\mathbf{p}_y = (0.65 \quad 0.35)^T, \quad \mathbf{P}_{xy} = \begin{pmatrix} 0.6 & 0.15 \\ 0.05 & 0.2 \end{pmatrix}$$



Jointly typical example (for any  $\varepsilon$ ):

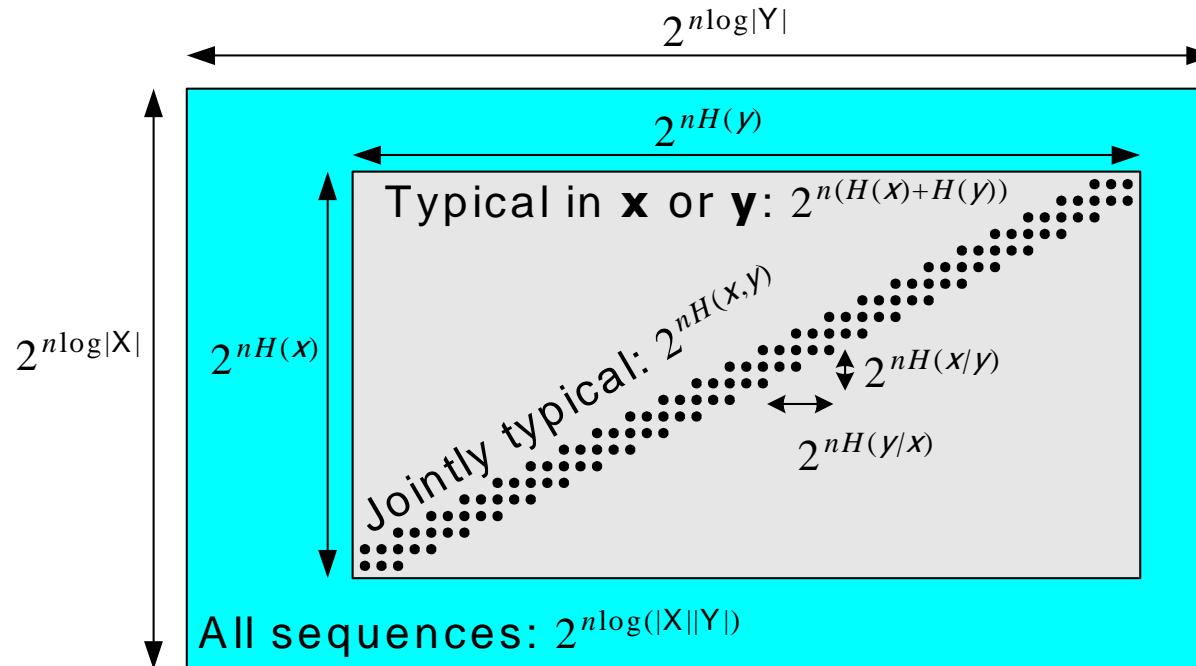
$\mathbf{x} = 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$

$\mathbf{y} = 1 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$

all combinations of  $x$  and  $y$  have exactly the right frequencies

# Jointly Typical Diagram

Each point defines both an **x** sequence and a **y** sequence



Dots represent  
jointly typical  
pairs  $(\mathbf{x}, \mathbf{y})$

Inner rectangle  
represents pairs  
that are typical  
in **x** or **y** but not  
necessarily  
jointly typical

- There are about  $2^{nH(X)}$  typical  $\mathbf{x}$ 's in all
- Each typical  $\mathbf{y}$  is jointly typical with about  $2^{nH(X|Y)}$  of these typical  $\mathbf{x}$ 's
- The jointly typical pairs are a fraction  $2^{-nI(X;Y)}$  of the inner rectangle
- Channel Code: choose  $\mathbf{x}$ 's whose J.T.  $\mathbf{y}$ 's don't overlap; use J.T. for decoding
- There are  $2^{nI(X;Y)}$  such codewords  $\mathbf{x}$ 's

# Joint Typical Set Properties

---

1. Indiv Prob:  $\mathbf{x}, \mathbf{y} \in J_{\varepsilon}^{(n)} \Rightarrow \log p(\mathbf{x}, \mathbf{y}) = -nH(x, y) \pm n\varepsilon$
2. Total Prob:  $p(\mathbf{x}, \mathbf{y} \in J_{\varepsilon}^{(n)}) > 1 - \varepsilon \quad \text{for } n > N_{\varepsilon}$
3. Size:  $(1 - \varepsilon)2^{n(H(x, y) - \varepsilon)} < |J_{\varepsilon}^{(n)}| \leq 2^{n(H(x, y) + \varepsilon)}$

**Proof 2:** (use weak law of large numbers)

Choose  $N_1$  such that  $\forall n > N_1, \quad p\left(\left|-n^{-1} \log p(\mathbf{x}) - H(x)\right| > \varepsilon\right) < \frac{\varepsilon}{3}$

Similarly choose  $N_2, N_3$  for other conditions and set  $N_{\varepsilon} = \max(N_1, N_2, N_3)$

**Proof 3:**  $1 - \varepsilon < \sum_{\mathbf{x}, \mathbf{y} \in J_{\varepsilon}^{(n)}} p(\mathbf{x}, \mathbf{y}) \leq |J_{\varepsilon}^{(n)}| \max_{\mathbf{x}, \mathbf{y} \in J_{\varepsilon}^{(n)}} p(\mathbf{x}, \mathbf{y}) = |J_{\varepsilon}^{(n)}| 2^{-n(H(x, y) - \varepsilon)} \quad n > N_{\varepsilon}$

$$1 \geq \sum_{\mathbf{x}, \mathbf{y} \in J_{\varepsilon}^{(n)}} p(\mathbf{x}, \mathbf{y}) \geq |J_{\varepsilon}^{(n)}| \min_{\mathbf{x}, \mathbf{y} \in J_{\varepsilon}^{(n)}} p(\mathbf{x}, \mathbf{y}) = |J_{\varepsilon}^{(n)}| 2^{-n(H(x, y) + \varepsilon)} \quad \forall n$$

# Properties

---

4. If  $p_x = p_{x'}$  and  $p_y = p_{y'}$  with  $x'$  and  $y'$  independent:

$$(1 - \varepsilon)2^{-n(I(x,y)+3\varepsilon)} \leq p(\mathbf{x}', \mathbf{y}' \in J_\varepsilon^{(n)}) \leq 2^{-n(I(x,y)-3\varepsilon)} \text{ for } n > N_\varepsilon$$

**Proof:**  $|J| \times (\text{Min Prob}) \leq \text{Total Prob} \leq |J| \times (\text{Max Prob})$

$$p(\mathbf{x}', \mathbf{y}' \in J_\varepsilon^{(n)}) = \sum_{\mathbf{x}', \mathbf{y}' \in J_\varepsilon^{(n)}} p(\mathbf{x}', \mathbf{y}') = \sum_{\mathbf{x}', \mathbf{y}' \in J_\varepsilon^{(n)}} p(\mathbf{x}') p(\mathbf{y}')$$

$$\begin{aligned} p(\mathbf{x}', \mathbf{y}' \in J_\varepsilon^{(n)}) &\leq |J_\varepsilon^{(n)}| \max_{\mathbf{x}', \mathbf{y}' \in J_\varepsilon^{(n)}} p(\mathbf{x}') p(\mathbf{y}') \\ &\leq 2^{n(H(x,y)+\varepsilon)} 2^{-n(H(x)-\varepsilon)} 2^{-n(H(y)-\varepsilon)} = 2^{-n(I(x,y)-3\varepsilon)} \end{aligned}$$

$$\begin{aligned} p(\mathbf{x}', \mathbf{y}' \in J_\varepsilon^{(n)}) &\geq |J_\varepsilon^{(n)}| \min_{\mathbf{x}', \mathbf{y}' \in J_\varepsilon^{(n)}} p(\mathbf{x}') p(\mathbf{y}') \\ &\geq (1 - \varepsilon) 2^{-n(I(x,y)+3\varepsilon)} \text{ for } n > N_\varepsilon \end{aligned}$$

# Channel Coding

---



- Assume Discrete Memoryless Channel with known  $\mathbf{Q}_{y|x}$
  - An  $(M, n)$  code is
    - A fixed set of  $M$  codewords  $\mathbf{x}(w) \in \mathcal{X}^n$  for  $w=1:M$
    - A deterministic decoder  $g(\mathbf{y}) \in 1:M$
  - The rate of an  $(M,n)$  code:  $R=(\log M)/n$  bits/transmission
  - Error probability  $\lambda_w = p(g(\mathbf{y}(w)) \neq w) = \sum_{\mathbf{y} \in \mathcal{Y}^n} p(\mathbf{y} | \mathbf{x}(w)) \delta_{g(\mathbf{y}) \neq w}$ 
    - Maximum Error Probability  $\lambda^{(n)} = \max_{1 \leq w \leq M} \lambda_w$
    - Average Error probability  $P_e^{(n)} = \frac{1}{M} \sum_{w=1}^M \lambda_w$
- $\delta_C = 1$  if  $C$  is true or 0 if it is false

# Shannon's ideas

---

- Channel coding theorem: the basic theorem of information theory
  - Proved in his original 1948 paper
- How do you correct all errors?
- Shannon's ideas
  - Allowing arbitrarily small but nonzero error probability
  - Using the channel many times in succession so that AEP holds
  - Consider a **randomly** chosen code and show the expected **average** error probability is small
    - Use the idea of typical sequences
    - Show this means  $\exists$  at least one code with small max error prob
    - Sadly it doesn't tell you how to construct the code

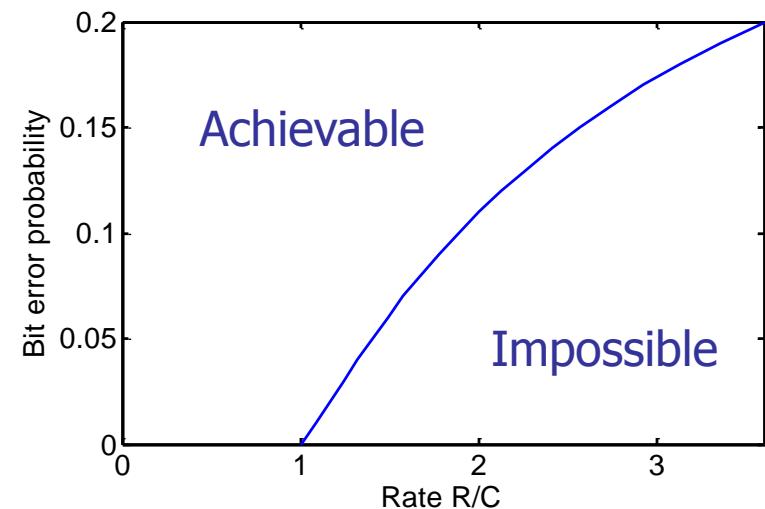
# Channel Coding Theorem

---

- A rate  $R$  is achievable if  $R < C$  and not achievable if  $R > C$ 
  - If  $R < C$ ,  $\exists$  a sequence of  $(2^{nR}, n)$  codes with max prob of error  $\lambda^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$
  - Any sequence of  $(2^{nR}, n)$  codes with max prob of error  $\lambda^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$  must have  $R \leq C$

A very counterintuitive result:

Despite channel errors you can get arbitrarily low bit error rates provided that  $R < C$



# Summary

---

- Jointly typical set

$$-\log p(\mathbf{x}, \mathbf{y}) = nH(X, Y) \pm n\varepsilon$$

$$p\left(\mathbf{x}, \mathbf{y} \in J_{\varepsilon}^{(n)}\right) > 1 - \varepsilon$$

$$\left|J_{\varepsilon}^{(n)}\right| \leq 2^{n(H(X, Y) + \varepsilon)}$$

$$(1 - \varepsilon)2^{-n(I(X, Y) + 3\varepsilon)} \leq p\left(\mathbf{x}', \mathbf{y}' \in J_{\varepsilon}^{(n)}\right) \leq 2^{-n(I(X, Y) - 3\varepsilon)}$$

- Machinery to prove channel coding theorem

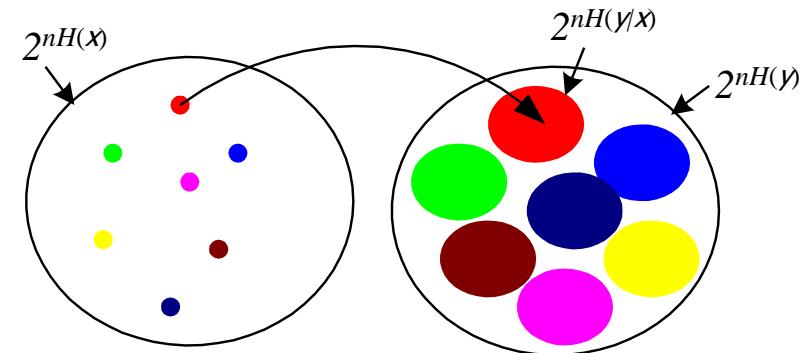
# Lecture 10

---

- Channel Coding Theorem
  - Proof
    - Using joint typicality
    - Arguably the simplest one among many possible ways
    - Limitation: does not reveal  $P_e \sim e^{-nE(R)}$
  - Converse (next lecture)

# Channel Coding Principle

- Consider blocks of  $n$  symbols:



- An average input sequence  $x_{1:n}$  corresponds to about  $2^{nH(y|x)}$  typical output sequences
- Random Codes:** Choose  $M = 2^{nR}$  ( $R \leq I(x,y)$ ) random codewords  $\mathbf{x}(w)$ 
  - their typical output sequences are unlikely to overlap much.
- Joint Typical Decoding:** A received vector  $\mathbf{y}$  is very likely to be in the typical output set of the transmitted  $\mathbf{x}(w)$  and no others. Decode as this  $w$ .

Channel Coding Theorem: for large  $n$ , can transmit at any rate  $R < C$  with negligible errors

# Random $(2^{nR}, n)$ Code

---

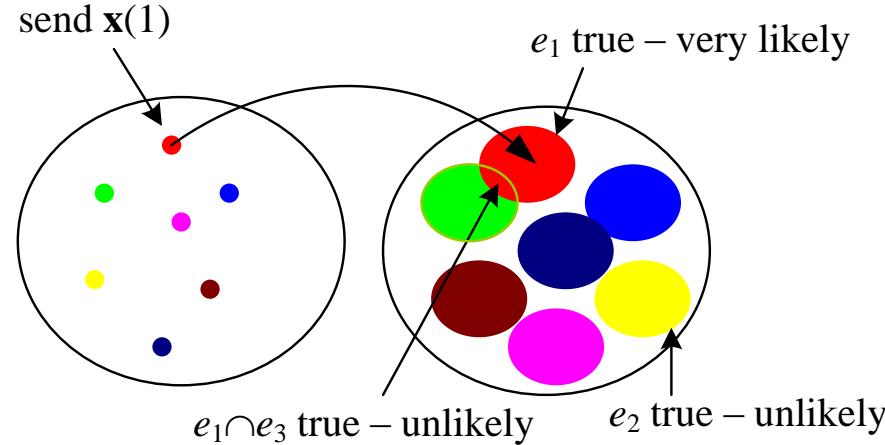
- Choose  $\varepsilon \approx$  error prob, joint typicality  $\Rightarrow N_\varepsilon$ , choose  $n > N_\varepsilon$
- Choose  $\mathbf{p}_x$  so that  $I(x; y) = C$ , the information capacity
- Use  $\mathbf{p}_x$  to choose a code  $\mathbf{C}$  with random  $\mathbf{x}(w) \in \mathcal{X}^n$ ,  $w=1:2^{nR}$ 
  - the receiver knows this code and also the transition matrix  $\mathbf{Q}$
- Assume the message  $W \in 1:2^{nR}$  is uniformly distributed
- If received value is  $y$ ; decode the message by seeing how many  $\mathbf{x}(w)$ 's are jointly typical with  $y$ 
  - if  $\mathbf{x}(k)$  is the only one then  $k$  is the decoded message
  - if there are 0 or  $\geq 2$  possible  $k$ 's then declare an error message 0
  - we calculate error probability averaged over all  $\mathbf{C}$  and all  $W$

$$p(E) = \sum_C p(C) 2^{-nR} \sum_{w=1}^{2^{nR}} \lambda_w(C) = 2^{-nR} \sum_{w=1}^{2^{nR}} \sum_C p(C) \lambda_w(C) \stackrel{(a)}{=} \sum_C p(C) \lambda_1(C) = p(E | w=1)$$

(a) since error averaged over all possible codes is independent of  $w$

# Decoding Errors

- Assume we transmit  $\mathbf{x}(1)$  and receive  $\mathbf{y}$
- Define the J.T. events  $e_w = \{(\mathbf{x}(w), \mathbf{y}) \in J_\varepsilon^{(n)}\}$  for  $w \in 1 : 2^{nR}$



- Decode using joint typicality
- We have an error if either  $e_1$  false or  $e_w$  true for  $w \geq 2$
- The  $\mathbf{x}(w)$  for  $w \neq 1$  are independent of  $\mathbf{x}(1)$  and hence also independent of  $\mathbf{y}$ . So  $p(e_w \text{ true}) < 2^{-n(I(x,y)-3\varepsilon)}$  for any  $w \neq 1$

Joint AEP

# Error Probability for Random Code

---

- Upper bound

$$p(A \cup B) \leq p(A) + p(B)$$

$$\begin{aligned}
 p(E) &= p(E | W = 1) = p(\overline{e_1} \cup e_2 \cup e_3 \cup \dots \cup e_{2^{nR}}) \leq p(\overline{e_1}) + \sum_{w=2}^{2^{nR}} p(e_w) \\
 &\leq \varepsilon + \sum_{i=2}^{2^{nR}} 2^{-n(I(x;Y)-3\varepsilon)} < \varepsilon + 2^{nR} 2^{-n(I(x;Y)-3\varepsilon)} \quad (1) \text{ Joint typicality} \\
 &\leq \varepsilon + 2^{-n(C-R-3\varepsilon)} \leq 2\varepsilon \quad \text{for } R < C - 3\varepsilon \text{ and } n > -\frac{\log \varepsilon}{C - R - 3\varepsilon} \quad (2) \text{ Joint AEP}
 \end{aligned}$$

we have chosen  $p(x)$  such that  $I(x; Y) = C$

- Since average of  $P(E)$  over all codes is  $\leq 2\varepsilon$  there must be at least one code for which this is true: this code has  $2^{-nR} \sum_w \lambda_w \leq 2\varepsilon$
- Now throw away the worst half of the codewords; the remaining ones must all have  $\lambda_w \leq 4\varepsilon$ . The resultant code has rate  $R - n^{-1} \cong R$ .

◆ = proved on next page

# Code Selection & Expurgation

---

- Since average of  $P(E)$  over all codes is  $\leq 2\epsilon$  there must be at least one code for which this is true.

**Proof:**

$$2\epsilon \geq K^{-1} \sum_{i=1}^K P_{e,i}^{(n)} \geq K^{-1} \sum_{i=1}^K \min_i \left( P_{e,i}^{(n)} \right) = \min_i \left( P_{e,i}^{(n)} \right)$$

$K$  = num of codes

- Expurgation: Throw away the worst half of the codewords; the remaining ones must all have  $\lambda_w \leq 4\epsilon$ .

**Proof:** Assume  $\lambda_w$  are in descending order

$$\begin{aligned} 2\epsilon &\geq M^{-1} \sum_{w=1}^M \lambda_w \geq M^{-1} \sum_{w=1}^{\frac{1}{2}M} \lambda_w \geq M^{-1} \sum_{w=1}^{\frac{1}{2}M} \lambda_{\frac{1}{2}M} \geq \frac{1}{2} \lambda_{\frac{1}{2}M} \\ \Rightarrow \quad \lambda_{\frac{1}{2}M} &\leq 4\epsilon \quad \Rightarrow \quad \lambda_w \leq 4\epsilon \quad \forall w > \frac{1}{2}M \end{aligned}$$

$$M' = \frac{1}{2} \times 2^{nR} \text{ messages in } n \text{ channel uses} \Rightarrow R' = n^{-1} \log M' = R - n^{-1}$$

# Summary of Procedure

---

- For any  $R < C - 3\epsilon$  set  $n = \max \left\{ N_\epsilon, -(\log \epsilon) / (C - R - 3\epsilon), \epsilon^{-1} \right\}$   
see (a),(b),(c) below
- Find the optimum  $p_X$  so that  $I(X; Y) = C$
- Choosing codewords randomly (using  $p_X$ ) to construct codes with  $2^{nR}$  (a) codewords and using joint typicality as the decoder
- Since average of  $P(E)$  over all codes is  $\leq 2\epsilon$  there must be at least (b) one code for which this is true.
- Throw away the worst half of the codewords. Now the worst (c) codeword has an error prob  $\leq 4\epsilon$  with rate  $= R - n^{-1} > R - \epsilon$
- The resultant code transmits at a rate as close to  $C$  as desired with an error probability that can be made as small as desired (but  $n$  unnecessarily large).

Note:  $\epsilon$  determines both error probability and closeness to capacity

# Remarks

---

- Random coding is a powerful method of proof, not a method of signaling
- Picking randomly will give a good code
- But  $n$  has to be large (AEP)
- Without a structure, it is difficult to encode/decode
  - Table lookup requires exponential size
- Channel coding theorem does not provide a practical coding scheme
- Folk theorem (but outdated now):
  - Almost all codes are good, except those we can think of

# Lecture 11

---

- Converse of Channel Coding Theorem
  - Cannot achieve  $R > C$
- Capacity with feedback
  - No gain for DMC but simpler encoding/decoding
- Joint Source-Channel Coding
  - No point for a DMC



# Converse of Coding Theorem

- Fano's Inequality: if  $P_e^{(n)}$  is error prob when estimating  $w$  from  $\mathbf{y}$ ,

$$H(w | \mathbf{y}) \leq 1 + P_e^{(n)} \log |W| = 1 + nR P_e^{(n)}$$

- Hence  $nR = H(w) = H(w | \mathbf{y}) + I(w; \mathbf{y})$

Definition of  $I$

$$\leq H(w | \mathbf{y}) + I(\mathbf{x}(w); \mathbf{y})$$

Markov :  $w \rightarrow \mathbf{x} \rightarrow \mathbf{y} \rightarrow \hat{w}$

$$\leq 1 + nR P_e^{(n)} + I(\mathbf{x}; \mathbf{y})$$

Fano

$$\leq 1 + nR P_e^{(n)} + nC$$

$n$ -use DMC capacity

$$\Rightarrow P_e^{(n)} \geq \frac{R - C - n^{-1}}{R} \xrightarrow[n \rightarrow \infty]{} 1 - \frac{C}{R} > 0 \text{ if } R > C$$

- For large (hence for all)  $n$ ,  $P_e^{(n)}$  has a lower bound of  $(R-C)/R$  if  $w$  equiprobable

- If achievable for small  $n$ , it could be achieved also for large  $n$  by concatenation.



# Minimum Bit-Error Rate



Suppose

- $w_{1:nR}$  is i.i.d. bits with  $H(w_i) = 1$
- The bit-error rate is  $P_b = E_i \left\{ p(w_i \neq \hat{w}_i) \right\} \stackrel{\Delta}{=} E_i \left\{ p(e_i) \right\}$

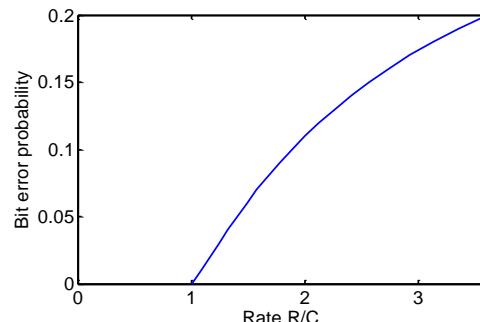
Then

$$\begin{aligned}
 nC &\stackrel{(a)}{\geq} I(X_{1:n}; Y_{1:n}) \stackrel{(b)}{\geq} I(w_{1:nR}; \hat{w}_{1:nR}) = H(w_{1:nR}) - H(w_{1:nR} | \hat{w}_{1:nR}) \\
 &= nR - \sum_{i=1}^{nR} H(w_i | \hat{w}_{1:nR}, w_{1:i-1}) \stackrel{(c)}{\geq} nR - \sum_{i=1}^{nR} H(w_i | \hat{w}_i) = nR \left( 1 - E_i \left\{ H(w_i | \hat{w}_i) \right\} \right) \\
 &\stackrel{(d)}{=} nR \left( 1 - E_i \left\{ H(e_i | \hat{w}_i) \right\} \right) \stackrel{(e)}{\geq} nR \left( 1 - E_i \left\{ H(e_i) \right\} \right) \geq nR \left( 1 - H(E_i P(e_i)) \right) = nR (1 - H(P_b))
 \end{aligned}$$

Hence

$$R \leq C(1 - H(P_b))^{-1}$$

$$P_b \geq H^{-1}(1 - C/R)$$



- (a) n-use capacity
- (b) Data processing theorem
- (c) Conditioning reduces entropy
- (d)  $e_i = w_i \oplus \hat{w}_i$
- (e) Jensen:  $E H(x) \leq H(E x)$

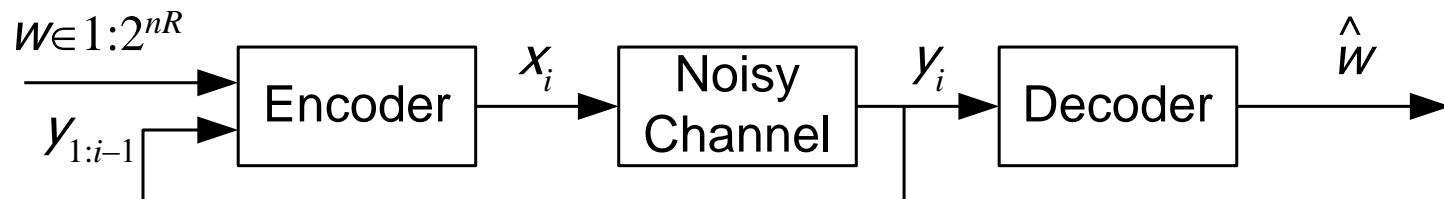
# Coding Theory and Practice

---

- Construction for good codes
  - Ever since Shannon founded information theory
  - **Practical**: Computation & memory  $\propto n^k$  for some  $k$
- Repetition code: rate  $\rightarrow 0$
- Block codes: encode a block at a time
  - Hamming code: correct one error
  - Reed-Solomon code, BCH code: multiple errors (1950s)
- Convolutional code: convolve bit stream with a filter
- Concatenated code: RS + convolutional
- Capacity-approaching codes:
  - Turbo code: combination of two interleaved convolutional codes (1993)
  - Low-density parity-check (LDPC) code (1960)
  - **Dream has come true for some channels today**

# Channel with Feedback

---



- Assume error-free feedback: does it increase capacity ?
- A  $(2^{nR}, n)$  feedback code is
  - A sequence of mappings  $x_i = x_i(w, y_{1:i-1})$  for  $i=1:n$
  - A decoding function  $\hat{w} = g(y_{1:n})$
- A rate R is **achievable** if  $\exists$  a sequence of  $(2^{nR}, n)$  feedback codes such that  $P_e^{(n)} = P(\hat{w} \neq w) \xrightarrow{n \rightarrow \infty} 0$
- Feedback capacity,  $C_{FB} \geq C$ , is the sup of achievable rates

# Feedback Doesn't Increase Capacity

$$\begin{aligned}
 I(W; \mathbf{y}) &= H(\mathbf{y}) - H(\mathbf{y} | W) \\
 &= H(\mathbf{y}) - \sum_{i=1}^n H(y_i | y_{1:i-1}, W) \\
 &= H(\mathbf{y}) - \sum_{i=1}^n H(y_i | y_{1:i-1}, W, x_i) \\
 &= H(\mathbf{y}) - \sum_{i=1}^n H(y_i | x_i) \\
 &\leq \sum_{i=1}^n H(y_i) - \sum_{i=1}^n H(y_i | x_i) = \sum_{i=1}^n I(x_i; y_i) \leq nC
 \end{aligned}$$



since  $x_i = x_i(w, y_{1:i-1})$

since  $y_i$  only directly depends on  $x_i$

cond reduces ent  
DMC

Hence

$$nR = H(W) = H(W | \mathbf{y}) + I(W; \mathbf{y}) \leq 1 + nRP_e^{(n)} + nC \quad \text{Fano}$$

$$\Rightarrow P_e^{(n)} \geq \frac{R - C - n^{-1}}{R} \quad \Rightarrow \text{Any rate } > C \text{ is unachievable}$$

The DMC does not benefit from feedback:  $C_{FB} = C$

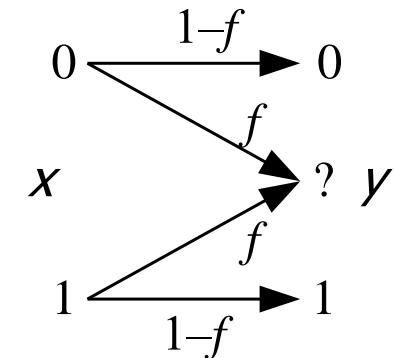
# Example: BEC with feedback

---

- Capacity is  $1 - f$
- Encode algorithm
  - If  $y_i = ?$ , tell the sender to retransmit bit  $i$
  - Average number of transmissions per bit:

$$1 + f + f^2 + \dots = \frac{1}{1 - f}$$

- Average number of successfully recovered bits per transmission =  $1 - f$ 
  - Capacity is achieved!
- Capacity unchanged but encoding/decoding algorithm much simpler.



# Joint Source-Channel Coding



- Assume  $w_i$  satisfies AEP and  $|W| < \infty$ 
  - Examples: i.i.d.; Markov; stationary ergodic
- Capacity of DMC channel is  $C$ 
  - if time-varying:  $C = \lim_{n \rightarrow \infty} n^{-1} I(\mathbf{x}; \mathbf{y})$
- Joint Source-Channel Coding Theorem:
 
$$\exists \text{ codes with } P_e^{(n)} = P(\hat{w}_{1:n} \neq w_{1:n}) \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{iff } H(W) < C$$
  - errors arise from two reasons
    - Incorrect encoding of  $\mathbf{w}$
    - Incorrect decoding of  $\mathbf{y}$

◆ = proved on next page

# Source-Channel Proof ( $\Leftarrow$ )

---

- Achievability is proved by using two-stage encoding
  - Source coding
  - Channel coding
- For  $n > N_\varepsilon$  there are only  $2^{n(H(W)+\varepsilon)}$  **w's** in the typical set: encode using  $n(H(W)+\varepsilon)$  bits
  - encoder error  $< \varepsilon$
- Transmit with error prob less than  $\varepsilon$  so long as  $H(W)+\varepsilon < C$
- Total error prob  $< 2\varepsilon$

# Source-Channel Proof ( $\Rightarrow$ )

---



**Fano's Inequality:**  $H(\mathbf{w} | \hat{\mathbf{w}}) \leq 1 + P_e^{(n)} n \log |\mathcal{W}|$

$$\begin{aligned}
 H(W) &\leq n^{-1} H(W_{1:n}) && \text{entropy rate of stationary process} \\
 &= n^{-1} H(W_{1:n} | \hat{W}_{1:n}) + n^{-1} I(W_{1:n}; \hat{W}_{1:n}) && \text{definition of } I \\
 &\leq n^{-1} (1 + P_e^{(n)} n \log |\mathcal{W}|) + n^{-1} I(X_{1:n}; Y_{1:n}) && \text{Fano + Data Proc Inequ} \\
 &\leq n^{-1} + P_e^{(n)} \log |\mathcal{W}| + C && \text{Memoryless channel}
 \end{aligned}$$

Let  $n \rightarrow \infty \Rightarrow P_e^{(n)} \rightarrow 0 \Rightarrow H(W) \leq C$

# Separation Theorem

---

- **Important result:** source coding and channel coding might as well be done separately since same capacity
  - Joint design is more difficult
- Practical implication: for a **DMC** we can design the source encoder and the channel coder separately
  - Source coding: efficient compression
  - Channel coding: powerful error-correction codes
- Not necessarily true for
  - Correlated channels
  - Multiuser channels
- Joint source-channel coding: still an area of research
  - Redundancy in human languages helps in a noisy environment

# Summary

---

- Converse to channel coding theorem
  - Proved using Fano's inequality
  - Capacity is a clear dividing point:
    - If  $R < C$ , error prob.  $\rightarrow 0$
    - Otherwise, error prob.  $\rightarrow 1$
- Feedback doesn't increase the capacity of DMC
  - May increase the capacity of memory channels (e.g., ARQ in TCP/IP)
- Source-channel separation theorem for DMC and stationary sources

# Lecture 12

---

- Polar codes
  - Channel polarization
  - How to construct polar codes
  - Encoding and decoding
- Polar source coding
- Extension

# About Polar Codes

---

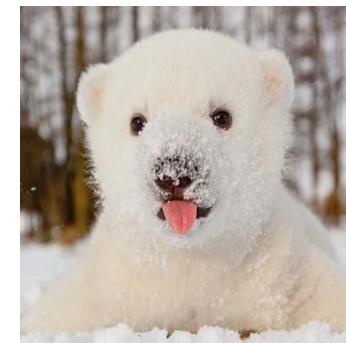
- Provably capacity-achieving
- Encoding complexity  $O(N \log N)$
- Successive decoding complexity  
 $O(N \log N)$
- Probability of error  $\approx 2^{-\sqrt{N}}$
- Main idea: channel polarization

# What Is Channel Polarization?

- Normal channel
- Extreme channel



sometimes cute,  
sometimes lazy,  
hard to manage



Useless  
channel

Perfect  
channel

# Channel Polarization

---

- Among all channels, there are two classes which are easy to communicate optimally
  - The perfect channels  
the output  $Y$  determines the input  $X$
  - The useless channels  
 $Y$  is independent of  $X$
- Polarization is a technique to convert noisy channels to a mixture of **extreme** channels
  - The process is **information-conserving**

# Generator Matrix

---

- Generator Matrix

$$\mathbf{F}_N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes n}, N = 2^n$$

$\otimes n$  denotes the  $n$ -fold Kronecker product.

- Example

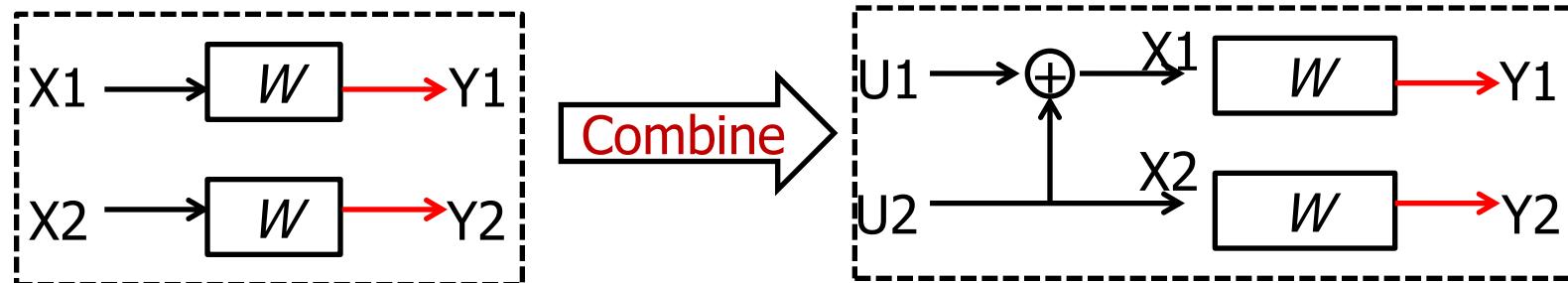
$$\mathbf{F}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{F}_4 = \begin{bmatrix} \mathbf{F}_2 & 0 \\ \mathbf{F}_2 & \mathbf{F}_2 \end{bmatrix} \text{ and so on.}$$

- Encoding

Let  $\mathbf{u}$  be the length- $N$  input to the encoder, then  $\mathbf{x} = \mathbf{u}\mathbf{F}_N$  is the codeword.

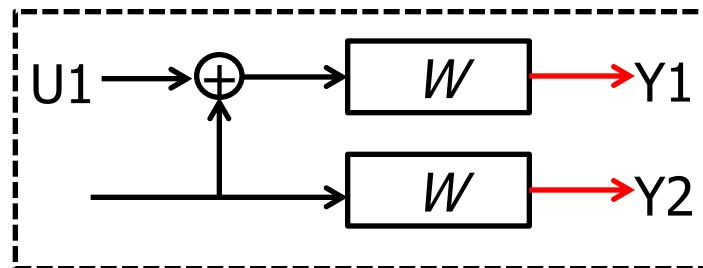
# Channel Combining and Splitting

- Basic operation ( $N = 2$ )

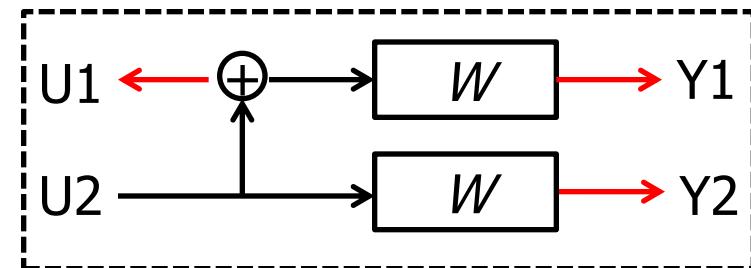


Channel splitting

$$(X_1, X_2) = (U_1, U_2) F_2$$



$$W^-: U_1 \rightarrow (Y_1, Y_2)$$



$$W^+: U_2 \rightarrow (Y_1, Y_2, U_1)$$

# What Happens?

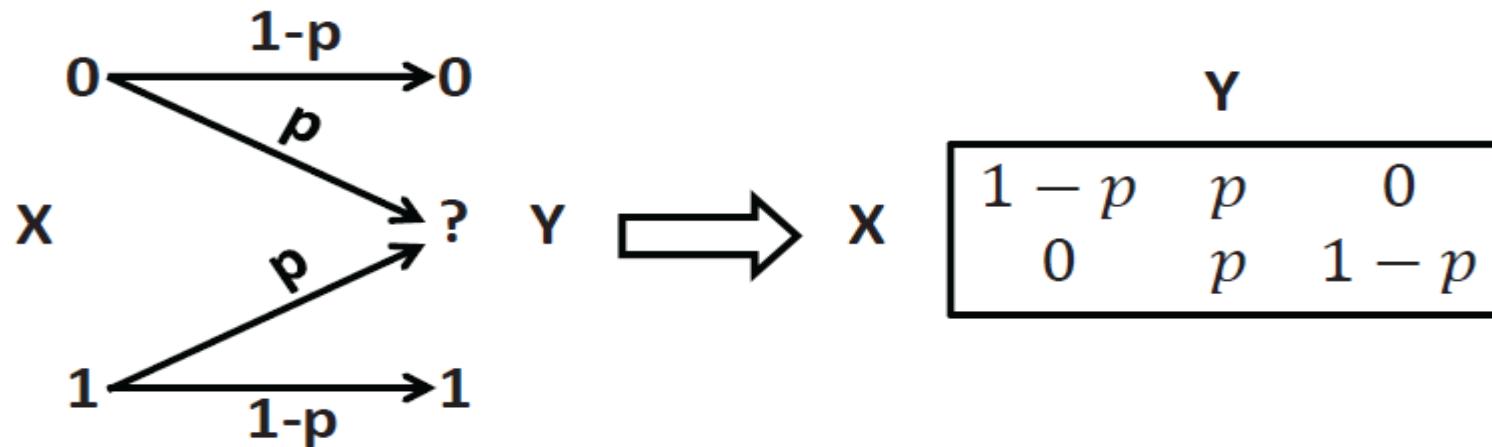
---

- Suppose  $W$  is a BEC( $p$ ), i.e.,  $Y=X$  with probability  $1-p$ , and  $Y=?$  (erasure) with probability  $p$ .
  - $W^-$  has input  $U_1$  and output  $(Y_1, Y_2)=(U_1+U_2, U_2)$  or  $(?, U_2)$  or  $(U_1+U_2, ?)$  or  $(?, ?)$ .
  - $W^-$  is a BEC( $2p - p^2$ )
  - $W^+$  has input  $U_2$  and output  $(Y_1, Y_2, U_1)=(U_1+U_2, U_2, U_1)$  or  $(?, U_2, U_1)$  or  $(U_1+U_2, ?, U_1)$  or  $(?, ?, U_1)$ .
  - $W^+$  is a BEC( $p^2$ )
- $W^-$  is **worse** than  $W$ , and  $W^+$  is **better** (recall capacity  $C(W)=1-p$ ).
  - $C(W^-) + C(W^+) = 2C(W)$
  - $C(W^-) \leq C(W) \leq C(W^+)$

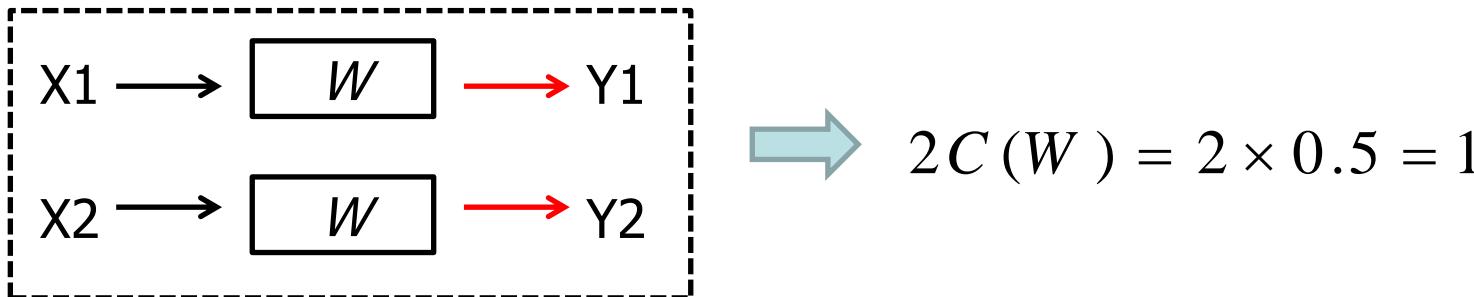
# Example: BEC(0.5)

---

- $W$  is a BEC with erasure probability  $p = 0.5$ .

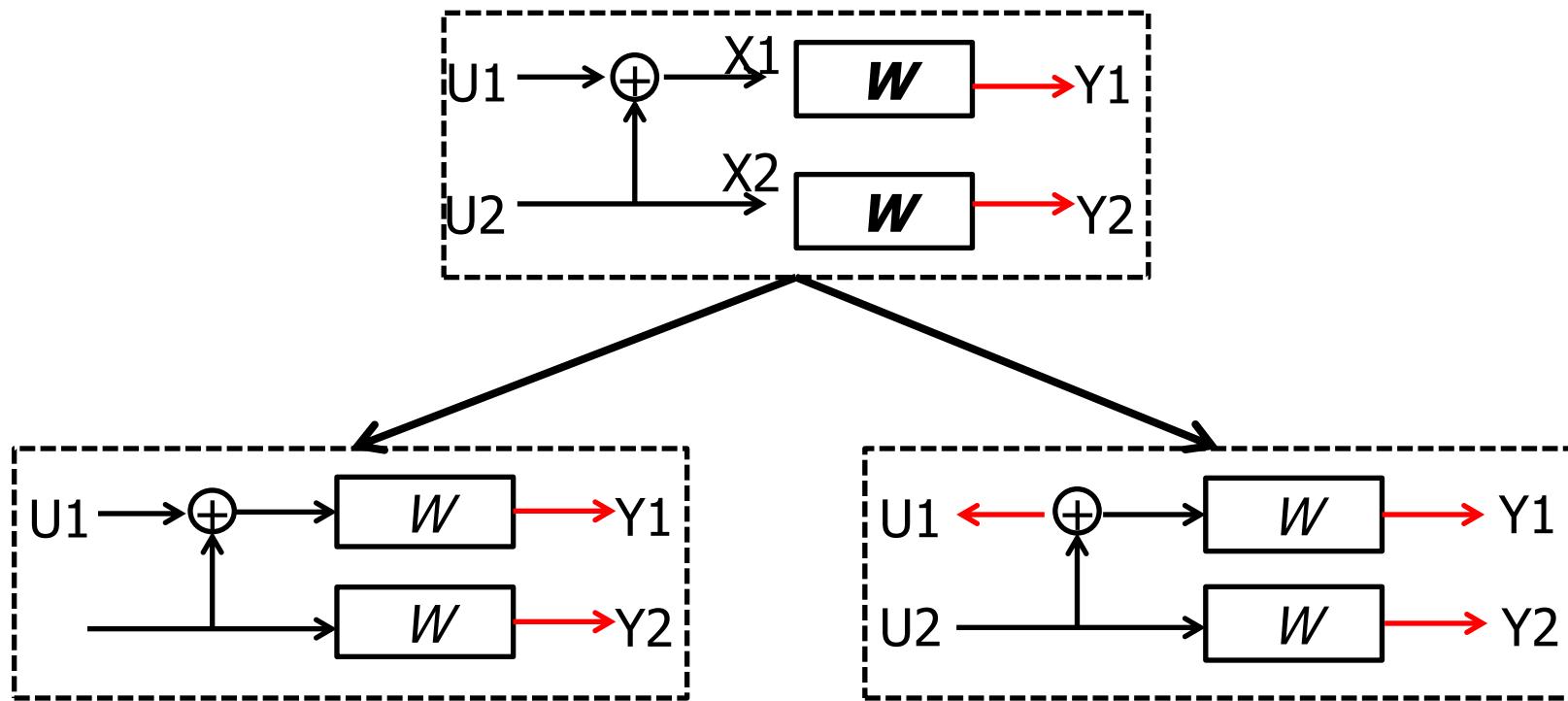


- If we use two copies of  $W$  separately



# Example: BEC(0.5)

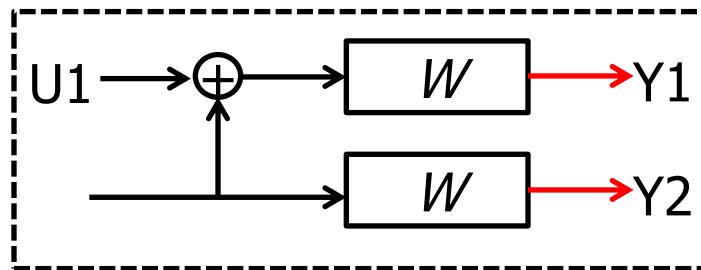
- Channel combining and splitting



# Example: BEC(0.5)

---

- Channel  $W^-$



$$C(W^-) = 4 \times \frac{1}{16} \log 2 = 0.25$$

(Y1,Y2)

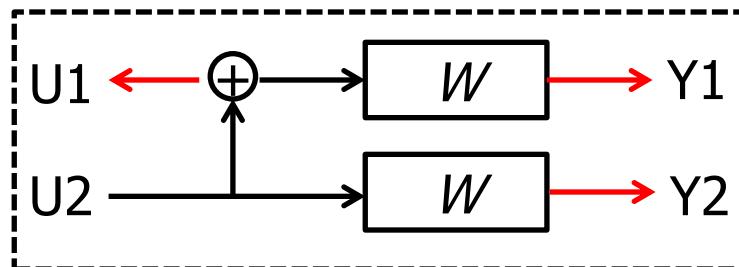
Transitional probabilities

	00	0?	01	?0	??	?1	10	1?	11	
U1	0	1/8	1/8	0	1/8	1/4	1/8	0	1/8	1/8
	1	0	1/8	1/8	1/8	1/4	1/8	1/8	1/8	0



# Example: BEC(0.5)

- Channel  $W^+$



$$C(W^+) = 12 \times \frac{1}{16} \log 2 = 0.75$$

$$C(W^-) + C(W^+) = 2C(W)$$

$$C(W^-) < C(W) < C(W^+)$$

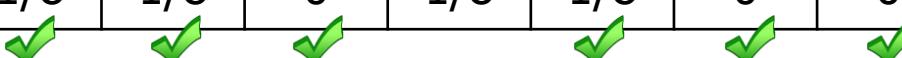
$(Y_1, Y_2, U_1)$

Transitional probabilities

	000	0?0	010	?00	?0?	?10	100	1?0	110
0	1/8	1/8	0	1/8	1/8	0	0	0	0
1	0	0	0	0	1/8	1/8	0	1/8	1/8

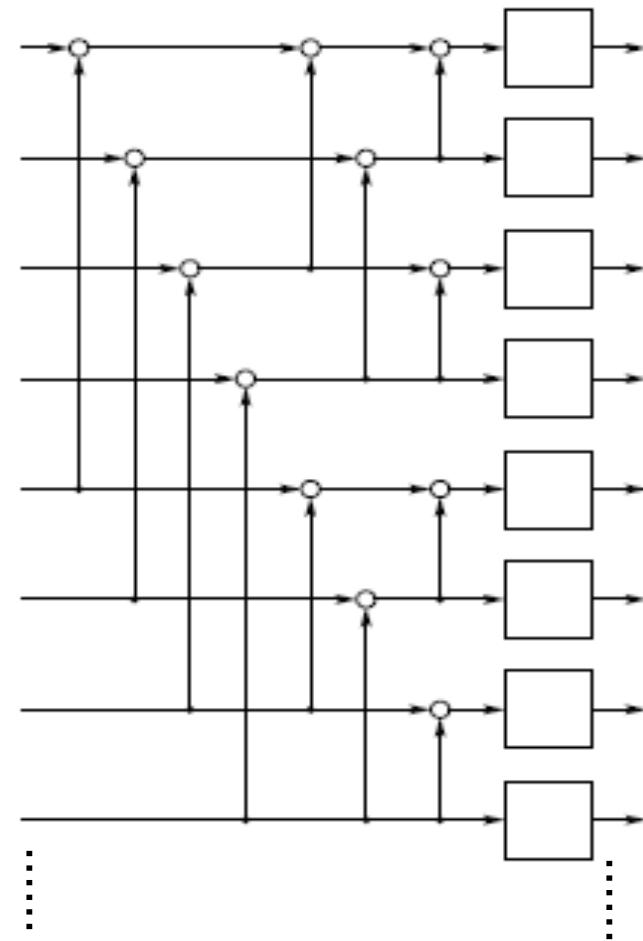
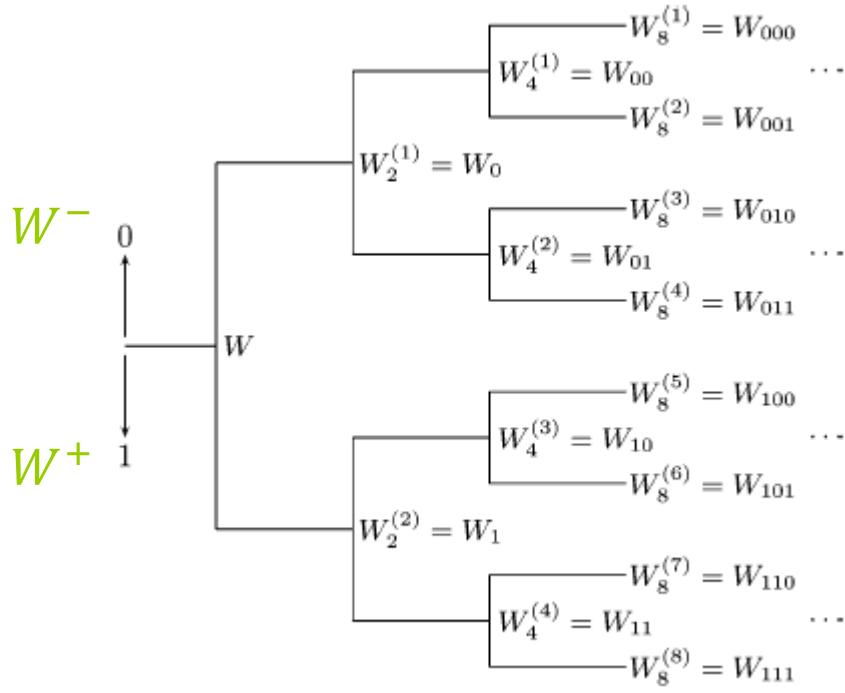
$U_2$

	001	0?1	011	?01	?11	?11	101	1?1	111
0	0	0	0	1/8	1/8	0	1/8	1/8	0
1	0	1/8	1/8	0	1/8	1/8	0	0	0



# More Polarization

- Repeating this, we obtain  $N$  'bit channels' at the  $n$ -th step.
- More conveniently, this process can be described as a binary tree.  
–Note how the 'bit channels'  $W_{b_1 b_2 \dots b_n}$  are labelled in the tree.



# Martingale

---

- Now pick a ‘bit channel’ uniformly at random on the  $n$ -th level of the tree, which is equivalent to a random traverse on the tree, namely, at each step the r.v  $b_i$  takes the value of 0 or 1 with equal probability.
- We claim capacity  $C_n$  at the  $n$ -th step is a **martingale**.
- **Proof:** By information-preserving

$$\begin{aligned} E[C_{n+1}|b_1, \dots, b_n] &= \frac{1}{2} C(W_{b_1 b_2 \dots b_n 0}) + \frac{1}{2} C(W_{b_1 b_2 \dots b_n 1}) \\ &= C(W_{b_1 b_2 \dots b_n}) = C_n \end{aligned}$$

- By the martingale convergence theorem,  $C_n$  converges to a random variable  $C_\infty$  such that  $E[C_\infty] = E[C_0] = C_0 = C(W)$ .
- In fact, the limit  $C_\infty = 0$  or  $1$  is a binary random variable (these are the fixed points of the polar transform).

# Review of Martingales

---

- Let  $\{X_n, n \geq 0\}$  be a random process. If
$$E[X_{n+1}|X_n, \dots, X_1, X_0] = X_n$$
then  $\{X_n\}$  is referred to as a **martingale**.
- **Martingale convergence theorem:** Let  $\{X_n, n \geq 0\}$  be a martingale with finite means. Then there exists a random variable  $X_\infty$  such that
$$X_n \rightarrow X_\infty \text{ almost surely}$$
as  $n \rightarrow \infty$ .

# How to construct polar codes

---

- To achieve  $C(W)$ , we need to identify the indices of those bit channels (branches in tree) with capacity  $\approx 1$ .
- For BEC, this can be computed recursively
$$C(W_{b_1 b_2 \dots b_n 0}) = C(W_{b_1 b_2 \dots b_n})^2$$
$$C(W_{b_1 b_2 \dots b_n 1}) = 2C(W_{b_1 b_2 \dots b_n}) - C(W_{b_1 b_2 \dots b_n})^2$$
- For other types of channels, it is difficult to obtain closed-form formulas. So numerical computation is often used.

# Polarization Speed

---

- For any positive real number  $\beta < 0.5$ ,

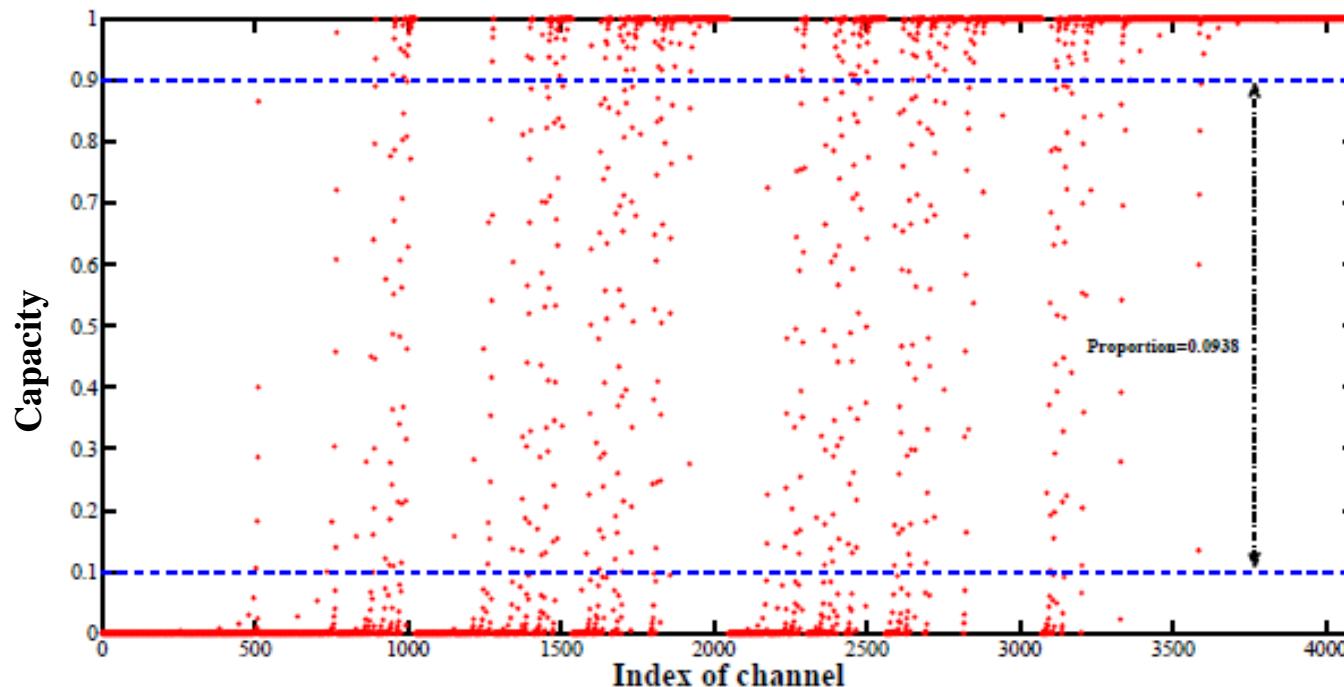
$$\lim_{n \rightarrow \infty} \frac{1}{N} \# \left\{ (b_1 \cdots b_n) : C(W_{b_1 b_2 \dots b_n}) \geq 1 - 2^{-N^\beta} \right\} = C(W).$$

$$\lim_{n \rightarrow \infty} \frac{1}{N} \# \left\{ (b_1 \cdots b_n) : C(W_{b_1 b_2 \dots b_n}) < 1 - 2^{-N^\beta} \right\} = 1 - C(W).$$

- The above statements do not hold for  $\beta > 0.5$ .
- Thus, the polarization speed is roughly  $2^{-\sqrt{N}}$ .

# Convergence

- The portion of almost perfect bit channels is  $C(W)$ , meaning that the capacity is achieved.
- Example: capacities for  $N = 2^{12}$  for BEC(0.5)



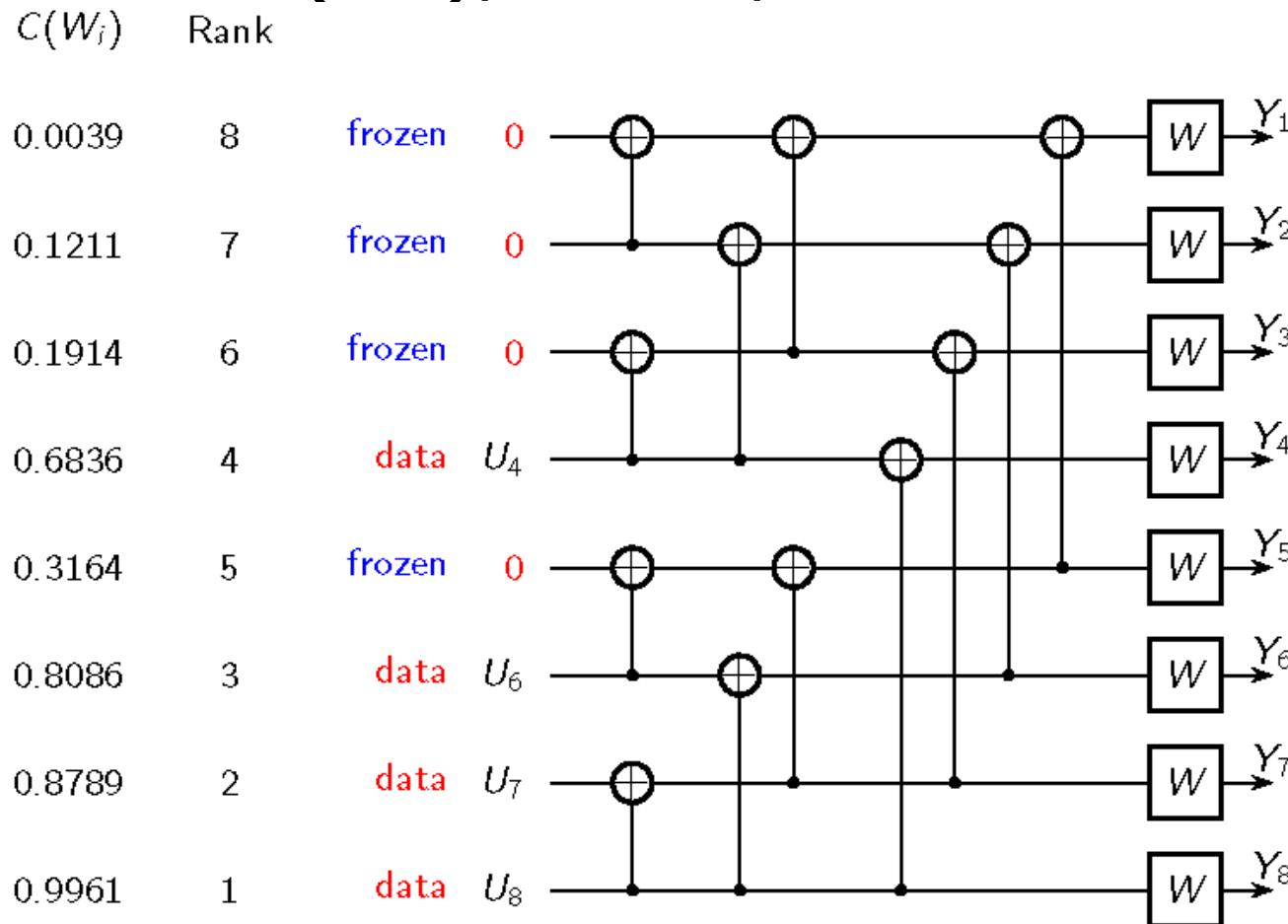
# Encoding

---

- Given  $N = 2^n$ , calculate  $C(W_{b_1 b_2 \dots b_n})$  for all synthetic bit channels.
- Given rate  $R < 1$  and  $K = NR$ , sort  $C(W_{b_1 b_2 \dots b_n})$  in descending order and define the union of the indices of the first  $K$  elements as the information set  $\Omega$ .
- Choose the information bits  $u^\Omega$  and freeze  $u^{\Omega^c}$  to be all-zero. Obtain the codeword  $\mathbf{x} = (u^\Omega, u^{\Omega^c}) \cdot \mathbf{F}_N$ .

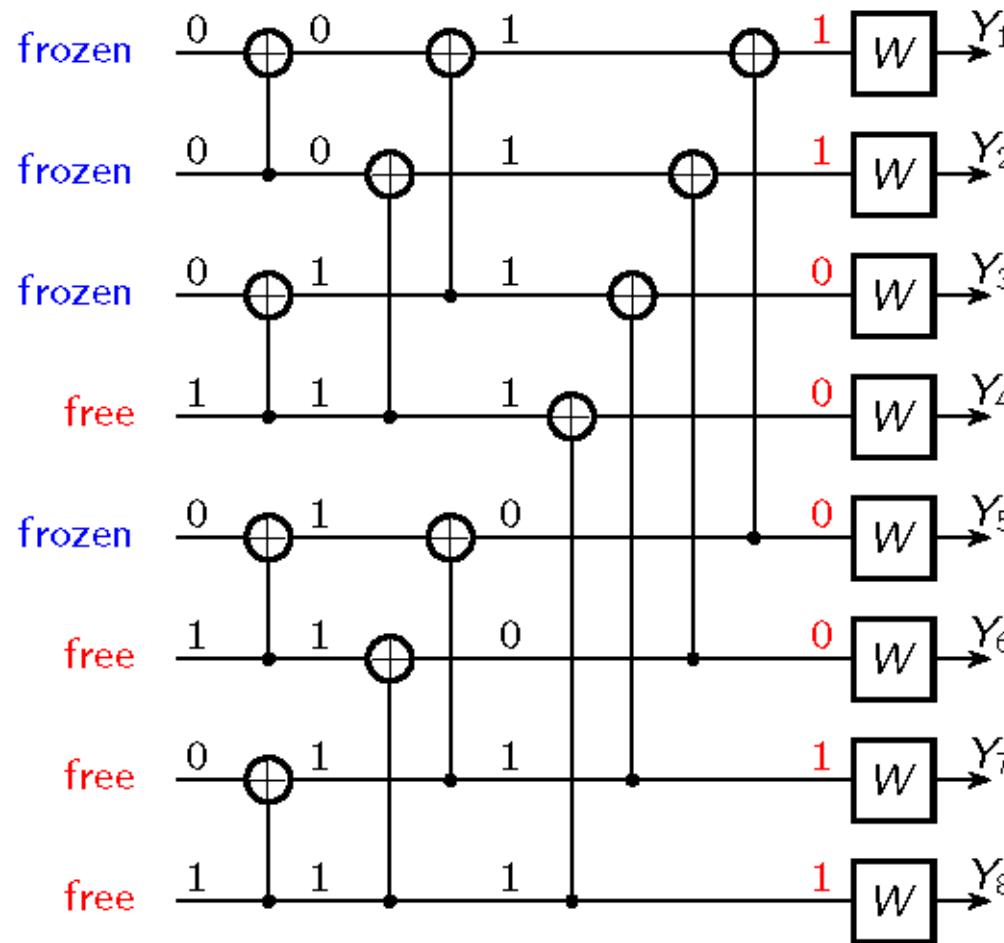
# Construction Example

- $W$  is a BEC(0.5),  $N = 8, R=0.5$ .



# Construction Example

- $W$  is a BEC(0.5),  $N = 8$ ,  $R=0.5$ .



Encoding  
complexity  
 $O(N \log N)$

# Successive decoding

---

- For the decoding we need to compute the likelihood ratio for  $u_i, i = (b_1 \cdots b_n)$

$$LR(u_i) = \frac{W_{b_1 b_2 \dots b_n}(\cdot | 1)}{W_{b_1 b_2 \dots b_n}(\cdot | 0)}$$

If  $i \in \Omega$ ,  $\hat{u}_i = 1$  if  $LR(u_i) > 1$ ; otherwise,  $\hat{u}_i = 0$ .

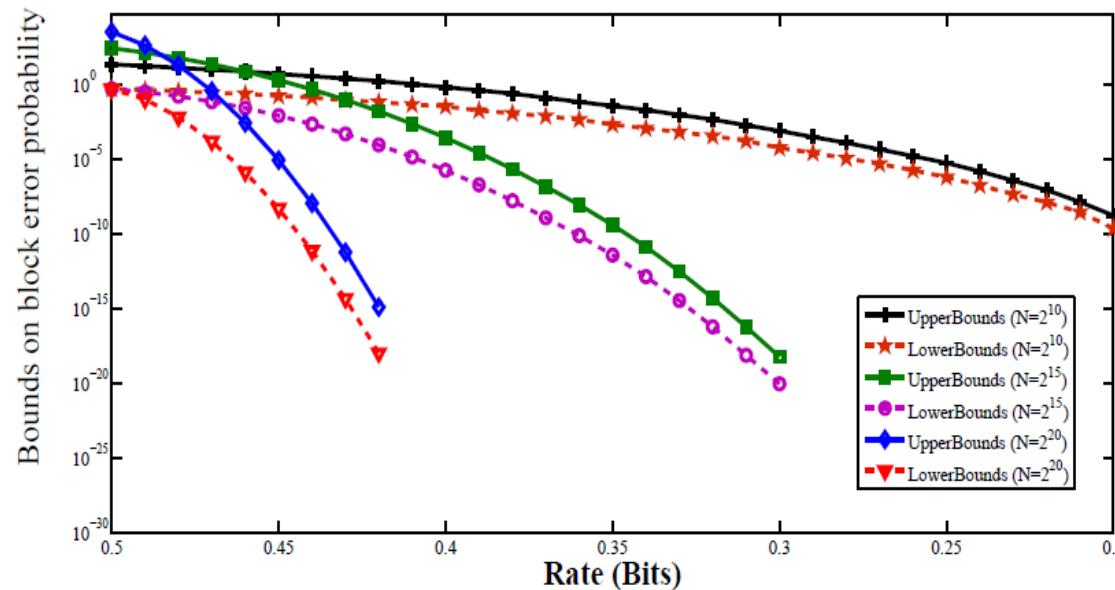
- Similar to  $C(W_{b_1 b_2 \dots b_n})$ ,  $LR(u_i)$  can also be calculated recursively.
- For more details of the decoding, see

E. Arikan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *IEEE Trans. on Information Theory*, vol. 55, no. 7, pp. 3051–3073, 2009.

# Probability of Error

- For a polar code with block length  $N$  and rate  $R < C(W)$ , the error probability under the successive cancellation decoding is given by

$$P_e \leq O(2^{-N^\beta}) \quad \beta < 0.5$$



**Error Bound of the SC decoding for BEC(0.5)**

# Polar Source Coding

---

- Let  $x$  be a random variable generated by a Bernoulli source  $\text{Ber}(p)$ , i.e.,

$$\Pr(x=0)=p \text{ and } \Pr(x=1)=1-p.$$

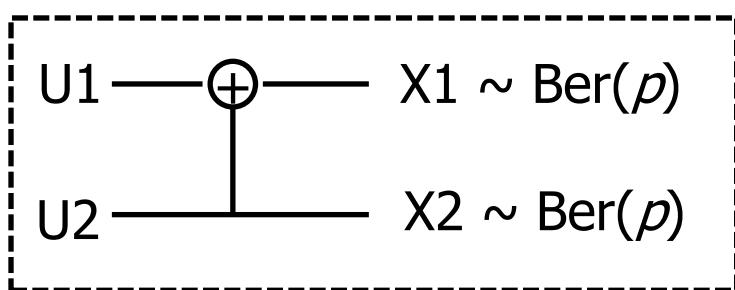
- The entropy (in bits) of  $x$  is

$$H(x) = -p \log_2 p - (1-p) \log_2 (1-p)$$

- If  $H(x) = 0$ , i.e.,  $p = 0$  or  $1$ ,  $x$  is a **constant**, no need for compression.
- If  $H(x) = 1$ , i.e.,  $p = 0.5$ ,  $x$  is **totally random**, we cannot do any compression.
- In other cases, can the polarization technique be used to achieve rate  $H(x)$ ?

# Source Polarization

- Similar idea applies to source coding:  
general sources  $\xrightarrow{\text{polarization}}$  extreme sources
- Basic source polarization



$$\begin{aligned}
 (U_1, U_2) &= (X_1, X_2)\mathbf{F}_2 \\
 H(U_1) + H(U_2|U_1) &= H(U_1, U_2) \\
 &= H(X_1, X_2) = 2H(x) \\
 H(U_1) &\geq H(x) \geq H(U_2|U_1)
 \end{aligned}$$

The process is **entropy-conserving**, but we obtain two new sources with **higher** and **lower** entropy than the original one.

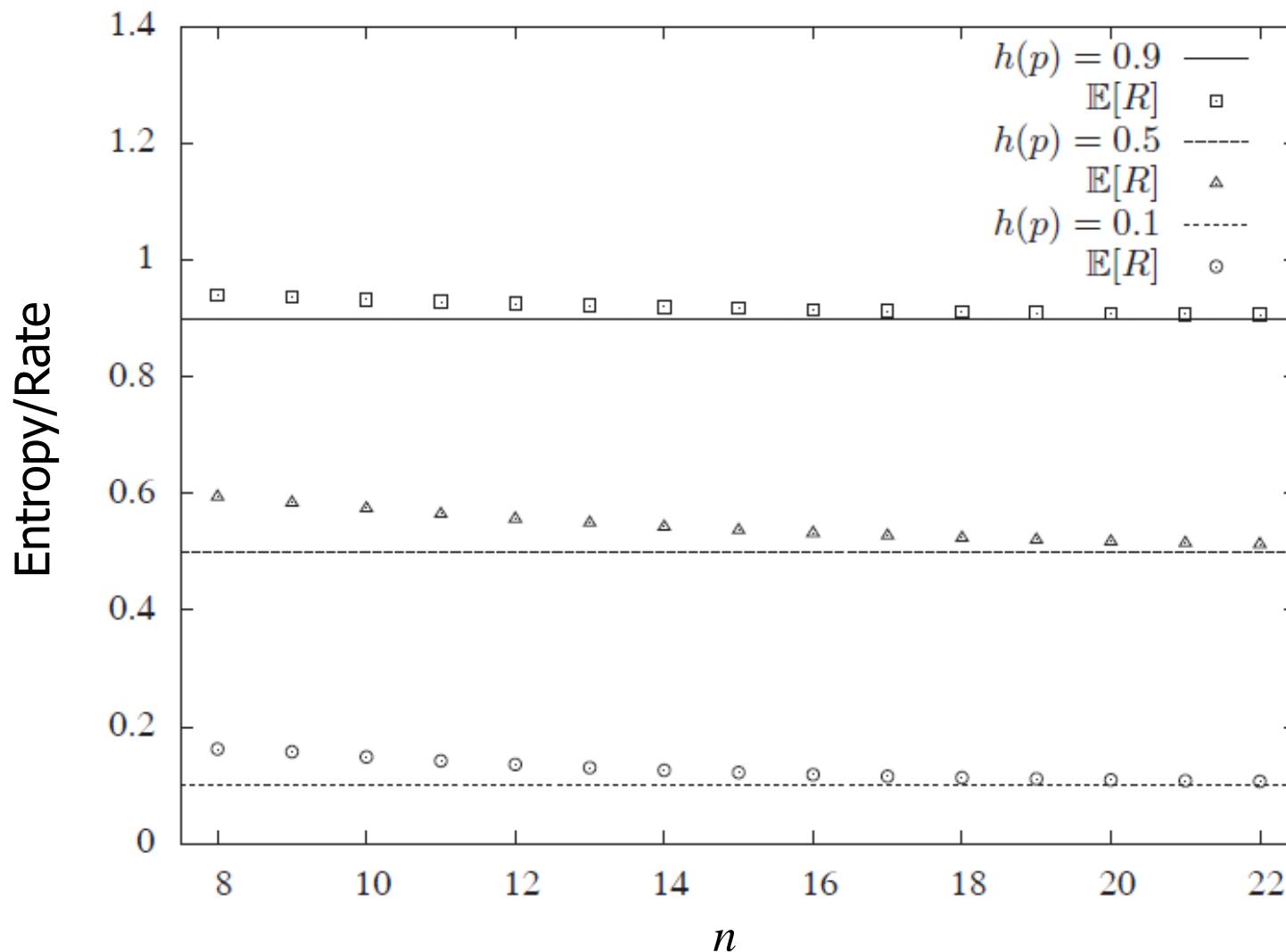
- Example: when  $p = 0.11$ ,  $H(x) = 0.5$ ,  $H(U_1) = 0.713$ ,  $H(U_2|U_1) = 0.287$ .

# Source Coding

---

- Keep polarizing by increasing  $N$ , the entropy of the synthetic sources tends to 0 or 1.
- Again, by the property of the **martingale**, the proportion of those sources with **entropy close to 1 is close to  $H(x)$** .
- Source coding is realized by recording the indexes with **entropy close to 1**, while the rest bits can be recovered with high probability because their associated entropy is **almost 0**.
- For me details, see  
E. Arikan, "Source polarization," *IEEE ISIT* 2010, pp. 899-903.

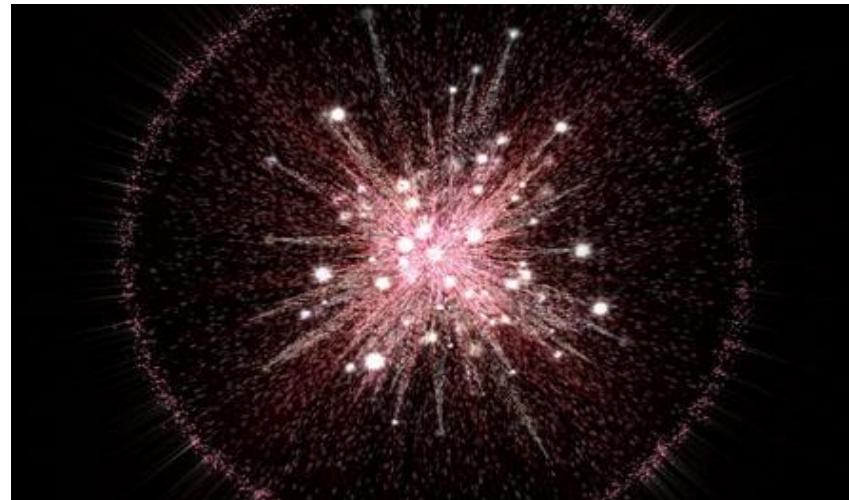
# Performance



# Extensions

---

- Polar codes also achieve capacity of other types of channels (discrete or continuous).
- Achieve entropy bound of other types of sources (lossless or lossy).
- Quantum polar codes, network information theory...



**Big bang in information theory**

# Lecture 13

---

- Continuous Random Variables
- Differential Entropy
  - can be negative
  - not really a measure of the information in  $x$
  - coordinate-dependent
- Maximum entropy distributions
  - Uniform over a finite range
  - Gaussian if a constant variance

# Continuous Random Variables

---

## Changing Variables

- pdf:  $f_x(x)$       CDF:  $F_x(x) = \int_{-\infty}^x f_x(t)dt$
- For  $g(x)$  monotonic:  $y = g(x) \Leftrightarrow x = g^{-1}(y)$ 

$$F_y(y) = F_x(g^{-1}(y)) \quad \text{or} \quad 1 - F_x(g^{-1}(y)) \quad \text{according to slope of } g(x)$$

$$f_y(y) = \frac{dF_y(y)}{dy} = f_x(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| = f_x(x) \left| \frac{dx}{dy} \right| \quad \text{where} \quad x = g^{-1}(y)$$
- Examples:

Suppose  $f_x(x) = 0.5$  for  $x \in (0,2)$   $\Rightarrow F_x(x) = 0.5x$

- (a)  $y = 4x \Rightarrow x = 0.25y \Rightarrow f_y(y) = 0.5 \times 0.25 = 0.125$  for  $y \in (0,8)$
- (b)  $z = x^4 \Rightarrow x = z^{1/4} \Rightarrow f_z(z) = 0.5 \times \frac{1}{4} z^{-3/4} = 0.125 z^{-3/4}$  for  $z \in (0,16)$

# Joint Distributions

---

Joint pdf:

$$f_{x,y}(x, y)$$

Marginal pdf:

$$f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x, y) dy$$

Independence:

$$\Leftrightarrow f_{x,y}(x, y) = f_x(x) f_y(y)$$

Conditional pdf:

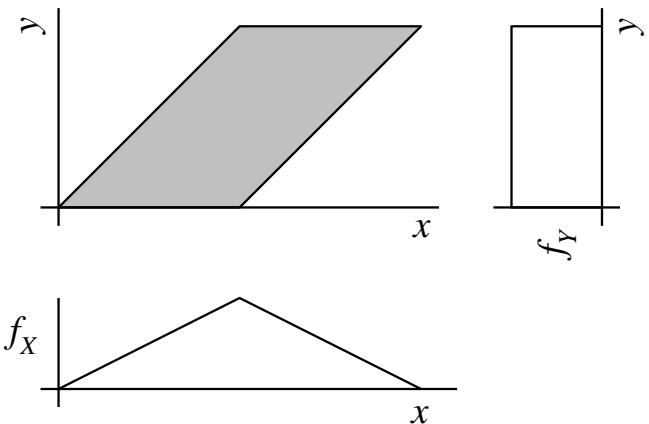
$$f_{x|y}(x) = \frac{f_{x,y}(x, y)}{f_y(y)}$$

Example:

$$f_{x,y} = 1 \text{ for } y \in (0,1), x \in (y, y+1)$$

$$f_{x|y} = 1 \text{ for } x \in (y, y+1)$$

$$f_{y|x} = \frac{1}{\min(x, 1-x)} \text{ for } y \in (\max(0, x-1), \min(x, 1))$$



# Entropy of Continuous R.V.

---

- Given a continuous pdf  $f(x)$ , we divide the range of  $x$  into bins of width  $\Delta$ 
  - For each  $i$ ,  $\exists x_i$  with  $f(x_i)\Delta = \int_{i\Delta}^{(i+1)\Delta} f(x)dx$  mean value theorem
- Define a discrete random variable  $Y$ 
  - $Y = \{x_i\}$  and  $p_Y = \{f(x_i)\Delta\}$
  - Scaled, quantised version of  $f(x)$  with slightly unevenly spaced  $x_i$
- $$\begin{aligned} H(Y) &= -\sum f(x_i)\Delta \log(f(x_i)\Delta) \\ &= -\log \Delta - \sum f(x_i) \log(f(x_i))\Delta \\ &\xrightarrow[\Delta \rightarrow 0]{} -\log \Delta - \int_{-\infty}^{\infty} f(x) \log f(x) dx = -\log \Delta + h(x) \end{aligned}$$
- Differential entropy:  $h(x) = -\int_{-\infty}^{\infty} f_x(x) \log f_x(x) dx$ 
  - Similar to entropy of discrete r.v. but there are differences

# Differential Entropy

---

Differential Entropy: 
$$h(x) \stackrel{\Delta}{=} - \int_{-\infty}^{\infty} f_x(x) \log f_x(x) dx = E - \log f_x(x)$$

## Bad News:

- $h(x)$  does not give the amount of information in  $x$
- $h(x)$  is not necessarily positive
- $h(x)$  changes with a change of coordinate system

## Good News:

- $h_1(x) - h_2(x)$  does compare the uncertainty of two continuous random variables **provided they are quantised to the same precision**
- Relative Entropy and Mutual Information still work fine
- If the range of  $x$  is normalized to 1 and then  $x$  is quantised to  $n$  bits, the entropy of the resultant discrete random variable is approximately  $h(x) + n$

# Differential Entropy Examples

---

- Uniform Distribution:  $x \sim U(a, b)$ 
  - $f(x) = (b - a)^{-1}$  for  $x \in (a, b)$  and  $f(x) = 0$  elsewhere
  - $h(X) = -\int_a^b (b - a)^{-1} \log(b - a)^{-1} dx = \log(b - a)$
  - Note that  $h(x) < 0$  if  $(b-a) < 1$
  
- Gaussian Distribution:  $x \sim N(\mu, \sigma^2)$ 
  - $f(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2}(x - \mu)^2 \sigma^{-2}\right)$
  - $$\begin{aligned} h(X) &= -(\log e) \int_{-\infty}^{\infty} f(x) \ln f(x) dx \\ &= -\frac{1}{2}(\log e) \int_{-\infty}^{\infty} f(x) \left( -\ln(2\pi\sigma^2) - (x - \mu)^2 \sigma^{-2} \right) dx \\ &= \frac{1}{2}(\log e) \left( \ln(2\pi\sigma^2) + \sigma^{-2} E((x - \mu)^2) \right) \\ &= \frac{1}{2}(\log e) \left( \ln(2\pi\sigma^2) + 1 \right) = \frac{1}{2} \log(2\pi e \sigma^2) \approx \log(4.1\sigma) \text{ bits} \end{aligned}$$

$$\log_x y = \frac{\log_e y}{\log_e x}$$

# Multivariate Gaussian

---

Given mean,  $\mathbf{m}$ , and symmetric positive definite covariance matrix  $\mathbf{K}$ ,

$$\begin{aligned}
 \mathbf{x}_{1:n} \sim \mathbf{N}(\mathbf{m}, \mathbf{K}) &\iff f(\mathbf{x}) = |2\pi\mathbf{K}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{K}^{-1}(\mathbf{x} - \mathbf{m})\right) \\
 h(f) &= -(\log e) \int f(\mathbf{x}) \times \left( -\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{K}^{-1}(\mathbf{x} - \mathbf{m}) - \frac{1}{2} \ln |2\pi\mathbf{K}| \right) d\mathbf{x} \\
 &= \frac{1}{2} \log(e) \times \left( \ln |2\pi\mathbf{K}| + E((\mathbf{x} - \mathbf{m})^T \mathbf{K}^{-1}(\mathbf{x} - \mathbf{m})) \right) \\
 &= \frac{1}{2} \log(e) \times \left( \ln |2\pi\mathbf{K}| + E \operatorname{tr}((\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^T \mathbf{K}^{-1}) \right) \quad \text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}) \\
 &= \frac{1}{2} \log(e) \times \left( \ln |2\pi\mathbf{K}| + \operatorname{tr}(E(\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^T \mathbf{K}^{-1}) \right) \quad E_f \mathbf{xx}^T = \mathbf{K} \\
 &= \frac{1}{2} \log(e) \times \left( \ln |2\pi\mathbf{K}| + \operatorname{tr}(\mathbf{KK}^{-1}) \right) \quad = \frac{1}{2} \log(e) \times (\ln |2\pi\mathbf{K}| + n) \\
 &= \frac{1}{2} \log(e^n) + \frac{1}{2} \log(|2\pi\mathbf{K}|) \quad \operatorname{tr}(\mathbf{I}) = n = \ln(e^n) \\
 &= \frac{1}{2} \log(|2\pi e \mathbf{K}|) = \frac{1}{2} \log((2\pi e)^n |\mathbf{K}|) \quad \text{bits}
 \end{aligned}$$

# Other Differential Quantities

---

## Joint Differential Entropy

$$h(x, y) = - \iint_{x, y} f_{x,y}(x, y) \log f_{x,y}(x, y) dx dy = E - \log f_{x,y}(x, y)$$

## Conditional Differential Entropy

$$h(x | y) = - \iint_{x, y} f_{x,y}(x, y) \log f_{x,y}(x | y) dx dy = h(x, y) - h(y)$$

## Mutual Information

$$I(x; y) = \iint_{x, y} f_{x,y}(x, y) \log \frac{f_{x,y}(x, y)}{f_x(x) f_y(y)} dx dy = h(x) + h(y) - h(x, y)$$

## Relative Differential Entropy of two pdf's:

$$\begin{aligned} D(f \| g) &= \int f(x) \log \frac{f(x)}{g(x)} dx \\ &= -h_f(x) - E_f \log g(x) \end{aligned}$$

(a) must have  $f(x)=0 \Rightarrow g(x)=0$

(b) continuity  $\Rightarrow 0 \log(0/0) = 0$

# Differential Entropy Properties

---

Chain Rules

$$h(x, y) = h(x) + h(y | x) = h(y) + h(x | y)$$

$$I(x, y; z) = I(x; z) + I(y; z | x)$$

Information Inequality:  $D(f \parallel g) \geq 0$

Proof: Define  $S = \{\mathbf{x} : f(\mathbf{x}) > 0\}$

$$-D(f \parallel g) = \int_{\mathbf{x} \in S} f(\mathbf{x}) \log \frac{g(\mathbf{x})}{f(\mathbf{x})} d\mathbf{x} = E_f \left( \log \frac{g(\mathbf{x})}{f(\mathbf{x})} \right)$$

$$\leq \log \left( E \frac{g(\mathbf{x})}{f(\mathbf{x})} \right) = \log \left( \int_S f(\mathbf{x}) \frac{g(\mathbf{x})}{f(\mathbf{x})} d\mathbf{x} \right) \quad \text{Jensen + log() is concave}$$

$$= \log \left( \int_S g(\mathbf{x}) d\mathbf{x} \right) \leq \log 1 = 0$$

all the same as for discrete r.v.  $H()$

# Information Inequality Corollaries

---

Mutual Information  $\geq 0$

$$I(x; y) = D(f_{x,y} \parallel f_x f_y) \geq 0$$

Conditioning reduces Entropy

$$h(x) - h(x | y) = I(x; y) \geq 0$$

Independence Bound

$$h(\mathbf{x}_{1:n}) = \sum_{i=1}^n h(\mathbf{x}_i | \mathbf{x}_{1:i-1}) \leq \sum_{i=1}^n h(\mathbf{x}_i)$$

all the same as for  $H()$

# Change of Variable

---

**Change Variable:**  $y = g(x)$

from earlier

$$f_y(y) = f_x(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

$$h(y) = -E \log(f_y(y)) = -E \log(f_x(g^{-1}(y))) - E \log \left| \frac{dx}{dy} \right|$$

$$= -E \log(f_x(x)) - E \log \left| \frac{dx}{dy} \right| = h(x) + E \log \left| \frac{dy}{dx} \right|$$

**Examples:**

- Translation:  $y = x + a \Rightarrow dy/dx = 1 \Rightarrow h(y) = h(x)$
- Scaling:  $y = cx \Rightarrow dy/dx = c \Rightarrow h(y) = h(x) + \log |c|$
- Vector version:  $y_{1:n} = \mathbf{A}x_{1:n} \Rightarrow h(\mathbf{y}) = h(\mathbf{x}) + \log |\det(\mathbf{A})|$

**not** the same as for  $H()$

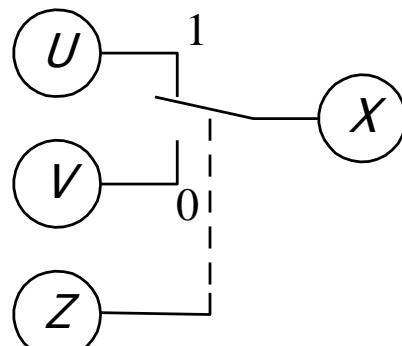
# Concavity & Convexity

---

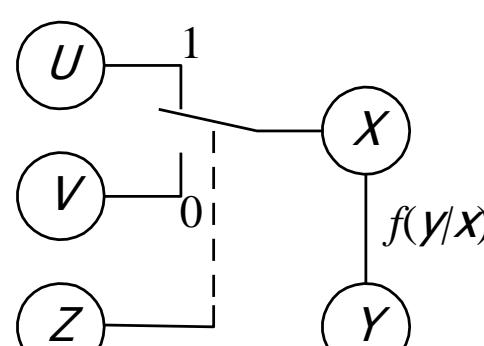
- Differential Entropy:
  - $h(x)$  is a **concave** function of  $f_x(x) \Rightarrow \exists$  a maximum
- Mutual Information:
  - $I(x; y)$  is a **concave** function of  $f_x(x)$  for fixed  $f_{y|x}(y)$
  - $I(x; y)$  is a **convex** function of  $f_{y|x}(y)$  for fixed  $f_x(x)$

## Proofs:

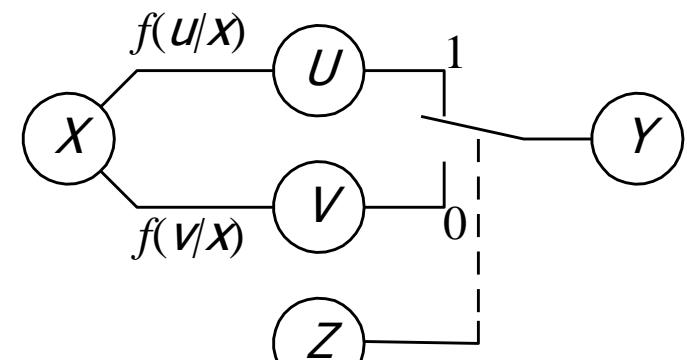
Exactly the same as for the discrete case:  $\mathbf{p}_z = [1-\lambda, \lambda]^T$



$$H(X) \geq H(X | Z)$$



$$I(X; Y) \geq I(X; Y | Z)$$



$$I(X; Y) \leq I(X; Y | Z)$$

# Uniform Distribution Entropy

---

What distribution over the finite range  $(a,b)$  maximizes the entropy ?

**Answer:** A uniform distribution  $u(x)=(b-a)^{-1}$

**Proof:**

Suppose  $f(x)$  is a distribution for  $x \in (a,b)$

$$\begin{aligned} 0 \leq D(f \parallel u) &= -h_f(x) - E_f \log u(x) \\ &= -h_f(x) + \log(b-a) \end{aligned}$$

$$\Rightarrow h_f(x) \leq \log(b-a)$$

# Maximum Entropy Distribution

---

What zero-mean distribution maximizes the entropy on  $(-\infty, \infty)^n$  for a given covariance matrix  $\mathbf{K}$  ?

**Answer: A multivariate Gaussian**  $\phi(\mathbf{x}) = |2\pi\mathbf{K}|^{-\frac{1}{2}} \exp(-\frac{1}{2}\mathbf{x}^T \mathbf{K}^{-1} \mathbf{x})$

**Proof:**  $0 \leq D(f \parallel \phi) = -h_f(\mathbf{x}) - E_f \log \phi(\mathbf{x})$

$$\begin{aligned} \Rightarrow h_f(\mathbf{x}) &\leq -(\log e)E_f \left( -\frac{1}{2} \ln (|2\pi\mathbf{K}|) - \frac{1}{2}\mathbf{x}^T \mathbf{K}^{-1} \mathbf{x} \right) \\ &= \frac{1}{2}(\log e) \left( \ln (|2\pi\mathbf{K}|) + \text{tr}(E_f \mathbf{x} \mathbf{x}^T \mathbf{K}^{-1}) \right) \\ &= \frac{1}{2}(\log e) \left( \ln (|2\pi\mathbf{K}|) + \text{tr}(\mathbf{I}) \right) \quad E_f \mathbf{x} \mathbf{x}^T = \mathbf{K} \\ &= \frac{1}{2} \log (|2\pi e \mathbf{K}|) = h_\phi(\mathbf{x}) \quad \text{tr}(\mathbf{I}) = n = \ln(e^n) \end{aligned}$$

Since translation doesn't affect  $h(X)$ , we can assume zero-mean w.l.o.g.

# Summary

---

- Differential Entropy:  $h(x) = - \int_{-\infty}^{\infty} f_x(x) \log f_x(x) dx$ 
  - Not necessarily positive
  - $h(x+a) = h(x)$ ,  $h(ax) = h(x) + \log|a|$
- Many properties are formally the same
  - $h(x|y) \leq h(x)$
  - $I(x; y) = h(x) + h(y) - h(x, y) \geq 0$ ,  $D(f||g) = E \log(f/g) \geq 0$
  - $h(x)$  concave in  $f_x(x)$ ;  $I(x; y)$  concave in  $f_x(x)$
- Bounds:
  - Finite range: Uniform distribution has max:  $h(x) = \log(b-a)$
  - Fixed Covariance: Gaussian has max:  $h(x) = \frac{1}{2}\log((2\pi e)^n |\mathbf{K}|)$

# Lecture 14

---

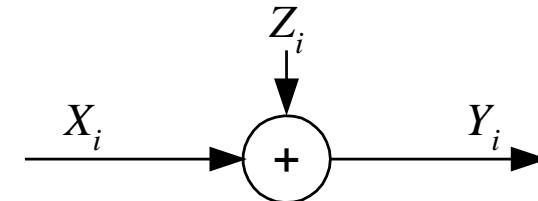
- Discrete-Time Gaussian Channel Capacity
- Continuous Typical Set and AEP
- Gaussian Channel Coding Theorem
- Bandlimited Gaussian Channel
  - Shannon Capacity

# Capacity of Gaussian Channel

Discrete-time channel:  $y_i = x_i + z_i$

- Zero-mean Gaussian i.i.d.  $z_i \sim N(0, N)$
- Average power constraint  $n^{-1} \sum_{i=1}^n x_i^2 \leq P$

$$EY^2 = E(x + z)^2 = Ex^2 + 2E(x)E(z) + Ez^2 \leq P + N$$



$X, Z$  indep and  $EZ=0$

## Information Capacity

- Define information capacity:  $C = \max_{Ex^2 \leq P} I(x; y)$

$$\begin{aligned} I(x; y) &= h(y) - h(y | x) = h(y) - h(x + z | x) \\ &\stackrel{(a)}{=} h(y) - h(z | x) = h(y) - h(z) \end{aligned}$$

(a) Translation independence

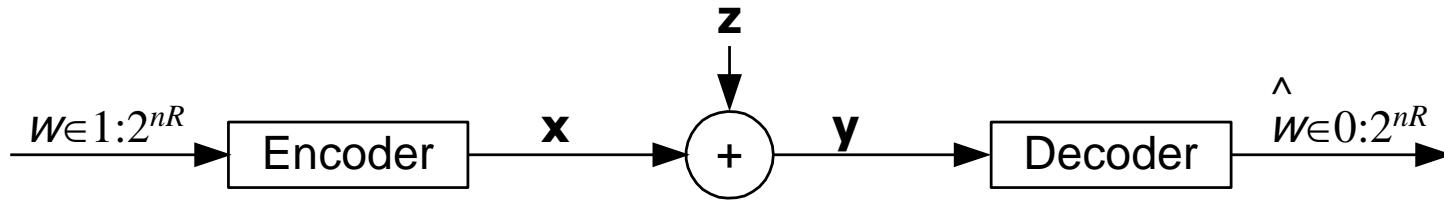
$$\begin{aligned} &\leq \frac{1}{2} \log 2\pi e(P + N) - \frac{1}{2} \log 2\pi e N \\ &= \frac{1}{2} \log \left( 1 + \frac{P}{N} \right) \end{aligned}$$

Gaussian Limit with equality when  $x \sim N(0, P)$

The optimal input is Gaussian

# Achievability

---



- An  $(M,n)$  code for a Gaussian Channel with power constraint is
  - A set of  $M$  codewords  $\mathbf{x}(w) \in \mathcal{X}^n$  for  $w=1:M$  with  $\mathbf{x}(w)^T \mathbf{x}(w) \leq nP \quad \forall w$
  - A deterministic decoder  $g(\mathbf{y}) \in 0:M$  where 0 denotes failure
  - Errors: codeword :  $\lambda_i$        $\max_i : \lambda^{(n)}$       average :  $P_e^{(n)}$
- Rate  $R$  is achievable if  $\exists$  seq of  $(2^{nR},n)$  codes with  $\lambda^{(n)} \xrightarrow[n \rightarrow \infty]{} 0$
- Theorem:  $R$  achievable iff  $R < C = \frac{1}{2} \log (1 + PN^{-1})$  ◆

◆ = proved on next pages

# Argument by Sphere Packing

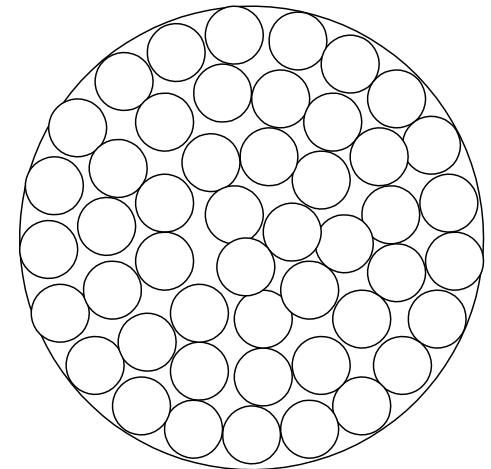
---

- Each transmitted  $\mathbf{x}_i$  is received as a probabilistic cloud  $\mathbf{y}_i$ 
  - cloud ‘radius’ =  $\sqrt{\text{Var}(\mathbf{y} \mid \mathbf{x})} = \sqrt{nN}$
- Energy of  $\mathbf{y}_i$  constrained to  $n(P+N)$  so clouds must fit into a hypersphere of radius  $\sqrt{n(P+N)}$
- Volume of hypersphere  $\propto r^n$
- Max number of non-overlapping clouds:

$$\frac{(nP + nN)^{\frac{1}{2}n}}{(nN)^{\frac{1}{2}n}} = 2^{\frac{1}{2}n \log(1+PN^{-1})}$$

- Max achievable rate is  $\frac{1}{2}\log(1+P/N)$

Law of large numbers



# Continuous AEP

---

**Typical Set:** Continuous distribution, discrete time i.i.d.

For any  $\varepsilon > 0$  and any  $n$ , the **typical set** with respect to  $f(\mathbf{x})$  is

$$T_{\varepsilon}^{(n)} = \left\{ \mathbf{x} \in S^n : \left| -n^{-1} \log f(\mathbf{x}) - h(\mathbf{x}) \right| \leq \varepsilon \right\}$$

where  $S$  is the **support** of  $f \Leftrightarrow \{\mathbf{x} : f(\mathbf{x}) > 0\}$

$$f(\mathbf{x}) = \prod_{i=1}^n f(x_i) \text{ since } x_i \text{ are independent}$$

$$h(\mathbf{x}) = E - \log f(\mathbf{x}) = -n^{-1} E \log f(\mathbf{x})$$

## Typical Set Properties

1.  $p(\mathbf{x} \in T_{\varepsilon}^{(n)}) > 1 - \varepsilon$  for  $n > N_{\varepsilon}$

Proof: LLN

2.  $(1 - \varepsilon) 2^{n(h(\mathbf{x}) - \varepsilon)} \leq \text{Vol}(T_{\varepsilon}^{(n)}) \leq 2^{n(h(\mathbf{x}) + \varepsilon)}$

Proof: Integrate  
max/min prob

where  $\text{Vol}(A) = \int d\mathbf{x}$

# Continuous AEP Proof

---

Proof 1: By law of large numbers

$$-n^{-1} \log f(\mathbf{x}_{1:n}) = -n^{-1} \sum_{i=1}^n \log f(\mathbf{x}_i) \xrightarrow{\text{prob}} E - \log f(\mathbf{x}) = h(\mathbf{x})$$

Reminder:  $\mathbf{x}_n \xrightarrow{\text{prob}} \mathbf{y} \Rightarrow \forall \varepsilon > 0, \exists N_\varepsilon$  such that  $\forall n > N_\varepsilon, P(|\mathbf{x}_n - \mathbf{y}| > \varepsilon) < \varepsilon$

Proof 2a:  $1 - \varepsilon \leq \int_{T_\varepsilon^{(n)}} f(\mathbf{x}) d\mathbf{x}$  for  $n > N_\varepsilon$  Property 1

$$\leq 2^{-n(h(\mathbf{x})-\varepsilon)} \int_{T_\varepsilon^{(n)}} d\mathbf{x} = 2^{-n(h(\mathbf{x})-\varepsilon)} \text{Vol}(T_\varepsilon^{(n)}) \quad \text{max } f(x) \text{ within } T$$

Proof 2b:  $1 = \int_{S^n} f(\mathbf{x}) d\mathbf{x} \geq \int_{T_\varepsilon^{(n)}} f(\mathbf{x}) d\mathbf{x}$   
 $\geq 2^{-n(h(\mathbf{x})+\varepsilon)} \int_{T_\varepsilon^{(n)}} d\mathbf{x} = 2^{-n(h(\mathbf{x})+\varepsilon)} \text{Vol}(T_\varepsilon^{(n)}) \quad \text{min } f(x) \text{ within } T$

# Jointly Typical Set

---

**Jointly Typical:**  $x_i, y_i$  i.i.d from  $\Re^2$  with  $f_{X,Y}(x_i, y_i)$

$$\begin{aligned} J_{\varepsilon}^{(n)} = \left\{ \mathbf{x}, \mathbf{y} \in \Re^{2n} : \right. & \left| -n^{-1} \log f_X(\mathbf{x}) - h(X) \right| < \varepsilon, \\ & \left| -n^{-1} \log f_Y(\mathbf{y}) - h(Y) \right| < \varepsilon, \\ & \left. \left| -n^{-1} \log f_{X,Y}(\mathbf{x}, \mathbf{y}) - h(X, Y) \right| < \varepsilon \right\} \end{aligned}$$

**Properties:**

1. Indiv p.d.:  $\mathbf{x}, \mathbf{y} \in J_{\varepsilon}^{(n)} \Rightarrow \log f_{X,Y}(\mathbf{x}, \mathbf{y}) = -nh(X, Y) \pm n\varepsilon$
2. Total Prob:  $p(\mathbf{x}, \mathbf{y} \in J_{\varepsilon}^{(n)}) > 1 - \varepsilon \quad \text{for } n > N_{\varepsilon}$
3. Size:  $(1 - \varepsilon)2^{n(h(X, Y) - \varepsilon)} \stackrel{n > N_{\varepsilon}}{\leq} \text{Vol}(J_{\varepsilon}^{(n)}) \leq 2^{n(h(X, Y) + \varepsilon)}$
4. Indep  $\mathbf{x}', \mathbf{y}'$ :  $(1 - \varepsilon)2^{-n(I(X;Y) + 3\varepsilon)} \stackrel{n > N_{\varepsilon}}{\leq} p(\mathbf{x}', \mathbf{y}' \in J_{\varepsilon}^{(n)}) \leq 2^{-n(I(X;Y) - 3\varepsilon)}$

Proof of 4.: Integrate max/min  $f(\mathbf{x}', \mathbf{y}') = f(\mathbf{x}')f(\mathbf{y}')$ , then use known bounds on  $\text{Vol}(J)$

# Gaussian Channel Coding Theorem

---

$R$  is achievable iff  $R < C = \frac{1}{2} \log (1 + P N^{-1})$

Proof ( $\Leftarrow$ ):

Choose  $\varepsilon > 0$

Random codebook:  $\mathbf{x}_w \in \Re^n$  for  $w = 1 : 2^{nR}$  where  $x_w$  are i.i.d.  $\sim N(0, P - \varepsilon)$

Use Joint typicality decoding

- Errors:
1. Power too big  $p(\mathbf{x}^T \mathbf{x} > nP) \rightarrow 0 \Rightarrow \leq \varepsilon$  for  $n > M_\varepsilon$
  2.  $\mathbf{y}$  not J.T. with  $\mathbf{x}$   $p(\mathbf{x}, \mathbf{y} \notin J_\varepsilon^{(n)}) < \varepsilon$  for  $n > N_\varepsilon$
  3. another  $\mathbf{x}$  J.T. with  $\mathbf{y}$   $\sum_{j=2}^{2^{nR}} p(\mathbf{x}_j, \mathbf{y}_i \in J_\varepsilon^{(n)}) \leq (2^{nR} - 1) \times 2^{-n(I(X;Y) - 3\varepsilon)}$

Total Err  $P_\varepsilon^{(n)} \leq \varepsilon + \varepsilon + 2^{-n(I(X;Y) - R - 3\varepsilon)} \leq 3\varepsilon$  for large  $n$  if  $R < I(X;Y) - 3\varepsilon$

Expurgation: Remove half of codebook\*:  $\lambda^{(n)} < 6\varepsilon$  now max error

We have constructed a code achieving rate  $R - n^{-1}$

\*:Worst codebook half includes  $\mathbf{x}_i$ :  $\mathbf{x}_i^T \mathbf{x}_i > nP \Rightarrow \lambda_i = 1$

# Gaussian Channel Coding Theorem

---

Proof ( $\Rightarrow$ ): Assume  $P_e^{(n)} \rightarrow 0$  and  $n^{-1} \mathbf{x}^T \mathbf{x} < P$  for each  $\mathbf{x}(w)$

$$\begin{aligned} nR &= H(\mathbf{W}) = I(\mathbf{W}; \mathbf{Y}_{1:n}) + H(\mathbf{W} | \mathbf{Y}_{1:n}) \xrightarrow{w \in 1:M} \text{Encoder} \xrightarrow{\mathbf{X}_{1:n}} \text{Noisy Channel} \xrightarrow{\mathbf{Y}_{1:n}} \text{Decoder } g(\mathbf{y}) \xrightarrow{\mathbf{W} \in 0:M} \\ &\leq I(\mathbf{X}_{1:n}; \mathbf{Y}_{1:n}) + H(\mathbf{W} | \mathbf{Y}_{1:n}) && \text{Data Proc Inequal} \\ &= h(\mathbf{Y}_{1:n}) - h(\mathbf{Y}_{1:n} | \mathbf{X}_{1:n}) + H(\mathbf{W} | \mathbf{Y}_{1:n}) \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^n h(\mathbf{Y}_i) - h(\mathbf{Z}_{1:n}) + H(\mathbf{W} | \mathbf{Y}_{1:n}) && \text{Indep Bound + Translation} \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^n I(\mathbf{X}_i; \mathbf{Y}_i) + 1 + nRP_e^{(n)} && \text{Z i.i.d + Fano, } |\mathcal{W}|=2^{nR} \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^n \frac{1}{2} \log(1 + PN^{-1}) + 1 + nRP_e^{(n)} && \text{max Information Capacity} \end{aligned}$$

$$R \leq \frac{1}{2} \log(1 + PN^{-1}) + n^{-1} + RP_e^{(n)} \rightarrow \frac{1}{2} \log(1 + PN^{-1})$$

# Bandlimited Channel

---

- Channel bandlimited to  $f \in (-W, W)$  and signal duration  $T$ 
  - Not exactly
  - Most energy in the bandwidth, most energy in the interval
- Nyquist: Signal is defined by  $2WT$  samples
  - white noise with double-sided p.s.d.  $\frac{1}{2}N_0$  becomes i.i.d gaussian  $N(0, \frac{1}{2}N_0)$  added to each coefficient
  - Signal power constraint =  $P \Rightarrow$  Signal energy  $\leq PT$ 
    - Energy constraint per coefficient:  $n^{-1}\mathbf{x}^T\mathbf{x} < PT/2WT = \frac{1}{2}W^{-1}P$
- Capacity:
 
$$C = \frac{1}{2} \log \left( 1 + \frac{1/2 \cdot P / W}{N_0 / 2} \right) \times \frac{2WT}{T} = W \log \left( 1 + \frac{P}{WN_0} \right) \text{ bits/second}$$
- More precisely, it can be represented in a vector space of about  $n=2WT$  dimensions with prolate spheroidal functions as an orthonormal basis

Compare discrete time version:  $\frac{1}{2}\log(1+PN^{-1})$  bits per channel use

# Limit of Infinite Bandwidth

---

$$C = W \log \left( 1 + \frac{P}{WN_0} \right) \text{ bits/second}$$

$$C \xrightarrow{W \rightarrow \infty} \frac{P}{N_0} \log e$$

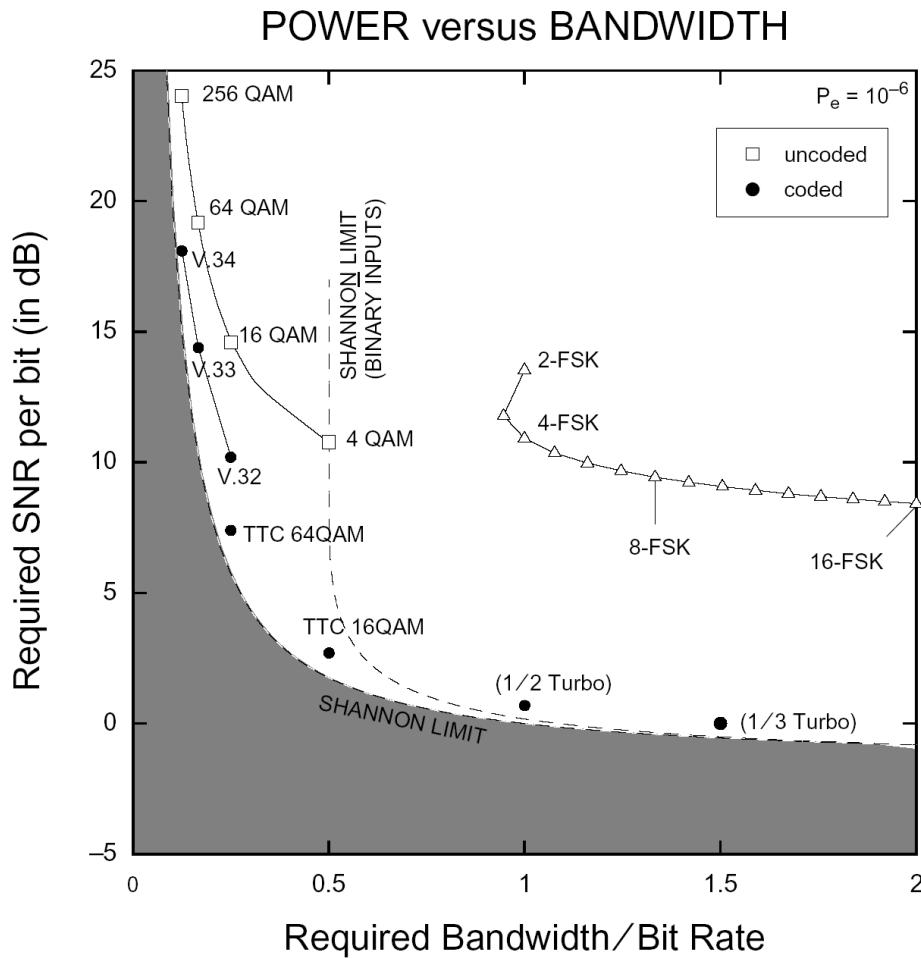
Minimum signal to noise ratio (SNR)

$$\frac{E_b}{N_0} = \frac{PT_b}{N_0} = \frac{P / C}{N_0} \xrightarrow{W \rightarrow \infty} \ln 2 = -1.6 \text{ dB}$$

Given capacity, trade-off between  $P$  and  $W$

- Increase  $P$ , decrease  $W$
- Increase  $W$ , decrease  $P$ 
  - spread spectrum
  - ultra wideband

# Channel Code Performance



- **Power Limited**
  - High bandwidth
  - Spacecraft, Pagers
  - Use QPSK/4-QAM
  - Block/Convolution Codes
- **Bandwidth Limited**
  - Modems, DVB, Mobile phones
  - 16-QAM to 256-QAM
  - Convolution Codes
- **Value of 1 dB for space**
  - Better range, lifetime, weight, bit rate
  - \$80 M (1999)

# Summary

---

- Gaussian channel capacity

$$C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right) \text{ bits/transmission}$$

- Proved by using continuous AEP
- Bandlimited channel

$$C = W \log \left( 1 + \frac{P}{WN_0} \right) \text{ bits/second}$$

- Minimum SNR =  $-1.6$  dB as  $W \rightarrow \infty$

# Lecture 15

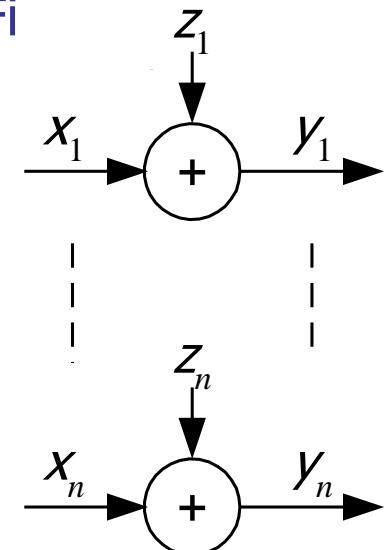
---

- Parallel Gaussian Channels
  - Waterfilling
- Gaussian Channel with Feedback
  - Memoryless: no gain
  - Memory: at most  $\frac{1}{2}$  bits/transmission

# Parallel Gaussian Channels

---

- $n$  independent Gaussian channels
  - A model for nonwhite noise wideband channel where each component represents a different frequency
  - e.g. digital audio, digital TV, Broadband ADSL, WiFi (multicarrier/OFDM)
- Noise is independent  $z_i \sim N(0, N_i)$
- Average Power constraint  $E\mathbf{x}^T\mathbf{x} \leq nP$
- Information Capacity:  $C = \max_{f(\mathbf{x}): E_f \mathbf{x}^T \mathbf{x} \leq nP} I(\mathbf{x}; \mathbf{y})$
- $R < C \Leftrightarrow R$  achievable
  - proof as before
- What is the optimal  $f(\mathbf{x})$  ?



# Parallel Gaussian: Max Capacity

---

Need to find  $f(\mathbf{x})$ :  $C = \max_{f(\mathbf{x}): E_f \mathbf{x}^T \mathbf{x} \leq nP} I(\mathbf{x}; \mathbf{y})$

$$I(\mathbf{x}; \mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y} | \mathbf{x}) = h(\mathbf{y}) - h(\mathbf{z} | \mathbf{x})$$

Translation invariance

$$= h(\mathbf{y}) - h(\mathbf{z}) = h(\mathbf{y}) - \sum_{i=1}^n h(z_i)$$

$\mathbf{x}, \mathbf{z}$  indep;  $Z_i$  indep

$$\stackrel{(a)}{\leq} \sum_{i=1}^n (h(y_i) - h(z_i)) \stackrel{(b)}{\leq} \sum_{i=1}^n \frac{1}{2} \log (1 + P_i N_i^{-1})$$

(a) indep bound;  
(b) capacity limit

Equality when: (a)  $y_i$  indep  $\Rightarrow x_i$  indep; (b)  $x_i \sim N(0, P_i)$

We need to find the  $P_i$  that maximise  $\sum_{i=1}^n \frac{1}{2} \log (1 + P_i N_i^{-1})$

# Parallel Gaussian: Optimal Powers

---

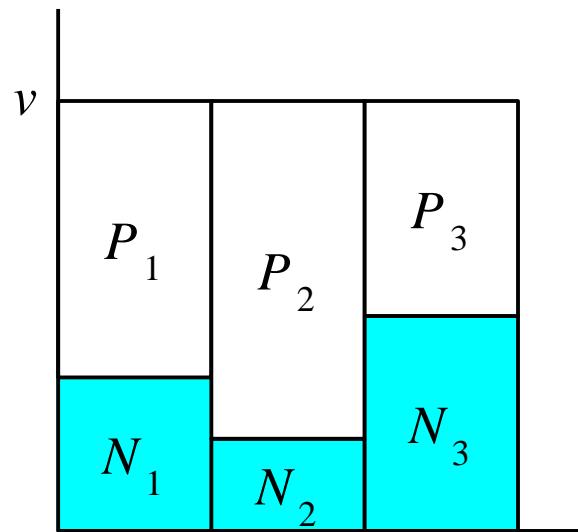
We need to find the  $P_i$  that maximise  $\log(e) \sum_{i=1}^n \frac{1}{2} \ln(1 + P_i N_i^{-1})$

- subject to power constraint  $\sum_{i=1}^n P_i = nP$
- use Lagrange multiplier

$$J = \sum_{i=1}^n \frac{1}{2} \ln(1 + P_i N_i^{-1}) - \lambda \sum_{i=1}^n P_i$$

$$\frac{\partial J}{\partial P_i} = \frac{1}{2} (P_i + N_i)^{-1} - \lambda = 0 \quad \Rightarrow \quad P_i + N_i = v$$

$$\text{Also } \sum_{i=1}^n P_i = nP \quad \Rightarrow \quad v = P + n^{-1} \sum_{i=1}^n N_i$$



**Water Filling:** put most power into least noisy channels to make equal power + noise in each channel

# Very Noisy Channels

---

- What if water is not enough?
- Must have  $P_i \geq 0 \forall i$
- If  $v < N_i$  then set  $P_i = 0$  and recalculate  
 $v$  (i.e.,  $P_i = \max(v - N_i, 0)$ )

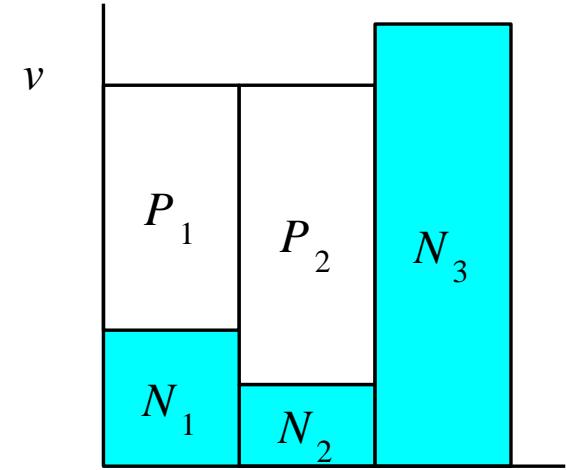
Kuhn-Tucker Conditions:

(not examinable)

- Max  $f(\mathbf{x})$  subject to  $\mathbf{Ax} + \mathbf{b} = \mathbf{0}$  and  

$$g_i(\mathbf{x}) \geq 0 \quad \text{for } i \in 1 : M \quad \text{with } f, g_i \text{ concave}$$
- set  $J(\mathbf{x}) = f(\mathbf{x}) - \sum_{i=1}^M \mu_i g_i(\mathbf{x}) - \boldsymbol{\lambda}^T \mathbf{Ax}$
- Solution  $\mathbf{x}_0, \boldsymbol{\lambda}, \mu_i$  iff

$$\nabla J(\mathbf{x}_0) = \mathbf{0}, \quad \mathbf{Ax} + \mathbf{b} = \mathbf{0}, \quad g_i(\mathbf{x}_0) \geq 0, \quad \mu_i \geq 0, \quad \mu_i g_i(\mathbf{x}_0) = 0$$



# Colored Gaussian Noise

---

- Suppose  $\mathbf{y} = \mathbf{x} + \mathbf{z}$  where  $E \mathbf{zz}^T = \mathbf{K}_z$  and  $E \mathbf{xx}^T = \mathbf{K}_x$
- We want to find  $\mathbf{K}_x$  to maximize capacity subject to power constraint:  $E \sum_{i=1}^n x_i^2 \leq nP \Leftrightarrow \text{tr}(\mathbf{K}_x) \leq nP$ 
  - Find noise eigenvectors:  $\mathbf{K}_z = \mathbf{Q} \Lambda \mathbf{Q}^T$  with  $\mathbf{QQ}^T = \mathbf{I}$
  - Now  $\mathbf{Q}^T \mathbf{y} = \mathbf{Q}^T \mathbf{x} + \mathbf{Q}^T \mathbf{z} = \mathbf{Q}^T \mathbf{x} + \mathbf{w}$   
where  $E \mathbf{ww}^T = E \mathbf{Q}^T \mathbf{zz}^T \mathbf{Q} = E \mathbf{Q}^T \mathbf{K}_z \mathbf{Q} = \Lambda$  is diagonal
    - $\Rightarrow W_i$  are now independent (so previous result on P.G.C. applies)
  - Power constraint is unchanged  $\text{tr}(\mathbf{Q}^T \mathbf{K}_x \mathbf{Q}) = \text{tr}(\mathbf{K}_x \mathbf{QQ}^T) = \text{tr}(\mathbf{K}_x)$
  - Use water-filling and indep. messages  $\mathbf{Q}^T \mathbf{K}_x \mathbf{Q} + \Lambda = v\mathbf{I}$
  - Choose  $\mathbf{Q}^T \mathbf{K}_x \mathbf{Q} = v\mathbf{I} - \Lambda$  where  $v = P + n^{-1} \text{tr}(\Lambda)$ 

$$\Rightarrow \mathbf{K}_x = \mathbf{Q} (v\mathbf{I} - \Lambda) \mathbf{Q}^T$$

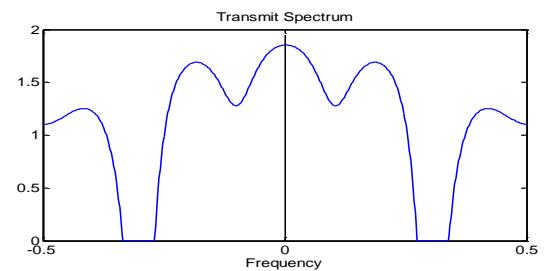
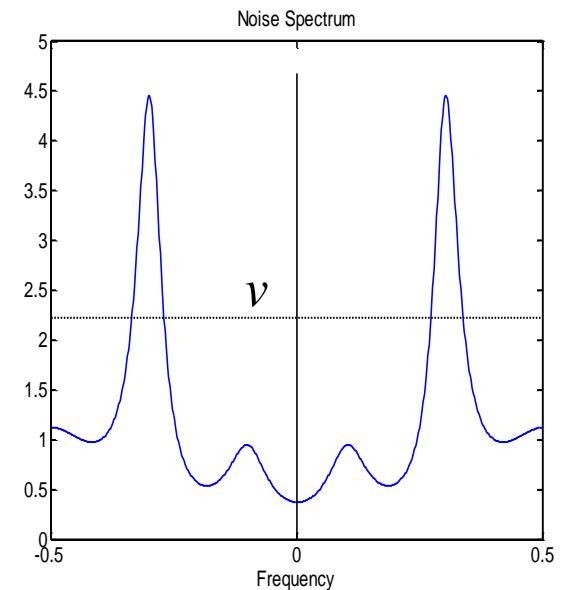
# Power Spectrum Water Filling

- If  $\mathbf{z}$  is from a stationary process then  $\text{diag}(\Lambda) \xrightarrow[n \rightarrow \infty]{} \text{power spectrum } N(f)$ 
  - To achieve capacity use waterfilling on noise power spectrum

$$P = \int_{-W}^W \max(v - N(f), 0) df$$

$$C = \int_{-W}^W \frac{1}{2} \log \left( 1 + \frac{\max(v - N(f), 0)}{N(f)} \right) df$$

- Waterfilling on spectral domain

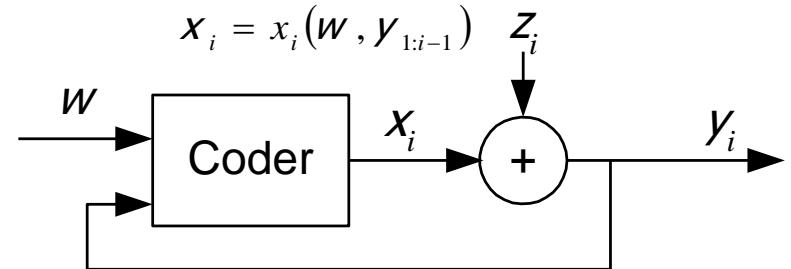


# Gaussian Channel + Feedback

Does Feedback add capacity ?

- White noise (& DMC) – No
- Coloured noise – Not much

$$I(w; \mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y} | w) = h(\mathbf{y}) - \sum_{i=1}^n h(y_i | w, y_{1:i-1})$$



Chain rule

$$= h(\mathbf{y}) - \sum_{i=1}^n h(y_i | w, y_{1:i-1}, x_{1:i}, z_{1:i-1})$$

$$\mathbf{x}_i = x_i(w, y_{1:i-1}), \mathbf{z} = \mathbf{y} - \mathbf{x}$$

$$= h(\mathbf{y}) - \sum_{i=1}^n h(z_i | w, y_{1:i-1}, x_{1:i}, z_{1:i-1})$$

$\mathbf{z} = \mathbf{y} - \mathbf{x}$  and translation invariance

$$= h(\mathbf{y}) - \sum_{i=1}^n h(z_i | z_{1:i-1})$$

$\mathbf{z}$  may be colored;  $z_i$  depends only on  $z_{1:i-1}$

Chain rule,  $h(\mathbf{z}) = \frac{1}{2} \log(|2\pi e \mathbf{K}_{\mathbf{z}}|)$  bits

$\Rightarrow$  maximize  $I(w; \mathbf{y})$  by maximizing  $h(\mathbf{y}) \Rightarrow \mathbf{y}$  gaussian

$$\leq \frac{1}{2} \log \frac{|\mathbf{K}_{\mathbf{y}}|}{|\mathbf{K}_{\mathbf{z}}|}$$

$\Rightarrow$  we can take  $\mathbf{z}$  and  $\mathbf{x} = \mathbf{y} - \mathbf{z}$  jointly gaussian

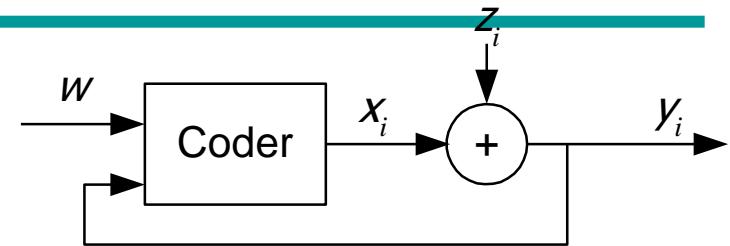
# Maximum Benefit of Feedback

$$C_{n,FB} = \max_{\text{tr}(\mathbf{K}_x) \leq nP} \frac{1}{2} n^{-1} \log \frac{|\mathbf{K}_y|}{|\mathbf{K}_z|}$$

$$\leq \max_{\text{tr}(\mathbf{K}_x) \leq nP} \frac{1}{2} n^{-1} \log \frac{|2(\mathbf{K}_x + \mathbf{K}_z)|}{|\mathbf{K}_z|}$$

$$= \max_{\text{tr}(\mathbf{K}_x) \leq nP} \frac{1}{2} n^{-1} \log \frac{2^n |\mathbf{K}_x + \mathbf{K}_z|}{|\mathbf{K}_z|}$$

$$= \frac{1}{2} + \max_{\text{tr}(\mathbf{K}_x) \leq nP} \frac{1}{2} n^{-1} \log \frac{|\mathbf{K}_x + \mathbf{K}_z|}{|\mathbf{K}_z|} = \frac{1}{2} + C_n \text{ bits / transmission}$$



Lemmas 1 & 2:

$$|2(\mathbf{K}_x + \mathbf{K}_z)| \geq |\mathbf{K}_y|$$

$$|k\mathbf{A}| = k^n |\mathbf{A}|$$

$\mathbf{K}_y = \mathbf{K}_x + \mathbf{K}_z$  if no feedback

$C_n$ : capacity without feedback

Having feedback adds at most  $\frac{1}{2}$  bit per transmission for colored Gaussian noise channels

# Max Benefit of Feedback: Lemmas

---

**Lemma 1:**  $\mathbf{K}_{\mathbf{x}+\mathbf{z}} + \mathbf{K}_{\mathbf{x}-\mathbf{z}} = 2(\mathbf{K}_{\mathbf{x}} + \mathbf{K}_{\mathbf{z}})$

$$\begin{aligned}\mathbf{K}_{\mathbf{x}+\mathbf{z}} + \mathbf{K}_{\mathbf{x}-\mathbf{z}} &= E(\mathbf{x} + \mathbf{z})(\mathbf{x} + \mathbf{z})^T + E(\mathbf{x} - \mathbf{z})(\mathbf{x} - \mathbf{z})^T \\ &= E(\mathbf{xx}^T + \mathbf{xz}^T + \mathbf{zx}^T + \mathbf{zz}^T + \mathbf{xx}^T - \mathbf{xz}^T - \mathbf{zx}^T + \mathbf{zz}^T) \\ &= E(2\mathbf{xx}^T + 2\mathbf{zz}^T) = 2(\mathbf{K}_{\mathbf{x}} + \mathbf{K}_{\mathbf{z}})\end{aligned}$$

**Lemma 2:** If  $\mathbf{F}, \mathbf{G}$  are positive definite then  $|\mathbf{F} + \mathbf{G}| \geq |\mathbf{F}|$

Consider two indep random vectors  $\mathbf{f} \sim N(0, \mathbf{F}), \mathbf{g} \sim N(0, \mathbf{G})$

$$\begin{aligned}\tfrac{1}{2} \log \left( (2\pi e)^n |\mathbf{F} + \mathbf{G}| \right) &= h(\mathbf{f} + \mathbf{g}) \\ &\geq h(\mathbf{f} + \mathbf{g} | \mathbf{g}) = h(\mathbf{f} | \mathbf{g}) \\ &= h(\mathbf{f}) = \tfrac{1}{2} \log \left( (2\pi e)^n |\mathbf{F}| \right)\end{aligned}$$

Conditioning reduces  $h()$   
Translation invariance

$\mathbf{f}, \mathbf{g}$  independent

Hence:  $|2(\mathbf{K}_{\mathbf{x}} + \mathbf{K}_{\mathbf{z}})| = |\mathbf{K}_{\mathbf{x}+\mathbf{z}} + \mathbf{K}_{\mathbf{x}-\mathbf{z}}| \geq |\mathbf{K}_{\mathbf{x}+\mathbf{z}}| = |\mathbf{K}_{\mathbf{y}}|$

# Gaussian Feedback Coder

$\mathbf{x}$  and  $\mathbf{z}$  jointly gaussian  $\Rightarrow$

$$\mathbf{x} = \mathbf{B}\mathbf{z} + \mathbf{v}(w)$$

where  $\mathbf{v}$  is indep of  $\mathbf{z}$  and

$\mathbf{B}$  is strictly lower triangular since  $x_i$  indep of  $z_j$  for  $j > i$ .

$$\mathbf{y} = \mathbf{x} + \mathbf{z} = (\mathbf{B} + \mathbf{I})\mathbf{z} + \mathbf{v}$$

$$\mathbf{K}_y = E\mathbf{yy}^T = E((\mathbf{B} + \mathbf{I})\mathbf{zz}^T(\mathbf{B} + \mathbf{I})^T + \mathbf{vv}^T) = (\mathbf{B} + \mathbf{I})\mathbf{K}_z(\mathbf{B} + \mathbf{I})^T + \mathbf{K}_v$$

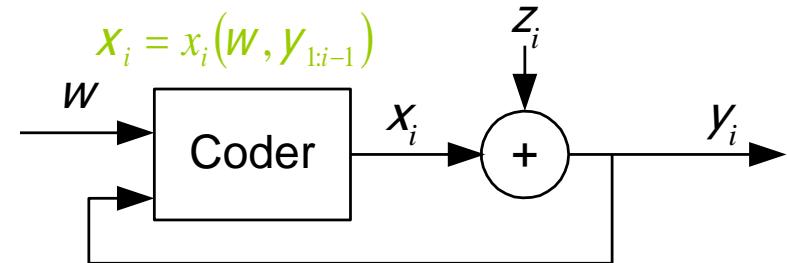
$$\mathbf{K}_x = E\mathbf{xx}^T = E(\mathbf{B}\mathbf{zz}^T\mathbf{B}^T + \mathbf{vv}^T) = \mathbf{BK}_z\mathbf{B}^T + \mathbf{K}_v$$

Capacity:  $C_{n,FB} = \max_{\mathbf{K}_v, \mathbf{B}} \frac{1}{2} n^{-1} \frac{|\mathbf{K}_y|}{|\mathbf{K}_z|} = \max_{\mathbf{K}_v, \mathbf{B}} \frac{1}{2} n^{-1} \log \frac{|(\mathbf{B} + \mathbf{I})\mathbf{K}_z(\mathbf{B} + \mathbf{I})^T + \mathbf{K}_v|}{|\mathbf{K}_z|}$

subject to  $\mathbf{K}_x = \text{tr}(\mathbf{BK}_z\mathbf{B}^T + \mathbf{K}_v) \leq nP$

hard to solve ☺

Optimization can be done numerically



# Gaussian Feedback: Toy Example

$$n = 2, \quad P = 2, \quad \mathbf{K}_z = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{B}\mathbf{z} + \mathbf{v} \Rightarrow x_1 = v_1, x_2 = bz_1 + v_2$$

Goal: Maximize (w.r.t.  $\mathbf{K}_v$  and  $b$ )

$$|\mathbf{K}_y| = |(\mathbf{B} + \mathbf{I})\mathbf{K}_z(\mathbf{B} + \mathbf{I})^T + \mathbf{K}_v|$$

Subject to:

$\mathbf{K}_v$  must be positive definite

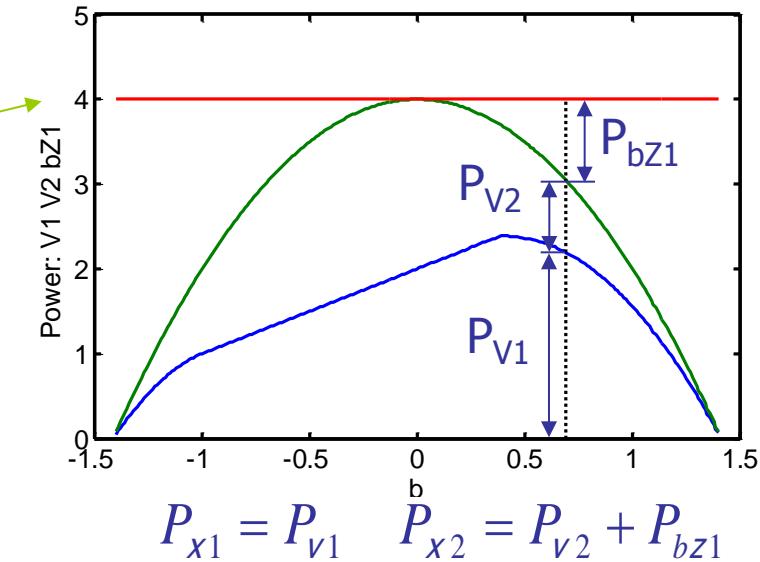
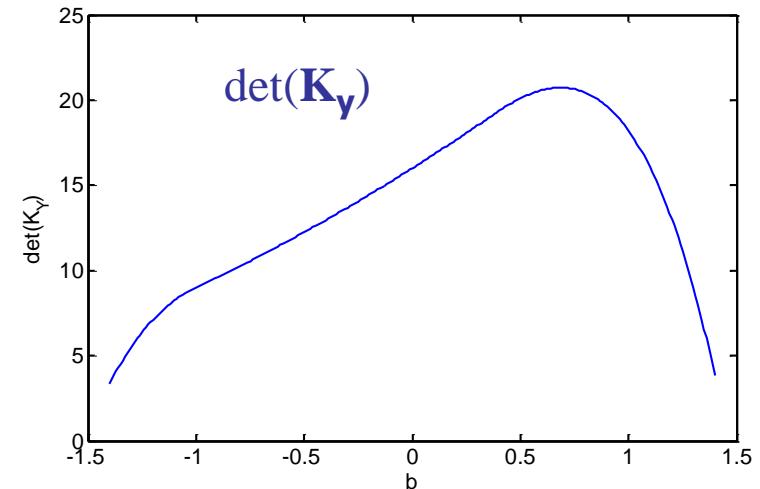
$$\text{Power constraint : } \text{tr}(\mathbf{B}\mathbf{K}_z\mathbf{B}^T + \mathbf{K}_v) \leq 4$$

Solution (via numerically search):

$$b=0: \quad |\mathbf{K}_y|=16 \quad C=0.604 \text{ bits}$$

$$b=0.69: \quad |\mathbf{K}_y|=20.7 \quad C=0.697 \text{ bits}$$

Feedback increases  $C$  by 16%



# Summary

---

- Water-filling for parallel Gaussian channel

$$C = \sum_{i=1}^n \frac{1}{2} \log \left( 1 + \frac{(v - N_i)^+}{N_i} \right) \quad x^+ = \max(x, 0)$$

$$\sum (v - N_i)^+ = nP$$

- Colored Gaussian noise

$$C = \sum_{i=1}^n \frac{1}{2} \log \left( 1 + \frac{(v - \lambda_i)^+}{\lambda_i} \right) \quad \lambda_i \text{ eigenvalues of } \mathbf{K}_z$$

$$\sum (v - \lambda_i)^+ = nP$$

- Continuous Gaussian channel

$$C = \int_{-W}^W \frac{1}{2} \log \left( 1 + \frac{(v - N(f))^+}{N(f)} \right) df$$

- Feedback bound

$$C_{n,FB} \leq C_n + \frac{1}{2}$$

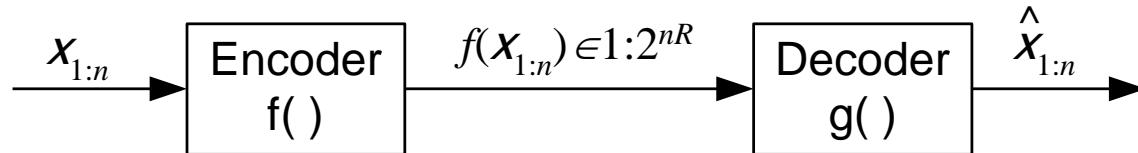
# Lecture 16

---

- Lossy Source Coding
  - For both discrete and continuous sources
  - Bernoulli Source, Gaussian Source
- Rate Distortion Theory
  - What is the minimum distortion achievable at a particular rate?
  - What is the minimum rate to achieve a particular distortion?
- Channel/Source Coding Duality

# Lossy Source Coding

---



**Distortion function:**  $d(x, \hat{x}) \geq 0$

- examples: (i)  $d_s(x, \hat{x}) = (x - \hat{x})^2$       (ii)  $d_H(x, \hat{x}) = \begin{cases} 0 & x = \hat{x} \\ 1 & x \neq \hat{x} \end{cases}$
- sequences:  $d(\mathbf{x}, \hat{\mathbf{x}}) = n^{-1} \sum_{i=1}^n d(x_i, \hat{x}_i)$

**Distortion of Code  $f_n( ), g_n( )$ :**  $D = E_{\mathbf{x} \in X^n} d(\mathbf{x}, \hat{\mathbf{x}}) = E d(\mathbf{x}, g(f(\mathbf{x})))$

**Rate distortion pair (R,D) is achievable for source  $X$  if**

$\exists$  a sequence  $f_n( )$  and  $g_n( )$  such that  $\lim_{n \rightarrow \infty} E_{\mathbf{x} \in X^n} d(\mathbf{x}, g_n(f_n(\mathbf{x}))) \leq D$

# Rate Distortion Function

---

Rate Distortion function for  $\{X_i\}$  with pdf  $p(\mathbf{x})$  is defined as

$$R(D) = \min\{R\} \text{ such that } (R, D) \text{ is achievable}$$

**Theorem:**  $R(D) = \min I(X; \hat{X})$  over all  $p(x, \hat{x})$  such that :

- (a)  $p(x)$  is correct
- (b)  $E_{x, \hat{x}} d(x, \hat{x}) \leq D$



- this expression is the Rate Distortion function for  $X$

Proof is not examinable

**Lossless coding:** If  $D = 0$  then we have  $R(D) = I(X; X) = H(X)$

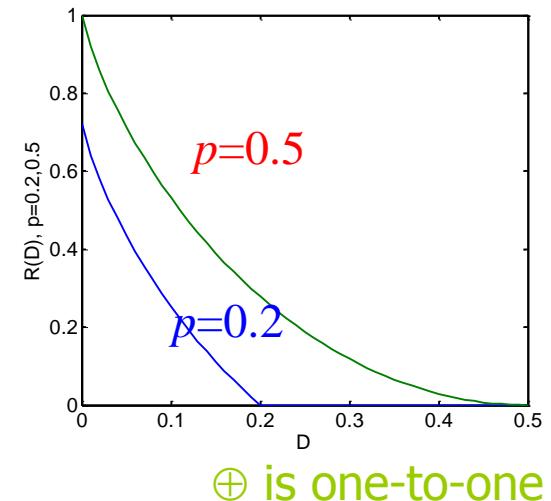
◆  $p(x, \hat{x}) = p(x)q(\hat{x} | x)$

# $R(D)$ bound for Bernoulli Source

Bernoulli:  $X = [0,1]$ ,  $p_X = [1-p, p]$  assume  $p \leq \frac{1}{2}$

- Hamming Distance:  $d(x, \hat{x}) = x \oplus \hat{x}$
- If  $D \geq p$ ,  $R(D) = 0$  since we can set  $g(\cdot) \equiv 0$
- For  $D < p \leq \frac{1}{2}$ , if  $E d(x, \hat{x}) \leq D$  then

$$\begin{aligned} I(x; \hat{x}) &= H(x) - H(x | \hat{x}) \\ &= H(p) - H(x \oplus \hat{x} | \hat{x}) \\ &\geq H(p) - H(x \oplus \hat{x}) \\ &\geq H(p) - H(D) \end{aligned}$$



Conditioning reduces entropy

Prob.( $x \oplus \hat{x} = 1$ )  $\leq D$  for  $D \leq \frac{1}{2}$

$H(x \oplus \hat{x}) \leq H(D)$  as  $H(p)$  monotonic

Hence  $R(D) \geq H(p) - H(D)$

# $R(D)$ for Bernoulli source

---

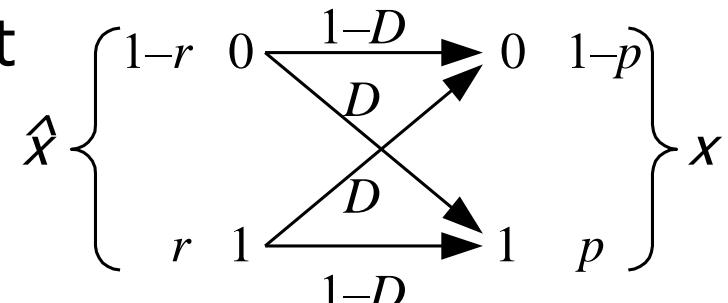
We know optimum satisfies  $R(D) \geq H(p) - H(D)$

- We show we can find a  $p(\hat{x}, x)$  that attains this.
- Peculiarly, we consider a **channel** with  $\hat{x}$  as the **input** and error probability  $D$

Now choose  $r$  to give  $x$  the correct probabilities:

$$r(1 - D) + (1 - r)D = p$$

$$\Rightarrow r = (p - D)(1 - 2D)^{-1}, \quad D \leq p$$



$$\text{Now } I(x; \hat{x}) = H(x) - H(x | \hat{x}) = H(p) - H(D)$$

$$\text{and } p(x \neq \hat{x}) = D \Rightarrow \text{distortion} \leq D$$

$$\text{Hence } R(D) = H(p) - H(D)$$

If  $D \geq p$  or  $D \geq 1-p$ , we can achieve  $R(D)=0$  trivially.

# $R(D)$ bound for Gaussian Source

---

- Assume  $X \sim N(0, \sigma^2)$  and  $d(x, \hat{x}) = (x - \hat{x})^2$
- Want to minimize  $I(X; \hat{X})$  subject to  $E(X - \hat{X})^2 \leq D$

$$\begin{aligned}
 I(X; \hat{X}) &= h(X) - h(X | \hat{X}) \\
 &= \frac{1}{2} \log 2\pi e \sigma^2 - h(X - \hat{X} | \hat{X}) && \text{Translation Invariance} \\
 &\geq \frac{1}{2} \log 2\pi e \sigma^2 - h(X - \hat{X}) && \text{Conditioning reduces entropy} \\
 &\geq \frac{1}{2} \log 2\pi e \sigma^2 - \frac{1}{2} \log (2\pi e \operatorname{Var}(X - \hat{X})) && \text{Gauss maximizes entropy} \\
 &\geq \frac{1}{2} \log 2\pi e \sigma^2 - \frac{1}{2} \log 2\pi e D && \text{require } \operatorname{Var}(X - \hat{X}) \leq E(X - \hat{X})^2 \leq D
 \end{aligned}$$

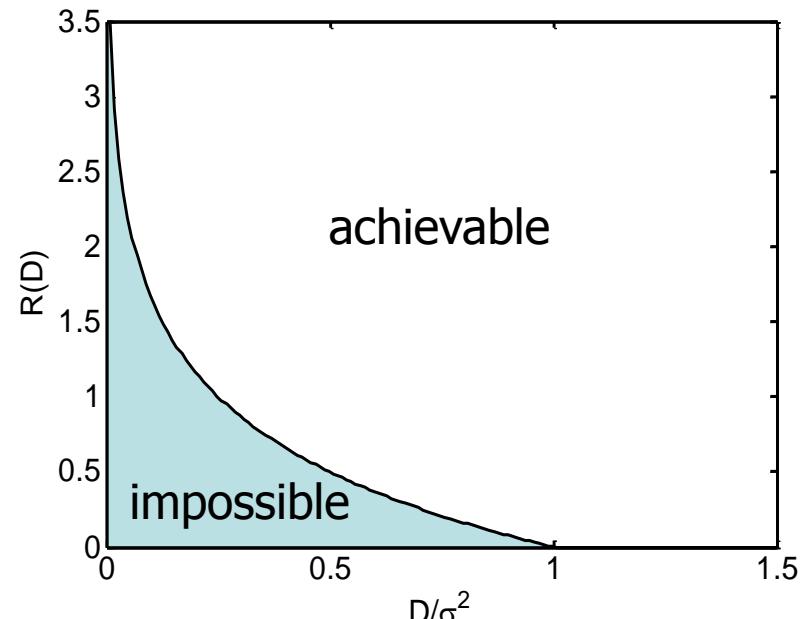
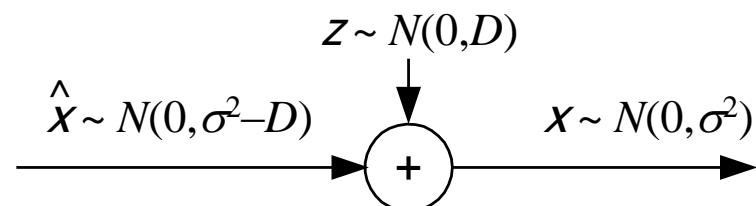
$$I(X; \hat{X}) \geq \max \left( \frac{1}{2} \log \frac{\sigma^2}{D}, 0 \right) \quad I(X; Y) \text{ always positive}$$

# $R(D)$ for Gaussian Source

To show that we can find a  $p(\hat{x}, x)$  that achieves the bound, we construct a **test channel** that introduces distortion  $D < \sigma^2$

$$\begin{aligned} I(X; \hat{X}) &= h(X) - h(X | \hat{X}) \\ &= \frac{1}{2} \log 2\pi e \sigma^2 - h(X - \hat{X} | \hat{X}) \\ &= \frac{1}{2} \log 2\pi e \sigma^2 - h(Z | \hat{X}) \\ &= \frac{1}{2} \log \frac{\sigma^2}{D} \\ \Rightarrow R(D) &= \max \left\{ \frac{1}{2} \log \frac{\sigma^2}{D}, 0 \right\} \end{aligned}$$

$$\Rightarrow D(R) = \frac{\sigma^2}{2^{2R}} \quad \text{cf. PCM} \quad D(R) = \frac{m_p^2 / 3}{2^{2R}} = \frac{16 / 3 \cdot \sigma^2}{2^{2R}}$$

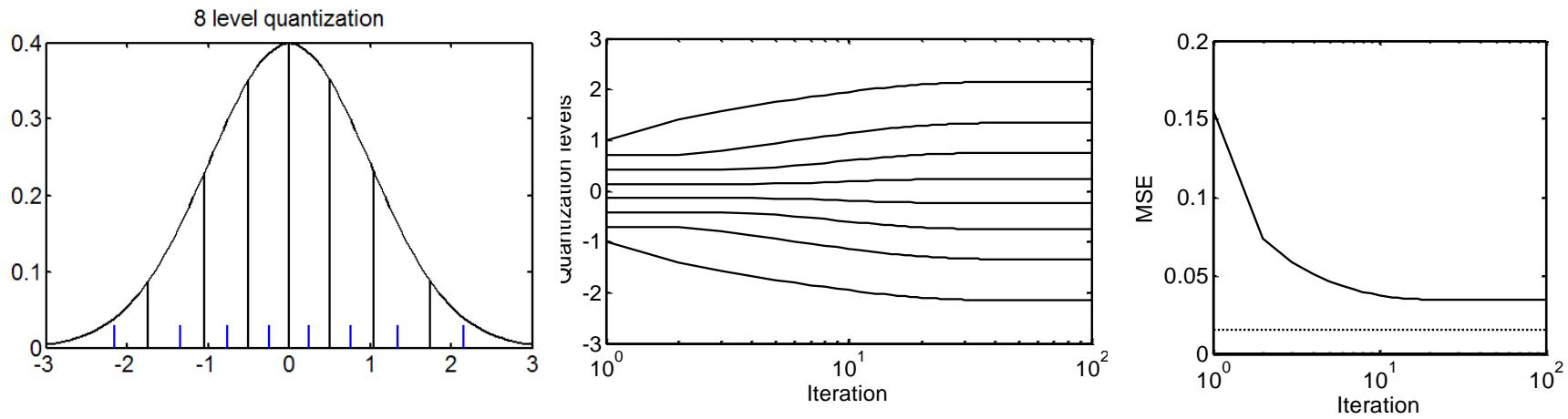


# Lloyd Algorithm

**Problem:** Find optimum quantization levels for Gaussian pdf

- a. Bin boundaries are midway between quantization levels
- b. Each quantization level equals the mean value of its own bin

**Lloyd algorithm:** Pick random quantization levels then apply conditions (a) and (b) in turn until convergence.



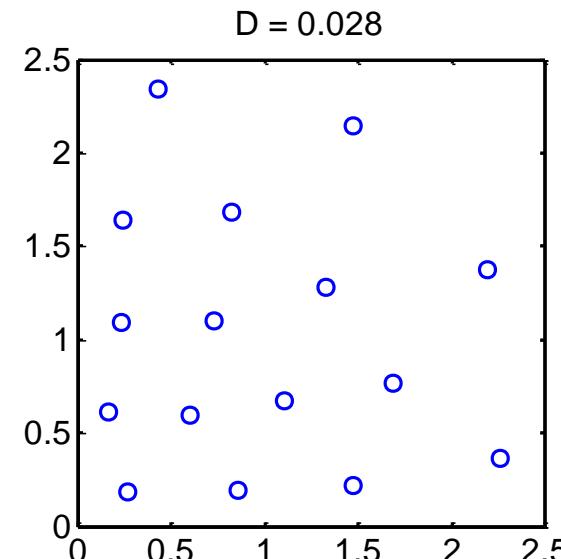
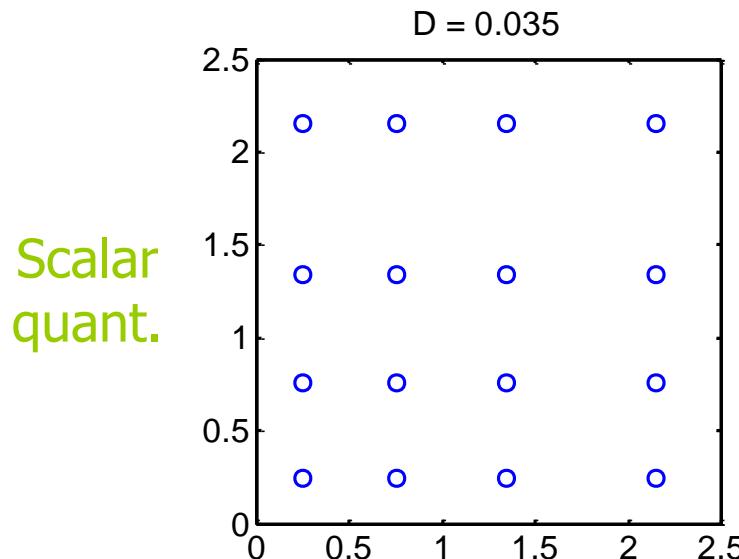
Solid lines are bin boundaries. Initial levels uniform in  $[-1, +1]$ .

Best mean sq error for 8 levels =  $0.0345\sigma^2$ . Predicted  $D(R) = (\sigma/8)^2 = 0.0156\sigma^2$

# Vector Quantization

To get  $D(R)$ , you have to quantize many values together

- True even if the values are independent



Two gaussian variables: one quadrant only shown

- Independent quantization puts dense levels in low prob areas
- Vector quantization is better (even more so if correlated)

# Multiple Gaussian Variables

---

- Assume  $x_{1:n}$  are independent gaussian sources with different variances. How should we apportion the available total distortion between the sources?
- Assume  $x_i \sim N(0, \sigma_i^2)$  and  $d(\mathbf{x}, \hat{\mathbf{x}}) = n^{-1}(\mathbf{x} - \hat{\mathbf{x}})^T(\mathbf{x} - \hat{\mathbf{x}}) \leq D$

$$I(x_{1:n}; \hat{x}_{1:n}) \geq \sum_{i=1}^n I(x_i; \hat{x}_i)$$

Mut Info Independence Bound  
for independent  $x_i$

$$\geq \sum_{i=1}^n R(D_i) = \sum_{i=1}^n \max\left(\frac{1}{2} \log \frac{\sigma_i^2}{D_i}, 0\right)$$

$R(D)$  for individual Gaussian

We must find the  $D_i$  that minimize

$$\sum_{i=1}^n \max\left(\frac{1}{2} \log \frac{\sigma_i^2}{D_i}, 0\right)$$

$$\Rightarrow D_i = \begin{cases} D_0 & \text{if } D_0 < \sigma_i^2 \\ \sigma_i^2 & \text{otherwise} \end{cases}$$

such that  $n^{-1} \sum_{i=1}^n D_i = D$

# Reverse Water-filling

Minimize  $\sum_{i=1}^n \max\left(\frac{1}{2} \log \frac{\sigma_i^2}{D_i}, 0\right)$  subject to  $\sum_{i=1}^n D_i \leq nD$

$$R_i = \frac{1}{2} \log \frac{\sigma_i^2}{D}$$

Use a Lagrange multiplier:

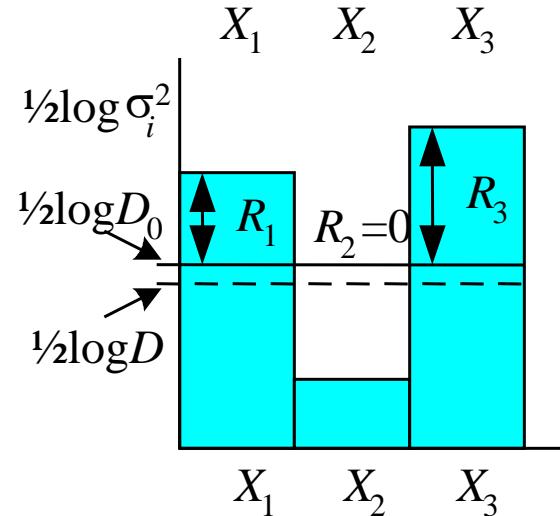
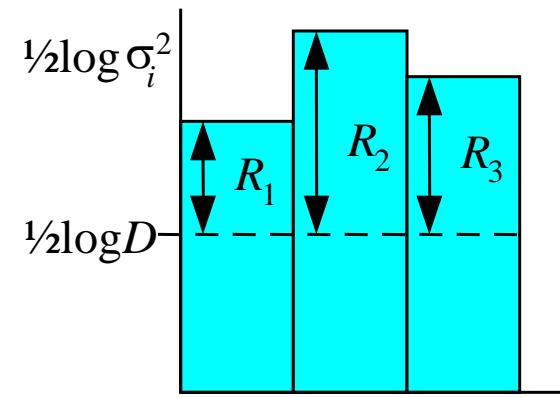
$$J = \sum_{i=1}^n \frac{1}{2} \log \frac{\sigma_i^2}{D_i} + \lambda \sum_{i=1}^n D_i$$

$$\frac{\partial J}{\partial D_i} = -\frac{1}{2} D_i^{-1} + \lambda = 0 \Rightarrow D_i = \frac{1}{2} \lambda^{-1} = D_0$$

$$\sum_{i=1}^n D_i = nD_0 = nD \Rightarrow D_0 = D$$

Choose  $R_i$  for equal distortion

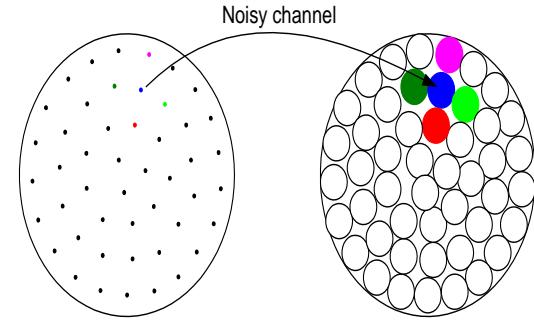
- If  $\sigma_i^2 < D$  then set  $R_i = 0$  (meaning  $D_i = \sigma_i^2$ ) and increase  $D_0$  to maintain the average distortion equal to  $D$



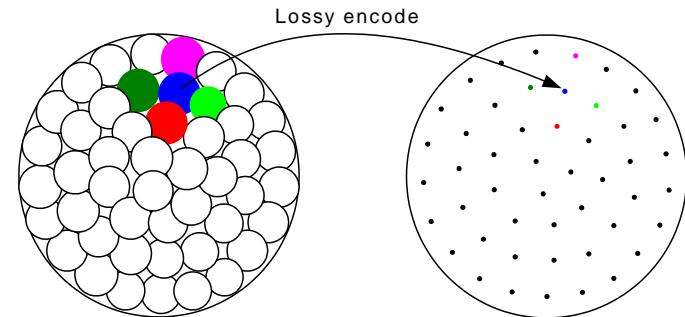
# Channel/Source Coding Duality

---

- **Channel Coding**
  - Find codes separated enough to give non-overlapping output images.
  - Image size = channel noise
  - The maximum number (highest rate) is when the images just don't overlap (some gap).
  
- **Source Coding**
  - Find regions that cover the sphere
  - Region size = allowed distortion
  - The minimum number (lowest rate) is when they just fill the sphere (**with no gap**).



Sphere Packing



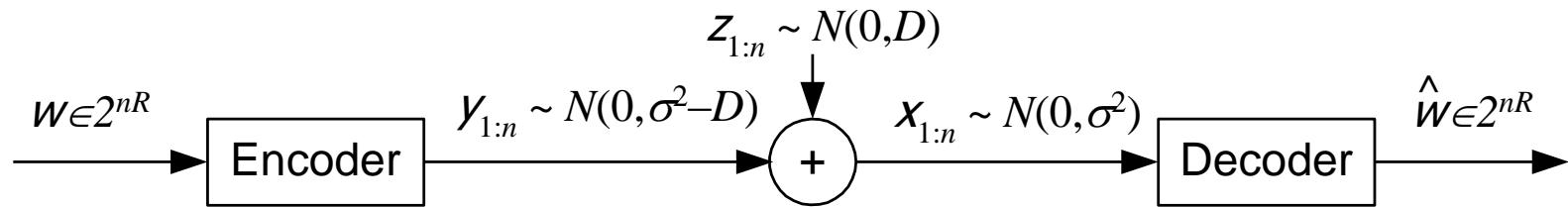
Sphere Covering

# Gaussian Channel/Source

---

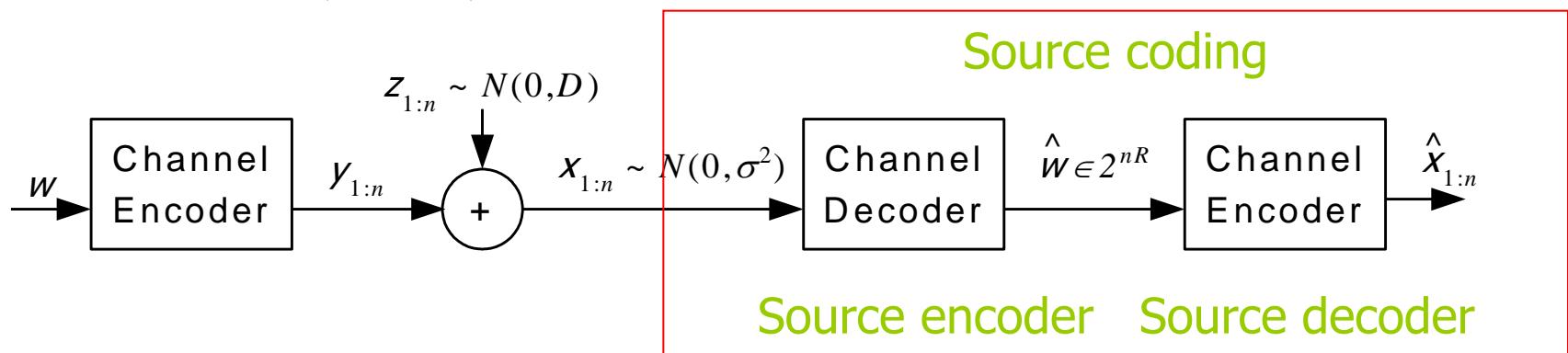
- Capacity of Gaussian channel ( $n$ : length)
  - Radius of big sphere  $\sqrt{n(P + N)}$
  - Radius of small spheres  $\sqrt{nN}$
  - Capacity  $2^{nC} = \frac{\sqrt{n(P + N)}^n}{\sqrt{nN}^n} = \left(\frac{P + N}{N}\right)^{n/2}$  Maximum number of small spheres packed in the big sphere
- Rate distortion for Gaussian source
  - Variance  $\sigma^2 \rightarrow$  radius of big sphere  $\sqrt{n\sigma^2}$
  - Radius of small spheres  $\sqrt{nD}$  for distortion  $D$
  - Rate  $2^{nR(D)} = \left(\frac{\sigma^2}{D}\right)^{n/2}$  Minimum number of small spheres to cover the big sphere

# Channel Decoder as Source Encoder



- For  $R \cong C = \frac{1}{2} \log \left( 1 + (\sigma^2 - D) D^{-1} \right)$ , we can find a channel encoder/decoder so that  $p(\hat{w} \neq w) < \varepsilon$  and  $E(x_i - y_i)^2 = D$
- Now reverse the roles of encoder and decoder. Since

$$p(\hat{x} \neq y) = p(w \neq \hat{w}) < \varepsilon \text{ and } E(x_i - \hat{x}_i)^2 \cong E(x_i - y_i)^2 = D$$



We have encoded  $x$  at rate  $R = \frac{1}{2} \log(\sigma^2 D^{-1})$  with distortion  $D$ !

# Summary

---

- Lossy source coding: tradeoff between rate and distortion
- Rate distortion function

$$R(D) = \min_{\mathbf{p}_{\hat{x}|x} \text{ s.t. } Ed(x, \hat{x}) \leq D} I(x; \hat{x})$$

- Bernoulli source:  $R(D) = (H(p) - H(D))^+$
- Gaussian source  
(reverse waterfilling):  
$$R(D) = \left( \frac{1}{2} \log \frac{\sigma^2}{D} \right)^+$$
- Duality: channel decoding (encoding)  $\Leftrightarrow$  source encoding (decoding)

# Nothing But Proof

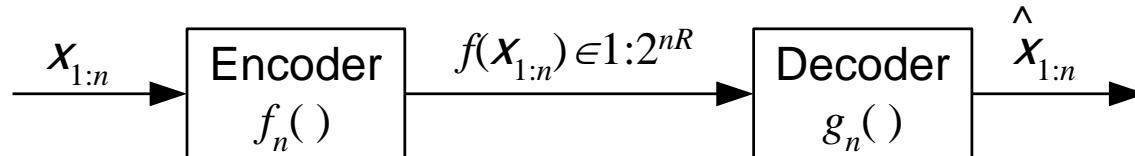
---

- Proof of Rate Distortion Theorem
  - Converse: if the rate is less than  $R(D)$ , then distortion of any code is higher than  $D$
  - Achievability: if the rate is higher than  $R(D)$ , then there exists a rate- $R$  code which achieves distortion  $D$

Quite technical!

# Review

---



Rate Distortion function for  $x$  whose  $p_x(\mathbf{x})$  is known is

$$R(D) = \inf R \text{ such that } \exists f_n, g_n \text{ with } \lim_{n \rightarrow \infty} E_{\mathbf{x} \in X^n} d(\mathbf{x}, \hat{\mathbf{x}}) \leq D$$

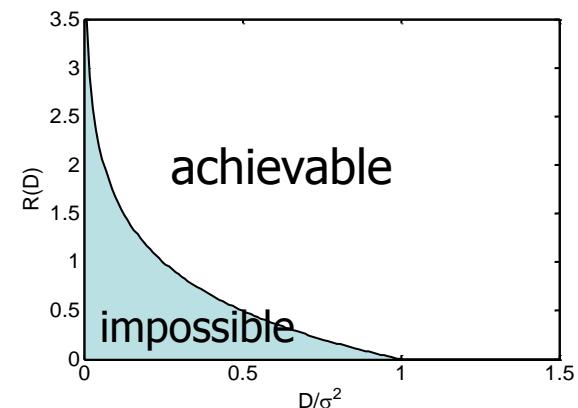
Rate Distortion Theorem:

$$R(D) = \min I(x; \hat{x}) \text{ over all } p(\hat{x} | x) \text{ such that } E_{x, \hat{x}} d(x, \hat{x}) \leq D$$

We will prove this theorem for discrete  $X$  and bounded  $d(x, y) \leq d_{\max}$

$R(D)$  curve depends on your choice of  $d(,)$

Decreasing and convex



# Converse: Rate Distortion Bound

---

Suppose we have found an encoder and decoder at rate  $R_0$  with expected distortion  $D$  for independent  $x_i$  (worst case)

We want to prove that  $R_0 \geq R(D) = R(E d(\mathbf{x}; \hat{\mathbf{x}}))$

- We show first that  $R_0 \geq n^{-1} \sum_i I(x_i; \hat{x}_i)$
- We know that  $I(x_i; \hat{x}_i) \geq R(E d(x_i; \hat{x}_i))$  Def<sup>n</sup> of  $R(D)$
- and use convexity of  $R(D)$  to show

$$n^{-1} \sum_i R(E d(x_i; \hat{x}_i)) \geq R\left(n^{-1} \sum_i E d(x_i; \hat{x}_i)\right) = R(E d(\mathbf{x}; \hat{\mathbf{x}})) = R(D)$$

We prove convexity first and then the rest

# Convexity of $R(D)$

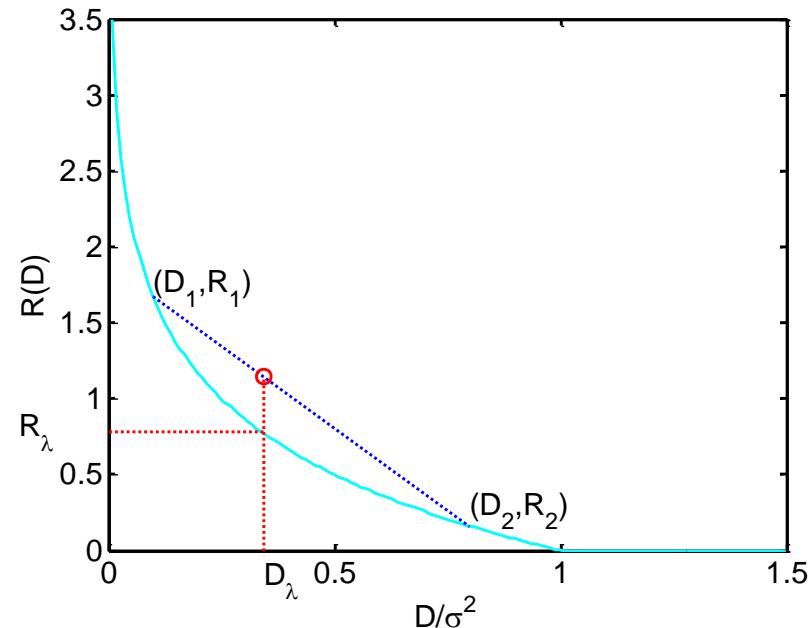
If  $p_1(\hat{x} | x)$  and  $p_2(\hat{x} | x)$  are associated with  $(D_1, R_1)$  and  $(D_2, R_2)$  on the  $R(D)$  curve we define

$$p_\lambda(\hat{x} | x) = \lambda p_1(\hat{x} | x) + (1 - \lambda) p_2(\hat{x} | x)$$

Then

$$E_{p_\lambda} d(x, \hat{x}) = \lambda D_1 + (1 - \lambda) D_2 = D_\lambda$$

$$\begin{aligned} R(D_\lambda) &\leq I_{p_\lambda}(x; \hat{x}) \\ &\leq \lambda I_{p_1}(x; \hat{x}) + (1 - \lambda) I_{p_2}(x; \hat{x}) \\ &= \lambda R(D_1) + (1 - \lambda) R(D_2) \end{aligned}$$



$$R(D) = \min_{p(\hat{x}|x)} I(X; \hat{X})$$

$I(X; \hat{X})$  convex w.r.t.  $p(\hat{x} | x)$

$p_1$  and  $p_2$  lie on the  $R(D)$  curve

# Proof that $R \geq R(D)$

---

$$nR_0 \geq H(\hat{X}_{1:n}) \geq H(\hat{X}_{1:n}) - H(\hat{X}_{1:n} | X_{1:n}) \quad \text{Uniform bound; } H(\hat{X} | X) \geq 0$$

$$= I(\hat{X}_{1:n}; X_{1:n}) \quad \text{Definition of } I();$$

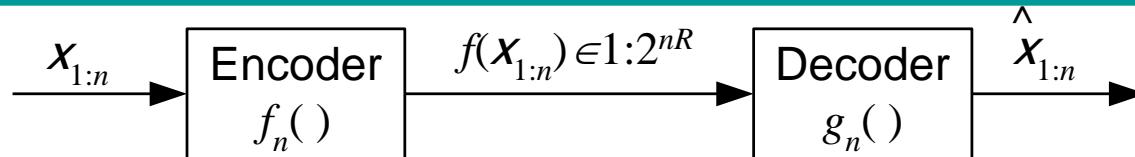
$$\geq \sum_{i=1}^n I(X_i; \hat{X}_i) \quad \begin{matrix} x_i \text{ indep: Mut Inf} \\ \text{Independence Bound} \end{matrix}$$

$$\geq \sum_{i=1}^n R(E d(X_i; \hat{X}_i)) = n \sum_{i=1}^n n^{-1} R(E d(X_i; \hat{X}_i)) \quad \text{definition of } R$$

$$\geq nR\left(n^{-1} \sum_{i=1}^n E d(X_i; \hat{X}_i)\right) = nR(E d(X_{1:n}; \hat{X}_{1:n})) \quad \begin{matrix} \text{convexity} \\ \text{defn of vector } d() \end{matrix}$$

$$\geq nR(D) \quad \begin{matrix} \text{original assumption that } E(d) \leq D \\ \text{and } R(D) \text{ monotonically decreasing} \end{matrix}$$

# Rate Distortion Achievability



We want to show that for any  $D$ , we can find an encoder and decoder that compresses  $x_{1:n}$  to  $nR(D)$  bits.

- $p_x$  is given
- Assume we know the  $p(\hat{x} | x)$  that gives  $I(x; \hat{x}) = R(D)$
- **Random codebook:** Choose  $2^{nR}$  random  $\hat{x}_i \sim p_{\hat{x}}$ 
  - There must be at least one code that is as good as the average
- **Encoder:** Use joint typicality to design
  - We show that there is almost always a suitable codeword

First define the typical set we will use, then prove two preliminary results.

# Distortion Typical Set

---

**Distortion Typical:**  $(x_i, \hat{x}_i) \in X \times \hat{X}$  drawn i.i.d.  $\sim p(x, \hat{x})$

$$\begin{aligned}
 J_{d,\varepsilon}^{(n)} = \left\{ \mathbf{x}, \hat{\mathbf{x}} \in X^n \times \hat{X}^n : \right. & \left| -n^{-1} \log p(\mathbf{x}) - H(X) \right| < \varepsilon, \\
 & \left| -n^{-1} \log p(\hat{\mathbf{x}}) - H(\hat{X}) \right| < \varepsilon, \\
 & \left| -n^{-1} \log p(\mathbf{x}, \hat{\mathbf{x}}) - H(X, \hat{X}) \right| < \varepsilon \\
 & \left. \left| d(\mathbf{x}, \hat{\mathbf{x}}) - E d(X, \hat{X}) \right| < \varepsilon \right\} \quad \text{new condition}
 \end{aligned}$$

**Properties of Typical Set:**

1. Indiv p.d.:  $\mathbf{x}, \hat{\mathbf{x}} \in J_{d,\varepsilon}^{(n)} \Rightarrow \log p(\mathbf{x}, \hat{\mathbf{x}}) = -nH(X, \hat{X}) \pm n\varepsilon$

2. Total Prob:  $p(\mathbf{x}, \hat{\mathbf{x}} \in J_{d,\varepsilon}^{(n)}) > 1 - \varepsilon \quad \text{for } n > N_\varepsilon$

weak law of large numbers;  $d(x_i, \hat{x}_i)$  are i.i.d.

# Conditional Probability Bound

---

**Lemma:**  $\mathbf{x}, \hat{\mathbf{x}} \in J_{d,\varepsilon}^{(n)} \Rightarrow p(\hat{\mathbf{x}}) \geq p(\hat{\mathbf{x}} | \mathbf{x}) 2^{-n(I(x; \hat{x}) + 3\varepsilon)}$

**Proof:** 
$$p(\hat{\mathbf{x}} | \mathbf{x}) = \frac{p(\hat{\mathbf{x}}, \mathbf{x})}{p(\mathbf{x})}$$

$$= p(\hat{\mathbf{x}}) \frac{p(\hat{\mathbf{x}}, \mathbf{x})}{p(\hat{\mathbf{x}})p(\mathbf{x})}$$

take max of top and min of bottom

$$\leq p(\hat{\mathbf{x}}) \frac{2^{-n(H(x, \hat{x}) - \varepsilon)}}{2^{-n(H(x) + \varepsilon)} 2^{-n(H(\hat{x}) + \varepsilon)}}$$

bounds from def<sup>n</sup> of  $J$

$$= p(\hat{\mathbf{x}}) 2^{n(I(x; \hat{x}) + 3\varepsilon)}$$

def<sup>n</sup> of  $I$

# Curious but Necessary Inequality

---

**Lemma:**  $u, v \in [0,1], m > 0 \Rightarrow (1 - uv)^m \leq 1 - u + e^{-vm}$

**Proof:  $u=0$ :**  $e^{-vm} \geq 0 \Rightarrow (1 - 0)^m \leq 1 - 0 + e^{-vm}$

**$u=1$ :** Define  $f(v) = e^{-v} - 1 + v \Rightarrow f'(v) = 1 - e^{-v}$

$f(0) = 0$  and  $f'(v) > 0$  for  $v > 0 \Rightarrow f(v) \geq 0$  for  $v \in [0,1]$

Hence for  $v \in [0,1]$ ,  $0 \leq 1 - v \leq e^{-v} \Rightarrow (1 - v)^m \leq e^{-vm}$

**$0 < u < 1$ :** Define  $g_v(u) = (1 - uv)^m$

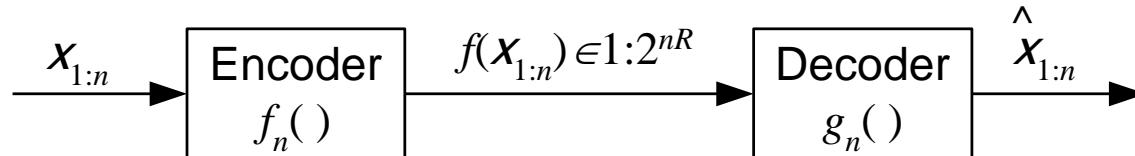
$\Rightarrow g''_v(x) = m(m-1)v^2(1 - uv)^{n-2} \geq 0 \Rightarrow g_v(u)$  convex for  $u, v \in [0,1]$

$(1 - uv)^m = g_v(u) \leq (1 - u)g_v(0) + ug_v(1)$  convexity for  $u, v \in [0,1]$

$$= (1 - u)1 + u(1 - v)^m \leq 1 - u + ue^{-vm} \leq 1 - u + e^{-vm}$$

# Achievability of $R(D)$ : preliminaries

---



- Choose  $D$  and find a  $p(\hat{x} | x)$  such that  $I(x; \hat{x}) = R(D); E d(x, \hat{x}) \leq D$   
Choose  $\delta > 0$  and define  $\mathbf{p}_{\hat{x}} = \{ p(\hat{x}) = \sum_x p(x) p(\hat{x} | x) \}$
- **Decoder:** For each  $w \in 1 : 2^{nR}$  choose  $g_n(w) = \hat{\mathbf{x}}_w$  drawn i.i.d.  $\sim \mathbf{p}_{\hat{x}}^n$
- **Encoder:**  $f_n(\mathbf{x}) = \min w$  such that  $(\mathbf{x}, \hat{\mathbf{x}}_w) \in J_{d, \varepsilon}^{(n)}$  else 1 if no such  $w$
- **Expected Distortion:**  $\overline{D} = E_{\mathbf{x}, g} d(\mathbf{x}, \hat{\mathbf{x}})$ 
  - over all input vectors  $\mathbf{x}$  and all random decoding functions,  $g$
  - for large  $n$  we show  $\overline{D} = D + \delta$  so there must be one good code

# Expected Distortion

---

We can divide the input vectors  $\mathbf{x}$  into two categories:

a) if  $\exists w$  such that  $(\mathbf{x}, \hat{\mathbf{x}}_w) \in J_{d, \varepsilon}^{(n)}$  then  $d(\mathbf{x}, \hat{\mathbf{x}}_w) < D + \varepsilon$

since  $E d(\mathbf{x}, \hat{\mathbf{x}}) \leq D$

b) if no such  $w$  exists we must have  $d(\mathbf{x}, \hat{\mathbf{x}}_w) < d_{\max}$   
 since we are assuming that  $d()$  is bounded. Suppose  
 the probability of this situation is  $P_e$ .

$$\begin{aligned}\text{Hence } \overline{D} &= E_{\mathbf{x}, g} d(\mathbf{x}, \hat{\mathbf{x}}) \\ &\leq (1 - P_e)(D + \varepsilon) + P_e d_{\max} \\ &\leq D + \varepsilon + P_e d_{\max}\end{aligned}$$

We need to show that the expected value of  $P_e$  is small

# Error Probability

---

Define the set of valid inputs for (random) code  $g$

$$V(g) = \left\{ \mathbf{x} : \exists w \text{ with } (\mathbf{x}, g(w)) \in J_{d,\varepsilon}^{(n)} \right\}$$

We have  $P_e = \sum_g p(g) \sum_{\mathbf{x} \notin V(g)} p(\mathbf{x}) = \sum_{\mathbf{x}} p(\mathbf{x}) \sum_{g: \mathbf{x} \notin V(g)} p(g)$  Change the order

Define  $K(\mathbf{x}, \hat{\mathbf{x}}) = 1$  if  $(\mathbf{x}, \hat{\mathbf{x}}) \in J_{d,\varepsilon}^{(n)}$  else 0

Prob that a random  $\hat{\mathbf{x}}$  does not match  $\mathbf{x}$  is  $1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}})$

Prob that an entire code does not match  $\mathbf{x}$  is  $\left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}}) \right)^{2^n R}$

Hence  $P_e = \sum_{\mathbf{x}} p(\mathbf{x}) \left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}}) \right)^{2^n R}$  Codewords are i.i.d.

# Achievability for Average Code

---

Since  $\mathbf{x}, \hat{\mathbf{x}} \in J_{d,\varepsilon}^{(n)} \Rightarrow p(\hat{\mathbf{x}}) \geq p(\hat{\mathbf{x}} | \mathbf{x}) 2^{-n(I(x;\hat{x})+3\varepsilon)}$

$$\begin{aligned} P_e &= \sum_{\mathbf{x}} p(\mathbf{x}) \left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}}) \right)^{2^{nR}} \\ &\leq \sum_{\mathbf{x}} p(\mathbf{x}) \left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}} | \mathbf{x}) K(\mathbf{x}, \hat{\mathbf{x}}) \cdot 2^{-n(I(x;\hat{x})+3\varepsilon)} \right)^{2^{nR}} \end{aligned}$$

Using  $(1 - uv)^m \leq 1 - u + e^{-vm}$

with  $u = \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}} | \mathbf{x}) K(\mathbf{x}, \hat{\mathbf{x}})$ ;  $v = 2^{-nI(x;\hat{x})-3n\varepsilon}$ ;  $m = 2^{nR}$

$$\leq \sum_{\mathbf{x}} p(\mathbf{x}) \left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}} | \mathbf{x}) K(\mathbf{x}, \hat{\mathbf{x}}) + \exp \left( -2^{-n(I(x;\hat{x})+3\varepsilon)} 2^{nR} \right) \right)$$

Note:  $0 \leq u, v \leq 1$  as required

# Achievability for Average Code

---

$$P_e \leq \sum_{\mathbf{x}} p(\mathbf{x}) \left( 1 - \sum_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}} | \mathbf{x}) K(\mathbf{x}, \hat{\mathbf{x}}) + \exp \left( - 2^{-n(I(X; \hat{X}) + 3\varepsilon)} 2^{nR} \right) \right)$$

$$= 1 - \sum_{\mathbf{x}, \hat{\mathbf{x}}} p(\mathbf{x}, \hat{\mathbf{x}}) K(\mathbf{x}, \hat{\mathbf{x}}) + \exp \left( - 2^{n(R - I(X; \hat{X}) - 3\varepsilon)} \right)$$

Mutual information does not involve particular  $\mathbf{x}$

$$= P\left\{(\mathbf{x}, \hat{\mathbf{x}}) \notin J_{d, \varepsilon}^{(n)}\right\} + \exp \left( - 2^{n(R - I(X; \hat{X}) - 3\varepsilon)} \right)$$

$$\xrightarrow[n \rightarrow \infty]{} 0$$

since both terms  $\rightarrow 0$  as  $n \rightarrow \infty$  provided  $nR > I(X, \hat{X}) + 3\varepsilon$

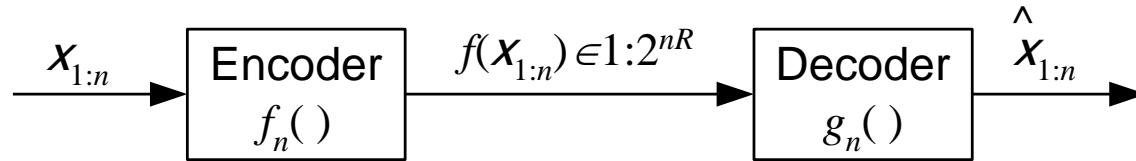
Hence  $\forall \delta > 0$ ,  $\bar{D} = E_{\mathbf{x}, g} d(\mathbf{x}, \hat{\mathbf{x}})$  can be made  $\leq D + \delta$

# Achievability

---

Since  $\forall \delta > 0$ ,  $\bar{D} = E_{\mathbf{x},g} d(\mathbf{x}, \hat{\mathbf{x}})$  can be made  $\leq D + \delta$   
 there must be at least one  $g$  with  $E_{\mathbf{x}} d(\mathbf{x}, \hat{\mathbf{x}}) \leq D + \delta$

Hence  $(R,D)$  is achievable for any  $R > R(D)$



that is  $\lim_{n \rightarrow \infty} E_{X_{1:n}} (\mathbf{x}, \hat{\mathbf{x}}) \leq D$

In fact a stronger result is true (proof in C&T 10.6):

$\forall \delta > 0, D$  and  $R > R(D), \exists f_n, g_n$  with  $p(d(\mathbf{x}, \hat{\mathbf{x}}) \leq D + \delta) \xrightarrow{n \rightarrow \infty} 1$

# Lecture 17

---

- Introduction to network information theory
- Multiple access
- Distributed source coding

# Network Information Theory

---

- System with **many senders and receivers**
- New elements: interference, cooperation, competition, relay, feedback...
- Problem: decide whether or not the sources can be transmitted over the channel
  - **Distributed source coding**
  - **Distributed communication**
  - The general problem has not yet been solved, so we consider various special cases
- Results are presented without proof (can be done using mutual information, joint AEP)

# Implications to Network Design

---

- Examples of large information networks
  - Computer networks
  - Satellite networks
  - Telephone networks
- A complete theory of network communications would have **wide implications** for the design of communication and computer networks
- Examples
  - **CDMA** (code-division multiple access): mobile phone network
  - **Network coding**: significant capacity gain compared to routing-based networks

# Network Models Considered

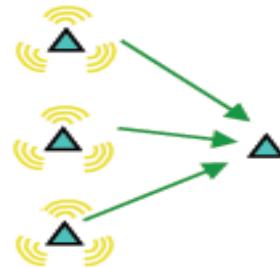
---

- Multi-access channel
- Broadcast channel
- Distributed source coding
- Relay channel
- Interference channel
- Two-way channel
- General communication network

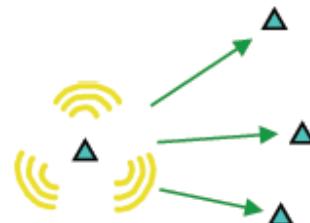
# State of the Art

- **Triumphs**

- Multi-access channel



- Gaussian broadcast channel

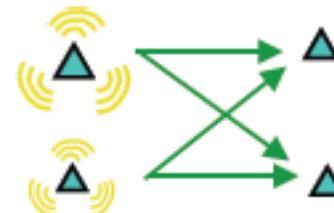


- **Unknowns**

- The simplest relay channel



- The simplest interference channel

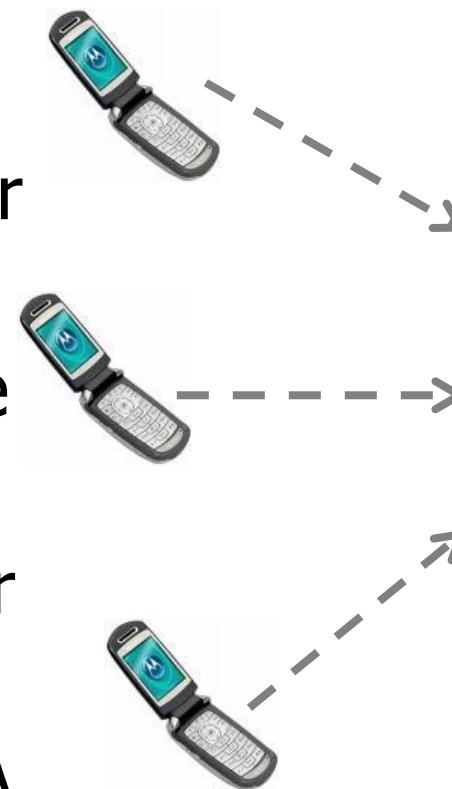


Reminder: Networks being built (ad hoc networks, sensor networks) are much more complicated

# Multi-Access Channel

---

- Example: many users communicate with a common base station over a common channel
- What rates are achievable simultaneously?
- Best understood multiuser channel
- Very successful: 3G CDMA mobile phone networks



# Capacity Region

---

- Capacity of single-user Gaussian channel

$$C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right) = C\left(\frac{P}{N}\right)$$

- Gaussian multi-access channel with  $m$  users

$$Y = \sum_{i=1}^m X_i + Z$$

$X_i$  has equal power  $P$   
noise  $Z$  has variance  $N$

- Capacity region

$$R_i < C\left(\frac{P}{N}\right)$$

$$R_i + R_j < C\left(\frac{2P}{N}\right)$$

$$R_i + R_j + R_k < C\left(\frac{3P}{N}\right)$$

⋮

$$\sum_{i=1}^m R_i < C\left(\frac{mP}{N}\right)$$

$R_i$ : rate for user  $i$

**Transmission:** independent and simultaneous  
(i.i.d. Gaussian codebooks)

**Decoding:** joint decoding, look for  $m$   
codewords whose sum is closest to  $Y$

The last inequality dominates when all rates  
are the same

The sum rate goes to  $\infty$  with  $m$

# Two-User Channel

- Capacity region

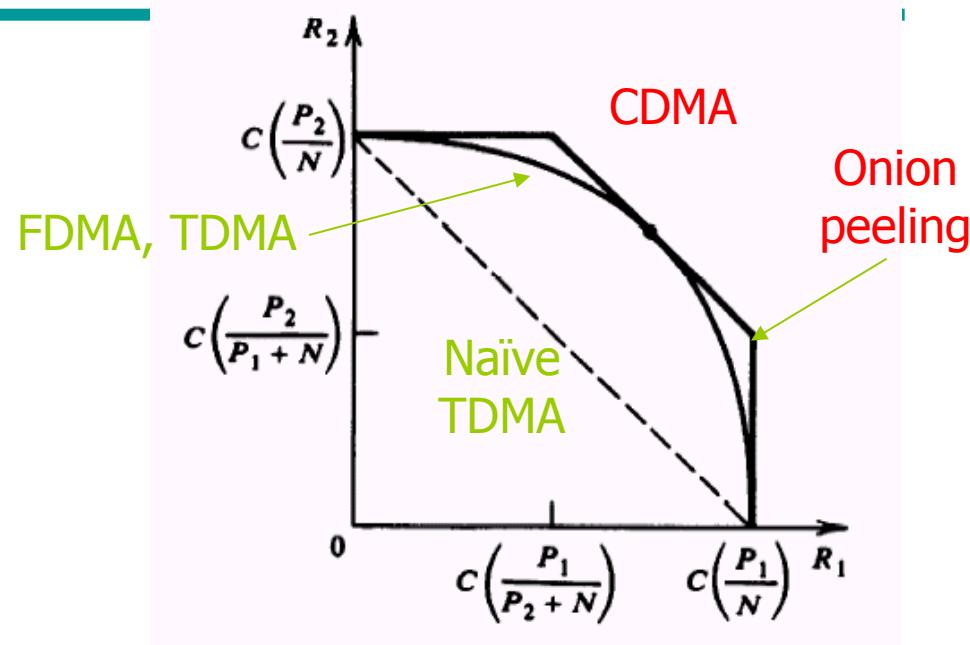
$$R_1 < C\left(\frac{P_1}{N}\right)$$

$$R_2 < C\left(\frac{P_2}{N}\right)$$

$$R_1 + R_2 < C\left(\frac{P_1 + P_2}{N}\right)$$

- Corresponds to CDMA
- Surprising fact: sum rate  
= rate achieved by a single sender with power  $P_1 + P_2$
- Achieves a higher sum rate than treating interference as noise, i.e.,

$$C\left(\frac{P_1}{P_2 + N}\right) + C\left(\frac{P_2}{P_1 + N}\right)$$



# Onion Peeling

---

- Interpretation of corner point: **onion-peeling**
  - First stage: decoder user 2, considering user 1 as noise
  - Second stage: subtract out user 2, decoder user 1
- In fact, it can achieve the entire capacity region
  - Any rate-pairs between two corner points achievable by time-sharing
- Its technical term is successive interference cancelation (SIC)
  - Removes the need for joint decoding
  - Uses a sequence of single-user decoders
- SIC is implemented in the uplink of CDMA 2000 EV-DO (evolution-data optimized)
  - Increases throughput by about 65%

# Comparison with TDMA and FDMA

---

- FDMA (frequency-division multiple access)

$$R_1 = W_1 \log \left( 1 + \frac{P_1}{N_0 W_1} \right)$$

Total bandwidth  $W = W_1 + W_2$

$$R_2 = W_2 \log \left( 1 + \frac{P_2}{N_0 W_2} \right)$$

Varying  $W_1$  and  $W_2$  tracing out the curve in the figure

- TDMA (time-division multiple access)

- Each user is allotted a time slot, transmits and other users remain silent
- Naïve TDMA: dashed line
- Can do better while still maintaining the same average power constraint; the same capacity region as FDMA

- CDMA capacity region is larger

- But needs a **more complex decoder**

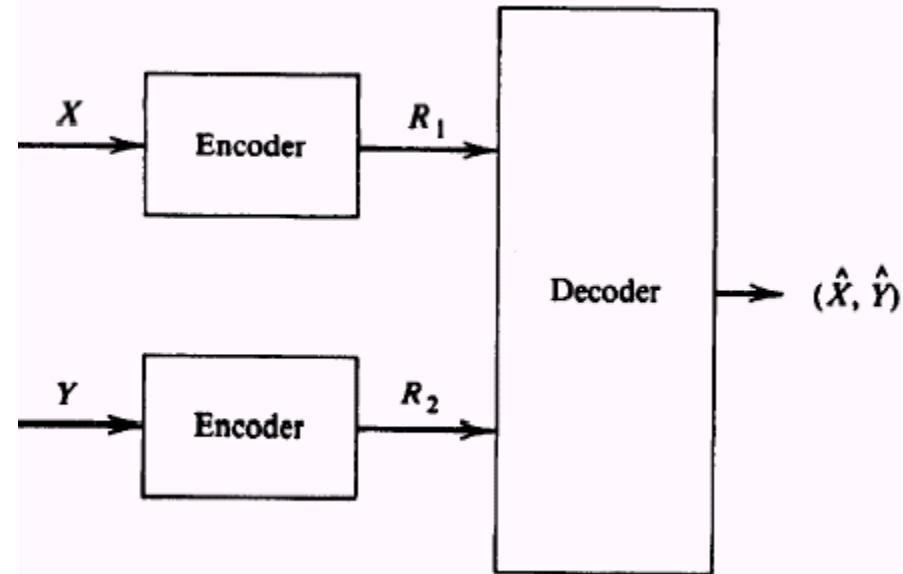
# Distributed Source Coding

---

- Associate with nodes are sources that are generally dependent
- How do we take advantage of the dependence to reduce the amount of information transmitted?
- Consider the special case where channels are noiseless and without interference
- Finding the set of rates associate with each source such that all required sources can be decoded at destination
- Data compression dual to multi-access channel

# Two-User Distributed Source Coding

- $X$  and  $Y$  are correlated
- But **the encoders cannot communicate**; have to encode independently
- A single source:  $R > H(X)$
- Two sources:  $R > H(X, Y)$  if encoding together
- What if encoding separately?
  - Of course one can do  $R > H(X) + H(Y)$
  - Surprisingly,  $R = H(X, Y)$  is sufficient (**Slepian-Wolf coding, 1973**)
  - Sadly, the coding scheme was not practical (again)



# Slepian-Wolf Coding

- Achievable rate region

$$R_1 \geq H(X | Y)$$

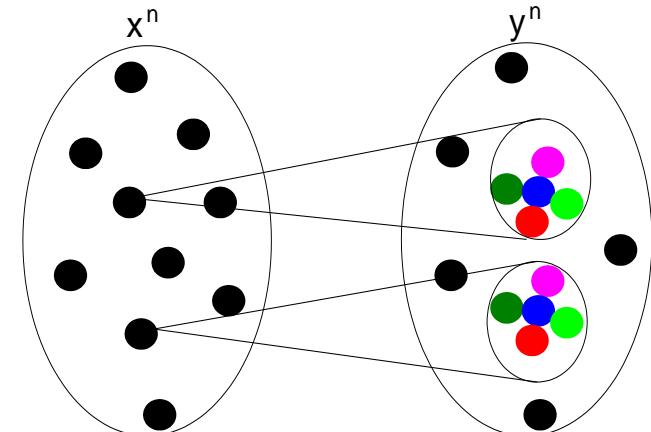
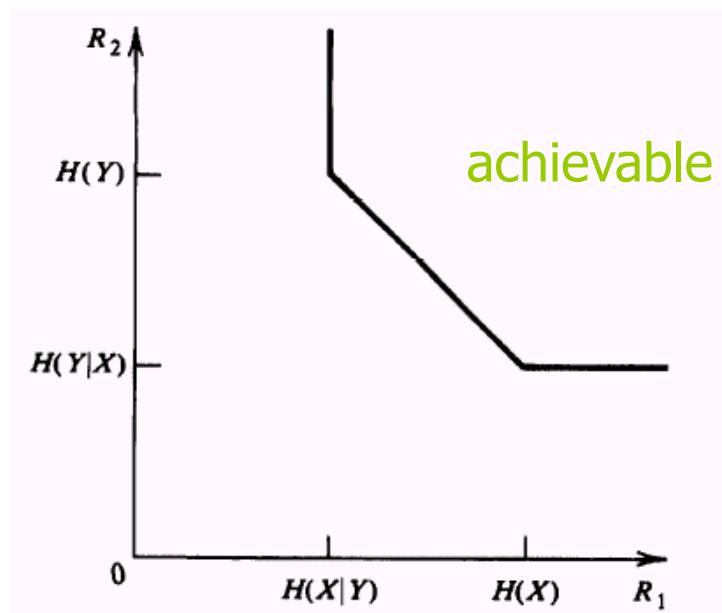
$$R_2 \geq H(Y | X)$$

$$R_1 + R_2 \geq H(X, Y)$$

- Core idea: joint typicality
- Interpretation of corner point  $R_1 = H(X), R_2 = H(Y|X)$

- X can encode as usual
- Associate with each  $x^n$  is a jointly typical fan (however Y doesn't know)
- **Y sends the color (thus compression)**
- Decoder uses the color to determine the point in jointly typical fan associated with  $x^n$

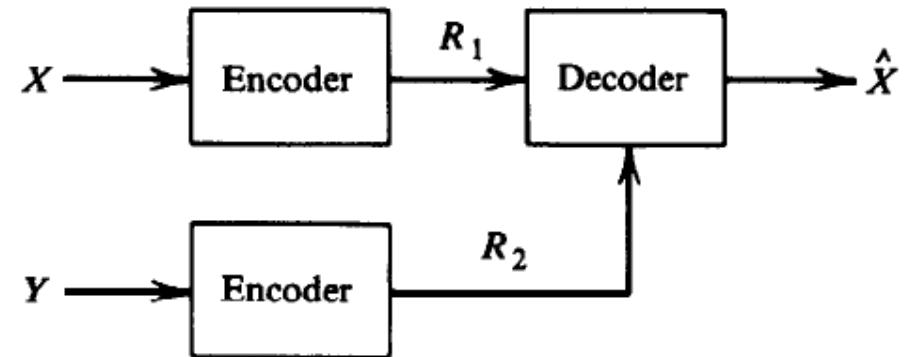
- Straight line: achieved by time-sharing



# Wyner-Ziv Coding

---

- Distributed source coding with **side information**
- $Y$  is encoded at rate  $R_2$
- Only  $X$  to be recovered
- How many bits  $R_1$  are required?
- If  $R_2 = H(Y)$ , then  $R_1 = H(X|Y)$  by Slepian-Wolf coding
- In general



$$R_1 \geq H(X | U)$$

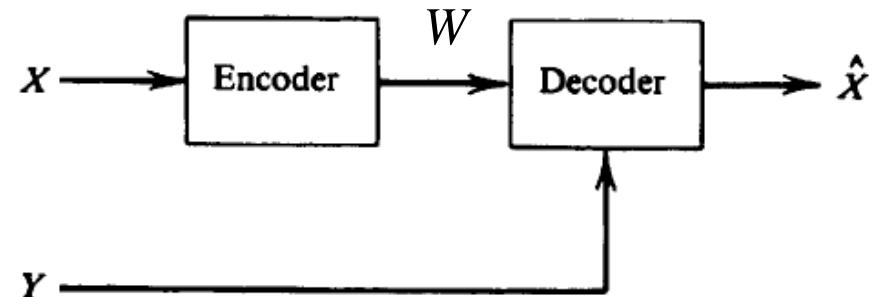
$$R_2 \geq I(Y; U)$$

where  $U$  is an auxiliary random variable (can be thought of as approximate version of  $Y$ )

# Rate-Distortion

- Given  $Y$ , what is the rate-distortion to describe  $X$ ?

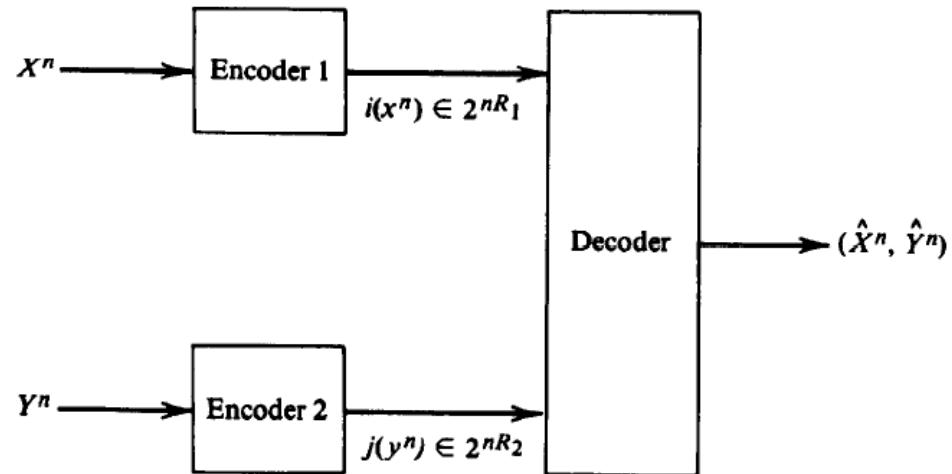
$$R_Y(D) = \min_{p(w|x)} \min_f \{I(X;W) - I(Y;W)\}$$



over all decoding functions  $f : Y \times W \rightarrow \hat{X}$

and all  $p(w|x)$  such that  $E_{x,w,y} d(x, \hat{x}) \leq D$

- The general problem of rate-distortion for correlated sources remains **unsolved**

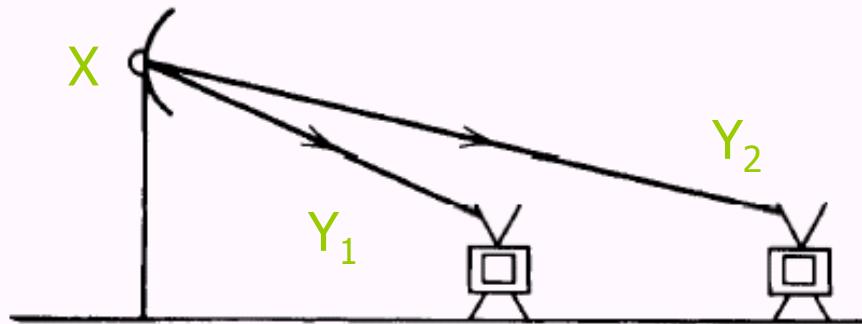


# Lecture 18

---

- Network information theory – II
  - Broadcast
  - Relay
  - Interference channel
  - Two-way channel
  - Comments on general communication networks

# Broadcast Channel



© Original Artist  
Reproduction rights obtainable from  
[www.CartoonStock.com](http://www.CartoonStock.com)

- One-to-many: HDTV station sending different information simultaneously to many TV receivers over a common channel; lecturer in classroom
- What are the achievable rates for all different receivers?
- How does the sender encode information meant for different signals in a common signal?
- Only partial answers are known.

# Two-User Broadcast Channel

---

- Consider a memoryless broadcast channel with one encoder and two decoders
- Independent messages at rate  $R_1$  and  $R_2$
- Degraded broadcast channel:  $p(y_1, y_2|x) = p(y_1|x)$   
 $p(y_2|y_1)$ 
  - Meaning  $X \rightarrow Y_1 \rightarrow Y_2$  (Markov chain)
  - $Y_2$  is a degraded version of  $Y_1$  (receiver 1 is better)
- Capacity region of degraded broadcast channel

$$R_2 \leq I(U; Y_2)$$

$$R_1 \leq I(X; Y_1 | U)$$

*U* is an auxiliary  
random variable

# Scalar Gaussian Broadcast Channel

- All scalar Gaussian broadcast channels belong to the class of degraded channels

$$Y_1 = X + Z_1$$

Assume variance

$$Y_2 = X + Z_2$$

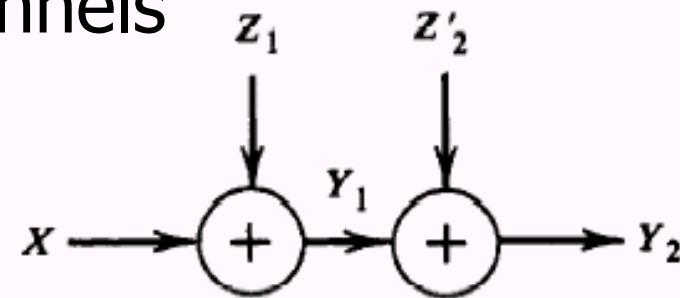
$$N_1 < N_2$$

- Capacity region

$$R_1 \leq C \left( \frac{\alpha P}{N_1} \right)$$

$$R_2 \leq C \left( \frac{(1-\alpha)P}{\alpha P + N_2} \right)$$

$$0 \leq \alpha \leq 1$$



## Coding Strategy

**Encoding:** one codebook with power  $\alpha P$  at rate  $R_1$ , another with power  $(1-\alpha)P$  at rate  $R_2$ , send the sum of two codewords

**Decoding:** Bad receiver  $Y_2$  treats  $Y_1$  as noise; good receiver  $Y_1$  first decode  $Y_2$ , subtract it out, then decode his own message

# Relay Channel

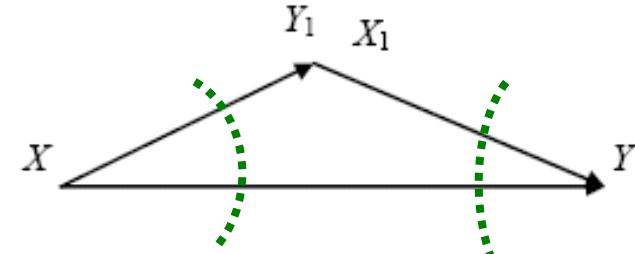
- One source, one destination, one or more intermediate relays

- Example: one relay

- A broadcast channel ( $X$  to  $Y$  and  $Y_1$ )
- A multi-access channel ( $X$  and  $X_1$  to  $Y$ )
- Capacity is unknown! Upper bound:

$$C \leq \sup_{p(x,x_1)} \min\{ I(X, X_1; Y), I(X; Y, Y_1 | X_1) \}$$

- Max-flow min-cut interpretation
  - First term: maximum rate from  $X$  and  $X_1$  to  $Y$
  - Second term: maximum rate from  $X$  to  $Y$  and  $Y_1$



# Degraded Relay Channel

---

- In general, the max-flow min-cut bound cannot be achieved
- Reason
  - Interference
  - What for the relay to forward?
  - How to forward?
- Capacity is known for degraded relay channel (i.e.,  $Y$  is a degradation of  $Y_1$ , or relay is better than receiver), i.e., the upper bound is achieved

$$C = \sup_{p(x, x_1)} \min\{I(X, X_1; Y), I(X; Y, Y_1 | X_1)\}$$

# Gaussian Relay Channel

---

- Channel model

$$Y_1 = X + Z_1 \quad \text{Variance}(Z_1) = N_1$$

$$Y = X + Z_1 + X_1 + Z_2 \quad \text{Variance}(Z_2) = N_2$$

- Encoding at relay:  $X_{1i} = f_i(Y_{11}, Y_{12}, \dots, Y_{1i-1})$

- Capacity

$$C = \max_{0 \leq \alpha \leq 1} \min \left\{ C \left( \frac{P + P_1 + 2\sqrt{(1-\alpha)PP_1}}{N_1 + N_2} \right), C \left( \frac{\alpha P}{N_1} \right) \right\}$$

X has power P  
X1 has power P1

- If

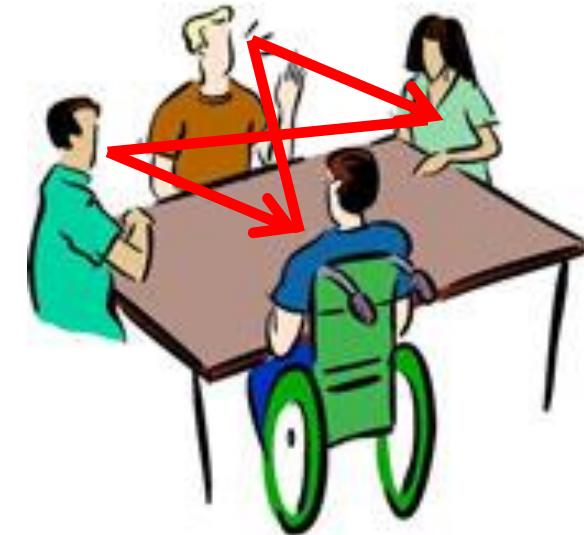
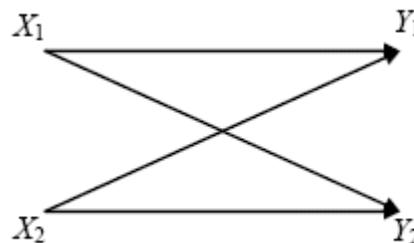
$$\text{relay - destination SNR} \quad \frac{P_1}{N_2} \geq \frac{P}{N_1} \quad \text{source - relay SNR}$$

then  $C = C(P/N_1)$  (capacity from source to relay can be achieved; exercise)

- Rate  $C = C(P/(N_1 + N_2))$  without relay is increased by the relay to  $C = C(P/N_1)$

# Interference Channel

- Two senders, two receivers, with crosstalk



- $Y_1$  listens to  $X_1$  and doesn't care what  $X_2$  speaks or what  $Y_2$  hears
- Similarly with  $X_2$  and  $Y_2$
- Neither a broadcast channel nor a multiaccess channel
- This channel has not been solved
  - Capacity is known to within one bit (Etkin, Tse, Wang 2008)
  - A promising technique — **interference alignment** (Camdenbe, Jafar 2008)

# Symmetric Interference Channel

---

- Model

$$Y_1 = X_1 + aX_2 + Z_1 \quad \text{equal power } P$$

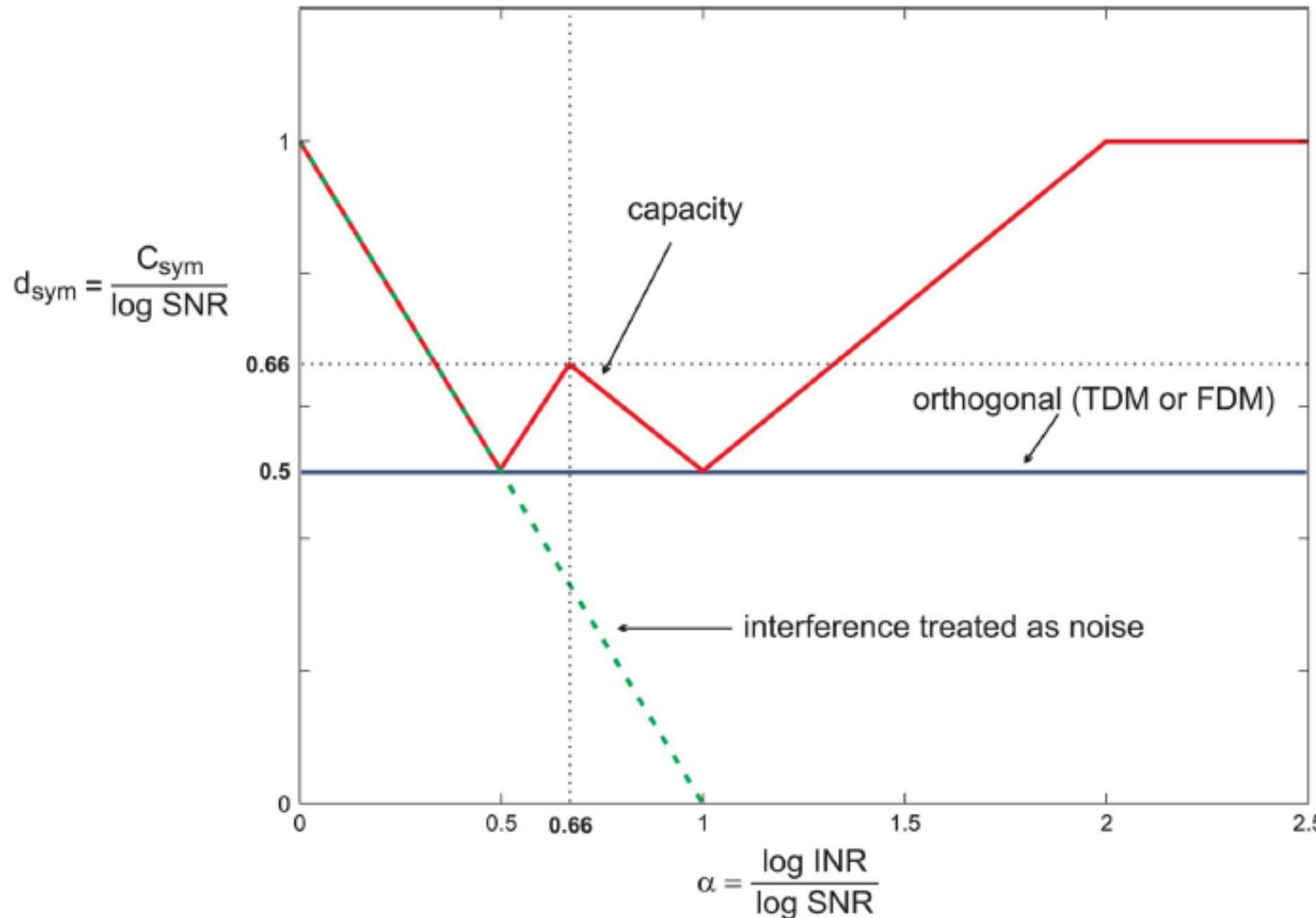
$$Y_2 = X_2 + aX_1 + Z_2 \quad \text{Var}(Z_1) = \text{Var}(Z_2) = N$$

- Capacity has been derived in the strong interference case ( $a \geq 1$ ) (Han, Kobayashi, 1981)
  - Very strong interference ( $a^2 \geq 1 + P/N$ ) is equivalent to no interference whatsoever
- Symmetric capacity (for each user  $R_1 = R_2$ )

$$d_{\text{sym}} = \begin{cases} 1 - \alpha, & 0 \leq \alpha < \frac{1}{2} \\ \alpha, & \frac{1}{2} \leq \alpha < \frac{2}{3} \\ 1 - \frac{\alpha}{2}, & \frac{2}{3} < \alpha \leq 1 \\ \frac{\alpha}{2}, & 1 \leq \alpha < 2 \\ 1, & \alpha \geq 2. \end{cases}$$

$d_{\text{sym}}(\alpha) := \lim_{\text{SNR, INR} \rightarrow \infty; \frac{\log \text{INR}}{\log \text{SNR}} = \alpha} \frac{C_{\text{sym}}(\text{INR, SNR})}{C_{\text{awgn}}(\text{SNR})}$   
 SNR =  $P/N$   
 INR =  $a^2 P/N$

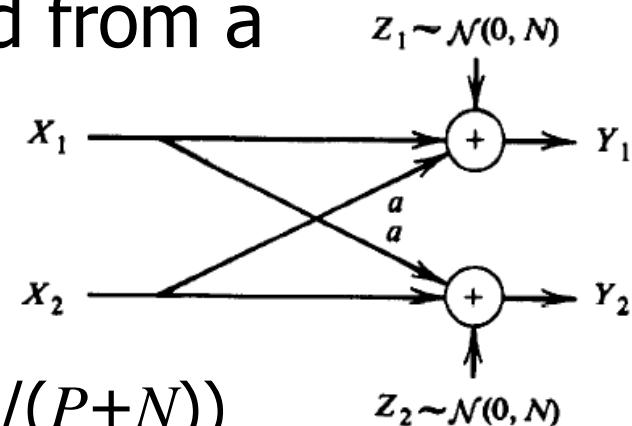
# Capacity



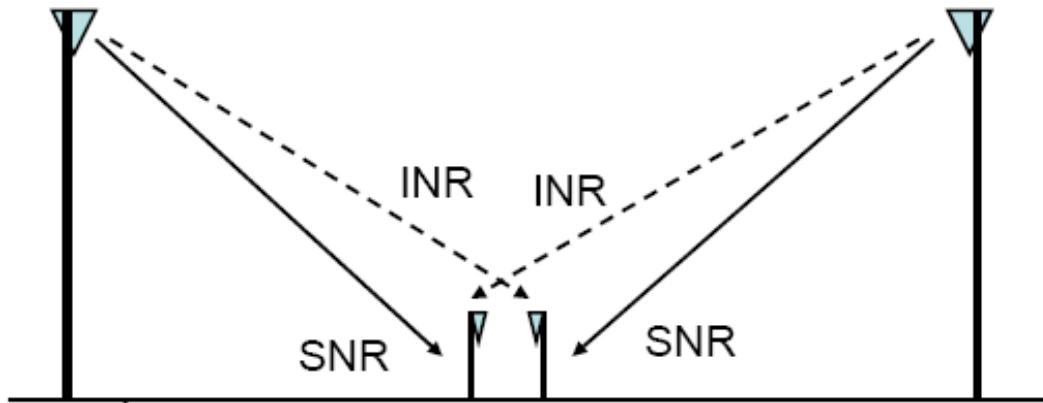
# Very strong interference = no interference

---

- Each sender has power  $P$  and rate  $C(P/N)$
- Independently sends a codeword from a Gaussian codebook
- Consider receiver 1
  - Treats sender 1 as interference
  - Can decode sender 2 at rate  $C(a^2P/(P+N))$
  - If  $C(a^2P/(P+N)) > C(P/N)$ , i.e.,
    - rate 2  $\rightarrow$  1  $>$  rate 2  $\rightarrow$  2 (crosslink is better)
    - he can perfectly decode sender 2
  - Subtracting it from received signal, he sees a clean channel with capacity  $C(P/N)$



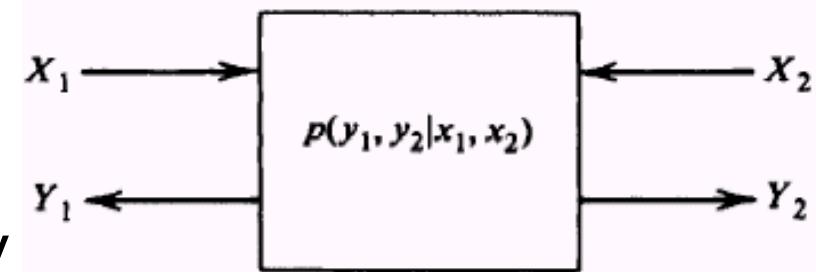
# An Example



- Two cell-edge users (bottleneck of the cellular network)
- No exchange of data between the base stations or between the mobiles
- Traditional approaches
  - Orthogonalizing the two links (reuse  $\frac{1}{2}$ )
  - Universal frequency reuse and treating interference as noise
- Higher capacity can be achieved by advanced **interference management**

# Two-Way Channel

- Similar to interference channel, but in both directions (Shannon 1961)
- Feedback
  - Sender 1 can use previously received symbols from sender 2, and vice versa
  - They can cooperate with each other
- Gaussian channel:
  - Capacity region is known (not the case in general)
  - Decompose into two independent channels



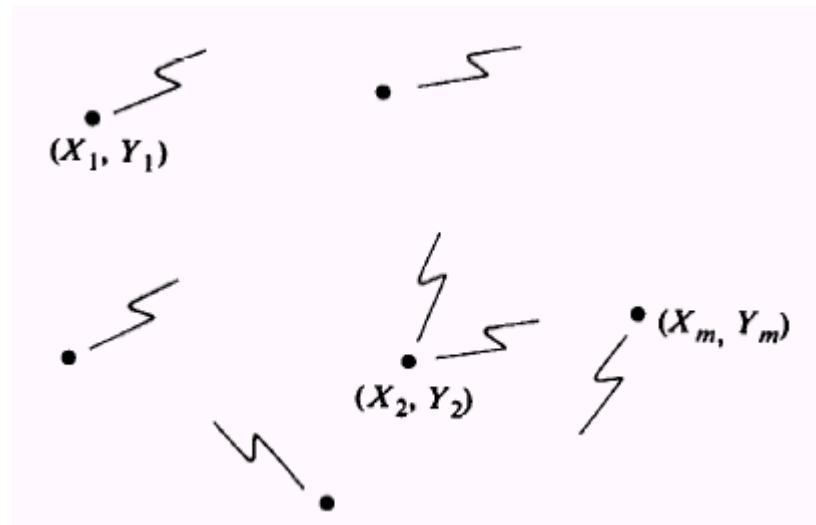
$$R_1 < C \left( \frac{P_1}{N_1} \right)$$

$$R_2 < C \left( \frac{P_2}{N_2} \right)$$

**Coding strategy:** Sender 1 sends a codeword; so does sender 2. Receiver 2 receives a sum but he can subtract out his own thus having an interference-free channel from sender 1.

# General Communication Network

- Many nodes trying to communicate with each other
- Allows computation at each node using its own message and all past received symbols
- All the models we have considered are special cases
- A comprehensive theory of network information flow is yet to be found

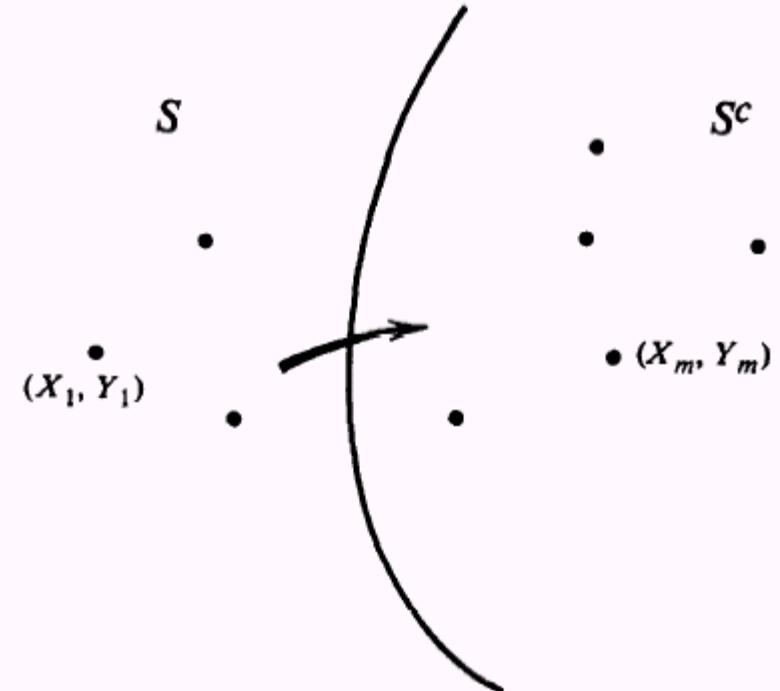
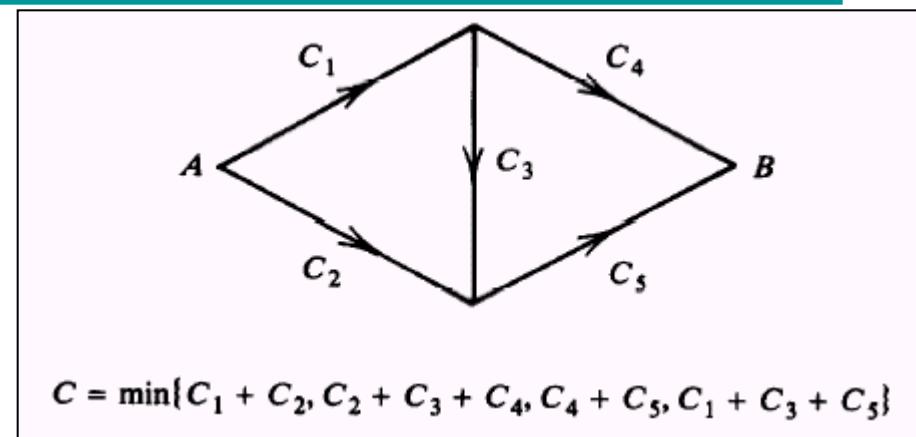


# Capacity Bound for a Network

- **Max-flow min-cut**
  - Minimizing the maximum flow across cut sets yields an upper bound on the capacity of a network
- Outer bound on capacity region

$$\sum_{i \in S, j \in S^c} R^{(i,j)} \leq I(X^{(S)}; Y^{(S^c)} | X^{(S^c)})$$

- Not achievable in general



# Questions to Answer

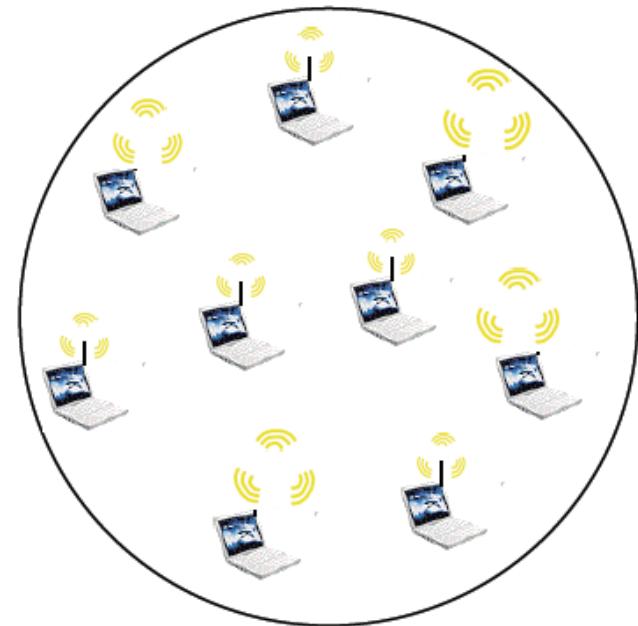
---

- Why multi-hop relay? Why decode and forward? Why treat interference as noise?
- Source-channel separation? Feedback?
- What is really the best way to operate wireless networks?
- What are the ultimate limits to information transfer over wireless networks?



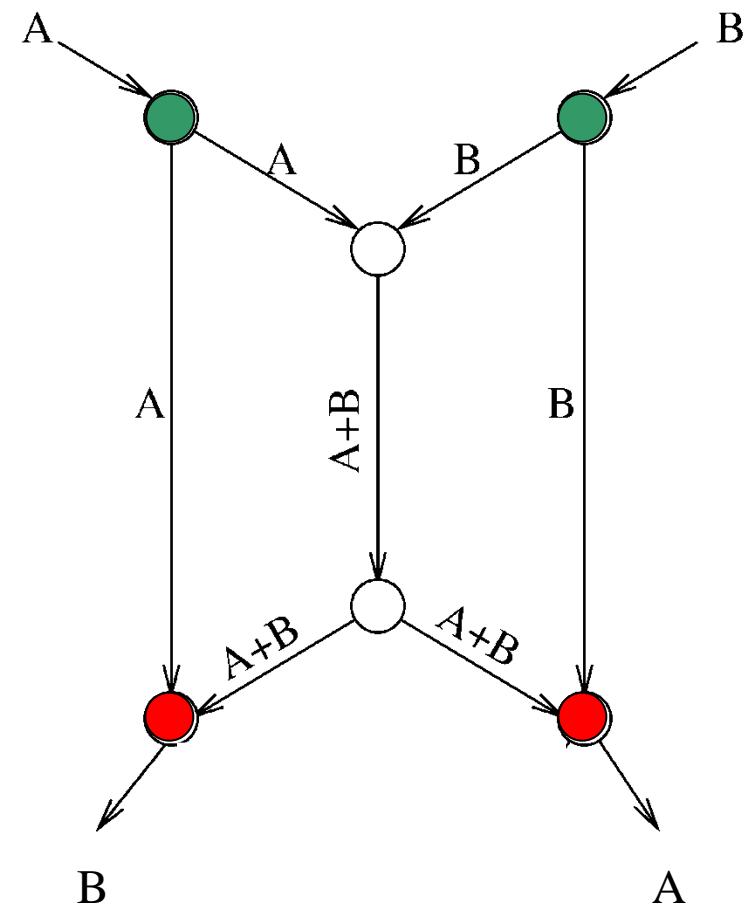
# Scaling Law for Wireless Networks

- High signal attenuation:  
(transport) capacity is  $O(n)$   
**bit-meter/sec** for a planar  
network with  $n$  nodes (Xie-  
Kumar'04)
- Low attenuation: capacity can  
grow superlinearly
- Requires cooperation between  
nodes
- Multi-hop relay is suboptimal  
but order optimal



# Network Coding

- Routing: store and forward (as in Internet)
- Network coding: recompute and redistribute
- Given the network topology, coding can increase capacity (Ahlswede, Cai, Li, Yeung, 2000)
  - Doubled capacity for butterfly network
- Active area of research



Butterfly Network

# Lecture 19

---

- Revision Lecture

# Summary (1)

---

- **Entropy:**  $H(x) = \sum_{x \in X} p(x) \times -\log_2 p(x) = E - \log_2(p_X(x))$ 
  - Bounds:  $0 \leq H(x) \leq \log |X|$
  - Conditioning reduces entropy:  $H(y|x) \leq H(y)$
  - Chain Rule:  $H(x_{1:n}) = \sum_{i=1}^n H(x_i | x_{1:i-1}) \leq \sum_{i=1}^n H(x_i)$   
 $H(x_{1:n} | y_{1:n}) \leq \sum_{i=1}^n H(x_i | y_i)$
- **Relative Entropy:**

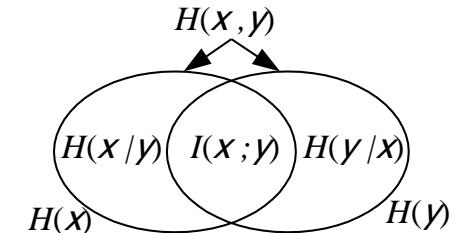
$$D(\mathbf{p} \parallel \mathbf{q}) = E_{\mathbf{p}} \log(p(x)/q(x)) \geq 0$$

# Summary (2)

---

- Mutual Information:

$$\begin{aligned} I(y; x) &= H(y) - H(y|x) \\ &= H(x) + H(y) - H(x,y) = D(\mathbf{p}_{x,y} \parallel \mathbf{p}_x \mathbf{p}_y) \end{aligned}$$



- Positive and Symmetrical:  $I(x;y) = I(y;x) \geq 0$
- $x, y$  indep  $\Leftrightarrow H(x,y) = H(y) + H(x) \Leftrightarrow I(x;y) = 0$

– Chain Rule:  $I(x_{1:n};y) = \sum_{i=1}^n I(x_i; y | x_{1:i-1})$

$x_i$  independent  $\Rightarrow I(x_{1:n};y_{1:n}) \geq \sum_{i=1}^n I(x_i; y_i)$

$$p(y_i | x_{1:n}; y_{1:i-1}) = p(y_i | x_i) \Rightarrow I(x_{1:n}; y_{1:n}) \leq \sum_{i=1}^n I(x_i; y_i)$$

*n-use DMC capacity*

# Summary (3)

---

- **Convexity:**  $f''(x) \geq 0 \Rightarrow f(x)$  convex  $\Rightarrow Ef(x) \geq f(Ex)$ 
  - $H(p)$  concave in  $p$
  - $I(X; Y)$  concave in  $p_x$  for fixed  $p_{y|x}$
  - $I(X; Y)$  convex in  $p_{y|x}$  for fixed  $p_x$
- **Markov:**  $x \rightarrow y \rightarrow z \Leftrightarrow p(z | x, y) = p(z | y) \Leftrightarrow I(X; Z | Y) = 0$   
 $\Rightarrow I(X; Y) \geq I(X; Z)$  and  $I(X; Y) \geq I(X; Y | Z)$
- **Fano:**  $x \rightarrow y \rightarrow \hat{x} \Rightarrow p(\hat{x} \neq x) \geq \frac{H(X | Y) - 1}{\log(|X| - 1)}$
- **Entropy Rate:**
  - Stationary process  $H(X) = \lim_{n \rightarrow \infty} n^{-1} H(X_{1:n})$
  - Markov Process:  $H(X) = \lim_{n \rightarrow \infty} H(X_n | X_{1:n-1})$
  - $H(X) = \lim_{n \rightarrow \infty} H(X_n | X_{n-1})$  if stationary

# Summary (4)

---

- **Kraft:** Uniquely Decodable  $\Rightarrow \sum_{i=1}^{|X|} D^{-l_i} \leq 1 \Rightarrow \exists$  instant code
- **Average Length:** Uniquely Decodable  $\Rightarrow L_C = E l(x) \geq H_D(x)$
- **Shannon-Fano:** Top-down 50% splits.  $L_{SF} \leq H_D(x) + 1$
- **Huffman:** Bottom-up design. Optimal.  $L_H \leq H_D(x) + 1$ 
  - Designing with wrong probabilities,  $\mathbf{q} \Rightarrow$  penalty of  $D(\mathbf{p}||\mathbf{q})$
  - Long blocks disperse the 1-bit overhead
- **Lempel-Ziv Coding:**
  - Does not depend on source distribution
  - Efficient algorithm widely used
  - Approaches entropy rate for stationary ergodic sources

# Summary (5)

---

- Typical Set
  - Individual Prob  $\mathbf{x} \in T_\varepsilon^{(n)} \Rightarrow \log p(\mathbf{x}) = -nH(\mathbf{x}) \pm n\varepsilon$
  - Total Prob  $p(\mathbf{x} \in T_\varepsilon^{(n)}) > 1 - \varepsilon \text{ for } n > N_\varepsilon$
  - Size  $(1 - \varepsilon)2^{n(H(\mathbf{x}) - \varepsilon)} < |T_\varepsilon^{(n)}| \leq 2^{n(H(\mathbf{x}) + \varepsilon)}$
  - No other high probability set can be much smaller
- Asymptotic Equipartition Principle
  - Almost all event sequences are equally surprising

# Summary (6)

---

- DMC Channel Capacity:  $C = \max_{\mathbf{p}_x} I(x; y)$
- Coding Theorem
  - Can achieve capacity: random codewords, joint typical decoding
  - Cannot beat capacity: Fano inequality
- Feedback doesn't increase capacity of DMC but could simplify coding/decoding
- Joint Source-Channel Coding doesn't increase capacity of DMC

# Summary (7)

---

- Polar codes are low-complexity codes directly built from information theory.
- Their constructions are aided by the polarization phenomenon.
- For channel coding, polar codes achieve channel capacity.
- For source coding, polar codes achieve the entropy bound.
- And much more.

# Summary (8)

---

- **Differential Entropy:**  $h(x) = E - \log f_x(x)$ 
  - Not necessarily positive
  - $h(x+a) = h(x)$ ,  $h(ax) = h(x) + \log|a|$ ,  $h(x|y) \leq h(x)$
  - $I(x; y) = h(x) + h(y) - h(x, y) \geq 0$ ,  $D(f||g) = E \log(f/g) \geq 0$
- **Bounds:**
  - **Finite range:** Uniform distribution has max:  $h(x) = \log(b-a)$
  - Fixed Covariance: Gaussian has max:  $h(x) = \frac{1}{2}\log((2\pi e)^n |K|)$
- **Gaussian Channel**
  - **Discrete Time:**  $C = \frac{1}{2}\log(1+PN^{-1})$
  - **Bandlimited:**  $C = W \log(1+PN_0^{-1}W^{-1})$ 
    - For constant C:  $E_b N_0^{-1} = PC^{-1}N_0^{-1} = (W/C)(2^{(W/C)^{-1}} - 1) \xrightarrow[W \rightarrow \infty]{} \ln 2 = -1.6 \text{ dB}$
  - **Feedback:** Adds at most  $\frac{1}{2}$  bit for coloured noise

# Summary (9)

---

- **Parallel Gaussian Channels:** Total power constraint  $\sum P_i = nP$ 
  - White noise: Waterfilling:  $P_i = \max(v - N_i, 0)$
  - Correlated noise: Waterfill on noise eigenvectors
- **Rate Distortion:**  $R(D) = \min_{\mathbf{p}_{\hat{x}|x} \text{ s.t. } Ed(x, \hat{x}) \leq D} I(x; \hat{x})$ 
  - Bernoulli Source with Hamming  $d$ :  $R(D) = \max(H(\mathbf{p}_x) - H(D), 0)$
  - Gaussian Source with mean square  $d$ :  $R(D) = \max(\frac{1}{2}\log(\sigma^2 D^{-1}), 0)$
  - Can encode at rate  $R$ : random decoder, joint typical encoder
  - Can't encode below rate  $R$ : independence bound

# Summary (10)

---

- Gaussian multiple access channel  $R_1 < C\left(\frac{P_1}{N}\right), \quad R_2 < C\left(\frac{P_2}{N}\right)$   
 $R_1 + R_2 < C\left(\frac{P_1 + P_2}{N}\right), \quad C(x) = \frac{1}{2} \log(1 + x)$
- Distributed source coding  
 – Slepian-Wolf coding  $R_1 \geq H(X | Y), \quad R_2 \geq H(Y | X)$   
 $R_1 + R_2 \geq H(X, Y)$
- Scalar Gaussian broadcast channel  
 $R_1 \leq C\left(\frac{\alpha P}{N_1}\right), \quad R_2 \leq C\left(\frac{(1-\alpha)P}{\alpha P + N_2}\right), \quad 0 \leq \alpha \leq 1$
- Gaussian Relay channel

$$C = \max_{0 \leq \alpha \leq 1} \min \left\{ C\left(\frac{P + P_1 + 2\sqrt{(1-\alpha)PP_1}}{N_1 + N_2}\right), C\left(\frac{\alpha P}{N_1}\right) \right\}$$

# Summary (11)

---

- Interference channel
  - Strong interference = no interference
- Gaussian two-way channel
  - Decompose into two independent channels
- General communication network
  - Max-flow min-cut theorem
  - Not achievable in general
  - But achievable for multiple access channel and Gaussian relay channel