E401: Advanced Communication Theory

Professor A. Manikas
Chair of Communications and Array Processing

Imperial College London

Multi-Antenna Wireless Communications

Part-C: Array Receivers for SIMO and MIMO

Table of Contents

General Objective

- By observing a vector-signal $\underline{x}(t) = \mathbb{S}\underline{m}(t) + \underline{n}(t)$ using an array system the aim is to obtain information about a signal environment.
- There are three general problems to solve.

M - number of transmitter

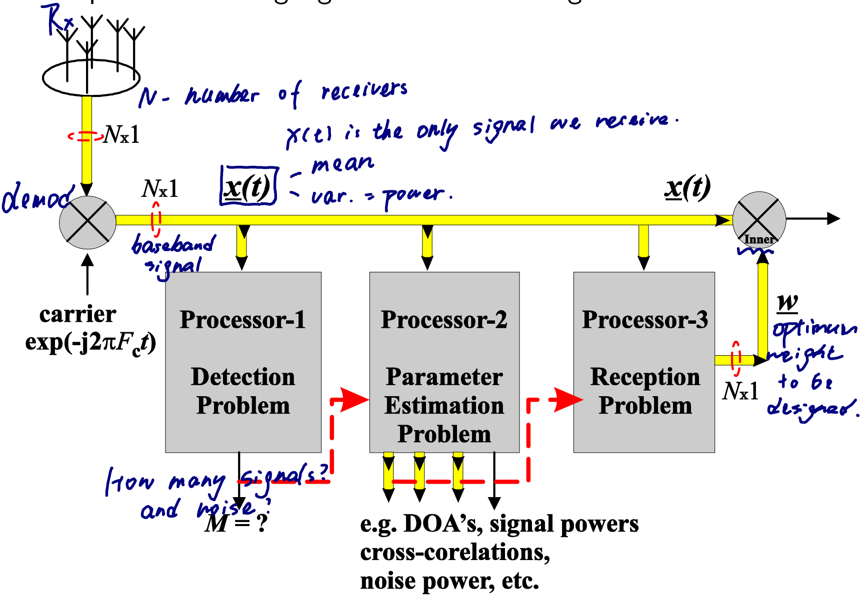
1. The Detection problem: $M = ?$
   (i.e. to detect the presence of M **co-channel** emitting sources)

2. The Estimation problem:
   to estimate various signal and channel parameters
   dir. of arrival
   e.g. DOAs$= ?\forall i$; $P_{m_i} = \varepsilon\{m_i^2(t)\} = ?\forall i$; $P_n = \sigma_n^2 = ?$;
   $\rho_{ij} = \varepsilon\{m_i(t)m_j^*(t)\} = ?\forall i, j$, with $i \neq j$
   polarization parameters, fading coefficients, signal spread.

3. The Reception problem:
   to receive one signal (desired signal) and suppress the remaining $M - 1$ as unwanted cochannel interference

4. The Classification problem: to tell apart signal & noise (different polarisation)

- These problems are highlighted in the following block structure:



General Problem Formulation M<N

- Consiner an observed $(N \times 1)$ complex signal-vector $\underline{x}(t)$ that is modelled as follows

known

$$\underset{\text{known}}{\underline{x}(t)} \triangleq \overbrace{\mathbb{S}(\underbrace{\underline{p}}_{\text{parameters (e.g. d.o.a.)}})}^{N\times M} \cdot \overbrace{\underline{m}(t)}^{M\times 1} + \overbrace{\underline{n}(t)}^{N\times 1} \quad (1)$$

Note that by observing $\underline{x}(t)$, its 2nd order statistics become known, i.e. the covariance matrix $\mathbb{R}_{xx}$ is known, where

$$\underset{N\times N}{\mathbb{R}_{xx}} = \mathcal{E}\{\underline{x}(t).\underline{x}(t)^H\} \quad (2)$$

redundancy - hardware inefficient

- Estimate $M, p_1, p_2, ..., p_M, \mathbb{R}_{mm}, \sigma_n^2$, etc.

where
$$\begin{cases} \mathbb{S} \triangleq \mathbb{S}(\underline{p}) = \left[S(p_1), \quad S(p_2), \quad ..., \quad S(p_M) \right] & \text{- (unknown)} \\ \underline{m}(t): & \text{message signal-vector - (unknown)} \\ \mathbb{R}_{mm}: & \text{2nd order statistics of } \underline{m}(t) \text{ - (unknown)} \\ \underline{n}(t): & \text{AWGN vector - (power } \sigma_n^2 \text{ unknown)} \end{cases}$$

with
$$\begin{cases} \underline{p} = & \text{the vector of generic (unknown) parameters } p_1, p_2, ..., p_M \\ N = & \textbf{known} \text{ (this is a system parameter)} \\ M = & \textbf{unknown} \text{ (this is a channel parameter - number of signals)} \\ & \text{with } M < N \text{ (later this condition will be removed)} \end{cases}$$

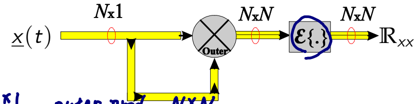
Theoretical Covariance Matrix

- If the $(N \times 1)$ vector-signal $\underline{x}(t) = \mathbb{S}\underline{m}(t) + \underline{n}(t)$ is observed over infinite observation interval then its 2nd order statistics can be calculated. These are given by the theoretical covariance matrix $\mathbb{R}_{xx}$ which is an $(N \times N)$ complex matrix - always Hermitian. That is,

$$x_i(t)\, x_j^*(t) = \begin{cases} \text{real number}, & i=j \\ \text{cross-correlation}, & i \neq j \end{cases}$$

$$\begin{aligned}
\mathbb{R}_{xx} &\triangleq \mathcal{E}\left\{\underline{x}(t)\underline{x}(t)^H\right\} \qquad (3) \\
&= \begin{bmatrix} \mathcal{E}\left\{x_1(t)x_1(t)^*\right\}, & \mathcal{E}\left\{x_1(t)x_2(t)^*\right\}, & \dots, & \mathcal{E}\left\{x_1(t)x_N(t)^*\right\} \\ \mathcal{E}\left\{x_2(t)x_1(t)^*\right\}, & \mathcal{E}\left\{x_2(t)x_2(t)^*\right\}, & \dots, & \mathcal{E}\left\{x_2(t)x_N(t)^*\right\} \\ \dots, & \dots, & \dots, & \dots \\ \mathcal{E}\left\{x_N(t)x_1(t)^*\right\}, & \mathcal{E}\left\{x_N(t)x_2(t)^*\right\}, & \dots, & \mathcal{E}\left\{x_N(t)x_N(t)^*\right\} \end{bmatrix} \\
&= \mathcal{E}\left\{(\mathbb{S}\underline{m}(t) + \underline{n}(t)) \,.\, (\mathbb{S}\underline{m}(t) + \underline{n}(t))^H\right\} \\
&= \mathcal{E}\left\{\mathbb{S}\underline{m}(t)\underline{m}(t)^H\mathbb{S}^H + \underline{n}(t)\underline{n}(t)^H + \mathbb{S}\underline{m}(t)\underline{n}(t)^H + \underline{n}(t)\underline{m}(t)^H\mathbb{S}^H\right\} \\
&= \mathbb{S}.\underbrace{\mathcal{E}\left\{\underline{m}(t)\underline{m}(t)^H\right\}}_{\triangleq \mathbb{R}_{mm}}.\mathbb{S}^H + \underbrace{\mathcal{E}\left\{\underline{n}(t)\underline{n}(t)^H\right\}}_{\triangleq \mathbb{R}_{nn}} + \mathbb{S}\underbrace{\mathcal{E}\left\{\underline{m}(t)\underline{n}(t)^H\right\}}_{=\mathbb{O}_{M,N}} + \underbrace{\mathcal{E}\left\{\underline{n}(t)\underline{m}(t)^H\right\}}_{O_{MN}}\mathbb{S}^H \\
&= \underbrace{\mathbb{S}\,.\mathbb{R}_{mm}.\,\mathbb{S}^H}_{\text{message}} + \underbrace{\mathbb{R}_{nn}}_{\text{noise}} \qquad (4)
\end{aligned}$$

- i.e.



$$\triangleq \mathcal{E}\left\{\underline{x}(t)\underline{x}(t)^H\right\}$$

$N\times 1$ $\underline{x}(t)$ $\xrightarrow[\text{average}]{\text{outer prod.}}$ $N\times N$ $\underline{\underline{R}}_{xx}$

$$= \mathbb{S}.\mathbb{R}_{mm}.\mathbb{S}^H + \mathbb{R}_{nn}$$

$$\boxed{\underline{\underline{R}}_{xx} = \mathbb{S}\,\underline{\underline{R}}_{mm}\,\mathbb{S}^H + \underline{\underline{R}}_{nn}}$$

where

'side' terms: cross-correlation = 0

$$\mathbb{R}_{mm} \triangleq \mathcal{E}\left\{\underline{m}(t).\underline{m}(t)^H\right\} = \text{2nd order statistics of } \underline{m}(t) \ \text{(unknown)}$$

if similar → complex number.

$$= \begin{bmatrix} \underbrace{\mathcal{E}\{m_1(t).m_1(t)^*\}}_{\mathcal{E}\{m_1(t)^2\}=P_1}, & \mathcal{E}\{m_1(t).m_2(t)^*\}, & \ldots, & \mathcal{E}\{m_1(t).m_M(t)^*\} \\ \mathcal{E}\{m_2(t).m_1(t)^*\}, & \underbrace{\mathcal{E}\{m_2(t).m_2(t)^*\}}_{\mathcal{E}\{m_2(t)^2\}=P_2}, & \ldots, & \mathcal{E}\{m_2(t).m_M(t)^*\} \\ \vdots & \vdots & \vdots, & \vdots \\ \mathcal{E}\{m_M(t).m_1(t)^*\}, & \mathcal{E}\{m_M(t).m_2(t)^*\}, & \ldots, & \underbrace{\mathcal{E}\{m_M(t).m_M(t)^*\}}_{\mathcal{E}\{m_M(t)^2\}=P_M} \end{bmatrix}$$

$m_i(t)\,m_j^*(t) = m_i^*(t)\,m_j(t)$

$= \text{an } (M \times M) \text{ complex matrix (always Hermitian) - unknown}$

$A = A^H$

$$
\begin{aligned}
\mathbb{R}_{\mathrm{nn}} &\triangleq \mathcal{E}\left\{\underline{\mathrm{n}}(t).\underline{\mathrm{n}}(t)^H\right\} \text{ is 2nd order statistics of } \underline{\mathrm{n}}(t) \\
&= \begin{bmatrix}
\underbrace{\mathcal{E}\{\mathrm{n}_1(t).\mathrm{n}_1(t)^*\}}_{\mathcal{E}\{\mathrm{n}_1(t)^2\}=P_{\mathrm{n}_1}}, & \underbrace{\mathcal{E}\{\mathrm{n}_1(t).\mathrm{n}_2(t)^*\}}_{0}, & \ldots, & \underbrace{\mathcal{E}\{\mathrm{n}_1(t).\mathrm{n}_N(t)^*\}}_{0} \\
\underbrace{\mathcal{E}\{\mathrm{n}_2(t).\mathrm{n}_1(t)^*\}}_{0}, & \underbrace{\mathcal{E}\{\mathrm{n}_2(t).\mathrm{n}_2(t)^*\}}_{\mathcal{E}\{\mathrm{n}_2(t)^2\}=P_{\mathrm{n}_2}}, & \ldots, & \underbrace{\mathcal{E}\{\mathrm{n}_2(t).\mathrm{n}_N(t)^*\}}_{0} \\
\ldots, & \ldots, & \ldots, & \ldots \\
\underbrace{\mathcal{E}\{\mathrm{n}_N(t).\mathrm{n}_1(t)^*\}}_{0}, & \underbrace{\mathcal{E}\{\mathrm{n}_N(t).\mathrm{n}_2(t)^*\}}_{0}, & \ldots, & \underbrace{\mathcal{E}\{\mathrm{n}_N(t).\mathrm{n}_N(t)^*\}}_{\mathcal{E}\{\mathrm{n}_N(t)^2\}=P_{\mathrm{n}_N}}
\end{bmatrix} \\
&= \sigma_{\mathrm{n}}^2 \mathbb{I}_N \qquad (5) \\
&= \text{an } (N \times N) \text{ complex matrix (always Hermitian) - unknown}
\end{aligned}
$$

delayed. cross-correlation = 0

equal power (only delayed)

- Note that, because we have assumed isotropic AWGN noise,

$$P_{\mathrm{n}_1} = P_{\mathrm{n}_2} = \ldots = P_{\mathrm{n}_N} = \sigma_{\mathrm{n}}^2 \qquad (6)$$

Practical Covariance Matrix

- Consider that the signal $\underline{x}(t) = \mathbb{S}\underline{m}(t) + \underline{\mathrm{n}}(t)$ is observed over finite observation interval equivalent to L snapshots.
- These L observations (snapshots) at times t_1, t_2, ..., t_L (i.e. finite observation interval) are denoted as $[\underline{x}(t_1), \underline{x}(t_2), ..., \underline{x}(t_L)]$ and represented by the $N \times L$ complex matrix $\mathbb{X}$ i.e.

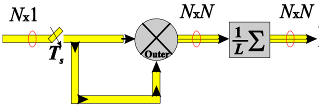
$$\begin{aligned}
\mathbb{X} &\triangleq [\underline{x}(t_1), \underline{x}(t_2), ..., \underline{x}(t_L)] && (7a)\\
&= [\mathbb{S}.\underline{m}(t_1) + \underline{\mathrm{n}}(t_1), \ \mathbb{S}.\underline{m}(t_2) + \underline{\mathrm{n}}(t_2), \ ..., \ \mathbb{S}.\underline{m}(t_L) + \underline{\mathrm{n}}(t_L)] \\
&= \mathbb{S}.\mathbb{M} + \mathbb{N} && (7b)
\end{aligned}$$

$$\text{with} \begin{cases} \mathbb{S} = [\underline{S}_1, \underline{S}_2, ..., \underline{S}_M] & (N \times M) \\ \mathbb{M} = [\underline{m}(t_1), \underline{m}(t_2), ..., \underline{m}(t_L)] & (M \times L) \\ \mathbb{N} = [\underline{\mathrm{n}}(t_1), \underline{\mathrm{n}}(t_2), ..., \underline{\mathrm{n}}(t_L)] & (N \times L) \end{cases} \quad (8)$$

where the matrices $\mathbb{S}$, $\mathbb{M}$ and $\mathbb{N}$ (as well as the dimension M) are unknown

- In this case the 2nd order statistics of $\underline{x}(t)$ are estimated by the practical covariance matrix $\mathbb{R}_{xx}$
- Practical Model:

$= \mathcal{E}\{\cdot\}$



$$\underline{x}(t) \rightarrow \mathbb{R}_{xx} = \frac{1}{L}\sum_{l=1}^{L}\underline{x}(t_l)\underline{x}(t_l)^H$$

$$\text{i.e. } \mathbb{R}_{xx} = \frac{1}{L}\mathbb{X}\mathbb{X}^H$$

$$= \underbrace{\mathbb{S}\frac{1}{L}\mathbb{M}\mathbb{M}^H\mathbb{S}^H}_{=\mathbb{R}_{mm}} + \underbrace{\frac{1}{L}\mathbb{N}\mathbb{N}^H}_{=\mathbb{R}_{nn}}$$

$$= \underset{\text{geo}}{\mathbb{S}}\,.\underset{\text{static}}{\mathbb{R}_{mm}}.\underset{\text{geo}}{\mathbb{S}^H} + \underset{\text{static}}{\mathbb{R}_{nn}}$$

- N.B.: In an array system the matrix $\mathbb{R}_{xx}$ (theoretical or practical) contains all the **geometrical information** about the various sources relative to the array.

Generating L Snapshots having a given Covariance Matrix

- To generate L snapshots of $\underline{x}(t)$ having a predefined covariance matrix $\mathbb{R}_{xx}$ the vectors $\underline{x}(t_l)$ for $l = 1, 2, .., L$ should be generated using the following expression

Relationship between $\underline{x}(t_l)$ and $\mathbb{R}_{xx}$:

$$\underline{x}(t_l) = \underbrace{\mathbb{E}}_{\text{eigenvector}}\, \underbrace{\mathbb{D}^{\frac{1}{2}}}_{\text{eigenvalue}}\, \underbrace{\underline{z}(t_l)}_{\text{Gaussian } N(0.1) \text{ vector}} \tag{9}$$

$\mathbb{R}_{xx}$

where

  - $\mathbb{E}$ and $\mathbb{D}$ are the eigenvector-matrix and eigenvalue-matrix of $\mathbb{R}_{xx}$, and
  - $\underline{z}(t_l) \in \mathcal{C}^N$ is a Gaussian random complex vector of N elements of zero mean and variance 1, i.e.

$$\mathcal{E}\left\{\underline{z}(t_l).\underline{z}(t_l)^H\right\} = \mathbb{I}_N \tag{10}$$

- That is,

$$\begin{aligned} \mathbb{X} &= \left[\underline{x}(t_1), \quad \underline{x}(t_2), \quad ..., \quad \underline{x}(t_L)\right] \\ &= \left[\mathbb{E}\,\mathbb{D}^{\frac{1}{2}}\underline{z}(t_1), \quad \mathbb{E}\,\mathbb{D}^{\frac{1}{2}}\underline{z}(t_2), \quad ..., \quad \mathbb{E}\,\mathbb{D}^{\frac{1}{2}}\underline{z}(t_L)\right] \end{aligned} \tag{11}$$

Proof:

- Let $\underline{z} \in \mathcal{C}^N$ such as $\mathcal{E}\{\underline{z}.\underline{z}^H\} = \mathbb{I}_N$. Then

$$\mathbb{R}_{xx} = \mathcal{E}\{\underline{x}.\underline{x}^H\} \tag{12}$$

- However,

$\boxed{\mathbb{R}_{xx} = \mathbb{E}\mathbb{D}\mathbb{E}^H}$

proof: $\because A\xi = \xi\lambda$ eigen-vector, eigen-value

$\therefore \mathbb{R}_{xx}\mathbb{E} = \mathbb{E}\mathbb{D}$

$\therefore \mathbb{R}_{xx} = \mathbb{E}\mathbb{D}\mathbb{E}^{-1}$ (orthogonal: $\mathbb{E}^{-1} = \mathbb{E}^H$)

$= \mathbb{E}\mathbb{D}\mathbb{E}^H$

$$\begin{aligned}\mathbb{R}_{xx} &= \mathbb{E}\mathbb{D}\mathbb{E}^H = \mathbb{E}\mathbb{D}^{\frac{1}{2}}\mathbb{D}^{\frac{1}{2}}\mathbb{E}^H = \mathbb{E}\mathbb{D}^{\frac{1}{2}}\mathbb{I}_N\mathbb{D}^{\frac{1}{2}}\mathbb{E}^H \\ &= \mathbb{E}\mathbb{D}^{\frac{1}{2}}\underbrace{\mathcal{E}\{\underline{z}.\underline{z}^H\}}_{=\mathbb{I}_N}\mathbb{D}^{\frac{1}{2}}\mathbb{E}^H \\ &= \mathcal{E}\{\underbrace{\mathbb{E}\mathbb{D}^{\frac{1}{2}}\underline{z}}_{\underline{x}}.\underbrace{\underline{z}^H\mathbb{D}^{\frac{1}{2}}\mathbb{E}^H}_{\underline{x}^H}\}\end{aligned} \tag{13}$$

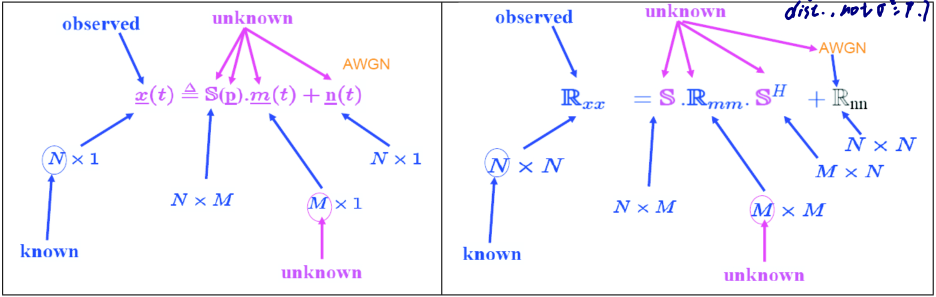
- By comparing Equation 13 and 12 we have

$$\underline{x} = \mathbb{E}\,\mathbb{D}^{\frac{1}{2}}\,\underline{z} \tag{14}$$

- Note: In a similar fashion L snapshots of $\underline{m}(t)$, $\underline{n}(t)$ or any other vector-signal can be generated.

Summary - General Problem Formulation

(only know Gaussian dist., not σ^2?.)



- Condition: $M < N$
- Estimate $M, \underline{p} = [p_1, p_2, ..., p_M]^T, \mathbb{R}_{mm}, \sigma_n^2$, etc
  where p_i is a parameter of interest associated with the i^{th} source.

The 'Detection' Problem

- This is to determine the parameter M

  i.e. to determine the number of signals and thus the dimensions of the vectors/matrices $\underline{m}(t), \mathbb{S}, \mathbb{R}_{mm}$ and $\mathbb{M}$

- In other words, to detect how many emitting sources/transmitters are present in an array environment

  i.e. to detect the presence of M sources

Detection Criteria: Infinite Observation Interval

finite → estimate (error)

- Based on $\underline{x}(t)$ we can form the matrix $\mathbb{R}_{xx}$ (representing the statistics of $\underline{x}(t)$):

column - manifold vector

$$\mathbb{R}_{xx} \triangleq \mathcal{E}\left\{\underline{x}(t)\underline{x}(t)^H\right\} = \underbrace{\overset{N\times M}{\mathbb{S}} \cdot \overset{M\times M}{\mathbb{R}_{mm}} \cdot \overset{M\times N}{\mathbb{S}^H}}_{=\mathbb{R}_{\text{signals}}} + \underbrace{\overset{N\times N}{\mathbb{R}_{\text{nn}}}}_{=\sigma_{\text{n}}^2\mathbb{I}_N} \qquad (15)$$

When the number of sources M is smaller than the number of system dimensions N (e.g. number of array-sensors) then the determinant of the $\mathbb{R}_{\text{signals}}$ is equal to zero
i.e.

$$\text{if } M < N \Rightarrow \det(\mathbb{R}_{\text{signals}}) = 0 \qquad (16)$$

This is due to the fact that the presence of an emitting source increases the rank of the matrix $\mathbb{R}_{\text{signals}}$ by one.i.e.

$$rank\left\{\mathbb{R}_{\text{signals}}\right\} = M \qquad (17)$$

$$\Rightarrow \quad rank\left\{\mathbb{R}_{xx} - \sigma_{\text{n}}^2\mathbb{I}_N\right\} = M \qquad (18)$$

$\underline{\underline{R}}_{xx} = \underline{\underline{E}}\,\underline{\underline{D}}\,\underline{\underline{E}}^H = \underline{\underline{E}}\,\underline{\underline{D}}^{\frac{1}{2}}\underline{\underline{I}}_N\underline{\underline{D}}^{\frac{1}{2}}\underline{\underline{E}}^H = \underline{\underline{E}}\,\underline{\underline{D}}^{\frac{1}{2}}\underline{\underline{Z}}\,\underline{\underline{Z}}^H\underline{\underline{D}}^{\frac{1}{2}}\underline{\underline{E}}^H$

- However, using eigen-decomposition of $\mathbb{R}_{\mathbf{xx}}$ we have

$$\mathbb{R}_{\mathbf{xx}} = \mathbb{E}.\mathbb{D}.\mathbb{E}^H \tag{19}$$

where

Tx

$$\begin{aligned} \mathbb{D} &= \begin{bmatrix} d_1, & 0, & \ldots, & 0, & 0, & 0, & \ldots, & 0 \\ 0, & d_2, & \ldots, & 0, & 0, & 0, & \ldots, & 0 \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots \\ 0, & 0, & \ldots, & d_M, & 0, & 0, & \ldots, & 0 \\ 0, & 0, & \ldots, & 0, & d_{M+1}, & 0, & \ldots, & 0 \\ 0, & 0, & \ldots, & 0, & 0, & d_{M+2}, & \ldots, & 0 \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots \\ 0, & 0, & \ldots, & 0, & 0, & 0, & \ldots, & d_N \end{bmatrix} \\ &= (N \times N) \text{ matrix} \end{aligned} \tag{20}$$

Rx

- An alternative way to express $\mathbb{R}_{\mathbf{xx}}$ as the addition of signals and noise covariance matrices and then to eigen-decompose the signal covariance matrix $\mathbb{R}_{signals}$. That is,

$$\begin{aligned}
\mathbb{R}_{xx} &= \mathbb{R}_{signals} + \mathbb{R}_{noise} \\
&= \mathbb{E}.\underline{\underline{\Lambda}}.\mathbb{E}^H + \sigma_{\mathrm{n}}^2 \mathbb{I}_N = \mathbb{E}.\underline{\underline{\Lambda}}.\mathbb{E}^H + \sigma_{\mathrm{n}}^2 \underbrace{\mathbb{E}.\mathbb{E}^H}_{\mathbb{I}_N} \\
&= \mathbb{E}.\underbrace{\left(\underline{\underline{\Lambda}} + \sigma_{\mathrm{n}}^2 \mathbb{I}_N\right)}_{\mathbb{D}}.\mathbb{E}^H
\end{aligned} \tag{21}$$

$$\text{where } \underline{\underline{\Lambda}} = \begin{bmatrix}
\lambda_1, & 0, & ..., & 0, & 0, & 0, & ..., & 0 \\
0, & \lambda_2, & ..., & 0, & 0, & 0, & ..., & 0 \\
..., & ..., & ..., & ..., & ..., & ..., & ..., & ... \\
0, & 0, & ..., & \lambda_M, & 0, & 0, & ..., & 0 \\
0, & 0, & ..., & 0, & 0, & 0, & ..., & 0 \\
0, & 0, & ..., & 0, & 0, & 0, & ..., & 0 \\
..., & ..., & ..., & ..., & ..., & ..., & ..., & ... \\
0, & 0, & ..., & 0, & 0, & 0, & ..., & 0
\end{bmatrix} \tag{22}$$

- That is, from Equ 21 in conjunction with Equs 20 and 22 we have

$$\mathbb{D} = \underline{\underline{\Lambda}} + \sigma_n^2 \mathbb{I}_N \tag{23}$$

$$\mathbb{D} = \begin{bmatrix} \underbrace{\lambda_1 + \sigma_n^2}_{=d_1} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & \underbrace{\lambda_2 + \sigma_n^2}_{=d_2} & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \underbrace{\lambda_M + \sigma_n^2}_{=d_M} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \underset{=d_{M+1}}{\underset{\uparrow}{\sigma_n^2}} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \underset{=d_{M+2}}{\underset{\uparrow}{\sigma_n^2}} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \underset{=d_N}{\underset{\uparrow}{\sigma_n^2}} \end{bmatrix}$$

$M = N$ - multiplicity.

mulplicity

- This implies that the eigenvalues of the data covariance matrix $\mathbb{R}_{xx}$ (i.e. the diagonal elements of $\mathbb{D}$) are related to the eigenvalues of the emitting signals covariance matrix $\mathbb{R}_{\text{signals}}$ (i.e. diagonal elements of $\underline{\underline{\Lambda}}$) as follows:

$$\begin{aligned} eig_i\left\{\mathbb{R}_{xx}\right\} &= eig_i\left\{\mathbb{R}_{\text{signals}}\right\} + \sigma_{\text{n}}^2 \\ d_i &= \lambda_i + \sigma_{\text{n}}^2 \end{aligned} \tag{24}$$

Now, since the smallest eigenvalue of $\mathbb{R}_{\text{signals}}$ is zero

$$eig_{\min}\left\{\mathbb{R}_{\text{signals}}\right\} = 0 \tag{25}$$

with multiplicity $N-M$, that means

$$eig_{\min}\left\{\mathbb{R}_{xx}\right\} = \sigma_{\text{n}}^{\mathbf{2}} \tag{26}$$

with multiplicity also $N-M$.

- Therefore, theoretically, the number of emitting sources M can be determined by the eigenvalues of the covariance matrix $\mathbb{R}_{xx}$ of the Rx signal-vector $\underline{x}(t)$, and more specifically by the following expression

(noise terms)

$$\boxed{M = N - (\text{multiplicity of } \underline{\textbf{minimum} \text{ eigenvalue}} \text{ of } \mathbb{R}_{xx})} \tag{27}$$

- Note: another useful expression is

$$\begin{aligned}\mathbb{R}_{\mathbf{xx}} &= \mathbb{E}.\mathbb{D}.\mathbb{E}^H = [\mathbb{E}_{\mathbf{s}}, \mathbb{E}_{\mathbf{n}}].\begin{bmatrix}\mathbb{D}_{\mathbf{s}} & \mathbb{O}\\ \mathbb{O} & \mathbb{D}_{\mathbf{n}}\end{bmatrix}[\mathbb{E}_{\mathbf{s}}, \mathbb{E}_{\mathbf{n}}]^H\\ &= \mathbb{E}_{\mathbf{s}}.\mathbb{D}_{\mathbf{s}}.\mathbb{E}_{\mathbf{s}}^H + \mathbb{E}_n\mathbb{D}_n\mathbb{E}_n^H \end{aligned} \quad (28)$$

where

$\mathbb{D}_{\mathbf{s}}$

$$\mathbb{D} = \begin{bmatrix}
\underbrace{\lambda_1+\sigma_n^2}_{=d_1} & 0 & \dots & 0 & 0 & 0 & \dots & 0\\
0 & \underbrace{\lambda_2+\sigma_n^2}_{=d_2} & \dots & 0 & 0 & 0 & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & \dots & \underbrace{\lambda_M+\sigma_n^2}_{=d_M} & 0 & 0 & \dots & 0\\
0 & 0 & \dots & 0 & \underset{=d_{M+1}}{\underset{\uparrow}{\sigma_n^2}} & 0 & \dots & 0\\
0 & 0 & \dots & 0 & 0 & \underset{=d_{M+2}}{\underset{\uparrow}{\sigma_n^2}} & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & \underset{=d_N}{\underset{\uparrow}{\sigma_n^2}}
\end{bmatrix}$$

$\mathbb{D}_{\mathbf{n}}$

Detection Criteria: Finite Observation Interval (L=finite)

AIC and MDL criteria

$$
\mathbb{D}. = \begin{bmatrix}
\underbrace{\lambda_1+\sigma_1^2}_{=d_1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & \underbrace{\lambda_2+\sigma_2^2}_{=d_2} & \dots & 0 & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & \underbrace{\lambda_M+\sigma_M^2}_{=d_M} & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & \underset{=d_{M+1}}{\underset{\uparrow}{\sigma_{M+1}^2}} & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & \underset{=d_{M+2}}{\underset{\uparrow}{\sigma_{M+2}^2}} & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & \underset{=d_N}{\underset{\uparrow}{\sigma_N^2}}
\end{bmatrix} \qquad (29)
$$

noise value not constant approximately!

- In theory,

$$\sigma_1^2 = \sigma_2^2 = ... = \sigma_M^2 = \sigma_{M+1}^2 = ... = \sigma_N^2 \triangleq \sigma_{\mathrm{n}}^2 \tag{30}$$

- However, in practice

$$\sigma_1^2 \neq \sigma_2^2 \neq ... \neq \sigma_M^2 \neq \sigma_{M+1}^2 \neq .... \neq \sigma_N^2 \tag{31}$$

but

$$\sigma_{\mathrm{n}}^2 \approx \sigma_1^2 \approx \sigma_2^2 \approx ... \approx \sigma_M^2 \approx \sigma_{M+1}^2 \approx .... \approx \sigma_N^2 \tag{32}$$

- if M is known then .

$$\begin{aligned} \widehat{\sigma}_{\mathrm{n}}^2 &= \text{the average of the } N-M \text{ smallest eigenvalues} \\ &= \frac{1}{N-M}\left(\sigma_{M+1}^2 + \sigma_{M+2}^2 + ... + \sigma_N^2\right) \end{aligned} \tag{33}$$

AIC - Detection Criterion

d_i - eigenvalue (ith)

$$\text{AIC} = -2L\left(\ln\begin{bmatrix}\prod_{i=1}^{N} d_i\\ \cdots \\ d_1 d_2 d_3 \\ d_1 d_2 \\ d_1\end{bmatrix} + \begin{bmatrix}N\\ N-1\\ \cdots\\ 3\\ 2\\ 1\end{bmatrix} \odot \left(\ln\begin{bmatrix}N\\ N-1\\ \cdots\\ 3\\ 2\\ 1\end{bmatrix} - \ln\begin{bmatrix}\sum_{i=1}^{N} d_i\\ \cdots\cdot \\ d_1+d_2+d_3\\ d_1+d_2\\ d_1\end{bmatrix}\right)\right)$$
$$+2\begin{bmatrix}0\\ 1\\ \cdots\\ N-2\\ N-1\end{bmatrix} \odot \begin{bmatrix}2N\\ 2N-1\\ \cdots\\ N+2\\ N+1\end{bmatrix} \quad \text{an } (N\times 1) \text{ real vector} \tag{34}$$

Minimum Description Length (MDL) Detection Criterion

$$\begin{aligned}
\text{MDL} &= -L\left(\ln\begin{bmatrix}\prod_{i=1}^{N} d_i\\ \dots\\ d_1 d_2 d_3\\ d_1 d_2\\ d_1\end{bmatrix} + \begin{bmatrix}N\\ N-1\\ N-2\\ \dots\\ 2\\ 1\end{bmatrix}\odot\left(\ln\begin{bmatrix}N\\ N-1\\ N-2\\ \dots\\ 2\\ 1\end{bmatrix} - \ln\begin{bmatrix}\sum_{i=1}^{N} d_i\\ \dots\\ d_1+d_2+d_3\\ d_1+d_2\\ d_1\end{bmatrix}\right)\right)\\
&+\frac{1}{2}\ln L\begin{bmatrix}0\\ 1\\ \dots\\ N-2\\ N-1\end{bmatrix}\odot\begin{bmatrix}2N\\ 2N-1\\ \dots\\ N+2\\ N+1\end{bmatrix} \quad \text{an } (N\times 1) \text{ real vector}
\end{aligned} \tag{35}$$

- remember: d_i , (for $i = 1$ to N with $d_1 < d_2 < \ldots < d_N$) denotes the i-th eigenvalue of $\mathbb{R}_{xx}$
- Notes:
    - if the **first element** of the vector AIC or MDL is minimum then $M = 0$
    - if the **second element** of the vector AIC or MDL is minimum then $M = 1$
    - if the **third element** of the vector AIC or MDL is minimum then $M = 2$
    - etc.
- **Reference**: M. Wax and T. Kailath, "Detection of Signals by Information Theoretic Criteria", *IEEE Transactions on ASSP*, vol. 33, pp. 387-392, Apr. 1985.

Detection Problem - Summary

- If $L = \infty$, i.e. Theoretical $\mathbb{R}_{xx} = \mathcal{E}\left\{\underline{x}(t).\underline{x}(t)^H\right\}$, then

$$M = N - (\text{multiplicity of min. eigenvalue of } \mathbb{R}_{xx}) \quad (36)$$

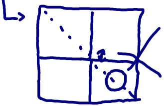
noise power $$\sigma_n^2 = \text{min.eigenvalue of } \mathbb{R}_{xx} \quad (37)$$

- If $L =$finite, i.e. Practical $\mathbb{R}_{xx} = \frac{1}{L}\sum_{l=1}^{L} \underline{x}(t_l).\underline{x}(t_l)^H = \frac{1}{L}\mathbb{X}.\mathbb{X}^H$

  then

$$M = \text{can be found using AIC or MDL} \quad (38)$$

$$\begin{aligned} \widehat{\sigma}_n^2 &= \text{the average of the } N - M \text{ smallest eigenvalues} \\ &= \frac{1}{N-M}\left(\sigma_{M+1}^2 + \sigma_{M+2}^2 + ... + \sigma_N^2\right) \end{aligned} \quad (39)$$



min eigenvalues
or
max gap

- Remember:
  - N =number of array elements
  - M = number of signals/sources

The Estimation Problem

The Maximum Likelihood (ML) approach

- Consider an observed $(N \times 1)$ complex signal-vector $\underline{x}(t)$ modelled as follows

$$\underline{x}(t) \triangleq \mathbb{S}(\underline{p}).\underline{m}(t) + \underline{n}(t) \tag{40}$$

- In this case the L observations at times t_1, t_2,...,t_L (i.e. finite observation interval) are

$$[\underline{x}(t_1), \underline{x}(t_2), \ldots, \underline{x}(t_L)] \text{ defined as the } N \times L \text{ complex matrix } \mathbb{X}$$

since the noise is modelled as a zero mean complex Gaussian random process, with a covariance matrix $\mathbb{R}_{nn} \triangleq \sigma^2 \mathbb{I}_N$ then the observed array signal $\underline{x}(t)$ has a mean vector and covariance matrix which are given as follows:

$$\mathcal{E}\{\underline{x}(t)\} = \mathbb{S}(\underline{p}).\underline{m}(t_l)$$

$$\mathcal{E}\{\underbrace{(\underline{x}(t) - \mathcal{E}\{\underline{x}(t)\}).(\underline{x}(t) - \mathcal{E}\{\underline{x}(t)\})^H}_{\mathbb{R}_{nn}}\} = \sigma_n^2 \mathbb{I}_N$$

- This implies that if there are L observations, which are independent, then the conditional probability density function (likelihood function - LF)

$$\mathrm{LF}_x \triangleq \mathrm{pdf}_x\left(\underline{x}(t_1), \underline{x}(t_2), \ldots, \underline{x}(t_L) \,\middle|\, \underline{p}, \mathbb{M}, \sigma_n^2\right) \tag{41}$$

- ML Solution:

$$\widehat{\mathbb{M}} = (\mathbb{S}(\underline{p})^H \mathbb{S}(\underline{p}))^{-1} \mathbb{S}(\underline{p})^H \mathbb{X} \tag{42}$$

$$\widehat{\underline{m}}(t_i) = \underbrace{(\mathbb{S}(\underline{p})^H \mathbb{S}(\underline{p}))^{-1} \mathbb{S}(\underline{p})^H}_{\triangleq \underline{\mathbb{W}}_{ML}^H} \underline{x}(t_i) \tag{43}$$

$$\begin{aligned} \widehat{\underline{p}}_{ML} &= \arg\max_{\underline{p}} \{\mathrm{LF}_x\} \\ &= \arg\max_{\underline{p}} \left\{ Tr\left(\mathbb{P}_{\mathbb{S}} . \mathbb{R}_{xx}\right)\right\} \end{aligned} \tag{44}$$

where

$$\mathbb{P}_{\mathbb{S}} = \mathbb{S}(\underline{p}) \left(\mathbb{S}^H(\underline{p}) \mathbb{S}(\underline{p})\right)^{-1} \mathbb{S}(\underline{p})^H \tag{45}$$

Space Channel Parameter Estimation:Subspace-type

- In this type of algorithms the parameter M is assumed known ($M < N$) and involves in some way, or another, two concepts:

  i) the concept of the "**signal-subspace**" associated with the observed signal-vector $\underline{x}(t)$ and its properties

  - ★ This is an *unknown linear subspace* of dimensionality equal M-embedded in an N-dimensional ($M < N$)

  ii) the concept of the "**manifold**" associated with the system/problem's characteristicsin the case of arrays known as "array manifold". It is independent of the noisy observed signal-vector $\underline{x}(t)$ and its properties.

  - ★ This is a *non-linear subspace* (e.g. a curve, surface etc) - embedded in an N-dimensional observation space

- Solution $= M$ points $\begin{cases} \overset{\text{(non-linear)}}{\in \text{system's manifold}} \\ \in \underset{\text{(linear)}}{\text{`signal subspace'}} \text{ of } \underline{x}(t) \end{cases}$

- As a result the objective is firstly, from the data, to *estimate the signal subspace* and then to **search the manifold** to find its intersection with the estimated signal-subspace.

data
↓
signal subspace
↓
manifold

The Concept of the "Signal Subspace"

- The first step is to utilize the observed (received) signal vector

$$
\begin{aligned}
\underline{x}(t) &= \overbrace{\beta_1 m_1(t)}^{\triangleq m_1(t)}\underline{S}_1 + \overbrace{\beta_2 m_2(t)}^{\triangleq m_2(t)}\underline{S}_2 + \ldots + \overbrace{\beta_M m_M(t)}^{\triangleq m_M(t)}\underline{S}_M \\
&\quad + n(t) \qquad (46) \\
&= \mathbb{S}\underline{m}(t) + \underline{\mathrm{n}}(t), \forall t \text{ (infinite observation interval)} \qquad (47)
\end{aligned}
$$

or, over L snapshots (finite observation interval)

to estimate the "**signal subspace** ".

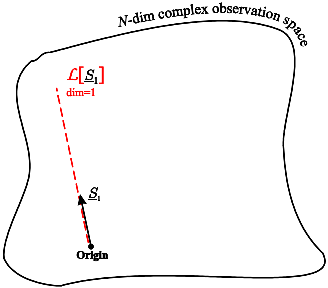
- The "**signal subspace** " should have **dimensionality** (in most cases) equal to M (known - or estimated) and the signal term $\mathbb{S}.\underline{m}(t)$ belongs always to this subspace.
  i.e.

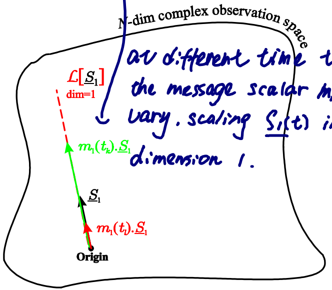
$$
\begin{aligned}
&\dim(\textbf{signal subspace}) = M \\
&\mathbb{S}\underline{m}(t) \in \textbf{signal subspace}
\end{aligned} \qquad (48)
$$

The Subspace of a Manifold (Response) Vector

manifold vector → $\underline{S}_\ell$ (ℓ-th path)

- Let us consider the subspace spanned by only one signal-term of Equation 46, for instance the first term $m_1(t)\underline{S}_1$ at $t = t_1$ and $t = t_2$ as well as the manifold vector $\underline{S}_1$ are shown as below:





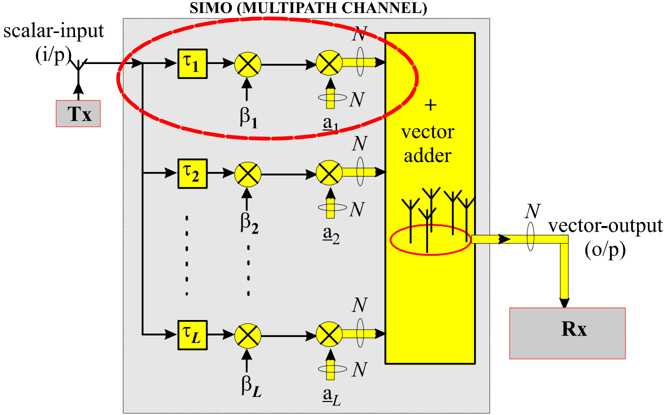
at different time t the message scalar $m_1(t)$ vary. scaling $\underline{S}_1(t)$ in dimension 1.

- It is clear from the above figures that

$$\mathcal{L}[\underline{S}_1] = \mathcal{L}[m(t)\underline{S}_1, \forall t] \quad (49)$$

The Subspace of a Manifold (Response) Vector

- For instance consider a SIMO multipath channel

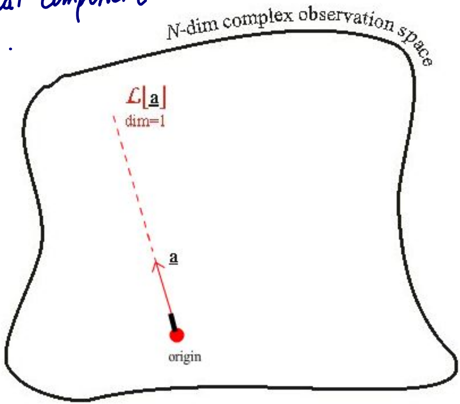


- Here the symbols/vectors $\underline{S}$ and $\underline{a}$ are equivalent and both denote an array manifold vector

The Subspace of a Single Signal/Source

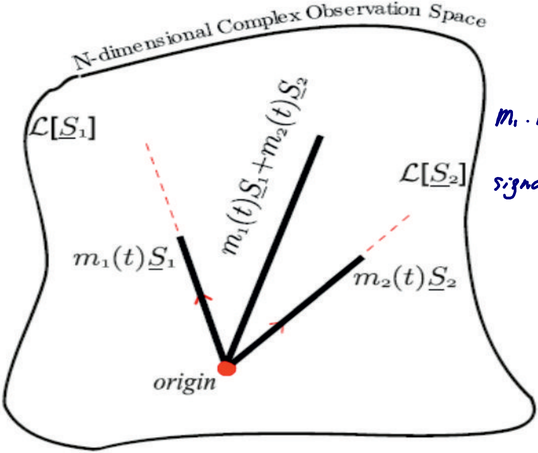
$$m(t)\underline{a} \Rightarrow \mathcal{L}[\beta m(t)\underline{a}] = \mathcal{L}[\underline{a}] \text{ where } \underline{a} \triangleq \underline{a}(p)$$

received signal component from path-ℓ.



The Subspace of two Signals/Sources

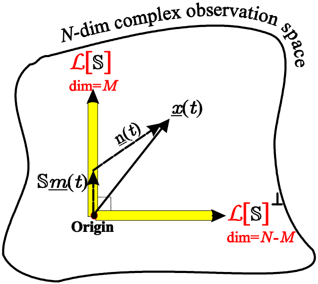
$$m_1(t)\underline{S}_1 + m_2(t)\underline{S}_2 \Rightarrow \mathcal{L}[m_1(t)\underline{S}_1 + m_2(t)\underline{S}_2] = \mathcal{L}[\underline{S}_1, \underline{S}_2]$$



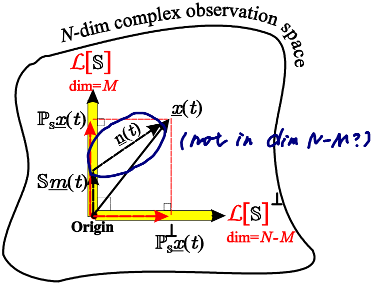
$m_1 \cdot m_2$ same variation
⇓
signal in one line.

The Subspace of the received signal-vector (t)

- Furthermore, Equation 47 can be represented as follows (at a specific time t):



$\Rightarrow$



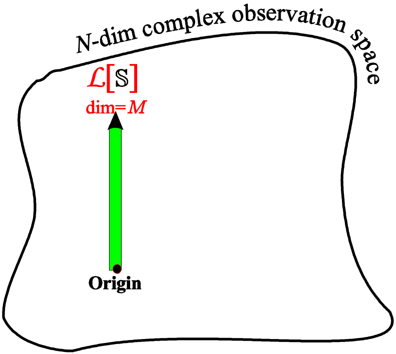
- That is,

$$\text{"subspace of } \underline{x}(t), \forall t\text{"} = \mathcal{L}[\underline{x}(t), \forall t] = \text{whole obs. space} \quad \text{with} \ \dim\left(\mathcal{L}[\underline{x}(t), \forall t]\right) = N \tag{50}$$

- As the dimensionality of this subspace is M and the number of columns of $\mathbb{S}$ is equal to M (remember $\mathbb{S} = [\underline{S}_1, \underline{S}_2, ..., \underline{S}_M]$) this implies that

$$\begin{aligned} \text{"signal subspace"} &= \mathcal{L}[\mathbb{S}] \\ \text{with } \dim(\mathcal{L}[\mathbb{S}]) &= M \end{aligned} \tag{51}$$

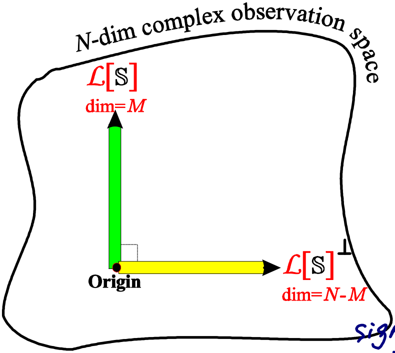
- That is, the signal subspace is spanned by the unknown M manifold vectors associated with the M signals (one signal - one vector)



- The complement subspace to the signal subspace is known as "noise subspace" i.e.

$$\begin{aligned} \text{"noise subspace"} &= \mathcal{L}[\mathbb{S}]^{\perp} \\ \text{with } \dim(\mathcal{L}[\mathbb{S}]^{\perp}) &= N - M \end{aligned}$$

- Signal-Subspace type techniques are based on **partitioning** the observation space into
  - the **Signal Subspace** $\mathcal{L}[\mathbb{S}]$ and
  - the Noise Subspace $\mathcal{L}[\mathbb{S}]^{\perp}$



- However, as the matrix $\mathbb{S}$ remains unknown, the signal subspace and consequently the noise subspace remain unknown.
- Note:

$$\text{observation-space} \triangleq L[\mathbb{X}] = L[\mathbb{R}_{xx}] = L[\underline{x}(t), \forall t] \tag{52}$$

Observation space (dim - N) $= L[\underline{x}(t), \forall t]$ / signal subspace (dim - M) $= L[\mathbb{S}]$ \ noise subspace (dim - N-M) $= L[\mathbb{S}]^{\perp}$

- *Estimation of the two subspaces:* This is achieved by performing an Eigenvector decomposition of the received data covariance matrix $\mathbb{R}_{xx}$, i.e.

$$[\mathbb{E}_s,\mathbb{E}_n]\begin{bmatrix}\mathbb{D}_s & 0\\ 0 & \mathbb{D}_n\end{bmatrix}\begin{bmatrix}\mathbb{E}_s^H\\ \mathbb{E}_n^H\end{bmatrix} = [\mathbb{E}_s\ \mathbb{E}_n]\begin{bmatrix}\mathbb{D}_s\mathbb{E}_s^H\\ \mathbb{D}_n\mathbb{E}_n^H\end{bmatrix} = \mathbb{E}_s\mathbb{D}_s\mathbb{E}_s^H + \mathbb{E}_n\mathbb{D}_n\mathbb{E}_n^H$$

$$\mathbb{R}_{xx}\mathbb{E}_n = [\mathbb{E}_s,\mathbb{E}_n]\begin{bmatrix}\mathbb{D}_s & 0\\ 0 & \mathbb{D}_n\end{bmatrix}\begin{bmatrix}\mathbb{E}_s^H\\ \mathbb{E}_n^H\end{bmatrix}\cdot\mathbb{E}_n$$

$$\mathbb{R}_{xx} = \mathbb{E}\mathbb{D}\mathbb{E}^H = [\mathbb{E}_s,\mathbb{E}_n].\begin{bmatrix}\mathbb{D}_s & \mathbb{O}\\ \mathbb{O} & \mathbb{D}_n\end{bmatrix}[\mathbb{E}_s,\mathbb{E}_n]^H \qquad (53)$$

$$= [\mathbb{E}_s\ \mathbb{E}_n]\begin{bmatrix}\mathbb{D}_s & 0\\ 0 & \mathbb{D}_n\end{bmatrix}\begin{bmatrix}0\\ I\end{bmatrix}$$

$$= \mathbb{E}_s.\mathbb{D}_s.\mathbb{E}_s^H + \mathbb{E}_n.\mathbb{D}_n.\mathbb{E}_n^H \qquad (54)$$

$$= [\mathbb{E}_s\ \mathbb{E}_n]\begin{bmatrix}0\\ \mathbb{D}_n\end{bmatrix}$$

$$= \mathbb{E}_n\mathbb{D}_n = \sigma^2\mathbb{E}_n$$

manifold space

$\mathcal{L}[\mathbb{S}]$: linear space spanned by manifold vectors.

$\therefore \mathbb{R}_{xx} = \mathbb{E}\Lambda\mathbb{E}^H + \sigma_n^2\mathbb{I}$

- This implies that

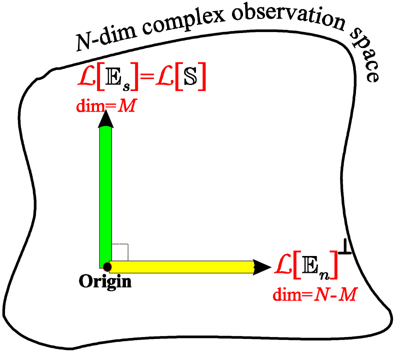
$\mathcal{L}[\mathbb{E}_s]$: linear space spanned by eigenvectors associated in signal eigenvalues.

$\therefore \mathbb{E}\Lambda\mathbb{E}^H\mathbb{E}_n = 0$ for all case

$\therefore \boxed{\mathbb{E}^H\mathbb{E}_n = 0}$

signal subspace = (signal subspace)	$\mathcal{L}[\mathbb{S}] = \mathcal{L}[\mathbb{E}_s] = \mathcal{L}[\mathbb{E}_n]^{\perp}$
noise subspace =	$\mathcal{L}[\mathbb{S}]^{\perp} = \mathcal{L}[\mathbb{E}_s]^{\perp} = \mathcal{L}[\mathbb{E}_n]$ (signal + noise eigenvalues)
remember:	$\mathbb{S} \neq \mathbb{E}_s$ although $\mathcal{L}[\mathbb{S}] = \mathcal{L}[\mathbb{E}_s]$
observation space =	$\mathcal{L}[\mathbb{R}_{xx}] = \mathcal{L}[\underline{x}(t), \forall t] = \mathcal{L}[\mathbb{X}]$

(55)

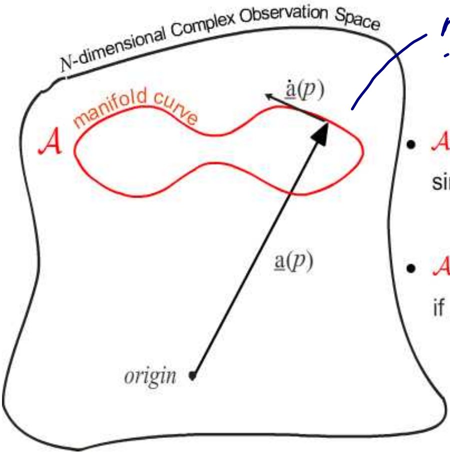

N-dim complex observation space
$\mathcal{L}[\mathbb{E}_s]=\mathcal{L}[\mathbb{S}]$
dim=M
$\mathcal{L}[\mathbb{E}_n]^\perp$
dim=N-M
Origin

The Concept of the "Manifold"

One-parameter Manifolds in Wireless Comms

- Consider the manifold vector (or array response vector) of a single parameter p representing for instance θ or ϕ or F_c
  i.e.
  $$\underline{a}(p) \in C^N$$
  This is a single-parameter vector function.

Locus of the Manifold Vectors



non-linear manifold space
'one-for-all'.

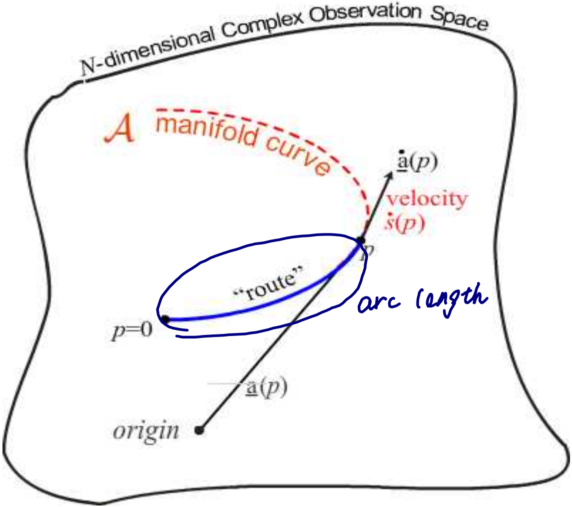
- $\mathcal{A} \triangleq \{\underline{a}(p) \in \mathcal{C}^N, \forall p : p \in \Omega_p\}$ single-parameter curve.

- $\mathcal{A}$ is a regular parameterised curve if $\dot{\underline{a}}(p)$ exists $\forall p$. (apart from end points!)

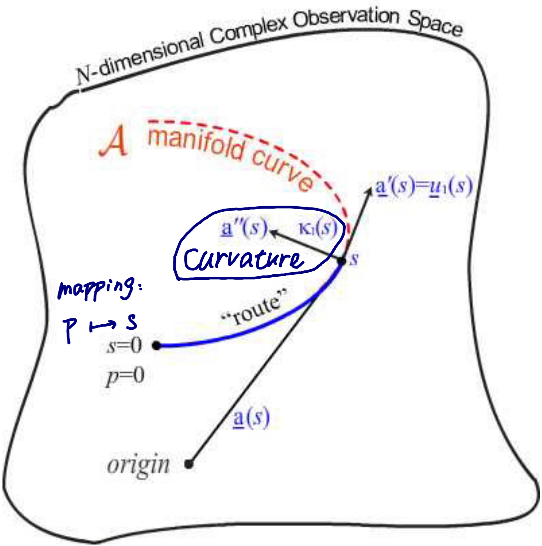
derivative exists
↓
smooth curve.

- By recording **the locus of the manifold vectors** as a function of the parameter p (e.g. direction), a "continuum" (i.e. a **geometrical object** such as a curve) is formed **lying in an N-dimensional space**.
- This **geometrical object** (locus of manifold vectors i.e. $\underline{S}(p), \forall p$) is known as **the array manifold**. (curve $\mathcal{A}$)
- In an array system the manifold (array manifold) can be calculated (and stored) from only the knowledge of the locations and directional characteristics of the sensors.
- Let $\underline{S}(p) \in C^N$ be the manifold vector of a system of N dimensions (e.g. of an array of N sensors) where p is a generic system parameter. This is a single-parameter vector function and as p varies the point $S(p)$ will trace out a curve $\mathcal{A}$ (see figure), embedded in an N-dimensional space C^N.

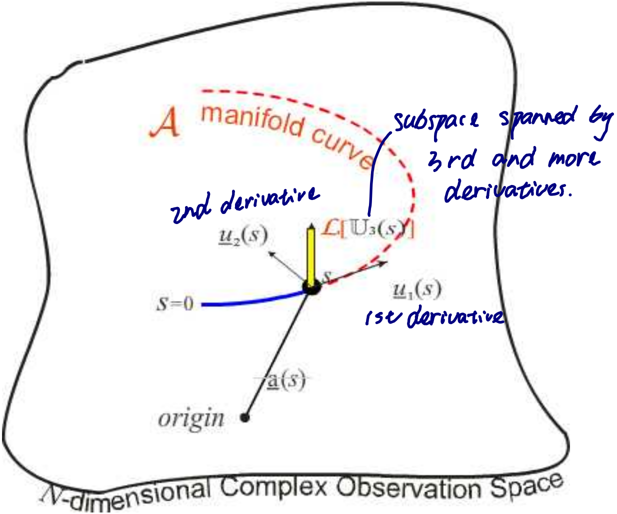
Important Parameters of a Curve



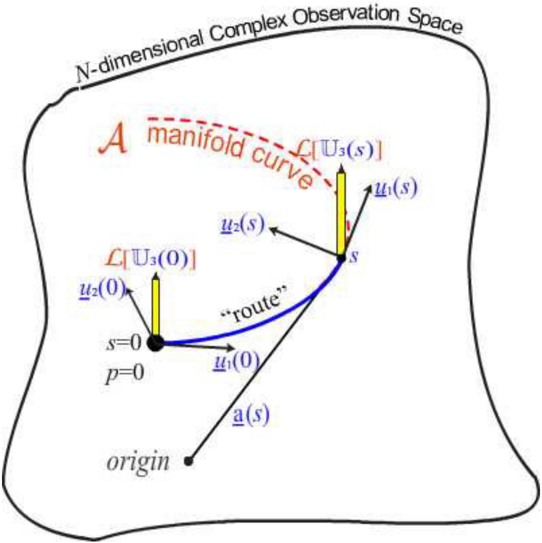
Important Parameters of a Curve



Important Parameters of a Curve

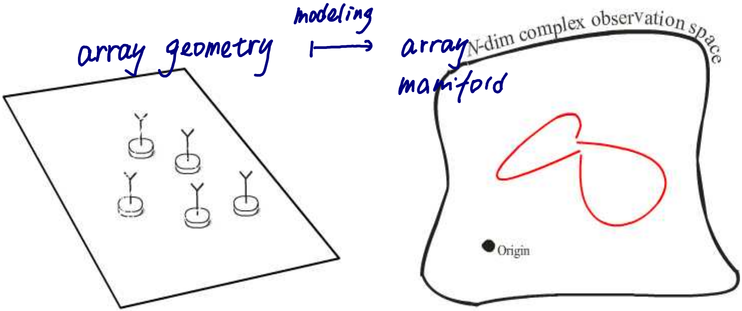


Important Parameters of a Curve



Array of Changing Geometry

- By changing the array geometry the corresponding curve will change

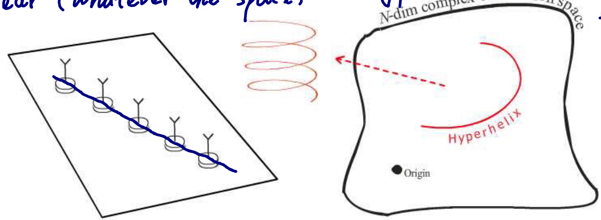


An Array of 5 Sensors with Changing Geometry

Array of Changing Geometry

- By changing the array geometry the corresponding curve will change

linear (whatever the space) → hyper-helix [constant curvatures every point!]



An Array of 5 Sensors with Changing Geometry

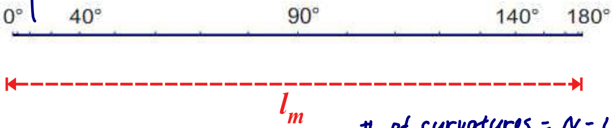
- why linear?

- Design by adding hyper-helixes

Length of a Curve

- Example:

'bad' area with more noise
(close dots)



# of curvatures = N - 1

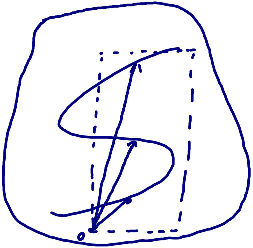
- **important parameters:**
  - **arclength**
    $s \triangleq s(p) = \int_0^p \left\| \dot{\underline{S}}(p) \right\| dp$
  - **rate-of-change of arclength**
    $\dot{s} \triangleq \dot{s}(p) = \left\| \dot{\underline{S}}(p) \right\|$
  - **length of manifold**
    l_m
  - **curvatures**
    a set of real numbers $\kappa_1, \kappa_2, \kappa_2$, etc (curve's shape)

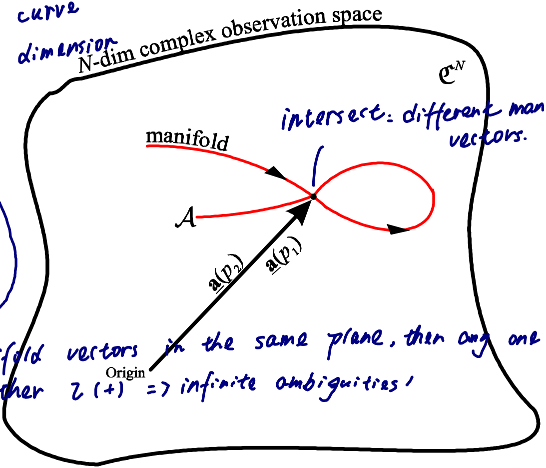
- A curve may have "bad" areas ($\dot{s}$=small) and "good" areas ($\dot{s}$=large)

"Bad" Curves

- There are "bad" and "good" curves (i.e. "bad" and "good" antenna geometries)
- Example of a "bad" curve:

Ambiguities ← long curve
← high dimension

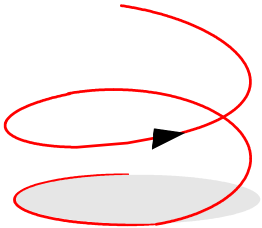




if 3 or more manifold vectors in the same plane, then any one can be represented by other 2 (+) => infinite ambiguities!

Linear Antenna Arrays

- All linear array geometries have manifolds of "hyperhelical" shape embedded in N-dimensional complex space.



Visualisation of a hyperhelix in 3D

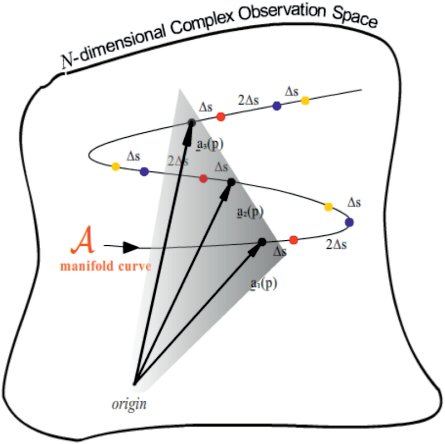
$$\mathbb{C} \triangleq \begin{bmatrix} 0 & -\kappa_1 & 0 & \cdots & 0 & 0 \\ \kappa_1 & 0 & -\kappa_2 & \cdots & 0 & 0 \\ 0 & \kappa_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\kappa_{d-1} \\ 0 & 0 & 0 & \cdots & \kappa_{d-1} & 0 \end{bmatrix}$$

κ_i don't vary.

- Curvatures: forming a matrix known as the **Cartan Matrix $\mathbb{C}$.**
- Hyperhelices: **curvatures=constant** for every s or p
- As we have infinite number of linear arrays we have infinite number of different hyperhelical curves - different set of curvatures

i.e. design 'modeling' flexibly for good array manifold.

Ambiguities



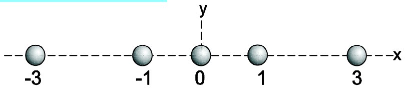
Linear Antenna Array Design

Frobenius Norm.

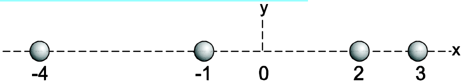
$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}$

- The Frobenius norm of the Cartan matrix (i.e. $\|\mathbb{C}\|_F$) is related to the array symmetricity as follows:

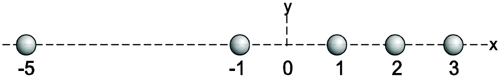
1) $\|\mathbb{C}\|_F = 1$ if array = **symmetric**



2) $\|\mathbb{C}\|_F = \sqrt{2}$ if array = **fully asymmetric**



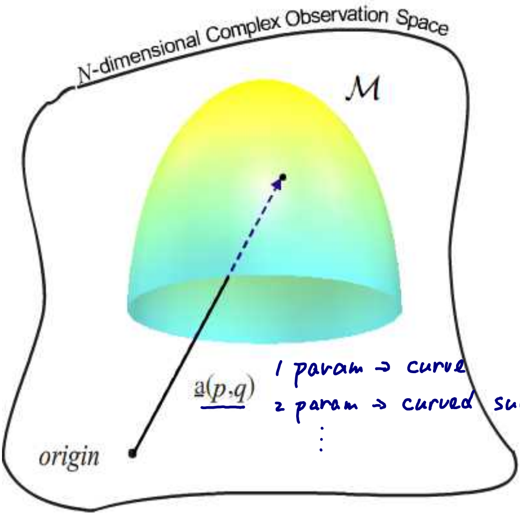
3) $1 < \|\mathbb{C}\|_F < \sqrt{2}$ if array = **partially symmetric**



$\text{eig}(\mathbb{C}) \Rightarrow$ antennas location $\underline{r}_x$

Manifold Surfaces in Wireless Comms

Two-Parameter Manifolds: Visualisation



1 param → curve

2 param → curved surface

⋮

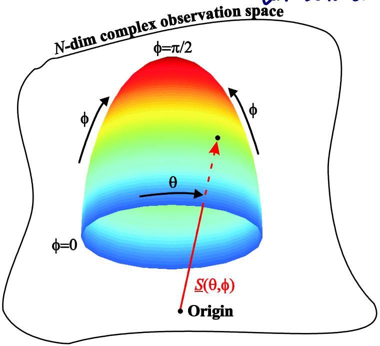
- In a similar fashion *if* there are **two** (unknown) parameters (p,q) per signal *then* $\underline{S}(p,q) \in \mathcal{C}^N$ is a **two-parameter manifold vector** (a vector function)
  and as (p,q) varies the point $\underline{S}(p,q) \in \mathcal{C}^N$ will form a surface $\mathcal{M}$ (see figure), formally defined as follows

$$\text{Array Manifold: } \mathcal{M} \triangleq \{\underline{S}(p,q) \in C^N, \forall (p,q) : p,q \in \Omega\} \tag{56}$$

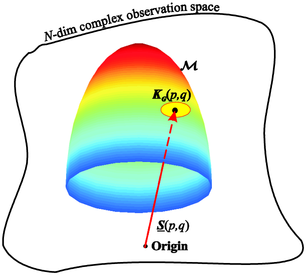
set of locus of manifold vectors

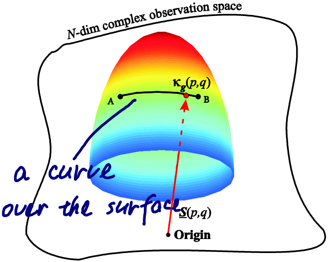
where Ω denotes the *parameter space*.

all solution in non-linear space



- This surface $\mathcal{M}$ is the locus of all manifold vectors $\underline{S}(p,q)$; $\forall p,q$





- For a point (p, q) on the manifold surface the most important parameters are:
  - *The Gaussian curvature*: $K_G(p, q)$
  - *The manifold metric*: $\mathbb{G}(p, q)$
  - *The Christoffel matrices*: $\underline{\underline{\Gamma}}(p, q)$

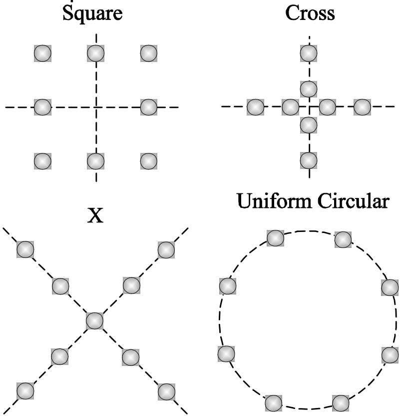
- For a curve on the manifold surface, the parameters of interest are:
  - *The arc length*: s
  - *The geodesic curvature*: κ_g (curve=geodesic$\Rightarrow \kappa_g = 0$)
    short curve $\Rightarrow$ $k_g = 0$

Some Important Results

- All **planar** antenna geometries (2-Dim arrays) have:

$K_G = 0$ (always) $\Rightarrow$ **flat or parabolic of conoid shape**
with the apex at point ϕ=90°

Examples:

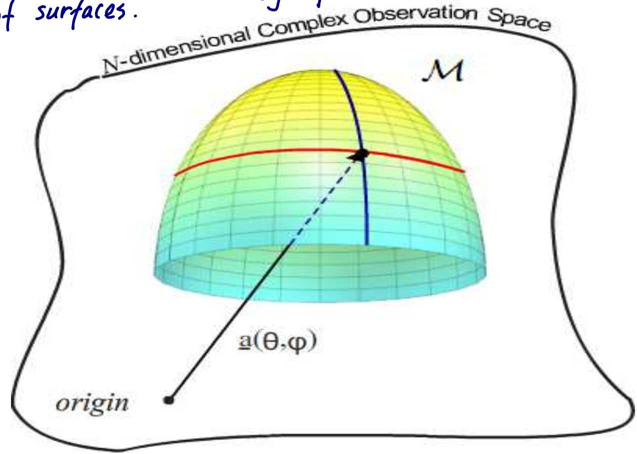


- Note: **'flatness' does not imply that there exist straight lines** , as in the case of a surface in $\mathcal{R}^3$.

  It means that such surfaces can be **generated** by rotating a **passing geodesic curve** around an apex point.

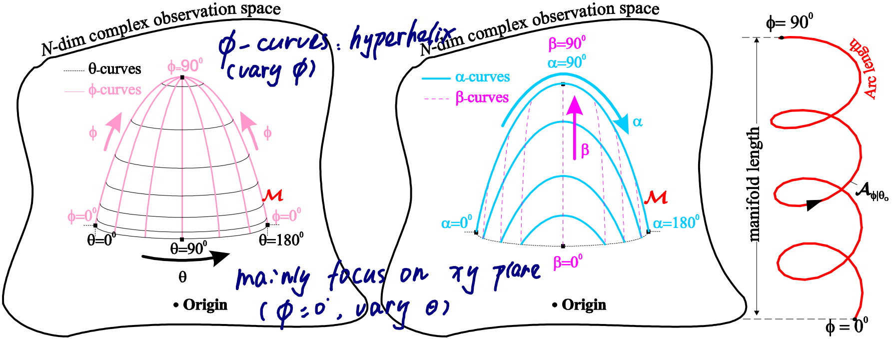
Manifold Surfaces as Families of Curves

analyse families of curves instead of surfaces. (eg. fix elevation first, then fix azimuth).



Manifold Surfaces: Parameterisation

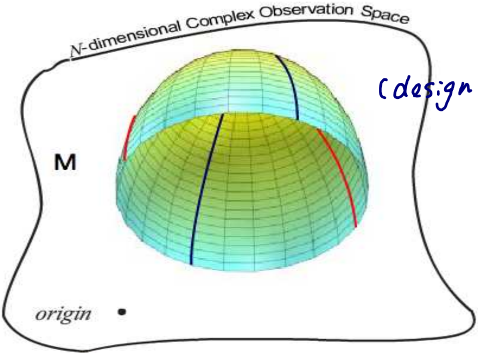
- There are **many parameterisations** of a hypersurface.
- We should always try to find a parameterization that makes solving the problem easier.
- e.g.:"Cone"-angles: (α, β) where $\begin{cases} \cos\alpha = \cos\phi\cos(\theta - \Theta) \\ \cos\beta = \cos\phi\sin(\theta - \Theta) \end{cases}$ where Θ =const



ϕ-curves: hyperhelix (vary ϕ)

mainly focus on xy plane (ϕ=0°, vary θ)

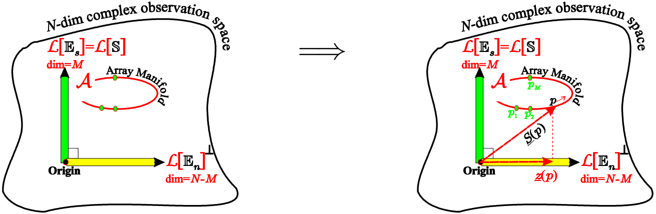
Planar Antenna Array Design Based on Curves



(design 2 basis curves)

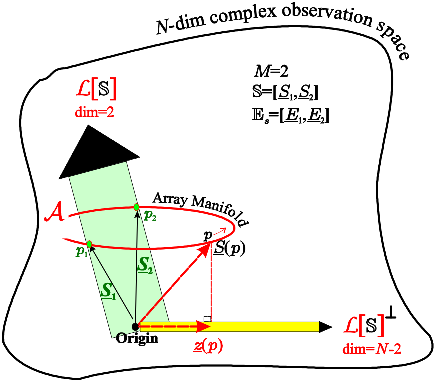
Intersections of Signal Subspace with the Array Manifold

- Both the **manifold** and $\mathcal{L}[\mathbb{E}_s]$ are embedded on the same N-dimensional observation space



- Therefore, the intersection of the manifold with $\mathcal{L}[\mathbb{E}_s]$ will provide the end-points of the columns of the matrix $\mathbb{S}$, i.e. it will provide the parameters $p_1, p_2, \ldots, p_M$.   # of intersects = dim(M)

- Example:



- Note:

$$\underline{S}_1, \underline{S}_2, \underline{E}_1, \underline{E}_2 \quad \perp \quad \mathcal{L}[\mathbb{E}_n]$$

$$\underline{S}_1, \underline{S}_2, \underline{E}_1, \underline{E}_2 \quad \in \quad \mathcal{L}[\mathbb{E}_s]$$

$\mathcal{L}[\mathbb{E}_s]$ is a plane which intersects the array manifold in 2 points S_1, S_2

The MUSIC Algorithm

- It estimates the intersection $\mathcal{L}[\mathbb{E}_s]$ and the array manifold by employing the following procedure:
  - Let p denote a parameter value. array manifold vector
  - Form the associated $\underline{S}(p)$ and then project $\underline{S}(p)$ on to the subspace $\mathcal{L}[\mathbb{E}_n]$. This will give us the vector

$$\underline{z}(p) = \mathbb{P}_{\mathbb{E}_n}.\underline{S}(p) \tag{57}$$

$\underline{z}(p)$: projection of $\underline{S}(p)$ onto the subspace $\mathcal{L}[\mathbb{E}_n]$.
(should be 0 since they are orthogonal).

$\mathbb{E}_s \cdot \mathbb{E}_n = 0$

- The norm-squared of $\underline{z}(p)$ can be written as

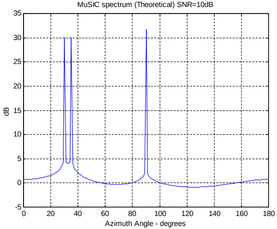
$$\begin{aligned}\xi(p) &= \underline{z}(p)^H \underline{z}(p) \\ &= \underline{S}(p)^H . \underbrace{\underbrace{\mathbb{P}_{\mathbb{E}_n}^H}_{=\mathbb{P}_{\mathbb{E}_n}} . \mathbb{P}_{\mathbb{E}_n}}_{=\mathbb{P}_{\mathbb{E}_n}} . \underline{S}(p) \\ &= \underline{S}(p)^H . \mathbb{P}_{\mathbb{E}_n} . \underline{S}(p) \\ &= \underline{S}(p)^H . \mathbb{E}_n . (\underbrace{\mathbb{E}_n^H \mathbb{E}_n}_{=\mathbb{I}_{N\times N}})^{-1} . \mathbb{E}_n^H . \underline{S}(p) \\ &= \underline{S}(p)^H . \mathbb{E}_n . \mathbb{E}_n^H . \underline{S}(p) \qquad (58)\end{aligned}$$

- It is obvious that $\xi(p) = 0$ iff $\begin{cases} p = p_1 & \text{or} \\ p = p_2 \end{cases}$
- Therefore, we search the array manifold, i.e. we evaluate the expression (58),$\forall p$, and we select as our estimates the p's which satisfy

$$\xi(p) = 0 \Rightarrow \boxed{\underline{S}(p)^H . \mathbb{E}_n . \mathbb{E}_n^H . \underline{S}(p) = 0, \forall p} \qquad (59)$$

- Equation-59 is known as the Multiple Signal Classification (MUSIC) algorithm and it is a Signal-Subspace type technique.

- Example of MUSIC used in conjunction with a Uniform Linear Array of 5 receiving elements (see AM1 experiment).
- The array operates in the presence of 3 unknown emitting sources with DOA's $(30^\circ, 0^\circ)$, $(35^\circ, 0^\circ)$, $(90^\circ, 0^\circ)$



- Equation-59 is known as the Multiple Signal Classification (MUSIC) algorithm and it is a Signal-Subspace type technique.
- MUSIC Algorithm in Step format:

step-0. assumptions: M and array geometry are known

step-1. receive the analogue signal vector $\underline{x}(t) \in C^{N\times 1}$or its discrete version $\underline{x}(t_l)$, for $l = 1, 2, ..., L$

step-2. find the covariance matrix

$$\mathbb{R}_{xx} = \begin{cases} \mathcal{E}\left\{\underline{x}(t).\underline{x}(t)^H\right\} \in C^{N\times N} & \text{in theory} \\ \frac{1}{L}\sum_{l=1}^{L} \underline{x}(t_l).\underline{x}(t_l)^H \in C^{N\times N} & \text{in practice} \end{cases}$$

step-3. find the Eigenvectors and Eigenvalues of $\mathbb{R}_{xx}$

step-4. form the matrix $\mathbb{E}_s$ with columns the eigevectors which correspond to the M largest eigenvalues of $\mathbb{R}_{xx}$

step-5. find the arg of the M minima of the function

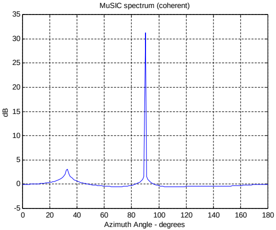
$$\xi(p) = \underline{S}(p)^H.\mathbb{P}_n.\underline{S}(p), \forall p$$

where $\mathbb{P}_n = \mathbb{I}_N - \mathbb{P}_{\mathbb{E}_s}$ and $\mathbb{P}_{\mathbb{E}_s}$is the projection operator onto $\mathcal{L}\{\mathbb{E}_s\}$,

$$\text{i.e.} \quad [p_1, p_2, .., p_M]^T = \arg\min_p \xi(p)$$

$\mathbb{S} = [\underline{S}_1 \ldots \underline{S}_M]$

- MUSIC Limitations:
  - MUSIC breaks down if some incident signals are coherent, i.e. fully correlated, (e.g. multipath situation or 'smart' jamming case)
  - Then $\mathcal{L}[\mathbb{E}_s] \neq \mathcal{L}[\mathbb{S}]$ or, to be more precise, $\mathcal{L}[\mathbb{E}_s] \in \mathcal{L}[\mathbb{S}]$
  - Therefore the 'intersection' argument cannot be used.
  - e.g. same environment as before but the $(30°, 0°)$ & $(35°, 0°)$ sources are coherent (fully correlated)



- $\exists$ algorithms which can handle coherent signals. (see AM1 experiment)

Estimation of Signal Powers, Cross-correlation etc

- Firstly estimate the DOA's and noise power and then use the concept of '**pseudo inverse**' to estimate $\mathbb{R}_{\mathbf{mm}}$

i.e.

DOA: θ or ϕ.

$\mathbb{R}_{xx} = \mathbb{S}\mathbb{R}_{mm}\mathbb{S}^H + \sigma_n^2 \mathbb{I}_N$

$\mathbb{S}\mathbb{R}_{mm}\mathbb{S}^H = \mathbb{R}_{xx} - \sigma_n^2 \mathbb{I}_N$

$\mathbb{S}^{\#}(\mathbb{R}_{xx} - \sigma_n^2 \mathbb{I}_N)\mathbb{S}^{\#H} = \mathbb{S}^{\#}\mathbb{S}\mathbb{R}_{mm}\mathbb{S}^H\mathbb{S}^{\#H}$

$= (\mathbb{S}^H\mathbb{S})^{-1}\mathbb{S}^H\mathbb{S}\mathbb{R}_{mm}\mathbb{S}^H\mathbb{S}(\mathbb{S}^H\mathbb{S})^{-1}$

$= \mathbb{R}_{mm}$

$$\begin{aligned}
\text{step}-1 \quad &: \quad \text{Based on } \mathbb{R}_{xx}, \text{ estimate } \underline{p} \text{ and } \sigma_{\mathrm{n}}^2 \\
\text{step}-2 \quad &: \quad \text{form } \mathbb{S} \\
\text{step}-3 \quad &: \quad \underbrace{\mathbb{R}_{mm} = \mathbb{S}^{\#} \cdot (\mathbb{R}_{xx} - \sigma_{\mathrm{n}}^2 \mathbb{I}_N) \cdot \mathbb{S}^{\#H}}
\end{aligned} \qquad (60)$$

$$\text{where } \mathbb{S}^{\#} = (\mathbb{S}^H \cdot \mathbb{S})^{-1} \cdot \mathbb{S}^H = \text{pseudo-inverse of } \mathbb{S} \qquad (61)$$

- Note that:

σ_n^2 =(min eigenvalue of $\mathbb{R}_{xx}$ or given by Equation-33)

=noise power

- **Proof of Equation-60** :

$$\begin{aligned} \mathbb{R}_{xx} &= \mathbb{S}.\mathbb{R}_{mm}.\mathbb{S}^H + \sigma_{\mathrm{n}}^2\mathbb{I}_N \\ \mathbb{R}_{xx} - \sigma_{\mathrm{n}}^2\mathbb{I}_N &= \mathbb{S}.\mathbb{R}_{mm}.\mathbb{S}^H \end{aligned} \tag{62}$$

By pre & post multiplying both sides of the previous equation with the pseudo inverse of $\mathbb{S}$ we have

$$\begin{aligned} & \mathbb{S}^{\#}.\left(\mathbb{R}_{xx} - \sigma_n^2\mathbb{I}_N\right).\mathbb{S}^{\#H} = \overbrace{\left(\mathbb{S}^H\mathbb{S}\right)^{-1}\mathbb{S}^H}^{=\mathbb{S}^{\#}}.\mathbb{S}.\mathbb{R}_{mm}.\mathbb{S}^H.\overbrace{\mathbb{S}\left(\mathbb{S}^H\mathbb{S}\right)^{-1}}^{=\mathbb{S}^{\#H}} \\ \Longrightarrow\quad & \mathbb{S}^{\#}\cdot\left(\mathbb{R}_{xx} - \sigma_n^2\mathbb{I}_N\right)\cdot\mathbb{S}^{\#H} = \mathbb{R}_{mm} \end{aligned}$$

Main Catergories of Beamformers

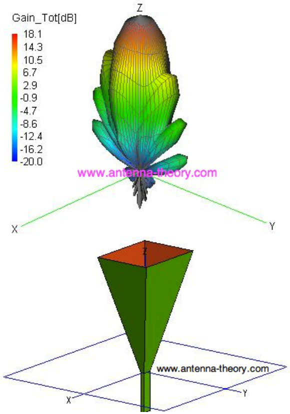
beam in certain direction.
physically rotate.

- **Category-1** : **Single sensor** with directional response.

- Green Bank Telescope, National Radio Astronomy Observatory, West Virginia.
- 100 m clear aperture; Largest fully steerable antenna in the world.

Main Catergories of Beamformers

- **Horn- Antenna** : Another example of **Category-1** (**Single sensor** with directional response).



Main Catergories of Beamformers

- **Category-2** : **Array of Sensor**
  - Used in SONAR, RADAR, communications, medical imaging, radio astronomy, etc.
  - Line array of directional sensors: Westerbork Synthesis Array Radio Telescope, (WSRT), the Neterlands.

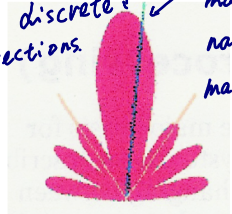
Main Catergories of Beamformers (Array of Sensors)

- **switched beamformer** : there is a finite number of fixed array patterns and the system chooses one of them to maximise signal strength (the one with main lobe closer to the desired user/signal) and switches from one to beam to another as the user/signal moves throughout the sector).
- **adaptive beamformer** (or adaptive array): array patterns are adjusted automatically (main lobe extending towards a user/signal with a null directed towards a cochannel user/signal)

fixed discrete directions.

massive users
↓
narrow beam
↓
massive MIMO

user: right at main lobe.
no cochannel sidelobes from other.
(null directed)

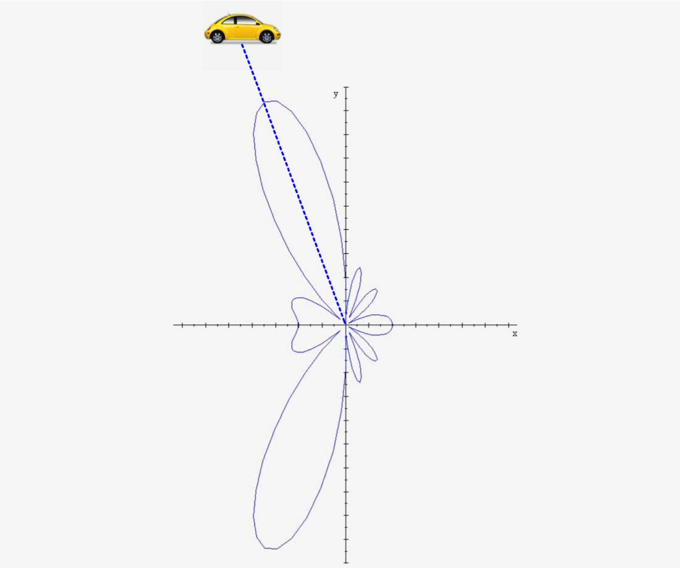


switched beamformer

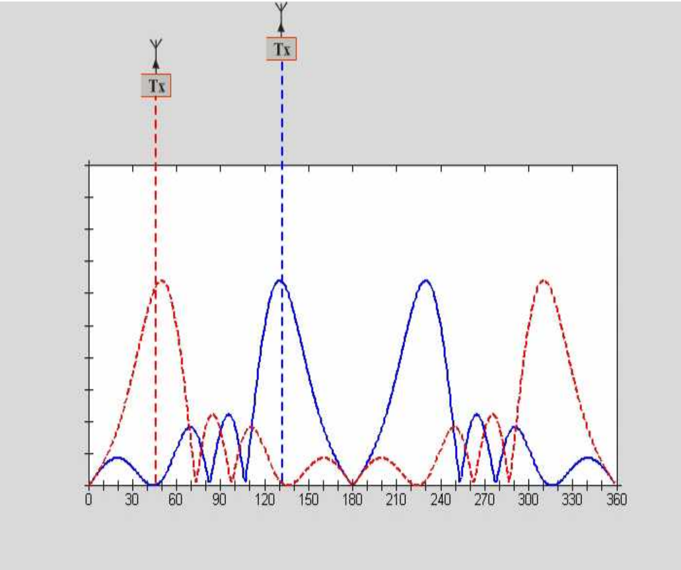
adaptive beamformer

Adaptive for 2 cochannel signals/users

Adaptive Beam (ULA, N=5, d=1)

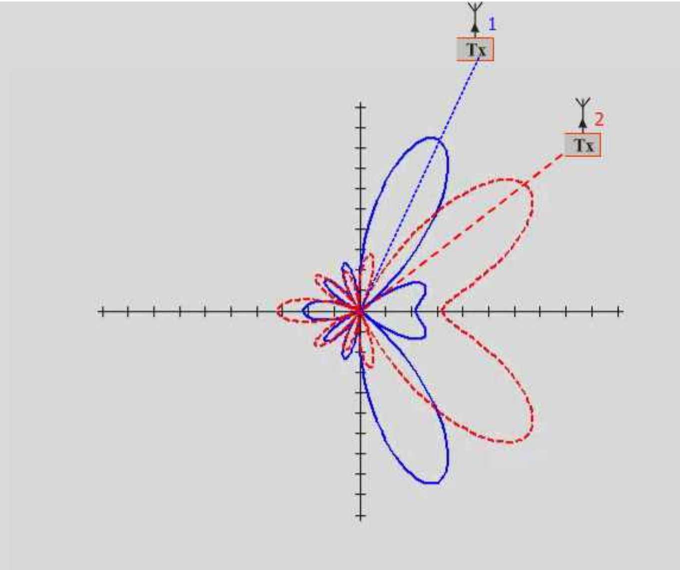


Adaptive Beam (Two co-Channel Signals)



ULA
$N = 5$
$d = 1$

Adaptive Beam (Two co-Channel Signals)



ULA

$N = 5$

$d = 1$

The 'Reception' Problem: Array Pattern & Beamforming

Definitions

- If the array elements are weighted by complex-weights then the **array pattern** provides the gain of the array as a function of DOAs

  e.g.

  complex number   fixed   vary - {array geo. / channel θ.

  $$\text{if } \theta \longmapsto \underline{S}(\theta) \text{ then } g(\theta) = \underline{w}^H \underline{S}(\theta) \quad (63)$$

  Array manifold vector (geometric)   gain in a certain direction

  where $g(\theta)$ denotes the gain of the array for a signal arriving from direction θ

  Then,

  $$\boxed{g(\theta), \forall\theta : \text{ is known as the array pattern}} \quad (64)$$

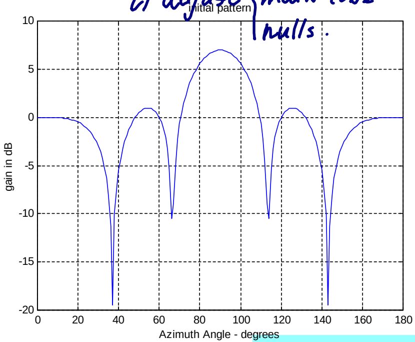
The array pattern is a function of the array manifold $\underline{S}(\theta)$ (i.e. array geometry and channel parameter θ) and the Rx weight vector $\underline{w}$

- N.B.: default pattern:

$$g(\theta) = \underline{1}_N^T \underline{S}(\theta) \quad (65)$$

i.e. $\underline{w} = \underline{1}_N$ (i.e. no weights)
e.g. Array Pattern of a Uniform Linear Array of 5 elements
($\underline{w} = \underline{1}_5$, i.e. no weights)

Require: 1) larger gain (main lobe).
2) adjust { main lobe
nulls.



- **Beamwidth**
The array pattern **has** a number of **lobes.**
  - the **largest** lobe is called the **'main lobe'** while
  - the **remaining** lobes are known as **'sidelobes'** .

$$\text{beamwidth}^\circ = 2\sin^{-1}\left(\frac{\lambda}{Nd}\right) \times \frac{180}{\pi} \quad (66)$$

$d = \frac{\lambda}{2} \Rightarrow$ default mainlobe at 90°.

- Note that d =inter-sensor-spacing, and

$$\text{if } d = \frac{\lambda}{2} \Rightarrow \text{beamwidth}^\circ = 2\sin^{-1}\left(\frac{2}{N}\right) \times \frac{180}{\pi} \tag{67}$$

- To **steer the main lobe** towards a specific (known) direction θ, a '**spatial correction weight**' $\underline{w}_{\text{main-lobe}}$ can be used which should be equal to

fix array

adjust main-lobe by weight control.

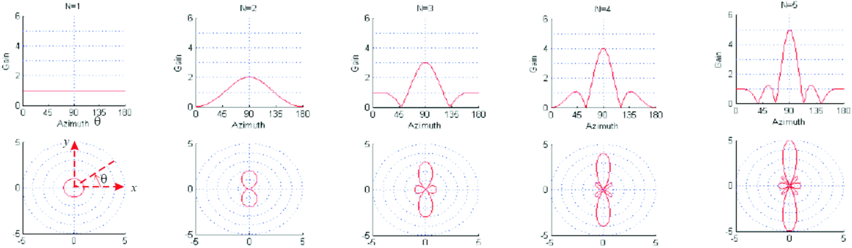
sensor location

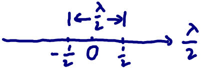
desired direction

$$\underline{w}_{\text{main-lobe}} = \exp(-j\underline{r}^T\underline{k}_{\text{main-lobe}}) \tag{68}$$

$$\underline{w}_{\text{main-lobe}} = S(\theta_{\text{main-lobe}}) \tag{69}$$

- ARRAY PATTERN: for arrays of $N = 1, 2, 3, 4$ and 5 sensors (Mainlobe at 90°, $d = 1$ half-wavelength)

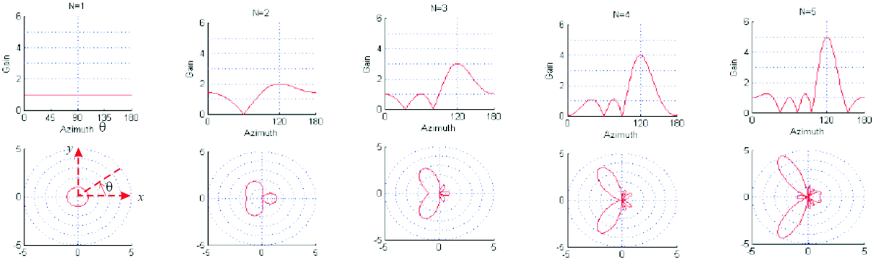




$$\underline{r}_x = [0] \quad = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}^T \quad = \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix}^T \quad = \begin{bmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{bmatrix}^T \quad = \begin{bmatrix} -1.5 \\ -0.5 \\ 0 \\ 0.5 \\ 1.5 \end{bmatrix}^T$$

$$\underline{w} \rightarrow = \quad = [1,1]^T \quad = [1,1,1]^T \quad = [1,1,1,1]^T \quad = [1,1,1,1,1]^T$$

- ARRAY PATTERN for ULA arrays of $N = 2, 3, 4, 5$ sensors (mainlobe at 120°, $d = 1$ half-wavelength)

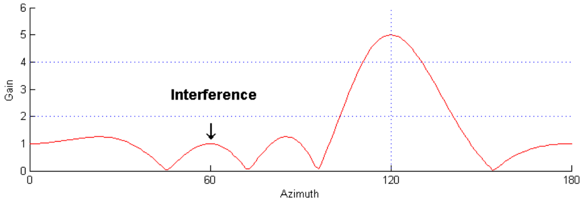


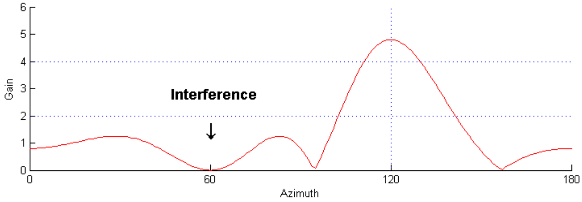
$$\underline{r}_x = [0] \quad = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}^T \quad = \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix}^T \quad = \begin{bmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{bmatrix}^T \quad = \begin{bmatrix} -1.5 \\ -0.5 \\ 0 \\ 0.5 \\ 1.5 \end{bmatrix}^T$$

$$\underline{w} = \underline{S}(120°, 0) = \exp(-j\underline{\underline{r}}^T \underline{k}(120°, 0°)) \tag{70}$$

$$(\text{simplified to}) = \exp(-j\pi \underline{r}_x \cos(120°)) \tag{71}$$

- A **beamformer** is an array system which **receives** a 'desired' signal and **suppresses** (according to a criterion) co-channel interference and noise effects, by **synthesizing an array pattern** with high-gain towards the DOA of the desired signal and deep nulls towards the DOAs of the interfering signals (adaptive arrays).





Some Popular Beamformers

need desired signal direction.

- WIENER-HOPF Beamformer:

Wiener:
- SNIR optimum
- don't need DOAs of interfering
- no guarantee for null interference
- for stationary environment
- if interference close enough, they can be regarded as one.

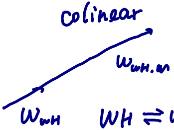
$$\boxed{\underline{w} = c.\mathbb{R}_{xx}^{-1}\underline{S}_{\substack{\text{desired}\\ \text{signal}}}} \tag{72}$$

$$\text{where } c = \text{a constant scalar}$$

  - Maximizes the SNIR at the array output.
  - It is optimum wrt SNIR criterion
  - It is a conventional beamformer (i.e. resolution is a function of the SNR_{in})
  - No need to know the DOAs of the interfering signals
  - (please try to prove Equation-72)

- Modified WIENER-HOPF Beamformer:

Covariance matrix of noise + jammer. (i.e interference)



$$\boxed{\underline{w} = c \cdot \mathbb{R}_{n+J}^{-1} \underline{S}_{\substack{\text{desired}\\ \text{signal}}}} \tag{73}$$

where c = a constant scalar

WH ⇄ WHM: only constant changes.

- comments similar to Wiener-Hopf
- robust to 'pointing' errors (i.e. robust to errors associated with the direction of the desired signal)

• if the direction of desired signal is not sure (i.e. with small error), WHM is better than WH. (remain in that direction).

- Minimum Variance Beamformer
  - It is also known as Capon's beamformer
  - It is the beamformer that solves the following optimisation problem

$$\boxed{\begin{aligned} \underline{w} = \arg \min_{\underline{w}} \left(\underline{w}^H \mathbb{R}_{xx} \underline{w} \right) \\ \text{subject to } \underline{w}^H \underline{S}(\theta) = 1 \end{aligned}} \quad \begin{aligned} (74) \\ (75) \end{aligned}$$

  - Equation 74 aims to minimise the effect of the desired signal plus noise. However, the constraint $\underline{w}^H \underline{S}(\theta) = 1$ (Equ 75) prevents the gain reduction in the direction of the desired signal.
  - Solution of Equations 74 and 75:

(same as wH

$$\underline{w} = c.\mathbb{R}_{xx}^{-1} \underline{S}_{\substack{\text{desired} \\ \text{signal}}} \quad (76)$$

but c ensure unit gain)

$$\text{where } c = \text{a constant scalar chosen such as } \underline{w}^H \underline{S}(\theta) = 1$$

need all directions (based on estimators)
e.g. MUSIC

- A Superresolution Beamformer based on DOA estimation:

Superresolution
· need DOAs of all signals
· interference cancellation
· optimum SIR
· superresolution (resolution ≠ SNR$_{in}$)

$$\boxed{\underline{w} = \mathbb{P}^{\perp}_{\mathbb{S}_J} \underline{S}_{\substack{\text{desired}\\ \text{signal}}}} \tag{77}$$

$$\text{where } \mathbb{S} = \left[\underline{S}_{\substack{\text{desired}\\ \text{signal}}}, \mathbb{S}_J\right]$$

interference manifold

  - Provides complete (asymptotically) interference cancellation.
  - Maximizes the SIR at the array output.
  - It is optimum wrt SIR criterion
  - It is a superresolution beamformer (i.e. resolution is not a function of the SNR$_{in}$)
  - Needs an estimation algorithm to provide the DOAs of all incident signals.

- What if estimation is hard?
- SB without DOA estimation?

- A Superresolution Beamformer not based on DOA estimation of interfering sources

$$\underline{w} = \mathbb{P}^{\perp}_{\mathbb{E}_{n_j}} \cdot \underline{S}_{\substack{\text{desired}\\ \text{signal}}} \tag{78}$$

where $\mathbb{E}_{n_j}$ =noise subspace of $\mathbb{R}_{n+J}$

Note: $\mathbb{R}_{n+J}$ =covariance matrix where the effects of the desired signal have been removed

- Maximum Likelihood (ML) Beamformer

$$\boxed{\underline{w} = \mathrm{col}_{\mathrm{des}}\left(\mathbb{S}^{\#}\right)} \tag{79}$$

$$\text{where } \mathbb{S}^{\#} = \mathbb{S}.(\mathbb{S}^{H}\mathbb{S})^{-1} \tag{80}$$

Examples of Array Patterns (Beamformers)

- Consider a uniform linear array of 5 elements operating in the presence of three signals with directions $(90^\circ, 0^\circ)$, $(30^\circ, 0^\circ)$ and $(35^\circ, 0^\circ)$. One of the signals is the 'desired' signal and the other two are unknown interferences.

- Initially, the 'desired' DOA is

$$(90^\circ, 0^\circ) = \text{known}$$

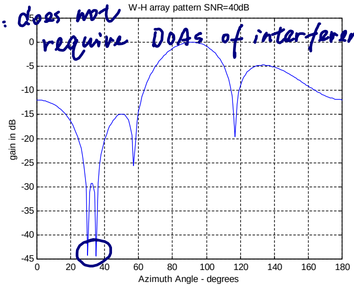
and the DOAs of interfering sources are:

$$\begin{aligned}(30^\circ, 0^\circ) &= \text{unknown}\\(35^\circ, 0^\circ) &= \text{unknown}\end{aligned}$$

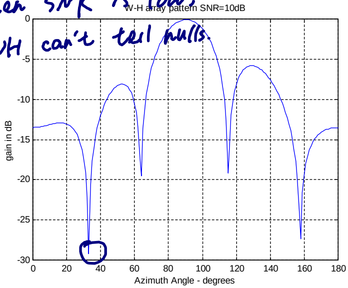
WIENER-HOPF Beamformer (Equ. 72):

SNR=40dB (high) or 10dB (low)

WH: does not require DOAs of interference.

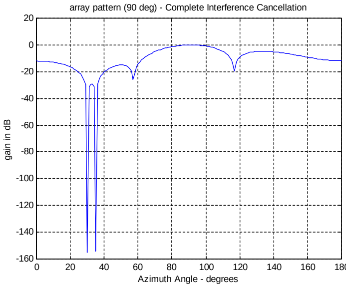


When SNR is low, WH can't tell nulls.



Superresolution Beamformer (Equ. 78)

SNR=10dB and 40dB

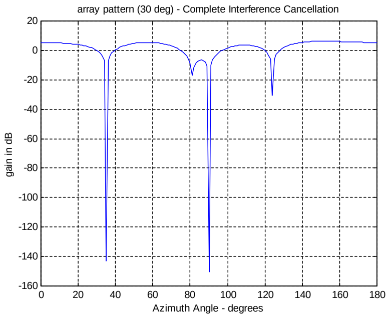


SB without DOAs:

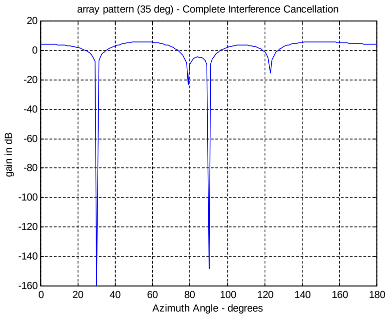
- can detect & cancel interference
- performance not relevant to noise level.

- Superresolution Beamformer (Equ.-77) (all DOAs known).

a) desired source=30°



b) desired source=35°



Beamformers in Mobile Communications

- Some Applications of Beamformers in Communications:

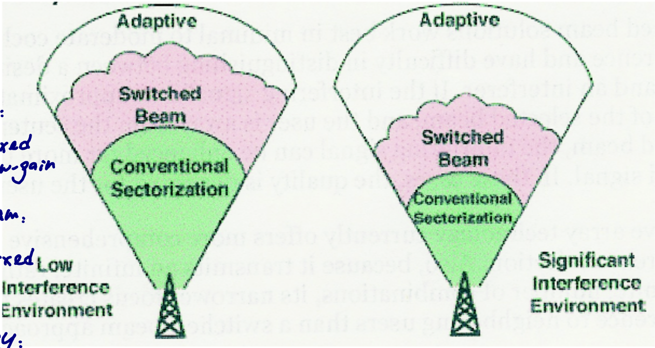
1	analogue access methods	FDMA (e.g. AMPS, TACS, NMT)
2	digital access methods	TDMA (e.g GSM, IS136), CDMA
3	duplex methods	FDD, TDD

- Main properties of beamforming in Communications

Δ

	Properties	Advantages
1	signal gain	better range/coverage(see figure below)
2	interference rejection	increase capacity
3	spatial diversity	multipath rejection
4	power efficiency	reduced expense

- Coverage patterns for switched beam and adaptive array antenna



Sectorization:

fixed
low-gain

Switched beam:

fixed

adaptive array:

flexible
no interference

Array Performance Criteria and Bounds

Introduction

- Two popular performance evaluation criteria are:
  - $\mathrm{SNIR}_{\mathrm{out}}$
  - Outage Probability
- Two Array Bounds
  - Detection Bound
  - Resolution Bound

SNIRout Criterion

- The signal at the output of the beamformer can be expressed as

$$\begin{aligned} y(t) &= \underline{w}^H \underline{x}(t) = \underline{w}^H (\mathbb{S}\underline{m}(t) + \underline{n}(t)) \\ &= \underline{w}^H (\underline{S}_1 m_1(t) + \mathbb{S}_J \underline{m}_J(t) + \underline{n}(t)) \end{aligned}$$

desired signal of user-1

where

$$\begin{aligned} \mathbb{S} &= [\underline{S}_1, \underbrace{\underline{S}_2, \ldots, \underline{S}_M}_{\triangleq \mathbb{S}_J}] \\ \underline{m}(t) &= [m_1(t), \underbrace{m_2(t), \ldots, m_M(t)}_{\triangleq \underline{m}_J^T}]^T \end{aligned}$$

jammer/interference

- Power of $y(t)$:

$$\begin{aligned}
P_y &= \mathcal{E}\left\{y(t)^2\right\} = \\
&= \mathcal{E}\left\{y(t)y(t)^*\right\} = \mathcal{E}\left\{\underline{w}^H \underline{x}(t)\underline{x}(t)^H \underline{w}\right\} \\
&= \underline{w}^H \underbrace{\mathcal{E}\left\{\underline{x}(t)\underline{x}(t)^H\right\}}_{=\mathbb{R}_{xx}} \underline{w} \\
&= \underline{w}^H \left(\underbrace{P_1 \underline{S}_1 \underline{S}_1^H}_{\triangleq \mathbb{R}_{dd}} + \underbrace{\mathbb{S}_J \mathbb{R}_{m_J m_J} \mathbb{S}_J^H}_{\triangleq \mathbb{R}_{JJ}} + \underbrace{\sigma^2 \mathbb{I}_N}_{\triangleq \mathbb{R}_{nn}} \right) \underline{w} \\
&= \underline{w}^H (\mathbb{R}_{dd} + \mathbb{R}_{JJ} + \mathbb{R}_{nn}) \underline{w}
\end{aligned}$$

(assuming desired, interfs & noise are uncorrelated)

- i.e.

$$P_y = \underbrace{\underline{w}^H \mathbb{R}_{dd} \underline{w}}_{=P_{d,out}} + \underbrace{\underline{w}^H \mathbb{R}_{JJ} \underline{w}}_{=P_{J,out}} + \underbrace{\underline{w}^H \mathbb{R}_{nn} \underline{w}}_{=P_{n,out}}$$

where

$$\begin{aligned} P_{d,out} &= \text{o/p desired term} \\ &= P_1 \underline{w}^H \underline{S}_1 \underline{S}_1^H \underline{w} = P_1 (\underline{w}^H \underline{S}_1)^2 \end{aligned}$$

$$\begin{aligned} P_{J,out} &= \text{o/p interf. term} \\ &= \sum_{i=2}^{M} \sum_{j=2}^{M} \rho_{ij} \underline{w}^H \underline{S}_i \underline{S}_j^H \underline{w} \text{ with } \rho_{ij} \triangleq \text{corr.coeff} \end{aligned}$$

$$\begin{aligned} P_{n,out} &= \text{o/p noise term} \\ &= \sigma_{\text{n}}^2 \underline{w}^H \underline{w} \end{aligned}$$

- Therefore, → capacity

$$\mathrm{SNIR}_{\mathrm{out}} = \frac{\overset{S}{P_{d,out}}}{\underset{I}{P_{J,out}} + \underset{N}{P_{n,out}}} = \frac{P_1 \left(\underline{w}^H \underline{S}_1\right)^2}{\underbrace{\sum_{i=2}^{M}\sum_{j=2}^{M} \rho_{ij} \underline{w}^H \underline{S}_i \underline{S}_j^H \underline{w}}_{\underline{w}^H \mathbb{S}_J \mathbb{R}_{m_J m_J} \mathbb{S}_J^H \underline{w}} + \sigma_n^2 \underline{w}^H \underline{w}} \tag{81}$$

- Note:
  - An alternative equivalent expression to Equ 81 is given below

beamformers → weights.

$$\boxed{\mathrm{SNIR}_{\mathrm{out}} = \frac{\underline{w}^H \mathbb{R}_{dd} \underline{w}}{\underline{w}^H \left(\mathbb{R}_{xx} - \mathbb{R}_{dd}\right) \underline{w}}} \tag{82}$$

  - Both Equations 81 and 82 are general expressions for any beamformer. However, for different beamformers (i.e. different weights) these equations give different values.

Outage Probability Criterion

- outage probability (OP) is defined as follows:

$$\text{OP} = \Pr(\text{SNIR}_{\text{out}} < \text{SNIR}_{pr}) \tag{83}$$

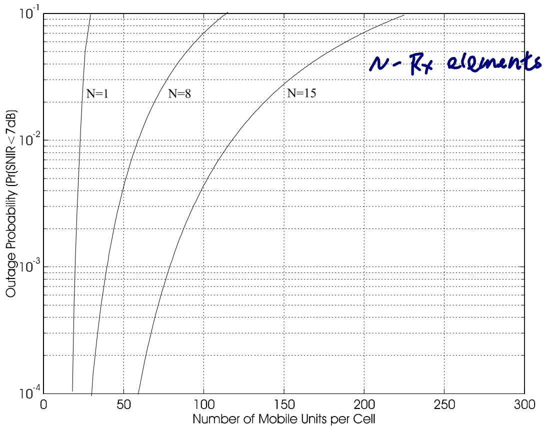
threshold (i.e. the minimum value that the system can function properly).

or

$$\text{OP} = \Pr(\text{SIR}_{\text{out}} < \text{SIR}_{pr}) \tag{84}$$

- It is a performance evaluation criterion.
- An example of an array-CDMA system's Outage Probability with $N = 1, 8, 15$ receiving elements is shown below clearly showing that by employing an antenna array and using, for instance, the beamformer of Equation 77 (complete interference cancellation beamformer) at the base station, the system capacity is increased.

- Outage Probability Examples/Graphs

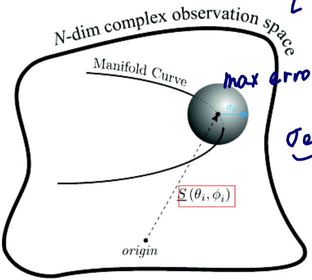


- For example, for 0.001 outage probability, the system capacity per cell increases from 20 mobiles, for a single antenna case (i.e. $N = 1$), to about 40 and 80 mobiles for N equal to 8 and 15, respectively.

Uncertainty Hyperspheres and CRB

Model the uncertainty remaining in the system after L snapshots as an *Uncertainty Hypersphere* of effective radius σ_e

L snapshots → estimation error



$L \uparrow \Rightarrow \sigma_e \downarrow$.

max error = radius σ_e

σ_e - accuracy

$$\sigma_e = \sqrt{\mathrm{CRB}[s]} \qquad (85)$$

$$= \sqrt{\frac{1}{2\,(\mathrm{SNR} \times L)\, C}}$$

$$\mathrm{CRB} = \text{Cramer-Rao Bound}$$

- [see chapter-8 of my book]: The uncertainty sphere represents the smallest achievable uncertainty (optimal accuracy) due to the presence of noise after L snapshots, when all the effects of the presence of other sources have been eliminated by an "ideal" parameter estimation algorithm Constant C : determined by estimation algorithm.
- The Parameter C $(0 < C \leq 1)$: models any additional uncertainty introduced by a practical parameter estimation algorithm. **N.B.: Ideal algorithm:** $C = 1$

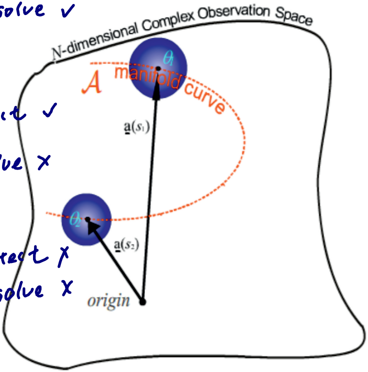
Detection and Resolution Bounds

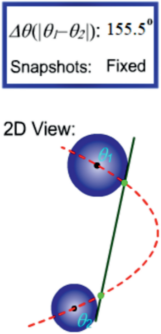
2x source are closed ?

① data line (→plane)
detect ✓
resolve ✓

② detect ✓
resolve ✗

③ detect ✗
resolve ✗





Detection and Resolution Bounds

Angular Separation: Resolution & Detection Laws

- **Square-root Law** :

$$\text{Detection: } \Delta p \propto^{-1} \left(\dot{s}, \sqrt[2]{L \times \mathrm{SNR}_{\mathrm{in}}}\right) \tag{85}$$

- **Fourth-root Law** :

$$\text{Resolution: } \Delta p_{th} \propto^{-1} \left(\dot{s}, \sqrt[4]{\kappa_1}, \sqrt[4]{L \times \mathrm{SNR}_{\mathrm{in}}}\right) \tag{86}$$

- Note: L =number of snapshots
- Remember - **Frequency Selective Channels** :
  - number of resolvable paths$=\left\lfloor \frac{\textit{Delay-Spread}}{\text{channel-symbol period}} \right\rfloor + 1$

    $\Downarrow$
  - two paths with a relative delay $<$ channel-symbol-period

    cannot be resolved
- for more info see Chapter 8 of my book

Detection and Resolution Bounds

Angular Separation: Resolution & Detection Laws

$\sigma_e \propto \frac{1}{\sqrt{SNR \cdot L}}$ - accuracy

- Note: $\sigma_e \propto \sqrt[-2]{SNR \times L}$; where L =number of snapshots
- **Square-root Law** : how close in space to detect them?

$$\text{Detection: } \Delta p = \mathrm{f}\left\{\overset{\text{geo data}}{\underset{\text{arc length}}{s, \sigma_e}}\right\} \tag{86}$$

- **Fourth-root Law** : to resolve them?

$$\text{Resolution: } \Delta p_{th} = \mathrm{f}\left\{\overset{\text{geo}}{s, \sqrt[4]{\kappa_1}}, \overset{\text{data}}{\sqrt{\sigma_e}}\right\} \tag{87}$$

- Remember - **Frequency Selective Channels** :

$T_c > T_{spread}$ flat

$T_c \leq T_{spread}$ f-selective

  - number of paths resolvable paths= $\left\lfloor \frac{\textit{Delay-Spread}}{\text{channel-symbol period}} \right\rfloor + 1$

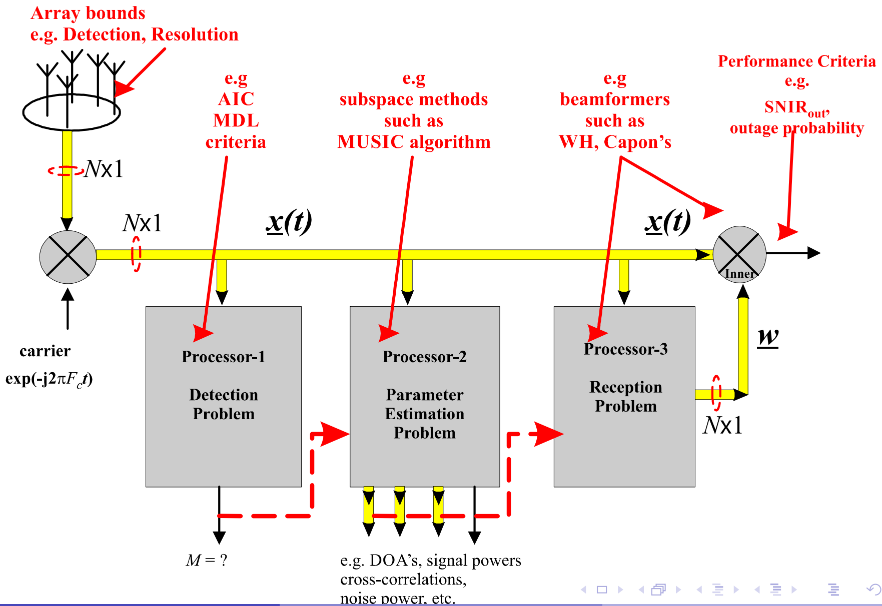
    $\Downarrow$

  - two paths with a relative delay < channel-symbol-period

    cannot be resolved

- for more info see Chapter 8 of my book

Summary


Array bounds
e.g. Detection, Resolution
e.g
AIC
MDL
criteria
e.g
subspace methods
such as
MUSIC algorithm
e.g
beamformers
such as
WH, Capon's
Performance Criteria
e.g.
SNIR$_{out}$,
outage probability
Nx1
Nx1
$\underline{x}(t)$
$\underline{x}(t)$
Inner
$\underline{w}$
carrier
exp(-j2$\pi F_c t$)
Processor-1
Detection
Problem
Processor-2
Parameter
Estimation
Problem
Processor-3
Reception
Problem
Nx1
M = ?
e.g. DOA's, signal powers
cross-correlations,
noise power, etc.