Homework 2

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1▷ Loop Invariant: $S_i = x$ is not equal to any of the first i elements of the array

Initialization: Before the start of the loop where $i = 0$, we know that x is not equal to the first 0 elements since $i = 0$. This is true because x cannot equal nothing.

Maintenance: Before the start of the j^{th} iteration, where $j = i+1$, x is not equal to any of the first i elements of the array. During the j^{th} iteration, the if statement $x = A[i]$ is computed, and if it is true, then i is returned and the loop terminates. If it is not true, then i is computed as $i+1$, so $j+1 = i+1+1$. The next iteration then requires x to not equal any of the first $j+1$ elements of the array. This will still hold true because the loop checked one element to see if it was the element x and then we added 1 to i if it wasn't. This means that we will have always checked all of the numbers before the j^{th} element in the array for x. When we iterate $j-1$ times where each iteration checks one element, we will have checked $j-1$ elements for x. Before the next iteration j, we will have checked i elements. This is true because $i = j-1$. Therefore, x is not equal to any of the first i elements. For example, the 1st iteration ($j=1$) of the loop will check the first element of the array. So when we iterate to the 2nd iteration ($j=2$) and before we enter the loop, we will have checked one element, $j=2$. $i = j-1$, $i = 2-1$, $i = 1$. So, x is not equal to any of the first 1 elements of the array.

Termination: Since initialization shows that the LI will be true before the loop begins and Maintenance shows

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that it will continue to be true before each iteration of j, so it must be true before the last iteration, the n^{th} iteration. The loop ends after the n^{th} iteration, so before the $(n+1)^{th}$ iteration, the LI must be true. Before the j^{th} iteration, i will equal j-1. Therefore, before the $(n+1)^{th}$ iteration, i will equal n because $j = n+1$ & $i = j-1 \Rightarrow i = n+1-1 = n$. The LI says that x will not equal to any of the first n elements of the array, since $i = n$. This means that the algorithm works correctly, because if the loop is never terminated by a a found x element, it will go through all n elements of the array to check.

2. Least Complex (Big-Oh)

$\frac{1}{n}$
2^{100}
$\log \log n$
$\sqrt{\log n}$
$\log^2 n$
$n^{0.01}$
$\lceil \sqrt{n} \rceil$
$3n^{0.5}$ } Θ
$\frac{2^{\log n}}{\sqrt{5n}}$ } Θ

$n \log_4 n$
$\frac{1}{2} n \log n$
$\lfloor 2n \log^2 n \rfloor$ } Θ
$4n^{3/2}$
$4^{\log n}$
$n^2 \log n$
n^3
2^n
$\frac{4^n}{2^{2n}}$ } Θ

Most Complex (Big-Oh)

3 ▷ A method for finding both the minimum and maximum of n numbers.

▷ We will have 2 variables, min and max. Min is initialized to the first element, A[0], and max is initialized to the second element, A[1],. We iterate through the array in pairs, with i and i+1 giving A[i] & A[i+1]. so each iteration will make i = i+2, initially starting with i=2. We first compare A[i] to A[i+1]. If A[i] is greater than A[i+1], then A[i] is a candidate for max and A[i+1] is a candidate for min. We would then compare A[i] with max and replace if its greater than max. Then we would compare A[i+1] with min and replace if it's less than min. If A[i+1] is greater than A[i], then we would do the same, just switching the comparisons for A[i] and A[i+1]. Each iteration will have 3 comparisons and it iterates by 2, resulting in less than $\frac{3n}{2}$ comparisons since we initialize max to A[0], min to A[1], and i to 2.

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4 What is wrong with this proof?

- $F(2)=2$ is not $O(2)$, it is $O(1)$. Since 2 is a constant, it will have a Big-Oh of 1 ($O(1)$). 2 is not a complexity used to describe Big-Oh. Then in the inductive step, we cannot apply changes to the Big-Oh complexities like we do to $F(n) \to F(n-1)$, when $F(n)$ goes to $F(n-1)$, its time complexity would stay $O(n)$ and not change to $O(n-1)$, as that is not a complexity used to describe Big-Oh either. The same applies for $F(n-2)$ and $O(n-2)$, it will be $O(n)$. Big-Oh is not a recursive function call like $F(n)$ and all the Fibonacci calls, it is a label/evaluation of the algorithm. Likewise, we cannot do $O((n-1)+(n-2))$ as it cannot be transformed that way. Overall, we get the correct time complexity through an incorrect inductive proof.

5 a) What does the recursive algorithm above compute?

- This algorithm will return the minimum element in the array passed to it.

b) $T(n) = 7 \quad n=1$

$T(n) = T(n/2) + T(n/2) + 17 \quad n \geq 2$

```
if i = j                                    cost   reasoning
    return A[i]                             3      read i, read j, compare
else                                        4      read i, find mem. loc., read value, return
    k = i + floor((j-i)/2)                  7      read i, read j, subtract, divide, floor, add, assign
    temp1 = Mystery(A[i...k])               1      assign
    temp2 = Mystery(A[(k+1)...j])           1      assign
    if temp1 < temp2                        3      read temp1, read temp2, compare
        return temp1                        2      read temp1, return
    else return temp2                       2      read temp2, return
```

5cont ▷ c)

Level	Level Number	Total #	Input size	Work done by each	Total work
Root	0	1	n	c	c
One level below	1	2	$n/2$	c	$2c$
Two levels below	2	4	$n/4$	c	$4c$
Above base case	$\lg n - 1$	$n/2$	2	c	$\frac{n}{2}c$
Base case	$\lg n$	n	1	c	nc

d) $c\sum_{i=0}^{\log n-1} 2^i = c\frac{2^{\lg n}-1}{2-1} = c\frac{n-1}{1} = c(n-1) = cn-c$

$T(n) = c\sum_{i=0}^{\log n-1} 2^i + cn = cn-c+cn = 2cn-c$

so $T(n) \Rightarrow O(n)$

6 ▷

Level	Level Number	Total #	Input size	Work done by each	Total work
Root	0	1	n	cn	cn
One level below	1	7	$n/8$	$cn/8$	$\frac{7}{8}cn$
Two levels below	2	49	$n/64$	$cn/64$	$\frac{49}{64}cn$
Above base case	$\lg n - 1$	$(\frac{n}{8})\log_8 7$	8	$8c$	$8c(\frac{n}{8})^{\log_8 7}$
Base Case	$\lg n$	$n^{\log_8 7}$	1	c	$cn^{\log_8 7}$

$\sum_0^\infty x^i = \frac{1}{1-x}; x<1$

$T(n) = cn\sum_0^{\infty}(7/8)^i + \Theta(n^{\log_8 7}) = \frac{cn}{1-7/8} + \Theta(n^{\log_8 7}) = 8cn + \Theta(n^{\log_8 7})$

so $T(n) \Rightarrow O(n)$

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7 Guess: $T(n) = O(n)$ $\quad T(n) = 3T(n/3) + 5 \quad T(1) = 5$

Prove: $T(n) \leq cn$
Base Case: $T(1) = 5 \leq c$ (if $c \geq 5$)
Inductive Hypothesis: $T(n/3) \leq cn/3$
Inductive Step: $T(n) \leq 3(\frac{cn}{3}) + 5$
$T(n) \leq cn + 5 \not\leq cn$ → Proof fails, change proof

⇓

Prove: $T(n) \leq cn - 5$
Base Case: $T(1) = 5 \leq c(1) - 5$ (if $c \geq 10$)
Inductive Hypothesis: $T(n/3) \leq \frac{cn}{3} - 5$
Inductive step: $T(n) \leq 3(\frac{cn}{3} - 5) + 5$
$\leq cn - 15 + 5$
$\leq cn - 10 \leq cn$ for all c

Values of c: $c \geq 10$

8 $T(n) = T(n/2) + T(n/4) + T(n/8) + T(n/8) + n \quad T(1) = c$

Recursion tree (recursive call : input size : workdone by each call)

Lvl
0 (1) $T(n) : n : n$

1 $T(\frac{n}{2}) : \frac{n}{2} : \frac{n}{2}$ $\quad T(\frac{n}{4}) : \frac{n}{4} : \frac{n}{4}$ $\quad T(\frac{n}{8}) : \frac{n}{8} : \frac{n}{8}$ $\quad T(\frac{n}{8}) : \frac{n}{8} : \frac{n}{8}$

2 $T(\frac{n}{4})$: $\frac{n}{4}$: $\frac{n}{4}$ $\quad T(\frac{n}{8})$: $\frac{n}{8}$: $\frac{n}{8}$ $\quad T(\frac{n}{16})$: $\frac{n}{16}$: $\frac{n}{16}$ $\quad T(\frac{n}{16})$: $\frac{n}{16}$: $\frac{n}{16}$ $\quad T(\frac{n}{8})$: $\frac{n}{8}$: $\frac{n}{8}$ $\quad T(\frac{n}{16})$: $\frac{n}{16}$: $\frac{n}{16}$ $\quad T(\frac{n}{32})$: $\frac{n}{32}$: $\frac{n}{32}$ $\quad T(\frac{n}{32})$: $\frac{n}{32}$: $\frac{n}{32}$ $\quad T(\frac{n}{16})$: $\frac{n}{16}$: $\frac{n}{16}$ $\quad T(\frac{n}{32})$: $\frac{n}{32}$: $\frac{n}{32}$ $\quad T(\frac{n}{64})$: $\frac{n}{64}$: $\frac{n}{64}$ $\quad T(\frac{n}{64})$: $\frac{n}{64}$: $\frac{n}{64}$ $\quad T(\frac{n}{16})$: $\frac{n}{16}$: $\frac{n}{16}$ $\quad T(\frac{n}{32})$: $\frac{n}{32}$: $\frac{n}{32}$ $\quad T(\frac{n}{64})$: $\frac{n}{64}$: $\frac{n}{64}$ $\quad T(\frac{n}{64})$: $\frac{n}{64}$: $\frac{n}{64}$

Lvl	total work
0	n
1	n
2	n

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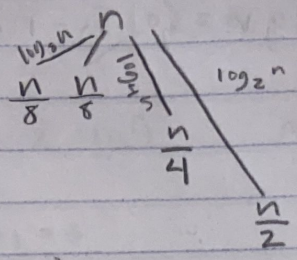
8cont (2) Shape of Tree



(3) Shallowest Part: $\log_8 n$

(4) Deepest Part: $\log_2 n$

(5) Guess: $O(n \log n)$

9 $T(n) = T(n/2) + T(n/4) + T(n/8) + T(n/8) + n \quad T(1) = c$

Guess: $O(n \log n)$

Prove: $T(n) \leq dn \log n$

Base Case: $T(1) = c \leq d \log 1 \rightarrow 0?$ $c \leq 0?$ X

↓

Base Case: $T(2) = c \leq 2d \log 2 = 2d$ if $(d \geq \frac{c}{2})$

Inductive Hypothesis: $T(n/2) \leq d(\frac{n}{2}) \log(n/2)$

$T(n/4) \leq d(\frac{n}{4}) \log(n/4)$

$T(n/8) \leq d(n/8) \log(n/8)$

Inductive Step: $T(n) \leq \frac{dn}{2} \log(\frac{n}{2}) + \frac{dn}{4} \log(\frac{n}{4}) + \frac{2dn}{8} \log(\frac{n}{8}) + n$

$T(n) \leq \frac{dn}{2}(\log n - \log 2) + \frac{dn}{4}(\log n - \log_2 4)^2 + \frac{dn}{4}(\log n - \log_2 8) + n$

$\leq dn(\frac{2}{4}\log n + \frac{1}{4}\log n + \frac{1}{4}\log n - \frac{2}{4} - \frac{2}{4} - \frac{3}{4}) + n$

$\leq dn(\log n - 7/4) + n$

$\leq dn \log n - \frac{7dn}{4} + n \leq dn \log n$ (when $d \geq 4/7$)

b/c then $dn \log n - n + n \leq dn \log n$

10 (a) $T(n) = 2T(\frac{99}{100} n) + 100n \quad a=2 \quad b=\frac{100}{99} \quad f(n) = 100n$

$n^{\log_{\frac{100}{99}} 2} \approx n^{.69-\varepsilon} = O(f(n)) = O(n)$ Case 1 ✓

$T(n) = \Theta(n^{\log_{\frac{100}{99}} 2})$

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10 cont (b) $T(n) = 16T(\frac{n}{2}) + n^3 \lg n$ $a=16$ $b=2$ $f(n)=n^3 \lg n$

$n^{\log_2 16} = n^{4-\epsilon} > n^3 \lg n$ Case 1 ✓

$T(n) = \Theta(n^4)$

(c) $T(n) = 16T(\frac{n}{4}) + n^2$ $a=16$ $b=4$ $f(n)=n^2$

$n^{\log_4 16} = n^2 = n^2 \Leftarrow f(n)$ Case 2 ✓

$T(n) = \Theta(n^2 \log n)$

11 $T(n) = 2T(n-1) + 1$ $T(0)=1$

- Backwards

(1) 3 expansions

$T(n-1) = 2T(n-2) + 1$

1 $T(n) = 2(2T(n-2)+1)+1$

$= 4T(n-2) + 3$

$T(n-2) = 2T(n-3) + 1$

2 $T(n) = 4(2T(n-3)+1)+3$

$= 8T(n-3) + 7$

$T(n-3) = 2T(n-4) + 1$

3 $T(n) = 8(2T(n-4)+1)+7$

$= 16T(n-4) + 15$

$T(n) \Rightarrow O(2^n)$

(2) $\sum_{k=0}^{n} x^k = \sum_{k=0}^{n} 2^k = \frac{2^{n+1}-1}{2-1} = T(n)$

$T(n) = 2^{n+1} - 1$

(3) LHS

$T(n) = 2^{n+1} - 1$

RHS

$2T(n-1) + 1 = 2(2^{(n-1)+1} - 1) + 1$

$= 2(2^n - 1) + 1 = 2(2^n) - 2 + 1$

$= 2^{n+1} - 1$

LHS = RHS

11 cont. $T(n) = 2T(n-1) + 1 \quad T(0) = 1$

- Forwards

(1) 3 expansions

$T(0) = 1$

$T(1) = 2T(0) + 1 = 2 + 1 = 3$

$T(2) = 2T(1) + 1 = 2(3) + 1 = 7$ $\Rightarrow T(n) = 2^{n+1} - 1$

$T(3) = 2T(2) + 1 = 2(7) + 1 = 15$

(2) $\sum_{k=0}^{n} x^k = \frac{x^{k+1} - 1}{x - 1}$ $\quad T(n) = \sum_{k=0}^{n} 2^k = \frac{2^{n+1} - 1}{2 - 1} = 2^{n+1} - 1$

(3) LHS

$T(n) = 2^{n+1} - 1$

RHS

$2T(n-1) + 1 = 2(2^{(n-1)+1} - 1) + 1 = 2(2^n) - 2 + 1 = 2^{n+1} - 1$

LHS = RHS