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An Instructor's Solutions Manual to Accompany

MECHANICAL VIBRATIONS: THEORY AND APPLICATIONS, 1ST EDITION

S. GRAHAM KELLY

CENGAGE Learning™

Mechanical Engineering/ College of Engineering

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TO ACCOMPANY

MECHANICAL VIBRATIONS
THEORY AND APPLICATIONS

FIRST EDITION

S. GRAHAM KELLY

Mechanical Engineering/ College of Engineering

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Preface

Mechanical vibrations is an applied engineering science. Students learn to apply previously learned sciences to realistic engineering problems. Students apply material learned in courses in statics, dynamics, mechanics of solids, fluid mechanics, calculus, differential equations and linear algebra to the solution of vibrations problems. The difference between a vibrations course and the aforementioned courses is the mathematical modeling aspect of vibrations. While studying vibrations students learn about mathematical modeling of systems with time as an independent variable. Students learn about assumptions made during the modeling process, coordinates and variables used, application of the basic laws of nature (including drawing of free-body diagrams), solving the mathematical problem, putting the solution in a form that can be used and most importantly how to interpret the solution to answer a given problem. Thus students learn the basic theory of mechanical vibrations and its application to problem solution and design. Mechanical vibrations is a precursor to engineering design.

The purpose of an Instructor's Solution Manual is to provide solutions to end-of-chapter problems in the manner solved in the text. That is why this Solutions Manual is so lengthy. The solution to the vast majority of the problems is presented in such a way that is easy to follow. The end-of-chapter problems are broken into two types. Short answer questions are concept questions addressing the reader's understanding of basic concepts. They are broken into four subtypes: true/false, short answer, short calculation, and dimensions. The true/false questions ask the reader to evaluate a statement for its veracity and either explain why it is true or rewrite it to make the statement true. Short answer problems require the reader to formulate a short answer to a question while short calculation questions require the reader to make short calculations to test understanding of the concepts. The questions on dimensions are included for most chapters to review dimensions of basic quantities used in the chapter. The solutions to short answer questions are presented without the problem statement.

Chapter problems involve lengthier calculations and may involve concepts form more than one chapter. The solutions of chapter problems are presented with a problem statement, a review of what is given in the problem statement with symbols assigned, a statement of what is desired in the solution, the solution of the problem and a statement regarding what the problem illustrates. The solution of the problem is presented in detail in most problems. This makes it easier for an instructor to decide what problems to assign.

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The author acknowledges the help of his wife, Seala Fletcher-Kelly in preparing the figures and organizing the manual. However any mistakes in the solutions are solely mine. I would appreciate being informed of mistakes as you find them at sgraham@uakron.edu. That way I can post errata on the website for the book.

S. Graham Kelly

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CHAPTER 1: INTRODUCTION

Short Answer Problems

1.1 True: The earth is taken to be non-accelerating for purposes of modeling systems on the surface of the earth.

1.2 False: Systems undergoing mechanical vibrations are not subject to nuclear reactions is an example of an implicit assumption.

1.3 True: Basic laws of nature can only be observed and postulated.

1.4 False: The point of application of surface forces is on the surface of the body.

1.5 False: The number of degree of freedom necessary to model a mechanical system is unique.

1.6 False: Distributed parameter systems are another name for continuous systems.

1.7 True: The Buckingham Pi theorem states that the number of dimensionless variables required in the formulation of a dimensional relationship is the number of dimensional variables, including the dependent variable, minus the number of dimensions involved in the dimensional variables.

1.8 True: The displacement of its mass center (x and y coordinates) and the rotation about an axis perpendicular to the mass center are degrees of freedom the motion of an unconstrained rigid body undergoing planar motion.

1.9 False: A particle traveling in a circular path has a velocity which is tangent to the circle.

1.10 False: The principle of work and energy is derived from Newton's second law by integrating the dot product of the law with a differential displacement vector as the particle moves from one location to another.

1.11 The continuum assumption treats all matter as a continuous material and implies that properties are continuous functions of the coordinates used in modeling the system.

1.12 An explicit assumption must be stated every time it is used, whereas an implicit assumption is taken for granted.

1.13 Constitutive equations are used to model the stress-strain relationships in materials. They are used in vibrations to model the force-displacement relationships in materials that behave as a spring.

1.14 A FBD is a diagram of a body abstracted from its surroundings and showing the effects of the surroundings as forces. They are drawn at an arbitrary time.

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1.15 The equation represents simple harmonic motion

1.16 (a) X is the amplitude of motion; (b) ω is the frequency at which the motion occurs (c) ϕ is the phase between the motion and a pure sinusoid.

1.17 The phase angle is positive for simply harmonic motion. Thus the response lags a pure sinusoid.

1.18 A particle has mass that is concentrated at a point. A rigid body has a distribution of mass about the mass center.

1.19 A rigid body undergoes planar motion if (1) the path of its mass center lies in a plane and (2) rotation occurs only about an axis perpendicular to the plane of motion of the mass center.

1.20 The acceleration of a particle traveling in a circular path has a tangential component that is the radius of the circle times the angular acceleration of the particle and a centripetal acceleration which is directed toward the center of the circle which is the radius time the square of the angular velocity.

1.21 An observer fixed at A observes, instantaneously that particle B is moving in a circular path of radius $|\mathbf{r}_{B/A}|$ about A.

1.22 It is applied to the FBD of the particle.

1.23 The effective forces for a rigid body undergoing planar motion are a force applied at the mass center equal to $m\bar{\mathbf{a}}$ and a moment equal to $\bar{I}\alpha$.

1.24 The two terms of the kinetic energy of a rigid body undergoing planar motion are $\frac{1}{2}m\bar{v}^2$, the translational kinetic energy, and $\frac{1}{2}\bar{I}\omega^2$, the rotational kinetic energy.

1.25 The principle of impulse and momentum states that a body's momentum (linear or angular) momentum at t_1 plus the external impulses applied to the body (linear or angular) between t_1 and t_2 is equal to the system's momentum (linear or angular) at t_2 .

1.26 One, let θ be the angular rotation of the bar, measured positive counterclockwise, from the system's equilibrium position.

1.27 Four, let x_1 be the absolute displacement of the cart, x_2 the displacement of the leftmost block relative to the cart, x_3 the displacement of the rightmost block away from the cart and θ the counterclockwise angular rotation of the bar.

1.28 Four, let x_1 represent the displacement of the center of the disk to the right, x_2 the downward displacement of the hanging mass, x_3 the displacement of the sliding mass to the left and θ the counterclockwise angular rotation of the rightmost pulley.

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1.29 Two, let θ be clockwise the angular displacement of the bar and x the downward displacement of the hanging mass.

1.30 Three, let x be the downward displacement of the middle of the upper bar, θ its clockwise angular rotation and ϕ the clockwise angular rotation of the lower bar.

1.31 Three, let θ represent the clockwise angular rotation of the leftmost disk, ϕ the clockwise angular rotation of the rightmost disk and x the upward displacement of the leftmost hanging mass.

1.32 Infinite, let x be a coordinate measured along the neutral axis of the beam measured for the fixed support. Then the displacement is a continuous function of x and t, w(x,t).

1.33 Three, let x_1 be the downward displacement of the hand, x_2 the downward displacement of the palm and x_3 the displacement of the fingers.

1.34 Given: Uniform acceleration, a=2 m/s. (a) $v(t) = at + v_0 \Longrightarrow v(5) = \left(2\frac{m}{s^2}\right)(5\text{ s}) + \left(0\frac{\text{m}}{\text{s}}\right) = 10\text{ m}$ (b) $x(t) = a\frac{t^2}{2} + v_0 t + x_0 \Longrightarrow x(5) = \frac{1}{2}\left(2\frac{\text{m}}{\text{s}^2}\right)(5\text{ s})^2 = 25\text{ m}$

1.35 Given: $\mathbf{v} = 2\cos 2t\,\mathbf{i} + 3\sin 2t\,\mathbf{j} + 0.4\,\boldsymbol{k}$ m/s. (a) $\mathbf{a} = \frac{d\mathbf{v}}{dt} = -4\sin 2t\ \mathbf{i} + 6\cos 2t\ \mathbf{j}\ \text{m/s}^2 \Longrightarrow a(\pi) = -4\sin 2\pi\ \mathbf{i} + 6\cos 2\pi\ \mathbf{j}\frac{\text{m}}{\text{s}^2} = 6\,\mathbf{j}\ \text{m/s}^2$ (b) $\mathbf{r} = \int \mathbf{v}dt = \left[(\sin 2t\ + C_1)\mathbf{i} + \left(-\frac{3}{2}\cos 2t + C_2\right)\mathbf{j} + (0.4t + C_3)\mathbf{k}\right]$ m. The particle starts at the origin at t = 0. Application of this condition leads to) $\mathbf{r}(\text{t}) = \left[(\sin 2t\ )\mathbf{i} + \left(-\frac{3}{2}\cos 2t + \frac{3}{2}\right)\mathbf{j} + 0.4t\mathbf{k}\right.$ m. Evaluation at π leads to $\mathbf{r}(\pi) = \sin 2\pi\,\mathbf{i} + -32\cos 2\pi + 32\mathbf{j} + 0.4\pi\mathbf{k}$ m $= 0.4\pi\mathbf{k}$ m.

1.36 Given: v=2 m/s, r=3 m, $\theta(0) = 0$ (a) $v = \frac{ds}{dt} \Longrightarrow s = \int v dt = 2t$ at t=2 s the particle has traveled 4 m. But $s = r\theta$ thus $\theta = \frac{4\text{ m}}{3\text{ m}} = 1.33\text{ rad} = 76.2°$. (b) The acceleration of a particle traveling on a circular path has two components. One is $\frac{dv}{dt}$ which is tangent to the circle and is zero for this problem. The other component is $\frac{v^2}{r} = \frac{(2\text{ m/s})^2}{3\text{ m}} = 1.33\text{ m}$ directed toward the center of the circle from the position of the particle.

1.37 Given: m=2 kg, $\bar{I} = 0.5\text{ kg}\cdot\text{m}^2$, $\bar{\mathbf{a}} = (5\mathbf{i} + 3\mathbf{j})\text{ m/s}^2$, $\alpha = 10\text{ rad/s}^2$. Effective forces are $m\bar{\mathbf{a}} = (2\text{ kg})\left[(5\mathbf{i} + 3\mathbf{j})\frac{\text{m}}{\text{s}^2}\right] = (10\mathbf{i} + 15\mathbf{j})\frac{\text{m}}{\text{s}^2}$ applied at the mass center and a couple $\bar{I}\alpha = (0.5\text{ kg}\cdot\text{m}^2)(10\text{ rad/s}^2) = 5\text{ N}\cdot\text{m}$.

1.38 Given: m = 0.1 kg, $\mathbf{v} = (9\mathbf{i} + 11\text{j})\text{ m/s}$. The kinetic energy of the particle is $T = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}(0.1\text{ kg})\left(\sqrt{9^2 + 11^2}\,m/s\right)^2 = 0.711\text{ J}$.

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1.39 Given: m=3 kg, $\bar{\mathbf{v}} = (3\mathbf{i} + 4\mathbf{j})$ m/s, d=0.2 m The angular velocity is calculated from $|\bar{\mathbf{v}}| = d\omega \Longrightarrow \omega = \frac{|\bar{\mathbf{v}}|}{d} = \frac{5\text{ m/s}}{0.2\text{ m}} = 20$ rad/s.

1.40 Given: $= 100$ J , $I = 0.03\text{ kg}\cdot\text{m}^2$ The kinetic energy of a rigid body which rotates about its centroidal axis is $T = \frac{1}{2}I\omega^2$. Thus $100\text{ J} = \frac{1}{2}(0.03\text{ kg}\cdot\text{m}^2)\omega^2$ which leads to $\omega = 81.65\ \frac{\text{rad}}{\text{sec}}$.

1.41 Given: m = 5 kg, $\bar{v} = 4$ m/s, $\omega = 20$ rad/s, $\bar{I} = 0.08\text{ kg}\cdot\text{m}^2$. The kinetic energy of a rigid body undergoing planar motion is $T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 = \frac{1}{2}(5\text{ kg})(4\text{ m/s})^2 + \frac{1}{2}(0.08\text{ kg}\cdot\text{m}^2)(20\text{ rad/s})^2 = 56$ J.

1.42 Given: F=12,000 N, $\Delta t = 0.03$ s. The impulse applied to the system is $I = F\Delta t = (12{,}000\text{ N})(0.03\text{ s}) = 360\text{ N}\cdot\text{s}$.

1.43 Given: m = 3 kg, $v_1 = 0$ m/s, force as given in Figure (a) The impulse imparted to the particle is $I = \int_0^{3\text{ s}} F dt = \frac{1}{2}(1)(100) + 2(100) + \frac{1}{2}(1)(100) = 300\text{ N}\cdot\text{s}$ (b) The velocity at t=2 s is given by the principle of impulse and momentum $mv = \int_0^{2\text{ s}} F dt \Longrightarrow v = \frac{\int_0^{2\text{ s}} F dt}{m} = \frac{250\text{ N}\cdot\text{s}}{3\text{ kg}} = 83.3$ m/s. (c) The velocity after 5 s is $v = \frac{\int_0^{5\text{ s}} F dt}{m} = \frac{300\text{ N}\cdot\text{s}}{3\text{ kg}} = 100$ m/s.

1.44 Given: m = 2 kg, F=6 N, t=10 s, $v_1 = 4$ m/s. The principle of work and energy is used to calculate how far the particle travels $T_1 + U_{1\to 2} = T_2$ after the velocity $v_2$is calculated from the principle of impulse and momentum $mv_1 + I = mv_2 \Longrightarrow v_2 = \frac{mv_1 + I}{m} = \frac{(2\text{ kg})(4\text{ m/s}) + (6\text{ N})(10\text{ s})}{2\text{ kg}} = 34$ m/s. Then letting x be the distance traveled application of work and energy gives $\frac{1}{2}(2\text{ kg})(4\text{ m/s})^2 + (6\text{ N})x = \frac{1}{2}(2\text{ kg})(34\text{ m/}$ s2which is solved to yield x=190 m.

1.45 (a) -(ii) (b)-(iv) (c)-(i) (d)-(v) (e)-(i) (f)-(v) (g)-(vi) (h)-(iii) (i)-(ix)

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Chapter Problems

1.1 The one-dimensional displacement of a particle is

$$x(t) = 0.5e^{-0.2t}\sin 5t \text{ m}$$

(a) What is the maximum displacement of the particle? (b) What is the maximum velocity of the particle? (c) What is the maximum acceleration of the particle?

Given: x(t)

Find:$(a)x_{max}$ $(b)v_{max}$ $(c)a_{max}$

Solution: (a) The maximum displacement occurs when the velocity is zero. Thus

$$\dot{x}(t) = 0.5e^{-0.2t}(-0.2\sin 5t + 5\cos 5t)$$

Setting the velocity to zero leads to

$$-0.2\sin 5t + 5\cos 5t = 0$$

or $\tan 5t = 25$. The first time that the solution is zero is t=0.3062. Substituting this value of t into the expression for x(t) leads to

$$x_{max} = 0.4699 \text{ m}$$

(b) The maximum velocity occurs when the acceleration is zero

$$\ddot{x}(t) = 0.5e^{-0.2t}[-0.2(-0.2\sin 5t + 5\cos 5t) - \cos 5t - 25\sin 5t]$$
$$= 0.5e^{-0.2t}(-24.96\sin 5t - 6\cos 5t)$$

The acceleration is zero when $24.96\sin 5t - 6\cos 5t = 0 \Longrightarrow \tan 5t = -0.240$. The first time that this is zero is t=0.5812 which leads to a velocity of

$$v_{min} = -2.185 \text{ m/s}$$

(c) The maximum acceleration occurs when $\dddot{x} = 0$,

$$\dddot{x} = 0.5e^{-0.2t}[-0.2(-24.96\sin 5t - 6\cos 5t) - (24.96)(5)\cos 5t + 30\sin 5t]$$
$$= 0.5e^{-0.2t}(34.992\sin 5t - 123.6\cos 5t)$$

The maximum acceleration occurs when $34.992\sin 5t - 123.6\cos 5t = 0 \Longrightarrow \tan 5t = 3.53$. The time at which the maximum acceleration occurs is t=0.2589 s which leads to

$$a_{max} = -12.18 \text{ m/s}^2$$

Problem 1.1 illustrates the relationships between displacement, velocity and acceleration.

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1.2 The one-dimensional displacement of a particle is

$$x(t) = 0.5\, e^{-0.2t} \sin(5t + 0.24) \text{ m} \tag{1}$$

(a) What is the maximum displacement of the particle? (b) What is the maximum velocity of the particle? (c) What is the maximum acceleration of the particle?

Given: x(t)

Find: $(a) x_{max}$ $(b) v_{max}$ $(c) a_{max}$

Solution: (a) The maximum displacement occurs when the velocity is zero. Thus

$$\dot{x}(t) = 0.5e^{-0.2t}[-0.2\sin(5t + 0.24) + 5\cos(5t + 0.24)]$$

Setting the velocity to zero leads to

$$-0.2\sin(5t + 0.24) + 5\cos(5t + 0.24) = 0$$

or $\tan(5t + 0.24) = 0.2582$. The first time that the solution is zero is t=0.3062. Substituting this value of t into the expression for x(t) leads to

$$x_{max} = 0.4745 \text{ m}$$

(b) The maximum velocity occurs when the acceleration is zero

$$\begin{aligned}\ddot{x}(t) = 0.5e^{-0.2t}\{&-0.2[(-0.2\sin(5t + 0.24) + 5\cos(5t + 0.24))] \\ &- \cos(5t + 0.24) - 25\sin(5t + 0.24)\} \\ &= 0.5e^{-0.2t}[-24.96\sin(5t + 0.24) - 6\cos(5t + 0.24)]\end{aligned}$$

The acceleration is zero when

$$-24.96\sin(5t + 0.24) - 6\cos(5t + 0.24) = 0 \Longrightarrow \tan(5t + 0.24) = -0.240.$$

The first time that this is zero is t = 0.5332 which leads to a velocity of

$$v_{min} = -2.0188 \text{ m/s}$$

(c) The maximum acceleration occurs when $\dddot{x} = 0$,

$$\begin{aligned}\dddot{x} = 0.5e^{-0.2t}\{&-0.2[-24.96\sin(5t + 0.24) - 6\cos(5t + 0.24)] \\ &- (24.96)(5)\cos(5t + 0.24) + 30\sin(5t + 0.24)\} \\ &= 0.5e^{-0.2t}[34.992\sin(5t + 0.24) - 123.6\cos(5t + 0.24)]\end{aligned}$$

The maximum acceleration occurs when

$$34.992\sin(5t + 0.24) - 123.6\cos(5t + 0.24) = 0 \Longrightarrow \tan(5t + 0.24) = 3.53.$$

The time at which the maximum acceleration occurs is t=0.2109 s which leads to

$$a_{max} = -12.30 \text{ m/s}^2$$

Problem 1.2 illustrates the relationships between displacement, velocity and acceleration.

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1.3 At the instant shown in Figure P1.3, the slender rod has a clockwise angular velocity of 5 rad/sec and a counterclockwise angular acceleration of 14 rad/sec^2. At the instant shown, determine (a) the velocity of point *P* and (b) the acceleration of point *P*.

FIGURE P1.3

Given: $\omega = 5$ rad/sec, $\alpha = 14$ rad/sec^2, $\theta = 10°$

Find: v_p, a_P

Solution: The particle at the pin support, call it O, is fixed. Hence its velocity and acceleration are zero. Using the relative velocity and acceleration equations between two particles on a rigid body

$$\mathbf{v}_P = \mathbf{v_o} + \boldsymbol{\omega} \times \mathbf{r}_{P/O} = -5\mathbf{k} \times (3\cos 10°\,\mathbf{i} - 3\sin 10°\,\mathbf{j}) = -15\sin 10°\,\mathbf{i} - 15\cos 10°\,\mathbf{j}$$
$$= -2.604\,\mathbf{i} - 14.772\,\mathbf{j}$$

and

$$\mathbf{a}_P = \mathbf{a}_O + \boldsymbol{\omega}\,\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r}_{P/O}) + \boldsymbol{\alpha}\mathbf{x}\mathbf{r}_{P/O}$$

$$\mathbf{a}_P = (-66.5\mathbf{i} + 54.3\mathbf{j})\ \frac{\text{m}}{\text{s}^2}$$

$$|\mathbf{a}_P| = 85.9\,\frac{\text{m}}{\text{s}^2}$$

Alternate solution: The bar is rotating about a fixed point. Thus any point on the bar moves on a circular arc about the point of support. The particle P has two components of acceleration, one directed between P and O (the normal acceleration), and one tangent to the path of P whose direction is determined using the right hand rule (the tangential component).

The component normal to the path of P is

$$a_n = 3\text{m}(5\,\frac{\text{rad}}{\text{s}})^2 = 75\,\frac{\text{m}}{\text{s}^2}$$

and is directed between P and O. The tangential acceleration is

$$a_t = (3\text{m})(14\,\frac{\text{rad}}{\text{s}^2}) = 42\,\frac{\text{m}}{\text{s}^2}$$

The normal and tangential components of acceleration are illustrated on the diagram below.

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Problem 1.3 illustrates the use of the relative acceleration equation of rigid body kinetics.

1.4 At t = 0, a particle of mass 1.2 kg is traveling with a speed of 10 m/s that is increasing at a rate of 0.5 m/s^2. The local radius of curvature at this instant is 50 m. After the particle travels 100 m, the radius of curvature of the particle's path is 50 m.

(a) What is the speed of the particle after it travels 100 m?
(b) What is the magnitude of the particle's acceleration after it travels 100 m?
(c) How long does it take the particle to travel 100 m?
(d) What is the external force acting on the particle after it travels 100 m?

Given: m = 1.2 kg, v(t=0) = 10 m/s, dv/dt= 0.5 m/s^2, and r = 25 m when s = 100 m
Find: (a) v when s = 100 m, (b) a when s = 3 m, (c) t when s = 3 m

Solution: Let s(t) be the displacement of the particle, measured from t = 0. The particle's velocity is

$$v(t) = \int_0^t \frac{dv}{dt}\,dt + v(0) = \int_0^t 0.5\,dt = 0.5t + 10$$

By definition v=ds/dt. Thus the displacement of the particle is obtained as

$$s(t) = \int_0^t v\,dt + s(0) = \int_0^t (0.5t + 10)\,dt = 0.25t^2 + 10t$$

When s = 100 m,

$$100\text{ m} = 0.25t^2 + 10t \Rightarrow t = 8.28\text{ s}$$

(a) The velocity when s = 100 m is

$$v = 0.5(8.28) + 10 = 14.14\text{ m/s}$$

(b) Since the particle is traveling along a curved path, its acceleration has two components: a tangential component equal to the rate of change of the velocity

$$a_t = \frac{dv}{dt} = 0.5\text{ m/s}^2$$

and a normal component directed toward the center of curvature

$$a_n = \frac{v^2}{r} = \frac{(14.14\text{ m/s})^2}{50\text{ m}} = 4.00\text{ m/s}^2$$

The magnitude of the acceleration at this instant is

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$$|\mathrm{a}| = \sqrt{a_t^2 + a_n^2} = \sqrt{(0.5\ \mathrm{m/s^2})^2 + (4.00\ \mathrm{m/s^2})^2}$$
$$|\mathrm{a}| = 4.03\ \mathrm{m/s^2}$$

(c) The time for the particle to travel 100 m is previously calculated as t = 8.28 s
(d) The external force equation written in terms of magnitudes is

$$\sum |\mathbf{F}| = m|a|$$

which upon application to the particle gives

$$\sum |\mathbf{F}| = (1.2\ \mathrm{kg})\left(4.03\ \frac{\mathrm{m}}{\mathrm{s}^2}\right) = 4.84\ \mathrm{N}$$

Problem 1.4 illustrates the kinematics of a particle traveling along a curved path.

1.5 The machine of Figure P1.15 has a vertical displacement, $x(t)$. The machine has component which rotates with a constant angular speed, ω. The center of mass of the rotating component is a distance e from its axis of rotation. The center of mass of the rotating component is as shown at $t = 0$. Determine the vertical component of the acceleration of the rotating component.

FIGURE P1.5

Given: e, ω, x (t)

Find: a_y

Solution: The particle of interest is on a component that moves relative to the machine. From the relative acceleration equation,

$$\mathbf{a}_G = \mathbf{a}_M + \mathbf{a}_{G/M}$$

where

$$\mathbf{a}_M = -\ddot{x}(t)\mathbf{j}$$

and

$$\mathbf{a}_{G/M} = e\omega^2\left(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\right)$$

Since the angular velocity of the rotating component is constant and θ = 0 when t = 0,

$$\theta = \omega t$$

Hence the vertical acceleration of the center of mass of the rotating component is

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$$a_y = -\ddot{x}(t) - e\omega^2 \sin \omega t$$

Problem 1.5 illustrates application of the relative acceleration equation. Vibrations of machines subject to a rotating unbalance are considered in Chapter 4.

1.6 The rotor of Figure P1.6 consists of a disk mounted on a shaft. Unfortunately, the disk is unbalanced, and the center of mass is a distance e from the center of the shaft. As the disk rotates, this causes a phenomena called "whirl", where the disk bows. Let r be the instantaneous distance from the center of the shaft to the original axis of the shaft and θ be the angle made by a given radius with the horizontal. Determine the acceleration of the mass center of the disk.

FIGURE P1.6

Given: e, r

Find: $\bar{\mathbf{a}}$

Solution: The position vector from the origin to the center of the disk is $r\mathbf{i}_r$ where r varies with time. The mass center moves in a circular path about the center of the disk. The relative acceleration equation gives

$$\mathbf{a}_c = \mathbf{a}_o + \boldsymbol{\alpha} \times r\mathbf{i}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times r\mathbf{i}_r) + \ddot{r}\mathbf{i}_r + 2\boldsymbol{\omega} \times \dot{r}\mathbf{i}_r$$

$$\mathbf{a}_c = (\ddot{r} - r\dot{\theta}^2)\mathbf{i}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{i}_\theta$$

The acceleration of the mass center is then

$$\bar{\mathbf{a}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{i}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{i}_\theta - e\omega^2[\cos(\omega t - \theta)\,\mathbf{i}_r + \sin(\omega t - \theta)\,\mathbf{i}_\theta]$$

Problem 1.6 illustrates application of the relative acceleration equation.

1.7 A 2 ton truck is traveling down an icy, 10º hill at 50 mph when the driver sees a car stalled at the bottom of the hill 250 ft away. As soon as he sees the stalled car, the driver applies his brakes, but due to the icy conditions, a braking force of only 2000 N is generated. Does the truck stop before hitting the car?

Given: W = 4000 lb., θ = 10º, d = 250 ft., F_b = 2000 N = 449.6 lb,
v_o = 50 mph = 73.33 ft/sec

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Find: v = 0 before x = 250 ft.

Solution: Application of Newton's Law to the free body diagram of the truck at an arbitrary instant

$$\left(\sum F_x\right)_{ext.} = \left(\sum F_x\right)_{eff.}$$

$$-F_b + W\sin\theta = \frac{W}{g}a$$

$$a = g\left(-\frac{F_b}{W} + \sin\theta\right)$$

$$a = 32.2\,\frac{\text{ft}}{\text{sec}^2}\left(-\frac{449.6\ \text{lb}}{4000\ \text{lb}} + \sin 10^0\right)$$

$$a = 1.973\,\frac{\text{ft}}{\text{sec}^2}$$

Since the acceleration is constant, the velocity and displacement of the truck are

$$v = at + v_0 = 1.973t + 73.33$$

$$x = a\frac{t^2}{2} + v_0 t = 0.986t^2 + 73.33t$$

The acceleration is positive thus the vehicle speeds up as it travels down the incline. The truck does not stop before hitting the car.

Problem 1.7 illustrates application of Newton's Law to a particle and kinematics of constant acceleration.

1.8 The contour of a bumpy road is approximated by $y(x) = 0.03\sin(0.125x)$ m. What is the amplitude of the vertical acceleration of the wheels of an automobile as it travels over this road at a constant horizontal speed of 40 m/s?

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Given: $y(x) = 0.03\sin(0.125x)$ m, $v = 40$ m/s

Find: A

Solution: Since the vehicle is traveling at a constant horizontal speed its horizontal distance traveled in a time t is $x = vt$. Thus the vertical displacement of the vehicle is

$$y(t) = 0.03\sin[0.125(40t)] = 0.03\sin(5t)\text{ m}$$

The vertical velocity and acceleration of the vehicle are calculated as

$$v(t) = 0.03(5)\cos(5t) = 0.15\cos(5t)\text{ m/s}$$
$$a(t) = -0.15(5)\sin(5t) = -0.75\sin(5t)\text{ m/s}^2$$

Thus the amplitude of acceleration is A=0.75 m/s^2.

Problem 1.8 illustrates the relationship between displacement, velocity, and acceleration for the motion of a particle.

1.9 The helicopter of Figure P1.9 has a horizontal speed of 110 ft/s and a horizontal acceleration of 3.1 ft/s^2. The main blades rotate at a constant speed of 135 rpm. At the instant shown, determine the velocity and acceleration of particle *A*.

Given: $v_h = 110$ ft/s, a_h=3ft/s^2, $\omega = 135$ rpm = 14.1 rad/s, r = 2.1 ft

Find: $\mathbf{v}_A$, $\mathbf{a}_A$

Solution: Construct a x-y coordinate system in the horizontal plane

As illustrated. Using this coordinate system

$$\mathbf{v} = -110\mathbf{i}\text{ ft/s},\ \ \mathbf{a} = -3\mathbf{i}\text{ ft/s}^2$$

The position vector of A relative to the helicopter at this instant is

$$\mathbf{r}_{A/h} = r[\cos(\pi/4)\mathbf{i} - \sin(\pi/4)\mathbf{j}] = 1.48\mathbf{i} - 1.48\mathbf{j}$$

The relative velocity equation is used to determine the velocity of particle A as

$$\mathbf{v}_A = \mathbf{v}_h + \omega\mathbf{k}\times\mathbf{r}_{A/h}$$
$$\mathbf{v}_A = -110\mathbf{i} + 14.1\mathbf{k}\times(1.48\mathbf{i} - 1.48\mathbf{j})$$
$$\mathbf{v}_A = -89.1\mathbf{i} + 20.9\mathbf{j}\text{ ft/s}$$

The relative acceleration equation is used to determine the acceleration of particle A as

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$$\mathbf{a}_A = \mathbf{a}_h + \alpha\mathbf{k} \times \mathbf{r}_{A/h} + \omega\mathbf{k} \times (\omega\mathbf{k} \times \mathbf{r}_{A/h})$$
$$\mathbf{a}_A = -3.1\mathbf{i} + 14.1\mathbf{k} \times (20.87\mathbf{i} + 20.87\mathbf{j})$$
$$\mathbf{a}_A = -297.4\mathbf{i} + 294.6\mathbf{j}\ \text{ft/s}^2$$

Problem 1.9 illustrates the use of the relative velocity and relative acceleration equations.

1.10 For the system shown in Figure P1.10, the angular displacement of the thin disk is given by $\theta(t) = 0.03\sin\left(30t + \frac{\pi}{4}\right)$ rad. The disk rolls without slipping on the surface. Determine the following as functions of time. (a) The acceleration of the center of the disk. (b) The acceleration of the point of contact between the disk and the surface. (c) The angular acceleration of the bar. (d) The vertical displacement of the block. (Hint: Assume small angular oscillations ϕ of the bar. Then $\sin\phi \approx \phi$.)

FIGURE P1.10

Given: $\theta(t)$, $r_d = 0.1$ m, $\ell = 0.3$ m, $d = 0.2$ m

Find: (a) $\bar{a}_d$ (b) $\bar{a}_c$ (c) α (d) x

Solution: (a) The angular acceleration of the disk is

$$\ddot{\theta}(t) = -(30)^2 0.03\sin\left(30t + \frac{\pi}{4}\right) = -27\sin\left(30t + \frac{\pi}{4}\right)\frac{\text{rad}}{\text{s}^2}$$

Since the disk rolls without slip the acceleration of the mass center is

$$a_d = r\alpha = (0.1\ \text{m})\left(-27\sin\left(30t + \frac{\pi}{4}\right)\frac{\text{rad}}{\text{s}^2}\right) = -2.7\sin\left(30t + \frac{\pi}{4}\right)\frac{\text{m}}{\text{s}^2}$$

(b) Since the disk rolls without slip the horizontal acceleration of the point of contact is zero. The vertical acceleration is $r\dot{\theta}^2$ towards the center

$$\mathbf{a}_c = (0.1\ \text{m})\left[(30)(0.03)\cos\left(30t + \frac{\pi}{4}\right)\frac{\text{rad}}{\text{s}}\right]^2\mathbf{j}$$

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$$= 0.081\left[\cos\left(30t + \frac{\pi}{4}\right)\right]^2 \frac{\text{m}}{\text{s}^2}\,\mathbf{j}$$

(c) Assuming small ϕ the angular displacement of the link is $\ell\phi = x_d$ or

$$\ddot{\phi} = \frac{\ddot{x}}{\ell} = \frac{-2.7\sin\left(30t+\frac{\pi}{4}\right)\frac{\text{m}}{\text{s}^2}}{0.3\text{ m}}$$

$$\ddot{\phi} = -9\sin\left(30t + \frac{\pi}{4}\right)\frac{\text{r}}{\text{s}^2}$$

(d) The displacement of the mass center of the block is $x = d\phi$ or

$$x = (0.2)\frac{9}{(30)^2}\sin\left(30t + \frac{\pi}{4}\right) = 2\sin\left(30t + \frac{\pi}{4}\right)\text{ mm}$$

Problem 1.10 illustrates angular acceleration and acceleration of a body rolling without slip.

1.11 The velocity of the block of the system of Figure P1.11 is $\dot{y} = 0.02\sin 20t$ m/s downward. (a) What is the clockwise angular displacement of the pulley? (b) What is the displacement of the cart?

FIGURE P1.11

Given: $\dot{y}$, $r_1 = 0.1$ m, $r_2 = 0.3$ m

Find: (a) $\theta(t)$ (b) $x(t)$

Solution: the displacement of the block is

$$y(t) = \int \dot{y}\,dt = 0.001(1 - \cos 20t)\text{ m}$$

(a) The angular displacement of the pulley is

$$\theta = \frac{y}{r_2} = \frac{0.001(1 - \cos 20t)\text{ m m}}{0.3\text{ m}} = 3.33 \times 10^{-4}(1 - \cos 20t)\text{ rad}$$

(b) The displacement of the cart is

$$x = \frac{r_1}{r_2}y = \frac{0.1\,m}{0.3\,m}[0.001(1 - \cos 20t)\text{ m}] = 3.33 \times 10^{-4}(1 - \cos 20t)\text{ m}$$

Problem 1.11 illustrates velocity and kinematics.

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1.12 A 60-lb block is connected by an inextensible cable through the pulley to the fixed surface, as shown in Figure P1.12. A 40-lb weight is attached to the pulley, which is free to move vertically. A force of magnitude $P = 100(1+ e^{-t})$ lb tows the block. The system is released from rest at $t = 0$.

FIGURE P1.12

(a) What is the acceleration of the 60 lb block as a function of time?

(b) How far does the block travel up the incline before it reaches a velocity of 2 ft/sec?

Given: $W_1 = 60$ lbs, $W_2 = 40$ lbs, $P = 100(1+e^{-t})$ lb, $\mu = 0.3$, $\theta = 45°$

Find: a(t), x(v = 2 ft/sec)

Solution: Let x be the distance the block travels from t = 0. Let y be the vertical distance traveled by the pulley from t = 0. The total length of the cable connecting the block, the pulley and the surface is constant as the block moves up the incline. Thus, referring to the diagrams below. At t = 0, $\ell = a + b + c$. At an arbitrary time, $\ell = a + x + b - y + c - y = a + b + c + x - 2y$. Hence $y = x/2$.

Free body diagrams of the blocks are shown at an arbitrary instant of time.

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From the free body diagrams of the pulley

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$
$$2T - m_2 g = m_2 \ddot{y} \tag{1}$$
$$T = \frac{m_2}{2}\left(\ddot{y} + g\right)$$

Summation of forces in the direction normal to the incline for the block yields

$$N = m_1 g \cos\theta \tag{2}$$

Summing forces in the direction along the incline on the block

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$
$$-T + P - F - m_1 g \sin\theta = m_1 \ddot{x} \tag{3}$$

Noting that F = μN and using eqs. (1) and (2) in eq. (3) gives

$$\ddot{x} = \frac{-\frac{m_2 g}{2} + P - \mu m_1 g \cos\theta - m_1 g \sin\theta}{m_1 + \frac{m_2}{4}} \tag{4}$$

Substituting given values leads to

$$\ddot{x} = 11.42 + 46.0e^{-t}\ \frac{\text{ft}}{\text{s}^2}$$

The velocity is calculated from

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$$\int_0^v dv = \int_0^t \left(11.42 + 46.9e^{-t}\right)\, dt$$
$$v = 46.0 + 11.42t - 46.0e^{-t} \tag{5}$$

Setting v = 2ft/sec in eq. (5) and solving the resulting equation for t by trial and error reveals that it takes 0.0354 sec for the velocity to reach 2 ft/sec. The displacement from the initial position is calculated from

$$\int_0^x dx = \int_0^t \left(46.0 + 11.42t - 46.0e^{-t}\right) dt$$
$$x(t) = -46.0 + 46.0t + 5.71t^2 + 46.0e^{-t} \tag{6}$$

Setting t = 0.0354 sec in eq.(6), leads to

$$x = 0.0356 \text{ ft}$$

Problem 1.12 illustrates the application of Newton's Law to a particle, the kinematics of pulley systems, and relationships between acceleration, velocity, and displacement. Note that the time required to attain a velocity of 2 ft/sec could have been attained using impulse and momentum.

1.13 Repeat Problem 1.12 for a force of $P = 100t$ N.

FIGURE P1.12

Given: W_1 = 60 lbs, W_2 = 40 lbs, P = 100t lbs, $\mu = 0.3$, θ = 45º

Find: a(t), x(v = 2 ft/sec)

Solution: Let x be the distance the block travels from t = 0. Let y be the vertical distance traveled by the pulley from t = 0. The total length of the cable connecting the block, the pulley and the surface is constant as the block moves up the incline. Thus, referring to the diagrams below. At t = 0, ℓ = a + b + c. At an arbitrary time, ℓ = a + x + b – y + c – y = a + b + c + x – 2y. Hence y = x/2.

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Free body diagrams of the blocks are shown at an arbitrary instant of time.

From the free body diagrams of the pulley

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$

$$2T - m_2 g = m_2 \ddot{y} \tag{1}$$

$$T = \frac{m_2}{2}\left(\ddot{y} + g\right)$$

Summation of forces in the direction normal to the incline for the block yields

$$N = m_1 g \cos\theta \tag{2}$$

Summing forces in the direction along the incline on the block

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$

$$-T + P - F - m_1 g \sin\theta = m_1 \ddot{x} \tag{3}$$

Noting that F = μN and using eqs.(1) and (2) in eq.(3) gives

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$$\ddot{x} = \frac{-\frac{m_2 g}{2} + P - \mu m_1 g \cos\theta - m_1 g \sin\theta}{m_1 + \frac{m_2}{4}} \tag{4}$$

Substituting given values leads to

$$\ddot{x} = 36.79t - 34.57$$

Note that the acceleration is initially negative, then becomes positive.

$$\int_o^v dv = \int_o^t (36.79t - 34.57)dt \tag{5}$$
$$v = 18.40t^2 - 34.57t$$

Setting v = 2 ft/sec in eq.(5) and solving the resulting quadratic equation reveals that it takes 2.07 sec for the velocity to reach 2 ft/sec. The displacement from the initial position is calculated from

$$\int_o^x dx = \int_o^t (18.4t^2 - 34.57t)dt \tag{6}$$

$$x(t) = 6.13t^3 - 17.28t^2$$
$$x(2.07\,\text{sec}) = -19.76\,\text{ft}$$

Problem 1.13 illustrates the application of Newton's Law to a particle, the kinematics of pulley systems, and relationships between acceleration, velocity, and displacement. Note that the time required to attain a velocity of 2 ft/sec could have been attained using impulse and momentum.

1.14 Figure P1.14 shows a schematic diagram of a one-cylinder reciprocating one-cylinder engine. If at the instant time shown the piston has a velocity *v* and an acceleration *a*, determine (a) the angular velocity of the crank and (b) the angular acceleration of the crank in terms of *v*, *a*, the crank radius *r*, the connecting rod length ℓ, and the crank angle θ.

Given: r, ℓ, θ, v, a

Find: α_{AB}

FIGURE P1.14

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Solution: (a) From the law of sines

$$\frac{r}{\sin\phi} = \frac{\ell}{\sin\theta}$$

or

$$\sin\phi = \frac{r}{\ell}\sin\theta \qquad (1)$$

Then from trigonometry

$$\cos\phi = \sqrt{1-\sin^2\phi}$$
$$= \sqrt{1-\left(\frac{r}{\ell}\sin\theta\right)^2} \qquad (2)$$

Using the relative velocity equation,

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} x \vec{r}_{B/A}$$
$$= \omega_{AB}\vec{k}x\left(-r\sin\theta\vec{i} + r\cos\theta\vec{j}\right)$$
$$= -r\omega_{AB}\cos\theta\vec{i} - r\omega_{AB}\sin\theta\vec{j}$$

and

$$\vec{v}_C = v\vec{j} = \vec{v}_B + \vec{\omega}_{BC} x \vec{r}_{C/B}$$
$$= \vec{v}_B + \omega_{BC}\vec{k}x\left(\ell\sin\phi\vec{i} + \ell\cos\phi\vec{j}\right)$$
$$= \left(-r\omega_{AB}\sin\theta + \ell\omega_{BC}\sin\phi\right)\vec{j} - \left(r\omega_{AB}\cos\theta + \ell\omega_{BC}\cos\phi\right)\vec{i}$$

The x component yields

$$\omega_{BC} = -\frac{r}{\ell}\omega_{AB}\frac{\cos\theta}{\cos\phi} \qquad (3)$$

which when substituted into the y component leads to

$$\omega_{AB} = -\frac{v}{r(\sin\theta + \cos\theta\tan\phi)} \qquad (4)$$

(b)The relative acceleration equations give

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$$\vec{a}_B = a_A + \alpha_{AB} x \vec{r}_{B/A} + \vec{\omega}_{AB} x (\vec{\omega}_{AB} x \vec{r}_{B/A})$$
$$= (-r\alpha_{AB}\cos\theta + r\omega_{AB}^2 \sin\theta)\vec{i} + (-r\alpha_{AB}\sin\theta - r\omega_{AB}^2\cos\theta)\vec{j}$$

and

$$\vec{a}_C = a\vec{j} = \vec{a}_B + \vec{\alpha}_{BC} x r_{C/B} + \vec{\omega}_{BC} x \vec{\omega}_{BC} x \vec{r}_{CB}$$
$$= (-r\alpha_{AB}\cos\theta - r\omega_{AB}^2\sin\theta - \ell\alpha_{BC}\cos\phi - \ell\omega_{BC}^2\sin\phi)\vec{i}$$
$$+ (-r\alpha_{AB}\sin\theta + r\omega_{AB}^2\cos\theta + \ell\alpha_{BC}\sin\phi - \ell\omega_{BC}^2\cos\phi)\vec{j}$$

The x component is used to determine

$$\alpha_{BC} = -\frac{1}{\ell\cos\phi}(r\omega_{AB}^2\sin\theta + r\alpha_{AB}\cos\theta + \ell\omega_{BC}^2\sin\phi)$$

Which when used in the y component leads to

$$\alpha_{AB} = -\frac{a - r\omega_{AB}^2\cos\theta + \ell\omega_{BC}^2\cos\phi - r\omega_{AB}^2\sin\theta\tan\phi + \ell\omega_{BC}^2\sin\phi\tan\phi}{r(\sin\theta + \cos\theta\tan\phi)} \quad (5)$$

Equation (5) is used to determine the angular acceleration of the crank using eqs.(1) - (4).

Problem 1.14 illustrates application of the relative velocity and relative acceleration equations for rigid body kinematics.

1.15 Determine the reactions at *A* for the two-link mechanism of Figure P1.15. The roller at C rolls on a frictionless surface.

FIGURE P1.15

Given : $\theta = 30°$, $L_{AB} = 2$ m, $L_{BC} = 3$ m, $m_{AB} = 2.4$ kg, $m_{BC} = 3.6$ kg, $v_C = 2.6$ m/sec, $a_C = -1.4$ m/sec^2

Find : A_x , A_y

Solution : From the law of sines

$$\frac{\sin\theta}{L_{BC}} = \frac{\sin\phi}{L_{AB}}$$
$$\sin\phi = \frac{L_{AB}}{L_{BC}}\sin\theta = 0.333$$

From trigonometry

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$$\cos\phi = \sqrt{1-\sin^2\phi} = 0.943$$

The relative position vectors are

$$\vec{r}_{B/A} = L_{AB}\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right) = 1.73\vec{i} + \vec{j}\ m$$
$$\vec{r}_{C/B} = L_{BC}\left(\cos\phi\vec{i} - \sin\phi\vec{j}\right) = 2.83\vec{i} - \vec{j}\ m$$

Using the relative velocity equation between two particles on a rigid body,

$$\mathbf{v}_B = \mathbf{v}_A + \omega_{AB}\mathbf{k}\mathbf{x}\mathbf{r}_{B/A}$$
$$\mathbf{v}_B = -\omega_{AB}\mathbf{i} + 1.73\,\omega_{AB}\mathbf{j}$$
$$\mathbf{v}_C = \mathbf{v}_{BC} + \omega_{BC}\mathbf{k}\mathbf{x}\mathbf{r}_{C/B}$$
$$2.6\,\mathbf{i} = \left(-\omega_{AB} + \omega_{BC}\right)\mathbf{i} + \left(1.73\,\omega_{AB} + 2.83\omega_{BC}\right)\mathbf{j}$$

Equating like components from both sides leads to

$$1.73\,\omega_{AB} + 2.83\omega_{BC} = 0$$
$$-\omega_{AB} + \omega_{BC} = 2.6$$

Simultaneous solution of the above equations leads to

$$\omega_{AB} = -1.61\frac{\text{rad}}{\text{s}},\ \omega_{BC} = 0.986\frac{\text{rad}}{\text{s}}$$

Use of the relative acceleration equation between two particles on a rigid body,

$$\mathbf{a}_B = \mathbf{a}_A + \alpha_{AB}\mathbf{k}\mathbf{x}\mathbf{r}_{B/A} + \omega_{AB}\mathbf{k}x\left(\omega_{AB}\mathbf{k}\mathbf{x}\mathbf{r}_{B/A}\right)$$
$$\mathbf{a}_B = \left(-\alpha_{AB} - 4.48\right)\mathbf{i} + \left(1.73\alpha_{AB} - 2.59\right)\mathbf{j}\frac{\text{m}}{\text{s}^2}$$
$$\mathbf{a}_C = \mathbf{a}_B + \alpha_{BC}\mathbf{k}\mathbf{x}\mathbf{r}_{C/B} + \omega_{BC}\mathbf{k}\mathbf{x}\left(\omega_{BC}\mathbf{k}\mathbf{x}\mathbf{r}_{C/B}\right)$$
$$-1.4\mathbf{i} = \left(-\alpha_{AB} + \alpha_{BC} + 7.23\right)\mathbf{i} + \left(1.72\alpha_{AB} + 2.83\alpha_{BC} - 1.62\right)\mathbf{j}$$

Equating like components from both sides leads to

$$1.72\alpha_{AB} + 2.83\alpha_{BC} = 1.62$$
$$-\alpha_{AB} + \alpha_{BC} = -8.63$$

Simultaneous solution of the above equations leads to

$$\alpha_{AB} = 5.72\frac{\text{rad}}{\text{s}^2},\ \alpha_{\mathbf{B}C} = -2.91\frac{\text{rad}}{\text{s}^2}$$

The relative acceleration equations are used to calculate the accelerations of the mass centers of the links as

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$$\mathbf{a}_{AB} = -5.09\mathbf{i} + 3.65\mathbf{j}\,\frac{\text{m}}{\text{s}^2}$$

$$\mathbf{a}_{BC} = -13.64\mathbf{i} + 3.66\mathbf{j}\,\frac{\text{m}}{\text{s}^2}$$

Free body diagrams of the two bar linkage at this instant of time are shown below

Summing moments about C

$$\left(\sum M_C\right)_{ext.} = \left(\sum M_C\right)_{eff.}$$

$$A_y\left(L_{AB}\cos\theta + L_{BC}\cos\phi\right) - m_{AB}g\left(\frac{L_{AB}}{2}\cos\theta + L_{BC}\cos\phi\right) - m_{BC}g\frac{L_{BC}}{2}\cos\phi$$

$$= -\bar{I}_{AB}\alpha_{AB} - \bar{I}_{BC}\alpha_{BC} + m_{AB}\bar{a}_{x_{AB}}\frac{L_{AB}}{2}\sin\theta + m_{BC}\bar{a}_{x_{BC}}\frac{L_{BC}}{2}\sin\phi$$

$$+ m_{AB}\bar{a}_{y_{AB}}\left(\frac{L_{AB}}{2}\cos\theta + L_{BC}\cos\phi\right) + m_{BC}\bar{a}_{y_{BC}}\frac{L_{BC}}{2}\cos\phi$$

Noting that

$$\bar{I}_{AB} = \frac{1}{12}m_{AB}L_{AB}^2 = 0.8\,\text{kg}\cdot\text{m}^2, \bar{I}_{BC} = \frac{1}{12}m_{BC}L_{BC}^2 = 2.7\,\text{kg}\cdot\text{m}^2$$

and substituting given and calculated values and solving for A_y leads to

$$A_y = 28.49\,\text{N}$$

Summing forces in the horizontal direction

$$\left(\sum F_x\right)_{ext.} = \left(\sum F_x\right)_{eff.}$$

$$A_x = m_{AB}\bar{a}_{x_{AB}} + m_{BC}\bar{a}_{x_{BC}}$$

Substituting given and calculated values leads to

$$A_x = -61.32\,\text{N}$$

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Problem 1.15 illustrates (a) application of the relative velocity equation for a linkage, (b) application of the relative acceleration equation for a linkage, and (c) application of Newton's laws to a system of rigid bodies. This problem is a good illustration of the effectiveness of the effective force method of application of Newton's Laws. Use of this method allows a free body diagram of the entire linkage to be drawn and used to solve for the unknown reactions. Application of Newton's Laws to a single rigid body exposes the reactions in the pin connection at B and complicates the solution.

1.16 Determine the angular acceleration of each of the disks in Figure P1.16.

FIGURE P1.16

Given: Disk of $I_P = 4$ kg-m^2, r = 60 cm with (a) m_1 = 30 kg and $m_2 = 20$ kg blocks attached or (b) F_1 = 270 N and F_2 = 180 N forces attached.

Find: α

Solution:

(a) Free body diagrams of the disk and the blocks are shown below

Summing moments about the center of the disk

$$(\Sigma M_O)_{ext.} = (\Sigma M_O)_{eff.}$$

$$m_1 gr - m_2 gr = I_p \alpha + m_1 r^2 \alpha + m_2 r^2 \alpha$$

$$\alpha = \frac{(m_1 - m_2)gr}{I_p + (m_1 + m_2) r^2} = 2.68 \frac{\text{rad}}{\text{s}^2}$$

(b) Free body diagrams of the disk are shown below

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Summing moments about the center of the disk

$$(\Sigma M_O)_{ext.} = (\Sigma M_O)_{eff.}$$

$$F_1 r - F_2 r = I_p \alpha$$

$$\alpha = \frac{(F_1 - F_2)r}{I_P} = 13.5 \frac{\text{rad}}{\text{s}^2}$$

Problem 1.16 illustrates application of Newton's Laws to systems of rigid bodies. It also illustrates the difference between an applied force and a mass.

1.17 Determine the reactions at the pin support and the applied moment if the bar of Figure P1.17 has a mass of 50 g.

FIGURE P1.17

Given: $\alpha = 14$ rad/sec^2, $\omega = -5$ rad/sec, m = 50 kg

L = 4 m, $\theta = 10°$

Find: M, O_x, O_y

Solution: The bar's centroidal moment of inertia of the bar

$$\bar{I} = \frac{1}{12} m L^2 = \frac{1}{12}(50\text{ kg})(4\text{ m})^2 = 66.67\text{ kg}\cdot\text{m}^2$$

Free body diagrams of the bar at this instant are shown below

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Summing moments about O

$$(\Sigma M_O)_{ext.} = (\Sigma M_O)_{eff.}$$

$$M - mg\frac{L}{4}\cos\theta = \bar{I}\alpha + m\frac{L}{4}\alpha\frac{L}{4}$$

$$M = (\bar{I} + m\frac{L^2}{16})\alpha + mg\frac{L}{4}\cos\theta$$

$$= [66.67\ \text{kg-m}^2 + \frac{(50\ \text{kg})(4\ \text{m})^2}{16}](14\frac{\text{rad}}{\text{sec}^2})$$

$$+ \frac{(50\ \text{kg})(9.81\frac{\text{m}}{\text{sec}^2})(4\ \text{m})}{4} = 2120\ \text{N}\cdot\text{m}$$

Summing forces in the horizontal direction

$$(\Sigma F_x)_{ext.} = (\Sigma F_x)_{eff.}$$

$$O_X = -m\frac{L}{4}\omega^2\cos\theta + m\frac{L}{4}\alpha\sin\theta$$

$$= -\frac{(50\ \text{kg})(4\ \text{m})}{4}(-5\frac{\text{rad}}{\text{s}})^2\cos 10°$$

$$+ (50\ \text{kg})\frac{4\ \text{m}}{4}(14\frac{\text{rad}}{\text{s}^2})\sin 10° = -1110\ \text{N}$$

Summing forces in the vertical direction

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$$(\Sigma F_y)_{ext.} = (\Sigma F_y)_{eff.}$$

$$O_y - mg = m\frac{L}{4}\omega^2 \sin\theta + m\frac{L}{4}\alpha\cos\theta$$

$$O_y = \frac{(50\text{ kg})(4\text{ m})}{4}(-5\frac{\text{rad}}{\text{s}})^2 \sin 10° + \frac{(50\text{ kg})(4\text{ m})}{4}(14\frac{\text{rad}}{\text{s}^2})\cos 10°$$
$$+ (50\text{ kg})(9.81\frac{\text{m}}{\text{s}^2}) = 1400\text{ N}$$

Problem 1.17 illustrates application of Newton's Laws to rigid bodies.

1.18 The disk of Figure P1.18 rolls without slipping. Assume if P = 18 N. (a) Determine the acceleration of the mass center of the disk. (b) Determine the angular acceleration of the disk.

FIGURE P1.18

Given: m = 18 kg, P = 18 N, r = 20 cm

Find: $\bar{a}$

Solution: (a) If the disk rolls without slip then its angular acceleration is related to the acceleration of the mass center by

$$\bar{a} = r\alpha$$

Free- body diagrams of the disk at an arbitrary instant are shown below

Summing moments about the contact point

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$$\left(\sum M_C\right)_{ext.} = \left(\sum M_C\right)_{eff.}$$
$$P\,r = \bar{I}\alpha + m\bar{a}r$$
$$P\,r = \frac{1}{2}mr^2\frac{\bar{a}}{r} + m\bar{a}r$$
$$\bar{a} = \frac{2P}{3m} = \frac{2(18\,\text{N})}{3(1.8\,\text{kg})} = 6.67\,\frac{\text{m}}{\text{s}^2}$$

(b) The angular acceleration of the disk is

$$\alpha = \frac{\bar{a}}{r} = 33.35\text{r/s}^2$$

Problem 1.18 illustrates application of Newton's Laws to a rigid body.

1.19 The coefficient of friction between the disk of Figure P1.18 and the surface is 0.12. What is the largest force that can be applied such that the disk rolls without slipping?

FIGURE P1.18

Given: m = 1.8 kg, r = 20 cm, $\mu = 0.12$
Find: $P_{max.}$ for no slip

Solution: Free body diagrams of the disk at an arbitrary instant are shown below.

Summing moments about the contact point,

$$\left(\sum M_c\right)_{ext.} = \left(\sum M_c\right)_{eff.}$$
$$Pr = \bar{I}\alpha + m\bar{a}r \qquad (1)$$

If the disk rolls without slip then

$$\alpha = \frac{\bar{a}}{r} \qquad (2)$$

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Substitution of eq.(2) into eq.(1) leads to

$$\bar{a} = \frac{2P}{3m} \tag{3}$$

Summing moments about the mass center of the disk

$$\left(\sum M_G\right)_{ext.} = \left(\sum M_G\right)_{eff.}$$

$$F\,r = \frac{1}{2}\,mr^2 = \frac{\bar{a}}{r}$$

$$F = \frac{1}{2}\,m\bar{a} = \frac{1}{2}\,m\,\frac{2P}{3m} = \frac{P}{3}$$

The maximum allowable friction force is μmg, thus in order for the no slip assumption to be valid,

$$\frac{P}{3} < \mu mg$$

$$P < 3\,\mu mg = 6.36\,\text{N}$$

Problem 1.19 illustrates application of Newton's Laws to a rigid body dynamics problem and rolling friction.

1.20 The coefficient of friction between the disk of Figure P1.18 and the surface is 0.12. If P = 22 N, what are the following? (a) Acceleration of the mass center. (b) Angular acceleration of the disk.

FIGURE P1.18

Given: r = 20 cm, m = 1.8 kg, P = 15 N, μ = 0.12

Find: $\bar{a}$, α

Solution: (a) Free body diagrams of the disk at an arbitrary instant of time are shown below

Summing moments about the contact point between the disk and the surface

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$$(\Sigma M_O)_{ext.} = (\Sigma M_O)_{eff.}$$
$$Pr = m\bar{a}r + \frac{1}{2}mr^2\alpha \qquad (1)$$

Summing moments about the mass center

$$(\Sigma M_G)_{ext.} = (\Sigma M_G)_{eff.}$$
$$Fr = \frac{1}{2}mr^2\alpha \qquad (2)$$

First assume the disk rolls without slipping. Then the velocity and the acceleration of the contact point are zero, which in turn implies that $\bar{a} = r\alpha$. Substituting into eq.(1) yields

$$\alpha = \frac{2P}{3mr} = 27.8\,\frac{\text{rad}}{\text{s}^2}$$

If the assumption of no slip is valid, then the friction force developed is less than the maximum allowable friction force,

$$F_{\max} = \mu mg = 2.12\text{N}$$

The friction force assuming no slip is calculated using eq.(2),

$$F = \frac{1}{2}mr\alpha = \frac{1}{2}(1.8\text{ kg})(0.2\text{ m})\left(27.8\,\frac{\text{rad}}{\text{s}}\right) = 5.0\text{ N}$$

(b) Thus the disk rolls and slides. The friction force takes on its maximum permissible value of 2.12 N. The velocity of the contact point is not zero and is independent of the velocity of the mass center implying that there is no kinematic relation between the angular acceleration and the acceleration of the mass center. Setting F = 2.12 N in eq.(2) leads to

$$\alpha = \frac{2F}{mr} = \frac{2(2.12\text{ N})}{(1.8\text{ kg})(0.2\text{ m})}11.8\,\frac{\text{rad}}{\text{s}^2}$$

Problem 1.20 illustrates application of Newton's Law to a rolling rigid body. Since it is not known whether the disk slides while rolling, an assumption of no slip is made. The assumption is proved false by checking the friction force. If an assumption of rolling and slipping is first made, there is no convenient way to check the assumption.

1.21 The 3 kg block of Figure P1.21 is displaced 10 mm downward and then released from rest. (a) What is the maximum velocity attained by the 3-kg block? (b) What is the maximum angular velocity attained by the disk?

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FIGURE P1.21

Given: $m_1 = 3$ kg, $m_2 = 5$ kg,

$I_P = 0.25$ kg-m^2, r = 20 cm,

K = 4000 N/m, δ = 10 mm

Find: $\dot{x}_{max}$, ω_{max}

Solution: Since gravity and spring forces are the only external forces doing work, energy is conserved. Let position 1 refer to the position when the 3 kg block is displaced 10 mm. Let position 2 refer to the position when the velocity is a maximum. Then

$$T_1 + V_1 = T_2 + V_2 \tag{1}$$

The spring is stretched when the system is in equilibrium, due to the gravity of the blocks. Thus when the spring is in equilibrium, it has a non-zero potential energy. However, when the system is in equilibrium its total energy is zero. Thus the potential energy due to gravity balances with the potential energy due to the static deflection in the spring. Neither must be included in the analysis. With this in mind

$$T_1 = 0$$

$$V_1 = \frac{1}{2} k\,\delta^2 = 0.2\text{N}\cdot\text{m}$$

$$V_2 = 0$$

$$T_2 = \frac{1}{2} m_1 r^2 \omega_2^2 + \frac{1}{2} m_2 r^2 \omega_2^2 + \frac{1}{2} I_P\, \omega_2^2 = 0.285\, \omega_2^2$$

Substitution into eq.(1) leads to

$$\omega_2 = \omega_{\text{max.}} = 0.837\,\frac{\text{rad}}{\text{s}}$$

Then

$$x_{max} = (0.2\text{ m})(0.837\text{ rad/s}) = 0.167\text{ m/s}$$

Problem 1.21 illustrates application of conservation of energy to a system of rigid bodies. It also illustrates that the potential energy present in a spring when a system is in equilibrium will balance with potential energies of the gravity forces that caused the static deflection. Neither must be included in a work-energy analysis as they cancel with each other.

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1.22 The center of the thin disk of Figure P1.22 is displaced 15 mm and released. What is the maximum velocity attained by the disk, assuming no slipping between the disk and the surface?

FIGURE P1.22

Given: m = 2 kg, r = 25 cm, k = 20,000 N/m, δ = 15 mm, no slip

Find: $v_{max.}$

Solution: Since the disk rolls without slipping, the velocity of the point of contact between the disk and the surface is zero, and hence the friction force does no work. Thus the spring force is the only external force which does work. The system is conservative. Let position 1 refer to the initial position of the system when the center is displaced 15 mm. Let position 2 refer to the position when the center attains its maximum velocity. Then from conservation of energy

$$T_1 + V_1 = T_2 + V_2 \tag{1}$$

where

$$T_1 = 0$$

$$V_1 = \frac{1}{2} k \delta^2 = \frac{1}{2}(20000 \frac{\text{N}}{\text{m}})(0.015\text{m})^2 = 2.25\text{N} \cdot \text{m}$$

$$T_2 = \frac{1}{2} m\bar{v}^2 + \frac{1}{2}(\frac{1}{2} mr^2)\omega_2^2$$

Since the disk rolls without slip

$$\bar{v}_2 = r\omega_2$$

and

$$T_2 = \frac{3}{4} m\bar{v}_2^2$$

Substituting into eq.(1) leads to

$$\bar{v}_2 = \bar{v}_{\text{max.}} = \sqrt{\frac{4(2.25\text{N} \cdot \text{m})}{3(2\text{ kg})}} = 1.22 \frac{\text{m}}{\text{s}}$$

Problem 1.22 illustrates application of conservation of energy to a system involving a rigid body. The time history of motion for this system is examined in Chapter 4.

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1.23 The block of Figure P1.23 is given a displacement δ and then released. (a) What is the minimum value of δ such that motion ensues? (b) What is the minimum value of δ such that the block returns to its equilibrium position without stopping?

FIGURE P1.23

Given: m, k

Find: (a) value of δ for motion, (b) value of δ such that block returns to equilibrium

Solution: (a) In order for motion to occur when the block is released, the spring force must be larger than the friction force. That is

$$k\delta > \mu mg$$

$$\delta > \frac{\mu mg}{k}$$

(b) In order for the block to return to equilibrium before motion ceases, the initial potential energy stored in the spring must not be dissipated due to friction before the block returns to equilibrium. Suppose the block is given a displacement just sufficient to return it to equilibrium before motion ceases. Let position 1correspond to the initial position and position 2 correspond to the position when the block returns to its equilibrium position. The principle of work energy states

$$T_1 + V_1 + U_{1-2} = T_2 + V_2$$

Since the system is released from rest, $T_1 = 0$. Since the displacement is just sufficient to return the block to equilibrium, it has a zero velocity when it returns to equilibrium and $T_2 = 0$. Since the block is in its equilibrium position in position 2, $V_2 = 0$. The work done by non-conservative forces is the work done by the friction force. Thus

$$\frac{1}{2}k\delta^2 - \mu mg\delta = 0$$

$$\delta = \frac{2\mu mg}{k}$$

Problem 1.23 illustrates motion of a mass-spring system when dry fiction is present. This problem is considered again in Chapter 3 in the discussion of Coulomb damping.

1.24 The five-blade ceiling fan of Figure P1.24 operates at 60 rpm. The distance between the mass center of a blade and the axis of rotation is 0.35 m. What is the total kinetic energy?

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Given: ω = 60 rpm, ceiling fan shown

Find: T

Solution: The rotational speed is converted to rad/sec by

FIGURE P1.24

$$\omega = \frac{60\,\text{rev}}{1\,\text{s}}\frac{2\pi\ \text{rad}}{1\ \text{rev}}\frac{1\ \text{min}}{60\ \text{sec}} = 6.283\ \frac{\text{rad}}{\text{sec}}$$

The velocity of the mass center of the motor

$$\bar{v} = (0.013\,\text{m})\omega = 0.082\,\frac{\text{m}}{\text{s}}$$

The kinetic energy of the motor is

$$T_m = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$
$$= \frac{1}{2}(4.7\,\text{kg})\left(0.082\,\frac{\text{m}}{\text{s}}\right)^2 + \frac{1}{2}(5.14\,\text{kg}-\text{m}^2)\left(6.283\,\frac{\text{rad}}{\text{s}}\right)^2$$
$$= 101.5\ \text{N}\cdot\text{m}$$

The velocity of the mass center of each blade is

$$\bar{v} = (0.35\,m)\omega = 2.20\,\frac{\text{m}}{\text{s}}$$

The kinetic energy of each blade is

$$T_b = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$
$$= \frac{1}{2}(1.21\,\text{kg})\left(2.20\,\frac{\text{m}}{\text{s}}\right)^2 + \frac{1}{2}(0.96\,\text{kg}\cdot\text{m}^2)\left(6.283\,\frac{\text{rad}}{\text{s}}\right)^2$$
$$= 21.88\ \text{N}\cdot\text{m}$$

The total kinetic energy of the ceiling fan is

$$T = T_m + 5T_b = 101.5\ \text{N}\cdot\text{m} + 5(21.88\,\text{N}\cdot\text{m})$$
$$= 210.9\,\text{N}\cdot\text{m}$$

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Problem 1.24 illustrates calculation of kinetic energy of a rigid body.

1.25 The U-tube manometer shown in Figure P1.25 rotates about axis *A-A* at a speed of 40 rad/sec. At the instant shown, the column of liquid moves with a speed of 20 m/sec relative to the manometer. Calculate the total kinetic energy of the column of liquid in the manometer.

FIGURE P1.25

Given: v = 20 m/sec, ω = 40 rad/sec, A = 0.0003 m^2 , S.G.=1.4

Find: T

Solution: The column of liquid is broken into three sections. The velocity of the fluid particles comprising each section is

$$\mathbf{v}_{AB} = v\mathbf{i} + r\omega\mathbf{k}$$
$$\mathbf{v}_{BC} = v\mathbf{i} + r\omega\mathbf{k}$$
$$\mathbf{v}_{CD} = v\mathbf{j} - 0.6\,\omega\mathbf{k}$$

Consider a differential mass, defined in each part of the manometer as shown. The kinetic energy of the differential mass is

$$dT = \frac{1}{2}|\mathbf{v}|^2 dm$$

The kinetic energy of the particles in each section is obtained by integrating dT over the liquid in that section.

Section AB: $dm_{AB} = \rho A dr$

$$T_{AB} = \int_0^{0.2\,m} \frac{1}{2}\rho A\left(v^2 + \omega^2 r^2\right)dr$$
$$= \frac{1}{2}\rho A\left(0.2v^2 + 0.00267\,\omega^2\right)$$

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Section BC: $dm_{BC} = \rho A dr$

$$T_{BC} = \int_0^{0.6\,m} \frac{1}{2}\rho A\left(v^2 + \omega^2 r^2\right)dr$$
$$= \frac{1}{2}\rho A\left(0.6\,v^2 + 0.072\,\omega^2\right)$$

Section CD: $dm_{CD} = \rho A dz$

$$T_{CD} = \int_0^{1\,m} \frac{1}{2}\rho A\left(v^2 + 0.36\,\omega^2\right)dz$$
$$= \frac{1}{2}\rho A\left(v^2 + 0.36\,\omega^2\right)$$

The total kinetic energy is

$$T = T_{AB} + T_{BC} + T_{CD}$$
$$= \frac{1}{2}\rho A\left(1.8\,v^2 + 0.435\,\omega^2\right)$$
$$= \frac{1}{2}(1.4)\left(1000\,\frac{\text{kg}}{\text{m}^3}\right)\left(0.0003\,\text{m}^2\right)\left[1.8\left(20\,\frac{\text{m}}{\text{s}}\right)^2 + 0.435\left(40\,\frac{\text{rad}}{\text{s}}\right)^2\right]$$
$$= 297.4\ \text{N}\cdot\text{m}$$

Problem 1.25 illustrates the kinetic energy calculation of a column of liquid in a manometer. The vibrations of the column of liquid in a manometer rotating about an axis other than an axis through its center are nonlinear if the rotational speed is large enough. The differential equations are formulated using energy methods and a kinetic energy calculation similar to that developed in the solution of this problem.

1.26 The displacement function for a simply supported beam of Figure P1.26 is

$$y(x,t) = c\sin\left(\frac{\pi x}{L}\right)\cos\left(\pi^2\sqrt{\frac{EI}{\rho A L^4}}\,t\right)$$

FIGURE P1.26

where c = 0.003 m and t is in seconds. Determine the kinetic energy of the beam.

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Given: $y(x,t)$

Find: T

Solution: The kinetic energy of the beam is

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial y}{\partial t}\right)^2 dx = \frac{1}{2}\int_0^L \rho A\left[c\sin\left(\frac{\pi x}{L}\right)\right]^2\left(\pi^2\sqrt{\frac{EI}{\rho AL^4}}\right)^2\left[\sin\pi^2\sqrt{\frac{EI}{\rho AL^4}}\right]^2 dx$$

$$= \frac{1}{2}c^2\pi^4\frac{EI}{L^4}\left[\sin\pi^2\sqrt{\frac{EI}{\rho AL^4}}\right]^2\int_0^L\left[\sin\left(\frac{\pi x}{L}\right)\right]^2 dx$$

$$= \frac{1}{4}c^2\pi^4\frac{EI}{L^3}\left[\sin\pi^2\sqrt{\frac{EI}{\rho AL^4}}\right]^2$$

Problem 1.26 illustrates the calculation of the kinetic energy of a continuous system.

1.27 The block of Figure P1.27 is displaced 1.5 cm from equilibrium and released.

(a) What is the maximum velocity attained by the block?
(b) What is the acceleration of the block immediately after it is released?

FIGURE P1.27

Given: m = 65 kg, k = 12,000 N/m, x_0 = 1.5 cm

Find: (a) v_{max} (b) a_0

Solution: When the system is in equilibrium the spring is stretched and has a static deflection Δ. Summing forces on the free-body diagram of the system's equilibrium position

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$$\sum F = 0$$
$$mg - k\Delta = 0$$
$$\Delta = \frac{mg}{k} = \frac{(65\text{ kg})(9.81\text{ m/s}^2)}{12000\text{ N/m}} = 0.053\text{ m}$$

(a) Let position 0 refer to the initial position of the system. Let position 1 refer to the system when the velocity of the block is a maximum. Since the system is is released from rest in position 0, T_0=0. The total stretching in the spring in position 0 is

$$\delta_0 = x_0 + \Delta = 0.015\text{ m} + 0.053\text{ m} = 0.068\text{ m}$$

If the equilibrium plane is chosen as the datum plane for referencing the potential energy due to gravity the potential energy in position 0 is

$$V_0 = -mgx_0 + \frac{1}{2}k\delta_0^2$$
$$V_0 = -(65\text{ kg})(9.81\text{ m/s}^2) + \frac{1}{2}(12000\text{ N/m})(0.068\text{ m})^2$$
$$V_0 = 18.18\text{ N}\cdot\text{m}$$

Since all forces are conservative, application of conservation of energy is applied leading to

$$18.18\text{ N - m} = T_1 + V_1$$

The maximum kinetic energy occurs when the potential energy is a minimum, which occurs when the system passes through its equilibrium position,

$$V_1 = \frac{1}{2}k\Delta^2 = \frac{1}{2}(12000\text{ N/m})(0.053\text{ m})^2 = 16.85\text{ N}\cdot\text{m}$$

Hence

$$18.18\text{ N}\cdot\text{m} = 16.85\text{ N}\cdot\text{m} + \frac{1}{2}mv_{\max}^2$$
$$v_{\max} = 0.202\text{ m/s}$$

(b) Application of Newton's law to the free-body diagram of the block in its initial position leads to

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$$\sum F = ma_0$$
$$mg - k(x_0 + \Delta)$$
$$a_0 = g - \frac{k}{m}(x_0 + \Delta)$$
$$a_0 = 9.81\,\text{m/s}^2 - \frac{12000\ \text{N/m}}{65\ \text{kg}}(0.068\ \text{m})$$
$$a_0 = -2.74\ \text{m/s}^2$$

Problem 1.27 illustrates (a) application of conservation of energy to a particle and (b) application of Newton's law to the free-body diagram of a particle.

1.28 The slender rod of Figure P1.28 is released from the horizontal position when the spring attached at *A* is stretched 10 mm and the spring attached at *B* is unstretched. (a) What is the angular acceleration of the bar immediately after it is released? (b) What is the maximum angular velocity attained by the bar?

FIGURE P1.28

Given: m = 1.2 kg, L = 1m, δ_1 = 10 mm, k_1 = 1200 N/m, k_2 = 1000 N/m

Find: (a) $Y_{max.}$, (b) $\omega_{max.}$

Solution: Consider first the system immediately after release.

Summing moments about B

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$$\left(\sum M_B\right)_{ext.} = \left(\sum M_B\right)_{eff.}$$

$$-k_1\delta_1 L + mg\frac{L}{2} = m\frac{L^2}{3}\alpha$$

$$\alpha = \frac{3}{mL}\left(\frac{mg}{2} - k_1\delta_1\right)$$

$$= -15.29\ \frac{\text{rad}}{\text{s}^2}$$

Hence α is clockwise and the bar moves upward. Now consider the geometry of the bar when it has moved a distance y upward. The horizontal displacement of B is

$$\delta_B = L - \sqrt{L^2 - y^2}$$

(b) Let ω be the angular velocity of the bar. Then using the relative velocity equation

$$\mathbf{v}_A = v_A\mathbf{j} = v_B\mathrm{i} + \omega\mathbf{k}\mathbf{x}\left(-L\cos\theta\mathbf{i} - L\sin\theta\mathbf{j}\right)$$

$$v_A\mathbf{j} = \left(v_B + L\omega\sin\theta\right)\mathbf{i} - L\omega\cos\theta\mathbf{j}$$

From the x component of the above equation

$$v_B = -L\sin\theta\,\omega$$

The velocity of the mass center of the bar is

$$\mathbf{v} = -L\omega\sin\theta\mathbf{i} + \omega\mathbf{k}\mathbf{x}\left(-\frac{L}{2}\cos\theta\mathbf{i} - \frac{L}{2}\sin\boldsymbol{\theta}\mathbf{j}\right)$$

$$\vec{v} = -\frac{L}{2}\omega\sin\theta\mathbf{i} + -\frac{L}{2}\omega\cos\theta\mathbf{j}$$

$$|\mathbf{v}| = \frac{L}{2}\omega$$

Let position 2 refer to the position of the system when the angular velocity is a maximum. Energy is conserved between position 1 and position 2.

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{2}k_1\delta_1^2 = mg\frac{y}{2} + \frac{1}{2}k_1(\delta_1 - y)^2 + \frac{1}{2}k_2\left(L - \sqrt{L^2 - y^2}\right) + \frac{1}{2}\frac{1}{12}mL^2\omega^2 + \frac{1}{2}m\left(\frac{L}{2}\omega\right)^2 \quad (3)$$

$$0 = \left(m\frac{g}{2} + k_1\delta_1\right)y + \frac{1}{2}k_1y^2 + \frac{1}{2}k_2\left(2L^2 - y^2 + 2L\sqrt{L^2 - y^2}\right) + \frac{1}{6}mL^2\omega^2$$

The above equation could be expressed as

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$$T_2 = \frac{1}{6} mL^2\omega^2 = V_1 - V_2$$

Thus the maximum angular velocity occurs when V_1-V_2 is maximized. To this end

$$\frac{d}{dy}(V_1 - V_2) = 0 = k_1\delta_1 - \frac{mg}{2} + (k_2 - k_1)y - \frac{k_2 Ly}{\sqrt{L^2 - y^2}}$$

$$6.114 - 200y - \frac{1000y}{\sqrt{1 - y^2}} = 0$$

A trial and error solution of the above equation reveals that the maximum angular velocity occurs for y = 0.0051 m. Then from eq. (3),

$$\omega = 0.322 \frac{\text{rad}}{\text{s}}$$

Problem 1.28 illustrates application of conservation of energy to a rigid body system.

1.29 Let x be the displacement of the left end of the bar of the system in Figure P1.29. Let θ represent the clockwise angular rotation of the bar. (a) Express the kinetic energy of the system at an arbitrary instant in terms of $\dot{x}$ and $\dot{\theta}$. (b) Express the potential energy of an arbitrary instant in terms of x and θ.

FIGURE P1.29

Given: x and θ as generalized coordinates

Find: (a) T (b) V

Solution: (a) The kinetic energy of a rigid body is

$$T = \frac{1}{2} m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

The angular velocity of the bar is $\omega = \dot{\theta}$. The displacement of the mass center in terms of the chosen generalized coordinates is

$$\bar{x} = x + \frac{L}{2}\sin\theta$$

Thus the velocity of the mass center is

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$$\dot{\bar{x}} = \dot{x} + \frac{L}{2}\dot{\theta}\cos\theta$$

Hence the kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\left(\dot{x} + \frac{L}{2}\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}\bar{I}\dot{\theta}^2$$

If the small-angle assumption is used the kinetic energy of the linearized system is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mL\dot{x}\dot{\theta} + \frac{1}{2}\left(m\frac{L^2}{4} + \bar{I}\right)\dot{\theta}^2$$

(b) The potential energy is due to the springs and is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k\left(x + \frac{3L}{4}\theta^2\right)$$

Problem 1.29 illustrates the evaluation of the kinetic energy and potential energy of a rigid body at an arbitrary instant in terms of chosen generalized coordinates.

1.30 Repeat problem 1.29 using coordinates x_1, which is the displacement of the mass center, and x_2, which is the displacement of the point of attachment of the spring that is a distance $3L/4$ from the left end.

FIGURE P1.29

Given: x_1 and x_2 as generalized coordinates.

Find: (a) T (b) V

Solution: The kinetic energy of the bar at an arbitrary instant is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}I\left[\left(\frac{4}{L}\right)(\dot{x}_2 - \dot{x}_1)\right]^2$$

The potential energy of the bar at an arbitrary instant is

$$V = \frac{1}{2}k(3x_2 - 2x_1)^2 + \frac{1}{2}k{x_2}^2$$

Problem 1.30 illustrates the evaluation of the kinetic and potential energy of a rigid body at an arbitrary instant in terms of chosen generalized coordinates.

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1.31 Let θ represent the clockwise angular displacement of the pulley system in Figure P1.31 from the system's equilibrium position.

(a) Express the potential energy of the system at an arbitrary instant in terms of θ.

(b) Express the kinetic energy of the system at an arbitrary instant in terms of $\dot{\theta}$.

FIGURE P1.31

Given: θ as generalized coordinate

Find: (a) V (b) T

Solution: Consider the free-body diagram of the system in its equilibrium position. Summing moments about the center of the pulley

$$\sum M_C = 0$$
$$-k\Delta_1 r - 2k\Delta_2(2r) + 2mg(2r) = 0$$

From the geometry of the system

$$\theta_{st} = \frac{\Delta_1}{r} = \frac{\Delta_2}{2r}$$

which when substituted into the previous equation leads to

$$\Delta_1 = \frac{4mg}{9k} \qquad \Delta_2 = \frac{8mg}{9k}$$

Let x_1 represent the displacement of the sliding block from the system's equilibrium position. Let x_2 represent the displacement of the hanging block from the system's equilibrium position. From geometry

$$x_1 = r\theta \qquad x_2 = 2r\theta$$

(a) Choosing the equilibrium position of the system as the datum for potential energy calculations, the potential energy at an arbitrary instant is

$$V = \frac{1}{2}k(x_1 + \Delta)^2 + \frac{1}{2}2k(x_2 + \Delta_2)^2 - 2mgx_2$$
$$V = \frac{1}{2}k\left(r\theta + \frac{4mg}{9k}\right)^2 + \frac{1}{2}2k\left(2r\theta + \frac{8mg}{9k}\right)^2 - 2mgx_2$$

Simplification leads to

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$$V = \frac{9}{2}kr^2\theta^2 + \frac{8}{9}\left(\frac{mg}{k}\right)^2$$

(b) The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}I_p\dot{\theta}^2 + \frac{1}{2}2m\dot{x}_2^2$$

$$T = \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}I_p\dot{\theta}^2 + \frac{1}{2}2m(2r\dot{\theta})^2$$

$$T = \frac{1}{2}(9mr^2 + I_p)\dot{\theta}^2$$

Problem 1.31 illustrates the calculation of a potential and kinetic energy of a system of rigid bodies at an arbitrary instant in terms of a chosen generalized coordinate.

1.32 A 20 ton railroad car is coupled to a 15 ton car by moving the 20 ton car at 5 mph toward the stationary 15 ton car. (a) What is the resulting speed of the two-car coupling? (b) What would the resulting speed be if the 15 ton car is moving at 5 mph toward a stationary 20 ton car?

Given: $W_1 = 40000$ lb, $W_2 = 30000$ lb, $v_1 = 5$ mph

Find: v_2

Solution: (a) Consider the impulse and momentum diagrams below

There are no external impulses acting on the two car system during coupling. Applying the principle of linear impulse and linear momentum

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$$\frac{W_1}{g}v_1 = \left(\frac{W_1}{g} + \frac{W_2}{g}\right)v_2$$

$$v_2 = \frac{W_1 v_1}{W_1 + W_2}$$

$$= \frac{(40000\,\text{lb})(5\,\text{mph})}{70000\,\text{lb}}$$

$$= 2.86\,\text{mph}$$

(b) If the 15 ton car has a velocity of 5 mph the velocity of the system after coupling is

$$v_2 = \frac{(30000\text{ lb})(5\text{ mph})}{(70000\text{ lb})} = 2.14\text{ mph}$$

Problem 1.32 illustrates application of the principle of linear impulse and linear momentum to a system when linear momentum is conserved. The couplings between railroad cars are actually elastic. Thus, after coupling the cars move relative to one another. The two car system will move together with a rigid body motion, but relative motion will occur. This is an example of a unrestrained system considered in Chapters 6 and 7.

1.33 The 15 kg block of Figure P1.33 is moving with a velocity of 3 m/s at $t = 0$ when the force $F(t)$ is applied to the block. (a) Determine the velocity of the block at $t = 2$ s. (b) Determine the velocity of the block at $t = 4$ s. (c) Determine the block's kinetic energy at $t = 4$ sec.

FIGURE P1.33

Given: $m = 15$ kg, $v_1 = 3$ m/s, $\mu = 0.08$, F(t)

Find: (a) v(t=2 s) (b) v(t=4 s) (c) T

Solution: (a) The principle of impulse and momentum is used to determine the velocity at t=2 s. Application of the principle leads to

$$mv_1 + \int_0^2 [F(t) - \mu mg]dt = mv_2$$

or substituting in given numbers yields

$$(15\text{ kg})\left(3\frac{\text{m}}{\text{s}}\right) + \frac{1}{2}(20\text{ N})(2\text{ s}) - (0.08)(15\text{ kg})\left(9.81\frac{\text{m}}{\text{s}^2}\right)(2\text{ s}) = (15\text{ kg})v_2$$

$$v_2 = 2.76\text{ m/s}$$

(b) The velocity at t = 4 s is determined from the principle of impulse and momentum

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$$mv_1 + \int_0^4 [F(t) - \mu mg]dt = mv_2$$

which upon substitution of given numbers yields

$$(15\text{ kg})\left(3\frac{\text{m}}{\text{s}}\right) + \frac{1}{2}(30\text{ N})(3\text{ s}) + (30\text{ N})(1\text{ s}) - (0.08)(15\text{ kg})\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)(4\text{ s}) = (15\text{ kg})v_2$$

$$v_2 = 4.86\text{ m/s}$$

(c) The kinetic energy is

$$T = \frac{1}{2}mv_2^2 = \frac{1}{2}(15\text{ kg})(4.86\text{ m/s})^2 = 177.1\text{ J}$$

Problem 1.33 illustrates application of the principle of impulse and momentum.

1.34 A 400 kg forging hammer is mounted on four identical springs, each of stiffness k = 4200 N/m. During the forging process, a 110 kg hammer, which is part of the machine, is dropped from a height of 1.4 m onto an anvil, as shown in Figure P1.34. (a) What is the resulting velocity of the entire machine after the hammer is dropped? (b) What is the maximum displacement of the machine?

FIGURE P1.34

Given: m = 400 kg, k = 42000 N/m, m_h = 110 kg, h = 1.4 m

Find: (a) v (b) x_{max}

Solution: (a) Application of the principle of conservation of energy to the hammer as it drops leads to the velocity of the hammer immediately before impacting the anvil

$$\frac{1}{2}mv^2 = mgh \Rightarrow v = \sqrt{2gh} = \sqrt{2\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)(1.4\text{ m})} = 5.24\ \frac{\text{m}}{\text{s}}$$

Applying the principle of impulse and momentum to the hammer and anvil as the hammer strikes leads to (assuming the hammer is part of the machine and the hammer sticks to and moves with the machine)

$$m_h v = mv_2 \Rightarrow v_2 = \frac{m_h v}{m} = \frac{(110\text{ kg})(5.24\text{ m/s})}{400\text{ kg}} = 1.44\text{ m/s}$$

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(b) Application of the principle of conservation of energy between the time immediately after impact to the time when the machine reaches its maximum displacement

$$\frac{1}{2}mv_2^2 = \frac{1}{2}kx_{max}^2$$

$$x_{max} = \frac{m}{k}v_2 = \frac{400\text{ kg}}{4200\text{ N/m}}1.44\frac{\text{m}}{\text{s}} = 0.137\text{ m}$$

Problem 1.34 illustrates application of the principle of impulse and momentum and the principle of conservation of energy.

1.35 The motion of a baseball bat in a ballplayer's hands is approximated as a rigid-body motion about an axis through the player's hands, as shown in Figure P1.35. The bat has a centroidal moment of inertia *I*. The player's "bat speed" is ω, and the velocity of the pitched ball is *v*. Determine the distance from the player's hand along the bat where the batter should strike the ball to minimize the impulse felt by his/her hands. Does the distance change if the player "chokes up" on the bat, reducing the distance from *G* to his/her hands?

FIGURE P1.35

Given: I, a, v, ω, m

Find: b

Solution: When the bat strikes the pitched ball, the ball exerts an impulse on the bat, call it B. Since the batter is holding the bat, he feels an impulse, call it P. The effect of the impulse on the bat is to change the "bat speed" from ω before hitting the ball to ω_2 after hitting the ball. Impulse-momentum diagrams of the bat during this time are shown below.

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Applying the principle of linear impulse and linear momentum

$$ma\omega + P - B = ma\omega_2$$
$$B = ma(\omega - \omega_2) + P \qquad (1)$$

Applying the principle of angular impulse and angular momentum about an axis through the batter's hands gives

$$\mathrm{I}\omega + ma^2\omega - Bb = \mathrm{I}\omega_2 + ma^2\omega_2$$
$$B = \frac{1}{b}\left(\mathrm{I} + ma^2\right)(\omega - \omega_2) \qquad (2)$$

Equating B from eqs. (l) and (2) leads to

$$P = (\omega - \omega_2)\left(\frac{\mathrm{I} + ma^2}{b} - ma\right)$$

Note that P = 0 if

$$b = a + \frac{\mathrm{I}}{ma}$$

Problem 1.35 illustrates application of the principle of linear impulse and momentum and angular impulse and momentum. The location where the bat should strike the ball to minimize the impulse felt by the batter is called the center of percussion.

1.36 A playground ride has a centroidal moment of inertia of 17 slug · ft^2. Three children of weights 50 lb, 50 lb, and 55 lb are on the ride, which is rotating at 60 rpm. The children are 30 in. from the center of the ride. A father stops the ride by grabbing it with his hands. What angular impulse is felt by the father?

Given: I = 17 slugs-ft^2, W_1 =50 lb, W_2 = 50 lb, W_3 = 55 lb, r = 20 in, ω = 60 rpm = 6.48 rad/sec

Find: J to stop the ride.

Solution: The father applies an angular impulse about the center of the ride of magnitude J to stop the ride. Consider the impulse and momentum diagrams

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SYSTEM MOMENTA BEFORE FATHER STOPS RIDE | EXTERNAL IMPULSE APPLIED BY FATHER | SYSTEM MOMENTA AFTER FATHER STOPS RIDE

The principle of angular impulse and angular momentum about the center of the ride is

$$\begin{pmatrix}\text{angular momentum}\\ \text{about O before}\\ \text{impulse}\end{pmatrix}+\begin{pmatrix}\text{applied angular}\\ \text{impulse about O}\end{pmatrix}=\begin{pmatrix}\text{angular mometum}\\ \text{about O after}\\ \text{impulse}\end{pmatrix}$$

$$\bar{I}\omega+\frac{W_1}{g}r\omega(r)+\frac{W_2}{g}r\omega(r)+\frac{W_3}{g}r\omega(r)-J=0$$

$$J=\left[\bar{I}+\frac{1}{g}(W_1+W_2+W_3)r^2\right]\omega$$

$$J=\left[17\,\text{sulgs}\cdot\text{ft}^2+\frac{155\text{ 1b}}{32.2\frac{\text{ft}}{\text{s}^2}}(1.667\,\text{ft})^2\right]\left(6.48\,\frac{\text{rad}}{\text{s}}\right)$$

$$=197.1\,\text{N}\cdot\text{sec}\cdot\text{m}$$

Problem 1.36 illustrates application of the principle of angular impulse and angular momentum.

1.37 The natural frequencies of a thermally loaded fixed-fixed beam (Figure P1.37) are a function of the material properties of the beam, including:

E, the elastic modulus of the beam

ρ, the mass density of the beam

α, the coefficient of thermal expansion

The geometric properties of the beam are

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A, its cross-sectional area

I, its cross section moment of inertia

L, its length

Also,

ΔT, the temperature difference between the installation and loading

(a) What are the dimensions involved in each of the parameters?
(b) How many dimensionless parameters does the Buckingham Pi theorem predict are in the non-dimensional formulation of the relation between the natural frequencies and the other parameters?
(c) Develop a set of dimensionless parameters.

FIGURE P1.37

Solution: (a) The dimensions of the parameters are

$$E: F/L^2 \quad \rho: \frac{M}{L^3} = \frac{FT^2}{L^4} \quad \alpha: \frac{L}{\Theta} \quad A: L^2 \quad I: L^4 \quad L: L \quad \Delta T: \Theta \quad \omega_n = \frac{1}{T}$$

where M represents mass, L represents length, T represents time, and Θ represents temperature.

(b) The Buckingham Pi theorem implies that there are n=m-k dimensionless parameters in the formulation where m is the number of dimensional parameters and k is the number of basic dimensions in those variables. There are 8 dimensional parameters and 4 basic dimensions in the parameters which implies there are 4 nondimensional parameters.

(c) Dimensionless parameters are $\Pi_1 = \frac{\varrho A \omega_n^2}{E}$, $\Pi_2 = \frac{\alpha \Delta T}{L}$, $\Pi_3 = \frac{AL^2}{I}$, $\Pi_4 = \frac{A}{L^2}$

Problem 1.37 illustrates application of the Buckingham Pi theorem.

1.38 The drag force F on a circular cylinder due to vortex shedding is a function of

U, the velocity of the flow

μ, the dynamic viscosity of the fluid

ρ, the mass density of the fluid

L, the length of the cylinder

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D, the diameter of the cylinder

(a) What are the dimensions involved in each of the parameters?
(b) How many dimensionless parameters does the Buckingham Pi theorem predict are in the non-dimensional formulation of the relation between the natural frequencies and the other parameters?
(c) Develop a set of dimensionless parameters.

Solution: (a) Dimensions of the parameters are

$$\mathrm{U}: \frac{L}{T} \quad \mu: \frac{FT}{L^2} \quad \rho: \frac{\mathrm{M}}{\mathrm{L}^3} = \frac{\mathrm{FT}^2}{\mathrm{L}^4} \quad L: \mathrm{L} \;\; D: \mathrm{L} \;\; F: \mathrm{F}$$

(b) The Buckingham Pi theorem implies that there are n=m-k dimensionless parameters in the formulation where m is the number of dimensional parameters and k is the number of basic dimensions in those variables. There are 6 dimensional parameters and 3 basic dimensions in the parameters which implies there are 3 nondimensional parameters.

(c) Dimensionless parameters are $\Pi_1 = \frac{\varrho AL}{\mu}$, $\Pi_2 = \frac{D}{L}$, $\Pi_3 = \frac{F}{\rho U^2 A}$

Problem 1.38 illustrates use of the Buckingham Pi Theorem.

1.39 The principal normal stress σ due to forcing of a beam with a concentrated harmonic excitation is a function of

F_0, the amplitude of loading

ω, the frequency of the loading

E, the elastic modulus of the beam

ρ, the mass density of the beam

A, the beam's cross-sectional area

I, the beam's cross-sectional moment of inertia

L, the beam's length

a, the location of the load along the axis of the beam

(a) What are the dimensions involved in each of the parameters?
(b) How many dimensionless parameters does the Buckingham Pi theorem predict are in the non-dimensional formulation of the relation between the natural frequencies and the other parameters?
(c) Develop a set of dimensionless parameters.

Solution: (a) Dimensions of the parameters are

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F_0:F ω: $\frac{1}{T}$ ρ: $\frac{M}{L^3} = \frac{FT^2}{L^4}$ E:$\frac{F}{L^2}$ L: L A: L^2 I:L^2 α:L σ: $\frac{F}{L^2}$

(b) The Buckingham Pi theorem implies that there are n=m-k dimensionless parameters in the formulation where m is the number of dimensional parameters and k is the number of basic dimensions in those variables. There are 9 dimensional parameters and 3 basic dimensions in the parameters which implies there are 6 nondimensional parameters.

(c) Dimensionless parameters are

$$\Pi_1 = \frac{\sigma}{E}, \Pi_2 = \frac{\alpha}{L}, \Pi_3 = \frac{F_0}{\rho I \omega^2}, \Pi_4 = \frac{A}{L^2}, \Pi_3 = \frac{I}{AL^2}, \Pi_6 = \frac{F_0}{EL^2}$$

Problem 1.39 illustrates use of the Buckingham Pi theorem.

1.40 A MEMS system is undergoing simple harmonic motion according to

$$x(t) = [3.1\sin(2\times 10^5 t + 0.48) + 4.8\cos(2\times 10^5 t + 1.74)]\mu\text{m}$$

(a) What is the period of motion? (b) What is the frequency of motion in Hz? (c) What is the amplitude of motion? (d) What is the phase and does it lead or lag? (e) Plot the displacement.

Given: $x(t)$

Find: (a) T (b) f (c) A (d) ϕ

Solution: (a) The period is $T = \frac{2\pi}{2\times 10^5} = 31.4\ \mu s$

(b) The frequency is the reciprocal of the period, $f = \frac{1}{31.4\ \mu s} = 3.183\times 10^4$ Hz.

(c) The amplitude is obtained by writing the response in the form of $x(t) = A\sin(2\times 105t+\phi$. To this end

$$\begin{aligned} 3.1\sin(2\times 10^5 t + 0.48) + 4.8\cos(2\times 10^5 t + 1.74) \\ = 3.1[\sin(2\times 10^5 \mathrm{t})\cos 0.48 + \cos(2\times 10^5 \mathrm{t})\sin 0.48] \\ + 4.8[\cos(2\times 10^5 \mathrm{t})\cos 1.74 - \sin(2\times 10^5 \mathrm{t})\sin 1.74] \end{aligned}$$

$$= (3.1\cos 0.48 - 4.8\sin 1.74)\sin(2\times 10^5 \mathrm{t}) + (3.1\sin 0.48 + 4.8\cos 1.74)\cos(2\times 10^5 \mathrm{t})$$

$$= -1.918\sin(2\times 10^5 \mathrm{t}) + 0.6232\cos(2\times 10^5 \mathrm{t})$$

$$= 2.0774\sin(2\times 10^5 \mathrm{t} - 0.3047)$$

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(d) The amplitude is 2.0774 μm. The phase is -0.3047 rad and is a phase lag.

(e)

Problem 1.40 illustrates simple harmonic motion.

1.41 The force that causes simple harmonic motion in the mass-spring system of Figure P1.31 is $F(t) = 35\sin 100t$ N. The resulting displacement of the mass is $x(t) = 0.002\sin(30t - \pi)$ m. (a) What is the period of the motion? (b) The amplitude of displacement is $X = \frac{F_0}{k}M$ where F_0 is the amplitude of the force and M is a dimensionless factor called the magnification factor. Calculate M. (c) M has the form

FIGURE P1.41

$$M = \frac{1}{\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|}$$

where ω_n is called the natural frequency. If $\omega_n < \omega$, then $\phi = \pi$; otherwise $\phi = 0$. Calculate ω_n.

Given: $F(t), x(t), k = 3.5 \times 10^4$ N/m

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Find: (a) T , (b) M , (c) ω_n

Solution: (a) The period of motion is $T = \frac{2\pi}{30} = 0.2094$ s.

(b) $M = \frac{kX}{F_0} = \frac{\left(3.5\times10^4 \frac{\text{N}}{\text{m}}\right)(0.002\text{ m})}{35\text{ N}} = 2$

(c) Since $\phi = \pi$, $\omega_n < \omega$ and

$$M = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} \Longrightarrow \left(\frac{\omega}{\omega_n}\right)^2 = 1 + \frac{1}{M} = 1.5 \Longrightarrow \omega_n = \frac{\omega}{\sqrt{1.5}} = \frac{30\ \frac{\text{rad}}{\text{s}}}{\sqrt{1.5}} = 24.49\ \frac{\text{rad}}{\text{s}}$$

Problem 1.41 illustrates simple harmonic motion.

1.42 The displacement vector of a particle is

$$\mathbf{r}(t) = [2\sin 20t\ \mathbf{i} + 3\cos 20t\ \mathbf{j}\,]\text{ mm}$$

(a) Describe the trajectory of the particle. (b) How long does it take the particle to make one circuit around the path?

Given: $\mathbf{r}(t)$

Find: path of particle, t

Solution: From the given information $x(t) = 2\sin 20t$ and $y(t) = 3\cos 20t$. Eliminating t between the equations leads to

$$x^2 + \frac{4}{9}y^2 = 1$$

The time it takes to make one circuit around the elliptical path is

$$t = \frac{2\pi}{20} = 0.314\text{ s}$$

Problem 1.42 illustrates the trajectory of a particle undergoing simple harmonic motion in x and y.

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CHAPTER 2: MODELING OF SDOF SYSTEMS

Short Answer Problems

2.1 True: the differential equations are the same because the resultant of gravity and the static spring force is zero for the case of the hanging mass-spring-viscous damper system.

2.2 False: The differential equation governing the motion of a SDOF linear system is second order.

2.3 False: Springs in parallel have an equivalent stiffness that is the sum of the individual stiffnesses of these springs.

2.4 False: The equivalent stiffness of a uniform simply supported beam at its middle is $\frac{48EI}{L^3}$.

2.5 True: Viscous damping is often added to a system to add a linear term in the governing differential equation.

2.6 False: When the equivalent systems method is used to derive the differential equation for a system with an angular coordinate used as the generalized coordinate the kinetic energy is used to derive the equivalent moment of inertia of the system.

2.7 True: The equivalent systems method applied only to linear systems.

2.8 False: The inertia effects of simply supported beam can be approximated by calculating the kinetic energy of the beam in terms of the velocity of the generalized coordinate and placing a particle of appropriate mass at the location whose displacement the generalized coordinate represents.

2.9 False: The static deflection of the spring in the system of Fig. SP2.9 is $\frac{mg}{2k}$.

FIGURE SP 2.09

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2.10 False: The springs in the system of Fig. SP2.10 are in parallel (the springs have the same displacement, x, and the resultant force on the FBD of the block is the sum of the spring forces).

FIGURE SP 2.10

2.11 True: A shaft is an elastic member in which an angular displacement occurs when acted on by a torque. The angular displacement has a value of $\theta = TL/(JG)$.

2.12 True: The equivalent viscous damping coefficient is calculated by comparing the energy dissipation in the combination of viscous dampers to that of an equivalent viscous damper.

2.13 False: The added mass of a fluid entrained by a vibrating system is determined by calculating the kinetic energy developed in the fluid.

2.14 False: If it is desired to calculate the reactions at the support of Fig SP2.14 the effects of the static spring force and gravity cancel and do need to be included on the FBD or in summing forces on the FBD (the cancelling of static spring forces with gravity only applies to the derivation of the differential equation).

FIGURE SP 2.14

2.15 False: Gravity does not cancel with the static spring force in the system of Figure SP2.15 and hence the potential energy of both is included in potential energy calculations. (Assuming small θ the potential energy in the spring is $\frac{1}{2}k\left(\frac{2L}{3}\theta\right)^2$. The potential energy due to gravity assuming the datum is the pin support is $mg\frac{L}{6}\sin\theta$).

FIGURE SP 2.15

2.16 The small angle assumption is used to linearize nonlinear systems a priori. If the angular displacement is small it is assumed that $\sin\theta \approx \theta, \cos\theta \approx 1, \tan\theta \approx \theta$ in derivation of the differential equation.

2.17 FBD's are drawn at an arbitrary instant for derivation of differential equations.

2.18 A quadratic form is form of kinetic energy equal to $\frac{1}{2}m\dot{x}^2$ when used to apply the equivalent systems method to derive a differential equation. The potential energy has a quadratic form of $\frac{1}{2}kx^2$.

2.19 The inertia effects of the spring in a mass-spring-viscous damper system can be approximated by adding a particle of 1/3 the mass of the spring to the point on the system where the spring is attached.

2.20 Each spring in a parallel combination has the same displacement.

2.21 The equivalent stiffness of a combination of springs is calculated by requiring the total potential energy of the combination when written in terms of the displacement of the

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particle where the equivalent spring is to be attached is equal to the potential energy of a spring of equivalent stiffness placed at that location.

2.22 The FBD is shown at an arbitrary instant.

FIGURE SP 2.22

2.23 At an arbitrary instant the upper bar has rotated through an angle θ, measured positive clockwise. The lower bar has an angular displacement ϕ, measure counterclockwise. The displacements of the particles must be the same where the rigid bar is attached, $\frac{L}{2}\theta = \frac{5L}{6}\phi$ or $\phi = \frac{3}{5}\theta$. The FBDs are shown at an arbitrary instant.

FIGURE SP 2.23

2.24 The equivalent systems method is used to derive the differential equation for linear SDOF systems. It can be used to model a linear SDOF system with an equivalent mass-spring-viscous damper model. Using a linear displacement as the generalized coordinate the equivalent mass, the equivalent stiffness, the equivalent damping viscous damping coefficient and the equivalent force are determined using the kinetic energy, potential energy, energy dissipated by viscous dampers and the work done by non-conservative forces.

2.25 Static spring forces not drawn on the FBD of external forces when they cancel with a source of potential energy for a linear system and the generalized coordinate is measured from the system's equilibrium position.

2.26 No, the equivalent systems method cannot be used for a nonlinear system.

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2.27 Given: Springs of individual stiffness's $k_1$and $k_2$placed in series. The equivalent stiffness of the combination is $\frac{1}{\frac{1}{k_1}+\frac{1}{k_2}}$.

2.28 Given: System of Figure SP2.28. The diagrams showing the reduction to a single spring of equivalent stiffness of $\frac{8k}{3}$.

FIGURE SP 2.28

2.29 Given: System of Figure 2.29. The aluminum shaft is in series with the steel shaft (angular displacements add). The stiffness of the aluminum shaft is $k_{Al} = \frac{J_{Al}G_{Al}}{L_{Al}} = \frac{\frac{\pi}{2}(0.02)^4(40\times10^9)}{0.5} = 2.10x10^4$ N·m/rad . The stiffness of the steel shaft is $k_{St} = \frac{J_{St}G_{St}}{L_{St}} = \frac{\frac{\pi}{2}(0.015)^4(80\times10^9)}{0.6} =$ 1.06×10^4 N·m/rad. The equivalent stiffness is $k_{eq} = \frac{1}{1/k_{Al}+1/k_{St}} = 6.94 \times 10^3$ N·m/rad.

FIGURE SP 2.29

2.30 Given: F = 300 N Δx =1 mm. The stiffness of the element is $k = \frac{F}{\Delta x} = \frac{300\text{ N}}{0.001\text{ m}} = 3 \times 10^5$ N/m.

2.31 Given: F=300 N Δx =1 mm. The potential energy is $V = \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}(3 \times$ 105 N/m*2*0.001 m*2=0.15* J.

2.32 Given: F=300 N Δx =1 mm. The potential energy is the same for a compressive force as for a tensile force. The potential energy is $V = \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}(3 \times 10^5\text{ N/m})^2(0.001\text{ m})^2 = 0.15$ J.

2.33 Given: $k_t = 250\text{ N}\cdot\frac{\text{m}}{\text{rad}}$, $\theta = 2°$. The potential energy developed in the spring is $V = \frac{1}{2}k_t\theta^2 = \frac{1}{2}\left(250\text{ N}\cdot\frac{\text{m}}{\text{rad}}\right)\left(2°\frac{2\pi\ rad}{360°}\right)^2 = 0.153$ J.

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2.34 Given: G = 80 × 10^9 N/m^2 L = 2.5 m , r_i = 10 cm, r_o = 15 cm The polar moment of inertia is $J = \frac{\pi}{2}(r_o^4 - r_i^4) = \frac{\pi}{2}[(0.15)^4 - (0.1)^4] = 6.38\text{x}10^{-4}\ \text{m}^4$. The torsional stiffness of the shaft is $k_t = \frac{JG}{L} = \frac{(6.38\times10^{-4}\ \text{m}^4)(80\times10^9\ \text{N/m}^2)}{2.5\ \text{m}} = 2.04 \times 10^7\ \text{N}\cdot\text{m/rad}$.

2.35 Given: G = 40 × 10^9 N/m^2, L = 1.8 m, r = 25 cm. The polar moment of inertia is $J = \frac{\pi}{2}r^4 = \frac{\pi}{2}(0.25)^4 = 6.12 \times 10^{-3}\ \text{m}^4$. The torsional stiffness of the shaft is $k_t = \frac{JG}{L} = \frac{(6.13\times10^{-3}\ \text{m}^4)(40\times10^9\ \text{N/m}^2)}{1.8\ \text{m}} = 1.36 \times 10^8\ \text{N}\cdot\text{m/rad}$.

2.36 Given: E = 200 × 10^9 N/m^2, L = 2.3 m, rectangular cross-section 5 cm × 6 cm. The longitudinal stiffness of the bar is $k = \frac{AE}{L} = \frac{(0.05\ \text{m})(0.06\ \text{m})(200\times10^9\ \text{N/m}^2)}{2.3\ \text{m}} = 2.61 \times 10^8\ \text{N/m}$.

2.37 Given: E = 200 × 10^9 N/m^2, L = 10 μm, beam of rectangular cross section of width 1μm and height 0.5μm. The stiffness of a cantilever beam at its end is $k = \frac{3EI}{L^3} = \frac{3(200\times10^9\ \text{N/m}^2)(1\ \mu m)(0.5\ \mu m)^3/12}{(10\ \mu m)^3} = 6.25$ N/m.

2.38 Given: k = 4000 N/m, m=20 kg. The static deflection of the spring is $\Delta_s = \frac{mg}{k} = \frac{(20\ \text{kg})(9.81\ \text{m/s}^2)}{4000\ \text{N/m}} = 4.91$ cm

2.39 Given: ℓ =10 cm, ρ = 2.3 g/cm, m = 150 g. The mass of the spring is $m_s = \rho\ell = (2.3\ \text{g/cm})\left(\frac{1\ \text{kg}}{1000\ \text{g}}\right)\left(\frac{100\ \text{cm}}{\text{m}}\right)(0.1\ \text{m}) = 0.023\ \text{kg} = 23\ \text{g}$. The added mass is $m_a = \frac{m_s}{3} = 7.67$ g.

2.40 Given: System of Figure SP2.40. The inertia effects of the springs are approximated by adding a particle of mass $m_s/3$ to the center of the disk and a particle of mass $m_s/3$ to the suspended block. The total kinetic energy of the system is $T = T_{disk} + T_{pulley} + T_{block} + T_{spring1} + T_{spring2}$. The kinetic energy of the block and the second spring is $T_{block} + T_{spring2} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2$ The angular displacement of the pulley is $\theta = \frac{x}{r_2}$ and its kinetic energy is $T_{pulley} = \frac{1}{2}I\left(\frac{\dot{x}}{r_2}\right)^2 = \frac{1}{2}\frac{I}{r_2^2}\dot{x}^2$ The displacement of the center of the disk is $y = r_1\theta = \frac{r_1}{r_2}x$. The disk rolls

FIGURE SP2.40

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without slipping, $T_{disk} = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 = \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}\frac{1}{2}m_2\dot{y}^2 = \frac{1}{2}\left(\frac{3}{2}m_2\right)\left(\frac{r_1}{r_2}\dot{x}\right)^2$. The kinetic energy of the first spring is $T_{spring1} = \frac{1}{2}\frac{m_s}{3}\dot{y}^2 = \frac{1}{2}\left[\frac{m_s}{3}\left(\frac{r_1}{r_2}\right)^2\right]\dot{x}^2$. The total kinetic energy of the system is $T = \frac{1}{2}\left[m_1 + \frac{m_s}{3} + \frac{I}{r_2^2} + \frac{3}{2}m_2\left(\frac{r_1}{r_2}\right)^2 + \frac{m_s}{3}\left(\frac{r_1}{r_2}\right)^2\right]\dot{x}^2$.

2.41 Given: System of Figure SP2.41. The work done by the viscous dampers as the system rotates through an angle θ is
$W_{1\to2} = -\int_0^\theta c\left(\frac{2L}{3}\dot{\theta}\right)d\left(\frac{2L}{3}\theta\right) - \int_0^\theta c\left(\frac{L}{3}\dot{\theta}\right)d\left(\frac{L}{3}\theta\right) - \int_0^\theta c\left(\frac{L}{3}\dot{\theta}\right)d\left(\frac{L}{3}\theta\right) = -\int_0^\theta c\left(\frac{2L^2}{3}\dot{\theta}\right)d\theta \Rightarrow c_{teq} = \frac{2L^2}{3}c.$

FIGURE SP2.41

2.42 (a) sin 0.05 = 0.05; (b) cos 0.05 = 1; (c) 1-cos 0.05 = $(0.05)^2/2$ = 0.00125; (d) tan 0.05 = 0.05; (e) cot 0.05 = 1/tan 0.05 = 1/0.05 = 20; (f) sec 0.05 = 1/cos 0.05 = 1; (g) csc 0.05 = 1/sin 0.05 = 20

2.43 (a) sin 3° = 6π/360 = $\pi/60$; (b) cos 3° = 1; (c) 1-cos 3° = $\left(\frac{\pi}{60}\right)^2$; (d) tan 3° = $\pi/60$

2.44 Given: System of Figure 2.44. The kinetic energy of the system is $T = \frac{1}{2}J_1(\dot{\theta}_1)^2 + \frac{1}{2}J_2\left(\frac{n_1}{n_2}\dot{\theta}_1\right)^2 + \frac{1}{2}J_3\left(\frac{n_1n_3}{n_2n_4}\dot{\theta}_1\right)^2 = \frac{1}{2}\left[J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2 + \left(\frac{n_1n_3}{n_2n_4}\right)^2 J_3\right]\dot{\theta}_1 \Rightarrow J_{eq} = J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2 + \left(\frac{n_1n_3}{n_2n_4}\right)^2 J_3$

FIGURE SP 2.44

2.45 (a)-(vi); (b)-(iii); (c)-(iv); (d)-(vii); (e)-(i); (f)-(iv); (g)-(v); (h)-(ii)

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Chapter Problems

2.1 Determine the equivalent stiffness of a linear spring when a SDOF mass-spring model is used for the system shown in Figure P2.1 with x being the chosen generalized coordinate.

FIGURE P 2.1

Given: $L = 2$ m, $E = 200 \times 10^9$ N/m^2, $I = 1.15 \times 10^{-4}$ m^4, m = 20 kg

Find: k_{eq}

Solution: The deflection of a pinned-pinned beam at its midspan is determined using Table D.2 with a = L/2, Z = L/2 as

$$y(Z = L/2) = \frac{L^3}{48EI}$$

The equivalent stiffness is the reciprocal of the deflection,

$$k_{eq} = \frac{48EI}{L^3}$$

$$= \frac{48(20\times10^9 \frac{\text{N}}{\text{m}^2})(1.15\times10^{-4}\,\text{m}^4)}{(2\text{m})^3}$$

$$= 1.38\times10^8 \frac{\text{N}}{\text{m}}$$

Problem 2.1 illustrates the determination of the equivalent stiffness of a structural member.

2.2 Determine the equivalent stiffness of a linear spring when a SDOF mass-spring model is used for the systems shown in Figure P2.2 with x being the chosen generalized coordinate.

FIGURE P 2.2

Given: k, E, I, L

Find: k_{eq}

Solution: The cantilever beam behaves as a linear spring. The displacement of the end of the upper spring and the end of the cantilever beam are the same. Thus the beam is in parallel with the upper spring. The equivalent stiffness of the cantilever beam at its end is

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$$k_b = \frac{3EI}{L^3}$$

Thus the equivalent stiffness of the beam and spring in parallel is

$$k_{eq_1} = \frac{3EI}{L^3} + k$$

The total deflection of the system is the deflection of the beam plus the change in length of the lower spring. Thus the lower spring is in series with the beam and upper spring. Using the equation for a series combination of springs

$$k_{eq} = \frac{1}{\frac{1}{k} + \frac{1}{k_{eq_1}}} = \frac{1}{\frac{1}{k} + \frac{1}{k + \frac{3EI}{L^3}}} = \frac{k\left(k + \frac{3EI}{L^3}\right)}{2k + \frac{3EI}{L^3}}$$

Problem 2.2 illustrates (a) principles for determining parallel and series combination of springs and (b) use of the formulas for series and parallel spring combinations.

2.3 Determine the equivalent stiffness of a linear spring when a SDOF mass-spring model is used for the the system shown in Figure P2.3 with x being the chosen generalized coordinate.

FIGURE P 2.3

Given: Fixed-pinned beam with overhang, dimensions shown

Find: $k_{eq.}$

Solution: The 20 kg machine is placed at A on the beam. Using the displacement of A as the generalized coordinate, the equivalent stiffness is the reciprocal of the displacement at A due to a unit concentrated load at A. From Table D2, with a = 0.6m, z_1 = 1.0 m, the displacement at A due to a unit concentrated load at A is

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$$EIy(z=a)=C_1\frac{a^3}{6}+C_2\frac{a^2}{2}+C_3a+C_4 \tag{1}$$

where

$$C_1=-\frac{3}{2}+\frac{3}{2}\frac{a}{z_1}+\frac{1}{2}\left(1-\frac{a}{z_1}\right)^3=-.568 \tag{2}$$

$$C_2=\frac{z_1}{2}\left(1-\frac{a}{z_1}\right)\left[1-\left(1-\frac{a}{z_1}\right)^2\right]=0.168 \tag{3}$$

$$C_3=0 \tag{4}$$

$$C_4=0 \tag{5}$$

Substituting eqs.(2)-(5) in eq.(l) leads to

$$EIy(z=0.6)=-.538\frac{(0.6)^3}{6}+0.168\frac{(0.6)^2}{2}=.01083$$

Hence the equivalent stiffness is

$$k_{eq.}=\frac{1}{y(z=0.6)}=\frac{1}{\frac{0.01083}{EI}}=92.3EI$$

Problem 2.3 illustrates the concept of equivalent stiffness for a one degree of freedom model of a mass attached to a beam. The equations and entries of Table D2 are used to determine the equivalent stiffness.

2.4 Determine the equivalent stiffness of a linear spring when a SDOF mass-spring model is used for the system shown in Figure P2.4 with x as the chosen generalized coordinate.

FIGURE P 2.4

Given: system shown

Solution: The stiffness of the fixed-free beam is

$$k_1=\frac{3EI}{L^3}=\frac{3(210\times10^9\ \mathrm{N/m^2})(6.1\times10^{-8}\ \mathrm{m^4})}{(2.5\ \mathrm{m})^3}=2.50\times10^5\ \mathrm{N/m}$$

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The stiffness of the pinned-pinned beam is

$$k_2 = \frac{48EI}{L^3} = \frac{48(210 \times 10^9 \text{ N/m}^2)(6.1 \times 10^{-8} \text{ m}^4)}{(2.5 \text{ m})^3} = 3.94 \times 10^6 \text{ N/m}$$

The equivalent stiffness is given by the model shown below. The upper beam acts in series with the upper spring (the displacements of the springs add to given the displacement of the midspan of the simply supported beam). The lower beam acts in series with the middle spring (their displacements add). The upper spring combination acts in parallel with the lower beam-spring combination. Both act in parallel with the spring below the mass. The equivalent stiffness of the upper beam and spring is

$$k_{1,eq} = \frac{1}{\dfrac{1}{3.94 \times 10^6} + \dfrac{1}{6 \times 10^4}} = 7.11 \times 10^4 \text{ N/m}$$

The equivalent stiffness of the lower spring and beam is

$$k_{2,eq} = \frac{1}{\dfrac{1}{2.5 \times 10^5} + \dfrac{1}{1 \times 10^5}} = 5.91 \times 10^4 \text{ N/m}$$

The equivalent stiffness of the combination is

$$k_{eq} = 7.11 \times 10^4 \frac{\text{N}}{\text{m}} + 5.91 \times 10^4 \frac{\text{N}}{\text{m}} + 8 \times 10^4 \frac{\text{N}}{\text{m}} = 2.10 \times 10^5 \text{N/m}$$

Problem 2.4 illustrates the equivalent stiffness of a combination of springs.

2.5 Determine the equivalent stiffness of a linear spring when a SDOF mass-spring model is used for the system shown in Figure P2.5 with x as the chosen generalized coordinate.

FIGURE P 2.5

Given: system shown

Given: k_{eq}

Solution: The potential energy of a spring of equivalent stiffness located at the point whose displacement is x is

$$V = \frac{1}{2} k_{eq} x^2$$

The potential energy of the system, using x as a generalized coordinate, is

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$$V = \frac{1}{2}3kx^2 + \frac{1}{2}k\left(\frac{5}{3}x\right)^2 + \frac{1}{2}k(4x)^2 = \frac{1}{2}\left(\frac{196}{9}k\right)x^2$$

Thus the equivalent stiffness is

$$k_{eq} = \frac{196}{9}k$$

Problem 2.5 illustrates the equivalence of two systems of springs using potential energy.

2.6 Determine the equivalent stiffness of a linear spring when a SDOF mass-spring model is used for the system shown in Figure P2.6 with x as the chosen generalized coordinate.

FIGURE P 2.6

Given: system shown

Given: k_{eq}

Solution: The potential energy of a spring of equivalent stiffness located at the point whose displacement is x is

$$V = \frac{1}{2}k_{eq}x^2$$

The potential energy of the system, using x as a generalized coordinate, is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}2kx^2 + \frac{1}{2}k\left(\frac{x}{3}\right)^2 = \frac{1}{2}\left(\frac{10}{9}k\right)x^2$$

Thus the equivalent stiffness is

$$k_{eq} = \frac{10}{9}k$$

Problem 2.6 illustrates the equivalence of two systems of springs using potential energy.

2.7 Determine the equivalent stiffness of a linear spring when a SDOF mass-spring model is used for the system shown in Figure P2.7 with x as the chosen generalized coordinate.

FIGURE P 2.7

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Given: system shown

Given: k_{eq}

Solution: The potential energy of a spring of equivalent stiffness located at the point whose displacement is x is

$$V = \frac{1}{2}k_{eq}x^2$$

The angular displacement of the upper bar is θ, measured positive clockwise. The angular displacement of the lower bar is ϕ, measured positive counterclockwise. The particles attached to the rigid link have the same displacement

$$\frac{2L}{3}\theta = L\phi$$

Noting that

$$x = \frac{4L}{3}\phi$$

thus

$$\theta = \frac{9}{8L}x$$

The potential energy of the system, using x as a generalized coordinate, is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k\left(\frac{3x}{8}\right)^2 + \frac{1}{2}k\left(\frac{3x}{4}\right)^2 + \frac{1}{2}2k\left(\frac{3x}{8}\right) = \frac{1}{2}\left(\frac{127k}{64}\right)x^2$$

Thus the equivalent stiffness is

$$k_{eq} = \frac{127}{64}k$$

Problem 2.7 illustrates the equivalence of two systems of springs using potential energy.

2.8 Determine the equivalent stiffness of a linear spring when a SDOF mass-spring model is used for the system shown in Figure P2.8 with x as the chosen generalized coordinate.

FIGURE P 2.8

Given: system shown

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Given: k_{eq}

Solution: The potential energy of a spring of equivalent stiffness located at the point whose displacement is x is

$$V = \frac{1}{2}k_{eq}x^2$$

The spring attached to the disk and around the pulley has a displacement of 3x, x from the displacement of the mass center and 2x (assuming no slip between the disk and the surface) from the angular rotation of the disk. The potential energy of the system, using x as a generalized coordinate, is

$$V = \frac{1}{2}3kx^2 + \frac{1}{2}k(3x)^2 = \frac{1}{2}(12k)x^2$$

Thus the equivalent stiffness is

$$k_{eq} = 12k$$

Problem 2.8 illustrates the equivalence of two systems of springs using potential energy.

2.9 Two helical coil springs are made from a steel $(\mathrm{E} = 200 \times 10^9 \mathrm{N/m^2})$ bar of radius 20 mm. One spring has a coil diameter of 7 cm; the other has a coil diameter of 10 cm. The springs have 20 turns each. The spring with the smaller coil diameter is placed inside the spring with the larger coil diameter. What is the equivalent stiffness of the assembly?

Given: $\mathrm{E} = 200 \times 10^9 \mathrm{N/m^2}$ (or $G = 80 \times 10^9\ \mathrm{N/m^2}$), r = 20 mm, $d_1 = 7$ cm, $d_2 = 10$ cm, $N_1 = N_2 = 20$

Find: k_{eq}

Solution: The stiffness of the inner spring is

$$k_1 = \frac{Gd^4}{64Nr^3} = \frac{(80 \times 10^9\ \mathrm{N/m^2})(0.07\ \mathrm{m})^4}{64(20)(0.02)^3} = 1.88 \times 10^8\ \mathrm{N/m}$$

The stiffness of the outer spring is

$$k_1 = \frac{Gd^4}{64Nr^3} = \frac{(80 \times 10^9\ \mathrm{N/m^2})(0.10\ \mathrm{m})^4}{64(20)(0.02)^3} = 7.81 \times 10^8\ \mathrm{N/m}$$

The springs act in parallel, the displacements are the same and the force on the block is the sum of the forces in the springs. Thus

$$k_{eq} = k_1 + k_2 = 1.88 \times 10^8 \frac{\mathrm{N}}{\mathrm{m}} + 7.81 \times 10^8 \frac{\mathrm{N}}{\mathrm{m}} = 9.69 \times 10^8 \frac{\mathrm{N}}{\mathrm{m}}$$

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Problem 2.9 illustrates springs acting in parallel.

2.10 A thin disk attached to the end of an elastic beam has three uncoupled modes of vibration. The longitudinal motion, the transverse motion, and the torsional oscillations are kinematically independent. Calculate the following of Figure P2.10. (a) The longitudinal stiffness; (b) The transverse stiffness; (c) The torsional stiffness

FIGURE P 2.10

Given: L = 65 cm, r = 10 mm, E = 200 × 10^9 N/m^2, G = 80 × 10^9 N/m^2

Find: k_ℓ, k_θ, and k_y

Solution: The geometric properties of the beam are

$$A = \pi r^2 = \pi(0.01\,\text{m})^2 = 3.14\times10^{-4}\ \text{m}^2$$

$$J = \frac{\pi}{2}r^4 = \frac{\pi}{2}(0.01\,\text{m})^4 = 1.57\times10^{-8}\,\text{m}^4$$

$$I = \frac{\pi}{4}r^4 = \frac{\pi}{4}(0.01\,\text{m})^4 = 7.58\times10^{-9}\,\text{m}^4$$

(a) The longitudinal stiffness is

$$k_\ell = \frac{AE}{L} = \frac{\left(3.14\times10^{-4}\ \text{m}^2\right)\left(200\times10^9\ \frac{\text{N}}{\text{m}^2}\right)}{0.65\,\text{m}} = 9.67\times10^7\ \frac{\text{N}}{\text{m}}$$

(b) The transverse stiffness is

$$k_y = \frac{3EI}{L^3} = \frac{3\left(200\times10^9\ \frac{\text{N}}{\text{m}^2}\right)\left(7.85\times10^{-9}\ \text{m}^4\right)}{(0.65\,\text{m})^3} = 1.72\times10^4\ \frac{\text{N}}{\text{m}}$$

(c) The torsional stiffness is

$$k_\theta = \frac{JG}{L} = \frac{\left(1.57\times10^{-8}\,\text{m}^4\right)\left(80\times10^9\ \frac{\text{N}}{\text{m}^2}\right)}{0.65\,\text{m}} = 1930\,\frac{\text{N}\cdot\text{m}}{\text{rad}}$$

Problem 2.10 illustrates three independent modes of vibration of a cantilever beam.

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2.11 Find the equivalent stiffness of the springs in Figure P2.11 in the x direction.

FIGURE P 2.11

Given: springs shown

Find: k_x

Solution: A FBD of the particle at an arbitrary instant is shown

Summing forces on the FBD in the x direction leads to

$$F_x = -4 \times 10^5 x(0.866) - 3 \times 10^5 x(0.707) - 5 \times 10^5 x(0.707) = -9.12 \times 10^5 x$$

Hence the equivalent stiffness in the x direction is

$$k_x = 9.12 \times 10^5 \ \frac{\mathrm{N}}{\mathrm{m}}$$

Problem 2.11 illustrates the determination of an equivalent stiffness when springs act on a particle at different angles.

2.12 A bimetallic strip used as a MEMS sensor is shown in Figure P2.12. The strip has a length of 20 μm. The width of the strip is 1 μm. It has an upper layer made of steel ($E = 210 \times 10^9\ \mathrm{N/m^2}$) and a lower layer made of aluminum ($E = 80 \times 10^9\ \mathrm{N/m^2}$). Each layer is 0.1 μm thick. Determine the equivalent stiffness of the strip in the axial direction.

FIGURE P 2.12

Given: L = 20 μm, w = 1 μm, $E_{st} = 210 \times 10^9\ \mathrm{N/m^2}$, $E_{Al} = 80 \times 10^9\ \mathrm{N/m^2}$, t = 0.1 μm

Find: k_{eq}

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Solution: The two layers behave as longitudinal springs in parallel. The layers have the same displacement and the forces from the layers add. The equivalent stiffness of a longitudinal spring is

$$k = \frac{EA}{L}$$

The strips have the same area and same length. The equivalent stiffness is the sum of the individual stiffnesses thus

$$k_{eq} = (E_{St} + E_{Al})\frac{wt}{L} = \left(210 \times 10^9 \frac{\text{N}}{\text{m}^2} + 80 \times 10^9 \text{ N/m}^2\right)\frac{(1\ \mu\text{m})(0.1\ \mu\text{m})}{20\ \mu\text{m}} = 1450\ \frac{\text{N}}{\text{m}}$$

Problem 2.12 illustrates equivalent stiffness of spring in series.

2.13 A gas spring consists of a piston of area A moving in a cylinder of gas. As the piston moves, the gas expands and contracts, changing the pressure exerted on the piston. The process occurs adiabatically (without heat transfer) so that

$$p = C\rho^{\gamma}$$

where p is the gas pressure, ρ is the gas density, γ is the constant ratio of specific heats, and C is a constant dependent on the initial state. Consider a spring when the initial pressure is p_0 and the initial temperature is T_0. At this pressure, the height of the gas column in the cylinder is h. Let $F = \rho_0 A + \delta F$ be the pressure force acting on the piston when it has displaced a distance x into the gas from its initial height.

(a) Determine the relation between δF and x.

(b) Linearize the relationship of part (a) to approximate the air spring by a linear spring. What is the equivalent stiffness of the spring?

(c) What is the required piston area for an air spring ($\gamma = 1.4$) to have a stiffness of 300 N·m for a pressure of 150 kPa (absolute) with h = 30 cm.

Given: A, p_0, T_0, h, γ (c) k=300 N/m, $p_0 = 150$ kPa, h=0.3 m, $\gamma = 1.4$

Find: (a) δF and x relation (b) k (c) A

Solution: (a) The ideal gas law is used to find the density in the initial state

$$p = \rho RT \Longrightarrow \rho_0 = \frac{p_0}{RT_0}$$

The initial volume of gas in the spring is

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$$V_0 = Ah$$

The total mass of the air is

$$m = \rho_o V_0 = \frac{p_0 A h}{R T_0}$$

When the piston has moved a distance x from its equilibrium position at an arbitrary time

$$V = A(h - x)$$

Since the total mass of the gas is constant the density becomes

$$\rho = \frac{m}{V} = \frac{p_0 h}{R T_0 (h - x)}$$

The initial state is defined by

$$p_0 = C\rho_o^\gamma \Longrightarrow C = \frac{p_0}{\rho_0^\gamma} = p_0^{\gamma-1}(RT_0)^\gamma$$

At an arbitrary time

$$p = p_0^{\gamma-1}(RT_0)^\gamma \left(\frac{p_0 h}{RT_0(h-x)}\right)^\gamma = p_0\left(\frac{h}{h-x}\right)^\gamma$$

(b) The force exerted on the piston is $pA = p_0 A + \delta F$. Thus

$$\delta F = p_0 A\left[\left(\frac{h}{h-x}\right)^\gamma - 1\right]$$

But from a binomial expansion

$$\left(\frac{h}{h-x}\right)^\gamma = h\left(1 - \frac{x}{h}\right)^{-\gamma} = h\left[1 + \frac{x}{h} + O\left(\left(\frac{x}{h}\right)^2\right)\right]$$

Thus

$$k_{eq} = \frac{\delta F}{x} = \frac{\gamma p_0 A}{h}$$

(c) Solving for A and substituting given values

$$A = \frac{k_{eq} h}{\gamma p_0} = \frac{(300\ \mathrm{N/m})(0.3\ \mathrm{m})}{1.4\ (150000\ \mathrm{N/m^2})} = 4.29 \times 10^{-4}\ \mathrm{m^2}$$

Problem 2.13 illustrates the linearized stiffness for an air spring.

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2.14 A wedge is floating stably on an interface between a liquid of mass density ρ, as shown in Figure P2.14. Let x be the displacement of the wedge's mass center when it is disturbed from equilibrium. (a) What is the buoyant force acting on the wedge? (b) What is the work done by the buoyant force as the mass center of the wedge moves from x_1 and x_2? (c) What is the equivalent stiffness of the spring if the motion of the mass center of the wedge is modeled by a mass attached to a linear spring?

FIGURE P 2.14

Given: ρ, ρ_w, r, L, h

Find: F_B, W, linear system

Solution: (a) Consider a free-body diagram of the wedge as it floats in equilibrium on the free surface. Let d be the depth of the wedge into the liquid. In this state the buoyant force must balance with the gravity force

$$F_B - W = 0$$

$$\rho L d r\left(1+\frac{d}{h}\right) = \rho_w g L h r$$

$$\rho_w h = \rho d\left(1+\frac{d}{h}\right)$$

(1)

Now consider the wedge as it oscillates on the free surface. The buoyant force at an arbitrary time is

$$F_B = \rho g L(d+x)\, r\left(1+\frac{d+x}{h}\right)$$

$$= \rho g L r\left[d\left(1+\frac{d}{h}\right)+2\frac{d}{h}x+x+\frac{x^2}{h}\right]$$

(b) The work done by the buoyant force as the center of mass moves between x_1 and x_2 is

$$W_{1\to 2} = \int_{x_1}^{x_2} F_B\, dx =$$

$$\int_{x_1}^{x_2} \rho g L r\left[d\left(1+\frac{d}{h}\right)+2\frac{d}{h}x+x+\frac{x^2}{h}\right] dx$$

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$$W_{1\to 2} = \rho g L r\left[d\left(1+\frac{d}{h}\right)(x_2 - x_1) + \frac{d}{h}\left(x_2^2 - x_1^2\right) + \frac{1}{2}\left(x_2^2 - x_1^2\right) + \frac{1}{3h}\left(x_2^3 - x_1^3\right)\right]$$

(c)The system cannot be modeled as a mass attached to a linear spring. The buoyant force is conservative. However when its potential energy function is formulated, it is not a quadratic function of the generalized coordinate.

Problem 2.14 illustrates the nonlinear oscillations of a wedge on the interface between a liquid and a gas.

2.15 Consider a solid circular shaft of length L and radius c made of an elastoplastic material whose shear stress–shear strain diagram is shown in Figure P2.15(a). If the applied torque is such that the shear stress at the outer radius of the shaft is less than τ_p, a linear relationship between the torque and angular displacement exists. When the applied torque is large enough to cause plastic behavior, a plastic shell is developed around an elastic core of radius $r < c$, as shown in Figure P2.15(b). Let

FIGURE P 2.15

$$T = \frac{\pi \tau_p c^2}{2} + \delta T \tag{1}$$

be the applied torque which results in an angular displacement of

$$\theta = \frac{\tau_p L}{cG} + \delta\theta \tag{2}$$

(a) The shear strain at the outer radius of the shaft is related to the angular displacement

$$\theta = \frac{\gamma_c L}{c} \tag{3}$$

The shear strain distribution is linear over a given cross section. Show that this implies

$$\theta = \frac{L\tau_p}{rG} \tag{4}$$

(b) The torque is the resultant moment of the shear stress distribution over the cross section of the shaft,

$$T = \int_0^c 2\pi\tau\rho^2\, d\rho \tag{5}$$

Use this to relate the torque to the radius of the elastic core.

(c) Determine the relationship between δT and $\delta\theta$.

(d) Approximate the stiffness of the shaft by a linear torsional spring. What is the equivalent torsional stiffness?

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Given: stress-strain diagram, $\tau > \tau_p$

Find: Show eq. (4), linear approximation to stiffness

Solution: (a) The shear stress is linear in the elastic core and at $\rho = r$, $\gamma = \tau_p / G$. The shear strain is linear throughout the cross section. Thus

$$\gamma = \frac{\tau_p \rho}{rG} \tag{6}$$

Then evaluating eq. (6) at $\rho = c$ and using eq. (3)

$$\gamma_c = \frac{\tau_p C}{rG} = \frac{c\theta}{L}$$

$$\theta = \frac{\tau_p L}{rG}$$

(b) The shear stress distribution over the cross section is shown. The resisting torque is the resultant moment of the shear stress distribution. But

$$\tau = \begin{cases} \tau_p \dfrac{\rho}{r}, 0 \le \rho \le r \\ \tau_p, r \le \rho \le c \end{cases}$$

Hence from eq.(5)

$$T = \int_0^r \left(\tau_p \frac{\rho}{r} \right) 2\pi \tau_p \rho^2 \, d\rho + \int_x^c \tau_p 2\pi \rho^2 \, d\rho$$
$$= 2\pi \tau_p \left(\frac{c^3}{3} - \frac{r^3}{12} \right) \tag{7}$$

(c) Equating the torques from eq. (1) and eq. (7)

$$2\pi \tau_p \left(\frac{c^3}{3} - \frac{r^3}{12} \right) = \pi \tau_p \frac{c^3}{2} + \delta T$$

$$\delta T = \pi \frac{\tau_p}{6} \left(c^3 - r^3 \right) \tag{8}$$

$$r = \left(c^3 - \frac{6 \delta T}{\pi \tau_p} \right)^{\frac{1}{3}}$$

Equating the angular displacement. from eqs. (2) and (4)

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$$\frac{\tau_p L}{cG} + \delta\theta = \frac{\tau_p L}{rG} \tag{9}$$

Substituting eq.(8) into eq.(9)

$$\frac{\tau_p L}{cG} + \delta\theta = \frac{\tau_p L}{G\left(c^3 - \frac{6\delta T}{\pi\tau_p}\right)^{\frac{1}{3}}} \tag{10}$$

(d) Note that

$$\left(c^3 - \frac{6\delta T}{\pi\tau_P}\right)^{-\frac{1}{3}} = \frac{1}{c}\left(1 - \frac{6\delta T}{\pi\tau_P c^3}\right)^{-\frac{1}{3}}$$

Then using the binomial theorem assuming small δT and keeping only the first two terms leads to

$$\left(c^3 - \frac{6\delta T}{\pi\tau_P}\right)^{-\frac{1}{3}} = \frac{1}{c}\left(1 + \frac{2\delta T}{\pi\tau_P c^3}\right) \tag{11}$$

Substituting eq.(11) in eq. (10) leads to

$$\frac{\tau_P L}{cG} + \delta\theta = \frac{\tau_P L}{cG}\left(1 + \frac{2\delta T}{\pi\tau_P c^3}\right)$$

or

$$\delta\theta = \frac{2\delta T L}{\pi c^4 G}$$

$$\frac{\delta T}{\delta\theta} = \frac{\pi c^4 G}{2L} = \frac{JG}{L}$$

The above approximation neglected terms involving powers of δT when the binomial expansion was performed. Thus, a linear approximation to the stiffness is the same as the linear stiffness.

Problem 2.15 illustrates a linear approximation to torsional stiffness for an elastoplastic material when the elastic shear stress is exceeded.

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2.16 A bar of length L and cross-sectional area A is made of a material whose stress-strain diagram is shown in Figure P2.16. If the internal force developed in the bar is such that $\sigma < \sigma_p$, then the bar's stiffness for a SDOF model is

$$k = \frac{AE}{L}$$

Consider the case when $\sigma > \sigma_p$. Let $P = \sigma_p A + \delta P$ be the applied load which results in a deflection $\Delta = \frac{\sigma_p L}{E} + \delta\Delta$.

(a) The work done by the applied force is equal to the strain energy developed in the bar. The strain energy per unit volume is the area under the stress–strain curve. Use this information to relate δP to $\delta\Delta$.

FIGURE P 2.16

(b) What is the equivalent stiffness when the bar is approximated as a linear spring for $\sigma > \sigma_p$?

Given: stress-strain curve, δP, E, σ_p

Find: $\delta\Delta = f(\delta P)$, linear stiffness approximation

Solution: The work done by application of a force P, resulting in a deflection Δ is

$$W = \frac{1}{2} P\Delta \tag{1}$$

When the stress exceeds the proportional limit, the work is written as

$$W = \frac{1}{2}(\sigma_P A + \delta P)\left(\frac{\sigma_P L}{E} + \delta\Delta\right)$$

The work is also the area under the P- Δcurve.

$$W = \frac{1}{2}(\sigma_P A)\left(\sigma_P \frac{L}{E}\right) + \int_{\frac{\sigma_P}{E}}^{\frac{\sigma_P}{E}+\frac{\delta\Delta}{L}} ALf(\in)d\in \tag{2}$$

Equating the work from eqs.(1) and (2) leads to

$$\frac{1}{2}\delta P\sigma_P \frac{L}{E} + \frac{1}{2}\sigma_P A\delta A + \frac{1}{2}\delta P\delta\Delta = \int_{\frac{\sigma_P}{E}}^{\frac{\sigma_P}{E}+\frac{\delta\Delta}{L}} ALf(\in)d\in \tag{3}$$

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(b) If δΔ is small, then so is δP. Hence the term with their product is much smaller than the other terms in eq. (3) and is neglected. In addition the mean value theorem is used to approximate the integral

$$\int_{\frac{\sigma_P}{E}}^{\frac{\sigma_P}{E}+\frac{\delta\Delta}{L}} ALf(\in)\ d\in = \frac{\delta\Delta}{L} ALf(\tilde{\in})$$

where

$$\frac{\sigma_P}{E} \le \tilde{\in} \le \frac{\sigma_P}{E} + \frac{\Delta\delta}{E}$$

Then eq. (5) becomes

$$\frac{1}{2}\delta P\sigma_P \frac{L}{E} + \frac{1}{2}\sigma_P A\delta\Delta + \frac{1}{2}\delta P\delta\Delta = A\delta\Delta f(\tilde{\in})$$

Dividing by δΔ leads to

$$\frac{\delta P}{\delta\Delta} = -\frac{AE}{L} + \frac{2AE}{L\sigma_P} f(\tilde{\in})$$

If the limit as δΔ →0 is taken then

$$\tilde{\in} \to \frac{\sigma_P}{E}$$
$$f(\tilde{\in}) \to \sigma_P$$

and

$$\frac{\delta P}{\delta\Delta} \to \frac{AE}{L}$$

Problem 2.16 illustrates the linear approximation to the stiffness when the elastic strength is exceed for a bar undergoing longitudinal oscillations.

2.17 Calculate the static deflection of the spring in the system of Figure P2.17.

FIGURE P 2.17

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Given: k, m_1, m_2, r_1, r_2

Find: Δ_{ST}

Solution: Summing moments about the center of the pulley using the free body diagram of the system when it is equilibrium,

$$\sum M_0 = 0$$
$$= m_1 g r_2 - k\Delta_{ST} r_1$$
$$\Delta_{ST} = \frac{m_1 g r_2}{k r_1}$$

Problem 2.17 illustrates calculation of the static deflection of a spring.

2.18 Determine the static deflection of the spring in the system of Figure P2.18.

FIGURE P 2.18

Given: L = 1.6 m, a = 1.2 m, m = 20 kg, k = 5×10^3 N/m, spring is stretched 20 mm when bar is vertical.

Find: $\Delta_{ST.}$

Solution: A free body diagram of the bar in its static equilibrium position is shown. It is assumed the spring force is horizontal. The equilibrium position is defined by θ_{ST}, the clockwise angle made by the bar with the vertical. Summing moments about the support

$$\sum M_0 = 0$$

leads to

$$-mg\left(a - \frac{L}{2}\right)\sin\theta_{ST.} + k\left(a\sin\theta_{ST.} - \delta\right)a\cos\theta_{ST.} = 0$$

Substituting given values and rearranging leads to

$$\tan\theta_{ST.} = 91.74\sin\theta_{ST.} - 1.53$$

The above equation is solved by trial and error for $\theta_{ST.}$ yielding

$$\theta_{ST.} = 0.965^\circ = 0.0168 \text{ rad}$$

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The static deflection in the spring is given by

$$\Delta_{ST.} = a\sin\theta_{ST.} - \delta = 0.22\text{mm}$$

Problem 2.18 illustrates the application of the equations of equilibrium to determine the static equilibrium position for a given system. The assumption that the spring force is horizontal is good, in light of the result. Equation (1) was solved by trial and error. An alternate method is to approximate $\tan\theta$ by θ and $\sin\theta$ by θ.

2.19 A simplified SDOF model of a vehicle suspension system is shown in Figure P2.19. The mass of the vehicle is 500 kg. The suspension spring has a stiffness of 100,000 N/m. The wheel is modeled as a spring placed in series with the suspension spring. When the vehicle is empty, its static deflection is measured as 5 cm.

(a) Determine the equivalent stiffness of the wheel
(b) Determine the equivalent stiffness of the spring combination

FIGURE P 2.19

Given: m = 500 kg, k_s = 100,000 N/m, δ = 5 cm

Find: (a) k_W (b) k_{eq}

Solution: (a) The wheel is in series with the suspension spring. The force developed in each spring is the same while the total displacement of the series combination is the sum of the displacements of the individual springs. When the system is in equilibrium, the springs are subject to the empty weight of the vehicle. Hence the force developed in each spring is equal to the weight of the vehicle W = mg = (500 kg)(9.81 m/s^2) = 4.905 × 10^3 N. The total displacement in the two springs is 5 cm,

$$\delta_w + \delta_s = 5\text{ cm}$$

But the force developed in a linear spring is kδ. Thus

$$\frac{mg}{k_s} + \frac{mg}{k_w} = 5\text{ cm}$$

Solving for k_w leads to

$$\frac{1}{k_w} = \frac{\delta}{mg} - \frac{1}{k_s} = \frac{0.05\text{ m}}{4.905\times10^3\text{ N}} - \frac{1}{100{,}000\text{ N/m}}$$

$$k_w = 5.16\times10^6\text{ N/m}$$

(b) The equivalent stiffness of the series combination is

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$$k_{eq} = \frac{1}{\frac{1}{k_s} + \frac{1}{k_w}} = \frac{1}{\frac{1}{1\times10^5 \text{ N/m}} + \frac{1}{5.16\times10^6 \text{ N/m}}}$$

$$k_{eq} = 9.63\times10^4 \text{ N/m}$$

Problem 2.19 illustrates the equivalent stiffness of two springs placed in series.

2.20 The spring of the system in Figure P2.20 is unstretched in the position shown. What is the deflection of the spring when the system is in equilibrium?

FIGURE P 2.20

Given: m = 150 kg, k = 2000 N/m,
E = 210 × 10^9 N/m^2, I = 8.2 × 10^{-7} m^4, L = 3 m

Find: Δ_{st}

Solution: The system behaves as two springs in parallel. The beam has the same displacement as the spring. The equivalent stiffness is

$$k_{eq} = k_b + k = \frac{3EI}{L^3} + k = \frac{3(210\times10^9 \text{ N/m}^2)(8.2\times10^{-7} \text{ m}^4)}{(3 \text{ m})^3} + 2000\frac{\text{N}}{\text{m}} = 2.11\times10^4 \text{ N/m}$$

The static deflection of the system is

$$\Delta_{st} = \frac{mg}{k_{eq}} = \frac{(150 \text{ kg})(9.81 \text{ m/s}^2)}{2.11\times10^4 \text{ N/m}} = 6.97 \text{ cm}$$

Problem 2.20 illustrates springs in parallel and static deflection.

2.21 Determine the static deflection of the spring in the system of Figure P2.21.

FIGURE P 2.21

Given: m, k, E, I, L

Find: Δ_{st}

Solution: The system behaves as two springs in parallel. The beam has the same displacement as the spring. The equivalent stiffness is

$$k_{eq} = k_b + k = \frac{48EI}{L^3} + k$$

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The static deflection is

$$\Delta_{st} = \frac{mg}{k_{eq}} = \frac{mg}{\frac{48EI}{L^3} + k} = \frac{mgL^3}{48EI + kL^3}$$

Problem 2.21 illustrates the concepts of springs in parallel and static deflection of springs.

2.22 Determine the static deflections in each of the springs in the system of Figure P2.22.

FIGURE P 2.22

Given: $k_1 = 1 \times 10^5$ N/m, $k_2 = 2 \times 10^5$ N/m, m = 4 kg, a = 0.4 m, b = 0.2 m

Find: Δ_{st1}, Δ_{st2}

Solution: A FBD of the system is shown when the system is in equilibrium

Summing forces on the FBD leads to

$$\sum F = 0 = k_1\Delta_{st1} + k_2\Delta_{st2} - mg$$

Summing moments about the mass center yields

$$\sum M_G = 0 = -k_1\Delta_{st1}(a) + k_2\Delta_{st2}(b)$$

Solution of the equations leads to

$$\Delta_{st1} = \frac{mg}{k_1\left(1 + \frac{a}{b}\right)} = \frac{(4\text{ kg})(9.81\text{ m/s}^2)}{(1 \times 10^5\text{ N/m})\left(1 + \frac{0.4\text{ m}}{0.2\text{ m}}\right)} = 0.131\text{ mm}$$

$$\Delta_{st2} = \frac{mg}{k_2\left(1 + \frac{b}{a}\right)} = \frac{(4\text{ kg})(9.81\text{ m/s}^2)}{(2 \times 10^5\text{ N/m})\left(1 + \frac{0.2\text{ m}}{0.4\text{ m}}\right)} = 0.131\text{ mm}$$

Problem 2.22 illustrates the determination of static deflections from the equations of static equilibrium.

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2.23 A 30 kg compressor sits on four springs, each of stiffness 1×10^4 N/m. What is the static deflection of each spring?

Given: m = 30 kg, $k = 1 \times 10^5$ N/m, n = 4

Find: Δ_{st}

Solution: The compressor sits on four identical springs. Thus the equivalent stiffness of the springs is that of four springs in parallel or

$$k_{eq} = 4k = 4(1 \times 10^5 \text{ N/m}) = 4 \times 10^5 \text{ N/m}$$

The static deflection of the compressor is

$$\Delta_{st} = \frac{mg}{k_{eq}} = \frac{(30 \text{ kg})(9.81 \text{ m/s}^2)}{4 \times 10^5 \text{ N/m}} = 0.736 \text{ mm}$$

Problem 2.23 illustrates the static deflection of a machine mounted on four springs in parallel.

2.24 The propeller of a ship is a tapered circular cylinder, as shown in Figure P2.24. When installed in the ship, one end of the propeller is constrained from longitudinal motion relative to the ship while a 500-kg propeller mass is attached to its other end. (a) Determine the equivalent longitudinal stiffness of the shaft for a SDOF model. (b) Assuming a linear displacement function along the shaft, determine the equivalent mass of the shaft to use in a SDOF model.

Given: $r_0 = 30$ cm, $r_1 = 20$ cm, $E = 210 \times 10^9$ N/m^2, $m_p = 500$ kg,

$\rho = 7350$ kg/m^3, L = 10 m

Find: k_{eq}, m_{eq}

FIGURE P 2.24

Solution: The equivalent system is that of a mass m_{eq} attached to a linear spring of stiffness k_{eq}. The equivalent mass is calculated to include inertia effects of the shaft.

The equivalent stiffness is the reciprocal of the deflection at the end of the shaft due to the application of a unit force. From strength of materials, the change in length of the shaft due to a unit load is

$$\delta = \int_0^L \frac{dx}{AE}$$

Let x be a coordinate along the axis of the shaft, measured from its fixed end. Then

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$$r(x) = r_0 - \frac{r_0 - r_1}{L} x = 0.3 - 0.01x$$

is the local radius of the shaft. Thus

$$\delta = \int_0^L \frac{dx}{\pi (0.3 - 0.01x)^2 E} = \frac{53.05}{E}$$

Hence the equivalent stiffness is

$$k_{eq} = \frac{E}{53.05} = 3.96 \times 10^9 \frac{\text{N}}{\text{m}}$$

Let u(x) represent the displacement of a particle in the cross section a distance x from the fixed end due to a load P applied at the end. From strength of materials

$$u(x) = \int_0^x \frac{P\,dx}{AE} = \int_0^x \frac{P\,dx}{\pi E (0.3 - 0.01x)^2}$$
$$= \frac{x}{0.3 - 0.01x} \frac{10P}{3\pi E}$$

Let z = u(L), then

$$z = \frac{10}{0.2} \frac{10P}{3\pi E} = 50 \frac{10P}{3\pi E}$$
$$\frac{10P}{3\pi E} = \frac{z}{50}$$
$$u(x) = \frac{z}{50} \frac{x}{0.3 - 0.01x}$$

Consider a differential element of mass dm = ρAdx, located a distance x from the fixed end. The kinetic energy of the differential element is

$$dT = \frac{1}{2} \dot{u}^2(x) \rho A(x) dx$$

The total kinetic energy of the shaft is

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$$T = \frac{1}{2}\int_0^L \dot{u}^2(x)\rho A(x)dx$$

$$= \frac{\rho}{2}\int_0^{10m}\left(\frac{\dot{z}}{50}\right)^2\left(\frac{x}{0.3-0.01x}\right)^2 \pi(0.3-0.01)^2\,dx$$

$$= \frac{\pi\rho\dot{z}^2}{5000}\int_0^{10m} x^2\,dx$$

$$= \frac{1}{2}\frac{1000\rho\pi}{3(2500)}\dot{z}^2$$

$$= \frac{1}{2}(3288\,\text{kg})\dot{z}^2$$

Hence the equivalent mass is 3288 kg.

Problem 2.24 illustrates the modeling of a non-uniform structural element using one-degree-of-freedom

2.25 (a) Determine the equivalent torsional stiffness of the propeller shaft of Problem 2.24. (b) Determine an equivalent moment of inertia of the shaft to be placed on the end of the shaft for a SDOF model of torsional oscillations.

FIGURE P 2.24

Given: $r_0 = 30$ cm, $r_1 = 20$ cm, $E = 80 \times 10^9$ N/m^2, $m_p = 500$ kg, $\rho = 7350$ kg/m^3, $L = 10$ m

Find: k_{teq}, I_{eq}

Solution: The equivalent system is that of a disk of moment of inertia I_{eq} attached to a torsional spring of stiffness k_{teq}. The equivalent mass is calculated to include inertia effects of the shaft.

The equivalent stiffness is the reciprocal of the deflection at the end of the shaft due to the application of a unit force. From strength of materials, the change in length of the shaft due to a unit load is

$$\theta = \int_0^L \frac{dx}{JG}$$

Let x be a coordinate along the axis of the shaft, measured from its fixed end. Then

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$$r(x) = r_0 - \frac{r_0 - r_1}{L}x = 0.3 - 0.01x$$

is the local radius of the shaft. Thus the moment of inertia of the shaft is $J(x) = \frac{\pi}{2}r^4(x)$

$$\theta = \int_0^L \frac{2dx}{\pi(0.3-0.01x)^4 G} = \frac{1866}{G}$$

Hence the equivalent stiffness is

$$k_{eq} = \frac{G}{1866} = 4.28\times10^7\ \mathrm{N\cdot m/rad}$$

Let θ(x) represent the displacement of a particle in the cross section a distance x from the fixed end due to a moment M applied at the end. From strength of materials

$$\theta(x) = \int_0^x \frac{M\,dx}{JG} = \int_0^x \frac{2M\,dx}{\pi G(0.3-0.01x)^4}$$
$$= \left(\frac{1}{(0.3-0.01x)^3} - \frac{1}{0.3^3}\right)\frac{20M}{3\pi G}$$

Let z = θ(L), then

$$z = 87.96\frac{20MP}{3\pi G}$$
$$\theta(x) = \frac{z}{87.96}\left(\frac{1}{(0.3-0.01x)^3} - \frac{1}{0.3^3}\right)$$

Consider a differential element of mass, located a distance x from the fixed end. The kinetic energy of the differential element is

$$dT = \frac{1}{2}\dot{\theta}^2(x)\rho J(x)dx$$

The total kinetic energy of the shaft is

$$T = \frac{1}{2}\int_0^L \left(\frac{\dot{z}}{87.96}\right)^2 \left[\frac{1}{(0.3-0.01x)^6} - \frac{2}{0.3^3(0.3-0.01x)^3} + \frac{1}{0.3^6}\right]\rho\frac{\pi}{2}(0.3-0.01x)^4 dx$$

The equivalent moment of inertia is determined from

$$T = \frac{1}{2}I_{eq}\dot{z}^2 = \frac{1}{2}(392.5\ \mathrm{kg\cdot m^2})\dot{z}^2$$

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Problem 2.25 illustrates the modeling of a non-uniform structural element using one-degree-of-freedom

2.26 A tightly wound helical coil spring is made from an 1.88-mm diameter bar made from 0.2 percent hardened steel ($G = 80 \times 10^9$ N/m^2, $\rho = 7600$ kg/m^3). The spring has a coil diameter of 1.6 cm with 80 active coils. Calculate (a) the stiffness of the spring, (b) the static deflection when a 100 g particle is hung from the spring, and ~~(b)~~ (c) the equivalent mass of the spring for a SDOF model.

Given: $G = 80 \times 10^9$ N/m^2, $\rho = 7600$ kg/m^3, D = 1.88 mm, r = 8 mm, N = 80, m = 100 g

Find: (a) Δ_{st} (b) m_{eq}

Solution: The stiffness of the helical coil spring is

$$k = \frac{GD^4}{64Nr^3}$$

$$k = \frac{(80\times10^9\ \text{N/m}^2)(0.00188\ \text{m})^4}{64(80)(0.008\ \text{m})^3}$$

$$k = 381.2\ \text{N/m}$$

When the 100-g particle is hung from the spring its static deflection is

$$\Delta_{st} = \frac{mg}{k} = 3.8\ \text{mm}$$

(b) The total mass of the spring is

$$m_s = \rho(2\pi Nr)\frac{1}{4}\pi D^2$$

$$m_s = 77.8\ \text{g}$$

The equivalent mass of the system is

$$m_{eq} = m + \frac{1}{3}m_s$$

$$m_{eq} = 125.9\ \text{g}$$

Problem 2.26 illustrates (a) the stiffness of a helical coil spring, (b) the static deflection of a spring, and (c) the equivalent mass of a spring used to approximate its inertia effects.

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2.27 One end of a spring of mass m_{s1} and stiffness k_1 is connected to a fixed wall, while the other end is connected to a spring of mass m_{s2} and stiffness k_2. The other end of the second spring is connected to a particle of mass m. Determine the equivalent mass of these two springs.

Given: k_1, m_{s1}, k_2, m_{s2}

Find: m_{eq}

Solution: Let x be the displacement of the block to which the series combination of springs is attached. The inertia effects of the left spring can be approximated by placing a particle of mass $m_{s1}/3$ at the joint between the two springs. Define a coordinate z_1, measured along the axis of the left spring and a coordinate z_2, measured along the axis of the right spring. Let $u_1(z_1)$ be the displacement function the left spring and $u_2(z_2)$ be the displacement function in the right spring. It is assumed that the springs are linear and the displacements are linear,

$$\begin{aligned} u_1(z_1) &= az_1 + b \\ u_2(z_2) &= cz_2 + d \end{aligned} \tag{1}$$

where the constants a, b, c, and d are determined from the following conditions

(a) Since the left end of the left spring is attached to the wall

$$u_1(0) = 0$$

This immediately yields b = 0.

(b) The right end of the right spring is attached to the block which has a displacement x

$$u_2(\ell_2) = x \tag{2}$$

where ℓ_2 is the unstretched length of the right spring.

(c) The displacement is continuous at the intersection between the two springs.

$$u_1(\ell_1) = u_2(0) \tag{3}$$

where ℓ_1 is the unstretched length of the left spring.

(d) Since the springs are in series, the forces developed in the springs must be the same.

$$k_1 u_1(\ell_1) = k_2 [u_2(\ell_2) - u_2(0)] \tag{4}$$

Using eq. (2)-(4) in eq. (l) leads to

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$$a = \frac{x}{\ell_1}\frac{k_2}{k_1+k_2}$$

$$c = \frac{x}{\ell_2}\frac{k_1}{k_1+k_2}$$

$$d = \frac{k_2 x}{k_1+k_2}$$

The kinetic energy of the left spring is

$$T_1 = \frac{1}{2}\frac{m_{s1}}{3}\dot{u}_1^{\,2}(\ell_2) = \frac{1}{2}\frac{m_{s1}}{3}\left(\frac{k_2}{k_1+k_2}\right)^2 \dot{x}^2$$

Thus the contribution to the equivalent mass from the left spring is

$$m_{eq1} = \frac{m_{s1}}{3}\left(\frac{k_2}{k_1+k_2}\right)^2$$

The displacement function in the right spring becomes

$$u_2(z_2) = \frac{x}{k_1+k_2}\left(k_1\frac{z}{\ell_2}+k_2\right)$$

Consider a differential element of length dz_2 in the right spring, a distance z_2 from the spring's left end. The kinetic energy of the element is

$$dT_2 = \frac{1}{2}\frac{m_{s2}}{\ell_2}\dot{u}_2^{\,2}(z_2)\,dz_2$$

The total kinetic energy of the spring is

$$T_2 = \frac{1}{2}\frac{m_{s2}}{\ell_2}\frac{\dot{x}^2}{(k_1+k_2)^2}\int_0^{\ell_2}\left(k_1\frac{z}{\ell_2}+k_2\right)^2 dz_2$$

$$= \frac{1}{2}\frac{m_{s2}}{3}\frac{(k_1+k_2)^3-k_2^3}{k_1(k_1+k_2)^2}$$

Hence the equivalent mass of the series spring combination is

$$m_{eq} = \frac{1}{3(k_1+k_2)^2}\left\{k_2^2 m_{s1} + k_1^2 m_{s2}\left[\left(1+\frac{k_2}{k_1}\right)^3 - \left(\frac{k_2}{k_1}\right)^3\right]\right\}$$

Problem 2.27 illustrates the equivalent mass of springs in series.

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2.28 A block of mass m is connected to two identical springs in series. Each spring has a mass m and a stiffness k. Determine the equivalent mass of the two springs at the mass.

Given: Two identical springs in series

Find: m_{eq}

Solution: Let x be the displacement of the block to which the series combination of springs is attached. The inertia effects of the left spring can be approximated by placing a particle of mass $m_{s1}/3$ at the joint between the two springs. Define a coordinate z_1, measured along the axis of the left spring and a coordinate z_2, measured along the axis of the right spring. Let $u_1(z_1)$ be the displacement function the left spring and $u_2(z_2)$ be the displacement function in the right spring. It is assumed that the springs are linear and the displacements are linear,

$$\begin{aligned} u_1(z_1) &= az_1 + b \\ u_2(z_2) &= cz_2 + d \end{aligned} \tag{1}$$

where the constants a, b, c, and d are determined from the following conditions

(a) Since the left end of the left spring is attached to the wall

$$u_1(0) = 0$$

This immediately yields b = 0.

(b) The right end of the right spring is attached to the block which has a displacement x

$$u_2(\ell) = x \tag{2}$$

where ℓ_2 is the unstretched length of the right spring.

(c) The displacement is continuous at the intersection between the two springs.

$$u_1(\ell) = u_2(0) \tag{3}$$

where ℓ_1 is the unstretched length of the left spring.

(d) Since the springs are in series, the forces developed in the springs must be the same.

$$ku_1(\ell) = k\left[u_2(\ell) - u_2(0)\right] \tag{4}$$

Using eqs. (2)-(4) in eq. (l) leads to

$$u_2(z) = \frac{2x}{3\ell}z + \frac{x}{3}$$

The kinetic energy of the second spring is

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$$T = \frac{1}{2}\int_0^{\ell} \dot{u}_2^2 \frac{m_2}{\ell} dz = \frac{\dot{x}^2 m_s}{18\ell}\int_0^{\ell}\left(\frac{2z}{\ell}+1\right)^2 dz = \frac{1}{2}\left(\frac{13}{27}m_s\right)\dot{x}^2$$

The total kinetic energy is

$$T = \frac{1}{2}\left(\frac{m_s}{3}\right)\left(\frac{\dot{x}}{3}\right)^2 + \frac{1}{2}\left(\frac{13}{27}m_s\right)\dot{x}^2 = \frac{1}{2}\left(\frac{14}{27}m_s\right)\dot{x}^2$$

Thus

$$m_{eq} = \frac{14}{27}m_s$$

Problem 2.28 illustrates the calculation of the equivalent mass of a system.

2.29 Show that the inertia effects of a torsional shaft of polar mass moment of inertia *J* can be approximated by adding a thin disk of moment of inertia *J*/3 at the end of the shaft.

Given: J

Find: I_{eq}

Solution: The angular displacement due to a moment M applied at the end of the shaft varies over the length of the shaft according to

$$\phi = \frac{Mx}{JG}$$

At the end of the shaft $\phi(L) = \theta = \frac{ML}{JG}$. Thus the moment at the end of the shaft is $M = \frac{\theta JG}{L}$ and

$$\phi = \frac{\theta x}{L}$$

The differential element of the shaft is $dI = \frac{J}{L}dx$ where J is the polar mass moment of inertia of the shaft. The kinetic energy is

$$T = \frac{1}{2}\int_0^{l} \dot{\phi}^2 dI = \frac{1}{2}\int_0^{L}\left(\frac{\theta\dot{x}}{L}\right)^2 \frac{J}{L}dx = \frac{1}{2}\frac{J}{3}\dot{x}^2$$

The kinetic energy of the shaft has the form $T = \frac{1}{2}I_{eq}\dot{\theta}^2$. Hence

$$I_{eq} = \frac{J}{3}$$

Problem 2.29 illustrates the equivalent moment of inertia of a shaft using a SDOF model of the shaft.

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2.30 Use the static displacement of a simply supported beam to determine the mass of a particle that should be added at the midspan of the beam to approximate inertia effects in the beam.

Given: m = 20 kg, m_b = 12 kg, E = 200 × 10^9 N/m^2,
I = 1.15 × 10^{-4} m^4, L = 2 m

Find: m_{eq}

Solution: the inertia effects of the beam are approximated by placing a particle of appropriate mass at the location of the block. The mass of the particle is determined by equating the kinetic energy of the beam to the kinetic energy of a particle placed at the location of the block. The kinetic energy of the beam is approximated using the static beam deflection equation. For a pinned-pinned beam, the deflection equation valid between the left support and the location of the block is obtained using Table D.2. In using Table D.2, set a = L/2. Note that Table D.2 gives results for unit loads which can be multiplied by the magnitude of the applied load to attain the deflection due to any concentrated load. Thus the deflection of a pinned-pinned beam due to a concentrated load P applied at a = L/2 is

$$y(z)=\frac{P}{EI}\left(-\frac{z^3}{12}+\frac{zL^2}{16}\right)$$

Let w be the deflection of the block, located at z = L/2. Thus

$$w=y(L/2)=\frac{PL^3}{48EI}$$

$$\frac{P}{EI}=\frac{48z}{L^3}$$

Hence

$$y(z)=\frac{wz}{L}\left(3-4\frac{z^2}{L^2}\right)$$

Consider a differential element of mass dm = ρAdz. The kinetic energy of the differential mass is

$$dT_b=\frac{1}{2}\dot{y}^2(z)\rho A dz$$

Since the beam is symmetric about its midspan the kinetic energy of the mass to the right of the midspan is equivalent to the kinetic energy of the mass to the left of the midspan. Thus the total kinetic energy of the beam is

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$$T_b = 2\int_0^{L/2} dT_b$$

$$= \frac{1}{2} 2\rho A \int_0^{\frac{L}{2}} \left(\frac{\dot{w}z}{L}\right)^2 \left(3 - 4\frac{z^2}{L^2}\right)^2 dz$$

Evaluation of the integral yields

$$T_b = \frac{1}{2}(0.492\rho AL)w^2 = \frac{1}{2}(0.492\, m_b)w^2$$

Hence the equivalent mass is

$$m = m + 0.486\, m_b$$

Problem 2.30 illustrates determination of the equivalent mass of a pinned-pinned beam.

2.31 Determine the equivalent mass or equivalent moment of inertia of the system shown in Figure P2.31 when the indicated generalized coordinate is used.

FIGURE P 2.31

Given: x, m, r

Find: m_{eq}

Solution: The kinetic energy of the system is the kinetic energy of the hanging block plus the kinetic energy of the sphere. The velocity of the mass center of the sphere is related to the velocity of the block by

$$v_s = \frac{\dot{x}}{2}$$

The total kinetic energy of the system assuming no slip between the sphere and the surface ($v_s = r_s\omega$) and knowing that the moment of inertia of a sphere is $\frac{2}{5}mr_s^2$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left(\frac{\dot{x}}{2}\right)^2 + \frac{1}{2}\left(\frac{2}{5}mr_s^2\right)\left(\frac{\dot{x}}{2r_s}\right) = \frac{1}{2}\left(\frac{27}{20}m\right)\dot{x}^2$$

The kinetic energy of the system is related to the equivalent mass by $T = \frac{1}{2}m_{eq}\dot{x}^2$. Thus

$$m_{eq} = \frac{27}{20}m$$

Problem 2.31 illustrates the equivalent mass of a system.

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2.32 Determine the equivalent mass or equivalent moment of inertia of the system shown in Figure P2.32 when the indicated generalized coordinate is used.

FIGURE P 2.32

Given: x, m, L

Find: m_{eq}

Solution: The total kinetic energy of the system is

$$T = \frac{1}{2}2m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2$$

where y is the displacement of the cart of mass m, z is the displacement of the mass center of the bar and θ measures the angular rotation of the bar. Kinematics is employed to obtain that if x is the displacement of the cart of mass 2m then assuming small θ

$$\frac{2L}{3}\theta = x$$

$$\frac{L}{3}\theta = y = \frac{x}{2}$$

$$\frac{L}{6}\theta = z = \frac{x}{4}$$

Thus the kinetic energy becomes noting that $I = \frac{1}{12}mL^2$

$$T = \frac{1}{2}2m\dot{x}^2 + \frac{1}{2}m\left(\frac{\dot{x}}{2}\right)^2 + \frac{1}{2}m\left(\frac{\dot{x}}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{3\dot{x}}{2L}\right)^2 = \frac{1}{2}\left(\frac{5m}{2}\right)\dot{x}^2$$

The kinetic energy of the system is related to the equivalent mass by $T = \frac{1}{2}m_{eq}\dot{x}^2$. Thus

$$m_{eq} = \frac{5}{2}m$$

Problem 2.32 illustrates the equivalent mass of a SDOF system.

2.33 Determine the equivalent mass or equivalent moment of inertia of the system shown in Figure P2.33 when the indicated generalized coordinate is used.

FIGURE P 2.33

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Given: m, L, θ

Find: I_{eq}

Solution: The relative velocity equation is used to relate the angular velocity of bar BC and the velocity of the collar at C to the angular velocity of bar AB.

$$\mathbf{v}_B = \dot{\theta}\mathbf{k} \times L(\cos\theta\ \mathbf{i} + \sin\theta\,\mathbf{j}) = -L\dot{\theta}\sin\ \theta\,\mathbf{i} + L\dot{\theta}\cos\theta\,\mathbf{j}$$

$$\begin{aligned}\mathrm{v}_c = \mathbf{v}_B + \omega_{BC}\mathbf{k} \times \frac{L}{2}(\cos\beta\,\mathbf{i} - \sin\beta\,\mathbf{j}) \\ = \left(-L\dot{\theta}\sin\ \theta + \omega_{BC}\frac{L}{2}\sin\beta\right)\mathbf{i} + \left(L\dot{\theta}\cos\theta + \omega_{BC}\frac{L}{2}\cos\beta\right)\mathbf{j}\end{aligned}$$

The law of sines is used to determine that

$$\sin\beta = 2\sin\theta$$

Then

$$\cos\beta = \sqrt{1 - 4\sin^2\theta}$$

Setting the j component to zero leads to

$$\omega_{BC} = \frac{2\dot{\theta}\cos\theta}{\cos\beta}$$

The x component leads to

$$v_C = -L\dot{\theta}\sin\ \theta + \omega_{BC}\frac{L}{2}\sin\beta = L\dot{\theta}(-\sin\theta + \cos\theta\tan\beta)$$

The relative velocity equation is used between particle B and the mass center of bar BC leading to

$$\bar{\mathbf{v}}_{BC} = \left(-L\dot{\theta}\sin\ \theta + \omega_{BC}\frac{L}{4}\sin\beta\right)\mathbf{i} + \left(L\dot{\theta}\cos\theta + \omega_{BC}\frac{L}{4}\cos\beta\right)\mathbf{j}$$

The kinetic energy of the system is

$$\begin{aligned}T = \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 \\ + \frac{1}{2}m\left[\left(-L\dot{\theta}\sin\ \theta + \omega_{BC}\frac{L}{4}\sin\beta\right)^2 + \left(L\dot{\theta}\cos\theta + \omega_{BC}\frac{L}{4}\cos\beta\right)^2\right] \\ + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left[L\dot{\theta}(-\sin\theta + \cos\theta\tan\beta)\right]^2\end{aligned}$$

The equivalent moment of inertia is calculated for a linear system by $T = \frac{1}{2}I_{eq}\dot{\theta}^2$. This system is linear only for small θ.

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Problem 2.33 illustrates that the concept of equivalent mass does not work for nonlinear systems.

2.34 Determine the equivalent mass or equivalent moment of inertia of the system shown in Figure 2.34 when the indicated generalized coordinate is used.

FIGURE P 2.34

Given: system shown

Find: I_{eq}

Solution: The total kinetic energy of the system is

$$T = \frac{1}{2}m\bar{v}_{AB}^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 + \frac{1}{2}m\bar{v}_{CD}^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\phi}_{CD}{}^2 + \frac{1}{2}mv_E^2$$

where ϕ is the angle made by the lower bar with the horizontal. The displacement of the particle on the upper bar that is connected to the rigid link in the same as the displacement of the lower bar that is connected to the link

$$L\theta = \frac{4L}{5}\phi \Longrightarrow \phi = \frac{5}{4}\theta$$

Substituting into the kinetic energy leads to

$$T = \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 + \frac{1}{2}m\left[\frac{L}{2}\left(\frac{5}{4}\dot{\theta}\right)\right]^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{5}{4}\dot{\theta}\right)^2 + \frac{1}{2}m\left[\frac{L}{3}\left(\frac{5}{4}\dot{\theta}\right)\right]^2$$
$$= \frac{1}{2}\left(\frac{37}{36}mL^2\right)\dot{\theta}^2$$

The equivalent moment of inertia when θ is used as the generalized coordinate is

$$I_{eq} = \frac{37}{36}mL^2$$

Problem 2.34 illustrates calculation of an equivalent moment of inertia.

2.35 Determine the equivalent mass or equivalent moment of inertia of the system shown in Figure P2.35 when the indicated generalized coordinate is used.

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Given: shafting system with rotors

Find: J_{eq}

Given: The relation between the angular velocities of the shafts is given by the gear equation

FIGURE P 2.35

$$\omega_2 = \frac{n_1}{n_2}\omega_1$$

$$\omega_3 = \frac{n_3}{n_4}\omega_2 = \frac{n_3}{n_4}\frac{n_1}{n_2}\omega_1$$

The kinetic energy of the shafting system is

$$T = \frac{1}{2}(J_1 + J_{G1})\omega_1^2 + \frac{1}{2}(J_{G2} + J_r + J_{G3})\left(\frac{n_1}{n_2}\omega_1\right)^2 + \frac{1}{2}J_{G4}\left(\frac{n_3}{n_4}\frac{n_1}{n_2}\omega_1\right)^2$$
$$= \frac{1}{2}\left[(J_1 + J_{G1}) + (J_{G2} + J_r + J_{G3})\left(\frac{n_1}{n_2}\right)^2 + J_{G4}\left(\frac{n_3}{n_4}\frac{n_1}{n_2}\right)^2\right]\omega_1^2$$

The equivalent moment of inertia is

$$J_{eq} = (J_1 + J_{G1}) + (J_{G2} + J_r + J_{G3})\left(\frac{n_1}{n_2}\right)^2 + J_{G4}\left(\frac{n_3}{n_4}\frac{n_1}{n_2}\right)^2$$

Problem 2.35 illustrates calculation of an equivalent moment of inertia of a shafting system.

2.36 Determine the kinetic energy of the system of Figure P2.36 at an arbitrary instant in terms of $\dot{x}$ including inertia effects of the springs.

FIGURE P 2.36

Given: system shown with x as generalized coordinate

Find: T

Solution: Let θ be the clockwise angular displacement of the pulley and let x_1 be the displacement of the center of the disk, both measured from the equilibrium position of the system. Inertia effects of a spring are approximated by imagining a particle of one-third of

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the mass of the spring at the location where the spring is attached to the system. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_p\dot{\theta}^2 + \frac{1}{2}2m\dot{x}_1^2 + \frac{1}{2}\frac{1}{2}2mr_D^2\omega_D^2 + \frac{1}{2}\frac{1}{3}m_s\dot{x}^2 + \frac{1}{2}\frac{1}{3}m_s\dot{x}_1^2$$

Kinematics leads to

$$\theta = \frac{x}{2r}$$

$$x_1 = \frac{x}{2}$$

Since the disk rolls without slip

$$\omega_D = \frac{\dot{x}_1}{r_D} = \frac{\dot{x}}{2r_D}$$

Substitution into the expression for kinetic energy leads to

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_p\left(\frac{\dot{x}}{2r}\right)^2 + \frac{1}{2}2m\left(\frac{\dot{x}}{2}\right)^2 + \frac{1}{2}\frac{1}{2}2mr_D^2\left(\frac{\dot{x}}{2r_D}\right)^2$$

$$+\frac{1}{2}\frac{1}{3}m_s\dot{x}^2 + \frac{1}{2}\frac{1}{3}m_s\left(\frac{\dot{x}}{2}\right)^2$$

$$T = \frac{1}{2}\left(\frac{7}{4}m + \frac{I_p}{4r^2} + \frac{1}{4}m_s\right)\dot{x}^2$$

Problem 2.36 illustrates the determination of the kinetic energy of a one-degree-of-freedom system at an arbitrary instant in terms of a chosen generalized coordinate and the approximation for inertia effects of springs.

2.37 The time-dependent displacement of the block of mass m of Figure P2.36 is $x(t) = 0.03e^{-1.35t}\sin(4t)$ m. Determine the time-dependent force in the viscous damper if c = 125 N·s/m.

FIGURE P 2.36

Given: x(t), c = 125 N·s/m

Find: F

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Solution: The viscous damper is attached to the center of the disk. If x_1 is the displacement of the center of the disk, then kinematics leads to $x_1 = x/2$. The force developed in the viscous damper is

$$F = c\dot{x}_1 = \frac{c}{2}\dot{x}$$

$$F = \frac{c}{2}\left[(0.03)e^{-1.35t}(-1.35\sin(4t) + 4\cos(4t))\right]$$

$$F = \frac{125 \text{ N - s/m}}{2}(0.03)e^{-1.35t}(-1.35\sin(4t) + 4\cos(4t))$$

$$F = 1.875e^{-1.35t}(-1.45\sin(4t) + 4\cos(4t)) \text{ N}$$

Problem 2.37 illustrates the force developed in a viscous damper.

2.38 Calculate the work done by the viscous damper of Problem 2.37 between $t = 0$ and $t = 1$ s.

FIGURE P 2.36

Given: x(t), c=125 N-s/m, 0 < t < 1 s

Find: W

$$F = 1.875e^{-1.35t}(-1.45\sin(4t) + 4\cos(4t)) \text{ N}$$

Solution: The time dependent force in the viscous damper is determined in Chapter Problem 2.37 as
The work done by the force is

$$W = -\int F(t)\, dx_1$$

where x_1 is the displacement of the point in the system where the viscous damper is attached. It is noted that

$$x_1(t) = \frac{1}{2}x(t) = 0.015e^{-1.35t}\sin 4t \text{ m}$$

Using the chain rule for differentials

$$dx_1 = \frac{dx_1}{dt}dt = \dot{x}_1 dt$$

It is noted that $F = c\dot{x}$. Thus

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$$W = -\int c\dot{x}_1^2 \, dt$$

$$W = -\int_0^1 0.0281 e^{-2.7t} \sin^2(4t) \, dt$$

$$W = -0.004211 \text{ N - m}$$

Problem 2.38 illustrates the work done by a viscous damping force.

2.39 Determine the torsional viscous-damping coefficient for the torsional viscous damper of Figure P2.39. Assume a linear velocity profile between the bottom of the dish and the disk.

FIGURE P 2.39

Given: θ, h, ρ, μ

Find: c_t

Solution: Assume the disk is rotating with an angular velocity $\dot{\theta}$. The velocity of a particle on the disk, a distance r away from the axis of rotation is

$$v = r\dot{\theta}$$

Solution: Assume the disk is rotating with an angular velocity $\dot{\theta}$. The velocity of a particle on the disk, a distance r away from the axis of rotation is

$$v = r\dot{\theta}$$

A velocity gradient exists in the fluid due to the rotation of the plate. Assume the depth of the plate is small enough such that the fluid velocity profile is linear between the bottom of the dish and the disk. The no-slip condition implies that a fluid particle adjacent to the disk, a distance r from the center of rotation has a velocity $r\theta$ while a fluid particle adjacent to the bottom of the dish has zero velocity. Hence the velocity gradient is

$$\frac{dv}{dy} = \frac{r\dot{\theta}}{h}$$

The velocity gradient leads to a shear stress from the fluid on the dish. The shear stress is calculated using Newton's viscosity law as

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$$\tau = \mu \frac{dv}{dy} = \frac{\mu r \dot{\theta}}{h}$$

The resisting moment acting on the disk due to the shear stress distribution is

$$M = \int \tau\, r\, dA = \int_0^{2\pi} \int_0^{R} \tau\, r \left(r\, dr d\theta \right)$$

$$= \int_0^{2\pi} \int_0^{R} \frac{\mu \dot{\theta}}{h} r^3\, dr d\theta$$

$$= \frac{\pi \mu R^4}{2h} \dot{\theta}$$

Hence the torsional damping coefficient is

$$C_t = \frac{\pi \mu R^4}{2h}$$

Problem 2.39 illustrates a type of torsional viscous damper.

2.40 Determine the torsional viscous-damping coefficient for the torsional viscous damper of Figure P2.40. Assume a linear velocity profile in the liquid between the fixed surface and the rotating cone.

FIGURE P 2.40

Given: h, d, r, ρ, μ

Find: c_t

Solution: Let y be a coordinate measured from the tip of the cone, positive upward. Assume the cone is rotating with an angular velocity $\dot{\theta}$. The velocity of a particle on the outer surface of the cone is

$$v = R(y)\dot{\theta}$$

where R(y) is the distance from the surface to the axis of the cone. From geometry

$$R(y) = \frac{ry}{h}$$

Hence,

$$v = \frac{ry\dot{\theta}}{h}$$

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Assume that d is small enough such that the velocity distribution in the fluid is linear. Let z be a coordinate normal to the surface of the cone. Then using the no-slip condition between the fluid and the cone's surface and between the fluid and the fixed surface gives

$$v(z) = \frac{ry\dot{\theta}}{h}\frac{z}{d}$$

The velocity gradient produces a shear stress on the surface of the cone. Using Newton's viscosity law

$$\tau = \mu\frac{dv}{dz} = \frac{\mu ry\dot{\theta}}{hd}$$

Consider a differential slice of the cone of thickness dy. The shear stress acts around the surface of the slice, causing a resisting moment about the center of the cone of

$$dM = y\left(2\pi R(y)\right)\tau\, dy$$
$$= \frac{2\pi r^2 \mu\dot{\theta}\, y^3}{h^2 d}dy$$

Thus the total resisting moment is

$$M = \int dM = \frac{2\pi r^2 \mu\dot{\theta}}{h^2 d}\int_0^h y^3 dy$$
$$= \frac{\pi r^2 \mu h^2}{2d}\dot{\theta}$$

Hence the torsional viscous damping coefficient for this configuration is

$$c_t = \frac{\pi r^2 \mu h^2}{2d}$$

Problem 2.40 illustrates determination of the torsional viscous damping coefficient for a specific configuration.

2.41 Shock absorbers and other forms of viscous dampers use a piston moving in a cylinder of viscous liquid as illustrated in Figure P2.41. For this configuration the force developed on the piston is the sum of the viscous forces acting on the side of the piston and the force due to the pressure difference between the top and bottom surfaces of the piston.

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(a) Assume the piston movers with a constant velocity v_p. Draw a free-body diagram of the piston and mathematically relate the damping force, the viscous force, and the pressure force.

FIGURE P 2.41

(b) Assume steady flow between the side of the piston and the side of the cylinder. Show that the equation governing the velocity profile between the piston and the cylinder is

$$\frac{dp}{dx} = \mu \frac{\partial^2 v}{\partial r^2} \tag{1}$$

(c) Assume the vertical pressure gradient is constant. Use the preceding results to determine the velocity profile in terms of the damping force and the shear stress on the side of the piston.

(d) Use the results of part (c) to determine the wall shear stress in terms of the damping force.

(e) Note that the flow rate between the piston and the cylinder is equal to the rate at which the liquid is displaced by the piston. Use this information to determine the damping force in terms of the velocity and thus the damping coefficient.

(f) Use the results of part (e) to design a shock absorber for a motorcycle that uses SAE 1040 oil and requires a damping coefficient of 1000 N·m/s.

Given: v_p, d, D, h, μ, ρ, (f) SAE 1040 oil, c = 1000 N·m/s
Find: (a) - (e) c_{eq}, (f) design damper

Solution: (a) The free body diagram of the piston at an arbitrary instant shown below illustrates the pressure force acting on the upper top and bottom surfaces of the piston, the viscous force which is the resultant of the shear stress distribution acting around the circumference of the piston, and the reaction force in the piston rod.

Assuming the inertia force of the piston is small, summation of forces acting on the piston leads to

$$F = F_{p\ell} - F_{pu} + F_v$$

where

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$$F_{p\ell} - F_{pu} = (p_\ell - p_u)\pi \frac{D^2}{4}$$
$$F_v = \tau_w \pi Dh$$

Hence

$$F = (p_\ell - p_u)\pi \frac{D^2}{4} + \tau_w \pi Dh$$

(b) Consider a differential ring of height dx and thickness dr, a distance r from the center of the position. Consider a free body diagram of the element

Summation of forces acting of the element leads to

$$\left(p + \frac{dp}{dx}dx - p\right)(2\pi r\, dr) + \left(\tau + \frac{\partial \tau}{\partial r}dr - \tau\right)(2\pi r\, dx) = 0$$
$$\frac{dp}{dx} = -\frac{\partial \tau}{\partial r}$$

If the fluid is Newtonian

$$\tau = -\mu \frac{\partial v}{\partial r}$$

where v(r, x) is the velocity distribution in the fluid. Thus

$$\frac{dp}{dx} = \mu \frac{\partial^2 v}{\partial r^2}$$

(c) Assume dp/dx = C, a constant. Then from the preceding equation

$$v = \frac{c}{2\mu} r^2 + c_1 r + c_2$$

where c_1 and c_2 are constants of integration. The boundary conditions are

$$v(R = D/2) = v$$
$$v(R + d) = 0$$

Application of the boundary conditions leads to

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$$c_1 = -\frac{v}{d} + \frac{C}{2\mu}(2R + d)$$

$$c_2 = v\left(1 + \frac{R}{d}\right) + \frac{C}{2\mu}\left(R^2 + Rd\right)$$

Using Newton's viscosity law

$$\tau_w = -\mu\frac{dv}{dr}(r = R) = \mu\frac{v}{d} + C\frac{d}{2}$$

$$C = \frac{2}{d}\left(\tau_w - \mu\frac{v}{d}\right)$$

Note that since the pressure is constant

$$\frac{dp}{dx} = \frac{p_\ell - p_u}{h}$$

Hence the damping force becomes

$$F = \tau_w \pi D h\left(1 + \frac{D}{2d}\right) - \frac{\mu v \pi D^2 h}{2d^2}$$

(d) Note that the flow rate must be equal to the velocity of the piston times the area of the piston

$$Q = \pi\frac{D^2}{4}v$$

The flow rate is also calculated by

$$Q = \int_R^{R+d} v(r)\, 2\pi r\, dr$$

$$= 2\pi\left[\frac{1}{\mu d}\left(\tau_w - \mu\frac{v}{d}\right)\left(-\frac{1}{6}Rd^3 + \frac{2}{3}d^4\right) + \frac{1}{6}vd^2\right]$$

Equating Q from the previous two equations and solving for the wall shear stress leads to

$$\tau_w = \frac{\mu v\left(3D^2 - 2dD - 12d^2\right)}{2d^2(D - 8d)}$$

and leads to

$$F = \frac{\mu D \pi h\left(3D^3 - dD^2 - 24d^3\right)}{4d^3(D - 8d)}v$$

which leads to the damping coefficient

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$$c = \frac{\mu \pi D h \left(3D^3 - D^2 d - 24 d^3\right)}{4 d^3 \left(D - 8d\right)}$$

If D>>d, the preceding equation is approximated by

$$c = \frac{3\pi \mu h D^3}{4 d^3}$$

Corrections to the above equation in powers of d/D can be obtained by expanding the reciprocal of the denominator in powers of d/D using a binomial expansion, multiplying by the numerator, simplifying and collecting coefficients on like powers of d/D.

(e) The viscosity of SAE 1040 oil is approximately 0.4 N·s / m^2

Assume h = 0.5 mm and d = 10 mm. Then setting c = 1000 N·s/m and assuming D >> d leads to

$$1000 = \frac{0.4\pi(0.0005)}{4(0.01)^3}\left(3D^3\right)$$
$$D = 0.374 \text{ m}$$

Problem 2.41 illustrates (a) the derivation of the viscous damping coefficient for a piston-cylinder dashpot, and (b) the use of the equation for the viscous damping coefficient to design a viscous damper for a given situation.

2.42 Derive the differential equation governing the motion of the one degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinates shown in Figure P2.42. Linearize nonlinear differential equations by assuming small displacements.

FIGURE P 2.42

Given: x as generalized coordinate, m, k

Find: differential equation

Solution: Free-body diagrams of the system at an arbitrary time are shown below.

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Summing forces acting on the block

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$

gives

$$-kx - 2kx = m\ddot{x}$$
$$m\ddot{x} + 3kx = 0$$
$$\ddot{x} + \frac{3k}{m}x = 0$$

Problem 2.42 illustrates application of Newton's law to derive the differential equation governing free vibration of a one-degree-of-freedom system.

2.43 Derive the differential equation governing the motion of the one degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinates shown in Figure P2.43. Linearize nonlinear differential equations by assuming small displacements.

FIGURE P 2.43

Given: x as generalized coordinate, k, m, I, r

Find: differential equation

Solution: Since x is measured from the system's equilibrium position, gravity cancels with the static spring forces in the governing differential equation. Thus, for purposes of deriving the differential equation, both are ignored. It is assumed there is no slip between the cable and the pulley. Thus the angular rotation of the pulley is kinematically related to the displacement of the block by

$$\theta = \frac{x}{2r}$$

Free-body diagrams of the system are shown below at an arbitrary instant.

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Summing moments about the center of the pulley

$$\left(\sum M_c\right)_{ext} = \left(\sum M_c\right)_{eff}$$

leads to

$$-\frac{1}{2}kx(r) = m\ddot{x}(2r) + \frac{I}{2r}\ddot{x}$$

$$\left(2rm + \frac{I}{2r}\right)\ddot{x} + \frac{1}{2}krx = 0$$

$$\ddot{x} + \frac{k}{2\left(2m + \frac{I}{2r^2}\right)}x = 0$$

Problem 2.43 illustrates application of Newton's law to derive the differential equation governing free vibration of a one-degree-of- freedom system. This problem also illustrates the benefits of using external and effective forces. Use of this method allows one free-body diagram to be drawn showing all effective forces. If this method were not used, one free-body diagram for the block and one free-body diagram of the pulley must be drawn. These free-body diagrams expose the tension in the pulley cable. Application of Newton's laws to the free-body diagrams yield equations involving the unknown tension. The tension must be eliminated between the equations in order to derive the differential equation.

2.44 Derive the differential equation governing the motion of the one degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinates shown in Figure P2.44. Linearize nonlinear differential equations by assuming small displacements.

Given: k, L ,m, c

Find: differential equation

FIGURE P 2.44

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Solution: The small angle assumption is used. Free-body diagrams of the bar at an arbitrary instant are shown below.

Summing moments about the point of support

$$\left(\sum M_0\right)_{ext} = \left(\sum M_0\right)_{eff}$$

leads to

$$-\frac{1}{4}Lc\dot{\theta}\left(\frac{1}{4}L\right)-\frac{3}{4}LK\theta\left(\frac{3}{4}L\right)-mg\frac{L}{4}\theta=\frac{1}{4}mL\ddot{\theta}\left(\frac{1}{4}L\right)+\frac{1}{12}mL^2\ddot{\theta}$$

$$\frac{7}{48}mL^2\ddot{\theta}+\frac{1}{16}cL^2\dot{\theta}+\left(\frac{9}{16}kL^2+mg\frac{L}{4}\right)\theta=0$$

$$\ddot{\theta}+\frac{3}{7}\frac{c}{m}\dot{\theta}+\left(\frac{27}{7}\frac{k}{m}+\frac{12}{7}\frac{g}{L}\right)\theta=0$$

Problem 2.44 illustrates application of Newton's law to derive the differential equation governing the free vibrations of a one-degree-of-freed- linear system with viscous damping.

2.45 Derive the differential equation governing the motion of the one degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinates shown in Figure P2.45. Linearize nonlinear differential equations by assuming small displacements.

FIGURE P 2.45

Given: m, c, k, L, θ as generalized coordinate

Find: differential equation

Solution: The small angle assumption is used. It is also noted that gravity, which causes static spring forces, causes with these static spring forces in the governing differential equation and hence both are ignored. Free-body diagrams of the bar at an arbitrary instant are shown below.

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Summing moments about the point of support,

$$\left(\sum M_o\right)_{ext} = \left(\sum M_o\right)_{eff}$$

leads to

$$-\frac{3}{4}kL\theta\left(\frac{3}{4}L\right)-\frac{L}{2}c\dot{\theta}\left(\frac{L}{2}\right)-\frac{L}{4}k\theta\left(\frac{L}{4}\right)-\frac{L}{2}c\dot{\theta}\left(\frac{L}{4}\right)=\frac{L}{4}m\ddot{\theta}\left(\frac{L}{4}\right)+\frac{1}{12}mL^2\ddot{\theta}$$

$$\frac{7}{48}mL^2\ddot{\theta}+\frac{3}{8}cL^2\dot{\theta}+\frac{5}{8}kL^2\theta=0$$

$$\ddot{\theta}+\frac{18}{7}\frac{c}{m}\dot{\theta}+\frac{30}{7}\frac{k}{m}\theta=0$$

Problem 2.45 illustrates application of Newton's law to derive the differential equation governing the free vibration of a one-degree-of-freedom system with viscous damping.

2.46 Derive the differential equation governing the motion of the one degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinates shown in Figure P2.46. Linearize nonlinear differential equations by assuming small displacements.

Thin disk of mass m radius r rolls without slip

FIGURE P 2.46

Given: m, k, c, x as generalized coordinate

Find: differential equation, ω_n

Solution: The effect of the incline is to cause a non-zero static deflection in the spring. Thus, neither the gravity force or the static spring force have any effect on the differential equation and both are ignored in drawing the free body diagrams. Assuming the disk rolls without slip, its angular acceleration is related to the acceleration of the mass center by

$$\alpha=\frac{\dot{x}}{r}$$

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Consider the free body diagrams drawn below at an arbitrary instant

Summing moments about the point of contact between the disk and the incline

$$\left(\sum M_c\right)_{ext} = \left(\sum M_c\right)_{eff}$$

leads to

$$-kxr - c\dot{x}r = \frac{1}{2}mr^2\frac{\ddot{x}}{r} + m\ddot{x}r$$

$$\frac{3}{2}m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{2c}{3m}\dot{x} + \frac{2k}{3m}x = 0$$

Problem 2.46 illustrates application of Newton's law to determine the governing differential equation for free vibrations of a one-degree-of-freedom system with viscous damping.

2.47 Derive the differential equation governing the motion of one-degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinate shown in Figure P.2.47. Linearize nonlinear differential equations by assuming small displacements.

Given: system shown

Find: differential equation

FIGURE P 2.47

Solution: Free-body diagrams of the system at an arbitrary instant are shown below.

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The displacement of each end of the rigid rod is the same. Using the small angle assumption

$$\frac{2L}{3}\theta = \frac{3L}{4}\theta_2$$

$$\theta_2 = \frac{8}{9}\theta$$

Summing moments about the pin support of the upper bar leads to

$$\left(\sum M_A\right)_{ext} = \left(\sum M_A\right)_{eff}$$

$$-c\left(\frac{L}{3}\dot{\theta}\right)\frac{L}{3} - 2k\left(\frac{2L}{3}\theta\right)\frac{2L}{3} + F\frac{2L}{3} = \frac{1}{12}mL^2\ddot{\theta} + m\frac{L}{6}\ddot{\theta}\frac{L}{6}$$

$$F = \frac{mL}{6}\ddot{\theta} + \frac{cL}{6}\dot{\theta} + \frac{4kL}{3}\theta$$

Summing moments about the pin support of the lower bar leads to

$$\left(\sum M_B\right)_{ext} = \left(\sum M_B\right)_{eff}$$

$$-kL\theta_2 L - c\left(\frac{L}{2}\dot{\theta}_2\right)\frac{L}{2} - F\frac{3L}{4} = \frac{1}{12}mL^2\ddot{\theta}_2 + m\left(\frac{L}{2}\ddot{\theta}_2\right)\frac{L}{2}$$

Substitution for F and θ_2 leads to

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$$\frac{91}{216}mL^2\ddot{\theta}+\frac{25}{72}cL^2\dot{\theta}+\frac{17}{9}kL^2\theta=0$$

Rewriting the equation in standard form

$$\ddot{\theta}+\frac{75c}{91m}\dot{\theta}+\frac{408k}{91m}\theta=0$$

Problem 2.47 illustrates the derivation of the differential equation governing the motion of a linear one-degree-of-freedom system using the free-body diagram method.

2.48 Derive the differential equation governing the motion of one-degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinate shown in Figure P2.48. Linearize nonlinear differential equations by assuming small displacements.

FIGURE P 2.48

Given: system shown

Find: differential equation

Solution: Free-body diagrams of the system at an arbitrary instant are shown below

Note that the force developed in the spring is proportional to the change in length of the spring. When the center of the disk is displaced a distance x from equilibrium, the end of the spring attached to the center of the disk compresses by x. When the center of the disk displaces x, the point on the disk to which the spring is attached has translated a distance x and rotated along the distance an angle θ. Assuming no slip between the disk and the

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surface, θ = x/r. Hence this end of the spring has displaced 2x. The total change in length of this spring is 3x.

Summing moments about the point of contact between the disk and surface leads to

$$\left(\sum M_C\right)_{ext} = \left(\sum M_C\right)_{eff}$$

$$-(kx + c\dot{x})r - 2k(3x)r - 2k(3x)(2r) = m\ddot{x}(r) + \frac{1}{2}mr^2\frac{\ddot{x}}{r}$$

$$\frac{3}{2}mr\ddot{x} + cr\dot{x} + 19krx = 0$$

The differential equation is put into standard form by dividing by the coefficient of $\ddot{x}$ leading to

$$\ddot{x} + \frac{2c}{3m}\dot{x} + \frac{38k}{3m}x = 0$$

Problem 2.48 illustrates derivation of the differential equation governing the motion of a one-degree-of-freedom system using the free-body diagram method, putting the differential equation into a standard form, and determination of the natural frequency from the differential equation.

2.49 Derive the differential equation governing the motion of the one-degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinate shown in Figure P2.49. Linearize nonlinear differential equations by assuming small displacements.

Slender bar of mass m connected to blocks through rigid links at A and B

FIGURE P 2.49

Given: system shown

Find: differential equation

Solution: Free-body diagrams of the system at an arbitrary instant are shown below

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Summing moments about the pin support leads to

$$\left(\sum M_O\right)_{ext} = \left(\sum M_O\right)_{eff}$$

$$-\left(k\frac{L}{2}\theta + c\frac{L}{2}\dot{\theta}\right)\frac{L}{2} - \left(k\frac{L}{2}\theta + c\frac{L}{2}\dot{\theta}\right)\frac{L}{2} = \frac{1}{12}mL^2\ddot{\theta} + m\frac{L}{2}\ddot{\theta}\frac{L}{2} + 2m\frac{L}{2}\ddot{\theta}\frac{L}{2}$$

$$\frac{5}{6}mL^2\ddot{\theta} + \frac{1}{2}cL^2\dot{\theta} + \frac{1}{2}kL^2\theta = 0$$

The differential equation is put into standard form by dividing by the coefficient of $\ddot{\theta}$ leading to

$$\ddot{\theta} + \frac{3c}{5m}\dot{\theta} + \frac{3k}{5m}\theta = 0$$

Problem 2.49 illustrates the use of the free-body diagram method to derive the differential equation governing the motion of a one-degree-of-freedom system.

2.50 Derive the differential equation governing the motion of the one degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinates shown in Figure P2.50. Linearize nonlinear differential equations by assuming small displacements.

FIGURE P 2.50

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Given: R, r, m, ϕ as generalized coordinate

Find: differential equation, ω_n

Solution: The generalized coordinate is chosen as ϕ, the angle made between the normal to the sphere and the surface at any instant of time. Let θ be an angular coordinate representing the angular displacement of the sphere. If the sphere rolls without slip, then the distance traveled by the mass center of the sphere is

$$x = r\theta \tag{1}$$

However, the mass center of the sphere is also traveling in a circular path of radius (R-r). Thus the distance traveled by the mass center is also equal to

$$x = (R-r)\phi \tag{2}$$

Equating x from eqs.(1) and (2) leads to

$$\theta = \frac{R-r}{r}\phi$$

Now consider free body diagrams of the sphere at an arbitrary instant.

Summing moments about the point of contact,

$$\left(\sum M_c\right)_{ext} = \left(\sum M_c\right)_{eff}$$

leads to

$$-mgr\sin\phi = \frac{2}{5}mr^2\left(\frac{R-r}{r}\right)\ddot{\phi} + m(R-r)\ddot{\phi}r$$

$$\frac{7}{5}(R-r)\ddot{\phi} + g\sin\phi = 0$$

Assuming small ϕ

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$$\frac{7}{5}(R-r)\ddot{\phi}+g\phi=0$$

$$\ddot{\phi}+\frac{5g}{7(R-r)}\phi=0$$

Problem 2.50 illustrates application of Newton's law to derive the differential equation governing free vibration of a one-degree-of-freedom system.

2.51 Derive the differential equation governing the motion of the one-degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinate shown in Figure P2.51. Linearize nonlinear differential equations by assuming small displacements.

FIGURE P 2.51

Given: system shown

Find: differential equation

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Solution: Free-body diagrams of the system at an arbitrary instant are shown below

Summing moments about the point of support of the bar using the small angle assumption leads to

$$\left(\sum M_O\right)_{ext} = \left(\sum M_O\right)_{eff}$$

$$-mg\frac{L}{6}\theta - 2k\frac{L}{3}\theta\frac{L}{3} + F\frac{L}{3} = \frac{1}{12}mL^2\ddot{\theta} + m\frac{L}{6}\ddot{\theta}\frac{L}{6}$$

$$F = \frac{1}{3}mL\ddot{\theta} + \left(2k\frac{L}{3} + \frac{1}{2}mg\right)\theta$$

Summing moments about the point of contact between the disk and the surface leads to

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$$\left(\sum M_C\right)_{ext} = \left(\sum M_C\right)_{eff}$$

$$-Fr - kxr - c\dot{x}r = m\ddot{x}r + \frac{1}{2}mr^2\left(\frac{\ddot{x}}{r}\right)$$

$$F = -\frac{3}{2}m\ddot{x} - c\dot{x} - kx$$

Kinematics is used to give

$$x = \frac{L}{3}\theta \qquad \theta = \frac{3x}{L}$$

Equating the two expressions for F and substituting for θ leads to

$$-\frac{3}{2}m\ddot{x} - c\dot{x} - kx = \frac{1}{3}mL\left(\frac{3\ddot{x}}{L}\right) + \left(2k\frac{L}{3} + \frac{1}{2}mg\right)\left(\frac{3x}{L}\right)$$

$$\frac{5}{2}m\ddot{x} + c\dot{x} + \left(3k + \frac{3mg}{2L}\right)x = 0$$

The differential equation is put into standard form by dividing by the coefficient of $\ddot{x}$ leading to

$$\ddot{x} + \frac{2c}{5m}\dot{x} + \left(\frac{6k}{5m} + \frac{3g}{5L}\right)x = 0$$

Problem 2.51 illustrates the application of the free-body diagram method to derive the differential equation governing the motion of a one-degree-of-freedom system.

2.52 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.52.

FIGURE P 2.52

Given: system shown

Find: differential equation using x as the generalized coordinate.

Solution: The springs attached to the mass act as two springs in parallel. The system can be modeled by a mass attached to a spring of equivalent stiffness 3k. Thus the governing differential equation is

$$m\ddot{x} + 3kx = 0$$

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or

$$\ddot{x}+3\frac{k}{m}x=0$$

Problem 2.52 illustrates the application of the equivalent system approach to derive the governing differential equation for a block attached to springs in parallel.

2.53 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.53.

FIGURE P 2.53

Given: x as generalized coordinate, k, m, I, r

Find: differential equation

Solution: Since x is measured from the system's equilibrium position, gravity cancels with the static spring forces in the governing differential equation. Thus, for purposes of deriving the differential equation, both are ignored. It is assumed there is no slip between the cable and the pulley. Thus the angular rotation of the pulley is kinematically related to the displacement of the block by

$$\theta=\frac{x}{2r}$$

The equivalent systems method is used. The system is modeled by a mass-spring system of an equivalent mass and equivalent stiffness, using the generalized coordinate, x. The kinetic energy of the equivalent system at an arbitrary time is

$$T=\frac{1}{2}m_{eq}\dot{x}^2$$

The kinetic energy of the system at an arbitrary instant is

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}I\left(\frac{\dot{x}}{2r}\right)^2$$
$$=\frac{1}{2}\left(m+\frac{I}{4r^2}\right)\dot{x}^2$$

Requiring the kinetic energy of the equivalent system to be equal to the kinetic energy of the original system at any instant leads to

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$$m_{eq} = m + \frac{I}{4r^2}$$

The potential energy of the equivalent system at an arbitrary instant is

$$V = \frac{1}{2} k_{eq} x^2$$

The potential energy of the system at hand at an arbitrary instant is

$$v = \frac{1}{2} k \left(\frac{x}{2} \right)^2$$

$$v = \frac{1}{2} \frac{k}{4} x^2$$

Requiring the potential energies to be equal at any instant leads to

$$k_{eq} = \frac{k}{4}$$

The differential equation governing free vibration is

$$m_{eq} \ddot{x} + k_{eq} x = 0$$

$$\left(m + \frac{I}{4r^2} \right) \ddot{x} + \frac{k}{4} x = 0$$

$$\ddot{x} + \frac{k}{4\left(m + \frac{I}{4r^2} \right)} x = 0$$

Problem 2.53 illustrates use of the equivalent system method to derive the differential equation governing free vibration of a one-degree-of-freedom system.

FIGURE P 2.54

2.54 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.54.

Given: k, m, c, θ as generalized coordinate

Find: differential equation, ω_n

Solution: The small angle assumption is used. Since the generalized coordinate is an angular displacement the

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system is modeled by a disk of mass moment of inertia I_{eq} attached to a shaft of torsional stiffness $k_{t,eq}$ and connected to a torsional viscous damper of torsional damping coefficient $c_{t,eq}$.

The kinetic energy of the system at an arbitrary time is

$$T = \frac{1}{2} m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$
$$= \frac{1}{2} m\left(\frac{L}{6}\dot{\theta}\right)^2 + \frac{1}{2}\frac{1}{12} mL^2\dot{\theta}^2$$
$$= \frac{1}{2}\frac{7}{48} mL^2\dot{\theta}^2$$

Hence,

$$I_{eq} = \frac{7}{48} mL^2$$

Using a horizontal plane through the pin support as the datum for potential energy calculations due to gravity, the potential energy of the system at an arbitrary time is

$$V = \frac{1}{2} k\left(\frac{3}{4} L\theta\right)^2 - mg\frac{L}{4}\cos\theta$$
$$= \frac{1}{2}\frac{9}{16} kL^2\theta^2 - mg\frac{L}{4}\cos\theta$$

Using the small angle assumption and the Taylor series expansion for $\cos\theta$, truncated after the quadratic term, leads to

$$V = \frac{1}{2}\frac{9}{16} kL^2\theta^2 - mg\frac{L}{4}\left(1 - \frac{1}{2}\theta^2\right)$$
$$= mg\frac{L}{2} + \frac{1}{2}\left(\frac{9}{16} kL^2 + mg\frac{L}{4}\right)\theta^2$$

Hence

$$k_{t_{eq}} = \frac{9}{16} kL^2 + mg\frac{L}{4}$$

The work done by the damping force between two arbitrary times is

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$$W = -\int c\left(\frac{L}{4}\dot{\theta}\right) d\left(\frac{L}{4}\theta\right)$$
$$= -\int \frac{1}{16} cL^2 \dot{\theta}\, d\theta$$

Hence

$$c_{t_{eq}} = \frac{1}{16} cL^2$$

The governing differential equation is

$$I_{eq}\ddot{\theta} + c_{t_{eq}}\dot{\theta} + k_{t_{eq}}\theta = 0$$
$$\frac{7}{48} mL^2\ddot{\theta} + \frac{1}{16} cL^2\dot{\theta} + \left(\frac{9}{16} kL^2 + mg\frac{L}{4}\right)\theta = 0$$
$$\ddot{\theta} + \frac{3}{7}\frac{c}{m}\dot{\theta} + \left(\frac{27}{7}\frac{k}{m} + \frac{12}{7}\frac{g}{L}\right)\theta = 0$$

Problem 2.54 illustrates application of the equivalent systems method to derive the differential equation governing the motion of a one-degree-of-freedom system with viscous damping.

2.55 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.55.

FIGURE P 2.55

Given: system shown

Find: differential equation using θ as the generalized coordinate

Solution: The small angle assumption is used. Since the generalized coordinate is an angular coordinate, the appropriate equivalent system model is a thin disk of mass moment-of inertia $I_{eq.}$ attached to a shaft of torsional stiffness $k_{t,eq.}$ and torsional viscous damper of damping coefficient $c_{t,eq.}$ .The kinetic energy of the equivalent system is

$$T = \frac{1}{2} I_{eq.}\dot{\theta}^2 \qquad (1)$$

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The kinetic energy of the system at hand is

$$T = \frac{1}{2} m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$
$$= \frac{1}{2} m\left(\frac{L}{4}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12} mL^2\right)\dot{\theta}^2 \qquad (2)$$
$$= \frac{1}{2}\left(\frac{7}{48} mL^2\right)\dot{\theta}^2$$

comparing eqs.(1) and (2) leads to

$$I_{eq.} = \frac{7}{48} mL^2 \qquad (3)$$

The potential energy of the equivalent system is

$$V = \frac{1}{2} k_{t_{eq.}} \theta^2 \qquad (4)$$

The potential energy of the system at hand, is

$$V = \frac{1}{2} k\left(\frac{3}{4} L\theta\right)^2 + \frac{1}{2} k\left(\frac{L}{4}\theta\right)^2$$
$$= \frac{1}{2}\left(\frac{5}{8} L^2\right)\theta^2 \qquad (5)$$

Comparing eqs. (4) and (5) leads to

$$k_{t_{eq.}} = \frac{5}{8} kL^2 \qquad (6)$$

The work done by the torsional viscous damper of the equivalent system is

$$U = -C_{t_{eq}} \int \dot{\theta} d\theta \qquad (7)$$

The work done by this viscous dampers in the system at hand is

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$$U = -\int c\dot{x}_A dx_A - \int 2c\dot{x}_B dx_B$$
$$= -\int c\left(\frac{L}{2}\dot{\theta}\right) d\left(\frac{L}{2}\theta\right) - \int 2c\left(\frac{L}{4}\dot{\theta}\right) d\left(\frac{L}{4}\theta\right) \tag{8}$$
$$= -\frac{3}{8}cL^2 \int \dot{\theta} d\theta$$

Comparing eqs.(7) and (8) leads to

$$c_{t_{eq}} = \frac{3}{8}cL^2 \tag{9}$$

The differential equation governing motion of the equivalent system is

$$I_{eq.}\ddot{\theta} + c_{t_{eq.}}\dot{\theta} + k_{t_{eq.}}\theta = 0 \tag{10}$$

Substituting eqs.(3), (6), and (9) in eq.(10) leads to the differential equation governing the system as

$$\frac{7}{48}mL^2\ddot{\theta} + \frac{3}{8}cL^2\dot{\theta} + \frac{5}{8}kL^2\theta = 0 \tag{11}$$

Dividing eq.(11) by the coefficient of its highest derivative gives

$$\ddot{\theta} + \frac{16}{7}\frac{c}{m}\dot{\theta} + \frac{30}{7}\frac{k}{m}\theta = 0 \tag{12}$$

Problem 2.55 illustrates use of the equivalent system method to derive the differential equation for a system with viscous damping when an angular coordinate is chosen as the generalized coordinate.

2.56 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.56.

Thin disk of mass m radius r rolls without slip

FIGURE P 2.56

Given: m, k, c, x as generalized coordinate

Find: differential equation

Solution: The system is modeled by a mass-spring-dashpot system of equivalent mass, stiffness, and viscous damping coefficient. The kinetic energy of the equivalent system is

$$T = \frac{1}{2}m_{eq}\dot{x}^2$$

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If the disk rolls without slip then its angular velocity is related to the velocity of its mass center by

$$\omega = \frac{\dot{x}}{r}$$

In this case the kinetic energy of the system is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{1}{2}mr^2\left(\frac{\dot{x}}{r}\right)^2$$
$$= \frac{1}{2}\frac{3}{2}m\dot{x}^2$$

and hence

$$m_{eq} = \frac{3}{2}m$$

The potential energy of the equivalent system is

$$V = \frac{1}{2}k_{eq}x^2$$

The gravity causes a static deflection in the spring, and does not contribute to any additional potential energy. Thus, ignoring gravity and the initial potential energy in the spring,

$$V = \frac{1}{2}kx^2$$

and

$$k_{eq} = k$$

The work done by the damping force in the equivalent system is

$$W = -\int c_{eq}\dot{x}dx$$

The work done by damping force in the system at hand is

$$W = -\int c\dot{x}dx$$

Hence, $c_{eq} = c$. Thus the governing differential equation is

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$$\frac{3}{2}m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{2c}{3m}\dot{x} + \frac{2k}{3m}x = 0$$

Problem 2.56 illustrates the use of the equivalent system method to derive the differential equation for a one-degree-of-freedom system.

2.57 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.57.

FIGURE P 2.57

Given: system shown

Find: differential equation

Solution: Let θ_2 be the counterclockwise angular displacement of the lower bar. Since the displacement of each end of the rigid rod is the same, use of the small angle approximation leads to

$$\frac{2L}{3}\theta = \frac{3L}{4}\theta_2$$

The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\left(\frac{L}{6}\dot{\theta}\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2 + \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}_2\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}_2^2$$

$$T = \frac{1}{2}m\left(\frac{L}{6}\dot{\theta}\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2 + \frac{1}{2}m\left(\frac{L}{2}\frac{8}{9}\dot{\theta}\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\left(\frac{8}{9}\dot{\theta}\right)^2$$

$$T = \frac{1}{2}mL^2\left(\frac{1}{36} + \frac{1}{12} + \frac{16}{81} + \frac{16}{243}\right)\dot{\theta}^2$$

$$T = \frac{1}{2}\frac{91}{243}mL^2\dot{\theta}^2$$

Since an angular coordinate is chosen as the generalized coordinate the torsional system is the appropriate model system with

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$$I_{eq} = \frac{91}{243} mL^2$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2} 2k\left(\frac{2L}{3}\theta\right)^2 + \frac{1}{2}k(L\theta_2)^2$$

$$V = \frac{1}{2} 2k\left(\frac{2L}{3}\theta\right)^2 + \frac{1}{2}k\left(L\frac{8}{9}\theta\right)^2$$

$$V = \frac{1}{2}kL^2\left(\frac{8}{9} + \frac{64}{81}\right)\theta^2$$

$$V = \frac{1}{2}\frac{136}{81}kL^2\theta^2$$

Thus,

$$k_{t_{eq}} = \frac{136}{81}kL^2$$

The work done by the viscous damper between two arbitrary times is

$$W_{1\to 2} = -\int c\left(\frac{L}{3}\dot{\theta}\right) d\left(\frac{L}{3}\theta\right) - \int c\left(\frac{L}{2}\right)\dot{\theta}_2\, d\left(\frac{L}{2}\theta_2\right)$$

$$W_{1\to 2} = -\int \frac{cL^2}{9}\dot{\theta}\, d\theta - \int \frac{cL^2}{4}\left(\frac{8}{9}\dot{\theta}\right) d\left(\frac{8}{9}\theta\right)$$

$$W_{1\to 2} = -\int cL^2\left(\frac{1}{9} + \frac{16}{81}\right)\dot{\theta}\, d\theta = -\int \frac{25cL^2}{81}\dot{\theta}\, d\theta$$

Thus the equivalent torsional damping coefficient is

$$c_{t_{eq}} = \frac{25cL^2}{81}$$

The differential equation governing the motion of the system is

$$\frac{91}{243}mL^2\ddot{\theta} + \frac{25}{81}cL^2\dot{\theta} + \frac{136}{81}kL^2\theta = 0$$

The equation is put into standard form by dividing through by the coefficient of the $\ddot{\theta}$ term leading to

$$\ddot{\theta} + \frac{75c}{91m}\dot{\theta} + \frac{324k}{91m}\theta = 0$$

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Problem 2.57 illustrates derivation of the governing differential equation using the equivalent systems method.

2.58 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.58.

FIGURE P 2.58

Given: system shown

Find: differential equation

Solution: It the disk rolls without slip then the velocity of the mass center is related to the angular velocity of the disk by $\dot{x} = r\omega$. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{\dot{x}}{r}\right)^2$$

$$T = \frac{1}{2}\left(\frac{3}{2}m\right)\dot{x}^2$$

Thus the equivalent mass of the system is

$$m_{eq} = \frac{3}{2}m$$

The potential energy developed in a spring is proportional to the square of the change in length of the spring. If the center of the disk displaces a distance x from equilibrium the end of the spring attached to the center of the disk displaces x. The point at the top of the disk where the spring is attached translates a distance x and rotates through an angle θ. Since the disk rolls without slip θ = x/r. Thus the total displacement of that end of the spring is x + rθ=2x. Then the total change in length of the spring is 3x. The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}2k(3x)^2$$

$$V = \frac{1}{2}(19k)x^2$$

Thus the equivalent stiffness of the system is

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$$k_{eq} = 19k$$

The work done by the viscous damper between two arbitrary positions is

$$W_{1\to 2} = -\int_{x_1}^{x_2} c\dot{x}\, dx$$

The equivalent viscous damping coefficient for the system is

$$c_{eq} = c$$

The differential equation governing the motion of the system is

$$\frac{3}{2}m\ddot{x} + c\dot{x} + 19kx = 0$$

The differential equation is put into standard form by dividing by the coefficient of $\ddot{x}$ leading to

$$\ddot{x} + \frac{2c}{3m}\dot{x} + \frac{38k}{3m}x = 0$$

Problem 2.58 illustrates derivation of the differential equation governing the motion of a linear one-degree-of-freedom system using the equivalent systems method.

2.59 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.59.

Slender bar of mass m connected to blocks through rigid links at A and B

FIGURE P 2.59

Given: system shown

Find: differential equation, ω_n

Solution: The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2 + \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}2m\left(\frac{L}{2}\dot{\theta}\right)^2$$

$$T = \frac{1}{2}\left(\frac{5}{6}mL^2\right)\dot{\theta}^2$$

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Since an angular coordinate is chosen as the generalized coordinate the equivalent system model is the torsional system. The equivalent moment of inertia of the system is

$$I_{eq} = \frac{5}{6} mL^2$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2} k \left(\frac{L}{2} \theta \right)^2 + \frac{1}{2} k \left(\frac{L}{2} \theta \right)^2$$

$$V = \frac{1}{2} \left(\frac{1}{2} kL^2 \right) \theta^2$$

The equivalent torsional stiffness is

$$k_{t_{eq}} = \frac{1}{2} kL^2$$

The work done by the viscous dampers between two arbitrary positions is

$$W_{1\to 2} = -\int c \left(\frac{L}{2} \dot{\theta} \right) d\left(\frac{L}{2} \theta \right) - \int c \left(\frac{L}{2} \dot{\theta} \right) d\left(\frac{L}{2} \theta \right)$$

$$W_{1\to 2} = -\int_{\theta_1}^{\theta_2} \left(c \frac{L^2}{2} \right) \dot{\theta} \, d\theta$$

The equivalent torsional viscous damping coefficient is

$$c_{t_{eq}} = \frac{1}{2} cL^2$$

The differential equation governing the motion of the system is

$$\frac{5}{6} mL^2 \ddot{\theta} + \frac{1}{2} cL^2 \dot{\theta} + \frac{1}{2} kL^2 \theta = 0$$

The differential equation is put into standard form by dividing by the coefficient of $\ddot{\theta}$ leading to

$$\ddot{\theta} + \frac{3c}{5m} \dot{\theta} + \frac{3k}{5m} \theta = 0$$

Problem 2.59 illustrates the application of the equivalent systems method to derive the differential equation governing the motion of a linear one-degree-of-freedom system.

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2.60 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.60.

FIGURE P 2.60

Given: system shown

Find: differential equation

Solution: The generalized coordinate is chosen as ϕ the angle made between the normal to the sphere and the surface at any instant. Let θ be an angular coordinate representing the angular displacement of the sphere. If the sphere rolls without slip, then the distance traveled by the mass center of the sphere is

$$x = r\theta$$

However the mass center of the sphere is also traveling in a circular path of radius (R-r). Thus the distance traveled by the mass center is also equal to

$$x = (R - r)\phi$$

Equating x between the two equations leads to

$$\theta = \frac{R - r}{r}\phi$$

The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{2}{5}mr^2\dot{\theta}^2$$

$$T = \frac{1}{2}m\left[(R - r)^2\dot{\phi}^2 + \frac{2}{5}r^2\left(\frac{R - r}{r}\right)^2\dot{\phi}^2\right]$$

$$T = \frac{1}{2}\left[\frac{7}{5}m(R - r)^2\right]\dot{\phi}^2$$

Hence the equivalent moment of inertia is

$$I_{eq} = \frac{7}{5}m(R - r)^2$$

The datum for potential energy calculations is taken as the position of the mass center of the sphere when it is in equilibrium at the bottom of the circular path. The potential energy at an arbitrary instant is

$$V = mg(R - r)\cos\phi$$

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Use of the small angle assumption leads to

$$V = \frac{1}{2} mg(R - r)\phi^2$$

Thus the equivalent torsional stiffness is

$$k_{t_{eq}} = mg(R - r)$$

The differential equation governing the motion of the system is

$$\frac{7}{5} m(R - r)^2 \ddot{\phi} + mg(R - r)\phi = 0$$

The differential equation is put into standard form by dividing by the coefficient multiplying the highest order derivative. This leads to

$$\ddot{\phi} + \frac{5g}{7(R - r)} \phi = 0$$

Problem 2.60 illustrates the application of the equivalent systems method to derive the differential equation governing the motion of a one-degree-of-freedom linear system with an angular displacement as the chosen generalized coordinate.

2.61 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figure P2.61.

FIGURE P 2.61

Given: system shown

Find: differential equation

Solution: The kinetic energy of the system is $T = T_b + T_s$ where T_b is the kinetic energy of the bar and T_s is the kinetic energy of the sphere. The kinetic energy of the sphere is assuming no slipping

$$T_s = \frac{1}{2} m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2} mr^2\right)\left(\frac{\dot{x}}{r}\right)^2 = \frac{1}{2}\frac{3m}{2}\dot{x}^2$$

Let θ (small) be the angular rotation of the bar. Both ends of the rigid link have the same displacement, thus

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$$x = \frac{L}{3}\theta \Longrightarrow \theta = \frac{3x}{L}$$

The kinetic energy of the bar is

$$T_b = \frac{1}{2}m\left[\frac{L}{6}\left(\frac{3\dot{x}}{L}\right)\right]^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{3\dot{x}}{L}\right)^2 = \frac{1}{2}m\dot{x}^2$$

Hence the total kinetic energy of the system is

$$T = \frac{1}{2}\frac{3m}{2}\dot{x}^2 + \frac{1}{2}m\dot{x}^2 = \frac{1}{2}\frac{5m}{2}\dot{x}^2$$

The equivalent mass of the system is $\frac{5m}{2}$. The potential energy of the system is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}2k\left[\frac{L}{3}\left(\frac{3x}{L}\right)\right]^2 + \frac{mgL}{6}(1 - \cos\theta)$$

Using the small angle assumption and approximating $1 - \cos\theta$as $\frac{1}{2}\theta^2 = \frac{1}{2}\left(\frac{3x}{L}\right)^2$leads to the potential energy of

$$V = \frac{1}{2}\left(3k + \frac{3mg}{2L}\right)x^2$$

The equivalent stiffness of the system is $3k + \frac{3mg}{2L}$. The work done by the viscous damping force is

$$U = -\int c\dot{x}\,dx$$

The equivalent viscous damping coefficient is c. The differential equation is

$$\frac{5m}{2}\ddot{x} + c\dot{x} + \left(3k + \frac{3mg}{2L}\right)x = 0$$

Problem 2.61 illustrates the application of the equivalent systems method to derive the differential equation governing the motion of a one-degree-of-freedom linear system with a liner displacement as the chosen generalized coordinate and gravity as a source of potential energy.

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CHAPTER 3: FREE VIBRATIONS OF SDOF SYSTEMS

Short Answer Problems

3.1 True: The period and natural frequency of a linear system are independent of initial conditions. They are functions of only the inertial and stiffness properties of the system, The period of a nonlinear system depends upon the initial conditions

3.2 False: The natural frequency determined directly from the differential equation of motion has units of rad/s.

3.3 False: A system with a natural frequency of $10\ \frac{\text{rad}}{\text{s}}$ has a longer period than a system of natural frequency $100\ \frac{\text{rad}}{\text{s}}$. (Period is proportional to reciprocal of natural frequency)

3.4 False: The free vibrations of an underdamped SDOF system are cyclic. (Overdamped free vibrations decay exponentially without completing one cycle)

3.5 True: An undamped system has no energy dissipation and returns to the same position at the end of every cycle.

3.6 True: Systems with damping ratios greater than one are overdamped.

3.7 False: The energy lost per cycle of motion for hysteretic damping is independent of the frequency but depends upon the square of the amplitude of motion.

3.8 True: All of the energy is never dissipated in one cycle of motion for an underdamped system. The energy dissipated is a constant fraction of the energy present at the beginning of the cycle.

3.9 False: Motion never ceases due to viscous damping for a system with underdamped free vibrations, it is exponentially decaying with infintesimally small amplitudes.

3.10 False: A system that has Coulomb damping is governed by two differential equations, one for positive velocity and another for negative velocity.

3.11 True: Motion with Coulomb damping ceases when the spring force is unable to overcome the friction force.

3.12 False: The period, measured in s, is the reciprocal of the natural frequency, measured in Hz.

3.13 True: The differential equation governing the motion of a SDOF system is second order in time and is homogeneous for free vibrations.

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3.14 True: The damping ratio is defined as the ratio of the damping coefficient to the critical damping coefficient.

3.15 False: The amplitude of an undamped SDOF system is time independent.

3.16 $\omega_n = \sqrt{\frac{k}{m}}$ is the system's natural frequency measured in rad/s. It is the frequency at which free vibrations occur. (b)$\zeta = \frac{c}{2\sqrt{mk}}$ is the system's damping ratio. It is the ratio of the actual damping coefficient to the critical damping coefficient.

3.17 (d) $x_0 = 1\text{ mm}, \dot{x}_0 = -1\text{ m/s}$

3.18 (a) plot b; (b) plot c; (c) plot a; (d) plot d

3.19 (1) Underdamped vibrations have exponentially decaying amplitude while Coulomb damped vibrations have a linear decay in amplitude; (2) Coulomb damped responses are at the natural frequency while underdamped responses are at the damped natural frequency; (3) Motion ceases for Coulomb damping while motion continues indefinitely for underdamped systems; and (4) Motion ceases with a permanent displacement from equilibrium for systems with Coulomb damping.

3.20 geometric

3.21 arithmetic

3.22 The equation is for the free response of an undamped system. A is the amplitude of motion, or the maximum displacement from equilibrium, ω_n is the natural frequency and ϕ is the phase angle, difference the response and a pure sinusoid.

3.23 Hysteresis occurs in engineering material due to energy lost as bonds between atoms break. As the material is unloaded it follows a force displacement curve. As the loading is removed it follows a different curve, usually parallel to the loading curve. The area under enclosed by the hysteresis loop is the energy lost due to hysteretic damping during the cycle of motion.

3.24 The concept of logarithmic decrement is based upon the cyclic motion of the system and the period of motion. If the system is overdamped there is no period of motion.

3.25 The critically damped system has less frictional resistance and hence returns to equilibrium faster.

3.26 $\zeta = \frac{c}{2\sqrt{mk}}$, thus given the same stiffness the system with the lower mass has the higher damping ratio. The system of mass 2 kg has a higher damping ratio than the system of mass 3 kg.

3.27 The period of vibration is the inverse of the frequency (in Hertz) of the motion. A system with viscous damping has a lower frequency than a corresponding system without

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damping thus it has a longer period. The period is longer reflecting the increased resistance due to the viscous damping.

3.28 $x(0)$ and $\dot{x}(0)$ must be specified for a SDOF system.

3.29 $x(0) = \delta$ and $\dot{x}(0) = 0$

3.30 $x(0) = 0$ and $\dot{x}(0) = \frac{I}{m}$

3.31 Total energy is the sum of the kinetic and potential energy present in the system at an arbitrary instant.

3.32 The energy dissipated per cycle of motion is calculated for aerodynamic drag assuming $x(t) = X \sin \omega t$ and compared to the energy dissipated on one cycle due to viscous damping assuming the same displacement which is $\Delta E = c\pi\omega X^2$. Thus the equivalent damping coefficient is $c_{eq} = \frac{\Delta E}{\pi\omega X^2}$.

3.33 The natural frequency of the pendulum is $\omega_n = \sqrt{\frac{g}{\ell}}$. Thus the period is $T = 2\pi\sqrt{\frac{\ell}{g}}$. Hence to lengthen the period move the pendulum mass farther from the support.

3.34 The decrease in amplitude per cycle of motion is $\Delta A = \frac{4\mu mg}{k}$. Thus an increase in k decreases the amplitude lost per cycle of motion. Given the same displacement an increase in k leads to an increase in the number of cycles executed.

3.35 The increase in k leads to an increase in the natural frequency. An increase in c leads to an increase in damping ratio (the damping ratio is proportional to c, but inversely proportional to the square root of the frequency). An increase in ζ leads to a decrease in damped natural frequency. The frequency may be less than the natural frequency of the first system.

3.36 Given: $2\ddot{x} + 40\dot{x} + 1800x = 0$, $x(0) = 0.001$ m , $\dot{x}(0) = 3\,\frac{\text{m}}{\text{s}}$ Putting the differential equation in its standard form $\ddot{x} + 20\dot{x} + 900x = 0$.

(a) $\omega_n = \sqrt{900} = 30$ rad/s.
(b) $2\zeta\omega_n = 20 \Longrightarrow \zeta = \frac{20}{2\cdot 30} = 1/3$.
(c) underdamped
(d) $T = \frac{2\pi}{\omega_n} = \frac{2\pi}{30} = 0.209$ s.
(e) $f = \frac{1}{T} = \frac{1}{0.209\text{ s}} = 4.77$ Hz.
(f) $\omega_d = \omega_n\sqrt{1-\zeta^2} = 30\sqrt{1-\left(\frac{1}{3}\right)^2} = 28.3$ rad/s.

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(g) $\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi\left(\frac{1}{3}\right)}{\sqrt{1-\left(\frac{1}{3}\right)^2}} = 2.22$

(h) $A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0+\zeta\omega_n x_0}{\omega_d}\right)^2} = \sqrt{(0.001)^2 + \left(\frac{3+0.333(30)(.001)}{28.3}\right)^2} = 0.106$ m

(i) $\phi = \tan^{-1}\left(\frac{x_0\omega_d}{\dot{x}_0+\zeta\omega_n x_0}\right) = \tan^{-1}\left(\frac{0.001(28.3)}{3+0.333(30)(.001)}\right) = 0.0094$ rad

(j) $x(t) = 0.106\, e^{-10t}\sin(30t + 0.0094)$m.

3.37 Given: $2\ddot{x} + 600\dot{x} + 9800x = 0$, $x(0) = 0.001$ m, $\dot{x}(0) = 3\frac{\text{m}}{\text{s}}$. Putting the differential equation in standard form $\ddot{x} + 300\dot{x} + 4900x = 0$.

(a) $\omega_n = \sqrt{4900} = 70$rad/s
(b) $2\zeta\omega_n = 300 \Longrightarrow \zeta = \frac{300}{2\cdot 70} = 2.143$
(c) overdamped
(h) The overdamped response is

$$x(t) = 0.264e^{-150t}(0.0469e^{132.67t} - 0.0431e^{-132.67t})$$

**Solutions for (d)-(g), (i), and (j) are not applicable for this problem.*

3.38 Given: $2\ddot{x} + 1800x = \begin{cases} 3 & \dot{x} < 0 \\ -3 & \dot{x} > 0 \end{cases}$, $x(0) = 0.02$ m, $\dot{x}(0) = 0$. Putting the differential equation in its standard form $\ddot{x} + 900x = \begin{cases} 3/2 & \dot{x} < 0 \\ -3/2 & \dot{x} > 0 \end{cases}$

(a) $\omega_n = \sqrt{900} = 30$ rad/s, $T = \frac{2\pi}{\omega_n} = \frac{2\pi}{30} = 0.209$ s,
(b) $\Delta A = \frac{4F_f}{m\omega_n^2} = \frac{4(3)}{2(900)} = 0.0067$m
(c) $x_f = \frac{4F_f}{m\omega_n^2} = 0.00167$ m
(d) $n > \frac{m\omega_n^2\delta}{4F_f} - \frac{1}{4} = \frac{0.02}{0.0067} - 0.25 = 2.75$, 3 cycles.

3.39 $\omega_n = \sqrt{\frac{2k}{m}}$

3.40 $\omega_n = \sqrt{\frac{3EI}{mL^3}}$

3.41 $\omega_n = \sqrt{\frac{EA}{mL}}$

3.42 $\omega_n = \sqrt{\frac{JG}{IL}}$

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3.43 $\omega_n = \sqrt{\frac{48EI}{mL^3}}$

3.44 Given: m = 12 kg, two springs each of stiffness k = 4000 N/m. The natural frequency is $\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2(4000)}{12}} = 25.8$ rad/s.

3.45 Given: m = 30 g, k = 150 N/m The damping coefficient such that the system is critically damped is $c_c = 2\sqrt{mk} = 2\sqrt{(0.03)(150)} = 4.25\ \mathrm{N \cdot s/m}$.

3.46 Given: m = 400 kg, Δ_s= 5 mm. The static deflection of the engine is $\Delta_s = \frac{mg}{k} = \frac{g}{\omega_n^2} \Longrightarrow$ $\omega_n = \sqrt{\frac{g}{\Delta_s}} = \sqrt{\frac{9.81}{0.005}} = 44.3$ rad/s.

3.47 Given: m = 2 kg, k = 1000 N/m, x(0) = 25 mm, $\Delta E = 0.06$ N·m. The hysteretic damping coefficient is $h = \frac{\Delta E}{\pi k X^2} = \frac{0.06}{\pi(1000)(0.025)^2} = 0.0306$. The equivalent viscous damping ratio is $\zeta = \frac{h}{2} = 0.0153$.

3.48 Given: m = 0.5 kg, ω_n = 100 rad/s, h = 0.06, $\dot{x}_0$ =2 m/s. The equivalent viscous damping ratio is $\zeta = \frac{h}{2} = 0.03$. The response is modeled by the underdamped free vibrations which is given by

$$x(t) = \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2} = 100\sqrt{1-(0.03)^2} = 99.95$rad/s. Thus

$x(t) = \frac{2}{99.95} e^{-(0.03)(100)t} \sin 99.95t = 0.02e^{-3t} sin\ 99.95t$ m

3.49 (a)-(iv); (b)-(iii); (c)-(iv); (d)-(iii); (e)-(ii); (f)-(vi); (g)-(i); (h)-(iii); (i)-(iv)

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Chapter Problems

3.1 The mass of the pendulum bob of a cuckoo clock is 30 g. How far from the pin support should the bob be placed such that its period is 1.0 sec?

Given: m = 30 g, T = 1.0 sec

Find: ℓ

Solution: The pendulum is modeled as a particle of mass m attached to a massless rod. Let ℓ be the distance between the particle and its axis of rotation. Let θ be the counterclockwise displacement of the rod, measured from the vertical. Free body diagrams of the particle are shown at an arbitrary instant.

Summing moments about the axis of rotation

$$\left(\sum M_0\right)_{ext} = \left(\sum M_0\right)_{eff}$$

$$-mg\ell\sin\theta = m\ell^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

Assuming small angular displacements

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

from which the natural frequency is determined as

$$\omega_n = \sqrt{\frac{g}{\ell}}$$

Requiring the period of motion to be 1 sec leads to

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$$T = 1\,\text{sec} = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\ell}{g}}$$

$$\ell = \frac{g}{4\pi^2} = \frac{9.81\frac{\text{m}}{\text{s}^2}}{4\pi^2\frac{1}{\text{s}^2}}$$

$$\ell = 0.248\,\text{m}$$

Problem 3.1 illustrates how knowledge of the period is used to determine information about the system.

3.2 A ceiling fan assembly of five blades is driven by a motor. The assembly is attached to the ceiling by a thin shaft fixed at the ceiling. What is the natural frequency of torsional oscillations of the fan of Figure P3.2?

FIGURE P3.2

Given: shaft: G = 80×10^9 N/m^2, L = 0.25 m, r = 6 mm
blades: m = 0.4 kg, I = 11 kg·m^2, r = 0.4 m;
motor: I = 10 kg·m^2

Find: ω_n

Solution: The shaft is assumed to be restrained from rotation at its fixed end. The torsional oscillations are modeled by those of a thin disk of an equivalent moment of inertia attached to a spring of an equivalent stiffness. The equivalent stiffness is

$$k_t = \frac{JG}{L} = \frac{\pi r^4 G}{2L} = \frac{\pi(0.006\,\text{m})^4(80\times10^9\,\text{N/m}^2)}{2(0.25\,\text{m})} = 650\frac{\text{N}\cdot\text{m}}{\text{rad}}$$

The moment of inertia of each blade about the axis of the shaft is

$$I_b = \bar{I}_b + m_b\bar{r}_b^2 = 11\text{kg}\cdot\text{m}^2 + (0.4\text{kg})(0.4\text{m})^2 = 11.064\text{kg}\cdot\text{m}^2$$

The moment of inertia of the ceiling fan about the axis of the shaft is

$$I_{eq} = \bar{I}_m + 5\bar{I}_b = 10\,\text{kg}\cdot\text{m}^2 + 5(11.064\,\text{kg}\cdot\text{m}^2) = 65.32\,\text{kg}\cdot\text{m}^2$$

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The natural frequency of torsional oscillations is

$$\omega_n = \sqrt{\frac{k_t}{I_{eq}}} = \sqrt{\frac{650\,\frac{\mathrm{N\cdot m}}{\mathrm{rad}}}{65.32\,\mathrm{kg\cdot m^2}}} = 3.15\,\frac{\mathrm{rad}}{\mathrm{s}}$$

Problem 3.2 illustrates (a) modeling of a system undergoing torsional oscillations and (b) determination of the natural frequency of torsional oscillations.

3.3 The cylindrical container of Figure P3.3 has a mass of 25 kg and floats stably on the surface of an unknown fluid. When disturbed, the period of free oscillations is measured as 0.2 s. What is the specific gravity of the liquid?

FIGURE P3.3

Given: m = 25 kg, D = 50 cm, L = 150 cm, T = 0.2 s

Find: S.G.

Solution: Let h be the length of the container above the surface when the cylinder is floating on the surface in equilibrium. Consider the free body diagram of the container in equilibrium. Summing forces to zero

$$\sum F = 0 = mg - \gamma(L-h)\pi\frac{D^2}{4} \qquad (1)$$

where γ is the specific weight of the fluid. Let x be the vertical displacement of the cylinder's center of mass from equilibrium after it is disturbed. Consider free body diagrams of the cylinder at an arbitrary time:

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Summing forces in the vertical direction

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$

$$mg - \gamma(L - h + x)\pi \frac{D^2}{4} = m\ddot{x}$$

which, in view of eq. (1), becomes

$$m\ddot{x} + \gamma\pi \frac{D^2}{4} x = 0$$

$$\ddot{x} + \frac{\pi\gamma D^2}{4m} x = 0 \tag{2}$$

Equation (2) is of the standard form of the differential equation governing free vibrations of an undamped linear one-degree-of-freedom system. The natural frequency is the square root of the term multiplying the displacement

$$\omega_n = \sqrt{\frac{\gamma\pi D^2}{4m}} \tag{3}$$

The natural frequency is determined from the natural period by

$$\omega_n = \frac{2\pi}{T} \tag{4}$$

Substituting eq. (4) into eq. (3) and solving for γ leads to

$$\gamma = \frac{16m\pi}{D^2T^2} = \frac{16(25\,\text{kg})\pi}{(0.5\,\text{m})^2(0.2\,\text{s})^2} = 125600\,\frac{\text{N}}{\text{m}^3}$$

The fluid density is

$$\rho = \frac{\gamma}{g} = \frac{125600\,\frac{\text{N}}{\text{m}^3}}{9.81\,\frac{\text{m}}{\text{s}^2}} = 12800\,\frac{\text{kg}}{\text{m}^3}$$

The density of water is 1000 kg/m^3. Hence the specific gravity of the fluid is

$$\text{S.G.} = \frac{\rho}{\rho_{H_2O}} = \frac{12800\,\frac{\text{kg}}{\text{m}^3}}{1000\,\frac{\text{kg}}{\text{m}^3}} = 12.8$$

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Problem 3.3 illustrates (a) that the buoyant oscillations at a cylinder on a free surface can be modeled as a linear one-degree-of-freedom system, (b) the relation between natural frequency and period, (c) the derivation of the differential equation governing free vibrations, and (d) determination of the natural frequency directly from the differential equation.

3.4 When the 5.1 kg connecting rod of Figure P3.4 is placed in the position shown, the spring deflects 0.5 mm. When the end of the rod is displaced and released, the resulting period of oscillation is observed as 0.15 sec. Determine the location of the center of mass of the connecting and the centroidal mass moment of inertia of the rod.

FIGURE P3.4

Given: m = 5.1 kg, $\Delta_{ST.}$ = 0.5 mm, T = 0.15 sec, L = 20 cm, k = 3×10^4 N/m

Find: ℓ, I

Solution: When the system is in equilibrium the moment of the spring force balances with the moment of the gravity force when moments are taken about the pin support,

$$\sum M_0 = 0$$

$$-mg\ell + k\Delta_{ST.}L = 0$$

$$\ell = \frac{k\Delta_{ST.}L}{mg}$$

$$= \frac{\left(3\times10^4\,\frac{\text{N}}{\text{m}}\right)(0.0005\,\text{m})(0.2\,\text{m})}{(5.1\,\text{kg})\left(9.81\,\frac{\text{m}}{\text{sec}^2}\right)}$$

$$= 0.060\,\text{m} = 6.0\,\text{cm}$$

Let θ be the clockwise angular displacement of the rod after it is released, measured from the system's equilibrium position. Assuming small displacements, free body diagrams of the system at an arbitrary instant are shown below

EXTERNAL FORCES EFFECTIVE FORCES

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Summing moments about O, noting that the static deflection terms cancel with gravity

$$\left(\sum M_0\right)_{ext.} = \left(\sum M_0\right)_{eff.}$$
$$-kL^2\theta = m\ell^2\ddot{\theta} + \bar{I}\ddot{\theta}$$
$$\left(m\ell^2 + \bar{I}\right)\ddot{\theta} + kL^2\theta = 0 \qquad (1)$$
$$\ddot{\theta} + \frac{kL^2}{m\ell^2 + \bar{I}}\theta = 0$$

The natural frequency is obtained from eq.(1) as

$$\omega_n = \sqrt{\frac{kL^2}{m\ell^2 + \bar{I}}} \qquad (2)$$

The natural frequency is calculated from the natural period by

$$\omega_n = \frac{2\pi}{T} \qquad (3)$$

Equations (2) and (3) are combined and used to solve for I as

$$I = \frac{kL^2T^2}{4\pi^2} - m\ell^2$$
$$= \frac{\left(3\times10^4\,\frac{\text{N}}{\text{m}}\right)(0.2\,\text{m})^2(0.15\,\text{sec})^2}{4\pi^2} - (5.1\,\text{kg})(0.060\,\text{m})^2$$
$$= 0.666\,\text{kg}\cdot\text{m}^2$$

Problem 3.4 illustrates the use of a free vibrations test to determine the moment of inertia of a connecting rod.

3.5 When a 2000 lb vehicle is empty, the static deflection of its suspension system is measured as 0.8 in. What is the natural frequency of the vehicle when it is carrying 700 pounds of passengers and cargo?

Given: W = 2000 lb., δ_{st} = 0.8 in, W_c =700 lb

Find: ω_n

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Solution: The vehicle is modeled as a one-degree-of-freedom mass-spring system. The stiffness is related to static deflection by

$$\delta_{st} = \frac{mg}{k}$$

$$k = \frac{mg}{\delta_{st}}$$

Using the data given for the empty vehicle

$$k = \frac{2000 \text{ lb}}{(0.8 \text{ in})(1 \text{ ft}/12 \text{ in})} = 3 \times 10^4 \text{ lb/ft}$$

When the vehicle is full its natural frequency is calculated from

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3 \times 10^4 \text{ N/m}}{(2700 \text{ lb})/(32.2 \text{ ft/s}^2)}} = 18.9 \text{ rad/s}$$

Problem 3.5 illustrates calculation of natural frequency from static deflection data.

3.6 A 400 kg machine is placed at the midspan of a 3.2-m simply supported steel ($E = 200 \times 10^9$ N/m^2) beam. The machine is observed to vibrate with a natural frequency of 9.3 Hz. What is the moment of inertia of the beam's cross section about its neutral axis?

Given: m = 400 kg, L = 3.2-m , E = 200×10^9 N/m^2 f = 9.3 Hz

Find: I

Solution: The natural frequency of the machine is

$$\omega_n = (9.3 \text{ cycles/s})(2\pi \text{ rad/cycle}) = 58.4 \text{ rad/s}$$

The natural frequency of the system is given by

$$\omega_n = \sqrt{\frac{k_b}{m}}$$

where k_b, the stiffness of the simply supported beam at its midspan, is

$$k_b = \frac{48EI}{L^3}$$

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The moment of inertia is determined in terms of the natural frequency as

$$I = \frac{m\omega_n^2 L^3}{48E}$$
$$I = 4.66 \times 10^{-4}\ \text{m}^4$$

Problem 3.6 illustrates the one-degree-of-freedom model of a machine on a simply supported beam.

3.7 A one degree-of-freedom model of a 9-m steel flagpole (ρ = 7400 kg/m^3, E = 200 × 10^9 N/m^2, G = 80 × 10^9 N/m^2) is that of a beam fixed at one end and free at one end. The flagpole has an inner diameter of 4 cm and an outer diameter of 5 cm.

(a) Approximate the natural frequency of transverse vibration.
(b) Approximate the natural frequency of torsional vibration.

Given: L = 9 m, ρ = 7400 kg/m^3, E = 200 × 10^9 N/m^2, G = 80 × 10^9 N/m^2, d_1 = 4 cm, d_2 = 5 cm

Find: (a) ω_n for transverse vibrations, (b) ω_n for torsional oscillations

Solution: The system is modeled using one-degree-of-freedom. All inertia effects will be lumped at the end of the beam. The moments of inertia of the beam are

$$J = \frac{\pi}{32}\left(d_2^4 - d_1^4\right) = 3.67 \times 10^{-8}\ \text{m}^4$$
$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 1.84 \times 10^{-8}\ \text{m}^4$$

(a) The transverse stiffness of the beam at its end is

$$k = \frac{3EI}{L^3} = 15.14\ \text{N/m}$$

The result for the equivalent mass at the end of the flagpole is

$$m_{eq} = 0.2357 m_b = 0.2357 \rho A L = 0.2357 \rho \pi \frac{d_2^2 - d_1^2}{4} L = 0.125\ \text{kg}$$

Thus the natural frequency approximation is

$$\omega_n = \sqrt{\frac{k}{m_{eq}}} = 11.1\ \text{rad/s}$$

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(b) The torsional stiffness of the beam is

$$k_t = \frac{JG}{L} = 316.2 \text{ N} \cdot \text{m/rad}$$

The equivalent mass moment of inertia of the shaft is

$$I_{eq} = \frac{1}{3}\rho JL = 8.15x10^{-4} \text{ kg} \cdot \text{m}^2$$

Thus the natural frequency for torsional oscillations is

$$\omega_n = \sqrt{\frac{k_t}{I_{eq}}} = 622.9 \text{ rad/s}$$

Problem 3.7 illustrates the one-degree-of-freedom models for transverse vibrations of beams and torsional oscillations of shafts.

3.8 A 250 kg compressor is to be placed at the end of a 2.5-m fixed-free steel ($E = 200\times10^9$ N/m^2) beam. Specify the allowable moment of inertia of the beam's cross section about its neutral axis such that the natural frequency of the machine is outside the range of 100 to 130 Hz.

Given: m = 250kg, L = 2.5-m, f < 100 Hz or f > 130 Hz, $E = 200 \times 10^9$ N/m^2

Find: I

Solution: The equivalent stiffness of a fixed-free beam at its end is

$$k = \frac{3EI}{L^3}$$

The natural frequency of the compressor is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{mL^3}}$$

Thus to require the natural frequency to be less than 100 Hz = 628 rad/s

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$$\sqrt{\frac{3EI}{mL^3}} < 628$$

$$I < (628)^2 \frac{mL^3}{3E}$$

$$I < 2.6 \times 10^{-3} \text{ m}^4$$

In order for the natural frequency to be greater than 130 Hz = 816 rad/s

$$I > (816)^2 \frac{mL^3}{3E}$$

$$I > 4.3 \times 10^{-3} \text{ m}^4$$

Thus in order for the natural frequency to be outside of the range from 100 Hz to 130 Hz

$$I < 2.6 \times 10^{-3} \text{ m}^4 \quad \text{or} \quad I > 4.3 \times 10^{-3} \text{ m}^4$$

Problem 3.8 illustrates the design of the cross-section of a beam such that the natural frequency of a machine attached to the beam is outside of a specified frequency range.

3.9 A 50 kg pump is to be placed at the midspan of a 2.8-m simply supported steel (E = 200 × 10^9 N/m^2) beam. The beam is of rectangular cross section of width 25 cm. What are the allowable values of the cross-sectional height such that the natural frequency is outside the range 50 to 75 Hz?

Given: m = 50 kg, L = 2.8 m, E = 200 × 10^9 N/m^2, w = 25 cm, 50 Hz < ω_n < 75 Hz

Find: h

Solution: The equivalent stiffness of the beam is

$$k = \frac{48EI}{L^3}$$

For a beam of rectangular cross-section of width w and height h

$$I = \frac{1}{12} wh^3$$

The natural frequency of the pump is

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$$\omega_n = \sqrt{\frac{k}{m}}$$

Combining the above equations leads to

$$\omega_n = \sqrt{\frac{4Ewh^3}{mL^3}} = 1.35\times10^4 h^{3/2}$$

In order for the natural frequency to be less than 50 Hz = 314.2 rad/s

$$314.2 > 1.35\times10^4 h^{3/2}$$
$$h < 0.0815 \text{ m}$$

In order for the natural frequency to be greater than 75 Hz = 471.2 rad/s

$$471.2 < 1.35\times10^4 h^{3/2}$$
$$h > 0.1068 \text{ m}$$

Problem 3.9 illustrates the natural frequency calculations for a one-degree-of-freedom model of a machine attached to a beam.

3.10 A diving board is modeled as a simply supported beam with an overhang. What is the natural frequency of a 140-lb diver at the end of the diving board of Figure P3.10?

FIGURE P3.10

Given: W = 140 lb, L = 10 ft, $E = 200 \times 10^9$ N/m^2 = 30×10^6 lb/in^2, w = 2 ft, t = 1 in.

Find: ω_n

Solution: The diver and diving board are modeled as a mass at the end of a pinned-pinned beam with an overhang. The moment of inertia of the cross section is

$$I = \frac{1}{12}wt^3 = \frac{1}{12}(2\text{ ft})\left(1\text{ in}\frac{1\text{ ft}}{12\text{ in}}\right)^3 = 9.65\times10^{-5}\text{ ft}^4$$

Appendix D is used to determine the stiffness of the beam. Entry 6 of Table D2 is used with

$$a = 10\text{ ft} \quad z_1 = 4\text{ ft}$$

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leading to

$$R_1 = -2.5$$
$$C_1 = 1.5$$
$$C_3 = -4$$

The deflection of the beam at its end due to a concentrated load at its end is

$$y(a) = \frac{1}{EI}\left[\frac{1}{6}R_1(a - z_1)^3 + \frac{1}{6}C_1 a^3 + C_3 a\right] = 2.88 \times 10^{-4} \text{ ft}$$

The equivalent stiffness of the beam is

$$k = \frac{1}{y(a)} = 3.47 \times 10^3 \text{ lb/ft}$$

The natural frequency of the diver is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3.47 \times 10^3 \text{ ft}}{(140 \text{ lb})/(32.2 \text{ ft/s}^2)}} = 28.3 \text{ rad/s}$$

Problem 3.10 illustrates the use of a one-degree-of-freedom approximation to obtain the natural frequency of a mass attached to a pinned-pinned beam with an overhang.

3.11 A diver is able to slightly adjust the location of the intermediate support on the diving board. What is the range of natural frequencies a 140 lb diver can attain if the distance between the supports can be adjusted between 4 and 6.5 ft?

FIGURE P3.10

Given: W = 140 lb, L = 10 ft, $E = 200 \times 10^9$ N/m^2 = 30×10^6 lb/in^2, w = 2 ft, t = 1 in, 4 ft < a < 6.5 ft

Find: range of ω_n

Solution: The diver and diving board are modeled as a mass at the end of a pinned-pinned beam with an overhang. The moment of inertia of the cross section is

$$I = \frac{1}{12}wt^3 = 9.65 \times 10^{-5} \text{ ft}^4$$

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The stiffness of the beam varies depending on the location of the pin support. Appendix D and Table D2 are used to determine the deflection of the end of the beam due to a concentrated load applied at its end. The equation for the deflection of the beam is

$$y(L) = \frac{1}{EI}\left[\frac{1}{6}R_1(L - z_1)^3 + \frac{1}{6}C_1L^3 + C_3L\right]$$

where z_1 is the distance to the right pin support from the left end of the diving board and

$$R_1 = -\frac{L}{z_1}$$

$$C_1 = \frac{L}{z_1} - 1$$

$$C_3 = \left(1 - \frac{L}{z_1}\right)\frac{z_1^2}{6}$$

For z_1=4.0 ft, y(10 ft) = 2.88×10^{-4} ft, k=1/y(10 ft) = 3.47×10^3 lb/ft.
For z_1=6.5 ft, y(10 ft) = 8.15×10^{-5} ft, k=1/y(10 ft) = 1.23×10^4 lb/ft.

The natural frequency of the diver is calculated by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}}$$

Thus the range of natural frequencies is

$$28.3\ \text{rad/s} < \omega_n < 53.1\ \text{rad/s}$$

Problem 3.11 illustrates the natural frequency calculation when a one-degree-of-freedom model is used for a mass on a pinned-pinned beam with an overhang.

3.12 A 60 kg drum of waste material is being hoisted by an overhead crane and winch system as illustrated in Figure P3.12. The system is modeled as a simply supported beam to which the cable is attached. The drum of waste material is attached to the end of the cable. When the length of the cable is 6 m, the natural period of the system is measured as 0.3 s. What is the mass of the waste material?

FIGURE P3.12

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Given: m = 60kg, T = 0.3 s

Beam: $E = 200 \times 10^9$ N/m^2, L = 2 m, I = 2.6×10^{-4} m^4

Cable: $E = 200 \times 10^9$ N/m^2, r = 8 cm, L = 6 m

Find: m_w

Solution: The equivalent stiffness of the beam is

$$k_b = \frac{48EI}{L^3} = \frac{48(200\times10^9\ \text{N/m}^2)(2.6\times10^{-4}\ \text{m}^4)}{(2\ \text{m})^3} = 3.18\times10^8\ \text{N/m}$$

The equivalent stiffness of the cable is

$$k_c = \frac{EA}{L} = \frac{(200\times10^9\ \text{N/m}^2)\pi(0.08\ \text{m})^2}{6\ \text{m}} = 6.70\times10^8\ \text{N/m}$$

The beam and cable act in series. The equivalent stiffness of the series combination is

$$k_{eq} = \frac{1}{\dfrac{1}{k_b}+\dfrac{1}{k_c}} = 2.16\times10^8\ \text{N/m}$$

The natural frequency of the system is

$$\omega_n = \frac{\mathbf{2}\pi}{T} = \frac{\mathbf{2}\pi}{\mathbf{0.3}\ \text{s}} = \mathbf{20.94}\ \text{rad/s}$$

The natural frequency in terms of the system parameters is

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$$

$$m_{eq} = \frac{k_{eq}}{\omega_n^2} = \frac{2.16\times10^8\ \text{N/m}}{(20.94\ \text{rad/s})^2} = 4.93\times10^6\ \text{kg}$$

Thus the mass of the waste material is

$$m_w = m_{eq} - m = 4.93\times10^6\ \text{kg}$$

Problem 3.12 illustrates the natural frequency of a SDOF system.

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3.13 A 200-kg package is being hoisted by a 120-mm-diameter steel cable ($E = 200 \times 10^9$ N/m^2) at a constant velocity v. What is the largest value of v such that the cable's elastic strength of 560×10^6 N/m^2 is not exceeded if the hoisting mechanism suddenly fails when the cable has a length of 10 m?

Given: m = 200 kg, $E = 200 \times 10^9$ N/m^2, d = 120 mm, L = 10 m, $\sigma_y = 560 \times 10^6$ N/m^2

Find: v

Solution: The stiffness of the cable when the mechanism fails is

$$k = \frac{AE}{L} = \frac{\pi d^2 E}{4L} = 2.26 \times 10^8 \text{ N/m}$$

After the hoisting mechanism fails the system is modeled as a one-degree-of-freedom undamped system of natural frequency

$$\omega_n = \sqrt{\frac{k}{m}} = 1.06 \times 10^3 \text{ rad/s}$$

Let x(t) be the displacement of the package from its equilibrium position when L = 10 m. Oscillations occur about the equilibrium position. The initial conditions for the oscillations are

$$x(0) = 0 \quad \dot{x}(0) = v$$

The oscillations are described by

$$x(t) = \frac{v}{\omega_n} \sin(\omega_n t)$$

The maximum force developed in the cable is the gravity force of the package plus the maximum force due to the oscillations

$$F = mg + k\frac{v}{\omega_n}$$

The maximum normal stress is

$$\sigma = \frac{F}{A} = \frac{4\left(mg + k\dfrac{v}{\omega_n}\right)}{\pi d^2} = 1.73 \times 10^5 + 1.88 \times 10^7 v$$

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Thus in order for the maximum yield stress to not be exceeded

$$\sigma_y = 560\times10^6 \text{ N/m} > 1.73\times10^5 + 1.88\times10^7 v$$
$$v < 29.8 \text{ m/s}$$

Problem 3.13 illustrates the free vibration response of an undamped one-degree-o-freedom system due to an initial velocity.

3.14 Determine the natural frequency of the system of Figure P2.43.

FIGURE P 2.43

Given: system shown

Find: ω_n

Solution: The differential equation governing the motion of the system is derived in the solution of Chapter Problem 2.43 as

$$\ddot{x} + \frac{k}{2\left(2m + \frac{I}{2r^2}\right)} x = 0$$

The natural frequency is obtained from the differential equation as

$$\omega_n = \sqrt{\frac{k}{2\left(2m + \frac{I}{2r^2}\right)}}$$

Problem 3.14 illustrates the determination of the natural frequency of a system from the differential equation governing the motion of the system.

3.15 Determine the natural frequency and damping ratio of the system of Figure P2.45.

FIGURE P 2.45

Given: system shown

Find: ω_n

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Solution: The differential equation governing the motion of the system is derived in the solution of Chapter Problem 2.45 as

$$\ddot{\theta}+\frac{18}{7}\frac{c}{m}\dot{\theta}+\frac{30}{7}\frac{k}{m}\theta=0$$

The natural frequency is obtained from the differential equation as

$$\omega_n=\sqrt{\frac{30\mathrm{k}}{7\mathrm{m}}}$$

The damping ratio is given by

$$2\zeta\omega_n=\frac{18c}{7m}\Rightarrow\zeta=\frac{9c}{7m\omega_n}=\frac{9c}{7m}\sqrt{\frac{7m}{30k}}=\frac{9c}{\sqrt{210mk}}$$

Problem 3.15 illustrates the determination of the natural frequency of a system from the differential equation governing the motion of the system.

3.16 Determine the natural frequency and damping ratio for the system of Figure P2.47.

Given: system shown

Find: ω_n

FIGURE P 2.47

Solution: The differential equation governing the motion of the system is derived in the solution of Chapter Problem 2.47 as

$$\ddot{\theta}+\frac{75c}{91m}\dot{\theta}+\frac{408k}{91m}\theta=0$$

The natural frequency is obtained from the differential equation as

$$\omega_n=\sqrt{\frac{408\mathrm{k}}{91\mathrm{m}}}$$

The damping ratio is

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$$2\zeta\omega_n = \frac{75c}{91m} \Rightarrow \zeta = \frac{75c}{182m\omega_n} = \frac{75c}{182m}\sqrt{\frac{91m}{408k}} = \frac{75c}{8\sqrt{9282mk}}$$

Problem 3.16 illustrates the determination of the natural frequency of a system from the differential equation governing the motion of the system.

3.17 Determine the natural frequency and damping ratio for the system of Figure P2.49.

Slender bar of mass m connected to blocks through rigid links at A and B

FIGURE P 2.49

Given: system shown

Find: ω_n

Solution: The differential equation governing the motion of the system is derived in the solution of Chapter Problem 2.49 as

$$\ddot{\theta} + \frac{3c}{5m}\dot{\theta} + \frac{3k}{5m}\theta = 0$$

The natural frequency is obtained from the differential equation as

$$\omega_n = \sqrt{\frac{3\mathrm{k}}{5\mathrm{m}}}$$

The damping ratio is

$$2\zeta\omega_n = \frac{3c}{5m} \Rightarrow \zeta = \frac{3c}{10m\omega_n} = \frac{3c}{10m}\sqrt{\frac{5m}{3k}} = \frac{3c}{2}\sqrt{\frac{3}{5km}}$$

Problem 3.17 illustrates the determination of the natural frequency of a system from the differential equation governing the motion of the system.

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3.18 Determine the natural frequency and damping ratio for the system of Figure P2.53.

FIGURE P 2.53

Given: system shown

Find: ω_n

Solution: The differential equation governing the motion of the system is derived in the solution of Chapter Problem 2.53 as

$$\ddot{x} + \frac{kr^2}{I + 4mr^2} x = 0$$

The natural frequency is obtained from the differential equation as

$$\omega_n = \sqrt{\frac{kr^2}{I + 4mr^2}}$$

The damping ratio is zero since the system is undamped.

Problem 3.18 illustrates the determination of the natural frequency of a system from the differential equation governing the motion of the system.

3.19 The inertia of the elastic elements is negligible. What is the natural frequency of the system assuming a SDOF model is used? See Figure P3.19.

FIGURE P3.19

Given: m = 150 kg, L = 0.8 m, $E = 210 \times 10^9$ N/m^2, $I = 1.6 \times 10^{-5}$ m^4

Find: ω_n

Solution: The system is modeled as a one-degree-of-freedom mass-spring system. The generalized coordinate is the displacement of the point on the beam where the mass is attached. The equivalent stiffness is the reciprocal of the beam's end deflection due to a concentrated unit load applied at its end

$$k = \frac{3EI}{L^3} = \frac{3\left(210\times10^9 \frac{\text{N}}{\text{m}^2}\right)\left(1.6\times10^{-5}\ \text{m}^4\right)}{(0.8\,\text{m})^3} = 1.97\times10^7 \frac{\text{N}}{\text{m}}$$

The differential equation governing free vibrations is

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$$150\ddot{x}+1.97\times10^{7}\,x=0$$

from which the natural frequency is determined as

$$\omega_n=\sqrt{\frac{1.97\times10^{7}\,\frac{\mathrm{N}}{\mathrm{m}}}{150\,\mathrm{kg}}}=362.8\,\frac{\mathrm{rad}}{\mathrm{s}}$$

Problem 3.19 illustrates the modeling of a concentrated mass on an elastic element by a one-degree-of-freedom system.

3.20 The inertia of the elastic elements is negligible. What is the natural frequency of the system assuming a SDOF model is used? See Figure P3.20.

FIGURE P3.20

Given: system shown

Find: ω_n

Solution: The longitudinal motion of the block is modeled as a block attached to two springs. The equivalent stiffnesses of the springs are

$$k_{AB}=\frac{A_{AE}E_{AB}}{L_{AB}}=\frac{\left(2.1\times10^{-4}\,\mathrm{m}^2\right)\left(210\times10^{9}\,\frac{\mathrm{N}}{\mathrm{m}^2}\right)}{0.65\mathrm{m}}=6.78\times10^{7}\,\frac{\mathrm{N}}{\mathrm{m}}$$

$$k_{BC}=\frac{A_{BC}E_{BC}}{L_{BC}}=\frac{\left(2.1\times10^{-4}\,\mathrm{m}^2\right)\left(180\times10^{9}\,\frac{\mathrm{N}}{\mathrm{m}^2}\right)}{0.35\mathrm{m}}=1.08\times10^{8}\,\frac{\mathrm{N}}{\mathrm{m}}$$

The two springs act in parallel and can be replaced by a single spring whose equivalent stiffness is

$$k_{eq.}=k_{AB}+k_{BC}=1.76\times10^{8}\,\frac{\mathrm{N}}{\mathrm{m}}$$

The equivalent system method is used to write the governing differential equation as

$$165\ddot{x}+1.76\times10^{8}\,x=0$$

or

$$\ddot{x}+1.07\times10^{6}\,x=0$$

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The system's natural frequency is calculated as

$$\omega_n = \sqrt{1.07 \times 10^6} = 1030 \frac{\text{rad}}{\text{sec}}$$

Problem 3.20 illustrates application of the equivalent systems method to derive the differential equation for systems where structural elements are used as springs.

3.21 The inertia of the elastic elements is negligible. What is the natural frequency of the system assuming a SDOF model is used? See Figure P3.21.

Given: Fixed-free beam with overhang, L = 1 m, $x_1 = 0.6$ m, $E = 180 \times 10^9$ N/m^2, $I = 4.6 \times 10^{-4}$ m^4, m = 65 kg

FIGURE P3.21

Find: ω_n

Solution: The equivalent stiffness of the beam at the location where the mass is attached is determined using Table D.2 with a = 1 m, $z_1 = 0.6$ m, and z = 1 m. The constants are evaluated as

$$C_1 = -\frac{3}{2} + \frac{3}{2}\frac{a}{z_1} = 1.0$$

$$C_2 = \frac{z_1}{2}\left(1 - \frac{a}{z_1}\right) = -0.2$$

$$C_3 = C_4 = 0$$

The reaction at the intermediate support is

$$R = \frac{1}{2} - \frac{3}{2}\frac{a}{z_1} = -2.0$$

The appropriate deflection equation for z =1 m is

$$y(z = 1\,\text{m}) = \frac{1}{EI}\left[\frac{1}{6}R(1 - z_1)^3 + \frac{C_1}{6} + \frac{C_2}{2}\right] = \frac{0.0454}{EI}$$

Hence the stiffness is

$$k = \frac{1}{y(1\,\text{m})} = \frac{EI}{0.0454} = 1.83 \times 10^9 \frac{\text{N}}{\text{m}^2}$$

The governing differential equation is

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$$65\ddot{x}+1.83\times10^{9}\,x=0$$

The natural frequency is

$$\omega_n=\sqrt{\frac{1.83\times10^{9}\,\frac{\text{N}}{\text{m}}}{65\,\text{kg}}}=5300\,\frac{\text{rad}}{\text{sec}}$$

Problem 3.21 illustrates (a) modeling of a mass attached to a beam using one degree of freedom and (b) use of Table D.2.

3.22 The inertia of the elastic elements is negligible. What is the natural frequency of the system assuming a SDOF model is used? See Figure P3.22.

FIGURE P3.22

Given: $E=200\times10^{9}\ \text{N/m}^2,\ I=4.23\times10^{-6}\ \text{m}^4,\ L=1.8\,\text{m}$
$m=200\,\text{kg},\ k_1=5\times10^{4}\ \text{N/m},\ k_2=8\times10^{4}\ \text{N/m}$

Find: ω_n

Solution: The equivalent stiffness of the beam is

$$k_b=\frac{3EI}{L^3}=\frac{3(200\times10^{9}\ \text{N/m}^2)(4.23\times10^{-6}\ \text{m}^4)}{(1.8\,\text{m})^3}=4.35\times10^{5}\ \text{N/m}$$

The upper spring is in parallel with the beam. The parallel combination is in series with the lower spring. The equivalent stiffness of the combination is

$$k_{eq}=\frac{1}{\dfrac{1}{k_2}+\dfrac{1}{k_b+k_1}}=6.87\times10^{4}\ \text{N/m}$$

The system is modeled as a 200-kg block suspended from a spring of stiffness k_{eq}. The natural frequency of the system is

$$\omega_n=\sqrt{\frac{k_{eq}}{m}}=\sqrt{\frac{6.87\times10^{4}\ \text{N/m}}{200\,\text{kg}}}=18.53\,\text{rad/s}$$

Problem 3.22 illustrates the natural frequency of a one-degree-of-freedom system.

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3.23 The inertia of the elastic elements is negligible. What is the natural frequency of the system assuming a SDOF model is used? See Figure P3.23.

FIGURE P3.23

Given:

$$r_1 = 8\text{ mm}, G_1 = 60\times10^9\text{ N/m}^2, L_1 = 60\text{ cm}, r_2 = 6\text{ mm}, G_2 = 80\times10^9\text{ N/m}^2, L_2 = 40\text{ cm},$$
$$I = 8.3\text{ kg-m}^2$$

Find: ω_n

Solution: The polar moments of inertia of the shafts are calculated as

$$J_1 = \frac{\pi}{2} r_1^4 = 6.43\times10^{-9}\text{ m}^4$$
$$J_2 = \frac{\pi}{2} r_2^4 = 2.04\times10^{-9}\text{ m}^4$$

The torsional stiffnesses of each shaft are

$$k_1 = \frac{J_1 G_1}{L_1} = 6.43\times10^2\text{ N}\cdot\text{m/rad}$$
$$k_2 = \frac{J_2 G_2}{L_2} = 4.07\times10^2\text{ N}\cdot\text{m/rad}$$

The two shafts act as torsional springs in series. The equivalent stiffness of the combination is

$$k_{eq} = \frac{1}{\frac{1}{k_1}+\frac{1}{k_2}} = 2.49\times10^2\text{ N}\cdot\text{m/rad}$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k_{eq}}{I}} = 5.48\text{ rad/s}$$

Problem 3.23 illustrates the natural frequency for a system with torsional springs in series.

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3.24 The center of the disk of Figure P3.24 is displaced a distance δ from its equilibrium and released. Determine $x(t)$ if the disk rolls without slip.

FIGURE P3.24

Given: δ, m, k

Find: $x(t)$

Solution: The differential equation governing $x(t)$ is derived as

$$\ddot{x} + \frac{2k}{3m}x = 0 \tag{1}$$

The system has a natural frequency of

$$\omega_n = \sqrt{\frac{2k}{3m}}$$

From the information given the initial conditions are

$$x(0) = \delta \tag{2}$$

and

$$\dot{x}(0) = 0 \tag{3}$$

The solution of eq.(1) is

$$x = A\cos\omega_n t + B\sin\omega_n t \tag{4}$$

where A and B are constants of integration. Application of eq. (2) to eq. (4) leads to

$$A = \delta$$

Application of eq. (3) to eq. (4) leads to

$$B = 0$$

Thus

$$x(t) = \delta\cos\sqrt{\frac{2k}{3m}}t$$

Problem 3.24 illustrates (a) the free vibration response of a one-degree-of-freedom system and (b) application of initial conditions to determine constants of integration.

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3.25 The coefficient of friction between the disk and the surface in Figure P3.24 is μ. What is the largest initial velocity of the mass center that can be imparted such that the disk rolls without slip for its entire motion?

FIGURE P3.24

Given: μ, m, k, r

Find: v_0 such that disk rolls without slip

Solution: The differential equation governing the free vibration response of this system, assuming no slip between the disk and the surface, is

$$\ddot{x} + \frac{2k}{3m}x = 0 \tag{1}$$

The system's natural frequency is

$$\omega_n = \sqrt{\frac{2k}{3m}}$$

The system is in equilibrium when the center of the disk is given an initial velocity v_0. The initial conditions are $x(0) = 0, \dot{x}(0) = v_0$. The solution of eq.(l) subject to these initial conditions is

$$x(t) = \frac{v_0}{\omega_n}\sin\omega_n t$$

Free-body diagrams of the system at an arbitrary instant are shown below

Summing moments about the mass center of the disk leads to

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$$\left(\sum M_C\right)_{ext} = \left(\sum M_C\right)_{eff}$$

$$Fr = \frac{1}{2}mr^2\frac{\ddot{x}}{r}$$

$$F = \frac{1}{2}m\ddot{x}$$

$$= -\frac{1}{2}mv_0\omega_n \sin\omega_n t$$

The maximum friction force that can be developed is

$$F_{\max} = \mu mg$$

If this maximum value is exceeded, the no slip assumption is incorrect. Thus, in order for the no slip assumption to be valid

$$\mu mg > v_0\sqrt{\frac{km}{6}}$$

$$v_0 < \mu g\sqrt{\frac{6m}{k}}$$

Problem 3.25 illustrates (a) application of initial conditions to determine the free vibration response of a one-degree-of-freedom system and (b) the slip assumption for rolling bodies.

3.26 For the system shown in Figure P3.26:

(a) Determine the damping ratio
(b) State whether the system is underdamped, critically damped, or overdamped
(c) Determine $x(t)$ or $\theta(t)$ for the given initial conditions

FIGURE P3.26

Given: m = 12.5 kg, c = 750 N-s/m, $k_1 = 4 \times 10^4$ N/m, $k_2 = 3 \times 10^4$ N/m, x(0) = 3 cm, $\dot{x}(0) = 0$

Find: (a) ζ (b) nature of damping (c) x(t)

Solution: The two springs act in parallel with an equivalent stiffness

$$k_{eq} = k_1 + k_2 = 7x10^4 \text{ N/m}$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 74.83 \text{ rad/s}$$

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(a) The damping ratio of the system is

$$\zeta = \frac{c}{2m\omega_n} = 0.40$$

(b) Since the damping ratio is less than 1 the system is underdamped.
(c) The response of an underdamped system is given by Eq.(2.63)

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi_d)$$

where the damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 68.6\ \text{rad/s}$$

Note that the initial conditions are given with a non-zero initial displacement and a zero initial velocity. For these conditions,

$$A = \sqrt{x_0^2 + \left(\frac{\zeta\omega_n x_o}{\omega_d}\right)^2} = x_0\sqrt{1+\frac{\zeta^2}{1-\zeta^2}} = \frac{x_0}{\sqrt{1-\zeta^2}} = \frac{0.03\ \text{m}}{\sqrt{1-(0.4)^2}} = 0.0327\ \text{m}$$

$$\phi_d = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right) = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{\sqrt{1-(0.4)^2}}{0.4}\right) = 1.153\ \text{rad}$$

Noting that $\zeta\omega_n = (0.4)(74.83) = 30$ the system response is

$$x(t) = 0.0327e^{-30t}\sin(68.56t + 1.153)\ \text{m}$$

Problem 3.26 illustrates the free-vibration response of an underdamped one-degree-of-freedom linear system.

3.27 For the system shown in Figure P3.27:

(a) Determine the damping ratio; (b) State whether the system is underdamped, critically damped, or overdamped; (c) Determine $x(t)$ or $\theta(t)$ for the given initial conditions

Given: $k = 3.2 \times 10^4$ N/m, $c = 150$ N · s/m,
$r = 10$ cm, $I_p = 0.3$ kg · m^2, $m_1 = 5$ kg,
$m_2 = 40$ kg, $\theta(0) = 0$, $\dot{\theta}(0) = 2.5$ rad/s

Find: (a) ζ (b) nature of damping (c) $\theta(t)$

FIGURE P3.27

Solution: The differential equation is derived using the equivalent systems method. Let $x_1 = 3r\theta$ be the downward displacement of the block of mass m_1 and $x_2 = r\theta$ be the upward displacement of the block of mass m_2. The kinetic energy of the system at an arbitrary instant is

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$$T = \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_1 (3r\dot{\theta})^2 + \frac{1}{2} m_2 (r\dot{\theta})^2$$

$$T = \frac{1}{2}\left(I_p + 9r^2 m_1 + r^2 m_2\right)\dot{\theta}^2$$

Since an angular displacement is chosen as the generalized coordinate the appropriate model is the torsional system. The equivalent moment of inertia is

$$I_{eq} = I_p + 9r^2 m_1 + r^2 m_2 = 1.15\,\mathrm{kg}\cdot\mathrm{m}^2$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2} k(3r\theta)^2 = \frac{1}{2}\left(9kr^2\right)\theta^2$$

from which the equivalent torsional stiffness if obtained as

$$k_{t_{eq}} = 9kr^2 = 2.88\times 10^3\ \mathrm{N}\cdot\mathrm{m/rad}$$

The work done by the viscous damper between two arbitrary positions is

$$W_{1\to 2} = -\int c\left(r\dot{\theta}\right)\mathrm{d}(\mathrm{r}\theta) = -\int cr^2\dot{\theta}\, d\theta$$

from which the equivalent viscous damping coefficient is

$$c_{t_{eq}} = cr^2 = 1.5\ \mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s/rad}$$

Thus the differential equation governing the motion of the system is

$$1.15\ddot{\theta} + 1.5\dot{\theta} + 2.88\times 10^3\theta = 0$$

$$\ddot{\theta} + 1.304\dot{\theta} + 2.504\times 10^3\theta = 0$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k_{t_{eq}}}{I_{eq}}} = \sqrt{2.504\times 10^3} = 50.043\ \mathrm{rad/s}$$

(a) The damping ratio is obtained from

$$\zeta = \frac{c_{eq}}{2I_{eq}\omega_n} = 0.013$$

(b) Since the damping ratio is less than 1 the system is underdamped.
(c) The free-vibration response of an underdamped one-degree-of-freedom system is

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$$\theta(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi_d)$$

where the damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$\omega_d = 50.0392 \text{ rad/s}$$

For the given initial conditions,

$$A = \dot{\theta}(0)/\omega_d = 0.050 \text{ rad}$$
$$\phi_d = \tan^{-1}(0) = 0$$

Noting that $\zeta\omega_n = (0.013)(50.043) = 0.651$, the free-vibration response is

$$\theta(t) = 0.050e^{-0.651t}\sin(50.0392t) \text{ rad}$$

Problem 3.27 illustrates the free-vibration response of an underdamped one-degree-of-freedom system.

3.28 For the system shown in Figure P3.28:

(a) Determine the damping ratio
(b) State whether the system is underdamped, critically damped, or overdamped
(c) Determine $x(t)$ or $\theta(t)$ for the given initial conditions

FIGURE P3.28

Given: G = 60×10^9 N/m^2, L = 1.3 m, J = 2.5×10^{-7} m^4, r = 40 cm, m = 10 kg, m_d = 22.5 kg, k = 1×10^5 N/m, M_0 = 280 N · m, c_t = 60 N · s · m/rad
Find: (a) ζ, (b) nature of damping, (c) $\theta(t)$

Solution: The torsional stiffness of the shaft is

$$k_t = \frac{JG}{L} = 1.15\times10^4 \text{ N}\cdot\text{m/rad}$$

The mass moment of inertia of the disk is

$$I = \frac{1}{2}m_d r^2 = 1.8 \text{ kg}\cdot\text{m}^2$$

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The differential equation governing θ(t), the clockwise angular displacement of the disk from the system's equilibrium position is derived using the equivalent systems method. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m(r\dot{\theta})^2 = \frac{1}{2}\left(I + mr^2\right)\dot{\theta}^2$$

Thus the equivalent moment of inertia for the system is

$$I_{eq} = I + mr^2 = 1.8\,\text{kg}\cdot\text{m}^2 + (10\,\text{kg})(0.4\,\text{m})^2 = 3.4\,\text{kg}\cdot\text{m}^2$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2} k_t\theta^2 + \frac{1}{2} k(r\theta)^2 = \frac{1}{2}(k_t + kr^2)\theta^2$$

Thus the equivalent torsional stiffness is

$$k_{t_{eq}} = k_t + kr^2 = 1.15\times10^4\ \text{N}\cdot\text{m/rad} + (1\times10^5\ \text{N/m})(0.4\,\text{m})^2 = 2.75\times10^4\ \text{N}\cdot\text{m/rad}$$

The work done by the torsional viscous damper between two arbitrary positions is

$$W_{1\to2} = -\int c_t\dot{\theta}\,\text{d}\theta$$

Thus the equivalent viscous damping coefficient is

$$c_{t_{eq}} = c_t = 60\,\text{N}\cdot\text{s}\cdot\text{m/rad}$$

The differential equation governing the motion of the system is

$$I_{eq}\ddot{\theta} + c_{t_{eq}}\dot{\theta} + k_{t_{eq}}\theta = 0$$
$$3.4\ddot{\theta} + 60\dot{\theta} + 2.75\times10^4\theta = 0$$

The differential equation is put in standard form by dividing by the coefficient of $\ddot{\theta}$ leading to

$$\ddot{\theta} + 17.65\dot{\theta} + 8.10\times10^3\theta = 0$$

The natural frequency is obtained from the differential equation as

$$\omega_n = \sqrt{8.10\times10^3} = 90.0\,\text{rad/s}$$

(a) The damping ratio is obtained from the differential equation

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$$2\zeta\omega_n = 17.65$$
$$\zeta = \frac{17.65}{2(90.0)} = 0.098$$

(b) Since $\zeta < 1$ the system is underdamped. Consider the free body diagrams of the disk and block when the system is in equilibrium in its initial position (noting that static deflection cancels with gravity). Note that if the initial angular displacement of the disk is θ_0 then the force developed in the spring is $F = kr\theta_0$ and the resisting moment from the shaft on the disk is $M = k_t\theta_0$. Thus summation of moments about the center of the disk leads to

$$\sum M_C = 0$$
$$-kr\theta_0 r - k_t\theta_0 + M_0 = 0$$
$$\theta_0 = \frac{M_0}{k_t + kr^2} = \frac{M_0}{k_{t_{eq}}}$$
$$\theta_0 = \frac{280\ \mathrm{N\cdot m}}{2.75\times10^4\ \mathrm{N\cdot m/rad}} = 0.0102\ \mathrm{rad}$$

Since the disk is released from rest from this position $\dot{\theta}(0) = 0$. The response of an underdamped system subject to these initial conditions is

$$\theta(t) = Ae^{-\zeta\omega t_n}\sin(\omega_d t + \phi_d)$$

where the damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 89.6\ \mathrm{rad/s}$$

and

$$A = \frac{\theta_0}{\sqrt{1-\zeta^2}} = 0.0102\ \mathrm{rad}$$
$$\phi_\mathrm{d} = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 1.47\ \mathrm{rad}$$

(c) The time dependent response of the system is

$$\theta(t) = 0.0102e^{-8.82t}\sin(89.6t + 1.47)\ \mathrm{rad}$$

Problem 3.28 illustrates the free-vibration response of an underdamped system.

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3.29 For the system shown in Figure P3.29:

(a) Determine the damping ratio
(b) State whether the system is underdamped, critically damped, or overdamped
(c) Determine $x(t)$ or $\theta(t)$ for the given initial conditions

FIGURE P3.29

Given: k = 50 N/m, m = 1.5 kg, L = 0.4 m, c = 100 N · s/m,
$\theta(0) = 0,\ \dot{\theta}(0) = 1.2$ rad/s

Find: (a) ζ (b) nature of damping (c) $\theta(t)$

Solution: Free-body diagrams of the system at an arbitrary instant are shown below.

Summing moments about the pin support

$$-c\frac{L}{4}\dot{\theta}\left(\frac{L}{4}\right) - k\frac{3L}{4}\theta\left(\frac{3L}{4}\right) + mg\frac{L}{4}\theta = \frac{1}{12}mL^2\ddot{\theta} + m\frac{L}{4}\ddot{\theta}\left(\frac{L}{4}\right)$$

$$\frac{7}{48}mL^2\ddot{\theta} + \frac{1}{16}cL^2\dot{\theta} + \left(\frac{9}{16}kL^2 - mg\frac{L}{4}\right)\theta = 0$$

The differential equation is put into standard form by dividing by the coefficient of the highest order derivative

$$\ddot{\theta} + \frac{3c}{7m}\dot{\theta} + \left(\frac{27k}{7m} - \frac{12g}{7L}\right)\theta = 0$$

The natural frequency is obtained as

$$\omega_n = \sqrt{\frac{27k}{7m} - \frac{12g}{7L}} = 9.30 \text{ rad.s}$$

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(a) The damping ratio is obtained from

$$2\zeta\omega_n = \frac{3c}{7m}$$

$$\zeta = \frac{3c}{14m\omega_n} = 1.54$$

(b) Since $\zeta > 1$ the system is overdamped
(c) The response of an overdamped system for the given initial conditions is

$$\theta(t) = \frac{\dot{\theta}(0)}{2\omega_n\sqrt{\zeta^2-1}} e^{-\zeta\omega_n t}\left(e^{\omega_n\sqrt{\zeta^2-1}t} - e^{-\omega_n\sqrt{\zeta^2-1}t}\right)$$

$$\theta(t) = 0.0553(e^{-3.39t} - e^{-25.19t})\text{ rad}$$

Problem 3.29 illustrates the free vibration response for an overdamped system.

3.30 For the system shown in Figure P3.30:

(a) Determine the damping ratio
(b) State whether the system is underdamped, critically damped, or overdamped
(c) Determine $x(t)$ or $\theta(t)$ for the given initial conditions

Given: 50 N force applied and released

Find: (a) ζ (b) nature of damping (c) $x(t)$

FIGURE P3.30

Solution: The link is assumed to be rigid and massless. The angular displacement of the link is assumed to be small such that when the 2 kg cart has moved a distance x to the right the 9 kg block has moved downward a distance $2x/3$. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}2\dot{x}^2 + \frac{1}{2}9\left(\frac{2\dot{x}}{3}\right)^2 = \frac{1}{2}6\dot{x}^2$$

Thus the equivalent mass is 6 kg. The potential energy is

$$V = \frac{1}{2}3000x^2 + \frac{1}{2}9000\left(\frac{2x}{3}\right)^2 = \frac{1}{2}7000x^2$$

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The equivalent stiffness is 7000 N/m. The equivalent viscous damping coefficient is $200\frac{\text{N}\cdot\text{s}}{\text{m}}$. The differential equation governing the motion of the system is

$$6\ddot{x} + 200\dot{x} + 7000x = 0$$

Putting the differential equation in standard form yields

$$\ddot{x} + 33.3\dot{x} + 1166.7x = 0$$

The natural frequency of the system is

$$\omega_n = \sqrt{1166.7} = 34.16\ \frac{\text{rad}}{\text{s}}$$

The damping ratio is determined as

$$\zeta = \frac{33.3}{2(34.16)} = 0.487$$

The system is underdamped since $\zeta < 1$. The work done in applying the initial force is

$$U = F\frac{2}{3}x(0) = \frac{1}{2}7000\,[x(0)]^2 \Longrightarrow x(0) = 4.77\ \text{mm}$$

With this initial condition and $\dot{x}(0) = 0$ the underdamped solution becomes

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi_d)$$

where

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 29.83\ \frac{\text{rad}}{\text{s}}$$

$$A = x(0)\sqrt{1 + \frac{\zeta^2}{1-\zeta^2}} = 5.46\ \text{mm}$$

$$\phi_d = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = 1.062\ \text{rad}$$

Thus the response is

$$x(t) = 5.46e^{-16.66t}\sin(29.83t + 1.062)\ \text{mm}$$

Problem 3.30 illustrates the free response of an underdamped system.

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3.31 For the system shown in Figure P3.31:

(a) Determine the damping ratio
(b) State whether the system is underdamped, critically damped, or overdamped
(c) Determine $x(t)$ or $\theta(t)$ for the given conditions

FIGURE P3.31

Solution: The vehicle has a natural frequency of

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{15000\ \frac{\text{N}}{\text{m}}}{150\ \text{kg}}} = 10\ \frac{\text{rad}}{\text{s}}$$

(a) The system's damping ratio is

$$\zeta = \frac{c}{2m\omega_n} = \frac{1000\ \frac{\text{N}\cdot\text{s}}{\text{m}}}{2(150\ \text{kg})\left(10\frac{\text{rad}}{\text{s}}\right)} = 0.333$$

(b) The system is underdamped.

(c) The initial conditions are

$$x(0) = -0.01\ \text{m}$$

$$\dot{x}(0) = 0$$

Subject to these initial conditions the solution for an underdamped system is

$$x(t) = x(0)\sqrt{\frac{1}{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-(0.333)^2}\ \frac{\text{rad}}{\text{s}} = 9.43\ \frac{\text{rad}}{\text{s}}$$

and

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-(0.333)^2}}{0.333} = 1.23\ \text{rad}$$

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Thus,

$$x(t) = \frac{-0.01}{\sqrt{1-(0.333)^2}} e^{-(0.333)(10)t} \sin(9.43t + 1.23)$$

$$= -0.0106 e^{-3.33t} \sin(9.43t + 1.23)$$

Problem 3.31 illustrates the underdamped free response of a SDOF system.

3.32 The amplitude of vibration of the system of Figure P3.32 decays to half of its initial value in 11 cycles with a period of 0.3 sec. Determine the spring stiffness and the viscous damping coefficient.

FIGURE P3.32

Given: $X_{11} = 1/2X_0$, T = 0.3 sec, J = 2.4 kg · m^2, m = 5 kg, $R_1 = 20$ cm, $R_2 = 40$ cm

Find: c, k

Solution: Let x represent the displacement of the block, measured positive downward from the system's equilibrium position. The equivalent system method is used to derive the differential equation governing free vibration. The angular rotation of the disk is

$$\theta = \frac{x}{R_1}$$

The change in length of the spring due to a displacement x is

$$x_2 = \frac{R_2}{R_1} x = 2x$$

The kinetic energy of the system is

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J\dot{\theta}^2 = \frac{1}{2}\left(m + \frac{J}{R_1^2}\right)\dot{\theta}^2$$

Hence the system's equivalent mass is

$$m_{eq} = m + \frac{J}{R_1^2} = 5\,kg + \frac{2.4\,kg \cdot m^2}{(0.2\ m)^2} = 65\ kg$$

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The potential energy of the system is

$$V = \frac{1}{2}kx_2^2 = \frac{1}{2}\left(4kx^2\right)$$

Hence the equivalent stiffness is

$$k_{eg} = 4k$$

The work done by the damping force is

$$W = -\int c\dot{x}dx$$

which implies that

$$c_{eq} = c$$

Thus the differential equation governing free vibrations of the system is

$$65\ddot{x} + c\dot{x} + 4kx = 0$$

$$\ddot{x} + \frac{c}{65}\dot{x} + \frac{4k}{65}x = 0$$

The natural frequency is determined from the differential equation as

$$\omega_n = \sqrt{\frac{4k}{65}}$$

The damping ratio is determined as

$$\zeta = \frac{c}{130\omega_n}$$

From the information given, the logarithmic decrement is

$$\delta = \frac{1}{11}\ln\left(\frac{x_0}{x_{11}}\right) = \frac{1}{11}\ln(2) = 0.0630$$

from which the damping ratio is calculated as

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.01$$

The damped natural frequency is

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$$\omega_d = \frac{2\pi}{T_d} = \frac{2\pi}{0.3\,\text{sec}} = 20.94\,\frac{\text{rad}}{\text{sec}}$$

The undamped natural frequency is obtained as

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 20.94\,\frac{\text{rad}}{\text{sec}}$$

The spring stiffness is

$$k = 65\,\frac{\omega_n^2}{4} = 7130\,\frac{\text{N}}{\text{m}}$$

The damping coefficient is

$$c = 130\zeta\omega_n = 27.2\,\frac{\text{N}\cdot\text{sec}}{\text{m}}$$

Problem 3.32 illustrates (a) application of the equivalent system method derive the differential equation governing free vibration of a one-degree-of-freedom system and (b) use of measured free vibration characteristics to determine system parameters.

3.33 The damping ratio of the system of Figure P3.33 is 0.3. How long will it take for the amplitude of free oscillation to be reduced to 2% of its initial value?

FIGURE P3.33

Given: $\zeta = 0.3$, $k = 2 \times 10^3$ N/m,
$m = 4.2$ kg,
$L = 1.1$ m

Find: t for $X = 0.2\,X_0$

Solution: Let θ be the clockwise angular displacement of the bar, measured from the systems equilibrium position. The equivalent system method is used to derive the governing differential equation. The kinetic energy of the system is

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

$$= \frac{1}{2}(4.2\,\text{kg})(0.45\,\text{m})^2\dot{\theta}^2 + \frac{1}{2}\frac{1}{12}(4.2\,\text{kg})(1.1\,\text{m})^2\dot{\theta}^2$$

$$= \frac{1}{2}\left(1.274\,\text{kg}\cdot\text{m}^2\right)\dot{\theta}^2$$

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The potential energy of the system is

$$V = \frac{1}{2}(0.4\,\text{m})^2\left(2\times10^3\,\frac{\text{N}}{\text{m}}\right)\theta^2 = \frac{1}{2}\left(320\,\frac{\text{N}\cdot\text{m}}{\text{rad}}\right)\theta^2$$

The work done by the damping force is

$$W = -\int c(1\,\text{m})\dot{\theta}\,d(1\,\text{m}\theta)$$

Using the equivalent system method, the governing differential equation becomes

$$1.274\ddot{\theta} + c\dot{\theta} + 320\theta = 0$$

The natural frequency is

$$\omega_n = \sqrt{\frac{320\,\frac{\text{N}\cdot\text{m}}{\text{rad}}}{1.274\,\text{kg}\cdot\text{m}^2}} = 15.85\,\frac{\text{rad}}{\text{s}}$$

The damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 15.85\,\frac{\text{rad}}{\text{s}}\sqrt{1-(0.3)^2} = 15.12\,\frac{\text{rad}}{\text{s}}$$

The logarithmic decrement is

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 1.98$$

The amplitude will be reduced to 2% of its initial value after n cycles where

$$\delta = \frac{1}{n}\ln(50)$$

The preceding equation gives n = 1.98. Thus the amplitude is reduced to less than 2% of its initial value in only 2 cycles. This occurs in

$$t = 2T_d = 2\frac{2\pi}{\omega_d} = 0.83\,\text{s}$$

Problem 3.33 illustrates (a) derivation of the differential equation governing free vibrations of a one-degree-of-freedom system, (b) the logarithmic decrement, and (c) the natural period of a damped system.

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3.34 When a 40-kg machine is placed on an elastic foundation, its free vibrations appear to decay exponentially with a frequency of 91.7 rad/s. When a 60-kg machine is placed on the same foundation, the frequency of the exponentially decaying oscillations is 75.5 rad/s. Determine the equivalent stiffness and equivalent viscous damping coefficient for the foundation.

Given : $m_1 = 40$ kg, $\omega_{d1} = 91.7$ rad/s, $m_2 = 60$ kg, $\omega_{d2} = 75.5$ rad/s

Find : k, c

Solution: When a machine is attached to the foundation, its free vibrations are modeled by a mass connected to a spring of stiffness k in parallel with a viscous damper of damping coefficient c. The frequency of damped free vibrations is

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \tag{1}$$

where the natural frequency and damping ratio are given by

$$\omega_n \sqrt{\frac{k}{m}} \tag{2}$$

and

$$\zeta = \frac{c}{2m\omega_n} \tag{3}$$

respectively. Substitution of eqs. (2) and (3) in eq.(1) leads to

$$\omega_d = \frac{1}{\sqrt{m}} \sqrt{k - \frac{c^2}{4m}} \tag{4}$$

Squaring eq. (4) and rearranging leads to

$$k - \frac{c^2}{4m} = m\omega_d^2 \tag{5}$$

When m = 40 kg, $\omega_d = 91.7$ rad/s. Substituting into eq. (5) leads to

$$k - \frac{c^2}{160} = 3.36 \times 10^5 \tag{6}$$

When m = 60 kg, $\omega_d = 75.5$ rad/s. Substituting into eq.(5) leads to

$$k - \frac{c^2}{240} = 3.42 \times 10^5 \tag{7}$$

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Equations (6) and (7) are solved simultaneously yielding

$$k = 3.53 \times 10^5 \ \frac{\text{N}}{\text{m}}$$

$$c = 1.65 \times 10^3 \ \frac{\text{kg}}{\text{s}}$$

Problem 3.34 illustrates (a) the relationship between damped natural frequency and undamped natural frequency, (b) the relationship between damping ratio and system parameters, and (c) the change of natural frequency and damping ratio with a change in system parameters.

3.35 A suspension system is being designed for a 1300-kg vehicle. When the vehicle is empty, its static deflection is measured as 2.5 mm. It is estimated that the largest cargo carried by the vehicle will be 1000 kg. What is the minimum value of the damping coefficient such that the vehicle will be subject to no more than 5 percent overshoot, whether it is empty or fully loaded?

Given: $m = 1300$ kg, $\Delta_{st} = 2.5$ mm, $m_c = 1000$ kg, $\eta/h = 0.05$

Find: c

Solution: The stiffness of the suspension system is determined from

$$\Delta_{st} = \frac{mg}{k}$$

$$k = \frac{mg}{\Delta_{st}} = \frac{(1300\,\text{kg})(9.81\,\text{m/s}^2)}{0.0025\,\text{mm}} = 5.1 \times 10^6 \ \text{N/m}$$

The damping ratio is given by

$$\zeta = \frac{c}{2\sqrt{mk}}$$

Since the damping ratio is smaller for larger masses, the maximum overshoot for a given suspension system will occur for the largest mass. Thus the suspension system is to be designed such that the overshoot is only 5 percent when it is carrying the maximum cargo. The damping ratio to limit the overshoot to five percent is

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$$\zeta = \frac{-\frac{1}{\pi}\ln\left(\frac{\eta}{h}\right)}{\sqrt{1+\left[-\frac{1}{\pi}\ln\left(\frac{\eta}{h}\right)\right]^2}} = \frac{-\frac{1}{\pi}\ln(0.05)}{\sqrt{1+\left[-\frac{1}{\pi}\ln(0.05)\right]^2}} = 0.69$$

Thus using the largest mass the damping coefficient is

$$c = 2\zeta\sqrt{mk}$$
$$c = 2(0.69)\sqrt{(2300\text{ kg})(5.1\times10^6\text{ N/m})}$$
$$c = 1.50\times10^5\text{ N}\cdot\text{s/m}$$

Problem 3.35 illustrates overshoot for an underdamped system.

3.36 During operation a 500-kg press machine is subject to an impulse of magnitude 5000 N · s. The machine is mounted on an elastic foundation that can be modeled as a spring of stiffness 8×10^5 N/m in parallel with a viscous damper of damping coefficient 6000 N · s/m. What is the maximum displacement of the press after the impulse is applied. Assume the press is at rest when the impulse is applied.

Given: m = 500 kg, I = 5000 N · s, k = 8×10^5 N/m, c =6 000 N-s/m

Find: x_{max}

Solution: The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = 40\text{ rad/s}$$

The damping ratio of the system is

$$\zeta = \frac{c}{2m\omega_n} = 0.15$$

The principle of impulse and momentum is used to determine the initial velocity imparted to the system due to the impulse

$$v = \frac{I}{m} = 10\text{ m/s}$$

The system is in equilibrium when the impulse is applied, thus x(0)=0. The free vibration response for this underdamped system is

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$$x(t) = \frac{v}{\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t)$$
$$x(t) = 0.253e^{-6t}\sin(39.55t)$$

The time at which the maximum occurs is obtained by setting

$$\frac{dx}{dt} = 0 = 0.253e^{-6t}\left[-6\sin(39.55t) + 39.55\cos(39.55t)\right]$$
$$0 = -6\sin(39.55t) + 39.55\cos(39.55t)$$
$$\tan(39.55t) = \frac{39.55}{6} = 6.59$$
$$t = 0.0359\text{ s}$$

Thus the maximum displacement is

$$x_{\max} = x(0.0359) = 0.253e^{-6(0.359)}\sin[39.55)(0.0359)]$$
$$x_{\max} = 0.202\text{ m}$$

Problem 3.36 illustrates the maximum displacement of an underdamped system.

3.37 For the press of Chapter Problem 3.36, determine (a) the force transmitted to the floor as a function of time, (b) the time at which the maximum transmitted force occurs, and (c) the value of the maximum transmitted force.

Given: $x(t) = 0.253e^{-6t}\sin(39.55t)$, k = 8 × 10^5 N/m, c = 6000 N-s/m

Find: $F(t), F_{max}, t_{max}$

Solution: The force transmitted to the floor is

$$\begin{aligned} F(t) &= c\dot{x}(t) + kx(t) \\ &= (8\times 10^5)[0.253\, e^{-6t}\sin(39.55t)] \\ &+ 6000(0.253)e^{-6t}[-6\sin(39.55t) + 39.55\cos(39.55t)] \\ &= e^{-6t}[1.93\times 10^5\sin(39.55t) + 6.00\times 10^4\cos(39.55t)] \end{aligned}$$

The transmitted force is a maximum when the derivative is equal to zero,

$$\begin{aligned} \frac{dF}{dt} &= e^{-6t}\{-6[1.93\times 10^5\sin(39.55t) + 6.00\times 10^4\cos(39.55t)] \\ &+ 39.55[1.93\times 10^5\cos(39.55t) - 6.00\times 10^4\sin(39.55t)]\} \\ &= e^{-6t}[-3.53\times 10^6\sin(39.55t) + 7.27\times 10^6\cos(39.55t)] \end{aligned}$$

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Setting the derivative to zero leads to

$$-3.53 \times 10^6 \sin(39.55t) + 7.27 \times 10^6 \cos(39.55t) = 0$$

$$\tan(39.55t) = \frac{7.27 \times 10^6}{3.53 \times 10^6} = 2.06$$

$$t = 0.0283 \text{ s}$$

The maximum force is

$$F_{max} = F(0.0283 \text{ s}) = e^{-6(0.0282)}[1.93 \times 10^5 \sin(1.12) + 6.00 \times 10^4 \cos(1.12)]$$
$$= 1.68 \times 10^5 \text{ N}$$

Problem 3.37 illustrates the use of the response of underdamped systems.

3.38 Repeat Chapter Problem 3.37 if the system has the same mass and stiffness but it is designed to be overdamped with a damping ratio of 1.3.

Given: m = 500 kg, k = 8 × 10^5 N/m, $\zeta = 1.3$, I = 5000 N · s

Find: (a) F(t), (b) t, (c) F_{max}

Solution: The principle of impulse and momentum is used to determine the initial velocity imparted to the system due to the impulse

$$v = \frac{I}{m} = 10 \text{ m/s}$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8 \times 10^5 \ \frac{\text{N}}{\text{m}}}{500 \text{ kg}}} = 40 \ \frac{\text{rad}}{\text{s}}$$

The damping coefficient is

$$c = 2\zeta m\omega_n = 2(1.3)(500 \text{ kg})\left(40 \ \frac{\text{rad}}{\text{s}}\right) = 5.2 \times 10^4 \ \frac{\text{N} \cdot \text{s}}{\text{m}}$$

The response of an overdamped system subject to the initial velocity condition is

$$x(t) = \frac{v}{2\omega_n\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_n t}\left(e^{\omega_n\sqrt{\zeta^2-1}t} - e^{-\omega_n\sqrt{\zeta^2-1}t}\right)$$

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Noting that

$$\omega_n\sqrt{\zeta^2-1} = 40\sqrt{(1.3)^2-1} = 33.23\ \frac{\text{rad}}{\text{s}}$$

the solution becomes

$$x(t) = \frac{10\ \frac{\text{m}}{\text{s}}}{2\left(33.23\ \frac{\text{rad}}{\text{s}}\right)} e^{-(1.3)(40)t}\left(e^{33.23t} - e^{-33.23t}\right)$$

$$= 0.150\left(e^{-18.77t} - e^{-85.23t}\right)\ \text{m}$$

(a) The transmitted force is

$$F(t) = kx + c\dot{x}$$
$$= 8\times 10^5[0.150(e^{-18.77t} - e^{-85.23t})] + 5.2 \times 10^4[0.150(-18.77e^{-18.77t} + 85.23e^{-85.23t})]$$

$$= -2.64\times 10^4 e^{-18.77t} + 5.45\times 10^5 e^{-85.23t}\ \text{N}$$

(b) The time at which the maximum occurs is obtained by differentiating the force

$$\frac{dF}{dt} = 0 = -2.64\times 10^4(-18.77)e^{-18.77t} + 5.45\times 10^5(-85.23)e^{-85.23t}$$

$$4.96\times 10^5 e^{-18.77t} = 4.64\times 10^7 e^{-85.23t}$$

$$0.106 = e^{-66.46t}$$

$$t = 0.0338\ \text{s}$$

(c) The maximum force is

$$F(0.0338) = -2.64\times 10^4 e^{-18.77(0.0338)} + 5.45\times 10^5 e^{-85.23(0.0338)}$$
$$= 16570\ \text{N}$$

Problem 3.38 Illustrates the free response of an overdamped system.

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3.39 One end of the mercury filled U-tube manometer of Figure P3.39 is open to the atmosphere while the other end is capped and under a pressure of 20 psig. The cap is suddenly removed. (a) Determine $x(t)$ as the displacement of the mercury-air interface from the column's equilibrium position if the column is undamped. (b) Determine $x(t)$ if it determined that the column of mercury has viscous damping with a damping ratio of 0.1. (c) Determine $x(t)$ if it is observed that after 5 cycles of motion the amplitude has decreased to one-third of its initial value.

FIGURE P3.39

Given: $p_0 = 20$ psig, $\ell = 12$ ft, Hg

Find: $x(t)$

Solution: (a) The differential equation governing $x(t)$ is derived using energy methods. The column is assumed to move as a rigid body. Thus the kinetic energy of the column is

$$T = \frac{1}{2}\rho A \ell \dot{x}^2$$

where ρ is the mass density of the mercury and A is the cross-sectional area of the manometer. Take the equilibrium position for potential energy calculations to be the bottom of the manometer. Let h be the length of the column in each leg when the column is in equilibrium. Then, the potential energy of the fluid in each leg is the instantaneous mass of the fluid in the leg times the instantaneous distance between the center of mass of that column and the bottom. To this end the potential energy of the fluid in the left leg is

$$V_L = \frac{1}{2}\rho g A (h - x)^2$$

and the potential energy of the fluid in the right leg is

$$V_R = \frac{1}{2}\rho g A (h + x)^2$$

The total potential energy is

$$V = V_L + V_R = \frac{1}{2}\rho g A \left(2h^2 + 2x^2\right)$$

The $2h^2$ term is present in the potential energy since the datum was not taken as the equilibrium position.

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Thus the equivalent system method is used to determine

$$m_{eq} = \rho A \ell$$
$$k_{eq} = 2\rho g A$$

Hence the governing differential equation is

$$\rho A \ell \ddot{x} + 2\rho A g x = 0$$
$$\ddot{x} + \frac{2g}{\ell} x = 0$$

from which the natural frequency is determined as

$$\omega_n = \sqrt{\frac{2g}{\ell}} = \sqrt{\frac{2\left(32.2\,\frac{\text{ft}}{\text{s}^2}\right)}{12\,\text{ft}}} = 2.32\,\frac{\text{rad}}{\text{s}}$$

Let t = 0 be measured immediately after the cap is removed. The fluid is at rest initially, thus $\dot{x}(0) = 0$ The difference between the level of the fluid in each leg initially determined by applying the basic principles of manometry. Let q be the distance between the two interfaces at t = 0. Then

$$p_L = p_R + \rho g q$$
$$q = \frac{p_L - p_R}{\rho g} = \frac{\left(20\,\frac{\text{lb}}{\text{in}^2}\right)\left(\frac{144\,\text{in}^2}{\text{ft}^2}\right)}{(12.6)\left(62.4\,\frac{\text{lb}}{\text{ft}^3}\right)} = 3.6\,\text{ft}$$

Since x is measured from equilibrium $x(0) = \frac{q}{2} = 1.83\,\text{ft}$ the solution to the differential equation subject to the initial conditions is

$$x(t) = 1.83 \sin 2.32t\ \text{ft}$$

(b) For a damping ratio of 0.1

$$x(t) = \frac{x_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$
$$= \frac{1.83}{\sqrt{1-(0.1)^2}} e^{-(0.1)(2.32)t} \sin\left(2.32\sqrt{1-(0.1)^2}\,t + \tan^{-1}\frac{\sqrt{1-(0.1)^2}}{0.1}\right)$$

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$$= 1.84e^{-0.232}\sin(2.31t + 1.471)$$

(c) After 5 cycles of motion the amplitude has reduced to one-third of its initial value, thus

$$\delta = \frac{1}{5}\ln 3 = 0.2197$$

and

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.035$$

The system is underdamped and has a solution of

$$x(t) = \frac{x_0}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$= \frac{1.83}{\sqrt{1-(0.03)^2}}e^{-(0.035)(2.32)t}\sin\left(2.32\sqrt{1-(0.035)^2}t + \tan^{-1}\frac{\sqrt{1-(0.035)^2}}{0.035}\right)$$

$$= 1.831e^{-0.0812t}\sin(2.32t + 1.535)$$

Problem 3.39 illustrates (a) application of the equivalent system method to derive the differential equations governing the motion of a column of liquid in a manometer, and (b) development and application of initial conditions.

3.40 The disk of Figure P3.40 rolls without slip.

(a) What is the critical damping coefficient, c_c, for the system?

FIGURE P3.40

(b) If $c = c_c/2$, plot the response of the system when the center of the disk is displaced 5 mm from equilibrium and released from rest.

(c) Repeat (b) if $c = 3c_c/2$.

(d) Repeat (b) if $c = c_c$.

Given: $k = 4000$ N/m, $r = 40$ cm, $m = 1$ kg, (a) $\zeta = 0.5$, (b) $\zeta = 1.5$ (c) $\zeta = 1$

Find: (a) c_c, (b) $x(t)$, (c) $x(t)$,

Solution: Let $x(t)$ be the displacement of the center of mass of the disk, measured from the system's equilibrium position. Assume the disk rolls without slip. Free-body diagrams of the system at an arbitrary instant are shown below.

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Summing moments about the point of contact between the disk and the surface,

$$\left(\sum M_C\right)_{ext} = \left(\sum M_C\right)_{eff}$$

leads to

$$-kxr - c\dot{x}r = \frac{1}{2}mr\ddot{x} + m\ddot{x}r$$

$$\frac{3}{2}m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{2c}{3m}\dot{x} + \frac{2k}{3m}x = 0$$

The system's natural frequency is

$$\omega_n = \sqrt{\frac{2k}{3m}} = \sqrt{\frac{2\left(4000\,\frac{\text{N}}{\text{m}}\right)}{3(1\text{kg})}} = 51.64\,\frac{\text{rad}}{\text{s}}$$

(a) The damping ratio is determined from

$$2\zeta\omega_n = \frac{2c}{3m}$$

$$c = 3m\zeta\omega_n$$

If the system is critically damped, then $\zeta = 1$, leading to

$$c_c = 3m\omega_n = 3(1\text{kg})\left(51.64\,\frac{\text{rad}}{\text{s}}\right) = 154.9\,\frac{\text{N}\cdot\text{s}}{\text{m}}$$

(b) If $c = 0.5c_c$, then $\zeta = 0.5$ and the system is underdamped. The damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \left(51.64\,\frac{\text{rad}}{\text{s}}\right)\sqrt{1-(0.5)^2} = 44.72\,\frac{\text{rad}}{\text{s}}$$

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The amplitude and phase are

$$A = x_0\sqrt{1+\left(\frac{\zeta\omega_n}{\omega_d}\right)^2} = 5.77\,\text{mm}$$

$$\phi = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right) = 1.047\,\text{rad}$$

The system response is

$$x(t) = 5.77e^{-25.82t}\sin(44.72t + 1.047)\text{mm}$$

The system response is sketched below

(c) If c = 1.5c_c, Then ζ = 1.5 and the system is overdamped. Application of the equation for the response of an overdamped system leads to

$$x(t) = 5.854e^{-8.54t} - 0.854e^{-146.4t}\ \text{mm}$$

A sketch of the system response follows.

(d) The system is critically damped, thus the response of the system is

$$x(t) = 0.005\,(1 + 51.64t)e^{-51.64t}\ \text{m}$$

Problem 3.40 illustrates (a) the derivation of the differential equation governing a system with viscous damping, (b) the free vibration response of an underdamped system, (c) the free vibration response of an overdamped system, and (d) rolling friction.

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3.41 A recoil mechanism of a gun is designed as a spring and viscous damper in parallel such that the system has critical damping. A 52-kg cannon has a maximum recoil of 50 cm after firing. Specify the stiffness and damping coefficient of the recoil mechanism such that the mechanism returns to within 5 mm of firing position within 0.5 s after firing.

Given: $m = 52$ kg, $\zeta = 1$, $x_0 = 50$ cm, $t_1 = 0.5$ s, $x_1 = 5$ mm

Find: k, c

Solution: Let t = 0 occur when the cannon has maximum recoil. The system is critically damped with initial conditions

$$x(0) = x_0 = 50\,\text{cm}, \quad \dot{x}(0) = 0\,\text{m/s}$$

The response of a critically damped system subject to these initial conditions is

$$x(t) = x_0 e^{-\omega_n t}(1 + \omega_n t)$$

It is desired to design the mechanism such that $x(0.5\,\text{s}) = 0.005$ m which leads to

$$0.005 = 0.5e^{-\omega_n(0.5\,\text{s})}\left[1 + (0.5\,\text{s})\omega_n\right]$$

A trial and error solution leads to

$$\omega_n = 13.3\,\text{rad/s}$$

The stiffness and damping coefficient are determined as

$$k = m\omega_n^2 = (52\,\text{kg})(13.3\,\text{rad/s})^2 = 9.20\times10^3\ \text{N/m}$$
$$\text{c} = 2\text{m}\omega_\text{n} = 2(52\,\text{kg})(13.3\,\text{rad/s}) = 1.38\times10^3\ \text{N}\cdot\text{s/m}$$

Problem 3.41 illustrates design of a system for critical damping

3.42 The initial recoil velocity of a 1.4-kg gun is 2.5 m/s. Design a recoil mechanism that is critically damped such that the mechanism returns to within 0.5 mm of firing within 0.5 s after firing.

Given: $m = 1.4$ kg, $v_0 = 2.5$ m/s, $t_1 = 0.5$ s, $x_1 = 0.5$ mm, $\zeta = 1$

Find: k, c

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Solution: The recoil mechanism is designed by specifying its stiffness and damping coefficient. Define t = 0 as immediately after firing. The system is critically damped with initial conditions

$$x(0) = 0 \quad \dot{x}(0) = v_0 = 2.5 \text{ m/s}$$

The response of a critically damped system with these initial conditions is

$$x(t) = v_0 t e^{-\omega_n t}$$

It is desired to design the mechanism such that

$$x(t_1) = x_1$$
$$x(0.5 \text{ s}) = 0.0005 \text{ m}$$

which leads to

$$0.0005 \text{ m} = 2.5(0.5 \text{ s}) e^{-\omega_n (0.5 \text{ s})}$$
$$\omega_n = 15.64 \text{ rad/s}$$

The stiffness and damping ratio are calculated as

$$k = m\omega_n^2 = (1.4 \text{ kg})(15.64 \text{ rad/s})^2 = 3.42 \times 10^2 \text{ N/m}$$
$$\text{c} = 2\text{m}\omega_\text{n} = 2(1.4 \text{ kg})(15.64 \text{ rad/s}) = 43.8 \text{ N} \cdot \text{s/m}$$

Problem 3.42 illustrates design of a system with critical damping.

3.43 A railroad bumper is modeled as a linear spring in parallel with a viscous damper. What is the damping coefficient of a bumper of stiffness 2×10^5 N/m such that the system has a damping ratio of 1.15 when it is engaged by a 22,000-kg railroad car?

Given: m = 22,000 kg, k = 2×10^5 N/m, $\zeta = 1.15$

Find: c

Solution: The natural frequency of the railroad car after it engages the bumper is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2 \times 10^5 \text{ N/m}}{22000 \text{ kg}}} = 3.015 \text{ rad/s}$$

In order for the system to have a damping ratio of 1.15, the required damping coefficient of the bumper is

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$$c = 2\zeta m\omega_n = 2(1.15)(22000\,\text{kg})(3.015\,\text{rad/s}) = 1.5\times10^5\ \text{kg/s}$$

Problem 3.43 illustrates the determination of the damping coefficient for an overdamped system.

3.44 Plot the responses of the bumper of Chapter Problem 3.43 when it is engaged by railroad cars traveling at 20 m/s when the mass of the railroad car is (a) 1500 kg (b) 22,000 kg, and (c) 30,000 kg.

Given: c =1.5×10^5 kg/s, k = 2×10^5 N/m

Find: x(t) when (a) m = 15,000 kg (b) m = 22,000 kg (c) m = 30,000 kg

Solution: The differential equation is

$$m\ddot{x} + 1.5 \times 10^5\dot{x} + 2 \times 10^5 x = 0$$

(a) The natural frequency and damping ratio are

$$\omega_n = \sqrt{\frac{2 \times 10^5}{15000}} = 3.65\ \frac{\text{rad}}{\text{s}}$$

$$\zeta = \frac{\omega_n\sqrt{1-\zeta^2}\ 1.5 \times 10^5}{2\sqrt{(15000)(2 \times 10^5)}} = 1.37$$

(b) The natural frequency and damping ratio are $\omega_n = 3.015$ rad/s and $\zeta = 1.15$

(c) The natural frequency and damping ratio are

$$\omega_n = \sqrt{\frac{2 \times 10^5}{30000}} = 2.58\ \frac{\text{rad}}{\text{s}}$$

$$\zeta = \frac{1.5 \times 10^5}{2\sqrt{(30000)(2 \times 10^5)}} = 0.968$$

The response of an overdamped system due to an initial velocity is

$$x(t) = \frac{v_0 e^{-\zeta\omega_n t}}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{\omega_n\sqrt{\zeta^2-1}t} - e^{-\omega_n\sqrt{\zeta^2-1}t}\right)$$

The response of an underdamped systm to an initial velocity is

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$$x(t) = \frac{v_0}{\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}t\right)$$

The responses of the systems in (a) –(c) are plotted using these equations

Problem 3.44 illustrates the equations for underdamped and overdamped free vibrations of SDOF systems.

3.45 Reconsider the restroom door of Example 3.9. The man, instead of kicking the door, pushes it so that it opens to 80º and then lets go. How long will it take the door after he lets go to close to within 5° of being shut if it is designed (a) with critical damping and (b) with a damping ration of 1.5?

Given: $(0) = 80°, \dot{\theta}(0) = 0, \omega_n = 1.14\ \frac{\text{rad}}{\text{s}}$; (a) $\zeta = 1$; (b) $\zeta = 1.5$

Find: time for $\theta = 5°$.

Solution: (a) The response of a critically damped system with these initial conditions is

$$\theta(t) = 80° e^{-1.14t}(1 + 1.14t)$$

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Setting $\theta = 5°$in the above and solving for t gives t = 16.3 s.

(b) The response of an overdamped system with these initial conditions is

$$\theta(t) = 80°\frac{e^{-1.71t}}{2.236}(2.62e^{1.27t} - 0.382e^{-1.27t}) = 80°(1.17e^{-0.44t} - 0.171e^{-2.98t})$$

Setting $\theta = 5°$in the above and solving for t gives t=91.6 s.

Problem 3.45 illustrates the response of critically damped and overdamped systems due to an initial displacement.

3.46 A block of mass *m* is attached to a spring of stiffness *k* and slides on a horizontal surface with a coefficient of friction μ. At some time *t*, the velocity is zero and the block is displaced a distance δ from equilibrium. Use the principle of work-energy to calculate the spring deflection at the next instant when the velocity is zero. Can this result be generalized to determine the decrease in amplitude between successive cycles?

Given: m, k, μ, δ

Find: x at next tine = 0, generalize result

Solution Let position 1 refer to the position of the system when the displacement is δ and the velocity is zero. Let position 2 refer to the position of the system at the next instant when the velocity is zero. From the principle of work and energy .

$$T_1 + V_1 + U_{1-2} = T_2 + V_2$$

where V represents the potential energy in the spring and U is the work done by the friction force. Since the velocity is zero in position 1 and position 2, $T_1 = T_2 = 0$. The potential energy in position 1 is

$$V_1 = \frac{1}{2}k\delta^2$$

Assuming δ is large enough, the block will pass through the equilibrium position before the velocity reaches zero. It will also travel a distance δ_1 past the equilibrium position. Thus the potential energy in position 2 is

$$V_2 = \frac{1}{2}k\delta_1^2$$

The magnitude and direction of the friction force is constant over this half cycle of motion. Since it always opposes the direction of motion its work is negative and

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$$U_{1-2} = -\mu mg(\delta + \delta_1)$$

Thus the principle of work and energy gives

$$\frac{1}{2}k\delta^2 - \mu mg(\delta + \delta_1) = \frac{1}{2}k\delta_1^2$$

$$\delta_1^2 + \frac{2\mu mg}{k}\delta_1 - \delta^2 + \frac{2\mu mg}{k}\delta = 0$$

The quadratic formula is used to solve for δ_1

$$\delta_1 = \frac{1}{2}\left[\frac{-2\mu mg}{k} \pm \sqrt{\frac{4\mu^2 m^2 g^2}{k^2} + 4\delta^2 - \frac{4\mu mg}{k}\delta}\right]$$

$$= \frac{1}{2}\left[\frac{-2\mu mmg}{k} \pm \sqrt{\left(2\delta - \frac{2\mu mmg}{k}\right)^2}\right] = -\frac{\mu mg}{k} \pm \left(\delta - \frac{\mu mg}{k}\right)$$

Choosing the positive roots gives

$$\delta_1 = \delta - \frac{2\mu mg}{k}$$

Thus the amplitude decreases by 2μmg/k during this half cycle. However, the procedure is independent of the value of δ, as long as δ is large enough f or the system to again pass through equilibrium. Thus, on every half cycle of motion the system experiences a decrease in amplitude of 2μmg/k or it experiences an amplitude loss of 4μmg/k on every cycle.

Problem 3.46 illustrates how the principle of work energy is used to determine the decrease in amplitude per cycle of motion for a system with Coulomb damping.

3.47 Reconsider Example 3.11 using a work-energy analysis. That is, assume the amplitude of the swing is θ at the end of an arbitrary cycle. Use the principle of work-energy to determine the amplitude at the end of the next half-cycle.

Given: swing system of Example 3.10.

Find: decrease in amplitude over one half cycle

Solution: Let position 1 be the position at the beginning of an arbitrary cycle, where the amplitude of the swing is θ. Let position 2 be the position of the swing at the next instant

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where its angular velocity is zero. At this instant the swing has an amplitude θ^*. Application of the principle of work-energy between position 1 and position 2 leads to

$$T_1 + v_1 + U_{1-2} = T_2 + V_2$$

However, since the angular velocity of the swing is zero in both position 1 and position 2

$$T_1 = T_2 = 0$$

Let the horizontal plane of the swing occupies when it is at rest in equilibrium be the datum for potential energy due to gravity calculations. Then

$$V_1 = mg\ell(1-\cos\theta)$$
$$V_2 = mg\ell(1-\cos\theta^*)$$

$U_{1\to 2}$ is the work done by the frictional moment as the swing moves between position 1 and position 2. Using the notation of Example-2.21, suppose in position 1, $T_1 > T_2$ Then the frictional moment is

$$M = (2T_1 - 2T_2)\frac{d}{2} = \frac{mgd}{2}\frac{e^{\mu\pi}-1}{1+e^{\mu\pi}}$$

The frictional moment is constant and opposes the direction of the velocity. Its total work is

$$U_{1\to 2} = -\int \frac{mgd}{2}\frac{1-e^{\mu\pi}}{1+e^{\mu\pi}}d\theta = \frac{mgd}{2}\frac{1-e^{\mu\pi}}{1+e^{\mu\pi}}(\theta+\theta^*)$$

Hence the principle of work-energy leads to

$$mg\ell(1-\cos\theta) - \frac{mgd}{2}\frac{1-e^{\mu\pi}}{1+e^{\mu\pi}}(\theta+\theta^*) = mg\ell(1-\cos\theta^*)$$

For small θ,

$$\cos\theta = 1 - \frac{\theta^2}{2} + \ldots \approx 1 - \frac{\theta^2}{2}$$

Thus,

$$\theta^{*^2} + C\theta^* + (c\theta - \theta^2) = 0$$

where

$$c = mgd\frac{e^{\mu\pi}-1}{1+e^{\mu\pi}}$$

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The above equation is a quadratic equation to solve for θ^*, given θ Use of the quadratic formula leads to

$$\theta^* = \frac{1}{2}\left[-c \pm \sqrt{c^2 - 4\left(c\theta - \theta^2\right)}\right]$$

$$\frac{1}{2} - c \pm \left(c - 2\theta\right) = \theta - c$$

Thus the decrease in amplitude over one-half cycle is

$$\delta\theta = c = mgd\,\frac{e^{\mu\pi} - 1}{1 + e^{\mu\pi}}$$

which agrees with the result of Example 3.19

Problem 3.47 illustrates the application of the principle of work-energy to determine the decrease in amplitude over a half cycle in the free vibrations of a system subject to Coulomb damping.

3.48 The center of the thin disk of Figure P3.48 is displaced a distance δ and the disk released. The coefficient of friction between the disk and the surface is μ. The initial displacement is sufficient to cause the disk to roll and slip.

FIGURE P3.48

(a) Derive the differential equation governing the motion when the disk rolls and slips.

(b) When the displacement of the mass center from equilibrium becomes small enough, the disk rolls without slip. At what displacement does this occur?

(c) Derive the differential equation governing the motion when the disk rolls without slip.

(d) What is the change in amplitude per cycle of motion?

Given: m, k, r, δ

Find: (a) differential equation when disk rolls and slips, (b) displacement for which disk rolls without slip, (c) differential equation when disk rolls without slip, (d) change in amplitude per cycle.

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Solution: (a) Consider free-body diagrams of the system at an arbitrary time.

Assume the velocity of the point-of contact is to the right. If the disk rolls and slips, then F = μmg and no kinematic relation exists between the displacement of the mass center and the angular rotation of the disk. Summing forces in the horizontal direction

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$
$$-kx - \mu mg = m\ddot{x}$$
$$m\ddot{x} + kx = -\mu mg$$

If the motion is in the opposite direction, the differential equation is

$$m\ddot{x} + kx = \mu mg$$

(b) If the disk rolls without slip, then the friction force is less than the maximum possible friction force of μmg and the acceleration of the mass center is related to the angular acceleration of the disk by

$$\ddot{x} = r\alpha$$

Summing moments about the point of contact leads to

$$\left(\sum M_c\right)_{ext} = \left(\sum M_c\right)_{eff}$$
$$-kxr = m\ddot{x}r + \frac{1}{2}mr^2\left(\frac{\ddot{x}}{r}\right)$$
$$\frac{3}{2}m\ddot{x} + kx = 0$$

The response of the system when the disk rolls without slip is

$$x(t) = A\sin(\omega_n t + \phi)$$

where

$$\omega_n = \sqrt{\frac{2k}{3m}}$$

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Summing forces in the horizontal direction gives

$$\left(\sum M_O\right)_{ext} = \left(\sum M_O\right)_{eff}$$

$$Fr = \frac{1}{2}mr^2\left(\frac{\ddot{x}}{r}\right)$$

$$F = \frac{1}{2}m\ddot{x}$$

But from the differential equation

$$\ddot{x} = -\frac{2k}{3m}$$

which leads to

$$F = -\frac{1}{3}kx$$

(c) The disk rolls without slip when the friction force is less than the maximum possible,

$$\frac{1}{3}kx < \mu mg$$

$$x < \frac{3\mu mg}{k}$$

(d) Consider the motion at the beginning of a cycle when the amplitude is

$$A_0 > \frac{3\mu mg}{k}$$

Thus the disk initially rolls and slips. Assume the disk begins the cycle at x = A with v = 0. The disk rolls with slip until x = (3 μmg)/k and will continue to roll without slip until x = -(3 μmg)/k. During this part of the motion, the system is conservative. The disk then rolls without slip. During the first part of its notion the system response is

$$x = \left(A_0 + \frac{\mu mg}{k}\right)\cos\sqrt{\frac{k}{m}}t - \frac{\mu mg}{k}$$

The disk continues to roll and slip until x = (3 μmg)/k when

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$$\frac{3\mu mg}{k} = \left(A_0 + \frac{\mu mg}{k}\right)\cos\sqrt{\frac{k}{m}}t - \frac{\mu mg}{k}$$

$$\cos\sqrt{\frac{k}{m}}t = \frac{\frac{4\mu mg}{k}}{A_0 + \frac{\mu mg}{k}}$$

At this time the velocity is

$$\dot{x} = \sqrt{\frac{k}{m}}\left(A_0 + \frac{\mu mg}{k}\right)\sqrt{\left(A_0 + \frac{\mu mg}{k}\right)^2 - \left(\frac{4\mu mg}{k}\right)^2}$$

The amplitude for at the end of the first quarter cycle as the disk begins to roll without slip is

$$A^2 = \left(\frac{3\mu mg}{k}\right)^2 + \frac{2}{3}\left(A_0 + \frac{\mu mg}{k}\right)^2\left[\left(A_0 + \frac{\mu mg}{k}\right)^2 - \left(\frac{4\mu mg}{k}\right)^2\right]$$

The loss in amplitude over the first quarter cycle is A_0 -A. This process continues to determine the amplitude loss over any cycle.

Problem 3.48 illustrates rolling friction. It is interesting to note that the natural frequency of the system is different when the system rolls with slip than when it rolls without slip. Thus over one cycle of motion, the natural frequency changes.

3.49 A 10-kg block is attached to a spring of stiffness 3×10^4 N/m. The block slides on a horizontal surface with a coefficient of friction of 0.2. The block is displaced 30 mm and released. How long will it take before the block returns to rest?

Given: $m = 10$ kg, $k = 3 \times 10^4$ N/m, $\mu = 0.2$, $\delta = 30$ mm

Find: t when block returns to rest

Solution: The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3\times10^4\,\frac{\text{N}}{\text{m}}}{10\,\text{kg}}} = 54.78\,\frac{\text{rad}}{\text{s}}$$

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The decrease in amplitude per cycle of motion is

$$\Delta A = \frac{4\mu mg}{k} = \frac{4(0.2)\left(9.81\frac{\text{m}}{\text{s}^2}\right)(10)}{3\times10^4\,\frac{\text{N}}{\text{m}}} = 0.0026\text{ m} = 2.6\,\text{mm}$$

The permanent displacement of the block when it comes to rest is

$$x_e = \frac{\mu\text{mg}}{\text{k}} = 0.654\,\text{mm}$$

The number of cycles before the system returns to rest is obtained from

$$\delta - n\Delta A = x_e$$

$$n = \frac{1}{\Delta A}(\delta - x_e) = 11.29\,\text{cycles}$$

The system can return to rest at the end of any half cycle. Hence n = 11.5.

The period per cycle is

$$T = \frac{2\pi}{\omega_n} = 0.115\,s$$

Thus the time for 11.5 cycles is

$$t = 11.5(0.115\,s) = 1.32\,s$$

Problem 3.49 illustrates (a) the decrease in amplitude per cycle of motion for a system with Coulomb damping (b) the natural frequency for a system with Coulomb damping, (c) the permanent displacement of a system with Coulomb damping.

3.50 The block of Chapter Problem 3.49 is displaced 30 mm and released. What is the range of values of the coefficient of friction such that the block comes to rest during the 14th cycle?

Given: m = 10 kg, k = 3 × 10^4 N/m, δ = 30 mm, 13 < n < 14

Find: μ

Solution: The block comes to rest after the nth cycle if

$$\delta - \frac{\mu mg}{k} = n\frac{4\mu mg}{k}$$

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Solving for μ

$$\mu = \frac{k\delta}{(4n+1)mg} = \frac{\left(3\times10^4\,\frac{\text{N}}{\text{m}}\right)(0.03\,\text{m})}{(4\text{n}+1)(10\text{kg})\left(9.81\,\frac{\text{m}}{\text{sec}^2}\right)}$$

$$= \frac{9.17}{4\text{n}+1}$$

For n = 13, μ = 0.173. For n = 14, μ = 0.161. Thus the system comes to rest during the 14th cycle is

$$0.161 < \mu < 0.173$$

Problem 3.50 illustrates the decrease in amplitude per cycle of motion for a system with Coulomb damping.

3.51 A 2.2-kg block is attached to a spring of stiffness 1000 N/m and slides on a surface that makes an angle of 7° with the horizontal. When displaced from equilibrium and released**,** the decrease in amplitude per cycle is observed to be 2 mm. Determine the coefficient of friction.

Given: m = 2.2 kg, k = 1000 N/m, θ = 7°, ΔA = 2 mm

Find: μ

Solution: The friction force developed when the block is in motion is

$$F_f = \mu mg\cos\theta \tag{1}$$

The change in amplitude per cycle of motion is given by

$$\Delta A = \frac{4F_f}{k} \tag{2}$$

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Combining eqs.(1) and (2) leads to

$$\mu = \frac{k\Delta A}{4mg\cos\theta}$$
$$= \frac{\left(1000\,\frac{\text{N}}{\text{m}}\right)(0.002\,\text{m})}{4(2.2\,\text{kg})\left(9.81\,\frac{\text{m}}{\text{sec}^2}\right)\cos 7°} \tag{3}$$
$$= 0.0233$$

Problem 3.51 illustrates (a) the constant decrease in amplitude per cycle of motion for a system with Coulomb damping.

3.52 A block of mass m is attached to a spring of stiffness k and viscous damper of damping coefficient c and slides on a horizontal surface with a coefficient of friction μ. Let $x(t)$ represent the displacement of the block from equilibrium.

(a) Derive the differential equation governing $x(t)$.

(b) Solve the equation and sketch the response over two periods of motion.

Given: m, k, c, μ

Find: (a) differential equation, (b) response over two cycles

Solution: (a) Consider free body diagrams of the system at an arbitrary instant when $\dot{x} > 0$,

Summing forces

$$\left(\sum F_{ext}\right) = \left(\sum F\right)_{eff}$$

leads to

$$-kx - c\dot{x} - \mu mg = m\ddot{x}$$
$$m\ddot{x} + c\dot{x} + kx = -\mu mg$$

Repeating the process when $\dot{x} < 0$ leads to

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$$m\ddot{x} + c\dot{x} + kx = \begin{cases} -\mu mg, \dot{x} > 0 \\ \mu mg, \dot{x} < 0 \end{cases}$$

(b) Assume the system is given an initial displacement $\delta > 0$ and released from rest. Then the velocity is negative over the first half cycle and the response is

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 \cos\omega_d t + C_2 \sin\omega_d t\right) + \frac{\mu mg}{k}$$

where

$$\omega_n = \sqrt{\frac{k}{m}}, \zeta = \frac{c}{2m\omega_n}, \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Application of the initial conditions leads to

$$x(0) = \delta = C_1 + \frac{\mu mg}{k}, C_1 = \delta - \frac{\mu mg}{k}$$

$$\dot{x}(0) = 0 = -\zeta\omega_n C_1 + \omega_d C_2, C_2 = \frac{\zeta}{\sqrt{1-\zeta^2}} C_1$$

Hence over the first half cycle,

$$x(t) = \left(\delta - \frac{\mu mg}{k}\right) e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t\right) + \frac{\mu mg}{k} \qquad (1)$$

The response given by eq. (1) is valid until the velocity becomes zero. From eq. (l)

$$\dot{x}(t) = -\left(\delta - \frac{\mu mg}{k}\right) e^{-\zeta\omega_n t}\left(\omega_d + \zeta\omega_n\right)\sin\omega_d t$$

Hence the velocity changes sign at

$$t = \frac{\pi}{\omega_d}$$

when

$$x\left(\frac{\pi}{\omega_d}\right) = -e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\left(\delta - \frac{\mu mg}{k}\right) + \frac{\mu mg}{k}$$

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The response over the second half cycle is given by

$$x(t) = e^{-\zeta\omega_n t}\left(C_3 \cos\omega_d t + C_4 \sin\omega_d t\right) - \frac{\mu mg}{k}$$

Application of the conditions at π/ω_d leads to

$$x\left(\frac{\pi}{\omega_d}\right) = -e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\left(\delta - \frac{\mu mg}{k}\right) + \frac{\mu mg}{k} = -e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} C_3 - \frac{\mu mg}{k}$$

$$C_3 = \left(\delta - \frac{\mu mg}{k}\right) - \frac{2\mu mg}{k} e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

and

$$\dot{x}\left(\frac{\pi}{\omega_d}\right) = 0 \rightarrow C_4 = \frac{\zeta}{\sqrt{1-\zeta^2}} C_3$$

Hence over the second half cycle

$$x(t) = \left[\left(\delta - \frac{\mu mg}{k}\right) - \frac{2\mu mg}{k} e^{\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right] e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t\right) - \frac{\mu mg}{k}$$

The velocity next changes sign at $t = 2\pi / \omega_d$ when

$$x\left(\frac{2\pi}{\omega_d}\right) = e^{-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}\left(\delta - \frac{\mu mg}{k}\right) - \frac{2\mu mg}{k} e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - \frac{\mu mg}{k}$$

The response over the third half cycle is given by

$$x = e^{-\frac{\zeta\omega_n}{\sqrt{1-\zeta^2}}}\left(C_5 \cos\omega_d t + C_6 \sin\omega_d t\right) + \frac{\mu mg}{k}$$

Application of conditions at $t = 2\pi / \omega_d$ lead to

$$x(t) = \left[\left(\delta - \frac{\mu mg}{k}\right) - \frac{2\mu mg}{k} e^{\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - \frac{2\mu mg}{k} e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}\right] e^{-\zeta\omega_n t}$$
$$\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t\right) + \frac{\mu mg}{k}$$

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The velocity next changes sign at t = 3π / ω_d, leading to the fourth half cycle being described by

$$x(t)=\left[\left(\delta-\frac{\mu mg}{k}\right)-\frac{2\mu mg}{k}e^{\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}-\frac{2\mu mg}{k}e^{\frac{2\zeta\pi}{\sqrt{1-\zeta^2}}}-\frac{2\mu mg}{k}e^{\frac{3\zeta\pi}{\sqrt{1-\zeta^2}}}\right]$$

$$e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)-\frac{\mu mg}{k}$$

The amplitude at the end of the second cycle is

$$x\left(\frac{4\pi}{\omega_d}\right)=\left(\delta-\frac{\mu mg}{k}\right)e^{-\frac{4\pi\zeta}{\sqrt{1-\zeta^2}}}-\frac{2\mu mg}{k}e^{-\frac{3\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$-\frac{2\mu mg}{k}e^{-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}-\frac{2\mu mg}{k}e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}-\frac{\mu mg}{k}$$

Note that the natural frequency for a system with both viscous and Coulomb damping is the system's damped natural frequency. The amplitude decays faster over each cycle than with viscous damping or Coulomb damping alone. A sketch of the motion over the first few cycles follows. The motion continues until the spring force and viscous damping force are insufficient to overcome friction. The motion then ceases with a permanent displacement.

Problem 3.52 illustrates the free-vibration response of a system with both viscous damping and Coulomb damping.

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3.53 A connecting rod is fitted around a cylinder with a connecting rod between the cylinder and bearing. The coefficient of friction between the cylinder and bearing is 0.08. If the rod is rotated 12° counterclockwise and then released, how many cycles of motion will it execute before coming to rest? The ratio of the diameter of the cylinder to the distance to the center of mass of the connecting rod from the center of the cylinder is 0.01.

Given: $\mu = 0.08$, $\theta_0 = 12°$, $d/\ell = 0.01$

Find: number of cycles

Solution: Let I be the mass moment of inertia of the connecting rod about its mass center, ℓ the distance between the mass center of the rod and the support, m the total mass of the rod, and d the diameter of the cylinder. As the connecting rod rotates about the center of the bearing, it is subject to a frictional moment

$$M = \mu mg\frac{d}{2}$$

opposing the direction of motion. Thus consider free body diagrams of the connecting rod at an arbitrary instant when the angular velocity is counterclockwise

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Summing moments about the center of the bearing leads to

$$\left(I + m\ell^2\right)\ddot{\theta} + mg\ell\sin\theta = \mu mg\frac{d}{2}$$

Assuming small θ,

$$\ddot{\theta} + \frac{mg\ell}{I + m\ell^2}\theta = -\frac{\mu mgd}{2\left(I + m\ell^2\right)}$$

If the velocity is clockwise, similar the governing differential equation is

$$\ddot{\theta} + \frac{mg\ell}{I + m\ell^2}\theta = \frac{\mu mg}{2\left(I + m\ell^2\right)}$$

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The above equations are analogous to eq. (2.78) with

$$F_f = \mu mg\frac{d}{2}, \tilde{m} = I + m\ell^2, \omega_n = \sqrt{\frac{mg\ell}{I + m\ell^2}}$$

The decrease in amplitude per cycle of motion is

$$\Delta A = \frac{4\mu mg\frac{d}{2}}{mg\ell}$$
$$= \frac{2\mu d}{\ell}$$

Given

$$\frac{d}{\ell} = 0.01$$

Then

$$\Delta A = 2(0.08)(0.01) = 1.6\times10^{-3}\ \frac{\text{rad}}{\text{cycle}}$$

Motion ceases when the moment of the gravity force is no longer greater than the frictional moment,

$$mg\ell\theta < \mu mg\frac{d}{2}$$
$$\theta < 0.005$$

Hence the number of cycles of motion is

$$n = \frac{12°\left(\frac{\pi\,\text{rad}}{180°}\right) - 0.005\,\text{rad}}{1.6\times10^{-3}\ \frac{\text{rad}}{\text{cycles}}} = 128\,\text{cycles}$$

Problem 3.53 illustrates (a) bearing friction, (b) the decrease in amplitude per cycle of motion due to a system with Coulomb damping, and (c) the number of cycles executed until motion ceases.

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3.54 A one-degree-of-freedom structure has a mass of 65 kg and a stiffness of 238 N/m. After 10 cycles of motion the amplitude of free vibrations amplitude is decreased by 75%. Calculate the hysteretic damping coefficient and the total energy lost during the first 10 cycles if the initial amplitude is 20 mm.

Given: m = 65 kg, k = 238 N/m, X_0 = 20 mm, $X_{10} = 0.25X_0$

Find: h, $\Delta E_{1\to 10}$

From the information given

$$X_{10} = 0.25X_0 = 0.25(20\text{mm}) = 5\,\text{mm}$$

From conservation of energy

$$E_0 = E_{10} + \Delta E_{0\to 10}$$

$$\Delta E_{0\to 10} = E_0 - E_{10} = \frac{1}{2}kX_0^2 - \frac{1}{2}kX_{10}^2 = \frac{1}{2}k\left(X_0^2 - X_{10}^2\right) = \frac{1}{2}(238\text{N})\left[(.02\,\text{m})^2 - (.005\,\text{m})^2\right]$$

$$\Delta E_{0\to 10} = 0.0446\,\text{N}\cdot\text{m}$$

The logarithmic decrement can be used to calculate the equivalent viscous damping ratio.

$$\delta = \frac{1}{10}\ln\left(\frac{X_0}{X_{10}}\right) = 0.1386$$

Hence

$$\zeta = \frac{\delta}{2\pi} = 0.0221$$

and

$$h = 2\zeta = 0.0442$$

Problem 3.54 illustrates the use of a viscous damping analogy to model systems with hysteretic damping.

3.55 The end of a steel cantilever beam ($E = 210 \times 10^9$ N/m^2) of $I = 1.5 \times 10^{-4}$m^4 is given an initial amplitude of 4.5 mm. After 20 cycles of motion the amplitude is observed as 3.7 mm. Determine the hysteretic damping coefficient and the equivalent viscous damping ratio for the beam.

Given: X_0 = 4.5 mm, X_{20} = 3.7 mm, E = 210×10^9 N/m^2, I = 1.5×10^{-4} m^4

Find: h, ζ

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Solution: The logarithmic decrement is

$$\delta = \frac{1}{20}\ln\left(\frac{X_0}{X_{20}}\right)$$
$$= \frac{1}{20}\ln\left(\frac{4.5\,\text{mm}}{3.7\,\text{mm}}\right) = 0.00979$$

The hysteretic damping coefficient is determined from

$$\delta = -\ln(1-\pi h)$$
$$h = \frac{1}{\pi}\left(1-e^{-\delta}\right) = 0.00310$$

The viscous damping ratio is

$$\zeta = \frac{h}{2} = 0.00155$$

Problem 3.55 illustrates (a) the modeling of hysteretic damping using an equivalent viscous damping coefficient and (b) the logarithmic decrement

3.56 A 500-kg press is placed at the midspan of a simply supported beam of length 3 m, elastic modulus 200×10^9 N/m^2, and cross-sectional moment of inertia 1.83×10^{-5} m^4. It is observed that the free vibrations of the beam decay to half of the initial amplitude in 35 cycles. Determine the response of the press, $x(t)$, if it is subject to an impulse of magnitude 10,000 N · s.

Given: W = 500-kg, L= 3 m, E = 200×10^9 N/m^2, I = 1.83×10^{-5} m^4, n = 35 cycles, I = 10,000 N · s

Find: $x(t)$

Solution: The logarithmic decrement is

$$\delta = \frac{1}{35}\ln\frac{1}{1/2} = 0.0198$$

The equivalent viscous damping ratio is

$$\zeta = \frac{\delta}{2\pi} = 0.00315$$

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The initial conditions are

$$x(0) = 0$$

$$\dot{x}(0) = \frac{I}{m} = \frac{10000\ \mathrm{N \cdot s}}{500\ \mathrm{kg}} = 20\ \frac{\mathrm{m}}{\mathrm{s}}$$

The stiffness of the beam is

$$k = \frac{48EI}{L^3} = \frac{48\left(200 \times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}\right)(1.83 \times 10^{-5}\ \mathrm{m}^4)}{(3\ \mathrm{m})^3} = 6.51 \times 10^6\ \frac{\mathrm{N}}{\mathrm{m}}$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6.51 \times 10^6\ \frac{\mathrm{N}}{\mathrm{m}}}{500\ \mathrm{kg}}} = 114.1\ \frac{\mathrm{rad}}{\mathrm{s}}$$

The damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \left(114.1\ \frac{\mathrm{rad}}{\mathrm{s}}\right)\sqrt{1-(0.00315)^2} = 114.1\ \frac{\mathrm{rad}}{\mathrm{s}}$$

The free vibration response is that of an underdamped system

$$x(t) = \frac{\dot{x}(0)}{\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t = \frac{20\ \frac{m}{s}}{114.1\ \frac{rad}{s}}e^{-(0.00315)(114.1)t}\sin 114.1t$$

$$= 0.175e^{-0.359t}\sin 114.1t\ \ \mathrm{m}$$

Problem 3.56 illustrates the use of a viscous damping model to determine the response of a system with hysteretic damping.

3.57 Use the theory of Section 3.9 to derive the equivalent viscous damping coefficient for Coulomb damping. Compare the response of a one-degree-of-freedom system of natural frequency 35 rad/s and friction coefficient 0.12 using the exact theory to that obtained using approximate theory with an equivalent viscous damping coefficient.
Note: Problem 3.58 in text should be 3.57

Given: $\omega_n = 35$ rad/s, $\mu = 0.12$

Find: c_{eq}

Solution: The friction force for Coulomb damping is given by

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$$F = -\mu mg \quad \text{for } \dot{x} > 0$$
$$F = +\mu mg \quad \text{for } \dot{x} < 0$$

If the motion is of the form

$$x(t) = X \sin \omega t$$

then over one cycle of motion

$$\dot{x}(t) = X\omega \cos \omega t$$
$$\dot{x} > 0 \quad \text{for } 0 < t < \frac{\pi}{2\omega} \quad \text{and } \frac{3\pi}{2\omega} < t < \frac{2\pi}{\omega}$$
$$\dot{x} < 0 \quad \text{for } \frac{\pi}{2\omega} < t < \frac{3\pi}{2\omega}$$

The energy dissipated over one cycle of motion is

$$\Delta E = \int_0^{2\pi/\omega} FX\omega \cos \omega t \, dt \int_0^{\pi/(2\omega)} (-\mu mg) X\omega \cos \omega t \, \mathrm{dt} +$$
$$\int_{\pi/(2\omega)}^{(3\pi)/(2\omega)} (\mu mg) X\omega \cos \omega t \, \mathrm{dt} + \int_{3\pi/(2\omega)}^{2\pi/\omega} (-\mu mg) X\omega \cos \omega t \, \mathrm{dt}$$
$$= (\mu mg) X \left[-\sin \omega t \Big|_{t=0}^{t=\pi/(2\omega)} + \sin \omega t \Big|_{t=\pi/(2\omega)}^{t=3\pi/(2\omega)} - \sin \omega t \Big|_{t=3\pi/(2\omega)}^{t=2\pi/\omega} \right]$$
$$= \mu mgX(-\sin \frac{\pi}{2} + \sin 0 + \sin \frac{3\pi}{2} - \sin \frac{\pi}{2} - \sin 2\pi + \sin \frac{3\pi}{2}) = -4\mu mgX$$

The negative sign indicates that the energy is dissipated. The equivalent viscous damping coefficient is

$$c_{eq} = \frac{\Delta E}{\pi \omega X^2} = \frac{4\mu mgX}{\pi \omega X^2} = \frac{4\mu mg}{\pi \omega X}$$

The equivalent viscous damping ratio for a cycle of motion is

$$\zeta_{eq} = \frac{c_{eq}}{2m\omega_n} = \frac{2\mu g}{\pi \omega_n^2 X}$$

leading to an amplitude change per cycle of motion given by

$$X_{n+1} = X_n e^{\frac{-4\mu g}{\omega_n^2 X_n}}$$

The amplitude loss per cycle of motion for a system with Coulomb damping, according to the exact theory is,

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$$\Delta A = \frac{4\mu mg}{k} = \frac{4\mu g}{\omega_n^2}$$

Problem 3.57 illustrates the determination of the equivalent viscous damping coefficient for Coulomb damping.

3.58 A 0.5-kg sphere is attached to a spring of stiffness 6000 N/m. The sphere is given an initial displacement of 8 mm from its equilibrium position and released. If aerodynamic drag is the only source of friction, how many cycles will the system execute before its amplitude is reduced to 1 mm? ***Note: Problem 3.59 in text should be 3.58***

Given: m = 0.5 kg, k = 6000 N/m, x_0 = 8 mm, x_f=1 mm

Find: n

Solution: The energy lost over one cycle of motion due to aerodynamic drag assuming a harmonic motion is

$$\Delta E = \frac{8}{3} C_D \omega^2 X^3$$

The equivalent viscous damping coefficient as

$$c_{eq} = \frac{8C_D}{3\pi}\omega X = 0.849\omega X$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6000\ \text{N/m}}{0.5\ \text{kg}}} = 109.5\ \text{rad/s}$$

If the viscous damping is small the frequency is approximated by the natural frequency. Then

$$c_{eq} = (0.849)(109.5\ \text{rad/s})\text{X} = 92.7X$$

The damping ratio for a given cycle is

$$\zeta_{eq} = \frac{c_{eq}}{2m\omega_n} = 0.846X$$

Using the logarithmic decrement a table is developed using the relationship

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$$X_n = X_{n-1} e^{-5.32 X_{n-1}}$$

Cycle	Amplitude at beginning of cycle
1	8.0 mm
2	7.7 mm
3	7.4 mm
4	7.1 mm
5	6.8 mm
6	6.6 mm
7	6.4 mm
8	6.2 mm
9	6.0 mm
10	5.8 mm
11	5.6 mm
12	5.4 mm
13	5.3 mm
14	5.1 mm
15	5.0 mm

The amplitude decreases to 1 mm after 155 cycles.

Problem 3.58 illustrates the use of the equivalent viscous damping coefficient to provide a linear approximation for the free vibration response of a system subject to aerodynamic drag.

3.59 A one-degree-of-freedom model of a suspension system is shown in Figure P3.59(a). For this model the mass of the vehicle is much greater than the axle mass, but the tire has characteristics which should be included in the analysis. In the model of Figure P3.59(b), the tire is assumed to be elastic with a stiffness k_t. The tire stiffness acts in series with the spring and viscous damper of the suspension system.

FIGURE P3.60

(a) Derive a third-order differential equation governing the displacement of the vehicle from the system's equilibrium position.
(b) Solve the differential equation to determine the response of the system when the wheel encounters a pothole of depth h.

Note: Problem 3.60 in text should be 3.59; Figure P3.60 should be Figure P3.59

Given: system shown, m, c, k_s, k_t, h

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Find: (a) differential equation, (b) system response

Solution: Let $x_1(t)$ be the displacement of the point where the suspension connects to the tire. Free-body diagrams of both the vehicle and the joint at an arbitrary instant are shown.

(a) Summing forces on the vehicle

$$\sum F_{ext} = \sum F_{eff}$$
$$-k_s(x - x_1) - c(\dot{x} - \dot{x}_1) = m\ddot{x}$$

Summing forces on the joint

$$\sum F = 0$$
$$k_s(x - x_1) + c(\dot{x} - \dot{x}_1) - k_t x_1 = 0$$
$$k_s(x - x_1) + c(\dot{x} - \dot{x}_1) = k_t x_1$$

Substitution of the last equation into the force equation from the vehicle leads to

$$-k_t x_1 = m\ddot{x}$$
$$x_1 = -\frac{m}{k_t}\ddot{x}$$

Substitution for x_1 into the force equation on the vehicle leads to

$$-k_s\left(x + \frac{m}{k_t}\ddot{x}\right) - c\left(\dot{x} + \frac{m}{k_t}\dddot{x}\right) = m\ddot{x}$$
$$\frac{cm}{k_t}\dddot{x} + \left(m + \frac{k_s m}{k_t}\right)\ddot{x} + c\dot{x} + k_s x = 0$$

(b) Solution of the third-order differential equation is obtained by assuming

$$x(t) = Ae^{\alpha t}$$

Substitution into the differential equation leads to the following cubic equation for α

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$$\frac{cm}{k_t}\alpha^3 + m\left(1+\frac{k_s}{k_t}\right)\alpha^2 + c\alpha + k_s = 0$$

The cubic equation will have three roots, one real and two complex conjugates

$$\alpha = \alpha_1, \alpha_{2r} \pm i\alpha_{2i}$$

which leads to a general solution of

$$x(t) = C_1 e^{\alpha_1 t} + e^{\alpha_{2r} t}\left(C_2 \cos\alpha_{2i} t + C_3 \sin\alpha_{2i} t\right)$$

The constants of integration are obtained by developing and applying appropriate initial conditions. The vehicle encounters the pothole at t = 0 with zero vertical velocity

$$x(0) = h$$
$$\dot{x}(0) = 0$$

Using these in the force balance on the joint leads to

$$k_s(h - x_1(0)) = k_t x_1(0)$$
$$x_1(0) = \frac{k_s h}{k_s + k_t}$$

From the derived relation between x and x_1

$$\ddot{x}(0) = -\frac{k_t}{m}x_1(0) = -\frac{k_s k_t}{k_s + k_t}h$$

Problem 3.59 illustrates the behavior of a system with a spring-dashpot system in series with another spring.

3.60 A one-degree-of-freedom model of a suspension system is shown in Figure P3.60(a). Consider a model in which the tire is modeled by a viscous damper of damping coefficient c_t and is placed in series with the spring and viscous damper modeling the suspension system, as illustrated in Figure P3.60(a) .

(a) Derive a third-order differential equation governing the displacement of the vehicle from the system's equilibrium position.
(b) A plot of the suspension system when the wheel encounters a pothole is given in Figure P3.61(b). The plot is made for a suspension system that is designed to have a damping ratio of 0.1. Use this information to find c_t.
Note: Problem 3.61 in text should be 3.60; Figure P3.61 should be Figure P3.60

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FIGURE P3.61

Given: system shown, $\zeta = 0.1$

Find: differential equation, c_t

Solution: (a) Let x(t) represent the displacement of the vehicle from its equilibrium position. Define $x_1(t)$ as the displacement of the joint between the suspension and the tire. Free body diagrams of the vehicle and the joint are shown below at an arbitrary instant.

Summing forces acting on the vehicle leads to

$$\sum F_{ext} = \sum F_{eff}$$
$$k_s(x_1 - x) + c_s(\dot{x}_1 - \dot{x}) = m\ddot{x}$$

Summing forces acting at the joint leads to

$$\sum F = 0$$
$$c_t\dot{x}_1 + k_s(x_1 - x) + c_s(\dot{x}_1 - \dot{x}) = 0$$
$$k_s(x_1 - x) + c_s(\dot{x}_1 - \dot{x}) = -c_t\dot{x}_1$$

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Substituting this result into the equation from the vehicle leads to

$$-c_t\dot{x}_1 = m\ddot{x}$$

$$\dot{x}_1 = -\frac{m}{c_t}\ddot{x}$$

Differentiating the force equation from the vehicle with respect to time

$$k_s(\dot{x}_1 - \dot{x}) + c_s(\ddot{x}_1 - \ddot{x}) = m\dddot{x}$$

Substituting for x_1 leads to

$$k_s\left(-\frac{m}{c_t}\ddot{x} - \dot{x}\right) + c_s\left(-\frac{m}{c_t}\dddot{x} - \ddot{x}\right) = m\dddot{x}$$

$$m\left(1 + \frac{c_s}{c_t}\right)\dddot{x} + \left(\frac{k_s m}{c_t} + c_s\right)\ddot{x} + k_s\dot{x} = 0$$

Dividing by m leads to

$$(1+\lambda)\dddot{x} + \left(\frac{\lambda\omega_n}{2\zeta} + 2\zeta\omega_n\right)\ddot{x} + \omega_n^2\dot{x} = 0$$

where

$$\lambda = \frac{c_s}{c_t}$$

$$\omega_n = \sqrt{\frac{k_s}{m}}$$

$$\zeta = \frac{c_s}{2m\omega_n} = \frac{c_s}{2\sqrt{mk_s}}$$

(b) The solution of the differential equation is assumed as

$$a(t) = Ae^{\alpha t}$$

Substitution of the assumed solution into the differential equation leads to the following algebraic equation for α

$$(1+\lambda)\alpha^3 + \left(\frac{\lambda\omega_n}{2\zeta} + 2\zeta\omega_n\right)\alpha^2 + \omega_n^2\alpha = 0$$

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The solution of the cubic equation is obtained by noting α=0 is a solution and solving the resulting quadratic equation leading to

$$\alpha = 0,\ \frac{\omega_n}{2(1+\lambda)}\left[-\left(\frac{\lambda}{2\zeta}+2\zeta\right)\pm\sqrt{\left(\frac{\lambda}{2\zeta}+2\zeta\right)^2-4(1+\lambda)}\right]$$

The solution can then be written in the form

$$x(t) = A + Be^{-\zeta_{eq}\omega_n t}\sin(\omega_{deq}t+\phi)$$

where A,B, and ϕ are determined in terms of initial conditions and

$$\zeta_{eq} = \frac{1}{2(1+\lambda)}\left(\frac{\lambda}{2\zeta}+2\zeta\right)$$

$$\omega_{deq} = \frac{\omega_n}{2(1+\lambda)}\sqrt{-\left(\frac{\lambda}{2\zeta}+2\zeta\right)^2+4(1+\lambda)}$$

From the information given

$$\zeta = 0.1$$

$$\frac{2\pi}{\omega_{deq}} = 0.85\ \text{s}$$

Note also that a logarithmic decrement can be defined in the form

$$\delta = \ln\left(\frac{x(t)-A}{x(t+2\pi/\omega_d)-A}\right)$$

Following the same procedure as for free vibrations of an underdamped one-degree-of-freedom system

$$\delta = \zeta_{eq}\omega_n\frac{2\pi}{\omega_{deq}}$$

From the graph of the system response

$$\delta = \ln\left(\frac{0.42}{0.2}\right) = 0.742$$

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Putting all of the information together

$$0.742 = 2\pi \frac{1}{2(1+\lambda)}\left(\frac{\lambda}{2\zeta}+2\zeta\right)\frac{2(1+\lambda)}{\sqrt{4(1+\lambda)-\left(\frac{\lambda}{2\zeta}+2\zeta\right)^2}}$$

$$0.742 = 2\pi \frac{\left(\frac{\lambda}{2\zeta}+2\zeta\right)}{\sqrt{4(1+\lambda)-\left(\frac{\lambda}{2\zeta}+2\zeta\right)^2}}$$

$$0.742 = 2\pi \frac{\lambda+.04}{\sqrt{0.16(1+\lambda)-(\lambda+.04)^2}}$$

A trial and error solution leads to λ=0.007. The natural frequency is obtained from

$$\omega_{deq} = \frac{2\pi}{0.85} = \frac{\omega_n}{2(1+\lambda)}\sqrt{4(1+\lambda)-\left(\frac{\lambda}{.2}+.2\right)^2}$$

$$\omega_n = 7.47 \text{ rad/s}$$

Problem 3.60 illustrates the modeling of a system with a viscous damper in series with a spring and viscous damper in parallel.

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CHAPTER 4: HARMONIC EXCITATION OF SDOF SYSTEMS

Short Answer Problems

4.1 True: The steady–state response for a linear system occurs at the same frequency as the input.

4.2 False: Resonance is characterized by a continual build up in amplitude (Beating is characterized by a periodic build up and decay of amplitude)

4.3 False: $\frac{mX}{m_0 e}$ for a machine with a rotating unbalance approaches one for large frequencies.

4.4 False: A decrease in damping leads to an increase of the percentage of isolation.

4.5 False: The phase angle for an undamped system is either zero (if the excitation frequency is less than the natural frequency) or π (if the excitation frequency is greater than the natural frequency).

4.6 False: The phase angle is independent of F_0, the amplitude of excitation.

4.7 False: If ϕ is positive in the equation $x(t) = X\sin(\omega t - \phi)$ the response leads the excitation.

4.8 True: $M(r, \zeta)$approaches zero for large r, for all values of ζ.

4.9 False: $\Lambda(r, \zeta)$ approaches 1 for large r for all values of ζ.

4.10 False: $T(r, \zeta)$ approaches 0 for large r for all values of ζ.

4.11 False: The amplitude of the acceleration response of a system is given by $R(r, \zeta)$ if its base is subject to a single frequency harmonic excitation.

4.12 True: Hysteretic damping is a nonlinear phenomena, but for a single frequency excitation the hysteretic damping can be approximated by viscous damping.

4.13 True: The linear differential equation is not valid when the system is subject to a multi-frequency excitation

4.14 True: A seismometer measures the displacement of the seismic mass relative to the body whose vibrations are to be measured.

4.15 True: A complex stiffness can be used to model hysteretic damping.

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4.16 True: For $\zeta < \frac{1}{\sqrt{2}}$, $M(r,\zeta)$ grows from 1 as r increase from zero. But $M(r,\zeta) \to 0$ for large r. Thus $M(r,\zeta)$reaches a maximum for a value of $r = \sqrt{1-2\zeta^2}$

4.17 Resonance occurs for an undamped system when the excitation frequency coincides with the natural frequency because the work done by the excitation force is not needed to sustain the motion at that frequency. Any initial energy sustains the free vibrations of an undamped system at the natural frequency.

4.18 The amplitude does not grow without bound for systems with viscous damping when the excitation frequency coincides with the natural frequency because the damping dissipates any initial energy. The work done by the excitation force is necessary to sustain the motion.

4.19 The response out of phase with the excitation for an undamped system when the frequency ratio is greater than one.

4.20 In the equation $x(t) = X\sin(\omega t - \phi)$, ϕ is negative when the frequency ratio is greater than one.

4.21 (a) zero (b) zero (c) two

4.22 (a) one (b) two (c) zero (d) all real values of r

4.23 (a) one (b) one (c) two

4.24 (a) two (b) one (c) one

4.25 Frequency response if the study of how the steady-state amplitude of vibration and the steady-state phase vary with the frequency of excitation. For SDOF systems the frequency response is studied by studying $M(r,\zeta)$ versus r for any value of ζ.

4.26 The frequency response for a system with a rotating unbalance is studied through $\Lambda(r,\zeta) = \frac{mX}{m_0 e}$.

4.27 The frequency response for a machine on a moveable foundation is studied through $T(r,\zeta) = \frac{X}{Y}$. The displacement of the machine relative to the foundation is studied by $\Lambda(r,\zeta) = \frac{Z}{Y}$.

4.28 Vibration isolation is difficult to achieve at low speeds because it requires a large static deflection of the isolator ($r > \sqrt{2}$, thus the required natural frequency for low speeds is small. The static deflection is inversely proportional to the square of the natural frequency.)

4.29 Percentage isolation is the percent by which an isolator reduces the transmitted force, it is equal to $100\left(1 - \frac{F_0}{F_T}\right)$.

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4.30 The transmitted force is $F_T = kx + c\dot{x}$. The force generated by the motion of a base that's transmitted to the body is $kx + c\dot{x}$. Thus, the systems behave the same, only in the first case the force is being transmitted to the foundation, in the second case the force is being transmitted to the body.

4.31 Seismometers have a <u>small</u> natural frequency and thus operate only for <u>large</u> frequency ratios.

4.32 Phase distortion during accelerometer measurements when a multi-frequency excitation is being measured. Since the accelerometer is actually measuring the displacement of the seismic mass relative to the body whose vibrations are to be measured and the accelerometer has damping different phase angles are involved in the measurement of the signal. In the range where accelerometers operate the phase angle is dependent of frequency. For seismometers the ratio of the measured frequency to the natural frequency of the seismometer is high, thus the phase angle is approximately π.

4.33 The principle of linear superposition states that for a linear differential equation the particular solution of a differential equation due to a summation on the right-hand side can be obtained by summing the solutions due to each individual term. Thus it allows the response due to multiple frequency input to be obtained as the sum of the responses due to the individual frequencies.

4.34 The principle of linear superposition applies to general periodic input because the input can be thought of as a summation of inputs applied over a very small interval of time.

4.35 Stick-slip may be present in the forced response of a system with Coulomb damping when the spring and inertia forces are temporarily unable to overcome the friction force and the motion stops.

4.36 $\iota < \pi/4$

4.37 Damping is used in vibration isolation because the operating speed is greater than the natural frequency. During start-up and stopping the natural frequency must be passed through. The damping limits the vibrations during these times.

4.38 (a) No, an undamped system has the same natural frequency as excitation frequency, thus a resonance condition exists. (b) Yes, the excitation frequency is the same as the natural frequency, but the system is damped. (c) Yes, the undamped system has a natural frequency that is different from the excitation frequency.

4.39 (a) Given: $M(r,0) = 1.4 \Longrightarrow \frac{1}{|1-r^2|} = 1.4$ Using the positive sign on $1 - r^2$ leads to $\frac{1}{1-r^2} = 1.4 \Longrightarrow r^2 = 1 - \frac{1}{1.4} \Longrightarrow r = 0.534$ Evaluating the absolute value as $r^2 - 1$ leads to $\frac{1}{r^2-1} = 1.4 \Longrightarrow r^2 = 1 + \frac{1}{1.4} \Longrightarrow r = 1.31$; (b) $M(r, 0.4) > 3$. There are no values of r which satisfy this equation. For $\zeta = 0.4$, $M_{max} = 1.36$; (c) $M(r, 0.8) < 1.2$. All values of r satisfy this equation. M does not reach a maximum for $\zeta = 0.8$. It starts out at 1 at r=0 and approaches 0 for large r.

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4.40 (a) Given: $T(r, 0.1) < 1 \Longrightarrow r > \sqrt{2}$; (b) Given: $T(r, 0.8) > 1 \Longrightarrow r < \sqrt{2}$; (c) Given: $T(r, 0.4) > T(r, 0.3) \Longrightarrow r > \sqrt{2}$

4.41 Given: $m = 30$ kg, ω =60 rad/s. Resonance occurs when the frequency of excitation coincides with the natural frequency, $\omega = \omega_n = \sqrt{\frac{k}{m}} \Longrightarrow k = m\omega^2 = (30\text{ kg})(60\text{ rad/s}2{=}1.08{\times}105\text{ N/m}$.

4.42 Given: $\omega_n = 98$ rad/s, ω =100 rad/s. (a) The period of response is $\frac{4\pi}{\omega_n+\omega} = \frac{4\pi}{100+98} =$ 0.0635 s; (b) The period of beating is $\frac{2\pi}{\omega_n-\omega} = \frac{2\pi}{100-98} = 3.1415$ s.

4.43 Given: ω =100 rad/s, m_0 = 5 kg, $e = 3$ cm. The amplitude of the harmonic excitation experienced by the machine is $F_0 = m_0 e\omega^2 = (5\text{ kg})(0.03\text{ m})(100\text{ rad/s})^2 = 1500$ N.

4.44 Given: $\omega = 1000$ rpm. The conversion to rad/s is $\left(1000\ \frac{\text{rev}}{\text{min}}\right)\left(2\pi\ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) =$ 104.7 rad/s.

4.45 Given: $F_0 = 15000$ N, $F_T = 3000$ N. The transmissibility ratio is $T = \frac{F_T}{F_0} = \frac{3000\text{ N}}{15000\text{ N}} =$ 0.2. The percentage isolation is $100(1 - T) = 100(1 - 0.2) = 80$ %.

4.46 Given: $m = 50$ kg, $k = 6.5 \times 10^5$ N/m, $\omega = 140$ rad/s (a) The frequency ratio is $r = \frac{\omega}{\omega_n} = \frac{\omega\sqrt{m}}{\sqrt{k}} = \frac{(140\text{ rad/s})\sqrt{50\text{ kg}}}{\sqrt{6.5{\times}10^5\text{N/m}}} = 1.23$ (b) No, because $r < \sqrt{2}$

4.47 (a) $b_i = 0$ (b) $a_i = 0$ $i = 1,2,3 \ldots$
(c) $a_i = 0$ (d) $b_i = 0$ (e) none.

FIGURE SP4.46

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4.48

4.49 Given: undamped accelerometer, E = 1 percent, $\omega_n = 200$ rad/s. The percent error in an accelerometer measurement for an undamped accelerometer is $100(M-1)$. Setting E = 1 percent leads to $M(r,0) = 1.01$ which gives r=0.0995 and $\omega = r\omega_n = 19.9$ rad/s.

4.50 Given: undamped seismometer, E = 1.5 percent, $\omega_n = 20$ rad/s. The percent error in an accelerometer measurement for an undamped accelerometer is $100(\Lambda-1)$. Setting E = 1.5 percent leads to $\Lambda(r,0) = 1.015$ which gives r = 8.10 and $\omega = r\omega_n = 162.1$ rad/s.

4.51 Given: $3\ddot{x} + 2700x = 20\sin 10t$. The differential equation is put into the standard form $\ddot{x} + 900x = \frac{20}{3}\sin 10t$. It is identified that $m = 3, \omega_n = 30, \omega = 10, \zeta = 0,\ F_0 = 20$. The frequency ratio is $\frac{\omega}{\omega_n}$ =1/3. The steady-state solution to the differential equation is $(t) = \frac{F_0}{m\omega_n^2} M\left(\frac{1}{3}, 0\right)\sin(10t - \phi)$, where $M\left(\frac{1}{3}, 0\right) = \frac{1}{|1-r^2|} = 1.125$ and $\phi = 0$. Thus, $x(t) = \frac{20}{3(30)^2} 1.125\ \sin 10t = 8.33 \times 10^{-3} \sin 10t$.

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4.52 Given: $3\ddot{x} + 2700x = 20\sin 60t$. The differential equation is put into the standard form $\ddot{x} + 900x = \frac{20}{3}\sin 60t$. It is identified that $m = 3, \omega_n = 30, \omega = 60, \zeta = 0, F_0 = 20$. The frequency ratio is $\frac{\omega}{\omega_n} = 2$. The steady-state solution to the differential equation is $(t) = \frac{F_0}{m\omega_n^2} M(2,0)\sin(10t - \phi)$, where $M(2,0) = \frac{1}{|1-r^2|} = 0.333$ and $\phi = \pi$. Thus, $x(t) = \frac{20}{3(30)^2} 0.333\sin(10t - \pi) = 2.47 \times 10^{-3}\sin(10t - \pi)$.

4.53 Given: $3\ddot{x} + 30\dot{x} + 2700x = 20\sin 10t$. The differential equation is put into the standard form $\ddot{x} + 10\dot{x} + 900x = \frac{20}{3}\sin 10t$. It is identified that $m = 3, \omega_n = 30, \omega = 10, \zeta = \frac{10}{2(30)} = \frac{1}{6}, F_0 = 20$. The frequency ratio is $\frac{\omega}{\omega_n} = \frac{1}{3}$. The steady-state solution to the differential equation is $(t) = \frac{F_0}{m\omega_n^2} M\left(\frac{1}{3}, \frac{1}{6}\right)\sin(10t - \phi)$, where

$$M\left(\frac{1}{3}, \frac{1}{6}\right) = \frac{1}{\sqrt{\left(1-\frac{1}{9}\right)^2 + \left[2\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\right]^2}} = 1.1163 \text{ and } \phi = \tan^{-1}\left[\frac{2\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)}{1-\frac{1}{9}}\right] = 0.1244.$$

Thus $x(t) = \frac{20}{3(30)^2} 1.1163\sin(10t - 0.1244) = 8.61 \times 10^{-3}\sin(10t - 0.1244)$.

4.54 Given: $3\ddot{x} + 30\dot{x} + 2700x = 0.01\omega^2\sin\omega t$. The differential equation is put into the standard form for a system subject to a frequency squared excitation: $\ddot{x} + 10\dot{x} + 900x = \frac{0.01\omega^2}{3}\sin 10t$. It is identified that $m = 3, \omega_n = 30, \zeta = \frac{10}{2(30)} = \frac{1}{6}, A = 0.01$. The frequency ratio is $\frac{\omega}{\omega_n}$. The steady-state solution to the differential equation is $(t) = \frac{0.01\omega^2}{3}\Lambda\left(r, \frac{1}{6}\right)\sin(\omega t - \phi)$, where $\Lambda(r, \zeta) = \frac{r^2}{\sqrt{(1-r^2)^2 + \left[2\left(\frac{1}{6}\right)r\right]^2}}$ and $\phi = \tan^{-1}\left[\frac{2\left(\frac{1}{6}\right)r}{1-r^2}\right]$.

4.55 Given: $3\ddot{x} + 30\dot{x} + 2700x = 30(0.002)(40)\cos 40t + 2700(0.002)\sin 40t$. The differential equation is put into the standard form for a system with a mass-spring viscous damper system attached to a moveable base: $\ddot{x} + 10\dot{x} + 900x = \frac{0.002}{3}(30(40)\cos 40t + 2700\sin 40t$. It is identified that *m=3, ωn=30, ω=40, ζ=102(30)=16, Y=0.002*. The frequency ratio is $\frac{\omega}{\omega_n} = \frac{4}{3}$. The steady-state solution to the differential equation is $(t) = YT\left(\frac{4}{3}, \frac{1}{6}\right)\sin(40t - \phi)$, where $T(r, \zeta) = \sqrt{\frac{1+\left[2\left(\frac{1}{6}\right)\left(\frac{4}{3}\right)\right]^2}{\left(1-\frac{1}{9}\right)^2 + \left[2\left(\frac{1}{6}\right)\left(\frac{4}{3}\right)\right]^2}} = 1.221$ and $\lambda = \tan^{-1}\left[\frac{2\left(\frac{1}{6}\right)\left(\frac{4}{3}\right)^3}{1+\left(4\left(\frac{1}{6}\right)^2 - 1\right)\left(\frac{4}{3}\right)^2}\right] = -0.9374$. Thus $x(t) = (0.002)(1.221)\sin(40t + 0.9374) = 2.44 \times 10^{-3}\sin(40t + 0.9374)$

4.56 Given: $3\ddot{x} + \frac{2700(0.002)}{\omega}\dot{x} + 2700x = 20\sin\omega t$. The differential equation is put into the standard form for a system with hysteretic damping where it is identified that

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$m = 3, \omega_n = 30,\ h = 0.002, F_0 = 20$. The frequency ratio is $r = \frac{\omega}{\omega_n}$. The steady-state solution of the differential equation is $x(t) = \frac{F_0}{m\omega_n^2} M_h(r, 0.002)\sin(\omega t - \phi_h) = 7.41 \times 10^{-3} M_h(r, 0.002)\sin(\omega t - \phi_h)$ where $M_h(r, 0.002) = \frac{1}{\sqrt{(1-r^2)^2 + (0.002)^2}}$ and $\phi_h = \tan^{-1}\left(\frac{0.002}{1-r^2}\right)$

4.57 Given: $3\ddot{x} + 30\dot{x} + 2700x = 30\sin 50t + 20\sin 20t$. The differential equation is put in the standard form for multi-frequency excitations: $\ddot{x} + 10\dot{x} + 900x = 10\sin 50t + \frac{20}{3}\sin 20t$. It is identified that $m = 3, \omega_n = 30,\ \zeta = \frac{10}{2(30)} = \frac{1}{6},\ \omega_1 = 50, F_{01} = 30, \omega_2 = 20, F_{02} = 20$. Hence, $r_1 = \frac{5}{3}$ and $r_2 = \frac{2}{3}$. The steady-state solution is

$$x(t) = \frac{F_{01}}{m\omega_n^2} M\left(\frac{5}{3}, \frac{1}{6}\right)\sin(50t - \phi_1) + \frac{F_{02}}{m\omega_n^2} M\left(\frac{2}{3}, \frac{1}{6}\right)\sin(20t - \phi_2)$$

where $M\left(\frac{5}{3}, \frac{1}{6}\right) = \frac{1}{\sqrt{\left(1-\frac{25}{9}\right)^2 + \left[2\left(\frac{1}{6}\right)\left(\frac{5}{3}\right)\right]^2}} = 0.1881$, $\phi_1 = \tan^{-1}\left[\frac{2\left(\frac{1}{6}\right)\left(\frac{5}{3}\right)}{1-\frac{25}{9}}\right] = -0.1574$,

$M\left(\frac{2}{3}, \frac{1}{6}\right) = \frac{1}{\sqrt{\left(1-\frac{4}{9}\right)^2 + \left[2\left(\frac{1}{6}\right)\left(\frac{2}{3}\right)\right]^2}} = 1.6713$, $\phi_2 = \tan^{-1}\left[\frac{2\left(\frac{1}{6}\right)\left(\frac{2}{3}\right)}{1-\frac{4}{9}}\right] = 0.3805$ Thus $x(t) = 1.39 \times 10^{-3}\sin(50t + 0.1574 + 1.238 \times 10^{-2}\sin(20t - 0.3805)$

4.58 Given: $3\ddot{x} + 2700x = \begin{cases} 50\sin 20t - 5 & \dot{x} < 0 \\ 50\sin 20t + 5 & \dot{x} > 0 \end{cases}$. The differential equation is in the standard form for a system with Coulomb damping. It is identified that $m = 3, \omega_n = 30, \omega = 20, \zeta = \frac{10}{2(30)} = \frac{1}{6}, F_0 = 50, F_f = 5$. Thus $r = \frac{2}{3}$ and $\iota = \frac{1}{10}$. Thus, the steady-state solution is given by $x(t) = \frac{F_0}{m\omega_n^2} M_c\left(\frac{2}{3}, \frac{1}{10}\right)\sin(20t - \phi_c)$ where $M_c\left(\frac{2}{3}, \frac{1}{10}\right) = \sqrt{\frac{1-\left(\frac{4(0.1)}{\pi}\right)}{\left(1-\frac{4}{9}\right)^2}} = 1.785$ and $\phi_c = \tan^{-1}\left[\frac{4(0.1)/\pi}{\sqrt{1-(4(0.1)/\pi)^2}}\right] = 0.1277$. Thus

$$x(t) = 0.01322\sin(20t - 0.1277)$$

4.59 (a)-(i) (b)-(v) (c)-(ii) (d)-(ii) (e)-(ix) (f)-(i) (g)-(ii) (h)-(vi) (i)-(ii) (j)-(ii) (k)-(x) (l)-(xii)

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Chapter Problems

4.1 A 40 kg mass hangs from a spring with a stiffness of 4×10^4 N/m. A harmonic force of magnitude of 100 N and frequency of 120 rad/sec is applied. Determine the amplitude of the forced response.

Given: $k = 4 \times 10^4$ N/m, m = 40 kg, $F_0 = 100$ N, $\omega = 120$ rad/sec

Find: X

Solution: The amplitude of the forced response for an undamped linear one-degree-of-freedom system is

$$X = \frac{F_0}{m\left(\omega_n^2 - \omega^2\right)}$$

where

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times10^4\,\frac{\text{N}}{\text{m}}}{40\,\text{kg}}} = 31.6\,\frac{\text{rad}}{\text{s}}$$

Substituting values yields

$$X = \frac{100\ \text{N}}{40\text{kg}\left[\left(31.6\,\frac{\text{rad}}{\text{s}}\right)^2 - \left(120\,\frac{\text{rad}}{\text{s}}\right)^2\right]} = -0.187\ \text{mm}$$

The negative sign indicates that the response is 180º out of phase with the excitation.

Problem 4.1 illustrates the determination of the amplitude of forced response for a one-degree-of-freedom undamped system subject to a single frequency harmonic excitation.

4.2 Determine the amplitude of the forced oscillations of the 30 kg block of Figure P4.2.

Given: $I_P = 0.68$ kg · m^2, m = 30 kg, k = 400 N/m, $F_0 = 200$ N, $\omega = 10$ rad/sec, r = 10 cm

Find: X

FIGURE P4.2

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Solution: Let $x(t)$ be the displacement of the block measured from its equilibrium position. The governing differential equation is derived by applying Newton's Laws to free body diagrams of the pulley and block at an arbitrary instant.

Summing moments about the center of the pulley

$$\left(\sum M_0\right)_{ext.} = \left(\sum M_0\right)_{eff.}$$

$$F_0 \sin \omega t\,(r) - kx\,(r) = m\ddot{x}\,(r) + I_P \frac{\ddot{x}}{r}$$

$$\left(m + \frac{I_P}{r^2}\right)\ddot{x} + kx = F_0 \sin \omega t$$

$$\ddot{x} + \frac{k}{m + \frac{I_P}{r^2}} x = \frac{F_0}{m + \frac{I_P}{r^2}} \sin \omega t$$

The equivalent mass is

$$\tilde{m} = m + \frac{I_P}{r^2} = 30\,\text{kg} + \frac{0.68\,\text{kg}\cdot\text{m}^2}{(0.1\,\text{m})^2} = 98\,\text{kg}$$

The natural frequency is obtained as

$$\omega_n = \sqrt{\frac{k}{\tilde{m}}} = \sqrt{\frac{400\,\frac{\text{N}}{\text{m}}}{98\,\text{kg}}} = 2.02\,\frac{\text{rad}}{\text{s}}$$

The amplitude of response calculated as

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$$x = \frac{F_0}{\tilde{m}\left(\omega_n^2 - \omega^2\right)} = \frac{200\text{ N}}{98\text{kg}\left[\left(2.02\frac{\text{rad}}{\text{s}}\right)^2 - \left(10\frac{\text{rad}}{\text{s}}\right)^2\right]} = -21.3\text{ mm}$$

The negative sign indicates that the response is 180º out of phase with the excitation.

Problem 4.2 illustrates the derivation of the differential equation governing the forced vibrations of a one-degree-of-freedom system and determination of the amplitude of response for a single frequency harmonic excitation.

4.3 For what values of M_0 will the forced amplitude of angular displacement of the bar in Figure P4.3 be less than 3° if $\omega = 25\ \frac{\text{rad}}{\text{s}}$?

FIGURE P4.3

Given: m = 0.8 kg, k = 1 × 10^4 N/s, L = 0.4 m, $\omega = 25\ \frac{\text{rad}}{\text{s}}$, $\theta_{\max} = 3°$

Find: M_0

Solution: The kinetic energy of the system is

$$T = \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 + \frac{1}{2}m\left(\frac{L}{4}\dot{\theta}\right)^2 = \frac{1}{2}\left(\frac{7}{48}mL^2\right)\dot{\theta}^2$$

Hence using θ as a generalized coordinate

$$I_{eq} = \frac{7}{48}mL^2 = \frac{7}{48}(0.8\text{ kg})(0.4\text{ m})^2 = 0.0187\text{ kg}\cdot\text{m}^2$$

The potential energy of the system is

$$V = \frac{1}{2}k\left(\frac{L}{4}\theta\right)^2 + \frac{1}{2}k\left(\frac{L}{4}\theta\right)^2 = \frac{1}{2}\left(k\frac{L^2}{2}\right)\theta^2$$

Hence the equivalent torsional stiffness is

$$k_{t,eq} = k\frac{L^2}{2} = \frac{1}{2}(1\times 10^4\text{ N/m})(0.4\text{ m})^2 = 800\text{ N}\cdot\text{m/rad}$$

The work done by the external force is

$$W = \int M_0 \sin\omega t\, d\theta$$

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Hence the equivalent moment is

$$M_{eq} = \int M_0 \sin \omega t$$

The governing differential equation describing the motion of the system is

$$0.0187\ddot{\theta} + 800\theta = M_0 \sin 25t$$

The differential equation is put into standard form by dividing by 0.0187 leading to

$$\ddot{\theta} + 4.278 \times 10^4 \theta = 53.48 M_0 \sin 25t$$

The natural frequency and frequency ratio are

$$\omega_n = \sqrt{4.278 \times 10^4} = 206.8 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{25 \text{ rad/s}}{206.8 \text{ rad/s}} = 0.121$$

The steady state amplitude is given by

$$\Theta = \frac{M_0}{k} M(0.121,0) \Longrightarrow M_0 = \frac{k\Theta}{M(0.121,0)}$$

Hence

$$M_0 < \frac{(3°)\left(\frac{2\pi \text{ rad}}{360°}\right)(1 \times 10^4 \text{ N} \cdot \text{m/rad})}{1 - (0.121)^2} = 531.4 \text{ N} \cdot \text{m}$$

Problem 4.3 illustrates the frequency-amplitude relation for an undamped system.

4.4 For what values of ω will the forced amplitude of the bar shown be less than 3° if $M_0 = 300 \text{ N} \cdot \text{m}$?

FIGURE P4.3

Given: m = 0.8 kg, k = 1 × 10⁴ N/s, L = 0.4 m, $M_0 = 300 \text{ N} \cdot \text{m}$, $\theta_{max} = 3°$

Find: ω

Solution: The kinetic energy of the system is

$$T = \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 + \frac{1}{2}m\left(\frac{L}{4}\dot{\theta}\right)^2 = \frac{1}{2}\left(\frac{7}{48}mL^2\right)\dot{\theta}^2$$

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Hence using θas a generalized coordinate

$$I_{eq} = \frac{7}{48} mL^2 = \frac{7}{48}(0.8\text{ kg})(0.4\text{ m})^2 = 0.0187\text{ kg}\cdot\text{m}^2$$

The potential energy of the system is

$$V = \frac{1}{2}k\left(\frac{L}{4}\theta\right)^2 + \frac{1}{2}k\left(\frac{L}{4}\theta\right)^2 = \frac{1}{2}\left(k\frac{L^2}{2}\right)\theta^2$$

Hence the equivalent torsional stiffness is

$$k_{t,eq} = k\frac{L^2}{2} = \frac{1}{2}(1\times 10^4\text{ N/m})(0.4\text{ m})^2 = 800\text{ N}\cdot\text{m/rad}$$

The work done by the external force is

$$W = \int M_0 \sin\omega t\, d\theta$$

Hence the equivalent moment is

$$M_{eq} = \int M_0 \sin\omega t$$

The governing differential equation describing the motion of the system is

$$0.0187\ddot{\theta} + 800\theta = M_0 \sin\omega t$$

The differential equation is put into standard form by dividing by 0.0187 leading to

$$\ddot{\theta} + 4.278\times 10^4\theta = 53.48 M_0 \sin 25t$$

The natural frequency is given by

$$\omega_n = \sqrt{4.278\times 10^4} = 206.8\text{ rad/s}$$

The steady state amplitude is given by

$$M(r,0) = \frac{k\Theta}{M_0}$$

Hence

$$M(r,0) < \frac{(3°)\left(\frac{2\pi\text{ rad}}{360°}\right)\left(1\times 10^4\text{ N}\cdot\frac{\text{m}}{\text{rad}}\right)}{300\text{ N}\cdot\text{m}} = 1.75$$

For $r < 1$ this implies

$$1.75 > \frac{1}{1-r^2} \Rightarrow r < 0.655$$

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and

$$\omega < 0.655\left(206.8\frac{\text{rad}}{\text{s}}\right) = 135.4\ \text{rad/s}$$

For $r > 1$ this implies

$$1.75 > \frac{1}{r^2 - 1} \Rightarrow r > 1.25$$

and

$$\omega > 1.25\left(206.8\frac{\text{rad}}{\text{s}}\right) = 258.5\ \text{rad/s}$$

Thus $\omega < 135.4\ \text{rad/s}$ or $\omega > 258.5\ \text{rad/s}$.

Problem 4.4 illustrates the frequency-amplitude relation for an undamped system.

4.5 A 2 kg gear with a radius 20 cm is mounted to the end of a 1-m long steel ($G = 80 \times 10^9\ \text{N/m}^2$) shaft. A moment M (t) = 100 sin 150t N-m is applied to the gear. For what shaft radii is the value of the forced amplitude of torsional oscillations less than 4°?

Given: m = 2 kg, r_G = 0.2 m, L= 1 m, G = 80 × 10^9 N/m^2
M(t) = 100 sin 150t N-m, $\Theta_{max.}$ = 4°

Find: r_S

Solution: The system is modeled using one degree of freedom. The amplitude of the forced torsional oscillations is given by

$$\frac{k_t \Theta}{M_0} = \frac{1}{1 - r^2} \tag{1}$$

where

$$r^2 = \frac{\omega^2}{\omega_n^2} = \frac{\omega^2}{\frac{k_t}{I_G}} = \frac{I_G \omega^2}{k_t} \tag{2}$$

Substituting eq.(2) into eq.(1) leads to

$$\frac{k_t \Theta}{M_0} = \frac{k_t}{\left|k_t - I_G \omega^2\right|} \tag{3}$$

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Requiring $\Theta < \Theta_{max.}$ from eq.(3) leads to

$$\frac{M_0}{\left|k_t - I_G\omega^2\right|} < \Theta_{max.} \tag{4}$$

Equation (4) is satisfied if

$$k_t > \frac{M_0}{\Theta_{max.}} + I_G\omega^2 \tag{5}$$

or

$$k_t < I_G\omega^2 - \frac{M_0}{\Theta_{max.}} \tag{6}$$

It is noted that

$$I_G = \frac{1}{2}m\,r_G^2 = \frac{1}{2}(2\text{kg})(0.2\text{m})^2 = 0.04\text{kg}\cdot\text{m}^2$$

$$M_0 = 100\ \text{N}\cdot\text{m}$$

$$\omega = 150\frac{\text{rad}}{\text{s}}$$

$$\Theta_{max.} = 4°\frac{2\pi}{360°} = 0.0698\text{rad}$$

$$\frac{M_0}{\Theta_{max.}} = \frac{100\text{N}\cdot\text{m}}{4°\left(\frac{\pi\text{rad}}{180°}\right)} = 1432.\text{N}\cdot\text{m}$$

$$I_G\omega^2 = \frac{1}{2}m_G r_G^2\omega^2 = \frac{1}{2}(2\text{kg})(0.2\text{m})^2(150\frac{\text{rad}}{\text{s}})^2 = 900\text{N}\cdot\text{m}$$

When these values are substituted into eq.(5),

$$k_t = \frac{\frac{\pi}{2}r_S^4 G}{L} > 2332\text{N}\cdot\text{m}$$

which gives

$$r_S > \left(\frac{2(2332\text{N}\cdot\text{m})(1\text{m})}{\left(80\times10^9\frac{\text{N}}{\text{m}^2}\right)\pi}\right)^{\frac{1}{4}} = 11.67\text{mm}$$

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When the values are substituted into eq.(6), the right hand side is negative and this case does not lead to any additional permissible values for the shaft radius.

Problem 4.5 illustrates application of the frequency response equation for undamped systems.

4.6 During operation, a 100 kg reciprocating machine is subject to a force $F(t) = 200 \sin 60t$ N. The machine is mounted on springs of an equivalent stiffness of 4.3×10^6 N/m. What is the machine's steady-state amplitude?

Given: m = 100 kg, k = $4.3 \times 10^6 \ \frac{\text{N}}{\text{m}}$, $F(t) = 200 \sin 60t$ N

Find: X

Solution: The natural frequency of the system and the frequency ratio are

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4.3 \times 10^6 \ \frac{\text{N}}{\text{m}}}{100 \text{ kg}}} = 207.4 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{60 \text{ rad/s}}{207.4 \text{ rad/s}} = 0.289$$

The steady-state amplitude of the machine is

$$X = \frac{F_0}{k} M(0.289,0) = \left(\frac{200 \ N}{4.3 \times 10^6 \ \frac{\text{N}}{\text{m}}}\right) \frac{1}{1 - (0.289)^2} = 50.8 \ \mu\text{m}$$

Problem 4.6 illustrates the frequency amplitude relation for undamped systems.

4.7 A 40 kg pump is to be placed at the midspan of a 2.5-m long steel ($E = 200 \times 10^9$ N/m^2) beam. The pump is to operate at 3000 rpm. For what values of the cross-sectional moment of inertia will the oscillations of the pump be within 3 Hz of resonance?

Given: m = 40-kg, L = 2.5 m, ω = 3000 rpm, $E = 200 \times 10^9$ N/m^2

Find: I such that ω is within 3 Hz of resonance

Solution: The excitation frequency in rad/s is

$$\omega = \left(3000 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 314.2 \text{ rad/s}$$

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It is noted that 3Hz = 18.85 rad/s. Thus the desired frequency range is

$$295.3 \text{ rad/s} < \omega < 333.0 \text{ rad/s}$$

The stiffness of a fixed-free beam at its midspan is

$$k = \frac{3EI}{(L/2)^3} = \frac{24EI}{L^3}$$

and its natural frequency is

$$\omega_n = \sqrt{\frac{24EI}{mL^3}}$$

or

$$I = \frac{mL^3\omega_n^2}{24E}$$

Using $\omega_n = 295.3$ rad/s leads to $I = 1.13 \times 10^{-5}$ m^4. Using $\omega_n = 333.0$ rad/s leads to $I = 1.45 \times 10^{-5}$ m^4

$$1.13\times10^{-5} \text{ m}^4 < I < 1.45\times10^{-5} \text{ m}^4$$

Problem 4.7 illustrates resonance of a machine attached to a fixed-free beam.

4.8 To determine the equivalent moment of inertia of a rigid helicopter component, an engineer decides to run a test in which she pins the component a distance of 40 cm and mounts the component on two springs of stiffness 3.6×10^5 N/m, as shown in Figure P4.8. She then provides a harmonic excitation to the component at different frequencies and finds that the maximum amplitude occurs at 50 rad/s. What is the equivalent centroidal moment of inertia predicted by the test?

FIGURE P4.8

Given: m = 4 kg, d = 0.4 m, $\ell = 0.5\ m$, $k = 3.6 \times 10^5\ \frac{\text{N}}{\text{m}}$, $\omega_{max} = 50$rad/s

Find: I

Solution: The differential equation governing the angular displacement of the helicopter from its equilibrium position assuming small θ is

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$$(I + md^2)\ddot{\theta} + 2k\ell^2\theta = M_0 \sin \omega t$$

For this undamped system the maximum displacement occurs when excited at the natural frequency which from the differential equation is

$$\omega_n = \sqrt{\frac{2k\ell^2}{I + md^2}}$$

Setting the natural frequency to 50 rad/s yields

$$\sqrt{\frac{2k\ell^2}{I + md^2}} = 50 \text{ rad/s}$$

$$I = \frac{2k\ell^2}{(50)^2} - md^2 = \frac{2(3.6 \times 10^5 \text{ N/m})(0.5 \text{ m})^2}{(50 \text{ rad/s})^2} - (4 \text{ kg})(0.4 \text{ m})^2 = 59.2 \text{ kg} \cdot \text{m}^2$$

Problem 4.8 illustrates how the natural frequency can be used to calculate system parameters.

4.9 The modeling of an airfoil requires at least two degrees-of-freedom. However, its torsional stiffness is unknown, so an engineer devises a test. She prevents the airfoil from motion in the transverse direction at A but still allows it to rotate as shown in Figure P4.9. She then places two springs of stiffness of 3×10^4 N/m at the tip of the airfoil and excites the airfoil with a harmonic excitation at the tip. She notices that the maximum amplitude of the tip occurs at a frequency 150 rad/sec. The mass of the airfoil is 15 kg. The distance between the mass center and A is 20 cm, and the tip is 60 cm from the A. What is the centroidal moment of inertia of the airfoil?

FIGURE P4.9

Given: m = 15 kg, d = 0.2 m, $\ell = 0.6\ m$, $k = 3 \times 10^4\ \frac{\text{N}}{\text{m}}$, $\omega_{max} = 150$rad/s

Find: I

Solution: The differential equation governing the angular displacement of the helicopter from its equilibrium position assuming small θ is

$$(I + md^2)\ddot{\theta} + 2k\ell^2\theta = F_0\ell \sin \omega t$$

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For this undamped system the maximum displacement occurs when excited at the natural frequency which from the differential equation is

$$\omega_n = \sqrt{\frac{2k\ell^2}{I + md^2}}$$

Setting the natural frequency to 150 rad/s yields

$$\sqrt{\frac{2k\ell^2}{I + md^2}} = 50 \text{ rad/s}$$

$$I = \frac{2k\ell^2}{(50)^2} - md^2 = \frac{2(3 \times 10^4 \text{ N/m})(0.6 \text{ m})}{(150 \text{ rad/s})^2} - (4 \text{ kg})(0.2 \text{ m})^2 = 8.04 \text{ kg} \cdot \text{m}^2$$

Problem 4.9 illustrates how the natural frequency can be used to calculate system parameters.

4.10 A machine with a mass of 50 kg is mounted on springs of equivalent stiffness 6.10×10^4 N/m and subject to a harmonic force of $370 \sin 35t$ N while operating. The natural frequency is close enough to the excitation frequency for beating to occur. (a) Write the overall response of the system, including the free response. (b) Plot the response of the system. (c) What is the maximum amplitude? (d) What is the period of beating?

Given: m = 50 kg, $k = 6.1 \times 10^4$ N/m, $F(t) = 370 \sin 35t$

Find: (a) x(t) (b) plot of response (c) X_{max} (d) T_b

Solution: The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = 34.929 \text{ rad/s}$$

which is close enough to $\omega = 35$rad/s for beating to occur.

(a) The general response is the sum of the free response and the forced response. The total response is

$$x(t) = \frac{2F_0}{m(\omega_n^2 - \omega^2)} \sin\left(\frac{\omega - \omega_n}{2} t\right) \cos\left(\frac{\omega + \omega_n}{2} t\right)$$

$$= \frac{2(370 \text{ N})}{(50 \text{ kg})[(34.929 \text{ rad/s})^2 - (35 \text{ rad/s})^2]} \sin(0.0358t) \cos(34.964t)$$

$$= -2.96 \sin(0.0358t) \cos(34.964t)$$

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(b) The time dependent plot is shown below

(c) The amplitude is 2.96 m

(d) The period of beating is $T_b = \frac{2\pi}{\omega - \omega_n}$=87.87 s.

4.11 A machine of mass 30 kg is mounted on springs of equivalent stiffness of 4.8×10^4 N/m. During operation, it is subject to a force of $200 \sin \omega t$. Determine and plot the response of the system if the machine is at rest in equilibrium when the forcing starts and (a) $\omega = 20$ rad/s, (b) $\omega = 40$ rad/s and (c) $\omega = 41$ rad/s.

Given: m = 30 kg, $k = 4.8 \times 10^4$ N/m , $F(t) = 200 \sin \omega t$ (a) $\omega = 20$ rad/s , (b) $\omega = 40$ rad/s and (c) $\omega = 41$ rad/s

Find: x(t)

Solution: The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = 40 \text{ rad/s}$$

(a) For $\omega = 20 \frac{\text{rad}}{\text{s}}$, the response including the free response is

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$$x(t) = \frac{-F_0\omega}{m\omega_n(\omega_n^2 - \omega^2)}\sin\omega_n t + \frac{F_0}{m(\omega_n^2 - \omega^2)}\sin\omega t$$
$$= \frac{200\ N}{(30\ kg)[(40\ \text{rad/s})^2 - (20\ \text{rad/s})^2]}\left(\sin 20t - \frac{20\ \text{rad/s}}{40\ \text{rad/s}}\sin 40t\right)$$
$$= 5.6(\sin 20t - \sin 40t)\ \text{mm}$$

(b) For $\omega = 40\frac{\text{rad}}{\text{s}}$, the response including the free response is

$$x(t) = \frac{F_0}{2m\omega_n^2}(\sin\omega_n t - \omega_n t\cos\omega_n t)$$
$$= \frac{200\ N}{2(30\ kg)(40\ rad/s)^2}(\sin 40t - 40t\cos 40t)$$
$$= 2.1(\sin 40t - 40t\cos 40t)\ \text{mm}$$

(c) For $\omega = 41\frac{\text{rad}}{\text{s}}$, the response including the free response is

$$x(t) = \frac{2F_0}{m(\omega_n^2 - \omega^2)}\sin\left(\frac{\omega - \omega_n}{2}t\right)\cos\left(\frac{\omega + \omega_n}{2}t\right)$$
$$= \frac{2(200\ N)}{(30\ kg)[(40\ \text{rad/s})^2 - (41\ \text{rad/s})^2]}\sin 0.5t\cos 40.5t$$
$$= -0.1646\sin 0.5t\cos 40.5t\quad \text{m}$$

Problem 4.11 illustrates the forced response of an undamped system away from resonance, at resonance, and at near resonance.

4.12 A 5 kg block is mounted on a helical coil spring such that the system's natural frequency is 50 rad/s. The block is subject to a harmonic excitation of amplitude 45 N at a frequency of 50.8 rad/s. What is the maximum displacement of the block from its equilibrium position?

Given: m = 5kg, ω_n = 50 rad/s, ω = 50.8 rad/s , F_0=45 N

Find**: X**

Solution: The frequency ratio is

$$r = \frac{\omega}{\omega_n} = 1.016$$

The magnification factor is

$$M = \frac{1}{\left|1 - r^2\right|} = 31.00$$

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from which the steady-state amplitude is obtained as

$$X = \frac{MF_0}{m\omega_n^2} = 0.112 \text{ m}$$

Problem 4.12 illustrates the steady-state response of an undamped system.

4.13 A 50-kg turbine is mounted on four parallel springs, each of stiffness of 3×10^5 N/m. When the machine operates at 40 Hz, its steady–state amplitude is observed as 1.8 mm. What is the magnitude of the excitation?

Given: m = 50 kg, k = 3×10^5 N/m, ω = 40 Hz, X = 1.8 mm

Find**:** F_0

Solution: Since the turbine is mounted on four springs in parallel the equivalent stiffness is

$$k_{eq} = 4k = 1.2 \times 10^6 \text{ N/m}$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 154.9 \text{ rad/s}$$

The frequency ratio and magnification factor are

$$r = \frac{\omega}{\omega_n} = \frac{(40 \text{ cycles/s})(2\pi \text{ rad/cycle})}{154.9 \text{ rad/s}} = 1.622$$

$$M = \frac{1}{\left|1 - r^2\right|} = 0.613$$

The excitation amplitude is then calculated by

$$F_0 = \frac{m\omega_n^2 X}{M} = \frac{(50 \text{ kg})(154.92 \text{ rad/s})^2 (0.0018 \text{ m})}{0.613} = 3.52 \times 10^3 \text{ N}$$

Problem 4.13 illustrates use of the magnification factor for an undamped system.

4.14 A system of equivalent mass 30 kg has a natural frequency 120 rad/sec and a damping ratio of 0.12 and is subject to a harmonic excitation of amplitude 2000 N and frequency 150 rad/sec. What is the steady–state amplitude and phase angle of the response?

Given: m = 30 kg, ω_n = 120 rad/sec, ζ = 0.12, F_0 = 2000 N, ω = 150 rad/sec

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Find: X, ϕ

Solution: The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{150\,\frac{\text{rad}}{\text{s}}}{120\,\frac{\text{rad}}{\text{s}}} = 1.25$$

The magnification factor is

$$M(1.25, 0.12) = \frac{1}{\sqrt{\left[1-(1.25)^2\right]^2 + \left[2(0.12)(1.25)\right]^2}} = 1.569$$

The steady-state amplitude is calculated

$$X = \frac{MF_0}{m\omega_n^2} = \frac{1.569\,(2000\,\text{N})}{(30\,\text{kg})\left(120\,\frac{\text{rad}}{\text{s}}\right)^2} = 7.27\,\text{mm}$$

The phase angle is

$$\phi = \tan^{-1}\left(\frac{2(0.12)(1.25)}{1-(1.25)^2}\right) = -0.49\,\text{rad}$$

Hence the steady-state response is given by

$$x(t) = 7.27\sin(150t + 0.49)\,\text{mm}$$

Problem 4.14 illustrates the application of the magnification factor to determine the steady-state amplitude of forced vibration of a one-degree-of-freedom system.

4.15 A 30-kg block is suspended from a spring with a stiffness of 300 N/m and attached to a dashpot of damping coefficient 120 N · s/m. The block is subject to a harmonic excitation of amplitude 1150 N at a frequency of 20 Hz. What is the block's steady–state amplitude ?

Given: m = 30 kg, k = 300 N/m, c = 1200 N·s/m, F_0 = 1150 N, ω = 450 Hz.

Find: X

Solution: The system's natural frequency is

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$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{300\frac{\text{N}}{\text{m}}}{30\,\text{kg}}} = 3.16\frac{\text{rad}}{\text{sec}}$$

The system's damping ratio is given by

$$\zeta = \frac{c}{2m\omega_n} = \frac{120\frac{\text{N}\cdot\text{s}}{\text{m}}}{2(30\,\text{kg})\left(3.16\frac{\text{rad}}{\text{s}}\right)} = 0.633$$

The frequency ratio is given by

$$r = \frac{\omega}{\omega_n} = \frac{20\frac{\text{cycles}}{\text{s}}\left(\frac{2\pi\,\text{rad}}{1\,\text{cycle}}\right)}{3.16\frac{\text{rad}}{\text{s}}} = 39.8$$

The magnification factor is

$$M(39.8,.633) = \frac{1}{\sqrt{\left(1-(39.8)^2\right)^2 + \left[2(39.8)(.633)\right]^2}} = 6.31\text{x}10^{-4}$$

The steady state amplitude is calculated from

$$X = \frac{MF_0}{k} = \frac{6.31\times10^{-4}(1150\,\text{N})}{300\frac{\text{N}}{\text{m}}} = 2.42\,\text{mm}$$

Problem 4.15 illustrates application of the frequency response equation to determine the steady state amplitude for a damped system.

4.16 What is the amplitude of steady–state oscillations of the 30 kg block of the system of Figure P4.16?

FIGURE P4.16

Given: m_1 = 40 kg, m_2 = 30 kg, k = 4 × 10^6 N/m, c = 2700 N · s/m, r_1 = 10 cm, r_2 = 20 cm, F_0 = 2000 N, ω = 100 rad/sec, I_p = 3 kg · m²

Find: X

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Solution: Let x represent the displacement of the 30 kg block, measured positive downward from the system's equilibrium position. The equivalent system method is used to derive the governing differential equation using x as the generalized coordinate.

The kinetic energy of the system is

$$T = \frac{1}{2} m_1 \left(\frac{r_1}{r_2} \dot{x} \right)^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I_P \left(\frac{\dot{x}}{r_2} \right)^2$$
$$= \frac{1}{2} \left(m_1 \frac{r_1^2}{r_2^2} + m_2 + \frac{I_P}{r_2^2} \right) \dot{x}^2$$

Hence the system's equivalent mass is

$$m_{eq.} = m_1 \frac{r_1^2}{r_2^2} + m_2 + \frac{I_P}{r_2^2} = 40\,\text{kg} \left(\frac{10\,\text{cm}}{20\,\text{cm}} \right)^2 + 30\,\text{kg} + \frac{3\,\text{kg} \cdot \text{m}^2}{(0.2\,\text{m})^2} = 115\,\text{kg}$$

The potential energy of the system is

$$V = \frac{1}{2} k \left(\frac{r_1}{r_2} x \right)^2$$

Hence the system's equivalent stiffness is

$$k_{eq.} = k \left(\frac{r_1}{r_2} \right)^2 = \left(4 \times 10^6 \frac{\text{N}}{\text{m}} \right) \left(\frac{10\,\text{cm}}{20\,\text{cm}} \right)^2 = 1 \times 10^6 \frac{\text{N}}{\text{m}}$$

The work done by the damping force is

$$W = -\int c \dot{x} dx$$

Hence the equivalent viscous damping coefficient is

$$c_{eq.} = c = 2700 \frac{\text{N} \cdot \text{s}}{\text{m}}$$

When the 30 kg block moves through a virtual displacement δx, the work done by the external force is

$$\delta W = F(t) \frac{r_1}{r_2} \delta x$$

Hence the generalized force is

$$F_{eq.} = F(t) \frac{r_1}{r_2} = \frac{10\,\text{cm}}{20\,\text{cm}} 2000 \sin 100t\, N = 1000 \sin 100t\, \text{N}$$

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The system parameters are calculated as

$$\omega_n = \sqrt{\frac{k_{eq.}}{m_{eq.}}} = \sqrt{\frac{1\times10^6\,\frac{\text{N}}{\text{m}}}{115\ \text{kg}}} = 93.25\,\frac{\text{rad}}{\text{s}}$$

$$\zeta = \frac{c_{eq.}}{2m_{eq.}\omega_n} = \frac{2700\,\frac{\text{N}\cdot\text{s}}{\text{m}}}{2(115\,\text{kg})\left(93.25\,\frac{\text{rad}}{\text{s}}\right)} = 0.126$$

$$r = \frac{\omega}{\omega_n} = \frac{100\,\frac{\text{rad}}{\text{s}}}{93.25\,\frac{\text{rad}}{\text{s}}} = 1.072$$

The magnification factor is

$$M(1.072, 0.126) = \frac{1}{\sqrt{\left(\left[1-(1.072)^2\right]\right)^2 + \left[2(0.126)(1.072)\right]^2}} = 3.24$$

The steady–state amplitude is calculated as

$$X = \frac{M(1.072, 0.126)F_{0_{eq.}}}{m_{eq.}\omega_n^2} = \frac{3.24(1000\,\text{N})}{(115\,\text{kg})\left(93.25\,\frac{\text{rad}}{\text{s}}\right)^2} = 3.24\,\text{mm}$$

Problem 4.16 illustrates application of the magnification factor to determine the steady-state amplitude of forced vibration when an equivalent system is used to model the original one-degree-of-freedom system.

4.17 If ω = 16.5 rad/s, what is the maximum value of M_0 such that the disk of Figure P4.17 rolls without slip?

Given: m = 20 kg, k = 4000 N/m,
c = 50 N · sec/m, r_D = 10 cm, ω = 16.5 rad/sec,
μ = 0.12

FIGURE P4.17

Find: M_0 such that disk rolls without slip

Solution: Let x be the displacement of the center of the disk, measured from equilibrium. Assume the disk rolls without slip. Free body diagrams of the disk at an arbitrary instant of time are shown below

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Summing moments about the point of contact

$$\left(\sum M_c\right)_{ext.} = \left(\sum M_c\right)_{eff.}$$

$$-kxr_D - c\dot{x}r_D + M_0 \sin\omega t = m\ddot{x}r_D + \frac{1}{2}mr_D^2\left(\frac{\ddot{x}}{r_D}\right)$$

$$\frac{3}{2}m\ddot{x} + c\dot{x} + kx = \frac{M_0}{r_D}\sin\omega t$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{2M_0}{3r_D}\sin\omega t$$

where

$$\omega_n = \sqrt{\frac{2k}{3m}} = \sqrt{\frac{2\left(4000\frac{\text{N}}{\text{m}}\right)}{3(20\,\text{kg})}} = 11.55\frac{\text{rad}}{\text{s}}$$

$$\zeta = \frac{2c}{3m\omega_n} = \frac{2\left(50\frac{\text{N}\cdot\text{s}}{\text{m}}\right)}{3(20\,\text{kg})\left(11.55\frac{\text{rad}}{\text{s}}\right)} = 0.144$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{16.5\frac{\text{rad}}{\text{s}}}{11.5\frac{\text{rad}}{\text{s}}} = 1.429$$

The steady-state response is given by

$$x(t) = X\sin(\omega t - \phi)$$

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where

$$X = \frac{\frac{M_0}{r_D} M(1.429, 0.144)}{\left(\frac{3m}{2}\right)\omega_n^2}$$

$$= \frac{2M_0}{3(20\,\text{kg})\left(11.55\frac{\text{rad}}{\text{sec}}\right)^2 (0.1\,\text{m})\sqrt{\left[1-(1.429)^2\right]^2 + \left[2(0.144)(1.429)\right]^2}}$$

$$= 2.23\times10^{-3} M_0$$

Using the free body diagrams to sum moments about the mass center gives

$$\left(\sum M_G\right)_{ext.} = \left(\sum M_G\right)_{eff.}$$

$$M_0 \sin\omega t + F r_D = m\ddot{x} r_D$$

$$F = m\ddot{x} - \frac{M_0}{r_D}\sin\omega t$$

$$F = -m\omega^2\left(2.23\times10^{-3}\,\text{M}_0\right)\sin(\omega t - \phi) - M_0 \sin\omega t$$

$$= -12.14 M_0 \sin(\omega t - \phi) - 10 M_0 \sin\omega t$$

$$F = 21.75 M_0 \sin(\omega t - k)$$

where k is a phase angle whose value is of no consequence.

If the disk rolls without slip, the friction force must be less than the maximum μmg. Thus

$$21.75 M_0 < \mu m g = 0.12(20\,\text{kg})\left(9.81\frac{\text{m}}{\text{s}^2}\right) = 23.54\ \text{N}$$

$$M_0 < 1.08\ \text{N}\cdot\text{m}$$

Problem 4.17 illustrates (a) application of Newton's Laws to free body diagrams to derive a governing differential equation, (b) the steady-state response of a one-degree-of-freedom system with viscous damping, (c) the no-slip condition.

4.18 If $M_0 = 2$ N · m, for what values of ω will the disk of Figure P4.17 roll without slip?

Given: m = 20 kg, k = 4000 N/m,
c = 50 N·sec/m, r_D = 10 cm, μ = 0.12 ,
$M_0 = 2$ N · m

FIGURE P4.17

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Find: ω such that disk rolls without slip

Solution: Let x be the displacement of the center of the disk, measured from equilibrium. Assume the disk rolls without slip. Free body diagrams of the disk at an arbitrary instant of time are shown below

Summing moments about the point of contact

$$\left(\sum M_c\right)_{ext.} = \left(\sum M_c\right)_{eff.}$$

$$-kxr_D - c\dot{x}r_D + M_0 \sin\omega t = m\ddot{x}r_D + \frac{1}{2}mr_D^2\left(\frac{\ddot{x}}{r_D}\right)$$

$$\frac{3}{2}m\ddot{x} + c\dot{x} + kx = \frac{M_0}{r_D}\sin\omega t$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{2M_0}{3r_D}\sin\omega t$$

where

$$\omega_n = \sqrt{\frac{2k}{3m}} = \sqrt{\frac{2\left(4000\frac{\text{N}}{\text{m}}\right)}{3(20\,\text{kg})}} = 11.55\frac{\text{rad}}{\text{s}}$$

$$\zeta = \frac{2c}{3m\omega_n} = \frac{2\left(50\frac{\text{N}\cdot\text{s}}{\text{m}}\right)}{3(20\,\text{kg})\left(11.55\frac{\text{rad}}{\text{s}}\right)} = 0.144$$

The steady-state response is given by

$$x(t) = X\sin(\omega t - \phi)$$

Using the free body diagrams to sum moments about the mass center gives

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$$\left(\sum M_G\right)_{ext.} = \left(\sum M_G\right)_{eff.}$$

$$M_0 \sin \omega t + F r_D = m\ddot{x} r_D$$

$$F = m\ddot{x} - \frac{M_0}{r_D} \sin \omega t$$

$$F = -m\omega^2 (X) \sin(\omega t - \phi) - \frac{M}{r_d} \sin \omega t$$

$$= -m\omega^2 X (\sin \omega t \cos \phi - \cos \omega t \sin \phi) - \frac{M}{r_D} \sin \omega t$$

$$= F \sin(\omega t - \kappa)$$

where κ is a phase angle whose value is of no consequence and

$$F = \sqrt{\left(m\omega^2 X \cos \phi + \frac{M}{r_d}\right)^2 + (m\omega^2 X \sin \phi)^2}$$

$$= \frac{M}{r_D} \sqrt{(r^2 M(r, 0.144) \cos \phi + 1)^2 + (r^2 M(r, 0.144) \sin \phi)^2}$$

$$= \frac{M}{r_D} \sqrt{r^4 M^2(r, 0.144) + 2r^2 M(r, 0.144) \cos \phi + 1}$$

If the disk rolls without slip, the friction force must be less than the maximum μmg. Thus

$$F < \mu m g$$

This is a trial and error equation to find r. Substituting given values and squaring

$$r^4 M^2(r, 0.144) + 2r^2 M(r, 0.144) \cos \phi + 1 < 1.3875$$

or

$$f(r) = r^4 M^2(r, 0.144) + 2r^2 M(r, 0.144) \cos \phi - 0.3875$$

The function above is plotted using MATLAB. The values or r where the plot is greater than zero yields the prohibited values of r.

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Problem 4.18 illustrates the use of the magnification factor.

4.19 For what values of d will the steady–state amplitude of angular oscillations be less than 1° for the rod of Figure P4.19?

FIGURE P4.19

Given: m = 20 kg, c = 100 N · s/m, a = 2/3 m, b = 4/3 m, F_0 = 1000 N, ω = 50 rad/sec, k = 4 × 10^4 N/m, $\Theta_{max.}$ = 1°

Find: d

Solution: Let θ be the angular displacement of the bar, measured positive clockwise with respect to the system's equilibrium position. The equivalent system method is used to derive the governing differential equation using θ as the generalized coordinate.

The kinetic energy of the system is

$$T = \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 + \frac{1}{2}m\left[\left(\frac{L}{2} - a\right)\dot{\theta}\right]^2$$

Thus the equivalent moment of inertia is

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$$I_{eq.} = \frac{1}{12}mL^2 + m\left(\frac{L}{2} - a\right)^2 = 8.90\,\text{kg}\cdot\text{m}^2$$

The potential energy of the system is

$$V = \frac{1}{2}k(d\theta)^2$$

Hence the equivalent torsional stiffness is

$$k_{t_{eq.}} = kd^2 = 4\times10^4 d^2$$

The work done by the damping force is

$$W = -\int c(b\dot{\theta})d(b\theta) = -\int cb^2\dot{\theta}d\theta$$

Hence the equivalent torsional viscous damping coefficient is

$$c_{t_{eq.}} = cb^2 = 177.8\frac{\text{N}\cdot\text{s}\cdot\text{m}}{\text{rad}}$$

The work done by the external force as the bar rotates through a virtual displacement $\delta\theta$ is

$$\delta W = aF(t)\delta\theta$$

Hence the generalized force is

$$\widetilde{F} = aF(t) = 667\sin 50t\,N$$

Since the equivalent torsional stiffness is in terms of d, the system properties can only be determined in terms of d

$$\omega_n = \sqrt{\frac{k_{t_{eq.}}}{I_{eq.}}} = \sqrt{\frac{4\times10^4 d^2}{8.9\,\text{kg}\cdot\text{m}^2}} = 67.04d$$

$$\zeta = \frac{c_{t_{eq.}}}{2I_{eq.}\omega_n} = \frac{177.8\frac{\text{N}\cdot\text{m}\cdot\text{s}}{\text{rad}}}{2(8.9\,\text{kg}\cdot\text{m}^2)(67.04d)} = \frac{0.149}{d}$$

$$r = \frac{\omega}{\omega_n} = \frac{50\frac{\text{rad}}{\text{s}}}{67.04d} = \frac{0.746}{d}$$

In order for the steady-state amplitude to be less than 1°

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$$\frac{I_{eq.}\omega_n^2\Theta_{\max.}}{\widetilde{F}_0} = M\left(\frac{0.746}{d}, \frac{0.149}{d}\right)$$

Substituting calculated values leads to

$$\frac{(8.9\,\text{kg}\cdot\text{m}^2)(67.04\text{d})^2(1^\circ)\left(\frac{2\pi\,\text{rad}}{360^\circ}\right)}{667\,\text{N}\cdot\text{m}}$$

$$> \frac{1}{\sqrt{\left[1-\left(\frac{0.746}{d}\right)^2\right]^2 + \left[2\left(\frac{0.149}{d}\right)\left(\frac{0.746}{d}\right)\right]^2}}$$

which simplifies to

$$1.047 > \frac{1}{\sqrt{\left[\left(d^2 - 0.557\right)^2 + 0.0494\right]}}$$

The appropriate solution of the above equation is

$$d > 1.22\,\text{m}$$

Hence,

$$1.22\,\text{m} < d < 1.33\,\text{m}$$

Problem 4.19 illustrates (a) derivation of differential equations for forced vibrations of a one-degree-of-freedom system, (b) calculation of system properties, and (c) relation between the steady-state amplitude and the magnification factor.

4.20 A 30-kg compressor is mounted on an isolator pad of stiffness 6×10^5 N/m. When subject to a harmonic excitation of magnitude 350 N and frequency 100 rad/sec, the phase difference between the excitation and the steady–state response is 24.3º. What is the damping ratio of the isolator and its maximum deflection due to this excitation?

Given: m = 30 kg, k = 6×10^5 N/m, F_0 = 350 N, ω = 100 rad/sec, ϕ = 24.3º

Find: ζ, X

Solution: The system's natural frequency and frequency ratio are

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$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6\times10^5\,\frac{\text{N}}{\text{m}}}{30\,\text{kg}}} = 141.4\,\frac{\text{rad}}{\text{sec}}$$

$$r = \frac{\omega}{\omega_n} = \frac{100\,\frac{\text{rad}}{\text{sec}}}{141.1\,\frac{\text{rad}}{\text{sec}}} = 0.707$$

The damping ratio is calculated from the phase angle

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

$$\zeta = \frac{1-r^2}{2r}\tan\phi$$

$$\zeta = \frac{1-(.707)^2}{2(.707)}\tan\left(24.3^\circ\right) = 0.160$$

The magnification factor is

$$M(0.707, 0.160) = \frac{1}{\sqrt{\left[1-(0.707)^2\right]^2 + \left[2(0.160)(0.707)\right]^2}} = 1.822$$

The amplitude is calculated using the magnification factor

$$\frac{m\omega_n^2 X}{F_0} = M(0.707, 0.160)$$

$$X = \frac{F_0 M(0.707, 0.160)}{m\omega_n^2}$$

$$= \frac{(350\,\text{N})(1.822)}{(30\,\text{kg})\left(141.4\,\frac{\text{rad}}{\text{sec}}\right)^2} = 1.06\,\text{mm}$$

Problem 4.20 illustrates (a) use of the phase angle to determine damping ratio, and (b) relation between steady-state amplitude and magnification factor.

4.21 A thin disk with a mass of 5 kg and a radius 10 cm is connected to a torsional damper of coefficient 4.1 N·s·m/rad and a solid circular shaft with a radius 10 mm, length 40 cm, and shear modulus 80×10^9 N/m^2. The disk is subject to a harmonic moment of magnitude 250 N·m and frequency 600 Hz. What is the amplitude of the steady–state torsional oscillations?

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Given: $m_D = 5$ kg, $r_D = 10$cm, $c_t = 4.1$ N·s·m/rad, $r_s = 10$ mm, $L_s = 40$ cm,
$G = 80 \times 10^9$ N/m^2, $M_0 = 250$ N-m, $\omega = 600$ Hz.

Find: Θ

Solution: The mass moment of inertia of the disk is

$$I_D = \frac{1}{2} m_D r_D^2 = \frac{1}{2}(5\,\text{kg})(0.1\,\text{m})^2 = 0.025\,\text{kg}\cdot\text{m}^2$$

The torsional stiffness of the shaft is

$$k_t = \frac{\pi r_s^4 G}{2L_s} = \frac{\pi(0.01\,\text{m})^4\left(80\times10^9\,\frac{\text{N}}{\text{m}^2}\right)}{2(0.4\,\text{m})} = 3140\,\frac{\text{N}\cdot\text{m}}{\text{rad}}$$

The system's parameters are

$$\omega_n = \sqrt{\frac{k_t}{I_D}} = \sqrt{\frac{3140\,\frac{\text{N}\cdot\text{m}}{\text{rad}}}{0.025\,\text{kg}\cdot\text{m}^2}} = 354.4\,\frac{\text{rad}}{\text{s}}$$

$$\zeta = \frac{c_t}{2I_D\omega_n} = \frac{4.1\,\frac{\text{N}\cdot\text{s}\cdot\text{m}}{\text{rad}}}{2(0.025\,\text{kg}\cdot\text{m}^2)\left(354.4\,\frac{\text{rad}}{\text{s}}\right)} = 0.231$$

$$r = \frac{\omega}{\omega_n} = \frac{\left(600\,\frac{\text{cycles}}{\text{sec}}\right)\left(\frac{2\pi\,\text{rad}}{1\,\text{cycle}}\right)}{354.4\,\frac{\text{rad}}{\text{s}}} = 10.64$$

The magnification factor is

$$M(10.64, 0.231) = \frac{1}{\sqrt{\left[1-(10.64)^2\right]^2 + \left[2(0.231)(10.64)\right]^2}} = 0.0089$$

The steady-state amplitude is calculated from

$$\Theta = \frac{M_0 M(10.64, 0.231)}{I_D\omega_n^2} = 0.00071\,\text{rad}$$

Problem 4.21 illustrates the relation between magnification factor and steady-state amplitude for a torsional system.

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4.22 A 50-kg machine tool is mounted on an elastic foundation. An experiment is run to determine the stiffness and damping properties of the foundation. When the tool is excited with a harmonic force of magnitude 8000 N at a variety of frequencies, the maximum steady–state amplitude obtained is 2.5 mm, occurring at a frequency of 32 Hz. Use this information to determine the stiffness and damping ratio of the foundation.

Given: m = 50kg, F_0 =8000 N, X_{max} =2.5 mm, ω_m = 32 Hz

Find**:** k, ζ

Solution: The maximum magnification factor is

$$M_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{m\omega_n^2 X_{\max}}{F_0}$$

$$\frac{(50\text{ kg})(0.0025\text{ m})\omega_n^2}{8000\text{ N}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$1.56\times10^{-5}\omega_n^2 = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

The frequency ratio at which the maximum displacement occurs is

$$r_{\max} = \sqrt{1-2\zeta^2} = \frac{\omega_m}{\omega_n}$$

$$\frac{(32\text{ cycles/s})(2\pi\text{ rad/cycle})}{\omega_n} = \sqrt{1-2\zeta^2}$$

$$\omega_n = \frac{201.1}{\sqrt{1-2\zeta^2}}$$

Eliminating the natural frequency between the two equations

$$1.56\times10^{-5}\left(\frac{201.1}{\sqrt{1-2\zeta^2}}\right)^2 = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\frac{0.631}{1-2\zeta^2} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Algebraic manipulation leads to

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$$(0.631)^2\left(4\zeta^2\right)(1-\zeta^2)=(1-2\zeta^2)^2$$
$$1.593\zeta^2-1.593\zeta^4=1-4\zeta^2+4\zeta^4$$
$$5.593\zeta^4-5.593\zeta^2+1=0$$

The quadratic formula is used to obtain

$$\zeta^2=\frac{1}{2(5.593)}\left[5.593\pm\sqrt{(5.593)^2-4(5.593)}\right]$$
$$\zeta^2=0.233,0.767$$
$$\zeta=0.483,0.876$$

Since a maximum occurs only for $\zeta < 0.707$ the appropriate damping ratio is $\zeta = 0.483$. The natural frequency is obtained as

$$\omega_n=\frac{201.1}{\sqrt{1-2\zeta^2}}=275.3\text{ rad/s}$$

And the system stiffness is

$$k=m\omega_n^2=3.79\times10^6\text{ N/m}$$

Problem 4.22 illustrates the maximum steady-state amplitude over a range of frequencies for a system with viscous damping.

4.23 A machine of mass 30 kg is placed on an elastic mounting of unknown properties. An engineer excites the machine with a harmonic force of magnitude 100 N at a frequency of 30 Hz. He measures the steady–state response as having an amplitude of 0.2 mm with a phase lag of 20°. Determine the stiffness and damping coefficient of the mounting.

Given: m = 30 kg, $f = 30$ Hz, $F_0 = 100$ N, $X = 0.2$ mm, $\phi = 20°$

Find: k, c

Solution: The amplitude is given by

$$X=0.0002\,m=\frac{F_0}{m\omega_n^2}M(r,\zeta)=\frac{r^2F_0}{m\omega^2}M(r,\zeta)$$
$$=\frac{(100\text{ N})}{(30\text{ kg})\left[\left(30\ \frac{\text{cycle}}{\text{s}}\right)\left(2\pi\ \frac{\text{rad}}{\text{cycle}}\right)\right]^2}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$
$$=9.382\text{x}10^{-5}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

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and the phase is given by

$$\phi = 20° = \tan^{-1}\frac{2\zeta r}{1-r^2} \Rightarrow \frac{2\zeta r}{1-r^2} = 0.364$$

The phase equation implies $2\zeta r = 0.364(1-r^2)$. Substitution into the amplitude equation leads to

$$0.0002 \text{ m} = 9.382 \times 10^{-5}\frac{1-\lambda}{\sqrt{\lambda^2 + 1.125\lambda^2}} = 8.818 \times 10^{-5}\frac{1-\lambda}{\lambda}$$

where $\lambda = 1 - r^2$. The equation is solved leading to $\lambda = 0.306 \Rightarrow r = 0.833$. Then $\zeta = 0.184$ .Since $r = \frac{\omega}{\omega_n}$, $\omega_n = \frac{(30)(2\pi)}{0.833} = 226.3 \text{ rad/s}$ and $k = m\omega_n^2 = (30)(226.3)^2 = 1.54 \times 10^6$ N/m. The damping ratio is $\zeta = \frac{c}{2m\omega_n} \Rightarrow c = \zeta(2m\omega_n) = 0.184(2)(30)(226.3) = 2.50 \times 10^3 \text{ N} \cdot \text{s/m}$.

Problem 4.23 illustrates the use of the phase and amplitude in calculating system properties.

4.24 A 80-kg machine tool is placed on an elastic mounting. The phase angle is measured as 35.5° when the machine is excited at 30 Hz. When the machine is excited at 60 Hz, the phase angle is 113°. Determine the equivalent damping coefficient and equivalent stiffness of the mounting.

Given: m = 80 kg, for f = 30 Hz. $\phi = 35.5°$, for f = 60 Hz. $\phi = 113°$

Find: c, k

Solution: The phase angle is

$$\phi = \tan^{-1}\frac{2\zeta r}{1-r^2}$$

The frequency ratio r varies with frequency but the damping ratio r is independent of frequency. For f = 30 Hz

$$\frac{2\zeta r}{1-r^2} = 0.713$$

For f = 60 Hz, r = 2r and

$$\frac{4\zeta r}{1-4r^2} = -2.36$$

Dividing the second equation by the first equation leads to

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$$\frac{1-4r^2}{2(1-r^2)} = -3.30$$

Solving for r yields r = 0.847. The using the first equation gives $\zeta = 0.119$. Noting that for f=30 Hz, $\omega_n = \frac{\omega}{r} = 222.5$ rad/s and $k = m\omega_n^2 = (30)(222.5)^2 = 1.49 \times 10^6$N/m. Also, $c = \zeta(2m\omega_n) = 0.119(2)(30)(222.5) = 1.59 \times 10^3\ \mathrm{N \cdot s/m}$.

Problem 4.24 illustrates the use of the phase angle in determining system parameters.

4.25 A 100-kg machine tool has a 2-kg rotating component. When the machine is mounted on an isolator and its operating speed is very large, the steady–state vibration amplitude is 0.7 mm. How far is the center of mass of the rotating component from its axis of rotation?

Given: m = 100 kg, m_0 = 2 kg, X (large r) = 0.7 mm

Find: e

Solution: When the frequency ratio is very large Λ is approximately 1 for all values of ζ. Thus from the information given

$$\Lambda = \frac{mX}{m_0 e}$$

$$1 = \frac{(100\,\mathrm{kg})(0.0007\,\mathrm{mm})}{(2\,\mathrm{kg})\mathrm{e}}$$

$$e = 0.035\,\mathrm{m}$$

Problem 4.25 illustrates the asymptotic limit of Λ.

4.26 A 1000 kg turbine with a rotating unbalance is placed on springs and viscous dampers in parallel. When the operating speed is 20 Hz, the observed steady–state amplitude is 0.08 mm. As the operating speed is increased, the steady–state amplitude increases with an amplitude of 0.25 mm at 40 Hz and an amplitude of 0.5 mm for much larger speeds. Determine the equivalent stiffness and damping coefficient of the system.

Given: m = 1000 kg, X(ω = 20 Hz.) = 0.08 mm, X(ω = 40Hz.) = 0.25 mm, X(large ω) = 0.5 mm

Find: $k_{eq.}$, $c_{eq.}$

Solution: Λ→1 for large r. Thus,

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$$\Lambda = \frac{mX}{m_0 e} \to 1 \text{ as } r \to \infty$$

$$\frac{(1000\,\text{kg})(0.5\,\text{mm})}{\text{m}_0\text{e}} = 1$$

$$m_0 e = 500\,\text{kg}\cdot\text{mm}$$

Let

$$r_1 = \frac{20\,\text{Hz.}}{\omega_n}$$

$$r_2 = \frac{40\,\text{Hz.}}{\omega_n} = 2r_1$$

Then

$$\Lambda(r_1,\zeta) = \frac{(1000\,\text{kg})(0.08\,\text{mm})}{500\,\text{kg}\cdot\text{mm}} = \frac{r_1^2}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} \tag{1}$$

and

$$\Lambda(r_2,\zeta) = \frac{(1000\,\text{kg})(0.25\,\text{mm})}{500\,\text{kg}\cdot\text{mm}} = \frac{r_2^2}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} \tag{2}$$

$$0.5 = \frac{4r_1^2}{\sqrt{(1-4r_1^2)^2 + (4\zeta r_1)^2}}$$

Solving for ζ in terms of r_1 from eq. (2) leads to

$$\zeta^2 = \frac{1}{16r_1^2}\left(48r_1^4 + 8r_1^2 - 1\right) \tag{3}$$

Substituting eq. (3) in eq. (1) and rearranging leads to

$$26.06r_1^4 = 0.75$$

whose solution is

$$r_1 = 0.4118$$

The system's natural frequency is calculated as

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$$\omega_n = \frac{20\,\text{Hz.}}{r_1} = \frac{40\pi\,\frac{\text{rad}}{\text{s}}}{0.4118} = 305.0\,\frac{\text{rad}}{\text{s}}$$

The equivalent stiffness is

$$k_{eq.} = m\omega_n^2 = (1000\,\text{kg})\left(305.0\,\frac{\text{rad}}{\text{s}}\right)^2 = 9.31\times10^7\,\frac{\text{N}}{\text{m}}$$

The damping ratio is calculated using eq. (3) as

$$\zeta = 0.800$$

from which the equivalent viscous damping coefficient is calculated as

$$c_{eq.} = 2\zeta m\omega_n = 2(0.800)(1000\,\text{kg})\left(305.0\,\frac{\text{rad}}{\text{s}}\right) = 4.88\times10^5\,\frac{\text{N}\cdot\text{s}}{\text{m}}$$

Problem 4.26 illustrates (a) the limit of $\Lambda(r,\zeta)$ for large r, (b) the use of Λ in calculations.

4.27 A 120-kg fan with a rotating unbalance of 0.35 kg · m is to be placed at the midspan of a 2.6-m simply supported beam. The beam is made of steel ($E = 210 \times 10^9$ N/m^2) with a uniform rectangular cross section of height of 5 cm. For what values of the cross-sectional depth will the steady–state amplitude of the machine be limited to 5 mm for all operating speeds between 50 and 125 rad/sec?

Given: m = 120 kg, m_0e = 0.35 kg-m, L = 2.6 m, $E = 210 \times 10^9$ N/m^2, h = 5 cm, X_{max} = 5 mm,

50 rad/s $\leq \omega \leq$ 125 rad/s

Find: appropriate values of d

Solution: The midspan deflection of a simply supported beam due to a concentrated unit load at its midspan is obtained using Table D.2 of Appendix D. This table is used with x = a = L/2,

$$\Delta = \frac{1}{EI}\left[-\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(\frac{L}{2}\right)^3 + \frac{3L^2}{48}\left(\frac{L}{2}\right)\right] = \frac{L^3}{48EI} \qquad (1)$$

Let x be the displacement of the machine from its equilibrium position. The vibrations of the machine are modeled using one degree of freedom using x as the generalized

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coordinate. Inertia effects of the beam are ignored. The equivalent stiffness is obtained from eq. (1) as

$$k_{eq.} = \frac{1}{\Delta} = \frac{48EI}{L^3} \tag{2}$$

The moment of inertia of the cross section is expressed as

$$I = \frac{1}{12} dh^3 \tag{3}$$

Substitution of eq. (3) into eq. (2) leads to

$$k_{eq.} = \frac{4Edh^3}{L^3} \tag{4}$$

The system's natural frequency is given by

$$\omega_n = \sqrt{\frac{k_{eq.}}{m}} = \sqrt{\frac{4Eh^3 d}{mL^3}} = \sqrt{\frac{4\left(210\times10^9 \frac{\mathrm{N}}{\mathrm{m}^2}\right)(.05\,\mathrm{m})^3 d}{(120\,\mathrm{kg})(2.6\,\mathrm{m})^3}} = 223.1\sqrt{d} \tag{5}$$

The machine's rotating unbalance causes a harmonic excitation whose amplitude is proportional to the square of its frequency. From the given information

$$\Lambda_{\max.} = \frac{mX_{\max.}}{m_0 e} = \frac{(120\,\mathrm{kg})(.005\,\mathrm{m})}{0.35\,\mathrm{kg}\cdot\mathrm{m}} = 1.714 \tag{6}$$

For an undamped system

$$\Lambda = \frac{r^2}{|1-r^2|} \tag{7}$$

Requiring $\Lambda < \Lambda_{max}$ when $r < 1$ leads to

$$r < \sqrt{\frac{\Lambda_{\max.}}{\Lambda_{\max.}+1}} = \sqrt{\frac{1.714}{2.714}} = 0.795$$

or

$$\frac{\omega}{\omega_n} < 0.795$$
$$\omega_n > \frac{\omega}{0.795} \tag{8}$$

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In order for eq. (8) to hold over the entire operating range,

$$\omega_n > \frac{125\,\frac{\text{rad}}{\text{s}}}{0.795} = 157.23\,\frac{\text{rad}}{\text{s}} \tag{9}$$

Using eq. (5) in eq. (9) leads to

$$\begin{gathered} 223.1\sqrt{d} > 157.23\,\frac{\text{rad}}{\text{s}} \\ \text{d} > 0.496\,\text{m} \end{gathered} \tag{10}$$

Referring to the adjacent graph, a second solution is obtained by requiring $\Lambda < \Lambda_{max}$ when r > 1. From eq.(7), this leads to

$$r > \sqrt{\frac{\Lambda}{\Lambda - 1}} = \sqrt{\frac{1.714}{0.714}} = 1.549 \tag{11}$$

or

$$\omega_n < \frac{\omega}{1.549} \tag{12}$$

Requiring $\Lambda < \Lambda_{max}$ over the entire operating range in eg.(12) leads to

$$\omega_n < 32.28\,\frac{\text{rad}}{\text{s}} \tag{13}$$

Using eq.(5) in eq.(13) leads to

$$\begin{gathered} 223.1\sqrt{d} < 32.28\,\frac{\text{rad}}{\text{s}} \\ d < 0.0209\,\text{m} \end{gathered} \tag{14}$$

Thus the acceptable values of d are

$$d < 20.9\text{mm} \;\; or \;\; d > 496.\text{mm}$$

Problem 4.27 illustrates the theory of rotating unbalance for an undamped system. It also illustrates the modeling of a mass attached to a beam using one degree of freedom.

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4.28 Solve Chapter Problem 4.27 assuming the damping ratio of the beam is 0.04.

Given: m = 120 kg, m_0e = 0.35 kg-m, L = 2.6 m, E = 210 × 10^9 N/m^2, h = 5 cm, ζ = 0.04, X_{max} = 5 mm, 50 rad/s < ω < 125 rad/s

Find: d

Solution: The maximum allowable value of Λ is obtained from

$$\Lambda_{\max} = \frac{mX_{\max}}{m_0 e} = \frac{(120\text{ kg})(0.005\text{ m})}{0.35\text{ kg}\cdot\text{m}} = 1.714$$

It is necessary to find the values of r for which Λ(r,0.04) < 1.714. To this end

$$1.714 > \frac{r^2}{\sqrt{(1-r^2)^2 + [2(0.04)r]^2}}$$

Squaring and rearranging leads to

$$1.9378r^4 - 5.857r^2 + 2.9378 = 0$$

The quadratic formula is used to solve for r^2 leading to r < 0.824 or r > 1.496. In order for r < 0.824 over the entire frequency range r = 0.824 should correspond to the highest frequency in the range, ω = 125 rad/s. To this end

$$\omega_n > \frac{\omega}{r} = \frac{125\text{ rad/s}}{0.824} = 151.7\text{ rad/s}$$

This leads to

$$k > (120\text{ kg})(151.7\text{ rad/s})^2 = 2.76\times10^6\text{ N/m}$$

For a simply supported beam

$$k = \frac{48EI}{L^3}$$

leading to

$$I > \frac{(2.76\times10^6)L^3}{48E} = 4.81\times10^{-6}\text{ m}^4$$

The moment of inertia of a rectangular cross section is

$$I = \frac{1}{12}dh^3$$

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which leads to d > 0.642 m.

If r > 1.496 over the entire frequency range, then r = 1.496 must correspond to the lowest frequency in the range, ω = 50 rad/s. Following the same procedure as above this leads to d < 0.0224 m.

Problem 4.28 illustrates the use of Λ(r,ζ) for machines with a rotating unbalance.

4.29 A 620-kg fan has a rotating unbalance of 0.25 kg·m. What is the maximum stiffness of the fan's mounting such that the steady–state amplitude is 0.5 mm or less at all operating speeds greater than 100 Hz? Assume a damping ratio of 0.08.

Given: m = 620 kg, m_0e = 0.25 kg-m, ζ = 0.08, $X_{max.}$ = 0.5 mm, $\omega_{min.}$ = 100 Hz.

Find: k

Solution: From the curve for Λ, for a fixed ζ, it is obvious that the steady–state amplitude is lower at higher operating speeds. Thus, if X_{max} = 0.5 mm for ω = 100 Hz, then X < 0.5 mm for all ω > 100 Hz. Using this information, at 100 Hz.

$$\Lambda = \frac{mX}{M_0 e} = \frac{(620\,\text{kg})(0.0005\,\text{m})}{0.25\,\text{kg}\cdot\text{m}} = 1.24$$

It is desired to find the frequency ratio corresponding to Λ = 1.24,

$$1.24 = \frac{r^2}{\sqrt{\left(1-r^2\right)^2 + \left(0.16\,r\right)^2}}$$

The greater solution of the above equation is

$$r = 2.26$$

Thus

$$\omega_n < \frac{\omega}{r} = \frac{\left(100\dfrac{\text{cycles}}{\text{sec}}\right)\left(\dfrac{2\pi\,\text{rad}}{1\,\text{cycle}}\right)}{2.26}$$

$$\omega_n < 278.0\frac{\text{rad}}{\text{s}}$$

$$k = m\omega_n^2 < (620\,\text{kg})\left(278.0\frac{\text{rad}}{\text{s}}\right)^2 = 4.8\times10^7\,\frac{\text{N}}{\text{m}}$$

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Problem 4.29 illustrates the application of Λ to rotating unbalance problems.

4.30 The tail rotor section of a helicopter consists of Figure P4.30 consists of four blades, each of mass 2.1 kg, and an engine box of mass 25 kg. The center of gravity of each blade is 170 mm from the rotational axis. The tail section is connected to the main body by an elastic structure. The natural frequency of the tail section has been observed as 150 rad/s. During flight the rotor operates at 900 rpm. Assume the system has a damping ratio of 0.05. During flight a 75-g particle becomes stuck to one of the blades, 25 cm from the axis of rotation. What is the steady–state amplitude of vibration caused by the resulting rotating unbalance?

FIGURE P4.30

Given: $m_b = 2.1$ kg, $m_e = 25$ kg, x = 170 mm, $\omega_n = 150$ rad/s, $\omega = 900$ rpm, $\zeta = 0.05$, $m_p = 75$ g, e = 25 cm

Find**:** X

Solution: When the particle is attached to a blade the total mass of the rotor is

$$m = 4m_b + m_e + m_p = 33.38 \text{ kg}$$

When the particle is attached to the blade it creates a rotating unbalance of magnitude

$$m_0 e = (0.075 \text{ kg})(0.25 \text{ m}) = 0.0188 \text{ kg} \cdot \text{m}$$

The frequency ratio of the system is

$$r = \frac{\omega}{\omega_n} = \frac{(900 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{150 \text{ rad/s}} = 0.628$$

The steady-state amplitude is calculated as

$$X = \frac{m_0 e}{m}\Lambda(0.628, 0.05)$$

$$X = \frac{0.0188 \text{ kg} \cdot \text{m}}{33.38 \text{ kg}} \frac{(0.628)^2}{\sqrt{\left[1-(0.628)^2\right]^2 + \left[2(0.05)(0.628)\right]^2}} = 0.36 \text{ mm}$$

Problem 4.30 illustrates the determination of the steady-state amplitude for a system with a rotating unbalance.

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4.31 The rotor tail rotor section of a helicopter consists of Figure P4.30 consists of four blades, each of mass 2.1 kg, and an engine box of mass 25 kg. The center of gravity of each blade is 170 mm from the rotational axis. The tail section is connected to the main body by an elastic structure. The natural frequency of the tail section has been observed as 150 rad/s. Determine the steady–state amplitude of vibration if one of the blades in Figure P4.30 snaps off during flight.

FIGURE P4.30

Given: $m_b = 2.1$ kg, $m_e = 25$ kg, $x = 170$ mm, $\omega_n = 150$ rad/s, $\omega = 900$ rpm, $\zeta = 0.05$

Find**:** X

Solution: The total mass of the rotor if one blade falls off is

$$m = 3m_b + m_e = 31.2 \text{ kg}$$

When one blade falls off, the system has a rotating unbalance of magnitude

$$m_0 e = (2.1 \text{ kg})(0.17 \text{ m}) = 0.357 \text{ kg} \cdot \text{m}$$

The equivalent stiffness of the tail section is determined from the natural frequency when all blades are attached

$$k_{eq} = m\omega_n^2 = (33.3 \text{ kg})(150 \text{ rad/s})^2 = 7.49 \times 10^5 \text{ N/m}$$

The natural frequency of the tail section when one blade is missing is

$$\omega_n = \sqrt{\frac{k}{m}} = 155.2 \text{ rad/s}$$

The frequency ratio of the system is

$$r = \frac{\omega}{\omega_n} = \frac{(900 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{155 \text{ rad/s}} = 0.607$$

The steady-state amplitude is calculated as

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$$X = \frac{m_0 e}{m}\Lambda(0.607, 0.05)$$

$$X = \frac{0.357\text{ kg - m}}{31.1\text{ kg}} \frac{(0.607)^2}{\sqrt{\left[1-(0.607)^2\right]^2 + \left[2(0.05)(0.607)\right]^2}} = 6.7\text{ mm}$$

Problem 4.31 illustrates the determination of the steady-state amplitude for a system with a rotating unbalance.

4.32 Whirling is a phenomenon that occurs in a rotating shaft when an attached rotor is unbalanced. The motion of the shaft and the eccentricity of the rotor causes an unbalanced inertia force, pulling the shaft away from its centerline, causing it to bow. Use Figure P4.32 and the theory of Section 4.5 to show that the amplitude of whirling is

$$X = e\Lambda(r, \zeta)$$

where e is the distance from the center of mass of the rotor to the axis of the shaft.

FIGURE P4.32

Given: e

Show: $X = e\Lambda(r, \zeta)$

Solution: The rotor is mounted on bearings of equivalent stiffness k and damping coefficient c. Free body diagrams of the rotor at an arbitrary instant are shown. The rotor is rotating at a constant angular speed ω. Let x(t) denote the distance between the geometric center of the rotor and the axis of the shaft. Using the relative acceleration equation the acceleration of the mass center is equal to the acceleration of the center of the rotor plus the relative acceleration, a term equal to $e\omega^2$ directed from G to C.

Summing forces on the rotor

$$\sum F_{ext} = \sum F_{eff}$$

$$-kx - c\dot{x} = m\ddot{x} + me\omega^2 \sin\theta$$

Since the angular speed is constant

$$\theta = \omega t$$

and the differential equation becomes

$$m\ddot{x} + c\dot{x} + kx = -me\omega^2 \sin\omega t$$

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The whirling leads to a harmonic excitation of the form Fsinωt where F = mω^2. Hence whirling leads to a frequency squared excitation with A = mω^2. Then using the theory of Sec. 3.5 the steady-state response is

$$x(t) = X\sin(\omega t - \phi)$$

where

$$\frac{mX}{me} = \Lambda(r, \zeta)$$

Problem 4.32 illustrates the amplitude of whirling.

4.33 A 30-kg rotor has an eccentricity of 1.2 cm. It is mounted on a shaft and bearing system whose stiffness is 2.8 × 10^4 N/m and damping ratio is 0.07. What is the amplitude of whirling when the rotor operates at 850 rpm?

Given: m = 30 kg, e = 1.2 cm, k = 2.8 × 10^4 N/m, ζ = 0.07, ω = 850 rpm

Find**:** X

Solution: The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = 30.6 \text{ rad/s}$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{(850 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{30.6 \text{ rad/s}} = 2.91$$

Using the results of Problem 4.32 the amplitude of whirling is

$$X = e\Lambda(2.91, 0.07)$$

$$X = (0.012 \text{ m})\frac{(2.91)^2}{\sqrt{\left[1 - (2.91)^2\right]^2 + \left[2(0.07)(2.91)\right]^2}} = 0.0136 \text{ m}$$

Problem 4.33 illustrates the steady-state amplitude due to whirling.

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4.34 An engine flywheel has an eccentricity of 0.8 cm and mass 38 kg. Assuming a damping ratio of 0.05, what is the necessary stiffness of the bearings to limit its whirl amplitude to 0.8 mm at all speeds between 1000 and 2000 rpm? Refer to Chapter Problem 4.32 for an explanation of whirling.

Given: m = 38 kg, e = 0.8 cm, ζ = 0.05, X = 0.8 mm, 1000 rpm < ω < 2000 rpm

Find**:** k

Solution: From Problem 4.32 the amplitude of whirl is

$$X = e\Lambda(r,\zeta)$$

From the information given the maximum allowable value of Λ is

$$\Lambda_{all} = \frac{X_{all}}{e} = \frac{0.0008 \text{ m}}{0.008 \text{ m}} = 0.1$$

It is noted that $\Lambda < 0.1$ only for small values of r. In order to find the appropriate values of the bearing stiffness set

$$0.1 = \Lambda(r0.05) = \frac{r^2}{\sqrt{(1-r^2)^2 + [2(0.05)r]^2}}$$

Squaring and rearranging leads to

$$0.99r^4 + 0.0199r^2 - 0.01 = 0$$

The quadratic formula is used to solve for r^2 leading to

$$r^2 = -0.111, 0.091$$

Only a positive root leads to a real solution r = 0.302. Thus the bearing stiffness must be chosen such that r < 0.302 over the entire range of frequencies. This occurs if r-0.302 corresponds to the highest frequency in the range ω = 2000 rpm = 209.4 rad/s. To this end

$$\frac{209.4}{\omega_n} < 0.302$$

$$\omega_n > 694.5 \text{ rad/s}$$

The stiffness must be chosen such that

$$k > (38 \text{ kg})(694.5 \text{ rad/s})^2 = 1.83 \times 10^7 \text{ N/m}$$

Problem 4.34 illustrates the choice of bearings to limit whirl amplitude.

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4.35 It is proposed to build a 6-m smokestack on the top of a 60-m factory. The smokestack will be made of steel (ρ = 7850 kg/m^3) and will have an inner radius of 40 cm and an outer radius of 45 cm. What is the maximum amplitude of vibration due to vortex shedding and at what wind speed will it occur? Use a SDOF model for the smokestack with a concentrated mass at its end to account for inertia effects. Use $\zeta = 0.05$.

Given: L = 6 m, $\rho_{st.}$= 7850 kg/m^3, r_i = 40 cm, r_o = 45 cm, $\zeta = 0.05$, h = 60 m

Find: $X_{max.}$

Solution: The smokestack is modeled as a cantilever beam with a concentrated mass on its end. The concentrated mass is the equivalent mass of the beam used to account for its inertia effects. The geometric properties of the smokestack are

$$A = \pi[(0.45\text{m})^2 - (0.4\text{m})^2] = 0.1335\ \text{m}^2$$

$$I = \frac{\pi}{4}[(0.45\text{m})^4 - (0.4\text{m})^4] = 0.0121\,\text{m}^4$$

$$m_b = \rho AL = (7850\frac{\text{kg}}{\text{m}^3})(0.1336\ \text{m}^2)(6\text{m}) = 6290\text{kg}$$

Let x be a coordinate along the axis of the smokestack. Let z be the deflection at the end of the smokestack. The deflection of a cantilever beam due to a concentrated load P applied at the end of the beam is

$$y(x) = \frac{Px^2}{6EI}(3L - x) \tag{1}$$

From eq.(1) the deflection at the end is calculated as

$$z = \frac{PL^3}{3EI} \tag{2}$$

Substituting eq.(2) into eq.(1) leads to

$$y(x) = \frac{x^2 z}{2L^3}(3L - x) \tag{3}$$

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Consider a differential element of length dx along the axis of the beam, a distance x from its fixed end. The mass of the differential element is

$$dm = \frac{m_b}{L} dx \qquad (4)$$

where m_b is the total mass of the beam. The kinetic energy of the differential element is

$$dT = \frac{1}{2} \dot{y}^2 \, dm \qquad (5)$$

Substitution of eqs.(3) and (4) in eq.(5) and integrating over the length of the beam leads to

$$\begin{aligned} T = \int dT &= \int_0^L \frac{1}{2} \left[\frac{x^2 \dot{z}}{2L^3} (3L - x) \right]^2 \frac{m_b}{L} dx \\ &= \frac{1}{2} (0.236 \, m_b) \dot{z}^2 \end{aligned} \qquad (6)$$

Hence the equivalent mass is

$$\tilde{m} = 0.236(6290\text{kg}) = 1484\text{kg}$$

The value of Λ corresponding to the maximum amplitude is

$$\Lambda_{\max.} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{2(.05)\sqrt{1-(.05)^2}} = 10.01$$

The corresponding maximum amplitude is calculated using

$$X_{\max.} = \frac{\rho 8 \, r_0^3 L \, \Lambda_{\max.}}{3.16 \tilde{m}}$$

Assuming air at 20° C

$$X_{\max.} = \frac{8(1.204 \frac{\text{kg}}{\text{m}^3})(0.45 \text{ m})^3 (6.0 \text{ m})(10.01)}{3.16(1484 \text{ kg})} = 11.2 \text{ mm}$$

The wind speed at which the maximum amplitude occurs is calculated from

$$r_{max} = \frac{1}{\sqrt{1 - 2\zeta^2}} = 1.002$$

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Thus

$$\omega = 1.002\omega_n = 1.002\sqrt{\frac{k}{m}} = 1.002\sqrt{\frac{3EI}{mL^3}} = 150.79\ \frac{\text{rad}}{\text{s}}$$

Then

$$v = \frac{\omega D}{0.4\pi} = \frac{\left(150.79\ \frac{\text{rad}}{\text{s}}\right)(0.45\ \text{m})}{0.4\pi} = 56.54\ \frac{\text{m}}{\text{s}}$$

Problem 4.35 illustrates modeling of a continuous system using one degree of freedom. It also illustrates application of the Λ theory to determine the maximum response of a circular cylinder sue to vortex shedding.

4.36 What is the steady–state amplitude of oscillation due to vortex shedding of the smokestack of Chapter Problem 4.35 if the wind speed is 22 mph?

Given: L = 6 m, ρ = 7850 kg/m^3, r_i = 40 cm, r_o = 45 cm, ζ = 0.05, v = 22 mph

Find: X

Solution: The smokestack is modeled as a cantilever beam with a concentrated end mass. The concentrated mass is the equivalent mass of the beam used to account for inertia effects. It is shown that the inertia effects of a fixed-free beam are approximated by using an equivalent mass of 0.236 times the mass of the beam. To this end

$$m_{eq} = 0.236m_b = 0.236\rho AL = 0.236\rho\pi(r_o^2 - r_i^2)L = 1484\,\text{kg}$$

The stiffness of the beam is

$$k = \frac{3EI}{L^3} = \frac{3E\pi(r_o^4 - r_i^4)}{4L^3} = 3.36\times10^7\ \text{N/m}$$

The natural frequency of the smokestack is

$$\omega_n = \sqrt{\frac{k}{m}} = 150.5\ \text{rad/s}$$

The frequency of vortex shedding is obtained from

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$$S = 0.2 = \frac{\omega D}{2\pi v}$$

$$\omega = \frac{0.4\pi v}{D}$$

$$\omega = \frac{0.4\pi(22\text{ mi/hr})(1.61\times10^3\text{ m/mi})(1\text{ hr}/3600\text{ s})}{0.9\text{ m}} = 13.74\text{ rad/s}$$

Thus the frequency ratio is

$$r = \frac{\omega}{\omega_n} = 0.0913$$

The steady-state amplitude is calculated from

$$X = \frac{\rho_a D_o^3 L}{0.316m}\Lambda(0.0913, 0.11)$$

$$X = \frac{(1.204\text{ kg/m}^3)(0.9\text{ m})^3(6\text{ m})}{0.316(1484\text{ kg})}\frac{(0.0913)^2}{\sqrt{[1-(0.0913)^2]^2 + [2(0.11)(0.0913)]^2}}$$

$$X = 9.43\times10^{-5}\text{ m}$$

Problem 4.36 illustrates the steady-state amplitude due to vortex shedding.

4.37 A factory is using the piping system of Figure P4.37 to discharge environmentally safe waste-water into a small river. The velocity of the river is estimated as 5.5 m/sec. Determine the allowable values of l such that the amplitude of torsional oscillations of the vertical pipe due to vortex shedding is less than 1°. Assume the vertical pipe is rigid and rotates about an axis perpendicular to the page through the elbow. The horizontal pipe is restrained from rotation at the river bank. Assume a damping ratio of $\zeta = 0.05$.

FIGURE P4.37

Given: $G = 80 \times 10^9$ N/m^2, $\rho = 7800$ kg/m^3, $D_i = 14$ cm, $t = 1$ cm, $v = 5.5$ m/sec., $\zeta = 0.05$, $\Theta < 1°$

Find: ℓ

Solution: Properties of water at 20° C are

$$\rho = 998\frac{\text{kg}}{\text{m}^3} \quad \mu = 1.003\times10^{-3}\frac{\text{N}\cdot\text{s}}{\text{m}^2}$$

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The vortex induced oscillations are modeled using one-degree-of-freedom. Vortex shedding occurs from the vertical pipe, which is free to rotate. The horizontal pipe acts as a torsional spring resisting the rotation of the pipe. The Reynolds number of the flow is

$$\mathrm{Re} = \frac{\rho v D_o}{\mu} = \frac{\left(998\,\frac{\mathrm{kg}}{\mathrm{m}^3}\right)\left(5.5\,\frac{\mathrm{m}}{\mathrm{s}}\right)(0.15\,\mathrm{m})}{1.003\times10^{-3}\,\frac{\mathrm{N}\cdot\mathrm{s}}{\mathrm{m}^2}} = 8.2\times10^5$$

and is approximately in the range where the frequency squared model of vortex induced oscillations is valid.

Free body diagrams of the vertical pipe at an arbitrary instant are shown below.

EXTERNAL FORCES EFFECTIVE FORCES

Summing moments about the axis of rotation

$$\left(\sum M_0\right)_{ext.} = \left(\sum M_0\right)_{eff.}$$

$$-mg\frac{L}{2}\theta - k_t\theta - c_t\dot{\theta} + F_0 h\sin\omega t = m\frac{L}{2}\ddot{\theta}\frac{L}{2} + \bar{I}\ddot{\theta}$$

$$\left(m\frac{L^2}{2} + \bar{I}\right)\ddot{\theta} + c_t\dot{\theta} + \left(k_t + mg\frac{L}{2}\right)\theta = F_0 h\sin\omega t$$

The inertia properties of the pipes are

$$m = \rho\pi\left(r_0^2 - r_i^2\right)L = \pi\left(7800\,\frac{\mathrm{kg}}{\mathrm{m}^3}\right)\left[(0.075\,\mathrm{m})^2 - (0.07\,\mathrm{m})^2\right](4\,\mathrm{m}) = 71.06\,\mathrm{kg}$$

$$\bar{I} = \frac{\rho\pi L}{12}\left[r_0^2\left(3r_0^2 + L^2\right) - r_i^2\left(3r_i^2 + L^2\right)\right] = 94.66\,\mathrm{kg}\cdot\mathrm{m}^2$$

Assuming the amplitude of the excitation is proportional to the square of the frequency and the drag coefficient is approximately 1.0, the magnitude of the exciting moment is

$$M_0 = F_0 h = 0.316\,\rho D^3 L h\,\omega^2$$

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Using the theory for frequency squared excitations,

$$\Lambda = \frac{3.16\left(m\frac{L^2}{4}+\bar{I}\right)\Theta}{\rho D^3 L h}$$

Requiring $\Theta < 1°$ leads to

$$\Lambda < \frac{3.16\left[\frac{1}{4}(284.3\,\text{kg})(4\,\text{m})^2 + 382\,\text{kg}-\text{m}^2\right](1°)\left(\frac{2\pi\,\text{rad}}{360°}\right)}{\left(998\frac{\text{kg}}{\text{m}^3}\right)(0.15\,\text{m})^3(4\,\text{m})(2.5\,\text{m})} = 0.62$$

which, in turn, leads to

$$0.62 > \frac{r^2}{\sqrt{\left(1-r^2\right)^2+\left(0.1r\right)^2}}$$

The solution of the above equation is $r < 0.384$. Note that

$$r = \frac{\omega}{\omega_n}$$

where the shedding frequency is

$$\omega = \frac{0.4\pi v}{D_o} = \frac{0.4\pi\left(5.5\frac{\text{m}}{\text{s}}\right)}{0.15\text{m}} = 46.1\frac{\text{rad}}{\text{s}}$$

The torsional stiffness is

$$k_t = \frac{JG}{\ell}$$

and thus the natural frequency is

$$\omega_n = \sqrt{\frac{\frac{JG}{\ell}+mg\frac{L}{2}}{\left(m\frac{L^2}{4}+\bar{I}\right)}}$$

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$$\sqrt{\frac{\frac{\pi}{2}\left[(0.075\text{m})^4-(0.07\text{m})^4\right]\left(80\times10^9\,\frac{\text{N}}{\text{m}^2}\right)}{\ell}+(71.06\text{kg})\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)\frac{4\text{m}}{4}}{1520\,\text{kg}\cdot\text{m}^2}$$

Then since r < 0.384

$$\omega_n > \frac{46.1\,\frac{\text{rad}}{\text{s}}}{0.384} = 120.05\,\frac{\text{rad}}{\text{s}}$$
$$\ell < 0.0436\,\text{m}$$

Problem 4.37 illustrates (a) torsional oscillations of a submerged vertical pipe induced by vortex shedding, (b) calculation of steady-state amplitude induced by vortex shedding, and (c) design calculations to avoid large oscillations.

4.38 Determine the amplitude of steady–state vibration for the system shown in Figure P4.38. Use the indicated generalized coordinate.

FIGURE P4.38

Given: $k_1 = 3 \times 10^4$ N/m, $k_2 = 1.5 \times 10^4$ N/m, m = 2.8 kg,
c = 100 N·s/m, Y = 0.02 m, ω = 100 rad/s

Find**:** X

Solution: Free-body diagrams of the block drawn for an arbitrary instant are drawn.

Summing forces on the free-body diagrams

$$\sum F_{ext} = \sum F_{eff}$$
$$-k_1 x - c\dot{x} + k_2(y - x) = m\ddot{x}$$
$$m\ddot{x} + c\dot{x} + (k_1 + k_2)x = k_2 y$$

Putting the equation in standard form

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k_1 + k_2}{m}x = \frac{k_2 Y}{m}\sin\omega t$$

Thus the natural frequency, damping ratio and frequency ratio are

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$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}} = 126.8 \text{ rad/s}$$

$$\zeta = \frac{\text{c}}{2\text{m}\omega_\text{n}} = 0.14$$

The steady-state amplitude is determined from

$$r = \frac{\omega}{\omega_n} = 0.789$$

$$\frac{m\omega_n^2 X}{k_2 Y} = M(0.789, 0.14) = \frac{1}{\sqrt{[1-(0.789)^2] + [2(0.14)(0.789)]^2}} = 2.28$$

Thus the steady-state amplitude is

$$X = \frac{2.28(k_2 Y)}{m\omega_n^2} = \frac{2.28(1.5\times 10^4 \text{ N/m})(0.02 \text{ m})}{(2.8 \text{ kg})(126.8 \text{ rad/s})^2} = 0.0152 \text{ m}$$

Problem 4.38 illustrates the derivation of the differential equation and the detemination of the steady-state amplitude for a system undergoing base motion.

4.39 Determine the amplitude of steady–state vibration for the system shown in Figure P4.39. Use the indicated generalized coordinate.

Given: m = 5 kg, k = 1 × 10^5 N/m, c = 400 N·sec/m, y(t) = 0.01sin250t m, L = 4 m

Find: Θ

FIGURE P4.39

Solution: Let θ be the clockwise angular rotation of the bar from its equilibrium position. Free body diagrams of the bar at an arbitrary instant are shown below.

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Summing moments about the point of support

$$\left(\sum M_0\right)_{ext.} = \left(\sum M_0\right)_{eff.}$$

$$-k\left(\frac{3}{4}L\theta + y\right)\frac{3}{4}L - c\frac{L}{4}\dot{\theta}\frac{L}{4} = \frac{1}{12}mL^2\ddot{\theta} + m\frac{L}{4}\ddot{\theta}\frac{L}{4}$$

$$\frac{7}{48}mL^2\ddot{\theta} + \frac{1}{16}cL^2\dot{\theta} + \frac{9}{16}kL^2\theta = \frac{3}{4}kLY\sin\omega t$$

The system parameters are

$$\tilde{m} = \frac{7}{48}mL^2 = \frac{7}{48}(5\,kg)(4\,m)^2 = 11.67\,kg - m^2$$

$$\omega_n = \sqrt{\frac{27\,k}{7\,m}} = \sqrt{\frac{27\left(1\times10^5\,\frac{\mathrm{N}}{\mathrm{m}}\right)}{7(5\,\mathrm{kg})}} = 277.7\,\frac{\mathrm{rad}}{\mathrm{s}}$$

$$\zeta = \frac{3c}{14m\omega_n} = \frac{3\left(400\,\frac{\mathrm{N\cdot s}}{\mathrm{m}}\right)}{14(5\,\mathrm{kg})\left(277.7\,\frac{\mathrm{rad}}{\mathrm{s}}\right)} = 0.062$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{250\,\frac{\mathrm{rad}}{\mathrm{s}}}{277.7\,\frac{\mathrm{rad}}{\mathrm{s}}}\,0.900$$

The magnification factor for this system is

$$M = \frac{1}{\sqrt{\left[1-(0.900)^2\right]^2 + \left[2(0.062)(0.900)\right]^2}} = 4.54$$

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The steady-state amplitude is calculated from

$$\frac{\tilde{m}\omega_n^2\Theta}{\frac{3kLY}{4}} = M$$

$$\Theta = \frac{3kLYM}{4\tilde{m}\omega_N^2}$$

$$= \frac{3\left(1\times10^5\,\frac{\text{N}}{\text{m}}\right)(4\,\text{m})(0.01\,\text{m})(4.54)}{4\left(11.67\,\text{kg}\cdot\text{m}^2\right)\left(277.7\,\frac{\text{rad}}{\text{s}}\right)^2} = 0.015\,\text{rad}$$

Problem 4.39 illustrates (a) derivation of the differential equation governing a system undergoing a base excitation, and (b) determination of the steady-state vibration amplitude using the magnification factor.

4.40 Determine the amplitude of steady–state vibration for the system shown in Figure P4.40. Use the indicated generalized coordinate.

FIGURE P4.40

Given: m = 115 kg, L = 1.5 m , E = 210 × 10^9 N/m^2, I = 4.6 × 10^{-5} m^4, y(t) = 0.08sin200t m

Find: X

Solution: Let x(t) be the absolute displacement of the point where the machine is attached. The system is modeled as a mass attached through an elastic element to a moveable support. The governing differential equation is

$$m\ddot{x} + kx = ky$$

The equivalent stiffness of the cantilever beam is

$$k = \frac{3EI}{L^3} = \frac{3\left(210\times10^9\,\frac{\text{N}}{\text{m}^2}\right)\left(4.6\times10^{-5}\,\text{m}^4\right)}{(1.5\text{m})^3} = 8.59\times10^6\,\frac{\text{N}}{\text{m}}$$

The system's natural frequency is

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$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8.59\times10^6\ \frac{\text{N}}{\text{M}}}{115\text{kg}}} = 273.3\frac{\text{rad}}{\text{s}}$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{200\frac{\text{rad}}{\text{s}}}{273.3\frac{\text{rad}}{\text{s}}} = 0.732$$

The steady-state amplitude is given by

$$X = YM(0.732,0) \quad \frac{m\omega_n^2 X}{kY} = M(0.732,0)$$

or

$$X = (.08\text{ m})\frac{1}{1-(0.732)^2} = 0.172\text{ m}$$

Problem 4.40 illustrates the use of the function Λ in the determination of steady-state amplitude of a system subject to harmonic base motion.

4.41 Determine the amplitude of steady–state vibration for the system shown in Figure P4.41. Use the indicated generalized coordinate.

FIGURE P4.41

Given: m = 4 kg, L = 50 cm, x(t) = 0.35sin10t m

Find: Θ

Solution: Free body diagrams of the system are shown below at an arbitrary instant. Note that the acceleration of the mass center of the bar is equal to the horizontal acceleration of the support plus the acceleration relative to the support.

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Summing moments about the point of support

$$\left(\sum M_0\right)_{ext.} = \left(\sum M_0\right)_{eff.}$$

$$-mg\frac{L}{2}\sin\theta = m\ddot{x}\frac{L}{2}\cos\theta + m\frac{L}{2}\ddot{\theta}\left(\frac{L}{2}\right) + \frac{1}{12}mL^2\ddot{\theta} \tag{1}$$

Assuming small θ, eq.(1) becomes

$$m\frac{L^2}{3}\ddot{\theta} + mg\frac{L}{2} = -m\ddot{x}\frac{L}{2}$$

$$\ddot{\theta} + \frac{3g}{2L}\theta = -\frac{3\ddot{x}}{2L}$$

$$\ddot{\theta} + \frac{3g}{2L}\theta = \frac{3\omega^2 X}{2L}\sin\omega t$$

The natural frequency is

$$\omega_n = \sqrt{\frac{3g}{2L}} = \sqrt{\frac{3\left(9.81\frac{\text{m}}{\text{s}^2}\right)}{2(0.5\text{m})}} = 5.42\frac{\text{rad}}{\text{s}}$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{10\frac{\text{rad}}{\text{s}}}{5.42\frac{\text{rad}}{\text{s}}} = 1.85$$

Since r>1 the magnification factor is calculated as

$$M(1.85,0) = \frac{1}{r^2-1} = 0.416$$

The steady-state amplitude is related to the magnification factor by

$$\frac{\omega_n^2\Theta}{\frac{3\omega^2 L}{2L}} = M(1.85,0)$$

or

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$$\Theta = \frac{3r^2 M(1.85,0)X}{2L} = \frac{3(1.85)^2(0.416)(0.035\text{m})}{2(0.5\text{m})} = 0.144\text{rad}$$

Problem 4.41 illustrates the derivation of governing differential equation for a base motion problem.

4.42 Determine the amplitude of steady-state vibration for the system shown in Figure P4.42. Use the indicated generalized coordinate.

Given: I = 1.5 kg-m^2, L = 1.1 m, G = 80 × 10^9 N/m^2, J = 4.6 × 10^{-6} m^4, Φ=0.1 rad, ω=300 rad/s

Find: Θ

FIGURE P4.42

Solution: The torsional stiffness of the shaft is

$$k_t = \frac{JG}{L} = 3.35\times10^5 \text{ N}\cdot\text{m/rad}$$

The natural frequency and frequency ratio are

$$\omega_n = \sqrt{\frac{k_t}{I}} = 472.6 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = 0.636$$

The magnification factor is

$$M = \frac{I\omega_n^2\Theta}{k_t\Phi} = \frac{\Theta}{\Phi}$$

For this undamped system

$$\frac{\Theta}{\Phi} = M(0.636,0) = \frac{1}{1=(0.636)^2} = 1.68$$

Hence

$$\Theta = 1.68\Phi = 1.68(0.1\text{rad}) = 0.168\text{rad}$$

Problem 4.42 illustrates the base rotation of a torsional system.

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4.43 A 40 kg machine is attached to a base through a spring stiffness 2×10^4 N/m in parallel with a dashpot of damping coefficient 150 N · s/m. The base is given a time-dependent displacement 0.15 sin 30.1*t*m. Determine the amplitude of the absolute displacement of the machine and the amplitude of displacement of the machine relative to the base.

Given: m = 40 kg, k = 2×10^4 N/m, c = 150 N·s/m, Y = 0.15 m, ω = 30.1 rad/sec

Find: Z, X

Solution: The system's natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2\times 10^4 \frac{\text{N}}{\text{m}}}{40\text{kg}}} = 22.36\frac{\text{rad}}{\text{s}}$$

Thus the frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{30.1\frac{\text{rad}}{\text{s}}}{22.36\frac{\text{rad}}{\text{s}}} = 1.346$$

The system's damping ratio is

$$\zeta = \frac{c}{2m\omega_n} = \frac{150\frac{\text{N}\cdot\text{s}}{\text{m}}}{2(40\text{kg})(22.36\frac{\text{rad}}{\text{s}})} = 0.0838$$

The amplitude of the relative displacement is

$$Z = Y\Lambda = \frac{Yr^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$= \frac{(0.15\text{m})(1.346)^2}{\sqrt{[(1-(1.346)^2)^2+[2(0.0838)(1.346)]^2}}$$

$$= 0.323\text{m}$$

The amplitude of the absolute displacement is

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$$X = TY = Y\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

$$= 0.15\text{m}\sqrt{\frac{1+[2(0.0838)(1.346)]^2}{[1-(1.346)^2)^2+[2(0.0838)(1.346)]^2}}$$

$$= 0.183\text{m}$$

Problem 4.43 illustrates the calculation of the absolute and relative amplitudes of a block undergoing harmonic base motion.

4.44 A 5-kg rotor-balancing machine is mounted on a table through an elastic foundation of stiffness 3.1×10^4 N/m and damping ratio 0.04. Transducers indicate that the table on which the machine is placed vibrates at a frequency of 110 rad/s with an amplitude of 0.62 mm. What is the steady–state amplitude of acceleration of the balancing machine?

Given: m = 5 kg, k = 3.1×10^4 N/m, $\zeta = 0.04$, $\omega = 110$ rad/s, Y = 0.62 mm

Find**:** A

Solution: The steady-state amplitude of acceleration is $A = \omega^2 X$ where X is the steady-state amplitude of the rotor- balancing machine. The natural frequency and frequency ratio for the system are

$$\omega_n = \sqrt{\frac{k}{m}} = 78.74 \text{ rad/s}$$

$$\text{r} = \frac{\omega}{\omega_n} = \frac{110 \text{ rad/s}}{78.74 \text{ rad/s}} = 1.40$$

The acceleration amplitude is calculated from

$$\frac{\omega^2 X}{\omega^2 Y} = T(1.40, 0.04) = \sqrt{\frac{1+[2(0.04)(1.40)]^2}{[9-(1.4)^2]^2+[2(0.04)(1.40)]^2}} = 1.04$$

which leads to

$$A = 1.04\omega^2 Y = 1.04(110 \text{ rad/s})^2(0.00062 \text{ m}) = 7.88 \text{ m/s}^2$$

Problem 4.44 illustrates the use of T(r,ζ) to determine the absolute displacement and acceleration of a system subject to a harmonic base excitation.

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4.45 During a long earthquake the one-story frame structure of Figure P4.45 is subject to a ground acceleration of amplitude 50 mm/s^2 at a frequency of 88 rad/s. Determine the acceleration amplitude of the structure. Assume the girder is rigid and the structure has a damping ratio of 0.03.

FIGURE P4.45

Given: $\omega^2 Y = 50$ mm/s^2, $\zeta = 0.03$, $\omega = 88$ rad/s,
$m = 2000$ kg, $k = 1.8 \times 10^6$ N/m

Find**:** $\omega^2 X$

Solution: The natural frequency and damping ratio for the system are

$$\omega_n = \sqrt{\frac{k}{m}} = 30 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = 2.933$$

The acceleration amplitude is calculated from

$$\frac{\omega^2 X}{\omega^2 Y} = T(2.933, 0.03) = \sqrt{\frac{1 + [2(0.03)(2.933)]^2}{[1-(2.933)^2]^2 + [2(0.03)(2.933)]^2}} = 0.133$$

The acceleration amplitude of the structure is

$$A = \omega^2 X = 0.133\omega^2 Y = 0.133(50 \text{ mm/s}^2) = 6.67 \text{ mm/s}^2$$

Problem 4.45 illustrates the absolute acceleration of a structure whose base is subject to a periodic motion.

4.46 What is the required column stiffness of a one-story structure to limit its acceleration amplitude to 2.1 m/s^2 during an earthquake whose acceleration amplitude is 150 mm/s^2 at a frequency of 50 rad/s? The mass of structure is 1800 kg. Assume a damping ratio of 0.05.

Given: $\omega^2 X = 2.1$ m/s^2, $\omega^2 Y = 150$ mm/s^2, $\zeta = 0.05$, $\omega = 50$ rad/s, $m = 1800$ kg

Find**:** k

Solution: The required acceleration ratio is

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$$T(r,\zeta) = \frac{\omega^2 X}{\omega^2 Y} = \frac{2.1\ \text{m/s}^2}{150\ \text{mm/s}^2} = 14.0$$

Using the definition of T(r,ζ)

$$14.0 = \sqrt{\frac{1+[2(0.05)r]^2}{(1-r^2)^2 + [2(0.05)r]^2}}\, r$$

Squaring and rearranging leads to

$$196(r^4 - 1.99r^2 + 1) = 1 + 0.01r^2$$

$$196r^4 - 390.05r^2 + 197 = 0$$

The quadratic formula is used to give

$$r^2 = \frac{390.05 \pm \sqrt{(390.05)^2 - 4(196)(197)}}{2(196)} = 0.995 \pm 0.725i$$

Since the roots of the equation are complex, all values of r lead to values of T < 14.0 for a damping ratio of 0.05. Hence any stiffness is OK.

Problem 4.46 illustrates the absolute acceleration of a system undergoing base excitation.

4.47 In a rough sea, the heave of a ship is approximated as harmonic of amplitude 20 cm at a frequency of 1.5 Hz. What is the acceleration amplitude of a 20-kg computer workstation mounted on an elastic foundation in the ship of stiffness 700 N/m and damping ratio 0.04?

Given: Y = 20 cm, ζ = 0.04, ω = 1.5 Hz, m = 20 kg, k = 700 N/m

Find**:** $\omega^2 X$

Solution: The natural frequency of the computer is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{700\ \text{N/m}}{20\ \text{kg}}} = 5.92\ \text{rad/s}$$

The frequency ratio for the excitation is

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$$r = \frac{\omega}{\omega_n} = \frac{(1.5\text{ cycles/s})(2\pi\text{ rad/cycle})}{5.92\text{ rad/s}} = 1.59$$

The acceleration amplitude is calculated from

$$\omega^2 X = \omega^2 Y T(1.59, 0.04)$$

$$\omega^2 X = (9.42\text{ rad/s})^2 (0.2\text{ m})\sqrt{\frac{1+[2(0.04)(1.59)]^2}{[1-(1.59)^2]^2 + [2(0.04)(1.59)]^2}} = 11.61\text{ m/s}^2$$

Problem 4.47 illustrates the use of T(r,ζ) for base excitation problems.

4.48 In a rough sea of Chapter Problem 4.47, what is the required stiffness of an elastic foundation of damping ratio 0.05 to limit the acceleration of a 5-kg radio set to 1.5 m/s^2?

Given: $\zeta = 0.05$, m = 5 kg, f = 1.5 Hz, Y = 20 cm, $\omega^2 X = 1.5$ m/s^2

Find**:** k

Solution: The frequency in rad/s is $\omega = 2\pi(1.5) = 9.42$ rad/s. The maximum of the ratio of acceleration amplitude is

$$\frac{\omega^2 X}{\omega^2 Y} = \frac{1.5\text{ m/s}^2}{(9.42\text{ rad/s})^2 (0.2\text{ m})} = 0.0844$$

Thus, in order to limit the acceleration amplitude to 1.5 m/s^2

$$0.0844 > T(r, 0.05) = \sqrt{\frac{1+(0.1r)^2}{(1-r^2)^2 + (0.1r)^2}}$$

Squaring and rearranging leads to

$$0.0071r^4 - 0.0242r^2 - 0.9929 = 0$$

The quadratic formula is used to obtain

$$r^2 = -10.24, 13.65$$

The negative root is rejected and r = 3.695. Since

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$$r = \frac{\omega}{\omega_n}$$

$$\omega_n = \frac{\omega}{r} = \frac{9.42 \text{ rad/s}}{3.695} = 2.55 \text{ rad/s}$$

$$k = m\omega_n^2 = (5 \text{ kg})(2.55 \text{ rad/s})^2 = 32.5 \text{ N/m}$$

Hence k < 32.5 N/m.

Problem 4.48 illustrates the acceleration of a mass subject to base excitation.

4.49 Consider the one degree-of-freedom model of a vehicle suspension system of Figure P4.49. Consider a motorcycle of mass 250 kg. The suspension stiffness is 70,000 N/m and the damping ratio is 0.15. The motorcycle travels over a terrain that is approximately sinusoidal with a distance between peaks of 10 m and the distance from peak to valley is 10 cm. What is the acceleration amplitude felt by the motorcycle rider when she is traveling at (a) 30 m/s; (b) 60 m/s; (c) 120 m/s

FIGURE P4.49

Given: l = 10 m, d = 5 cm, ζ = 0.15, m = 250 kg, k = 70,000 N/m, (a) v = 30 m/s, (b) v = 60 m/s, (c) v = 120 m/s

Find**:** A

Solution: The natural frequency of the vehicle is

$$\omega_n = \sqrt{\frac{k}{m}} = 16.73 \text{ rad/s}$$

If v is the horizontal speed of the vehicle the road contour provides a harmonic base motion to the vehicle. The amplitude of the excitation is d and the frequency of the excitation is

$$\omega = \frac{2\pi}{\ell} v = 0.628v$$

The acceleration amplitude is given by

$$A = \omega^2 (d) T(r, 0.15)$$

(a) For v = 30 m/s the frequency and frequency ratio are

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$$\omega = 0.628(30\text{ m/s}) = 18.85\text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{18.85\text{ rad/s}}{16.73\text{ rad/s}} = 1.1265$$

The acceleration amplitude is

$$A = (18.85\text{ rad/s})^2(0.05\text{ m})\sqrt{\frac{1+[2(0.15)(1.1265)]^2}{[1-(1.1265)^2]^2+[2(0.15)(1.1265)]^2}} = 43.4\text{ m/s}^2$$

(b) For v = 60 m/s the frequency and frequency ratio are

$$\omega = 0.628(60\text{ m/s}) = 37.68\text{ rad/s}$$

$$\text{r} = 2.252$$

The acceleration amplitude is

$$A = \omega^2(d)T(2.252,0.15) = 20.8\text{ m/s}^2$$

(c) For v = 120 m/s the frequency and frequency ratio are

$$\omega = 0.628(120\text{ m/s}) = 75.4\text{ rad/s}$$

$$\text{r} = 4.50$$

The acceleration amplitude is

$$A = \omega^2(d)T(4.50,0.15) = 24.7\text{ m/s}^2$$

Problem 4.49 illustrates the acceleration amplitude for a harmonic base excitation problem.

4.50 For the motorcycle of Chapter Problem 4.49 determine (a) the "frequency response" of the motorcycle's suspension system by plotting the amplitude of acceleration versus motorcycle speed and (b) determine and plot the amplitude of displacement of the motorcycle versus speed.

FIGURE P4.49

Given: motorcycle of Chapter Problem 4.49

Find: (a) A versus v (b) X versus v

Solution: (a) It is determined in Chapter Problem 4.49 that

$$A = \omega^2(d)T(r,0.15)$$

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where

$$\omega = 2\pi v$$

$$r = \frac{\omega}{\omega_n} = \frac{2\pi v}{16.73 \text{ rad/s}}$$

The plot of A versus v is shown below

(b) The steady state amplitude of displacement is

$$X = dT(r, 0.15)$$

which is illustrated below

Problem 4.50 illustrates the principle of frequency response.

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4.51 What is the minimum static deflection of an undamped isolator that provides 75 percent isolation to a 200-kg washing machine at 5000 rpm?

Given: m = 200 kg, $\omega = 5000$ rpm, 75 percent isolation

Find: Δ_{st}

Solution: For 75 percent isolation, T = 0.25 or for an undamped isolator

$$T(r,0) = 0.25 = \frac{1}{r^2 - 1} \Rightarrow r^2 = 1 + \frac{1}{0.25} = 5 \Rightarrow r = 1.495$$

$$r = \frac{\omega}{\omega_n} \Rightarrow \omega_n = \frac{\omega}{r} = \frac{\left(5000\ \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\ \text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{s}}\right)}{1.495} = 350.1\ \frac{\text{rad}}{\text{s}}$$

$$\Delta_{st} = \frac{g}{\omega_n^2} = \frac{9.81\ \frac{\text{m}}{\text{s}^2}}{\left(350.1\ \frac{\text{rad}}{\text{s}}\right)^2} = 8.00 \times 10^{-5}\ \text{m}$$

Problem 4.51 illustrates the minimum static deflection of an isolator.

4.52 What is the maximum allowable stiffness of an isolator of damping ratio 0.05 that provides 81% isolation to a 40-kg printing press operating at 850 rpm?

Given: m = 40 kg, ω = 850 rpm, ξ = 0.05, T = 0.19

Find: maximum k

Solution: Requiring the isolator to provide 81% isolation leads to T = 0.19. The minimum required frequency ratio for a damping ratio of 0.05 is calculated from

$$T(r, 0.05) = 0.19 = \sqrt{\frac{1 + (0.1r)^2}{(1 - r^2)^2 + (0.1r)^2}}$$

Rearranging leads to the following equation

$$r^4 - 2.267r^2 - 26.7 = 0$$

whose real positive solution is

$$r = 2.53$$

The maximum allowable natural frequency is calculated from

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$$\omega_n = \frac{\omega}{r} = \frac{\left(850\ \frac{\text{rev}}{\text{min}}\right)\left(2\pi\ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)}{2.53} = 35.18\ \frac{\text{rad}}{\text{sec}}$$

The maximum allowable isolator stiffness is

$$k = m\omega_n^2 = (40\ \text{kg})\left(35.18\ \frac{\text{rad}}{\text{sec}}\right)^2 = 49500\ \frac{\text{N}}{\text{m}}$$

Problem 4.52 illustrates calculation of the maximum allowable stiffness of a damped isolator.

4.53 When set on a rigid foundation and operating at 800 rpm, a 200-kg machine tool provides a harmonic force with a magnitude 18,000 N to the foundation. An engineer has determined that the maximum magnitude of a harmonic force to which the foundation should be subjected to is 2600 N. (a) What is the maximum stiffness of an undamped isolator that provides sufficient isolation between the tool and the foundation? (b) What is the maximum stiffness of an isolator with a damping ratio of 0.11?

Given: m = 200 kg, ω = 800 rpm, F_0 = 18,000 N, $F_{T,max.}$ = 2600 N

Find: k

Solution: The maximum transmissibility ratio is

$$T = \frac{F_T}{F_0} = \frac{2600\ \text{N}}{18000\ \text{N}} = 0.144$$

(a) Requiring $T < T_{max.}$ leads to

$$0.144 > \frac{1}{r^2 - 1}$$

$$r = \frac{\omega}{\omega_n} > 2.82$$

$$\omega_n < \frac{\omega}{2.82} = \frac{800\frac{\text{rev}}{\text{min}}\left(2\pi\ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)}{2.82} = 29.73\ \frac{\text{rad}}{\text{sec}}$$

The maximum isolator stiffness is given by

$$k = m\omega_n^2 = (200\ \text{kg})\left(29.72\ \frac{\text{rad}}{\text{sec}}\right)^2 = 1.763 \times 10^5\ \frac{\text{N}}{\text{m}}$$

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(b) The minimum frequency ratio for an isolator with ξ = 0.11 is calculated from

$$0.144 = T(r, 0.11) = \sqrt{\frac{1 + (0.22r)^2}{(1 - r^2)^2 + (0.22r)^2}}$$

which can be rearranged to

$$r^4 - 4.27r^2 - 46.9 = 0$$

The real, positive solution of the above equation is

$$r = 3.05$$

The maximum natural frequency is calculated as

$$\omega_n = \frac{\omega}{r} = \frac{\left(800\ \frac{\text{rev}}{\text{min}}\right)\left(2\pi\ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)}{3.05} = 27.46\ \frac{\text{rad}}{\text{sec}}$$

The maximum isolator stiffness is

$$k = m\omega_n^2 = (200\ \text{kg})\left(27.01\ \frac{\text{rad}}{\text{sec}}\right)^2 = 1.46\text{x}10^5\ \frac{\text{N}}{\text{m}}$$

Problem 4.53 illustrates determination of the maximum stiffness for a damped isolator.

4.54 A 150-kg engine operates at 1500 rpm. (a) What percent isolation is achieved if the engine is mounted on four identical springs each of stiffness 1.2×10^5 N/m? (b) What percent isolation is achieved if the springs are in parallel with a viscous damper of damping coefficient 1000 N · s/m?

Given: m = 150 kg, ω = 1500 rpm = 314.2 rad/s, 4 springs, k = 1.2×10^5 N/m, c = 1000 N · s/m?

Find: percent isolation

Solution: (a) The equivalent stiffness of the four springs in parallel is

$$k_{eq} = 4k = 4.8 \times 10^5\ \text{N/m}$$

The natural frequency of the engine is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 56.6\ \text{rad/s}$$

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The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{314.2 \text{ rad/s}}{566 \text{ rad/s}} = 5.55$$

For the undamped system,

$$T(5.55,0) = \frac{1}{(5.55)^2 - 1} = 0.0335$$

The percent isolation achieved is

$$I = 100(1 - T) = 96.6 \text{ percent}$$

(b) The equivalent stiffness of the four springs in parallel is

$$k_{eq} = 4k = 4.8 \times 10^5 \text{ N/m}$$

The natural frequency of the engine is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 56.6 \text{ rad/s}$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{314.2 \text{ rad/s}}{566 \text{ rad/s}} = 5.55$$

The damping ratio for the system is

$$\zeta = \frac{c}{2m\omega_n} = \frac{1000 \text{ N - s/m}}{2(150 \text{ kg})(56.6 \text{ rad/s})} = 0.0589$$

The transmissibility ratio is,

$$T(5.55,00589) = \sqrt{\frac{1 + [2(0.0589)(5.55)]^2}{[1 - (5.55)^2]^2 + [2(0.0589)(5.55)]^2}} = 0.040$$

The percent isolation achieved is

$$I = 100(1 - T) = 96.0 \text{ percent}$$

Problem 4.54 illustrates the percentage isolation achieved using a damped isolator.

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4.55 A 150 kg engine operates at speeds between 1000 and 2000 rpm. It is desired to achieve at least 85 percent isolation at all speeds. The only readily available isolator has a stiffness of 5×10^5 N/m. How much mass must be added to the engine to achieve the desired isolation?

Given: m = 150 kg, 1000 rpm < ω < 2000 rpm, k = 5×10^5 N/m, 85 percent isolation

Find: m_{add}

Solution: Higher isolation is achieved at higher speeds. Thus better than 85 percent isolation is achieved at all speeds if the system is designed such that 85 percent isolation is achieved at ω = 1000 rpm = 104.7 rad/s. For an undamped isolator

$$T = \frac{1}{r^2 - 1}$$

$$0.15 = \frac{1}{r^2 - 1}$$

$$r = \sqrt{1 + \frac{1}{0.15}}$$

$$r = 2.77$$

The maximum natural frequency of the system is

$$\omega_n = \frac{\omega}{r} = \frac{104.7 \text{ rad/s}}{2.77} = 37.8 \text{ rad/s}$$

If the isolator is used without added mass the system's natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5x10^5 \text{ N/m}}{150 \text{ kg}}} = 57.8 \text{ rad/s}$$

Since the natural frequency exceeds the maximum allowable natural frequency, the isolator can be used only if mass is added to the system. The required mass is

$$m = \frac{k}{\omega_n^2} = \frac{5x10^5 \text{ N/m}}{(37.8 \text{ rad/s})^2} = 350 \text{ kg}$$

Thus 200 kg must be added to the machine to achieve the desired isolation.

Problem 4.55 illustrates the addition of mass to a system as a means of vibration control.

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4.56 Cork pads of stiffness of 6×10^5 N/m and a damping ratio of 0.2 are used to isolate a 40-kg machine tool from its foundation. The machine tool operates at 1400 rpm and produces a harmonic force of magnitude 80,000 N. If the pads are placed in series, how many are required such that the magnitude of the transmitted force is less than 10,000 N?

Given: m = 40 kg, F_0 = 80000 N, ω = 1400 rpm, $F_{T,max}$ = 10000 N, k = 6×10^5 N/m

Find: n (number of pads)

Solution: The maximum transmissibility ratio is

$$T = \frac{F_T}{F_0} = \frac{10000\ \text{N}}{80000\ \text{N}} = 0.125$$

The minimum frequency ratio is determined from

$$T(r, 0.2) = 0.125 = \sqrt{\frac{1 + (0.4r)^2}{(1 - r^2)^2 + (0.4r)^2}}$$

which can be rearranged to

$$r^4 - 4.40r^2 - 15.0 = 0$$

The real, positive root of the above equation is

$$r = 2.58$$

The maximum natural frequency is

$$\omega_n = \frac{\omega}{r} = \frac{\left(1400\ \frac{\text{rev}}{\text{min}}\right)\left(2\pi\ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)}{2.58} = 56.82\ \frac{\text{rad}}{\text{sec}}$$

The maximum allowable isolator stiffness is

$$k = m\omega_n^2 = (40\ \text{kg})\left(56.82\ \frac{\text{rad}}{\text{sec}}\right)^2 = 1.29 \times 10^5\ \frac{\text{N}}{\text{m}}$$

When n pads are placed in series the equivalent stiffness is k/n. Thus in order to achieve sufficient isolation

$$\frac{6 \times 10^5\ \frac{\text{N}}{\text{m}}}{\text{n}} = 1.29 \times 10^5\ \frac{\text{N}}{\text{m}}$$

which leads to

$$n = 5$$

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Problem 4.56 illustrates that isolator pads can be placed in series to help achieve sufficient isolation.

4.57 A 100-kg machine operates at 1400 rpm and produces a harmonic force of magnitude 80,000 N. The magnitude of the force transmitted to the foundation is to be reduced to 20,000 N by mounting the machine on four identical undamped isolators in parallel. What is the maximum stiffness of each isolator?

Given: m = 100 kg, ω = 1400 rpm , F_0 = 80,000 N, $F_{T,max}$ = 20,000 N, ξ = 0

Find: k

Solution: The maximum transmissibility ratio is

$$T = \frac{F_T}{F_0} = \frac{20{,}000\ \text{N}}{80{,}000\ \text{N}} = 0.25$$

The minimum frequency ratio is determined from

$$T(r, 0) = 0.25 = \frac{1}{r^2 - 1}$$

which leads to

$$r = \sqrt{\frac{1.25}{0.25}} = 2.24$$

The maximum natural frequency is

$$\omega_n = \frac{\omega}{r} = \frac{\left(1400\ \frac{\text{rev}}{\text{min}}\right)\left(2\pi\ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)}{2.24} = 65.44\,\frac{\text{rad}}{\text{sec}}$$

The maximum total stiffness of the isolation system is calculated from

$$k = m\omega_n^2 = (100\ \text{kg})\left(65.44\ \frac{\text{rad}}{\text{sec}}\right)^2 = 4.28 \times 10^5\ \frac{\text{N}}{\text{m}}$$

Since the isolation system consists of four isolators in parallel, the maximum stiffness of each isolator is

$$\frac{k}{4} = \frac{4.3 \times 10^5\ \frac{\text{N}}{\text{m}}}{4} = 1.075 \times 10^5\ \frac{\text{N}}{\text{m}}$$

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Problem 4.57 illustrates design of an undamped isolation system.

4.58 A 10-kg laser flow-measuring device is used on a table in a laboratory. Because of operation of other equipment, the table is subject to vibration. Accelerometer measurements show that the dominant component of the table vibrations is at 300 Hz and has an amplitude of 4.3 m/s^2. For effective operation, the laser can be subject to an acceleration of 0.7 m/s^2. (a) Design an undamped isolator to reduce the transmitted acceleration, to an acceptable amplitude. (b) Design the isolator such that it has a damping ratio of 0.04.

Given: m = 10 kg, ω = 300 Hz, a = 4.3 m/s^2, a_{max} = 0.7 m/s^2, ξ = 0

Find: k

Solution: (a) The isolation of the flow measuring device from the table's vibrations is a similar problem to the isolation of a foundation from the forces produced in a reciprocating machine. The transmissibility ratio is

$$T = \frac{a_{max}}{a} = \frac{0.7\ \frac{\text{m}}{\text{sec}^2}}{4.3\ \frac{\text{m}}{\text{sec}^2}} = 0.163$$

The minimum frequency ratio for an undamped isolator to achieve this transmissibility is determined from

$$T(r, 0) = 0.163 = \frac{1}{r^2 - 1}$$

which gives

$$r = \sqrt{\frac{1.163}{0.1623}} = 2.67$$

The maximum natural frequency is calculated as

$$\omega_n = \frac{\omega}{r} = \frac{\left(300\ \frac{\text{cycles}}{\text{sec}}\right)\left(2\pi\ \frac{\text{rad}}{1\ \text{cycle}}\right)}{2.67} = 706.0\,\frac{\text{rad}}{\text{sec}}$$

The maximum isolator stiffness is

$$k = m\omega_n^2 = (10\ \text{kg})\left(706.0\ \frac{\text{rad}}{\text{sec}}\right)^2 = 4.98 \times 10^6\ \frac{\text{N}}{\text{m}}$$

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(b) For a damping ratio of 0.4

$$T(r, 0.04) = 0.163$$

which leads to r=2.70,

$$\omega_n = \frac{\omega}{r} = \frac{\left(300\ \frac{\text{cycles}}{\text{sec}}\right)\left(2\pi\ \frac{\text{rad}}{1\ \text{cycle}}\right)}{2.70} = 698.1\,\frac{\text{rad}}{\text{sec}}$$

and

$$k = m\omega_n^2 = (10\ \text{kg})\left(698.1\ \frac{\text{rad}}{\text{sec}}\right)^2 = 4.87 \times 10^6\ \frac{\text{N}}{\text{m}}$$

Problem 4.58 illustrates isolation from surrounding vibration.

4.59 Rough seas cause a ship to heave with an amplitude of 0.4 m at a frequency of 20 rad/s. Design an isolation system with a damping ratio of 0.13 such that a 45 kg navigational computer is subject to an acceleration of only 20 m/sec^2.

Given: ω = 20 rad/s, Y = 0.4 m, ξ = 0.13, m = 45 kg , $a_{,max}$ = 20 m/sec^2

Find: ω_n

Solution: The acceleration amplitude of the ship is

$$a = \omega^2 Y = \left(20\ \frac{\text{rad}}{\text{sec}}\right)^2 (0.4\ \text{m}) = 160\ \frac{\text{m}}{\text{sec}^2}$$

The maximum transmissibility ratio is

$$T = \frac{a_{max}}{a} = \frac{20\ \frac{\text{m}}{\text{sec}^2}}{160\ \frac{\text{m}}{\text{sec}^2}} = 0.125$$

The minimum frequency ratio is determined from

$$T(r, 0.13) = 0.125 = \sqrt{\frac{1 + (0.26r^2)}{(1 - r^2)^2 + (0.26r)^2}}$$

which can be rearranged to

$$r^4 - 6.259r^2 - 63 = 0$$

The real, positive root of the above equation is

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$$r = 3.42$$

The maximum natural frequency is

$$\omega_n = \frac{\omega}{r} = \frac{\left(20\ \frac{\text{rad}}{\text{sec}}\right)}{3.42} = 5.52\ \frac{\text{rad}}{\text{sec}}$$

Problem 4.59 illustrates isolation from periodic motion.

4.60 A sensitive computer is being transported by rail in a boxcar. Accelerometer measurements indicate that when the train is travelling at its normal speed of 85 m/s the dominant component of the boxcar's vertical acceleration is 8.5 m/s^2 at a frequency of 36 rad/s. The crate in which the computer is being transported is tied to the floor of the boxcar. What is the required stiffness of an isolator with a damping ratio of 0.05 such that the acceleration amplitude of the 60 kg computer is less than 0.5 m/s^2? With this isolator, what is the displacement of the computer relative to the crate?

Given: a = 8.5 m/s^2, ω = 36 rad/s, $\xi = 0.05$, $a_{,max} = 0.05$ m/s^2, m = 60 kg

Find: k, z

Solution: The maximum transmissibility ratio is

$$T = \frac{a_{max}}{a} = \frac{0.5\ \frac{m}{sec^2}}{8.5\frac{m}{sec^2}} = 0.0588$$

The minimum frequency ratio is determined from

$$T(r, 0.05) = 0.0588 = \sqrt{\frac{1 + (0.1r^2)}{(1 - r^2)^2 + (0.1r)^2}}$$

which can be rearranged to

$$r^4 - 4.88r^2 - 288.2 = 0$$

The real, positive root of the above equation is

$$r = 4.426$$

The maximum natural frequency is

$$\omega_n = \frac{\omega}{r} = \frac{\left(36\ \frac{\text{rad}}{\text{sec}}\right)}{4.426} = 8.133\ \frac{\text{rad}}{\text{sec}}$$

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The maximum isolator stiffness is

$$k = m\omega_n^2 = (60\text{ kg})\left(8.133\ \frac{\text{rad}}{\text{sec}}\right)^2 = 3970\ \frac{\text{N}}{\text{m}}$$

When using this isolator the amplitude of the relative displacement is

$$Z = Y\Lambda(4.426,0.05)$$

$$= Yr^2M(4.426,0.05)$$

$$= \frac{\omega^2 Y}{\omega_n^2}M(4.426,0.05)$$

$$= \frac{a}{\omega_n^2}M(4.426,0.05)$$

$$\frac{8.5\ \frac{\text{m}}{\text{sec}^2}}{\left(8.133\ \frac{\text{rad}}{\text{sec}}\right)^2}\frac{1}{\sqrt{(1-(4.436)^2)^2+\left(2(0.05)(4.426)\right)^2}}$$

$$= 6.9\text{ m}$$

Problem 4.60 illustrates isolation from harmonic excitation.

4.61 A 200 kg engine operates at 1200 rpm. Design an isolator such that the transmissibility ratio during start-up is less than 4.6 and the system achieves 80 percent isolation.

Given: m = 200 kg, ω = 1200 rpm, T_{start} = 4.6, T = 0.2

Find: k, ζ

Solution: The maximum transmissibilty during start up is determined by the damping ratio of the system.

$$T_{\max} = 4\zeta^2\left[\frac{\sqrt{1+8\zeta^2}}{2+16\zeta^2+(16\zeta^4-8\zeta^2-2)\sqrt{1+8\zeta^2}}\right]^{1/2}$$

Setting ζ = 0.15 leads to T_{max} = 3.51. Hence an isolator with a damping ratio of 0.15 is acceptable. Eighty percent isolation is then achieved when

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$$T(r,0.15) < 0.2$$

$$0.2 > \sqrt{\frac{1+[2(0.15)r]^2}{(1-r^2)^2+[2(0.15)r]^2}}$$

The minimum r to achieve 80 percent isolation is calculated as 2.72. The maximum allowable natural frequency is

$$\omega_n = \frac{\omega}{r} = \frac{(1200\text{ rev/min})(2\pi\text{ rad/rev})(1\text{ min}/60\text{ s})}{2.72} = 46.2\text{ rad/s}$$

The isolator stiffness is calculated as

$$k = m\omega_n^2 = (200\text{ kg})(46.2\text{ rad/s})^2 = 4.27\times10^5\text{ N/m}$$

Problem 4.61 illustrates design of a damped isolator.

4.62 A 150 kg machine tool operates at speeds between 500 and 1500 rpm. At each speed a harmonic force of magnitude 15,000 N is produced. Design an isolator such that the maximum transmitted force during start-up is 60,000 N and the maximum transmitted steady–state force is 2000 N.

Given: m = 150 kg, 500 rpm < ω < 1500 rpm, F_0= 15,000 N, $F_{max\text{-}start}$ = 60,000 N,
F_{max} = 2000 N

Find: k, ζ

Solution: the maximum transmissibility during start up is

$$T_{start} = \frac{F_{\max-start}}{F_0} = \frac{60000\text{ N}}{15000\text{ N}} = 4$$

The maximum transmissibilty during start up is determined by the damping ratio of the system. From Eq.(3.75)

$$T_{\max} = 4\zeta^2\left[\frac{\sqrt{1+8\zeta^2}}{2+16\zeta^2+(16\zeta^4-8\zeta^2-2)\sqrt{1+8\zeta^2}}\right]^{1/2}$$

Setting ζ = 0.15 leads to T_{max} = 3.51. Hence an isolator with a damping ratio of 0.15 is acceptable. Since the magnitude of the excitation is the same for all operating speeds, the maximum transmitted force will occur at the lowest speed. The maximum transmissibilty ratio is

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$$T = \frac{F_{\max}}{F_0} = \frac{2000\text{ N}}{15000\text{ N}} = 0.1333$$

The required isolation is achieved by requiring

$$T(r, 0.15) < 0.1333$$

$$0.1333 > \sqrt{\frac{1 + [2(0.15)r]^2}{(1 - r^2)^2 + [2(0.15)r]^2}}$$

The minimum r required is calculated as r = 3.43. The maximum allowable natural frequency is calculated as

$$\omega_n = \frac{\omega_{\min}}{r} = \frac{(500\text{ rev/min})(2\pi\text{ rad/rev})(1\text{ min}/60\text{ s})}{3.43} = 15.27\text{ rad/s}$$

The maximum allowable stiffness is calculated as

$$k = m\omega_n^2 = (150\text{ kg})(15.27\text{ rad/s})^2 = 3.50 \times 10^4\text{ N}$$

Problem 4.62 illustrates undamped isolator design.

4.63 A 200 kg testing machine operates at 500 rpm and produces a harmonic force of magnitude 40,000 N. An isolation system for the machine consists of a damped isolator and a concrete block for mounting the machine. Design the isolation system such that all of the following are met:

(i) The maximum transmitted force during start-up is 100,000 N.
(ii) The maximum transmitted force in the steady–state is 5000 N.
(iii) The maximum steady–state amplitude of the machine is 2 cm.

Given: m = 200 kg, ω = 500 rpm, F_0 = 40,000 N, $F_{\max,\text{ start up}}$ = 100,000 N, $F_{\max}$ = 5000 N, $x_{\max}$ = 2 cm

Find: isolation system

Solution: The maximum force during start up is given by

$$F_{\max,startup} = F_0 T_{\max} = F_0 4\zeta^2 \left[\frac{\sqrt{1 + 8\zeta^2}}{2 + 16\zeta^2 + (16\zeta^4 - 8\zeta^2 - 2)\sqrt{1 + 8\zeta^2}} \right]$$

The minimum damping ratio is obtained from

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$$100000 = 4(40000)\zeta^2\left[\frac{\sqrt{1+8\zeta^2}}{2+16\zeta^2+(16\zeta^4-8\zeta^2-2)\sqrt{1+8\zeta^2}}\right]$$

A trial and error solution leads to $\zeta = 0.225$. In order to set the maximum steady-state force to 5000 N

$$\frac{F_{T\max}}{F_0} = T(r, 0.225)$$

$$\frac{5000}{40000} = \sqrt{\frac{1+[2(0.225)r]^2}{(1-r^2)^2+[2(0.225)r]^2}}$$

which is solved for r= 4.2. The natural frequency is calculated as

$$\omega_n = \frac{\omega}{r} = \frac{(500\text{ rev/min})(2\pi\text{ rad/rev})(1\text{min}/60\text{ sec})}{4.2} = 12.47\text{ rad/s}$$

The minimum mass required to limit the steady-state amplitude to 2 cm is obtained from

$$m = \frac{F_0}{X_{\max}\omega_n^2}M(4.2, 0.225)$$

$$m = \frac{40000\text{ N}}{(0.02\text{ m})(12.47\text{ rad/s})^2}\frac{1}{\sqrt{[1-(4.2)^2]^2+[2(0.225)(4.2)]^2}} = 767\text{ kg}$$

The isolator stiffness, damping ratio, and added mass are

$$m_a = 767\text{ kg} - 200\text{ kg} = 567\text{ kg}$$

$$k = m\omega_n^2 = (767\text{ kg})(12.47\text{ rad/s})^2 = 1.19\times10^5\text{ N/m}$$

$$c = 2\zeta m\omega_n = 4.3\times10^3\text{ N - s/m}$$

Problem 4.63 illustrates design of an isolator for multiple constraints.

4.64 A 150-kg washing machine has a rotating unbalance of 0.45 kg · m. The machine is placed on isolators of equivalent stiffness 4×10^5 N/m and damping ratio 0.08. Over what range of operating speeds will the transmitted force between the washing machine and the floor be less than 3000 N?

Given: m = 150 kg, m_0e = 0.45 kg · m, k = 4×10^5 N/m, ξ = 0.08, $F_{T,max}$ = 3000 N

Find: range of ω

Solution: The system's natural frequency is

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$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \times 10^5 \ \frac{\text{N}}{\text{m}}}{150 \text{ kg}}} = 51.6 \ \frac{\text{rad}}{\text{sec}}$$

The maximum allowable value of R is

$$R_{max.} = \frac{F_{Tmax.}}{m_0 e \omega_n^2} = \frac{3000 \text{ N}}{(0.45 \text{ kg} \cdot \text{m}) \left(51.6 \ \frac{\text{rad}}{\text{sec}}\right)^2} = 2.5$$

Since $\xi < 0.353$ and $R_{max} > 2$, there are two values of r such that

$$R_{max.} = R(r, 0.08)$$

$$2.5 = r^2 \sqrt{\frac{1 + (.16r)^2}{(1 - r^2)^2 + (.16r)^2}}$$

The real solutions of the above equation are

$$r = 1.28, \quad 14.21$$

Thus the range of operating speeds for which the transmitted force is less than 3000 N is

$$66.0 \ \frac{\text{rad}}{\text{sec}} < \omega < 733.2 \ \frac{\text{rad}}{\text{sec}}$$

Problem 4.64 illustrates the use of $R(r,\xi)$ to determine the effective operating range of a machine with a rotating unbalance.

4.65 A 54-kg air compressor operates at speeds between 800 and 2000 rpm and has a rotating unbalance of 0.23 kg · m. Design an isolator with a damping ratio of 0.15 such that the transmitted force is less than 1000 N at all operating speeds.

Given: m = 54 kg, m_0e = 0.23 kg · m, ξ = 0.15,800 rpm ≤ ω ≤ 2000 rpm, $F_{T,max}$ = 1000 N

Find: k

Solution: From Figure 4.8 the value of r for which the minimum of $R(r,\xi)$ occurs for ξ = 0.15 is r = 2.5. As a first trial select ω_n such that r = 2.5 corresponds to the midpoint of the range. That is

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$$\frac{\left(1400\ \frac{\text{rev}}{\text{min}}\right)\left(2\pi\ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)}{\omega_n} = 2.5$$

$$\omega_n = 58.6\ \frac{\text{rad}}{\text{sec}}$$

Now check the transmitted force at ω = 800 rpm

$$r = \frac{\left(800\ \frac{\text{rev}}{\text{min}}\right)\left(2\pi\ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)}{58.6\ \frac{\text{rad}}{\text{sec}}} = 1.43$$

$$F_T = m_0 e\omega_n^2 R(1.43{,}0.15)$$

$$= (0.23\ \text{kg}\cdot -\text{m})\left(58.6\ \frac{\text{rad}}{\text{sec}}\right)^2 (1.43)^2 \sqrt{\frac{1+[2(0.15)(1.43)]^2}{[1-(1.43)^2]^2+[2(0.15)(1.43)]^2}}$$

Since F_T > 1000 N, the solution is unacceptable. It is imperative to require a larger value of r corresponding to ω = 800 rpm.

Now, find the value of r such that the transmitted force is exactly 1000 N when the machine operates at 800 rpm,

$$T = \frac{F_{T_{max.}}}{m_0 e\omega^2} = \frac{1000\ \text{N}}{(0.23\ \text{kg}\cdot\text{m})\left(83.8\ \frac{\text{rad}}{\text{sec}}\right)^2} = 0.619$$

$$0.619 = \sqrt{\frac{1+(0.3\,r)^2}{(1-r^2)^2+(0.3r)^2}}$$

The solution of the above equation is r = 1.65, which leads to a natural frequency of 50.8 rad/sec. Checking the transmitted force at 2000 rpm,

$$r = \frac{209.4\ \frac{\text{rad}}{\text{sec}}}{50.8\ \frac{\text{rad}}{\text{sec}}} = 4.12$$

$$F_T = m_0 e\omega_n^2 R(4.12{,}0.15)$$

$$= (0.23\ \text{kg}\cdot\text{m})\left(50.8\ \frac{\text{rad}}{\text{sec}}\right)^2 (4.12)^2 \sqrt{\frac{1+[2(.15)(4.12)]^2}{[1-(4.12)^2]^2+[2(.15)(4.12)]^2}}$$

$$= 1000\ \text{N}$$

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Thus an isolator with a natural frequency of 50.8 rad/sec is acceptable. The required isolator stiffness is

$$k = m\omega_n^2 = (54\text{ kg})\left(50.8\ \frac{\text{rad}}{\text{sec}}\right)^2 = 1.39 \times 10^5\ \frac{\text{N}}{\text{m}}$$

Note that if a more flexible isolator is chosen, then the value of r corresponding to 2000 rpm is greater and the transmitted force when the machine is operating at 2000 rpm is greater than 1000 N. If a stiffer isolator is chosen, the value of r corresponding to 800 rpm is less than 1.65 and the transmitted force when the machine is operating is greater than 1000 N.

Problem 4.65 illustrates the logical process for the design of an isolator for a system with a rotating unbalance.

4.66 A 1000 kg turbomachine has a rotating unbalance of 0.1 kg · m. The machine operates at speeds between 500 and 750 rpm. What is the maximum isolator stiffness of an undamped isolator that can be used to reduce the transmitted force to 300 N at all operating speeds?

Given: m = 1000 kg, 500 rpm < ω < 750 rpm, m_0e=0.1 kg · m, F_{max} = 300 N

Find: k_{max}

Solution: Without isolation the transmitted force is

$$F_0 = m_0 e\omega^2$$

At the upper end of the operating range the force is

$$F_0 = (0.1\text{ kg - m})\left[(750\text{ rev/min})(2\pi\text{ rad/rev})(1\text{min}/60\text{ s})\right]^2 = 616.8\text{ N}$$

Isolation at this speed requires

$$\frac{F_{\max}}{F_0} = T(r,0)$$

$$\frac{300\text{ N}}{616.8\text{ N}} = \frac{1}{r^2 - 1}$$

$$r = 1.748$$

which leads to a natural frequency and a stiffness of

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$$\omega_n = \frac{\omega}{r} = 44.92 \text{ rad/s}$$
$$k = m\omega_n^2 = 2.02\times10^6 \text{ N/m}$$

If this isolator is used at the lowest operating speed

$$r = \frac{\omega}{\omega_n} = \frac{(500 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{44.92 \text{ rad/s}} = 1.1655$$

$$F_T = m_0 e\omega^2 T(1.1655,0) = (0.1 \text{ kg}\cdot\text{m})(52.36 \text{ rad/s})^2 \frac{1}{(1.1655)^2 - 1} = 765 \text{ N}$$

Obviously this isolator does not work appropriately. Requiring the transmitted force to be 300 N at the lowest operating speed leads to

$$\frac{300 \text{ N}}{(0.1 \text{ kg}\cdot\text{m})(52.36 \text{ rad/s})^2} = \frac{1}{r^2 - 1}$$
$$r = 1.383$$
$$\omega_n = \frac{\omega}{r} = 37.85 \text{ rad/s}$$
$$k = m\omega_n^2 = 1.43\times10^6 \text{ N/m}$$

Then at the highest operating speed

$$r = \frac{\omega}{\omega_n} = 2.075$$
$$F_T = (612.8 \text{ N})\frac{1}{(2.075)^2 - 1}$$
$$F_T = 186 \text{ N}$$

Hence the maximum allowable stiffness of the isolator is 1.43×10^6 N/m.

Problem 4.66 illustrates the design of an isolator to be used over a range of frequencies.

4.67 A motorcycle travels over a road whose contour is approximately sinusoidal, $y(z) = 0.2 \sin(0.4z)$ m where z is measured in meters. Using a SDOF model, design a suspension system with a damping ratio of 0.1 such that the acceleration felt by the rider is less than 15 m/s^2 at all horizontal speeds between 30 and 80 m/s. The mass of the motorcycle and the rider is 225 kg.

Given: $y(z) = 0.2 \sin(0.4z)$ m, 30 m/s $\le$ v $\le$ 80 m/s, $\xi = 0.1$, m = 225 kg, $a_{max.} = 15$ m/s^2

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Find: k

Solution: If the horizontal speed is constant,

$$z = vt$$

The time dependent vertical displacement felt by the rider is

$$y(t) = 0.2\sin 0.4vt \text{ m}$$

The frequency of excitation is

$$\omega = 0.4v$$

Hence the frequency range is

$$12\ \frac{\text{rad}}{\text{sec}} \leq \omega \leq 32\ \frac{\text{rad}}{\text{sc}}$$

The transmissibility ratio for acceleration is

$$\frac{a}{\omega^2 Y} = T(r,\xi)$$

$$\frac{a}{\omega_n^2 Y} = r^2 T(r,\xi) = R(r,\xi)$$

Hence the suspension system can be designed using knowledge of R(r,ξ). For ξ = 0.1, R(r,ξ) has a minimum corresponding to r = 2.94. Since the value of R increases faster with decreasing r, it is best to choose r = 2.94 to correspond to an excitation frequency less than halfway into the operating range. Thus, let r = 2.94 correspond to ω = 20 rad/sec,

$$2.94 = \frac{\omega}{\omega_n}$$

$$\omega_n = \frac{\omega}{r} = \frac{20\ \frac{\text{rad}}{\text{sec}}}{2.94} = 6.8\ \frac{\text{rad}}{\text{sec}}$$

For ω = 12 rad/sec

$$r = \frac{12\ \frac{\text{rad}}{\text{sec}}}{6.8\ \frac{\text{rad}}{\text{sec}}} = 1.77$$

and

$$a = \omega_n^2 Y R(1.77, 0.1)$$

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$$= \left(6.8\ \frac{\text{rad}}{\text{sec}}\right)^2 (0.2\ \text{m})(1.77)^2 \sqrt{\frac{1 + [2(0.1)(1.77)]^2}{[1 - (1.77)^2]^2 + [2(0.1)(1.77)]^2}}$$

$$= 14.2\ \frac{\text{m}}{\text{sec}^2}$$

For ω = 32 rad/sec

$$r = \frac{32\ \frac{\text{rad}}{\text{sec}}}{6.8\ \frac{\text{rad}}{\text{sec}}} = 4.71$$

$$a = \omega_n^2 Y R(4.71, 0.1) = 13.2\ \frac{\text{m}}{\text{sec}^2}$$

Problem 4.67 illustrates how the function R(r,ξ) is used to design isolators to provide protection from harmonic base excitation.

4.68 A suspension system is being designed for a 1000 kg vehicle. A first model of the system used in the design process is a spring of stiffness *k* in parallel with a viscous damper of damping coefficient *c*. The model is being analyzed as the vehicle traverses a road with a sinusoidal contour, $y(z) = Y \sin(2\pi\ z/d)$ when the vehicle has a constant horizontal speed *v*. The suspension system is to be designed such that the maximum acceleration of the passengers is 2.5 m/s^2 for all vehicle speeds less than 60 m/s for all reasonable road contours. It is estimated that for such contours, Y < 0.01 m and 0.2 m < d < 1 m. Specify *k* and *c* for such a design.

Given: m = 1000 kg, A_{max}=2.5 m/s^2, v < 60 m/s, Y < 0.01 m, 0.2 m < d < 1 m

Find: *k*, *c*

Solution: If the vehicle is moving at a constant horizontal speed *v*, the time taken to travel a distance *z* is $t = z/d$. Then the vertical displacement to the vehicle is

$$y(t) = Y \sin\left(\frac{2\pi v}{d} t\right)$$

which is a sinusoidal contour with a frequency

$$\omega = \frac{2\pi v}{d}$$

From the given information, it is desired to isolate the passengers from frequencies ranging from 0 to

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$$\omega_{\max} = \frac{2\pi v_{\max}}{d_{\min}} = \frac{2\pi(60\text{ m/s})}{(0.2\text{ m})} = 1.88\times 10^3\text{ rad/s}$$

If X is the amplitude of absolute displacement of the vehicle the acceleration felt by the passengers in the vehicle is

$$A = \omega^2 X$$

Thus the acceleration response is similar to that of an isolation system subject to a frequency squared excitation. In this case define

$$R = r^2 T = \frac{\omega^2}{\omega_n^2}\frac{X}{Y} = \frac{A}{\omega_n^2 Y}$$

If the damping ratio is greater than 0.354 R increases without bound with r. Since the acceleration must be limited over a wide range of frequencies, one solution may be to choose the damping ratio greater to or equal to 0.354. Suppose it is chosen exactly as 0.354. A trial and error procedure is now used to find an appropriate value of the natural frequency. Using r = 10000 leads to A = 2.5 m/s^2. Then

$$\omega_n = \frac{\omega}{r} = 0.1885\text{ rad/s}$$

$$k = m\omega_n^2 = 35.5\text{ N/m}$$

$$c = 2\zeta m\omega_n = 133.4\text{ N}\cdot\text{s/m}$$

Problem 4.68 illustrates the use of R.

4.69 The coefficient of friction between the block and the surface is 0.15. What is the steady–state amplitude?

Given: m = 20 kg, k = 1 × 10^5 N/m, F_0 = 300 N, ω = 80 rad/sec, μ = 0.15, θ = 30°

Find: X

Solution: Assume the friction force is small enough such that the equivalent viscous damping theory of section 3.7 can be used. The normal force developed between the block and the surface is

$$N = mg\cos\theta = (20\text{ kg})\left(9.81\frac{\text{m}}{\text{sec}^2}\right)\cos 30° = 170\text{ N}$$

The friction force developed between the block and the surface is

$$F_f = \mu N = 0.15(170\text{ N}) = 25.48\text{ N}$$

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The force ratio becomes

$$\iota = \frac{F_f}{F_0} = \frac{25.48\,\text{N}}{300\,\text{N}} = 0.085$$

The natural frequency and frequency ratio are

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1\times 10^5\,\frac{\text{N}}{\text{m}}}{20\,\text{kg}}} = 70.7\,\frac{\text{rad}}{\text{sec}}$$

$$r = \frac{\omega}{\omega_n} = \frac{80\,\frac{\text{rad}}{\text{sec}}}{70.7\,\frac{\text{rad}}{\text{sec}}} = 1.13$$

The magnification factor and amplitude are calculated as

$$M_C(1.13,.085) = \sqrt{\frac{1-\left[\frac{4(.085)}{\pi}\right]^2}{\left[1-(1.13)^2\right]^2}} = 3.59$$

$$\frac{m\omega_n^2 X}{F_0} = M_C(1.13,.085)$$

$$X = \frac{M_c F_0}{m\omega_n^2} = 10.8\,\text{mm}$$

Problem 4.69 illustrates calculation of steady-state amplitude for a system with Coulomb damping.

4.70 A 20 kg block is connected to a spring of stiffness 1×10^5 N/m and placed on a surface which makes an angle of 30º with the horizontal. A force of 300 sin 80*t* N is applied to the block. The steady–state amplitude is measured as 10.6 mm. What is the coefficient of friction between the block and the surface?

Given: m = 20 kg, k = 1×10^5 N/m, F_0 = 300 N, ω = 80 rad/sec, X = 10.6 mm, θ = 30°

Find: μ

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Solution: The natural frequency and frequency ratio are calculated as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1\times 10^5 \frac{\text{N}}{\text{m}}}{20\,\text{kg}}} = 70.7\frac{\text{rad}}{\text{sec}}$$

$$r = \frac{\omega}{\omega_n} = \frac{80\frac{\text{rad}}{\text{sec}}}{70.7\frac{\text{rad}}{\text{sec}}} = 1.13$$

The value of the magnification factor is

$$M_c = \frac{m\omega_n^2 X}{F_0} = \frac{(20\,\text{kg})\left(70.7\frac{\text{rad}}{\text{sec}}\right)^2(.0106\,\text{m})}{300\,\text{N}} = 3.533$$

Thus from eq. (3.85)

$$3.533 = \sqrt{\frac{1-\left(\frac{4\iota}{\pi}\right)^2}{\left[1-(1.13)^2\right]^2}}$$

Which is solved yielding

$$\iota = 0.1157$$

Then

$$\iota = \frac{\mu mg\cos 30°}{F_0}$$

$$\mu = \frac{\iota F_0}{mg\cos 30°} = 0.204$$

Problem 4.70 illustrates calculation of the steady-state amplitude of a one-degree-of-freedom system subject to a single frequency excitation and Coulomb damping.

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4.71 A 40 kg block is connected to a spring of stiffness 1×10^5 N/m and slides on a surface with a coefficient of friction of 0.2. When a harmonic force of frequency 60 rad/sec is applied to the block, the resulting amplitude of steady–state vibrations is 3 mm. What is the amplitude of the excitation?

Given: m = 40 kg, k = 1×10^5 N/m, ω = 60 rad/sec, μ = 0.2, X = 3 mm

Find: F_0

Solution: The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1\times 10^5 \frac{\text{N}}{\text{m}}}{40\,\text{kg}}} = 50\frac{\text{rad}}{\text{sec}}$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{60\frac{\text{rad}}{\text{sec}}}{50\frac{\text{rad}}{\text{sec}}} = 1.2$$

The friction force is

$$F_f = \mu mg = (0.2)(40\,\text{kg})\left(9.81\frac{\text{m}}{\text{sec}^2}\right) = 78.4\ \text{N}$$

The value of the magnification factor is

$$M_c = \frac{m\omega_n^2 X}{F_0} = \frac{(40\,\text{kg})\left(50\frac{\text{rad}}{\text{sec}}\right)^2(0.003\,\text{m})}{F_0} = \frac{300}{F_0} \qquad (1)$$

The magnification factor is also equal to

$$M = \sqrt{\frac{1-\left[\frac{4(78.5\,N)}{\pi F_0}\right]^2}{\left[1-(1.2)^2\right]^2}} \qquad (2)$$

Equating M from eqs.(l) and (2) and solving for F_0 leads to

$$F_0 = 165.5\ \text{N}$$

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Problem 4.71 illustrates the forced response of a system with Coulomb damping.

4.72 Determine the steady–state amplitude of motion of the 5-kg block. The coefficient of friction between the block and surface is 0.11.

FIGURE P4.72

Given: m = 5 kg, μ = 0.11, k = 2 × 10^5 N/m, Y = 2.7 × 10^{-4} m, ω = 180 rad/s

Find**:** X

Solution: Free-body diagrams at an arbitrary instant are shown below.

Summing forces on the free-body diagrams leads to

$$\sum F_{ext} = \sum F_{eff}$$
$$-k(x-y) \pm \mu mg = m\ddot{x}$$
$$m\ddot{x} + kx = \pm\mu mg + kY \sin \omega t$$

The natural frequency and frequency ratio for the system are

$$\omega_n = \sqrt{\frac{k}{m}} = 200 \text{ rad/s}$$
$$\text{r} = \frac{\omega}{\omega_\text{n}} = \frac{180 \text{ rad/s}}{200 \text{ rad/s}} = 0.9$$

The force ratio is

$$\iota = \frac{\mu mg}{kY} = \frac{(0.11)(5 \text{ kg})(9.81 \text{ m/s}^2)}{(2\times10^5 \text{ N/m})(2.7\times10^{-4} \text{ m})} = 0.10$$

The steady-state amplitude of the block is calculated from

$$\frac{m\omega_n X}{kY} = \frac{kX}{kY} = \frac{X}{Y} = M_c(0.9, 0.1)$$

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where

$$M_c(0.9,0.1) = \sqrt{\frac{1-\left(\frac{4\iota}{\pi}\right)^2}{(1-r^2)^2}} = 5.22$$

The steady-state amplitude is

$$X = 5.22Y = 5.22(2.7\times10^{-4}\text{ m}) = 1.4\times10^{-3}\text{ m}$$

Problem 4.72 illustrates the determination of the steady-state amplitude of a system with Coulomb damping subject to a harmonic excitation.

4.73 Determine the steady–state amplitude of motion of the 5-kg block. The coefficient of friction between the block and surface is 0.11.

FIGURE P4.73

Given: m = 5 kg, μ = 0.11, k = 1 × 10^5 N/m, Y = 3.2 × 10^{-4} m, ω = 220 rad/s

Find**:** X

Solution: Free-body diagrams of the system at an arbitrary instant are shown below.

Summing forces on the free-body diagrams leads to

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$$\sum F_{ext} = \sum F_{eff}$$
$$-kx - k(x - y) \pm \mu mg = m\ddot{x}$$
$$m\ddot{x} + 2kx = \pm\mu mg + kY \sin \omega t$$

The natural frequency and frequency ratio for the system are

$$\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2(1\times10^5 \text{ N/m})}{5 \text{ kg}}} = 200 \text{ rad/s}$$
$$r = \frac{\omega}{\omega_n} = \frac{220 \text{ rad/s}}{200 \text{ rad/s}} = 1.1$$

The force ratio is

$$\iota = \frac{\mu mg}{kY} = \frac{(0.11)(5 \text{ kg})(9.81 \text{ m/s}^2)}{(1\times10^5 \text{ N/m})(3.2\times10^{-4} \text{ m})} = 0.169$$

The steady-state amplitude of the block is calculated from

$$\frac{m\omega_n^2 X}{kY} = M_c(1.1, 0.169)$$
$$\frac{2kX}{kY} = \sqrt{\frac{1 - \left(\frac{4(0.169)}{\pi}\right)^2}{[1 - (1.1)^2]^2}}$$

$$X = \frac{Y}{2}(4.65) = \frac{1}{2}(3.2\times10^{-4} \text{ m})(4.65) = 7.44\times10^{-4} \text{ m}$$

Problem 4.73 illustrates the determination of the steady-state amplitude for a system with Coulomb damping subject to a harmonic excitation.

4.74 Use the equivalent viscous damping approach to determine the steady–state response of a system subject to both viscous damping and Coulomb damping.

Given: system with viscous damping and Coulomb damping

Find: x(t)

Solution: Consider a one-degree-of-freedom mass-spring

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system subject to both viscous damping and Coulomb damping. Free body diagrams for the system at an arbitrary instant are shown below.

Summing forces on the free body diagrams leads to the following differential equation

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t + \begin{cases} \mu mg, \dot{x} < 0 \\ -\mu mg, \dot{x} > 0 \end{cases} \tag{1}$$

The total damping force is the sum of the viscous damping force and the Coulomb damping force. The equivalent viscous damping coefficient is calculated by requiring the energy dissipated over one cycle of motion by the total damping force, when the system executes harmonic motion of frequency ω and amplitude X, to the energy dissipated over one cycle of motion by an viscous damping force of an equivalent damping coefficient. Thus when the equivalent viscous damping coefficient is calculated,

$$c_{eq.} = c + \frac{4\mu mg}{\pi \omega X} \tag{2}$$

Using this method of linearization, eq. (1) is replaced by the approximate equation

$$m\ddot{x} + \left(c + \frac{4\mu mg}{\pi \omega X} \right) \dot{x} + kx = F_0 \sin \omega t \tag{3}$$

or

$$\ddot{x} + 2\zeta_{eq.} \omega_n \dot{x} + \omega_n^2 x = \frac{F_0}{m} \sin \omega t \tag{4}$$

where

$$\zeta_{eq.} = \frac{c}{2m\omega_n} + \frac{2\iota}{\pi Mr} \tag{5}$$

The solution of eq.(4) is

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$$x(t) = X\sin(\omega t - \phi)$$

where

$$X = \frac{MF_0}{m\omega_n^2}$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta_{eq.}r}{1-r^2}\right)$$

The magnification factor is given by

$$M = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta_{eq.}r)^2}} \tag{6}$$

Substituting eq. (5) into eq. (6) leads to

$$M = \frac{1}{\sqrt{(1-r^2)^2 + \left[2r\left(\frac{c}{2m\omega_n} + \frac{2\iota}{\pi Mr}\right)\right]^2}}$$

Squaring and rearranging leads to

$$\left[(1-r^2)^2 + \left(\frac{cr}{2m\omega_n}\right)^2\right]M^4 + \frac{8\iota cr}{\pi m\omega_n}M^3 + \left(\frac{4\iota}{\pi}\right)M^2 - 1 = 0 \tag{7}$$

Let

$$\zeta_1 = \frac{c}{2m\omega_n}$$

Then eq. (7) becomes

$$\left[(1-r^2)^2 + (\zeta_1 r)^2\right]M^4 + \frac{16\zeta_1 \iota r}{\pi}M^3 + \left(\frac{4\iota}{\pi}\right)^2 M^2 - 1 = 0 \tag{8}$$

Equation (8) is a quartic equation whose solution yields the appropriate value of $M(r, \zeta_1, \xi)$.

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Problem 4.74 illustrates the approximate steady-state response of a system subject to both viscous damping and Coulomb damping when excited by a single frequency harmonic excitation.

4.75 The area under the hysteresis curve for a particular helical coil spring is 0.2 N · m when subject to a 350 N load. The spring has a stiffness of 4×10^5 N/m. If a 44 kg block is hung from the spring and subject to an excitation force of 350 sin 35*t* N, what is the amplitude of the resulting steady–state oscillations?

Given: $\Delta E = 0.2$ N · m, $k = 4 \times 10^5$ N/m, F = 350 N, m = 44 kg, $F_0 = 350$ N, $\omega = 35$ rad/sec

Find: X

Solution: The hysteretic damping coefficient is related to the area under the hysteresis curve by

$$\Delta E = \pi k h X^2$$
$$h = \frac{\Delta E}{\pi k X^2}$$

The displacement is given by

$$X = \frac{F}{k} = \frac{350\text{N}}{4 \times 10^5 \frac{\text{N}}{\text{m}}} = 8.75 \times 10^{-4} m$$

Substituting given values the hysteretic damping coefficient is

$$h = \frac{0.2\,\text{N} \cdot \text{m}}{\pi\left(4 \times 10^5 \frac{\text{N}}{\text{m}}\right)\left(8.75 \times 10^{-4}\,\text{m}\right)^2} = 0.208$$

For the system at hand

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$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times10^5\,\frac{\text{N}}{\text{m}}}{44\,\text{kg}}} = 95.35\,\frac{\text{rad}}{\text{sec}}$$

$$r = \frac{\omega}{\omega_n} = \frac{35\,\frac{\text{rad}}{\text{sec}}}{95.35\,\frac{\text{rad}}{\text{sec}}} = 0.367$$

The magnification factor is calculated as

$$M_h = \frac{1}{\sqrt{\left(1-r^2\right)^2 + h^2}} = \frac{1}{\sqrt{\left[1-(0.367)^2\right]^2 + (0.208)^2}} = 1.124$$

The steady-state amplitude is

$$X = \frac{MF_0}{m\omega_n^2} = \frac{1.124(350\,\text{N})}{(44\,\text{kg})\left(95.35\,\frac{\text{rad}}{\text{sec}}\right)^2} = 0.983\,\text{mm}$$

Problem 4.75 illustrates calculation of the hysteretic damping coefficient and steady-state amplitude of a system with hysteretic damping.

4.76 When a free-vibration test is run on the system of Figure P4.76, the ratio of amplitudes on successive cycles is 2.8 to 1. Determine the response of the pump when it has an excitation force of magnitude 3000 N at a frequency of 2000 rpm. Assume the damping is hysteretic.

FIGURE P4.76

Given: m = 215 kg, E = 200 × 10 ^{9} N/m^2, I = 2.4 × 10^{-4} m^4, F_0 = 3000 N, ω = 2000 rpm, ratio of amplitudes on successive cycles is 2.8 to 1, L = 3.1 m

Find**:** x(t)

Solution: The stiffness of the beam is

$$k = \frac{3EI}{L^3} = 4.83\times10^6\ \text{N/m}$$

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The natural frequency and frequency ratio are

$$\omega_n = \sqrt{\frac{k}{m}} = 149.9 \text{ rad/s}$$

$$\mathrm{r} = \frac{\omega}{\omega_{\mathrm{n}}} = \frac{(2000 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{149.9 \text{ rad/s}} = 1.40$$

The information about amplitude of successive cycles is used to determine the logarithmic decrement

$$\delta = \ln\left(\frac{X_1}{X_2}\right) = \ln\left(\frac{2.8}{1}\right) = 1.03$$

The hysteretic damping coefficient is calculated as

$$h = \frac{\delta}{\pi} = 0.328$$

The steady-state response of a system with hysteretic damping subject to a harmonic excitation is

$$x(t) = X_h \sin(\omega t - \phi_h)$$

where

$$\frac{m\omega_n^2 X_h}{F_0} = M_h(1.40, 0.328)$$

$$X_h = \frac{F_0}{m\omega_n^2} \frac{1}{\sqrt{[1-(1.40)^2]^2 + (0.328)^2}}$$

$$X = \frac{3000 \text{ N}}{(215 \text{ kg})(149.9 \text{ rad/s})^2}(0.994)$$

$$X = 6.17 \times 10^{-4} \text{ m}$$

and

$$\varphi_h = \tan^{-1}\left(\frac{h}{1-r^2}\right) = \tan^{-1}\left(\frac{0.328}{1-(1.40)^2}\right)$$

$$\varphi_h = -0.329 \text{ rad}$$

Thus the steady-state response is

$$x(t) = 6.17 \times 10^{-4} \sin(209.4t + 0.329) \text{ m}$$

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Problem 4.76 illustrates the steady-state response of a system with hysteretic damping.

4.77 When a free-vibration test is run on the system of Figure P4.76, the ratio of amplitudes on successive cycles is 2.8 to 1. When operating, the engine has a rotating unbalance of magnitude 0.25 kg · m. The engine operates at speeds between 500 and 2500 rpm. For what value of ω within the operating range will the pump's steady–state amplitude be largest? What is the maximum amplitude? Assume the damping is hysteretic.

FIGURE P4.76

Given: m = 215 kg, E = 200 × 10^9 N/m^2, I = 2.4 × 10^{-4} m^4, m_0e = 0.25 kg · m, 500 rpm < ω < 2500 rpm, ratio of amplitudes on successive cycles is 2.8 to 1, L = 3.1 m

Find**:** ω_m, X_m

Solution: The stiffness of the beam is

$$k = \frac{3EI}{L^3} = 4.83 \times 10^6 \text{ N/m}$$

The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = 149.9 \text{ rad/s}$$

The information about amplitude of successive cycles is used to determine the logarithmic decrement

$$\delta = \ln\left(\frac{X_1}{X_2}\right) = \ln\left(\frac{2.8}{1}\right) = 1.03$$

The hysteretic damping coefficient is calculated as

$$h = \frac{\delta}{\pi} = 0.328$$

The analysis can be extended to frequency squared excitations

$$\frac{mX}{m_0 e} = \Lambda_h(r, h)$$

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where, as in the case for the magnification factor, to use $\Lambda(r, \zeta)$ to determine $\Lambda_h(r,h)$, ζ is replaced by h/2r leading to

$$\Lambda_h(r,h) = \frac{r^2}{\sqrt{(1-r^2)^2 + h^2}}$$

For a fixed h the maximum of Λ_h is obtained by

$$\frac{d\Lambda_h^2}{dr} = 0 = \frac{4r^3\left[(1-r^2)^2 + h^2\right] - r^4\left[2(1-r^2)(-2r)\right]}{(1-r^2)^2 + h^2}$$

$$0 = (1-r^2)^2 + h^2 + r^2(1-r^2)$$

$$0 = 1 + h^2 - r^2$$

$$r = \sqrt{1+h^2}$$

Hence the maximum steady-state amplitude occurs for a frequency ratio of

$$r = \sqrt{1+h^2} = \sqrt{1+(0.328)^2} = 1.052$$

which corresponds to a frequency of

$$\omega = r\omega_n = 1.052(149.9 \text{ rad/s}) = 157.8 \text{ rad/s} = 1507 \text{ rpm}$$

The maximum steady-state amplitude is

$$X_m = \frac{m_0 e}{m}\Lambda_h(1.052, 0.328)$$

$$X_m = \frac{0.25 \text{ kg - m}}{215 \text{ kg}} \frac{(1.052)^2}{\sqrt{\left[1-(1.052)^2\right]^2 + (0.328)^2}}$$

$$X_m = 0.0037 \text{ m}$$

Problem 4.77 illustrates solution of frequency squared excitation problems for systems with hysteretic damping.

4.78 When the pump at the end of the beam of Figure P4.76 operates at 1860 rpm, it is noted that the phase angle between the excitation and response is 18º. What is the steady–state amplitude of the pump if it has a rotating unbalance of 0.8 kg · m and operates at 1860 rpm? Assume hysteretic damping.

Given: m = 215 kg , E = 200 × 10^9 N/m^2, I = 2.4 × 10^{-4} m^4, m_0e = 0.8 kg · m,

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ω = 1860 rpm, L = 3.1 m, ϕ_h=18°

Find**:** X

Solution: The stiffness of the beam is

$$k = \frac{3EI}{L^3} = 4.83 \times 10^6 \text{ N/m}$$

The natural frequency and frequency ratio are

$$\omega_n = \sqrt{\frac{k}{m}} = 149.9 \text{ rad/s}$$

$$\text{r} = \frac{\omega}{\omega_\text{n}} = \frac{(1860 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{149.9 \text{ rad/s}} = 1.30$$

The phase angle for hysteretic damping is

$$\phi_h = \tan^{-1}\left(\frac{h}{1-r^2}\right)$$

Note that since r > 1 for this situation it is assumed the phase angle is actually negative. Using the given information the hysteretic damping coefficient is calculated from

$$h = (1-r^2)\tan\phi_h = [1-(1.30)^2]\tan(-18°) = 0.224$$

The steady–state amplitude is calculated as

$$X = \frac{m_0 e}{m}\Lambda_h(1.30, 0.224)$$

$$X = \frac{0.8 \text{ kg - m}}{215 \text{ kg}} \frac{(1.30)^2}{\sqrt{\left[1-(1.30)^2\right]^2 + (0.224)^2}}$$

$$X = 8.67 \text{ mm}$$

Problem 4.78 illustrates the determination of the steady–state amplitude for a system with hysteretic damping subject to a frequency squared excitation and the use of the phase angle in determination of the hysteretic damping coefficient.

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4.79 A schematic of a single-cylinder engine mounted on springs and a viscous damper is shown in Figure P4.79. The crank rotates about O with a constant speed ω. The connecting rod of mass m_r connects the crank and the piston of mass m_p such that the piston moves in a vertical plane. The center of gravity of the crank is at its axis of rotation.

(a) Derive the differential equation governing the absolute vertical displacement of the engine including the inertia forces of the crank and piston, but ignoring forces due to combustion. Use an exact expression for the inertia forces in terms of m_r, m_p, ω, the crank length r, and the connecting rod length ℓ.

(b) Since $F(t)$ is periodic, a Fourier series representation can be used. Set up, but do not evaluate, the integrals required for a Fourier series expansion for $F(t)$.

FIGURE P4.79

(c) Assume r/ ℓ << 1. Rearrange $F(t)$ and use a binomial expansion such that

$$F(t) = \sum_{i=1}^{\infty} \alpha_i \left(\frac{r}{\ell}\right)^i \tag{1}$$

(d) Truncate the preceding series after $i = 3$. Use trigonometric identities to approximate

$$F(t) \approx b_1 cos\omega t + b_2 cos2\omega t + b_3 cos3\omega t \tag{2}$$

(e) Find an approximation to the steady–state form of $x(t)$

Given: m, m_r, m_p, r, ℓ, k, c, ω

Find: Differential equation, $F(t)$ in the form of eq.(2), $x(t)$

Solution: (a) Let y(t) represent the displacement of the piston relative to the engine, and let $y_G(t)$. Represent the vertical displacement of the mass center of the connecting rod relative to the engine. Consider the free body diagrams of the engine at an arbitrary instant.

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Summing forces in the vertical direction

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$
$$-c\dot{x} - kx = \left(m - m_p - m_r\right)\ddot{x} + m_p\left(\ddot{x} + \ddot{y}\right) + m_r\left(\ddot{x} + \ddot{y}_G\right)$$
$$m\ddot{x} + c\dot{x} + kx = -m_p\ddot{y} - m_r\ddot{y}_G$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -\frac{m_p}{m}\ddot{y} - \frac{m_r}{m}\ddot{y}_G$$

Referring to the geometry in the figure to the right

$$r\sin\theta = \ell\sin\phi \qquad (3)$$

$$y = r\cos\theta + \ell\cos\phi \qquad (4)$$

$$y_G = y - \frac{\ell}{2}\cos\phi \qquad (5)$$

Note that since only steady-state is considered and the crank speed is constant,

$$\theta = \omega t \qquad (6)$$

Differentiating eq. (3) twice with respect to time

$$r\omega\cos\theta = \ell\dot{\phi}\cos\phi$$
$$\dot{\phi} = \frac{r}{\ell}\omega\frac{\cos\theta}{\cos\phi} \qquad (7)$$

$$-r\omega^2\sin\theta = -\ell\dot{\phi}^2\sin\phi + \ell\ddot{\phi}\cos\phi$$
$$\ddot{\phi} = \frac{1}{\ell\cos\phi}\left[\ell\omega^2\left(\frac{r}{\ell}\right)^2\cos^2\theta\frac{\sin\phi}{\cos^2\phi} - r\omega^2\sin\theta\right] \qquad (8)$$

Differentiating eq. (4) twice with respect to time, and using eqs. (7) and (8) leads to

$$\dot{y} = -\ell\dot{\phi}\sin\phi - r\omega\sin\theta$$
$$\ddot{y} = -\ell\ddot{\phi}\sin\phi - \ell\dot{\phi}^2\cos\phi - r\omega^2\cos\theta$$

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$$\ddot{y} = -\ell\omega^2\left[\frac{r}{\ell}\cos\theta + \left(\frac{r}{\ell}\right)^2 \frac{\cos^2\theta - \sin^2\theta}{\cos\phi} + \left(\frac{r}{\ell}\right)^2 \frac{\sin^2\theta\cos^2\theta}{\cos^3\phi}\right]$$
$$= -\ell\omega^2\left[\frac{r}{\ell}\cos\theta + \left(\frac{r}{\ell}\right)^2 \frac{\cos 2\theta}{\cos\phi} + \left(\frac{r}{\ell}\right)^4 \frac{\sin^2 2\theta}{4\cos^3\phi}\right] \tag{9}$$

Differentiating eq. (5) twice with respect to time and using eqs. (7) and (8) leads to

$$\dot{y}_G = \dot{y} + \frac{\ell}{2}\dot{\phi}\sin\phi$$
$$\ddot{y}_G = \ddot{y} + \frac{\ell}{2}\ddot{\phi}\sin\phi + \frac{\ell}{2}\dot{\phi}^2\cos\phi \tag{10}$$
$$\ddot{y}_G = \ddot{y} + \frac{\ell}{2}\omega^2\left[\left(\frac{r}{\ell}\right)^2 \frac{\cos 2\theta}{\cos\phi} + \left(\frac{r}{\ell}\right)^4 \frac{\sin^2 2\theta}{4\cos^3\phi}\right]$$

From eq. (3) the excitation, force is

$$F(t) = -m_p\ddot{y} - m_r\ddot{y}_G$$
$$= \ell\omega^2\left[m_p\cos\theta\frac{r}{\ell} + \left(m_p + \frac{m_r}{2}\right)\frac{\cos 2\theta}{\cos\phi}\left(\frac{r}{\ell}\right)^2 + \left(m_p + \frac{m_r}{2}\right)\frac{\sin^2 2\theta}{\cos^3\phi}\right] \tag{11}$$

It is noted that

$$\cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-\left(\frac{r}{\ell}\right)^2\sin^2\theta}$$

(b) F(t) is periodic of period 2π/ω and thus a Fourier series representation could be used for F(t). If this is tried

$$F(t) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty}\left(a_\ell\cos\frac{2\pi\ell}{\omega}t + b_\ell\sin\frac{2\pi\ell}{\omega}t\right)$$

where

$$a_0 = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} F(t)\,dt$$

$$a_\ell = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} F(t)\cos\frac{2\pi\ell}{\omega}t\,dt$$

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$$b_\ell = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} F(t)\sin\frac{2\pi\ell}{\omega}t\,dt$$

(c) An alternative to the formal Fourier series development is to use binomial expansions and trigonometric identities. To this end

$$\frac{1}{\cos\phi} = \left[1-\left(\frac{r}{\ell}\right)^2\sin^2\theta\right]^{-\frac{1}{2}} = 1+\frac{1}{2}\left(\frac{r}{\ell}\right)^2\sin^2\theta+\ldots \tag{12}$$

$$\frac{1}{\cos^3\phi} = \left[1-\left(\frac{r}{\ell}\right)^2\sin^2\theta\right]^{-\frac{3}{2}} = 1+\frac{3}{2}\left(\frac{r}{\ell}\right)^2\sin^2\theta+\ldots \tag{13}$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \tag{14}$$

Substituting eqs. (12)-(14) in eq.(11), collecting terms on like powers of r/ℓ and truncating after $(r/\ell)^4$ leads to

$$F(t) = \ell\omega^2\left[m_p\cos\omega t\frac{r}{\ell}+\left(m_p+\frac{m_r}{2}\right)\cos 2\omega t + \left(m_p+\frac{m_r}{2}\right)\left(\frac{3}{8}+\frac{1}{4}\cos 2\omega t+\frac{1}{8}\cos 4\omega t\right)\left(\frac{r}{\ell}\right)^4+\ldots\right]$$

or

$$F(t) = \ell\omega^2\left\{\frac{3}{8}\left(m_p+\frac{m_r}{2}\right)\left(\frac{r}{\ell}\right)^4+m_p\frac{r}{\ell}\cos\omega t+\left(m_p+\frac{m_r}{2}\right)\left[\left(\frac{r}{\ell}\right)^2+\frac{1}{4}\left(\frac{r}{\ell}\right)^4\right]\cos 2\omega t + \frac{1}{8}\left(m_p+\frac{m_r}{2}\right)\left(\frac{r}{\ell}\right)^4\cos 4\omega t+\ldots\right\}$$

(d) The system response is given by

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$$x(t)=\frac{\ell}{m}\frac{\omega^2}{\omega_n^2}\left\{\frac{3}{8}\left(m_p+\frac{m_r}{2}\right)\left(\frac{r}{\ell}\right)^4+\frac{r}{\ell}m_pM_1\cos(\omega t-\phi_1)\right.$$
$$\left.+\left(m_p+\frac{m_r}{2}\right)\left[\left(\frac{r}{\ell}\right)^2+\frac{1}{4}\left(\frac{r}{\ell}\right)^4\right]M_2\cos(2\omega t-\phi_2)+\frac{1}{8}\left(m_p+\frac{m_r}{2}\right)\left(\frac{r}{\ell}\right)^4M_4\cos(4\omega t-\phi_4)+\ldots\right\}$$

where

$$r_i=\frac{i\omega}{\omega_n}$$
$$M_i=\frac{1}{\sqrt{\left(1-r_i^2\right)^2+\left(2\zeta r_i\right)^2}}$$
$$\phi_i=\tan^{-1}\left(\frac{2\zeta r_i}{1-r_i^2}\right)$$

Problem 4.79 illustrates (a) determination of the excitation provided to a one-degree-of-freedom system by a slider-crank mechanism (b) development of the harmonic form of the excitation using kinematics and the binomial expansion, and (c) response due to a multi-frequency excitation.

4.80 Using the results of Problem 4.79, determine the maximum steady–state response of a single-cylinder engine with m_r = 1.5 kg, m_p = 1.7 kg, r = 5.0 cm, ℓ = 15.0 cm, ω = 800 rpm, $k = 1 \times 10^5$ N/m, c = 500 N · sec/m, and total mass 7.2 kg.

Given: single cylinder engine, m_r = 1.5 kg, m_p = 1.7 kg, r = 5.0 cm, ℓ = 15 cm, ω = 800 rpm, $k = 1 \times 10^5$ N/m, c = 500 N · sec/m, m = 7.2 kg

Find: $x_{max.}$

Solution: The approximate response of the engine, as obtained in problem 3.66 is

$$x(t)=\frac{\ell}{m}\frac{\omega^2}{\omega_n^2}\left\{\frac{3}{8}\left(m_p+\frac{m_r}{2}\right)\left(\frac{r}{\ell}\right)^4+\frac{r}{\ell}m_rM_1\cos(\omega t-\phi_1)\right.$$
$$\left.+\left(m_p+\frac{m_r}{2}\right)\left[\left(\frac{r}{\ell}\right)^2+\frac{1}{4}\left(\frac{r}{\ell}\right)^4\right]M_2\cos(2\omega t-\phi_2)+\frac{1}{8}\left(m_p+\frac{m_r}{2}\right)\left(\frac{r}{\ell}\right)^4M_4\cos(4\omega t-\phi_4)+\ldots\right\}$$

Calculation of parameters for the values given leads to

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$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1\times10^5\,\frac{\text{N}}{\text{m}}}{7.2\,\text{kg}}} = 117.9\,\frac{\text{rad}}{\text{sec}}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{500\,\frac{\text{N}-\text{sec}}{\text{m}}}{2(7.2\,\text{kg})\left(117.9\,\frac{\text{rad}}{\text{sec}}\right)} = 0.294$$

$$r_1 = \frac{\omega}{\omega_n} = \frac{\left(800\,\frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\,\text{rad}}{1\,\text{rev}}\right)\left(\frac{1\,\text{min}}{60\,\text{sec}}\right)}{117.9\,\frac{\text{rad}}{\text{sec}}} = 0.711$$

Also,

$$m_p + \frac{m_r}{2} = 2.45\,kg$$

$$\frac{r}{\ell} = \frac{1}{3}$$

The magnification factors are calculated as

$$M_1 = \frac{1}{\sqrt{\left[1-(0.711)^2\right]^2 + \left[2(.294)(0.711)\right]^2}} = 1.544$$

$$M_2 = \frac{1}{\sqrt{\left[1-(1.422)^2\right]^2 + \left[2(.294)(1.422)\right]^2}} = 0.757$$

$$M_4 = \frac{1}{\sqrt{\left[1-(2.844)^2\right]^2 + \left[2(.294)(2.844)\right]^2}} = 0.1373$$

The phase angles are

$$\phi_1 = \tan^{-1}\left(\frac{2(.294)(.711)}{1-(.711)^2}\right) = 0.702\,\text{rad}$$

$$\phi_2 = \tan^{-1}\left(\frac{2(.294)(1.422)}{1-(1.422)^2}\right) = -0.686\,\text{rad}$$

$$\phi_3 = \tan^{-1}\left(\frac{2(.294)(2.844)}{1-(2.844)^2}\right) = -0.232\,\text{rad}$$

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Substituting the calculated values into eq. (1) leads to

$$x(t) = 0.119 + 8.13\cos(83.78t - 0.702) + 2.41\cos(167.56t + 0.686) + 0.005\cos(335.0t + 0.232)\,\text{mm}$$

An approximation of the maximum displacement is

$$x_{\max.} < (0.119\text{ mm}) + (8.13\text{ mm}) + (2.41\text{ mm}) + (0.005\text{ mm}) = 10.664\text{ mm}$$

Problem 4.80 illustrates the calculations in the determination of the response due to a multi frequency excitation.

4.81 A 5-kg rotor-balancing machine is mounted to a table through an elastic foundation of stiffness 10,000 N/m and damping ratio 0.04. Use of a transducer reveals that the table's vibration has two main components: an amplitude of 0.8 mm at a frequency of 140 rad/s and an amplitude of 1.2 mm at a frequency of 200 rad/s. Determine the steady–state response of the rotor balancing machine.

Given: m = 5 kg, k = 10,000 N/m, $\zeta = 0.04$, $Y_1 = 0.8$ mm, $\omega_1 = 140$ rad/s, $X_2 = 1.2$ mm, $\omega_2 = 200$ rad/s

Find**:** x(t)

Solution: The natural frequency and frequency ratios are

$$\omega_n = \sqrt{\frac{k}{m}} = 44.72\text{ rad/s}$$

$$r_1 = \frac{\omega_1}{\omega_n} = \frac{140\text{ rad/s}}{44.72\text{ rad/s}} = 3.13$$

$$r_2 = \frac{\omega_2}{\omega_n} = \frac{200\text{ rad/s}}{44.72\text{ rad/s}} = 4.47$$

The steady–state response of the system is

$$x(t) = X_1\sin(\omega_1 t - \phi_1) + X_2\sin(\omega_2 t - \phi_2)$$

where the steady-state amplitudes are

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$$X_1 = Y_1 T(r_1, \zeta) = 0.0008\text{ m}\sqrt{\frac{1+[2(0.04)(3.13)]^2}{[1-(3.13)^2]^2+[2(0.04)(3.13)]^2}} = 9.37\times10^{-5}\text{ m}$$

$$X_2 = Y_2 T(r_2, \zeta) = 0.0012\text{ m}\sqrt{\frac{1+[2(0.04)(4.47)]^2}{[1-(4.47)^2]^2+[2(0.04)(4.47)]^2}} = 6.71\times10^{-5}\text{ m}$$

The phase angles are

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r_1}{1-r_1^2}\right) = \tan^{-1}\left(\frac{2(0.04)(3.13)}{1-(3.13)^2}\right) = -0.0285\text{ rad}$$

$$\phi_2 = \tan^{-1}\left(\frac{2\zeta r_2}{1-r_2^2}\right) = \tan^{-1}\left(\frac{2(0.04)(4.47)}{1-(4.47)^2}\right) = -0.0188\text{rad}$$

Thus the steady-state response of the rotor-balancing machine is

$$x(t) = 9.37\times10^{-5}\sin(140t+0.0285) + 6.71\times10^{-5}\sin(200t+0.0188)\text{ m}$$

Problem 4.81 illustrates the steady-state response of a system subject to a two-frequency base excitation.

4.82 During operation a 100-kg press is subject to the periodic excitations shown. The press is mounted on an elastic foundation of stiffness 1.6×10^5 N/m and damping ratio 0.2. Determine the steady–state response of the press and approximate the maximum displacement from equilibrium. Each excitation is shown over one period.

FIGURE P4.82

Given: m = 100 kg, $\omega_n = \sqrt{\frac{k}{m}}$ =40 rad/sec, ζ = 0.2, F(t)

Find: x(t), $x_{max.}$

Solution: Since the response is periodic, it is first necessary to develop its Fourier series representation. It is noted that the given excitation is an odd function of period 0.3 sec. Hence,

$$a_\ell = 0,\ \ \ell = 0,1,2,\ldots$$

$$\omega_\ell = \frac{2\pi\ell}{T} = \frac{20\pi\ell}{3}$$

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The mathematical form of F(t) is

$$F(t)=\begin{cases} F_0, & 0 \le t \le 0.1 \text{ sec} \\ 0, & 0.1 \text{ sec} < t \le 0.2 \text{ sec} \\ -F_0, & 0.2 \text{ sec} < t < 0.3 \text{ sec} \end{cases}$$

where F_0=10,000 N. The Fourier sine coefficients are given by

$$b_\ell = \frac{2}{0.3 \text{ sec}} \int_0^{0.3 \text{ sec}} F(t) \sin \frac{20\pi\ell}{3} t \, dt$$

$$= \frac{20}{3} \int_0^{0.1 \text{ sec}} F_0 \sin \frac{20\pi\ell}{3} t \, dt + \int_{0.2 \text{ sec}}^{0.3 \text{ sec}} (-F_0) \sin \frac{20\pi\ell}{3} t dt$$

$$= -\frac{F_0}{\pi\ell} \left(\cos \frac{4\pi\ell}{3} + \cos \frac{2\pi\ell}{3} \right)$$

The response of the system is given by

$$x(t) = \frac{1}{m\omega_n^2} \sum_{i=1}^{\infty} b_\ell M_\ell \sin\left(\frac{20\pi\ell}{3} t - \phi_\ell\right)$$

where

$$M_\ell = \frac{1}{\sqrt{(1-r_\ell^2)^2 + (0.4 r_\ell)^2}}$$

$$\phi_\ell = \tan^{-1}\left(\frac{0.4 r_\ell}{1-r_\ell^2}\right)$$

$$r_\ell = \frac{\omega_\ell}{\omega_n} = \frac{20\pi\ell}{3(40)} = \frac{\pi\ell}{6}$$

The Fourier coefficients, frequency ratios, magnification factors, and response amplitudes for the first 10 terms are given in the table below. Note that

$$X_\ell = \frac{b_\ell M_\ell}{m\omega_n^2}$$

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ℓ	b_ℓ (N)	r_ℓ	M_ℓ	X_ℓ (m)	ϕ_ℓ (rad)
1	3183.	0.5236	1.3237	0.02633	0.2809
2	1591.	1.0472	2.3264	0.02314	-1.3449
3	-2122.	1.5708	0.6265	-0.00831	-0.4046
4	795.8	2.0944	0.2867	0.00143	-0.2425
5	636.6	2.6180	0.1682	0.00070	-0.1770
6	-1061.	3.1416	0.1163	-0.00074	-0.1407
7	454.7	3.6652	0.0799	0.00023	-0.1174
8	397.8	4.1188	0.0601	0.00015	-0.1010
9	-707.3	4.7124	0.0470	-0.00021	-0.08866
10	318.3	5.2360	0.0377	0.00008	-0.0791

An upper bound on the maximum displacement is obtained by

$$x_{\max.} < \sum_{i=1}^{10} |X_i| = 0.06113 \text{ m}$$

Problem 4.82 illustrates calculation of the Fourier series for an odd function and determination of the response of a one-degree-of-freedom system to a periodic excitation.

4.83 During operation a 100-kg press is subject to the periodic excitations shown. The press is mounted on an elastic foundation of stiffness 1.6×10^5 N/m and damping ratio 0.2. Determine the steady–state response of the press approximate the maximum displacement from equilibrium. Each excitation is shown over one period.

FIGURE P4.83

Given: m = 100 kg, ω_n = 40 rad/sec, ζ = 0.2, F(t)

Find: x(t), $x_{max.}$

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Solution: F(t) is periodic of period T = 0.2 sec. F(t) is neither even or odd. The mathematical definition of F(t) is

$$F(t)=\begin{cases} F_0, 0<t<0.1\sec \\ 0, 0.1\sec \le t \le 0.2\sec \end{cases}$$

where $F_0 = 10000$ N. The Fourier coefficients are calculated as

$$\begin{aligned} a_0 &= \frac{2}{0.2\sec}\int_0^{0.2\sec} F(t)dt \\ &= 10\int_0^{0.1\sec} F_0\,dt \\ &= F_0 \end{aligned}$$

$$\begin{aligned} a_\ell &= \frac{2}{0.2\sec}\int_0^{0.2\sec} F(t)\cos 10\pi\ell t\,dt \\ &= 10\int_0^{0.1\sec} F_0\cos 10\pi\ell t\,dt \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_\ell &= \frac{2}{0.2\sec}\int_0^{0.2\sec} F(t)\sin 10\pi\ell t\,dt \\ &= 10\int_0^{0.1\sec} F_0\sin 10\pi\ell t\,dt \\ &= \frac{F_0}{\pi\ell}(\cos\pi\ell - 1) \end{aligned}$$

The response of the system is given by

$$X(t)=\frac{a_0}{2m\omega_n^2}+\sum_{\ell=1}^{\infty} X_\ell \sin(10\pi\ell t-\phi_\ell)$$

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where

$$X_\ell = \frac{b_\ell M_\ell}{m\omega_n^2}$$

$$\phi_\ell = \tan^{-1}\left(\frac{0.4r_\ell}{1-r_\ell^2}\right)$$

$$M_\ell = \frac{1}{\sqrt{\left(1-r_\ell^2\right)^2 + \left(0.4r_\ell\right)^2}}$$

$$r_\ell = \frac{10\pi\ell}{40\frac{rad}{\sec}} = \frac{\pi\ell}{4}$$

Note that

$$\frac{a_0}{2m\omega_n^2} = \frac{(10000\,N)}{2(100\,kg)\left(40\frac{rad}{\sec}\right)^2} = 0.03125\,m$$

Values of the constants corresponding to the first 10 terms in the response are given in the table below.

ℓ	b_ℓ (N)	r_ℓ	M_ℓ	X_ℓ (m)	ϕ_ℓ (rad)
1	-6366.	0.7854	2.0183	-0.08030	0.6868
2	0	1.5708	0.6265	0.	-0.4046
3	-2122.	2.3562	0.2151	-0.00285	-0.2042
4	0	3.1416	0.1116	0.	-0.1407
5	-1273.	3.9270	0.0689	-0.00055	-0.1085
6	0	4.7124	0.0470	0.	-0.0887
7	-909.	5.4978	0.0341	-.00019	-0.0751
8	0	6.2832	0.0259	0.	-0.0652
9	-707.	7.0686	0.0204	-0.00009	-0.0577
10	0	7.8540	0.0165	0.	-0.0517

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Thus the response of the system is

$$x(t) = 0.03125 - 0.08030\sin(10\pi t - 0.6868)$$
$$- 0.00285\sin(30\pi t + 0.2042) - 0.00055\sin(50\pi t + 0.10849) - \ldots \text{m}$$

The maximum displacement is approximated by

$$X_{\max.} < \frac{a_0}{2m\omega_n^2} + \sum_{\ell=1}^{\infty} |X_\ell| \approx 0.115\,\text{m}$$

Problem 4.83 illustrates (a) development of the Fourier series for a periodic function that is neither even or odd, (b) determination of the response of a system to a periodic excitation, and (c) approximation of the maximum displacement due to a periodic excitation.

4.84 During operation a 100-kg press is subject to the periodic excitations shown. The press is mounted on an elastic foundation of stiffness 1.6×10^5 N/m and damping ratio 0.2. Determine the steady–state response of the press and approximate the maximum displacement from equilibrium. Each excitation is shown over one period.

FIGURE P4.84

Given: m = 100 kg, ω_n = 40 rad/sec, ζ = 0.2, F(t)

Find: x(t), $x_{max.}$

Solution: The excitation is periodic of period T = 0.3 sec. Its mathematical form is

$$F(t) = 10\,F_0 \begin{cases} t, & 0 \le t \le 0.1\,\text{sec} \\ -t + 0.2, & 0.1\,\text{sec} < t \le 0.2\,\text{sec} \\ 0, & 0.2\,\text{sec} < t < 0.3\,\text{sec} \end{cases}$$

F(t) is neither even or odd, thus all Fourier coefficients must be calculated. To this end

$$a_0 = \frac{2}{0.3\,\text{sec}} \int_0^{0.3\,\text{sec}} F(t)\,dt$$
$$= \frac{200\,F_0}{3}\left[\int_0^{0.1\,\text{sec}} t\,dt + \int_{0.1\,\text{sec}}^{0.2\,\text{sec}} (-t + 0.2)\,dt\right]$$
$$= \frac{2\,F_0}{3}$$

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$$a_\ell = \frac{2}{0.3\sec}\int_0^{0.3\sec} F(t)\cos\frac{20\pi\ell}{3}t\,dt$$

$$= \frac{200\,F_0}{3}\left[\int_0^{0.1\sec} t\cos\frac{20\pi\ell}{3}t\,dt + \int_{0.1\sec}^{0.2\sec}(-t+0.2)\cos\frac{20\pi\ell}{3}t\,dt\right]$$

$$= \frac{3\,F_0}{2\,\pi^2\,\ell^2}\left(2\cos\frac{2\pi\ell}{3} - \cos\frac{4\pi\ell}{3} - 1\right)$$

$$b_\ell = \frac{2}{0.3\sec}\int_0^{0.3\sec} F(t)\sin\frac{20\pi\ell}{3}t\,dt$$

$$= \frac{200\,F_0}{3}\left[\int_0^{0.1\sec} t\sin\frac{20\pi\ell}{2}t\,dt + \int_{0.1\sec}^{0.2\sec}(-t+0.2)\sin\frac{20\pi\ell}{3}t\,dt\right]$$

$$= \frac{3\,F_0}{2\,\pi^2\,\ell^2}\left(2\sin\frac{2\pi\ell}{3} - \sin\frac{4\pi\ell}{3}\right)$$

The Fourier series can be represented as

$$F(t) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} c_\ell \sin\left(\frac{20\pi\ell}{3}t + \kappa_\ell\right)$$

where

$$c_\ell = \sqrt{a_\ell^2 + b_\ell^2}$$

$$\kappa_\ell = \tan^{-1}\left(\frac{a_\ell}{b_\ell}\right)$$

The system response is given by

$$x(t) = \frac{a_0}{2m\,\omega_n^2} + \sum_{\ell=1}^{\infty} x_\ell \sin\left(\frac{20\pi\ell}{3}t + \kappa_\ell - \phi_\ell\right)$$

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where

$$X_\ell = \frac{c_\ell M_\ell}{m\omega_n^2}$$

$$\phi_\ell = \tan^{-1}\left(\frac{0.4\, r_\ell}{1 - r_\ell^2}\right)$$

$$M_\ell = \frac{1}{\sqrt{(1 - r_\ell^2)^2 + (0.4 r_\ell)^2}}$$

$$r_\ell = \frac{20\pi\ell}{3(40\,\frac{rad}{\sec})} = \frac{\pi\ell}{6}$$

The Fourier coefficients and system response parameters for the first 10 terms are shown in the table below.

ℓ	a_ℓ	b_ℓ	c_ℓ	κ_ℓ	r_ℓ	M_ℓ	X_ℓ (mm)	ϕ_ℓ
1	-2279	3948	4559	-.523	.523	1.324	37.7	.281
2	-569	-987	1139	.523	1.05	2.326	16.6	-1.35
3	0	0	0	0	1.57	0.625	0	-.405
4	-142	246	284	-.523	2.09	0.287	0.51	-.242
5	-91	-158	182	.523	2.62	0.168	0.19	-.177
6	0	0	0	0	3.14	0.112	0	-.141
7	-47	80	93	-.523	3.67	.080	.05	-.117
8	-35	-61	71	.523	4.19	0.060	.03	-.101
9	0	0	0	0	4.71	0.047	0	-.089
10	-23	39	45	-.523	5.24	0.038	.01	-.079

It is noted that

$$\frac{a_0}{2m\,\omega_n^2} = 20.83 \text{ mm}$$

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Hence the system response is

$$x(t) = 20.83 + 37.72\sin(\frac{20\pi}{3}t - .804)$$
$$+ 16.57\sin(\frac{40\pi}{3}t + 1.867) + 0.51\sin(\frac{80\pi}{3}t - .280) + \ldots \text{mm}$$

The maximum displacement is approximated by

$$x_{\max.} < \frac{a_0}{2m\omega_n^2} + \sum_{\ell=1}^{\infty} x_\ell \approx 75.89\text{mm}$$

Problem 4.84 illustrates (a) development of the Fourier series for a periodic function that is neither even or odd, (b) determination of the response of a system due to a periodic excitation, and (c) approximation of the maximum displacement due to a periodic excitation.

4.85 During operation a 100-kg press is subject to the periodic excitations shown. The press is mounted on an elastic foundation of stiffness 1.6×10^5 N/m and damping ratio 0.2. Determine the steady–state response of the press and approximate the maximum displacement from equilibrium. Each excitation is shown over one period.

FIGURE P4.85

Given: m = 100 kg, $\omega_n = 40$ rad/sec, $\zeta = 0.2$

Find: x(t), $x_{\max.}$

Solution: F(t) is periodic of period T = 0.2 sec. The mathematical form of F(t) over one period is

$$F(t) = 10F_o \begin{cases} t, & 0 \le t \le 0.1\,\text{sec} \\ -t + 0.2, & 0.1\,\text{sec} < t < 0.2\,\text{sec} \end{cases}$$

where $F_0 = 10{,}000$ N. F(t) is an even function, thus

$$b_\ell = 0, \quad \ell = 1,2,\ldots$$

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The Fourier cosine coefficients are calculated as

$$a_0 = \frac{2}{0.2\,\text{sec}} \int_0^{0.2\,\text{sec}} F(t)\,dt$$

$$= 10\,F_o \left[\int_0^{0.1\,\text{sec}} t\,dt + \int_{0.1\,\text{sec}}^{0.2\,\text{sec}} (-t + 0.2)\,dt \right]$$

$$= F_o$$

$$a_\ell = \frac{2}{0.2\,\text{sec}} \int_0^{0.2\,\text{sec}} F(t) \cos 10\pi\ell t\,dt$$

$$= 10\,F_o \left[\int_0^{0.1\,\text{sec}} t \cos 10\pi\ell t\,dt + \int_{0.1\,\text{sec}}^{0.2\,\text{sec}} (-t + 0.2) \cos 10\pi\ell t\,dt \right]$$

$$= \frac{2\,F_o}{\pi^2\,\ell^2} (\cos \pi\ell - 1)$$

The system response is

$$x(t) = \frac{a_0}{2m\omega_n^2} + \sum_{\ell=1}^{\infty} X_\ell \sin(10\pi\ell t + \frac{\pi}{2} - \phi_\ell)$$

where

$$X_\ell = \frac{a_\ell M_\ell}{m\omega_n^2}$$

$$\phi_\ell = \tan^{-1}\left(\frac{0.4\,r_\ell}{1 - r_\ell^2} \right)$$

$$M_\ell = \frac{1}{\sqrt{(1 - r_\ell^2)^2 + (0.4\,r_\ell)^2}}$$

$$r_\ell = \frac{10\pi\ell}{40\,\frac{rad}{\text{sec}}} = \frac{\pi\ell}{4}$$

The Fourier coefficients and the system response constants for the first 10 terms are given in the table below.

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ℓ	a_ℓ (N)	r_ℓ	M_ℓ	X_ℓ (mm)	ϕ_ℓ (rad)
1	-4053.	0.7854	2.0183	-51.12	0.6868
2	0	1.5708	0.6247	0	-0.4057
3	-450.	2.3562	0.2151	-0.61	-0.2042
4	0	3.1416	0.1116	0	-0.1407
5	-162.	3.9270	0.0689	-.07	-0.1085
6	0	4.7124	0.0469	0	-0.0887
7	-83.	5.4988	0.0341	-0.02	-0.0751
8	0	6.2832	0.0259	0	-0.0652
9	-50.	7.0686	0.0204	-0.01	-0.0577
10	0	7.8540	0.0165	0	-0.0517

It is noted that

$$\frac{a_0}{2m\ \omega_n^2} = 31.25 \text{ mm}$$

Hence

$$x(t) = 31.25 - 51.12\sin(\frac{20\pi}{3}t + 0.8830)$$
$$-0.61\sin(\frac{40\pi}{3}t + 1.775) - 0.07\sin(\frac{80\pi}{3}t + 1.679) + \ldots \text{ mm}$$

The maximum displacement is approximated by

$$x_{\max.} < \frac{a_0}{2m\,\omega_n^2} + \sum_{\ell=1}^{\infty}|X_\ell| \approx 83.07 \text{ mm}$$

Problem 4.85 illustrates (a) development of the Fourier series for an even function, (b) determination of the response of a one-degree-of-freedom system subject to a periodic

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excitation, and (c) approximation of the maximum displacement of a system subject to a periodic excitation.

4.86 During operation a 100-kg press is subject to the periodic excitations shown. The press is mounted on an elastic foundation of stiffness 1.6×10^5 N/m and damping ratio 0.2. Determine the steady–state response of the press and approximate the maximum displacement from equilibrium. Each excitation is shown over one period.

FIGURE P4.86

Given: m = 100 kg, $\omega_n = 40$ rad/sec, $\zeta = 0.2$, F(t)

Find: x(t), $x_{max.}$

Solution: F(t) is periodic of period T = 0.4 sec. The mathematical form of F(t) over one period is

$$F(t) = 10\,F_0 \begin{cases} t, & 0 \le t \le 0.1\,\text{sec} \\ -t + 0.2, & 0.1\,\text{sec} \le t \le 0.3\,\text{sec} \\ t - 0.4, & 0.3\,\text{sec} \le t \le 0.4\,\text{sec} \end{cases}$$

F(t) is an odd function, thus

$$a_\ell = 0, \quad \ell = 0,1,2,\ldots$$

where $F_0 = 10{,}000$ N. The Fourier sine coefficients are

$$b_\ell = \frac{2}{0.4\,\text{sec}} \int_0^{0.4\,\text{sec}} F(t) \sin 5\pi\ell t\, dt$$

$$= 50\,F_0 \left[\int_0^{0.1\,\text{sec}} t \sin 5\pi\ell t\, dt \right.$$

$$\left. + \int_{0.1\,\text{sec}}^{0.3\,\text{sec}} (-t + 0.2) \sin 5\pi\ell t\, dt + \int_{0.3\,\text{sec}}^{0.4\,\text{sec}} (t - 0.4) \sin 5\pi\ell t\, dt \right]$$

$$= \frac{4\,F_0}{\pi^2 \ell^2} \left(\sin\frac{\pi\ell}{2} - \sin\frac{3\pi\ell}{2} \right)$$

The system response is given by

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$$x(t) = \sum_{\ell=1}^{\infty} X_\ell \sin(5\pi\ell t - \phi_\ell)$$

where

$$X_\ell = \frac{b_\ell M_\ell}{m\omega_n^2}$$

$$\phi_\ell = \tan^{-1}\left(\frac{0.4 r_\ell}{1 - r_\ell^2}\right)$$

$$M_\ell = \frac{1}{\sqrt{(1 - r_\ell^2)^2 + (0.4 r_\ell)^2}}$$

$$r_\ell = \frac{5\pi\ell}{40\,\frac{rad}{\sec}} = \frac{\pi\ell}{8}$$

The Fourier coefficients and the system response constants for the first 10 terms are given in the table below.

ℓ	b_ℓ (N)	r_ℓ	M_ℓ	X_ℓ (mm)	ϕ_ℓ (rad)
1	8106.	0.3927	1.1625	58.89	0.1836
2	0	0.7854	2.0183	0	0.6868
3	-901.	1.1781	1.6383	-9.22	-0.8821
4	0	1.5708	0.6265	0	-0.4046
5	324.	1.9635	0.3377	0.68	-0.2684
6	0	2.3562	0.2154	0	-0.2042
7	-165.	2.7489	0.1504	-0.16	-0.1662
8	0	3.1416	0.1116	0	-0.1407
9	100.	3.5343	0.0864	0.05	-0.1224
10	0	3.9270	0.0689	0	-0.1085

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The system response is given by

$$x(t) = 58.59\sin(5\pi t - .1836) - 9.22\sin(10\pi t + .8828) + \ldots \text{ mm}$$

The maximum displacement is approximated by

$$x_{\max.} < \sum_{\ell=1}^{\infty} |X_\ell| \approx 69.01 \text{ mm}$$

Problem 4.86 illustrates (a) development of the Fourier series for an odd function, (b) determination of the response of a one-degree-of-freedom system subject to a periodic excitation, and (c) approximation of the maximum displacement of a system subject to a periodic excitation.

4.87 Use of an accelerometer of natural frequency 100 Hz and a damping ratio of 0.15 reveals that an engine vibrates at a frequency of 20 Hz. and has an acceleration amplitude of 14.3 m/sec^2. Determine (a) The percent error in the measurement; (b) The actual acceleration amplitude; (c) The displacement amplitude.

Given: $\omega_n = 100$ Hz., $\zeta = 0.15$, $\omega = 20$ Hz., $\omega_n^2 Z = 14.3$ m/sec^2

Find: E, $\omega^2 Y$, Y

Solution: (a) The percent error in amplitude measurement in using an accelerometer with a damping ratio less than 0.707 is

$$E = 100|1 - M| \tag{1}$$

where, for this situation

$$r = \frac{\omega}{\omega_n} = \frac{20\text{Hz.}}{100\text{Hz.}} = 0.2$$

$$M(0.2, 0.15) = \frac{1}{\sqrt{\left[1 - (.2)^2\right]^2 + \left[2(.15)(.2)\right]^2}} = 1.0396$$

Thus using eq.(l),

$$E = 100|1 - 1.0396| = 3.96\%$$

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(b) The amplitude measured by the accelerometer is the amplitude of the displacement of the seismic mass relative to the accelerometer's point of attachment, multiplied by the square of the natural frequency of the accelerometer. The actual acceleration amplitude is

$$\omega^2 Y = \omega_n^2 \frac{Z}{M}$$

$$\frac{14.3 \dfrac{\text{mm}}{\text{sec}^2}}{1.0396} = 13.76 \frac{\text{mm}}{\text{sec}^2}$$

(c) The actual displacement amplitude is given by

$$Y = \frac{\omega^2 Y}{\omega^2}$$

$$= \frac{13.76 \dfrac{\text{mm}}{\text{sec}^2}}{\left[\left(20 \dfrac{\text{cycles}}{\text{sec}}\right)\left(\dfrac{2\pi \text{ rad}}{1 \text{ cycle}}\right)\right]^2} = 8.7\text{x}10^{-4}\,\text{mm}$$

Problem 4.87 illustrates (a) the error in using an accelerometer to measure vibration amplitude and (b) calculation of the actual amplitude using the measured amplitude and accelerometer properties.

4.88 An accelerometer with a natural frequency 200 Hz and damping ratio 0.7 is used to measure the vibrations of a system whose actual displacement is $x(t) = 1.6\sin 45.1t\,\text{mm}$. What is the accelerometer output?

Given: $\omega_n = 200$ Hz., $\zeta = 0.7$, x(t)

Find: $\omega_n^2 z$ (t)

Solution: The accelerometer measures the displacement of its seismic mass relative to the accelerometer's point of attachment. This is multiplied by the square of the natural frequency of the accelerometer to approximate the acceleration of the point of attachment. Thus the accelerometer actually records

$$\omega_n^2 z(t) = \omega_n^2 \Lambda X \sin(\omega t - \phi)$$
$$= \omega^2 M X \sin(\omega t - \phi)$$

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For the situation at hand

$$r = \frac{\omega}{\omega_n} = \frac{45.1\dfrac{\text{rad}}{\text{sec}}}{\left(200\dfrac{\text{cycles}}{\text{sec}}\right)\left(\dfrac{2\pi\,\text{rad}}{1\,\text{cycle}}\right)} = .0359$$

$$M = \frac{1}{\sqrt{\left[1-(.0359)^2\right]^2 + \left[2(0.7)(.0359)\right]^2}} = 1.00007$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2(0.7)(.0359)}{1-(.0359)^2}\right) = 0.0502\,\text{rad}$$

Thus

$$\omega_n^2(t) = \left(1256.4\frac{\text{rad}}{\text{sec}}\right)^2(1.00007)(1.6\,\text{mm})\sin(45.1t-.0502)$$

$$= 2.52x10^6\sin(45.1t-.0502)\frac{\text{mm}}{\text{sec}^2}$$

Problem 4.88 illustrates the use of an accelerometer to measure a one frequency vibration.

4.89 An accelerometer with a natural frequency 200 Hz and damping ratio of 0.2 is used to measure the vibrations of an engine operating at 1000 rpm. What is the percent error in the measurement?

Given ω_n = 200 Hz, ζ = 0.2, ω = 1000 rpm

Find**:** E

Solution: The percent error in an accelerometer measurement is

$$E = 100\left|1 - M(r,\zeta)\right|$$

where the frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{(1000\text{ rev/min})}{(200\text{ rev/s})(60\text{ s/1 min})} = 0.0833$$

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The magnification factor is

$$M(0.0833,0.2) = \frac{1}{\sqrt{[1-(0.0833)^2]^2 + [2(0.2)(0.0833)]^2}} = 1.0064$$

Thus the percent error in the accelerometer measurement is

$$E = 100|1 - 1.0064| = 0.64$$

Problem 4.89 illustrates the percent error in an accelerometer measurement.

4.90 When a machine tool is placed directly on a rigid floor, it provides an excitation of the form

$$F(t) = (4000sin100t + 5100sin150t)\ \text{N} \qquad (1)$$

to the floor. Determine the natural frequency of the system with an undamped isolator with the minimum possible static deflection such that when the machine is mounted on the isolator the amplitude of the force transmitted to the floor is less than 3500 N.

Given: F(t), $T_{Tmax} = 3500$ N

Find: ω_n

Solution: The isolator with the minimum static deflection leads to the largest possible natural frequency. The excitation is a two-frequency excitation. The total transmitted force through an undamped isolator is

$$F_t = 4000T_1 + 5000T_2 \qquad (2)$$

where

$$T_1 = \frac{1}{\left(\frac{100}{\omega_n}\right)^2 - 1}, \qquad T_2 = \frac{1}{\left(\frac{150}{\omega_n}\right)^2 - 1}$$

Hence, in order for the transmitted force to be equal to 3500 N,

$$3500 = \frac{4000}{\left(\frac{100}{\omega_n}\right)^2 - 1} + \frac{5100}{\left(\frac{150}{\omega_n}\right)^2 - 1}$$

The above equation is rearranged, leading to

$$126\omega_n^4 - 2.55 \times 10^6\omega_n^2 + 7.88 \times 10^9 = 0$$

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which yields

$$\omega_n = 61.72\ \frac{\text{rad}}{\text{sec}},\quad 128.5\ \frac{\text{rad}}{\text{sec}}$$

The larger solution is invalid, since 100/128< 1. Then the first term in eq.(2) should be negative. Hence

$$\omega_n = 61.7\ \frac{\text{rad}}{\text{sec}}$$

Problem 4.90 illustrates the design of an undamped isolator for a multi-frequency excitation.

4.91 Use the force shown in Figure P4.91 as an approximation to the force provided by the punch press during its operation. Rework Example 4.17 for the excitation.

FIGURE P4.91

Given: F(t), $\xi = 0.1$, m = 500 kg, $F_{T,max,} = 1000$ N

Find: ωn, $\Delta_{ST.}$, Δ

Solution: The period of the excitation is 1.0 sec. Over one period the mathematical form of the excitation is

$$F(t) = F_0 \begin{cases} 10t, & 0 < t < 0.1 \text{ sec} \\ 1, & 0.1 \text{ sec} < t < 0.3 \text{ sec} \\ -10t + 4, & 0.3 \text{ sec} < t < 0.4 \text{ sec} \\ 0, & 0.4 \text{ sec} < t < 1.0 \text{ sec} \end{cases}$$

where F_0 = 4000 N. The Fourier coefficients for the Fourier series representation of the excitation are

$$a_0 = \frac{2}{1 \text{ sec}} \int_0^{1 \text{ sec}} F(t)dt$$

$$= 2F_0 \left[\int_0^{0.1 \text{ sec}} 10t\, dt + \int_{0.1 \text{ sec}}^{0.3 \text{ sec}} dt + \int_{0.3 \text{ sec}}^{0.4 \text{ sec}} (-10t + 4)dt \right]$$

$$= 0.6\, F_0$$

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$$a_\ell = \frac{2}{1\ \text{sec}}\int_0^{1\ \text{sec}} F(t)cos2\pi\ell t\ dt$$

$$= 2F_0\left[\int_0^{0.1\ sec} 10tcos2\pi\ell t\ dt + \int_{0.1\ sec}^{0.3\ sec} cos2\pi\ell t\ dt + \int_{0.3sec}^{0.4\ sec}(-10t+4)cos2\pi\ell t\ dt\right]$$

$$= \frac{5F_0}{2\pi^2\ell^2}(-1 + cos0.2\pi\ell + cos0.6\pi\ell - cos0.8\pi\ell)$$

$$b_\ell = \frac{2}{1\ sec}\int_0^{1\ sec} F(t)sin2\pi\ell t\ dt$$

$$= 2F_0\left[\int_0^{0.1\ sec} 10tsin2\pi\ell t\ dt + \int_{0.1\ sec}^{0.3\ sec} sin2\pi\ell t\ dt + \int_{0.3\ sec}^{0.4\ sec}(-10t+4)sin2\pi\ell t\ dt\right]$$

$$= \frac{5F_0}{2\pi^2\ell^2}(sin0.2\pi\ell + sin0.6\pi\ell - sin0.8\pi\ell)$$

The Fourier series representation for F(t) is

$$F(t) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} c_\ell \sin\,(2\pi\ell t + \kappa_\ell)$$

where

$$c_\ell = \sqrt{a_\ell^2 + b_\ell^2}$$

$$\kappa_\ell = \tan^{-1}\left(\frac{a_\ell}{b_\ell}\right)$$

The system response is given by

$$x(t) = \frac{a_0}{2m\omega_n^2} + \sum_{\ell=1}^{\infty}\frac{M_\ell c_\ell}{m\omega_n^2} sin(2\pi\ell t + \kappa_\ell - \phi_\ell)$$

The repeating component of the force transmitted between the isolator and the foundation is

$$F_t(t) = \sum_{\ell=1}^{\infty} c_\ell T_\ell sin(2\pi\ell t + \kappa_\ell + \lambda_\ell)$$

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where

$$M_\ell = \frac{1}{\sqrt{\left(1 - r_\ell^2\right)^2 + (2\xi r_\ell)^2}}$$

and

$$T_\ell = \sqrt{\frac{1 + (2\xi r_\ell)^2}{\left(1 - r_\ell^2\right)^2 + (2\xi r_\ell)^2}}$$

An upper bound on the magnitude of the transmitted force is

$$F_T < \sum_{\ell=1}^{\infty} c_\ell T_\ell \tag{1}$$

Equation (1) is an equation that can be solved to determine an upper bound on the natural frequency of the system when the machine is placed on an isolator. The solution of the above equation is by trial and error. One convenient method is to develop a spreadsheet of the relevant equations, programming ω_n, to be changed. The value of ω_n is changed and the spreadsheet is recalculated until FT=1000 N. For this problem, that occurs when ω_n=3.94 rad/sec. The spreadsheet for the first 20 terms is shown below.

ℓ	a_ℓ	b_ℓ	r_ℓ	c_ℓ	M_ℓ	T_ℓ	$c_\ell T_\ell$	$c_\ell M_\ell/(m\omega_n^2)$
1	939.3	939.6	1.595	1345.7	0.634	0.666	896.38	0.110
2	-458.2	332.9	3.189	566.4	0.108	0.129	73.07	0.008
3	-273.2	-66.2	4.785	281.1	0.045	0.063	17.76	0.002
4	-43.7	134.6	6.38	141.6	0.025	0.041	5.78	0.0005
5	-81.1	0	7.97	81.06	0.016	0.030	2.44	0.0002
6	-19.45	-59.85	9.56	62.93	0.011	0.023	1.50	0.0001
7	-50.2	12.1	11.2	51.6	0.008	0.020	1.02	0.000
8	-28.6	-20.8	12.8	35.4	0.006	0.017	0.60	0.000
9	11.6	-11.9	14.3	16.5	0.005	0.015	0.25	0.000
10	0	0	15.95	0	0.004	0.013	0	0.000
11	7.8	8.0	17.54	11.12	0.003	0.012	0.13	0.000
12	-12.7	9.2	10.13	15.73	0.003	0.011	0.17	0.000
13	-14.6	-3.5	20.73	14.97	0.002	0.010	0.15	0.000
14	-3.6	11.0	22.3	11.56	0.002	0.009	0.10	0.000
15	-9.0	0	23.92	9.00	0.002	0.008	0.08	0.000
16	-2.7	-8.4	25.52	8.85	0.002	0.008	0.07	0.000
17	-8.5	2.1	27.11	8.76	0.001	0.008	0.07	0.000
18	-5.7	-4.1	28.70	6.99	0.001	0.007	0.05	0.000
19	2.6	-2.7	30.30	3.73	0.001	0.007	0.02	0.000
20	0	0	31.89	0	0.001	0.006	0	0.000

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The $a_0/2$ term in the excitation force contributes a static load to the system. Thus the static deflection of the isolator is the deflection produced by a force

$$W + \frac{a_0}{2} = (500\text{ kg})\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right) + \frac{2400\text{ N}}{2} = 6105\text{ N}$$

The required static deflection is

$$\Delta_{ST.} = \frac{6105\text{ N}}{(500\text{ kg})\left(3.94\ \frac{\text{rad}}{\text{sec}}\right)^2} = 0.786\text{ m}$$

The total dynamic deflection is calculated from the spreadsheet as 0.1205 m which leads to a total deflection of 0.907 m.

Problem 4.91 illustrates (a) development of the Fourier series for a periodic excitation and (b) vibration isolation calculations for a periodic excitation.

4.92 A 550-kg industrial sewing machine has a rotating unbalance of 0.24 kg · m. The machine operates at speeds between 2000 and 3000 rpm. The machine is placed on an isolator pad of stiffness 5×10^6 N/m and damping ratio 0.12. What is the maximum natural frequency of an undamped seismometer that can be used to measure the steady–state vibrations at all operating speeds with an error less than 4%? If this seismometer is used, what is its output when the machine is operating at 2500 rpm?

Given: Machine-isolator system: m = 550 kg, m_0e = 0.24 kg · m, k = 5×10^6 N/m, ζ = 0.12, 2000 rpm ≤ ω ≤ 3000 rpm;
Seismometer: E = 4%, Assume undamped

Find: Seismometer: ω_n, output for ω = 2500 rpm

Solution: For an undamped seismometer, the percent error in the amplitude measurement is

$$E = 100(\Lambda - 1) \qquad (1)$$

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$$4 > 100(\Lambda - 1)$$
$$\Lambda = \frac{r^2}{r^2 - 1} < 1.04 \tag{2}$$

Equation (2) is solved yielding

$$r > 5.099 \tag{3}$$

The seismometer must be designed such that the error is less than 4% over the entire operating range. Thus

$$r = \frac{\omega}{\omega_n} > 5.099, \;\; 2000\ rpm \le \omega \le 3000\ rpm \tag{4}$$

Equation (4) is satisfied if

$$\omega_n < \frac{2000\ \text{rpm}}{5.099} = \frac{(2000\ \text{rpm})(2\pi \frac{\text{rad}}{\text{rev}})(\frac{1\ \text{min}}{60\ \text{sec}})}{5.099} = 41.07 \frac{\text{rad}}{\text{sec}}$$

The seismometer output is the displacement of its seismic mass relative to the displacement of the sewing machine. Since the seismometer is undamped, it is

$$z(t) = Z \sin \omega t \tag{5}$$

where

$$Z = \Lambda_1 Y \tag{6}$$

where Y is the steady-state amplitude of the sewing machine and

$$\Lambda_1 = \frac{r_1^2}{r_1^2 - 1} \tag{7}$$

When the machine operates at 2500 rpm

$$r_1 = \frac{\omega}{\omega_n} = \frac{(2500\ \text{rpm})(2\pi \frac{\text{rad}}{\text{rev}})(\frac{1\text{min}}{60\text{sec}})}{41.07 \frac{\text{rad}}{\text{sec}}} = 6.37$$

and thus $\Lambda_1 = 1.025$.

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The vibration amplitude of the machine is given by

$$\frac{mY}{m_0 e} = \Lambda_2 = \frac{r_2^2}{\sqrt{(1-r_2^2)^2 + (2\zeta_2 r_2)^2}}$$

The machine's natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5\times 10^6 \frac{\text{N}}{\text{m}}}{550\text{kg}}} = 95.35\frac{\text{rad}}{\text{sec}}$$

The frequency ratio is

$$r_2 = \frac{(2500\text{rpm})(2\pi\frac{\text{rad}}{\text{rev}})(\frac{1\text{min}}{60\text{sec}})}{93.35\frac{\text{rad}}{\text{sec}}} = 2.745$$

Hence

$$\Lambda_2 = \frac{(2.745)^2}{\sqrt{[1-(2.745)^2]^2 + [2(.12)(2.745)]^2}} = 1.147$$

Then

$$Y = \frac{m_0 e \Lambda_2}{m} = \frac{(0.24\text{ kg}\cdot\text{m})(1.147)}{550\text{ kg}} = 0.501\text{ mm}$$

The phase difference between the steady state vibrations of the machine and the excitation is

$$\phi = \tan^{-1}\left(\frac{2\zeta r_2}{1-r_2^2}\right) = -0.100\,\text{rad}$$

Thus the seismometer output is

$$\begin{aligned} z(t) &= (1.0125)(0.501\text{ mm})\sin(261.8t + 0.100) \\ &= 0.507\sin(261t + 0.100)\ \text{mm} \end{aligned}$$

Problem 4.92 illustrates design of a seismometer used to measure the vibrations of a machine subject to a rotating unbalance. It also illustrates the seismometer output.

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4.93 The system of Figure P4.93 is subject to the excitation

$F(t) = 1000\sin 25.4t + 800\sin(48t + 0.35) - 300\sin(100t + 0.21)\ N$

What is the output in mm/s^2 of an accelerometer of natural frequency 100 Hz and damping ratio 0.7 placed at A?

FIGURE P4.93

Given: System shown with F(t) as above, ω_n = 100 Hz., ζ =0.7, m = 16.2 kg

Find: accelerometer output

Solution: In order to predict the accelerometer response, the response of point A must first be determined. To this end let θ(t) represent the clockwise angular displacement of the bar measured with respect to the system's equilibrium position. Let x(t) be the displacement of point A. Then

$$x(t) = (0.2m)\theta(t) \tag{1}$$

The equivalent system method is used to derive the governing differential equation. The kinetic energy of the system is

$$T = \frac{1}{2}\left[\frac{1}{12}mL^2\dot{\theta}^2 + m(0.15\,m)^2\dot{\theta}^2\right] = \frac{1}{2}\left(1.026\,\text{kg}\cdot\text{m}^2\right)\dot{\theta}^2$$

The potential energy of the system is

$$V = \frac{1}{2}k\left(2\times10^5\,\frac{\text{N}}{\text{m}}\right)(.2\,\text{m})\theta^2 = \frac{1}{2}\left(8000\,\frac{\text{N}\cdot\text{m}}{\text{rad}}\right)\theta^2$$

The work done by the damping force is

$$W = -\int\left(400\,\frac{\text{N}\cdot\text{sec}}{\text{m}}\right)(0.5\,\dot{\theta})d(.5\theta) = -\int\left(100\,\frac{\text{N}\cdot\text{sec}\cdot\text{m}}{\text{rad}}\right)\dot{\theta}\,d\theta$$

When the bar moves through a virtual displacement $\delta\theta$, the work done by the external force is

$$\delta W = F(t)\delta(0.5\theta) = 0.5F(t)\delta\theta$$

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Hence the governing differential equation is

$$1.026\ddot{\theta} + 100\dot{\theta} + 8000\theta = 0.5F(t)$$

The system's natural frequency and damping ratio are calculated as

$$\omega_n = \sqrt{\frac{8000\,\mathrm{N}\cdot\mathrm{m}}{1.026\,\mathrm{kg}\cdot\mathrm{m}^2}} = 88.3\frac{\mathrm{rad}}{\mathrm{sec}}$$

$$2\zeta\omega_n = \frac{100\dfrac{\mathrm{N}\cdot\mathrm{sec}\cdot\mathrm{m}}{\mathrm{rad}}}{1.026\,\mathrm{kg}\cdot\mathrm{m}^2}$$

$$\zeta = 0.552$$

The response of the system due to the multi-frequency excitation is obtained using the principle of linear superposition

$$\theta(t) = \sum_{i=1}^{3} \frac{0.5F_iM_i}{(1.026\,kg - m^2)\omega_n^2}\sin(\omega_i t + \psi_i - \phi_i)$$

$$= \sum_{1=i}^{3} \frac{F_iM_i}{16000}\sin(\omega_i t + \psi_i - \phi_i)$$

where

$$r_i = \frac{\omega_i}{\omega_n}$$

$$M_i = \frac{1}{\sqrt{(1-r_i^2)^2 + (2\zeta r_i)^2}}$$

$$\phi_i = \tan^{-1}\left(\frac{2\zeta r_i}{1-r_i^2}\right)$$

The calculations and results for the excitation given are summarized in the table below

$F_i(t)$	100sin25.4t	800sin(48t+.35)	300sin(100t+.21)
F_i	100 N	800 N	-300 N
ω_i	25.4 rad/sec	48 rad/sec	100 rad/sec

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Ψ_i	0	0.35 rad	0.21 rad
r_i	0.288	0.544	1.13
M_i	1.030	1.080	0.612
ϕ_i	0.334 rad	0.706 rad	-1.35 rad
θ_i	0.00644 rad	0.0540 rad	0.0115 rad
$X_i=0.2\theta_i$	.001288 m	.0180 m	.00230 m

Thus the response of point A is

$$x(t) = .001288\sin(25.4t - 0.334)$$
$$+ .0108\sin(48t - .356) - .0023\sin(100t + 1.56)\text{m}$$

The accelerometer measures the displacement of its seismic mass relative to point A. It multiplies by the square of its natural frequency to produce an output approximating the acceleration of A. That is the output of the accelerometer for this response is

$$\omega_n^2 z(t) = \sum_{i=1}^{3} \omega_i^2 M_{1,i} X_i \sin(\omega_i t + \psi_{1,i} - \phi_{1,i})$$

where the magnification factor and phase angles are now calculated using the accelerometer properties,

$$r_i = \frac{\omega_i}{628.3\dfrac{\text{rad}}{\text{sec}}}$$

The accelerometer calculations are summarized in the table below

$x_i(t)$	.001288sin(25.4t-.334)	.0108sin(48t-.356)	-.0023sin(100t+1.56)
X_I	.001288 m	.0108 m	-.0023 m
ω_I	25.4 rad/sec	48 rad/sec	100 rad/sec

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$\psi_{1,i}$	-.334 rad	-.356 rad	1.56 rad
r_I	.0404	.0764	.159
M_i	1.000	1.000	1.000
$\omega_i^2 M_i X_i$	0.831 m/sec^2	24.88 m/sec^2	230. m/sec^2
$\phi_{1,i}$	0.0566 rad	0.1076 rad	0.428 rad

Thus the response measured by the accelerometer is

$$\omega_n^2 z(t) = 831\sin(25.4t - .391)$$
$$+ 24880\sin(48t - .464) - 230000\sin(100t + 1.13)\frac{\text{mm}}{\text{sec}^2}$$

Problem 4.93 illustrates (a) the derivation of the differential equation governing forced vibration of a one-degree-of-freedom system, (b) determination of the response due to a multi-frequency excitation, and (c) accelerometer measurement of a multi-frequency vibration.

4.94 What is the output, in mm, of a seismometer with a natural frequency of 2.5 Hz and a damping ratio of 0.05 placed at *A* for the system of Figure P4.93?

Given: System shown, $\omega_n = 2.5$ Hz., $\zeta = 0.05$

Find: z(t)

Solution: The time dependent response of point *A* is determined in the solution of Problem 4.93 as

$$x(t) = 1.29\sin(25.4t - .334)$$
$$+ 10.8\sin(48t - .356) - 2.30\sin(100t + 1.56)\text{mm} \qquad (1)$$

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The seismometer measures the displacement of its seismic mass relative to the instrument's point of attachment. For the multi frequency vibration of the form of eq.(1), its output is

$$z(t)=\sum_{i=1}^{3}\Lambda_i X_i \sin(\omega_i t+\psi_i-\phi_i)$$

where

$$r_i=\frac{\omega_i}{\left(2.5\frac{\text{cycles}}{\text{sec}}\right)\left(2\pi\frac{\text{rad}}{\text{cycle}}\right)}=\frac{\omega_i}{15.7\frac{\text{rad}}{\text{sec}}}$$

$$\Lambda_i=\frac{r_i^2}{\sqrt{\left(1-r_i^2\right)^2+\left(0.1r_i\right)^2}}$$

$$\phi_i=\tan^{-1}\left(\frac{0.1r_i}{1-r_i^2}\right)$$

The calculations are summarized in the table below.

$x_i(t)$	1.288sin(25.4t-.334) mm	10.8sin(48t-.356) mm	2.3sin (100t+1.56) mm
ω_i	25.4 rad/sec	48 rad/sec	100 rad/sec
X_i	1.288 mm	10.8 mm	2.3 mm
ψ_i	-.334 rad	-.356 rad	1.56 rad
r_i	1.618	3.057	6.37
Λ_i	1.610	1.119	1.025
$\Lambda_i X_i$	2.07 mm	12.1 mm	2.36 mm
ϕ_i	-.0996 rad	-.0336 rad	-.0161 rad

Thus the seismometer output is

$$z(t)=2.07\sin(25.4t-.2334)$$
$$+12.1\sin(48t-.393)-2.36\sin(100t+1.58)\text{mm}$$

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Problem 4.94 illustrates (a) derivation of the differential equation governing forced vibrations of a one-degree-of-freedom system, (b) response of a one-degree-of-freedom system to a multi-frequency excitation, and (c) measurement of a multi-frequency vibration using a seismometer.

4.95 A 20 kg block is connected to a moveable support through a spring of stiffness 1×10^5 N/m in parallel with a viscous damper of damping coefficient 600 N · s/m. The support is given a harmonic displacement of amplitude 25 mm and frequency 40 rad/s. An accelerometer of natural frequency 25 Hz and damping ratio 0.2 is attached to the block. What is the output of the accelerometer in mm/s^2?

Given: m = 20 kg, $k_1 = 1 \times 10^5$ N/m, c_1 = 600 N-s/m, Y = 25 mm, ω = 40 rad/sec,
ω_{na} = 25 Hz, ζ_a = 0.2

Find: $\omega_{na}^2 z(t)$

Solution: Let y(t) denote the displacement of the support, x(t) denote the absolute displacement of the 20 kg block, and z(t) denote the displacement of the accelerometer's seismic mass with respect to the block. The accelerometer actually measures z(t). However it is calibrated such that it multiplies z by ω_{na}^2 before output. Then

$$y(t) = Y\sin\omega t = 25\sin 40t \text{ mm} \tag{1}$$

$$x(t) = X\sin(\omega t - \phi) \tag{2}$$

$$Z(t) = Z\sin(\omega t - \phi - \lambda) \tag{3}$$

The system parameters are calculated as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1\times 10^5 \frac{\text{N}}{\text{m}}}{20 \text{ kg}}} = 70.7 \frac{\text{rad}}{\text{sec}}$$

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$$\zeta_1 = \frac{c_1}{2m\,\omega_n} = \frac{600\,\dfrac{\text{N}\cdot\text{sec}}{\text{m}}}{2(20\text{ kg})\left(70.7\,\dfrac{\text{rad}}{\text{sec}}\right)} = 0.212$$

$$r_1 = \frac{\omega}{\omega_n} = \frac{40\,\dfrac{\text{rad}}{\text{sec}}}{70.7\,\dfrac{\text{rad}}{\text{sec}}} = 0.566$$

Then

$$X = YT(.566,.212)$$
$$= 25\text{ mm}\sqrt{\frac{1+[2(0.212)(0.566)]^2}{[1-(0.566)^2]^2+[2(0.212)(0.566)]^2}}$$
$$= 25\text{ mm}(1.427) = 35.7\text{mm}$$

and

$$\lambda = \tan^{-1}\left(\frac{2\zeta_1 r_1^3}{1+(4\zeta_1^2-1)r_1^2}\right) = 0.104\text{rad}$$

The parameters used in calculating the displacement of the seismic mass are $\zeta_2 = 0.2$ and

$$r_2 = \frac{\omega}{\omega_{na}} = \frac{40\,\dfrac{\text{rad}}{\text{sec}}}{25\,\dfrac{\text{cycles}}{\text{sec}}(2\pi\,\dfrac{\text{rad}}{\text{cycle}})} = 0.255$$

Then

$$Z = X\Lambda(.255,.2)$$
$$= 35.7\text{ mm}\frac{(0.255)^2}{\sqrt{[1-(0.255)^2]^2+[2(0.2)(0.255)]^2}}$$
$$= 35.7\text{ mm}\,(0.069) = 2.47\text{mm}$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta_2 r_2}{1-r_2^2}\right) = 0.109\text{rad}$$

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Then

$$z(t) = 2.47\sin(40t - 0.213)\text{mm}$$

The accelerometer output is

$$\omega_{na}^2 z(t) = 60945\sin(40t - 0.213)\frac{\text{mm}}{\text{sec}^2}$$

Problem 4.95 illustrates the use of an accelerometer to measure the motion of a block excited by harmonic base motion.

4.96 An accelerometer has a natural frequency of 80 Hz and a damping coefficient of 8.0 N · s/m. When attached to a vibrating structure, it measures an amplitude of 8.0 m/s^2 and a frequency of 50 Hz. The true acceleration of the structure is 7.5 m/s^2. Determine the mass and stiffness of the accelerometer.

Given: ω_n = 80 Hz, c = 8.0 N · s/m, ω = 50 Hz, $\omega_n^2 Z$ = 8.0 m/s^2, $\omega^2 X$ = 7.5 m/s^2

Find: m, k

Solution: The error in an accelerometer measurement is defined as

$$E = \frac{\text{measured acceleration} - \text{true acceleration}}{\text{true acceleration}}$$

which from the information given is

$$E = \frac{8.0\frac{\text{m}}{\text{sec}^2} - 7.5\frac{\text{m}}{\text{sec}^2}}{7.5\frac{\text{m}}{\text{sec}^2}} = .0667$$

The error is also given by

$$E = |1 - M| \qquad (1)$$

Setting E = .0667 in eq.(1) leads to M = 1.071. Noting that the frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{50\,\text{Hz.}}{80\,\text{Hz.}} = 0.625$$

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then

$$1.071 = \frac{1}{\sqrt{\left[1-(.625)^2\right]^2 + \left[2(\zeta)(.625)\right]^2}}$$

which is solved yielding $\zeta = 0.566$. Then the value of the seismic mass is

$$m = \frac{c}{2\zeta\omega_n} = \frac{8.0\dfrac{\text{N}\cdot\text{sec}}{\text{m}}}{2(0.566)\left(80\dfrac{\text{cycles}}{\text{sec}}\right)\left(\dfrac{2\pi\,\text{rad}}{1\,\text{cycle}}\right)} = 0.014\,\text{kg}$$

The accelerometer stiffness is

$$k = m\omega_n^2 = (0.014\,\text{kg})\left[\left(80\frac{\text{cycles}}{\text{sec}}\right)\left(\frac{2\pi\,\text{rad}}{1\,\text{cycle}}\right)\right]^2 = 3540\frac{\text{N}}{\text{m}}$$

Problem 4.96 illustrates error in accelerometer measurement and calculation of accelerometer parameters.

4.97 Vibrations of a 30 kg machine occur at 150 rad/s with an amplitude of 0.003 mm.

(a) Design an energy harvester of damping ratio 0.2 that harvests theoretical maximum power over one cycle of vibrations from the body. (b) What is the power harvested by this harvester in one hour?

Given: m = 30 kg, $\omega = 150\ \frac{\text{rad}}{\text{s}}$, $\zeta = 0.2$, X = 0.003 mm

Find: ω_n, P

Solution: (a) The natural frequency is obtained from Eq.(4.208) as

$$r = \left[\frac{1}{3}\left(1 - 2\zeta^2 \pm \sqrt{4 - \zeta^2 + \zeta^4}\right)\right]^{0.5} = \left[\frac{1}{3}\left(1 - 2(0.2)^2 \pm \sqrt{4 - (0.2)^2 + (0.2)^2}\right)\right]^{0.5} = 0.983$$

Hence

$$\omega_n = \frac{\omega}{r} = \frac{150\ \frac{\text{rad}}{\text{s}}}{0.983} = 152.6\ \frac{\text{rad}}{\text{s}}$$

(b) The theoretical power harvester over one cycle is

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$$\bar{P} = m\omega^3 Y^2 \Phi(0.983, 0.2) = (30\text{ kg})\left(150\ \frac{\text{rad}}{\text{s}}\right)^3 (3\times 10^{-6}\text{ m})^2 \left[\frac{0.2}{0.982}\left(\frac{(0.982)^4}{(1-0.982^2)^2 + (2(0.2)(0.982))^2}\right)\right]$$

$$= 0.0011\text{ W}$$

The number of cycles executed in one hour is

$$n = (1\text{ hr})\left(\frac{3600\text{ s}}{\text{hr}}\right)\left(150\ \frac{\text{rad}}{\text{s}}\right)\left(\frac{1\text{ cycle}}{2\pi\text{ rad}}\right) = 8.59\times 10^4\text{ cycles}$$

The power harvested in one hour is

$$P = n\bar{P} = (8.59\times 10^4\text{ cycles})(0.0011\text{ W}) = 94.6\text{ W}$$

Problem 4.97 illustrates an energy harvester.

4.98 An energy harvester is being designed to harvest the vibrations from a 200 kg machine that has a rotating unbalance of 0.1 kg·m which operates at 1000 rpm. The harvester is to have a mass of 1 kg and a damping ratio of 0.1. (a) What is the stiffness of the harvester? (b) What is the power harvested from the machine if it operates continuously in one day.

Given: m = 200 kg, $m_0 e = 0.1\text{ kg}\cdot\text{m}$, ω = 1000 rpm, $\zeta = 0.1$

Find: k, P

Solution: The frequency ratio of the harvester is determined from Fig. 4.46 as r = 0.9962 which gives

$$\omega_n = \frac{\omega}{r} = \frac{(1000\text{ rpm})\left(\frac{2\pi}{60}\frac{\text{rad/s}}{\text{rpm}}\right)}{0.9962} 105.1\ \frac{\text{rad}}{\text{s}}$$

Then

$$k = m\omega_n^2 = (200\text{ kg})\left(105.1\ \frac{\text{rad}}{\text{s}}\right)^2 = 2.21\times 10^6\ \frac{\text{N}}{\text{m}}$$

(b) The amplitude provided by the rotating unbalance is

$$Y = \frac{0.1}{200} = 5\times 10^{-4}$$

The average power harvested per cycle of motion is

$$\bar{P} = m\omega^3 Y^2 \Phi(0.9962, 0.1)$$

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$$= (200\text{ kg})\left((1000\text{ rpm})\left(\frac{2\pi}{60}\frac{\text{rad}}{\text{s}}\right)\right)^3 (5\times10^{-4}\text{ m})^2\left[\frac{0.1}{0.9963}\left(\frac{(0.9963)^4}{(1-0.9963^2)^2+(2(0.1)(0.9963))^2}\right)\right]$$

$$= 142.9\text{ W}$$

The number of cycles in one day is

$$n = (1\text{ day})\left(\frac{24\text{ hr}}{\text{day}}\right)\left(\frac{3600\text{ s}}{\text{hr}}\right)\left(104.7\ \frac{\text{rad}}{\text{s}}\right)\left(\frac{1\text{ cycle}}{2\pi\text{ rad}}\right) = 1.44\times10^6\text{ cycles}$$

The power harvested in one day is

$$P = n\bar{P} = (1.44\times10^6\text{ cycles})(142.9\text{ W}) = 2.06\times10^5\text{ kW}$$

Problem 4.98 illustrates the use of an energy harvester.

4.99 An energy harvester is being designed for a vehicle with a simplified suspension system similar to that in the benchmark examples. The harvester, which is to be mounted on the vehicle, is to harvest energy as the vehicle vibrates while traveling. The harvester will have a mass of 0.1 kg, damping ratio 0.1 and natural frequency 30 rad/s. Estimate how much power is harvested over one cycle of a sinusoidal road with a spatial period of 10 m and amplitude of 5 mm while the vehicle is traveling at 50 m/s.

Given: m = 0.1 kg, $\zeta = 0.1, \omega = 30\text{ rad/s}$, d = 10 m, A = 5 mm, v = 50 m/s

Find: Energy harvested in one hour

Solution: The frequency of the vehicle traveling over the road is

$$\omega = \frac{2\pi v}{d} = \frac{2\pi\left(50\ \frac{\text{m}}{\text{s}}\right)}{10\text{ m}} = 31.41\ \frac{\text{rad}}{\text{s}}$$

The frequency ratio is

$$r = \frac{31.41\ \frac{\text{rad}}{\text{s}}}{30\ \frac{\text{rad}}{\text{s}}} = 1.05$$

If the amplitude of the road is 5 mm, then the suspension system of the vehicle reduces that to 1 mm. Thus the energy harvested over 1 cycle is

$$\bar{P} = m\omega^3 Y^2 \frac{\zeta}{r}\Lambda^2(1.05,0.1)$$

$$= (0.1\text{ kg})\left(31.41\ \frac{\text{rad}}{\text{s}}\right)^3 (0.001\text{ m})^2 \frac{0.1}{1.05}\frac{(1.05)^4}{(1-1.05^2)^2+[2(0.1)(1.05)]^2}$$

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$$= 2.2 \times 10^{-3}\ \text{W}$$

The energy harvested over one hour is

$$P = \left(\frac{31.41}{2\pi}\frac{\text{cycles}}{\text{s}}\right)\left(\frac{3600\ \text{s}}{1\ \text{hr}}\right)\left(2.2 \times 10^{-3}\ \frac{\text{W}}{\text{cycle}}\right) = 39.6\ \frac{\text{W}}{\text{hr}}$$

Problem 4.99 illustrates the use of energy harvesters.

4.100 How much energy is harvested over one period by the energy harvester of Problem 4.99 if the vehicle is traveling over 50 m/s over a road whose contour is shown in Figure P4.100?

FIGURE P4.100

Given: Y(ξ)

Find: Energy harvested

Solution: Since the vehicle is traveling at 50 m/s the fundamental frequency of the Fourier series is

$$\boldsymbol{\omega_1} = \frac{2\pi(50\ \text{m/s})}{2.8\ \text{m}} = 112.2\ \text{rad/s}$$

with a period of

$$T = \frac{2\pi}{\omega_1} = 0.056\ \text{s}$$

The Fourier coefficients are

$$a_i = \frac{1}{0.056}\int_0^{0.002} h\cos(112.2it)\,dt = \frac{h}{112.2i(0.056)}\sin(0.224it)$$
$$= \frac{0.159h}{i}\sin(0.224it)$$

$$\boldsymbol{b_i} = \frac{1}{0.056}\int_0^{0.002} h\sin(112.2it)\,dt = \frac{h}{112.2i(0.056)}[1-\cos(0.224\text{it})]$$
$$= \frac{0.159h}{i}[1-\cos(0.224\text{it})]$$

The Fourier series is represented by

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$$y(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} c_i \sin(\omega_i t + \kappa_i)$$

where

$$c_i = \sqrt{a_i^2 + b_i^2}$$

The relative displacement of the seismic mass relative to the body is

$$z(t) = \sum_{i=1}^{\infty} c_i \, \Lambda_{\mathrm{i}}(\mathrm{r}, \zeta)$$

The power dissipated by the viscous damper over one period is

$$P = \int_0^{\frac{2\pi}{\omega_1}} c \left(\sum_{i=1}^{\infty} c_i \, \Lambda_{\mathrm{i}}(\mathrm{r}, \zeta) \omega_{\mathrm{i}} \cos(\omega_i t + \kappa_i) \right)^2 dt$$

$$= c \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \Lambda(\mathrm{r_i}, \zeta) \Lambda(\mathrm{r_j}, \zeta) \omega_{\mathrm{i}} \omega_{\mathrm{j}} \int_o^{\frac{2\pi}{\omega_1}} \cos(\omega_j t + \kappa_j) \cos(\omega_i t + \kappa_i) \, dt$$

$$= \frac{c}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \Lambda(\mathrm{r_i}, \zeta) \Lambda(\mathrm{r_j}, \zeta) \left[\frac{\sin\left((\mathrm{i}+\mathrm{j})2\pi + \kappa_{\mathrm{i}} + \kappa_{\mathrm{j}}\right)}{(\mathrm{i}+\mathrm{j})\omega_1} + \frac{\sin\left((\mathrm{i}-\mathrm{j})2\pi + \kappa_{\mathrm{i}} - \kappa_{\mathrm{j}}\right)}{(\mathrm{i}-\mathrm{j})\omega_1} \right]$$

The above equation can be evaluated for the harvester.

Problem 4.100 illustrates energy harvesting for a periodic motion.

4.101 An energy harvester is being designed to harvest energy from a MEMS system. The harvester consists of a micro-cantilever beam vibrating in a viscous liquid such that its damping ratio is 0.2. The micro-cantilever is made of silicon ($E = 1.9 \times 10^{11}$ N/m^2) is 30 μm long, is rectangular in cross section, has a base width of 2 μm, and a height of 0.5 μm. The mass density of silicon is 2.3 g/cm^3. (a) What is the natural frequency of the energy harvester using a SDOF model? Use the equivalent mass of a cantilever beam at its end. (b) What energy is harvested over one cycle of motion if the harvesting occurs at the natural frequency with a vibration amplitude of 1 μm? (c) What is the average power harvested over one cycle? (d) What is the power harvested over one hour?

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Given: $E = 1.9 \times 10^{11}\ N/m^2$, L = 30 μm, w = 2 μm h = 0.5 μm, $\rho = 2.3$ g/cm^3

Find: ω_n, E, $\bar{P}$

Solution: (a) The equivalent mass of a fixed-free beam is approximately

$$m_{eq} = 0.29\rho AL = 0.29\left(2300\ \frac{\text{kg}}{\text{m}^3}\right)[(2 \times 10^{-9}\ \text{m})(0.5 \times 10^{-9}\ \text{m})](30 \times 10^{-9}\ \text{m})$$
$$= 2 \times 10^{-23}\ \text{kg}$$

The stiffness is

$$k = \frac{3EI}{L^3} = \frac{3\left(1.9 \times 10^{11}\ \frac{\text{N}}{\text{m}^2}\right)\left[\frac{1}{12}(2 \times 10^{-9}\ \text{m})(0.5 \times 10^{-9}\ \text{m})^3\right]}{(30 \times 10^{-9}\ \text{m})^3} = 4.05 \times 10^{-4}\ \frac{\text{N}}{\text{m}}$$

The natural frequency is

$$\omega_n = \sqrt{\frac{4.05 \times 10^{-4}\ \frac{\text{N}}{\text{m}}}{2 \times 10^{-23}\ \text{kg}}} = 4.5 \times 10^9\ \frac{\text{rad}}{\text{s}}$$

(b) The average power harvested over one cycle is

$$\bar{P} = m\omega^3 Y^2 \frac{\zeta}{r}\Lambda^2(1,0.1) = (2 \times 10^{-23}\ \text{kg})\left(4.5 \times 10^9\ \frac{\text{rad}}{\text{s}}\right)^3 (1 \times 10^{-6})^2 \frac{0.2}{1}\frac{1}{[2(0.2)]^2}$$
$$= 5.69 \times 10^{-5}\ \frac{\text{W}}{\text{cycle}}$$

(c) The energy harvested is related to the power by

$$E = \frac{2\pi}{\omega}\bar{P} = \frac{2\pi}{4.5 \times 10^9\ \frac{\text{rad}}{\text{s}}} 5.69 \times 10^{-5}\ \frac{\text{W}}{\text{cycle}} = 7.94 \times 10^{-32}\ \text{J}$$

(d) The power harvested in one hour is

$$P = \left(\frac{4.5 \times 10^9}{2\pi}\frac{\text{cycles}}{\text{s}}\right)\left(\frac{3600\ \text{s}}{1\ \text{hr}}\right)\left(5.69 \times 10^{-5}\ \frac{\text{W}}{\text{cycle}}\right) = 146\ \frac{\text{MW}}{\text{hr}}$$

Problem 4.101 illustrates energy harvesters.

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CHAPTER 5: TRANSIENT VIBRATIONS OF SDOF SYSTEMS

Short Answer Problems

5.1 True: The convolution integral solves the differential equation for a SDOF system.

5.2 True: Alternate derivations of the convolution integral use variation of parameters (assuming a particular solution which is a linear combination of the homogenous solution with coefficients that vary with time and deriving integrals for the coefficients) or the Laplace transform method (applying the Laplace transform method to the differential equation, reducing the solution for the transform of the differential equation to the product of two transforms and using the convolution property to invert)

5.3 False: The effect of an impulse applied to a SDOF system is to cause a discrete change in velocity.

5.4 False: The Laplace transform method involves the initial conditions in the solution of the differential equation.

5.5 True: Numerical integration of the convolution integral can be obtained by interpolating the forcing function and exactly integrating the interpolation times $h(t-\tau)$. The method uses piecewise impulses, piecewise constants or piecewise linear functions to interpolate the forcing function.

5.6 True: Self-starting methods use initial conditions to start the integration process.

5.7 False: The transfer function for a SDOF system is the ratio of the Laplace transform of the output to the Laplace transform of the input.

5.8 False: The transfer function is the Laplace transform of the impulsive response of a system.

5.9 False: The maximum force transferred to the foundation from a machine mounted on an isolator due to an impulsive force is minimized by selecting the damping ratio of the system to be 0.25.

5.10 True: The transmitted force is flat for a damping ratio between 0.23-0.3.

5.11 The function $h(t)$is the response of a system due to a unit impulse, or the system's impulsive response.

5.12 The principle of impulse and momentum is used in the derivation of the convolution integral.

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5.13 The convolution integral represents the solution to the differential equation governing the motion of a SDOF system due to any type of excitation.

5.14 $x(1) = \int_0^1 F(\tau)h(1-\tau)d\tau$ represents the response of a system with an impulsive response h(t) at a time of 1 second due to an applied force F(t).

5.15 A pulse of short duration is applied over a short enough time such that the shape of the pulse has little effect on the response of the system. Only the total impulse imparted to the system by the pulse has an effect. The system is then modeled as a system undergoing free vibrations with an initial displacement equal to zero and an initial velocity equal to I/m where I is the impulse imparted by the pulse.

5.16 The response spectrum of a pulse is a plot of kx_{max}/F_0versus the nondimensional time $\omega_n t_0/(2\pi)$.

5.17 The impulsive response of a system due to motion input does not exist because an impulsive motion input means an instantaneous change in motion input which is not possible.

5.18 Given: m = 2 kg, k = 1000 N/m, I =12 N·s. The velocity imparted to the system is $v = \frac{I}{m} = \frac{12\text{ N·s}}{2\text{ kg}} = 6$ m/s.

5.19 Given: system shown, $G(s) = \frac{1}{5s^2+30s+100}$

FIGURE SP5.19

5.20 Given: system shown, $G(s) = \frac{100s+1000}{10s^2+100s+1000}$

FIGURE SP5.20

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5.21 Given: system shown, $G(s) = \frac{1}{5s^2+10s+250}$

FIGURE SP5.21

5.22 Given: m = 5 kg, k = 1000 N/m, $\zeta = 0$ The natural frequency of the system is $\omega_n = \sqrt{\frac{1000\ \text{N/m}}{5\ \text{kg}}} = 14.1\ \text{rad/s}$. The impulsive response of an undamped system is $x(t) = \frac{1}{m\omega_n}\sin\omega_n t = \frac{1}{(5\ \text{kg})(14.1\ \text{rad/s})}\sin 14.1t = 0.141\sin 14.1t$.

5.23 Given: I =15 N · s, m = 0.5 kg, k = 200 N/m. The natural frequency of the system is $\omega_n = \sqrt{\frac{200\ \text{N/m}}{0.5\ \text{kg}}} = 20\ \text{rad/s}$. The response of the system is $x(t) = \frac{I}{m\omega_n}\sin\omega_n t = \frac{15\ \text{N·s}}{(0.5\ \text{kg})(20\ \text{rad/s})}\sin 20t = 1.5\sin 20t$.

5.24 (a)-(vi); (b)-(iii); (c)-(i); (d)-(i); (e)-(iii); (f)-(ii)

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Chapter Problems

5.1 A SDOF system with m = 20 kg, k = 10000 N/m and c = 540 $\mathrm{N \cdot s/m}$ is at rest in equilibrium when a 50 $\mathrm{N \cdot s}$ impulse is applied. Determine the response of the system.

Given: m = 20 kg, k = 10000 N/m and c = 540 $\mathrm{N \cdot s/m}$, I = 50 $\mathrm{N \cdot s}$

Find: $x(t)$

Solution: The natural frequency and damping ratio of the system are

$$\omega_n = \sqrt{\frac{k}{m}} = 22.36\ \frac{\mathrm{rad}}{\mathrm{s}}$$

$$\zeta = \frac{c}{2m\omega_n} = 0.604$$

The system's damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 17.83\ \ \mathrm{rad/s}$$

The response of an underdamped SDOF system to an impulse is

$$x(t) = \frac{I}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t = \frac{50\ \mathrm{N \cdot s}}{(20\ \mathrm{kg})(17.83\ \ \mathrm{rad/s})}e^{-0.604(22.36t)}\sin 17.83t$$
$$= 0.1402e^{-13.5t}\sin 17.83t$$

Problem 5.1 illustrates the response of a SDOF system subject to an impulse.

5.2 A SDOF system is with m = 10 kg, k = 40,000 N/m, and c = 300 $\mathrm{N \cdot s/m}$ is at rest in equilibrium when a 80 $\mathrm{N \cdot s}$ impulse is applied. This is followed by a 40 $\mathrm{N \cdot s}$ impulse 0.02 sec later. Determine the response of the system.

Given: m = 10 kg, k = 40,000 N/m and c = 300 $\mathrm{N \cdot s/m}$, I_1 = 80 $\mathrm{N \cdot s}$, I_2 = 40 $\mathrm{N \cdot s}$, $t_0 = 0.02$ s

Find: $x(t)$

Solution: The natural frequency and damping ratio of the system are

$$\omega_n = \sqrt{\frac{k}{m}} = 63.25\ \frac{\mathrm{rad}}{\mathrm{s}}$$

$$\zeta = \frac{c}{2m\omega_n} = 0.237$$

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The system's damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 61.44 \text{ rad/s}$$

The response of an underdamped SDOF system to the two impulses is obtained using the method of superposition as

$$\begin{aligned} x(t) &= \frac{I_1}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t + \frac{I_2}{m\omega_d}e^{-\zeta\omega_n(t-t_0)}\sin\omega_d(t-t_0)\,u(t-t_0) \\ &= \frac{80 \text{ N}\cdot\text{s}}{(10 \text{ kg})\left(61.44 \frac{\text{rad}}{\text{s}}\right)}e^{-0.237(63.25t)}\sin 61.44t \\ &\quad + \frac{40 \text{ N}\cdot\text{s}}{(10 \text{ kg})\left(61.44 \frac{\text{rad}}{\text{s}}\right)}e^{-0.237(63.25)(t-0.02)}\sin 61.44(t-0.02)\,u(t-0.02) \\ &= 0.1302e^{-15t}\sin 61.44t + 0.0651e^{-15(t-0.02)}\sin 61.44(t-0.02)\,u(t-0.02) \end{aligned}$$

Problem 5.2 illustrates application of multiple impulses to an underdamped SDOF system.

5.3 A SDOF system with m = 1.3 kg, k = 12,000 N/m, and c = 400 N · s/m is at rest in equilibrium when a 100 N · s impulse is applied. This is followed by a 150 N · s impulse 0.12 sec later. Determine the response of the system.

Given: m = 1.3 kg, k = 12,000 N/m, and c = 400 N · s/m, I_1 = 100 N · s, I_2 = 150 N · s, $t_0 = 0.12$ s

Find: $x(t)$

Solution: The natural frequency and damping ratio of the system are

$$\omega_n = \sqrt{\frac{k}{m}} = 96.07 \frac{\text{rad}}{\text{s}}$$

$$\zeta = \frac{c}{2m\omega_n} = 1.60$$

The response of an overdamped SDOF system to the two impulses is obtained using the method of superposition as

$$\begin{aligned} x(t) &= \frac{I_1}{2\omega_n\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}\left(e^{\omega_n\sqrt{\zeta^2-1}t} - e^{-\omega_n\sqrt{\zeta^2-1}t}\right) \\ &\quad + \frac{I_2}{2\omega_n\sqrt{\zeta^2-1}}e^{-\zeta\omega_n(t-t_0)}\left(e^{\omega_n\sqrt{\zeta^2-1}(t-t_0)} - e^{-\omega_n\sqrt{\zeta^2-1}(t-t_0)}\right)u(t-t_0) \end{aligned}$$

Noting that $\omega_n\sqrt{\zeta^2-1} = 120.2$ rad/s

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$$= \frac{10\ \mathrm{N\cdot s}}{2\left(120.2\ \frac{\mathrm{rad}}{\mathrm{s}}\right)} e^{-1.6(96.07)t}\left(e^{(120.2)t} - e^{-(120.2)t}\right)$$
$$+ \frac{150\ \mathrm{N\cdot s}}{2\left(120.2\ \frac{\mathrm{rad}}{\mathrm{s}}\right)} e^{-1.6(96.07)t}\left(e^{(120.2)(t-0.12)} - e^{-(120.2)(t-0.12)}\right)u(t-0.12)$$
$$= 0.4161(e^{-33.5t} - e^{-273.7t}) + 0.6242\left(e^{-33.5(t-0.12)} - e^{0273.3(t-0.12)}\right)u(t-0.12)$$

Problem 5.3 illustrates application of multiple impulses to an overdamped SDOF system.

5.4 Use the method of variation of parameters to obtain the general solution of Equation (5.1)

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m} \tag{1}$$

and show that it can be written in the form of the convolution integral, Equation (5.25).

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n t}\sin\omega_d(t-\tau)d\tau \tag{2}$$

Given: Equation (1)

Show: Equation (2) using variation of parameters

Solution: The homogeneous solution of eq. (1) is

$$x_h(t) = C_1 e^{-\zeta\omega_n t}\cos\omega_d t + C_2 e^{-\zeta\omega_n t}\sin\omega_d t$$

Application of the method of variation of parameters involves assuming a particular solution of the form of the homogeneous solution, but with the constants replaced by unknown functions of time,

$$x(t) = C_1(t)e^{-\zeta\omega_n t}\cos\omega_d t + C_2(t)e^{-\zeta\omega_n t}\sin\omega_d t \tag{3}$$

Differentiating eq. (3) with respect to time

$$\dot{x}(t) = C_1\left(-\zeta\omega_n e^{-\zeta\omega_n t}\cos\omega_d t - \omega_d e^{-\zeta\omega_n t}\sin\omega_d t\right) + \dot{C}_1 e^{-\zeta\omega_n t}\cos\omega_d t$$
$$+ C_2\left(-\zeta\omega_n e^{-\zeta\omega_n t}\sin\omega_d t + \omega_d e^{-\zeta\omega_n t}\cos\omega_d t\right) + \dot{C}_2 e^{-\zeta\omega_n t}\sin\omega_d t$$

The algebra is simplified by choosing

$$e^{-\zeta\omega_n t}\cos\omega_d t\dot{C}_1 + e^{-\zeta\omega_n t}\sin\omega_d t\dot{C}_2 = 0 \tag{4}$$

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Then

$$\begin{aligned}\ddot{x}(t) &= \dot{C}_1\left(-\zeta\omega_n e^{-\zeta\omega_n t}\cos\omega_d t - \omega_d e^{-\zeta\omega_n t}\sin\omega_d t\right)\\ &+ C_1\left(\zeta^2\omega_n^2 e^{-\zeta\omega_n t}\cos\omega_d t + 2\zeta\omega_n\omega_d e^{-\zeta\omega_n t}\sin\omega_d t - \omega_d^2 e^{-\zeta\omega_n t}\cos\omega_d t\right)\\ &+ \dot{C}_2\left(-\zeta e^{-\zeta\omega_n t}\sin\omega_d t + \omega_d e^{-\zeta\omega_n t}\cos\omega_d t\right)\\ &+ C_2\left(\zeta^2\omega_n^2 e^{-\zeta\omega_n t}\sin\omega_d t - 2\zeta\omega_n\omega_d e^{-\zeta\omega_n t}\cos\omega_d t - \omega_d^2 e^{-\zeta\omega_n t}\sin\omega_d t\right)\end{aligned} \tag{5}$$

Substituting into eq.(1) and simplifying leads to

$$\begin{aligned}&\dot{C}_1 e^{-\zeta\omega_n t}\left(-\zeta\omega_n\cos\omega_d t - \omega_d\sin\omega_d t\right)\\ &+ \dot{C}_2 e^{-\zeta\omega_n t}\left(-\zeta\omega_n\sin\omega_d t + \omega_d\cos\omega_d t\right) = \frac{F(t)}{m}\end{aligned} \tag{6}$$

Equation (6) is simplified by using eq. (4)

$$-\omega_d\dot{C}_1 e^{-\zeta\omega_n t}\sin\omega_d t + \omega_d\dot{C}_2 e^{-\zeta\omega_n t}\cos\omega_d t = \frac{F(t)}{m\omega_d} \tag{7}$$

Equations (5) and (8) can be solved simultaneously yielding

$$\begin{aligned}\dot{C}_1 &= \frac{F(t)}{m\omega_d}e^{\zeta\omega_n t}\sin\omega_d t\\ \dot{C}_2 &= \frac{F(t)}{m\omega_d}e^{\zeta\omega_n t}\cos\omega_d t\\ C_1(t) &= -\int_0^t \frac{F(\tau)}{m\omega_d}e^{\zeta\omega_n\tau}\sin\omega_d\tau\ d\tau\\ C_2(t) &= \int_0^t \frac{F(\tau)}{m\omega_d}e^{\zeta\omega_n\tau}\cos\omega_d\tau\ d\tau\end{aligned} \tag{8}$$

Substitution of eq. (8) into eq. (3) leads to

$$\begin{aligned}x(t) &= \int_0^t \frac{F(\tau)}{m\omega_d}e^{-\zeta\omega_n(t-\tau)}\left(-\cos\omega_d t\sin\omega_d\tau + \sin\omega_d t\cos\omega_d\tau\right)d\tau\\ &= \int_0^t \frac{F(\tau)}{m\omega_d}e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau\end{aligned}$$

Problem 5.4 illustrates derivation of the convolution integral using the method of variation of parameters.

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5.5 Use the convolution integral to determine the response of an underdamped SDOF system of mass *m* and natural frequency ω_n when the excitation is the unit step function, $u(t)$.

Given: m, ω_n, $\zeta < 1$, $F(t) = F_0 u(t)$

Find: x(t)

Solution: Using the convolution integral

$$x(t) = \int_0^t F(\tau) h(t-\tau) d\tau$$
$$= \frac{1}{\tilde{m}\omega_d} \int_0^t F_0 u(\tau) e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau \tag{1}$$

The integral is easiest evaluated by letting

$$u = t - \tau \tag{2}$$

Changing the variable of integration from τ to u leads to

$$x(t) = -\frac{F_0}{\tilde{m}\omega_d} \int_t^0 e^{-\zeta\omega_n u} \sin \omega_d u\, du \tag{3}$$

The integral in eq. (3) can be evaluated by referring to a table of integrals or integration by parts twice. Either method leads to

$$x(t) = \frac{F_0}{\tilde{m}\omega_d} \left[\frac{\omega_d - e^{-\zeta\omega_n u} (\zeta \sin \omega_d t + \omega_d \cos \omega_d t)}{\omega_n^2} \right]_{u=t}^{u=0}$$
$$= \frac{F_0}{\tilde{m}\omega_n\omega_d} \left[\sqrt{1-\zeta^2} - e^{-\zeta\omega_n t} \left(\zeta \sin \omega_d t + \sqrt{1-\zeta^2} \cos \omega_d t \right) \right]$$

Problem 5.5 illustrates the application of the convolution integral to determine the response of an underdamped one-degree-of-freedom system to the unit step function.

5.6 Let $g(t)$ be the response of an underdamped system to a unit step function and $h(t)$ the response of an underdamped system to the unit impulse function. Show

$$h(t) = \frac{dg}{dt} \tag{1}$$

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Given: h(t), g(t)

Show: eq.(1)

Solution: The response of an underdamped system to a unit impulse is

$$h(t) = \frac{1}{\tilde{m}\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{2}$$

From problem 5.5 the response of an underdamped system to the unit step function is

$$g(t) = \frac{F_0}{\tilde{m}\omega_n\omega_d}\left[\sqrt{1-\zeta^2} - e^{-\zeta\omega_n t}\left(\zeta \sin \omega_d t + \sqrt{1-\zeta^2} \cos \omega_d t\right)\right] \tag{3}$$

Differentiating eq.(3) with respect to t, using the product rule

$$\frac{dg}{dt} = -\frac{F_0}{\tilde{m}\omega_n\omega_d}\left[-\zeta\omega_n e^{-\zeta\omega_n t}\left(\zeta \sin \omega_d t + \sqrt{1-\zeta^2} \cos \omega_d t\right)\right.$$
$$\left. + e^{-\zeta\omega_n t}\left(\zeta\omega_d \cos \omega_d t - \omega_d \sqrt{1-\zeta^2} \sin \omega_d t\right)\right]$$

$$\frac{dg}{dt} = \frac{F_0}{\tilde{m}\omega_n\omega_d} e^{-\zeta\omega_n t}\omega_n\left[\left(1-\zeta^2+\zeta^2\right)\sin \omega_d t + \left(\zeta\sqrt{1-\zeta^2} - \zeta\sqrt{1-\zeta^2}\right)\cos \omega_d t\right]$$

Hence

$$\frac{dg}{dt} = \frac{F_0}{\tilde{m}\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$
$$= h(t)$$

Problem 5.6 illustrates that the response of a system due to a unit impulse function equals the derivative of the response of the system due to the unit step function.

5.7 Use the convolution integral and the notation and results of Chapter Problem 5.6 to derive the following alternative expression for the response of a system subject to an excitation, $F(t)$:

$$x(t) = F(0)g(t) + \int_0^t \frac{dF(\tau)}{d\tau} g(t-\tau)\, d\tau \tag{1}$$

Given: x(t)

Find: Show eq. (1)

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Solution: The response of a one-degree-of-freedom system subject to the excitation F(t) is determined using the convolution integral as

$$x(t)=\int_0^t F(\tau)h(t-\tau)d\tau \tag{2}$$

The general integration by parts formula is

$$\int u\,dv = uv - \int v\,du$$

Let

$$u = F(\tau)$$
$$dv = h(t-\tau)d\tau$$

The results of problem 5.6 show that

$$h(t)=\frac{dg}{dt}$$

Hence

$$du = \frac{dF}{d\tau}d\tau$$
$$v = -g(t-\tau)$$

Application of integration by parts to eq. (2) results in

$$x(t)=\left[-F(\tau)g(t-\tau)\right]_{\tau=0}^{\tau=t}+\int_0^t \frac{dF}{d\tau}g(t-\tau)d\tau$$

or

$$x(t)=F(0)g(t)-F(t)g(0)+\int_0^t \frac{dF}{d\tau}g(t-\tau)d\tau$$

But g(0) = 0, hence eq.(1) is obtained

Problem 5.7 illustrates an alternate form of the convolution integral.

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5.8 A SDOF undamped system is initially at rest in equilibrium and subject to a force $F(t) = F_0 te^{-t/2}$. Use the convolution integral to determine the response of the system.

Given: m, ω, F(t)

Find: x(t)

Solution: The convolution integral yields the response of an undamped system as

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

For the excitation given

$$x(t) = \frac{1}{m\omega_n}\int_0^t F_0\tau e^{-\frac{\tau}{2}}\sin\omega_n(t-\tau)\,d\tau$$

Evaluation of the integral leads to

$$x(t) = -\frac{F_0}{m\omega_n}\left\{-\frac{\omega_n t e^{-\frac{t}{2}}}{\omega_n^2+\frac{1}{4}} - \frac{1}{2}\frac{\omega_n e^{-\frac{t}{2}}}{\left(\omega_n^2+\frac{1}{4}\right)^2}\right.$$

$$\left. + \frac{1}{\left(\omega_n^2+\frac{1}{4}\right)^2}\left[\left(\frac{1}{4}-\omega_n^2\right)\sin\omega_n t + \frac{1}{2}\omega_n\cos\omega_n t\right]\right\}$$

Problem 5.8 illustrates use of the convolution integral to determine the transient response of an undamped one-degree-of-freedom system.

5.9 The mass of Figure P5.9 has a velocity v when it engages the spring-dashpot system. Let $x(t)$ be the displacement of the mass from the position where the mechanism is engaged. Use the convolution integral to determine $x(t)$. Assume the system is underdamped.

FIGURE P5.9

Given: m, v, k, c, θ

Find: x(t)

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Solution: Since x(t) is measured from when the mechanism is engaged, the force in the spring in the initial position is zero. Thus, consider free-body diagrams of the system at an arbitrary time.

Summing forces in the direction along the surface

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$

leads to

$$mg\sin\theta - kx - c\,\dot{x} = m\,\ddot{x}$$
$$m\ddot{x} + c\dot{x} + kx = mg\sin\theta$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = mg\sin\theta$$

where

$$\omega_n = \sqrt{\frac{k}{m}},\ \zeta = \frac{c}{2\sqrt{mk}}$$

The initial conditions for the motion are

$$x(0) = 0,\quad \dot{x}(0) = v$$

The convolution integral with non-zero initial conditions, is used with

$$F(t) = mg\sin\theta$$

leading to

$$x(t) = \frac{v}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t + \frac{1}{m\omega_d}\int_0^t mg\sin\theta e^{-\zeta\omega_n\tau}\sin\omega_d(t-\tau)\,d\tau$$
$$= \frac{v}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t + \frac{g\sin\theta}{\omega_n^2}\left[1 + e^{-\zeta\omega_n t}\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t - \cos\omega_d t\right)\right]$$

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Problem 5.9 illustrates (a) that when the generalized coordinate is measured from a position other than the system's equilibrium position, nonhomogeneous terms occur in the governing equation and, (b) the use of the convolution integral with non-zero initial conditions.

5.10 Use the convolution integral to determine the response of the system of Figure P5.10.

FIGURE P5.10

Given: System shown

Find: x(t)

Solution: Free body diagrams of the system at an arbitrary instant are shown below.

EXTERNAL FORCES EFFECTIVE FORCES

Summing moments about the point of support

$$\left(\sum M_0\right)_{ext.} = \left(\sum M_0\right)_{eff.}$$

$$-k\frac{L}{3}\theta\frac{L}{3} - 2k\frac{L}{3}\theta\frac{L}{3} + M_0e^{-t/5} = \frac{1}{12}mL^2\ddot{\theta} + m\frac{L}{6}\ddot{\theta}\frac{L}{6}$$

$$m\frac{L^2}{9}\ddot{\theta} + k\frac{L^2}{3} = M_0e^{-\frac{t}{5}}$$

The system is undamped with a natural frequency of

$$\omega_n = \sqrt{\frac{3k}{m}}$$

The convolution integral is used to write the solution as

$$\theta(t) = \frac{9M_0}{mL^2\omega_n}\int_0^t e^{-\frac{\tau}{5}}\sin\omega_n(t-\tau)$$

Integration and application of limits results in

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$$\theta(t)=\frac{9M_0}{mL^2\omega_n}\frac{1}{\omega_n^2+\frac{1}{25}}\left(\omega_n e^{-\frac{t}{5}}+\frac{1}{5}\sin\omega_n t-\omega_n\cos\omega_n t\right)$$

Problem 5.10 illustrates application of the convolution integral to determine the response of a one-degree-of-freedom system.

5.11 Use the convolution integral to determine the response of an underdamped SDOF system of natural frequency ω_n and damping ratio ζ when subject to a harmonic excitation $F(t) = F_0 \sin \omega t$.

Given: $F(t) = F_0 \sin \omega t$

Find: x(t)

Solution: The convolution integral for the response of the system is

$$x(t)=\frac{1}{m\omega_d}\int_0^t F_0\sin(\omega\tau)e^{-\zeta\omega_n(t-\tau)}\sin[\omega_d(t-\tau)]d\tau$$

The integral is evaluated to yield

$$x(t)=\frac{F_0}{mD}\left[-2\zeta\omega\omega_n\cos(\omega t)+(\omega_n^2-\omega^2)\sin\omega t\right]$$
$$-\frac{F\omega_0}{m\omega_d D}e^{-\zeta\omega_n t}\left[-2\zeta\omega\omega_d\cos(\omega_d t)+(\omega_n^2(1-2\zeta^2)-\omega^2)\sin(\omega t)\right]$$

where

$$D=(\omega_n^2+\omega^2+2\omega\omega_d)(\omega_n^2+\omega^2-2\omega\omega_d)$$

Problem 5.11 illustrates the use of the convolution integral to determine the forced response of a damped system.

5.12 A machine tool with a mass of 30 kg is mounted on an undamped foundation of stiffness 1500 N/m. During operation, it is subject to the machining forces shown in Figure P5.12. Use the principle of superposition and the convolution integral to determine the response of the system to each force.

FIGURE P5.12

Given: m = 30 kg, k = 1500 N/m, F(t)

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Find: x(t)

Solution: The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500\,\frac{\mathrm{N}}{\mathrm{m}}}{30\,\mathrm{kg}}} = 7.07\,\frac{\mathrm{rad}}{\mathrm{sec}}$$

The graphical breakdown of F(t) into functions whose response is presented in Table 5.1 is shown below.

The mathematical form of F(t) is

$$F(t) = 6000t[u(t) - u(t-0.5)] + (-2000t + 4000)[u(t-0.5) - u(t-2.)]N$$

The response of the system is given by

$$x(t) = x_1(t) - x_2(t) + x_3(t) - x_4(t)$$

where each $x_i(t)$, i = 1, 2, 3, 4 is determined using Table 5.1 as shown below.

$x_1(t)$: Ramp function, A = 6000 N/sec, B = 0, $t_0 = 0$

$$x_1(t) = \frac{6000\,\frac{\mathrm{N}}{\mathrm{sec}}}{1500\,\frac{\mathrm{N}}{\mathrm{m}}}\left[t - \frac{1}{7.07}\sin 7.07t\right]u(t)$$
$$= 4[t - 0.141\sin 7.07t]u(t)\mathrm{m}$$

$x_2(t)$: Ramp function, A = 6000 N/sec, B=0, $t_0 = 0.5$ sec

$$x_2(t) = \frac{6000\,\frac{\mathrm{N}}{\mathrm{sec}}}{1500\,\frac{\mathrm{N}}{\mathrm{m}}}[t - 0.5\cos 7.07(t-0.5) - 0.141\sin 7.07(t-0.5)]u(t-0.5)$$
$$= 4[t - 0.5\cos(7.07t - 3.54) - 0.141\sin(7.07t - 3.54)]u(t-0.5)\mathrm{m}$$

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$x_3(t)$: Ramp function, A = -2000 N/sec, B = 4000 N, $t_0 = 0.5$ sec

$$x_3(t) = \frac{-2000\frac{\text{N}}{\text{sec}}}{1500\frac{\text{N}}{\text{m}}}\left[t-2.-(0.5-2)\cos 7.07(t-0.5)\right.$$
$$\left.-0.141\sin 7.07(t-0.5)\right]u(t-0.5)$$
$$=-1.33\left[t-2.+1.5\cos(7.07t-3.54)-0.141\sin(7.07t-3.54)\right]u(t-0.5)\text{m}$$

$x_4(t)$: Ramp function, A = -2000 N/sec, B = 4000 N, $t_0 = 2.0$ sec

$$x_4(t) = \frac{-2000\frac{\text{N}}{\text{sec}}}{1500\frac{\text{N}}{\text{m}}}\left[t-2.-0.141\sin 7.07(t-2.)\right]u(t-2.)$$
$$=-1.33\left[t-2.-0.141\sin(7.07t-14.14)\right]u(t-2.)$$

Problem 5.12 illustrates (a) graphical breakdown of an excitation whose form changes at discrete times and (b) use of superposition and Table 5.1 to determine the response of an undamped system.

5.13 A machine tool with a mass of 30 kg is mounted on an undamped foundation of stiffness 1500 N/m. During operation, it is subject to the machining forces shown in Figure P5.13. Use the principle of superposition and the convolution integral to determine the response of the system to each force.

Given: m = 30 kg, k = 1500 N/m, F(t)

Find: x(t)

Solution: The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500\frac{\text{N}}{\text{m}}}{30\,\text{kg}}} = 7.07\frac{\text{rad}}{\text{sec}}$$

FIGURE P5.13

The graphical breakdown of the excitation into functions whose responses are available in Table 4.1 is shown below.

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The mathematical from of the excitation is

$$F(t) = 1000 \sin \pi t [u(t) - u(t-2)] \text{N}$$

or since sinπt is periodic of period 2,

$$F(t) = 1000 \sin \pi t u(t) - 1000 \sin \pi (t-2) u(t-2) \text{N}$$

The response of the system is

$$x(t) = x_1(t) - x_2(t)$$

where $x_1(t)$ and $x_2(t)$ are obtained using Table 5.1

$x_1(t)$: Sinusoidal excitation, A = 1000 N, ω = π rad/sec, $t_0 = 0$

Note that $\omega/\omega_n = 0.444$, then

$$x_1(t) = \frac{1000\,\text{N}}{2\left(1500\,\frac{\text{N}}{\text{m}}\right)}\left[\frac{1}{0.444-1}(\sin \pi t - \sin 7.07t) - \frac{1}{0.444+1}(\sin \pi t + \sin 7.07t)\right]$$

$$= \frac{1}{3}(-1.799 \sin \pi t + 1.7999 \sin 7.07t - 0.693 \sin \pi t - 0.693 \sin 7.07t)$$

$$= \frac{1}{3}(-2.49 \sin \pi t + 1.097 \sin 7.07t)\text{m}$$

$x_2(t)$: Sinusoidal excitation, A = 1000 N, ω = π rad/sec, $t_0 = 2$ sec

$$x_2(t) = \frac{1000\,\text{N}}{2\left(1500\,\frac{\text{N}}{\text{m}}\right)}\{-1.799[\sin \pi(t-2) - \sin 7.07(t-2)]$$

$$-0.693[\sin \pi(t-2) + \sin 7.07(t-2)]\}u(t-2)$$

$$= \frac{1}{3}[-2.49 \sin \pi t + 1.097 \sin(7.07t - 14.14)]u(t-2)\text{m}$$

Problem 5.13 illustrates (a) the graphical breakdown of an excitation whose form changes with time and (b) application of superposition and Table 5.1 to determine the response of an undamped system to an excitation whose form changes at discrete times.

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5.14 A machine tool with a mass of 30 kg is mounted on an undamped foundation of stiffness 1500 N/m. During operation, it is subject to the machining forces shown in Figure P5.14. Use the principle of superposition and the convolution integral to determine the response of the system to each force.

FIGURE P5.14

Given: m = 30 kg, k = 1500 N/m, F(t) as shown

Find: x(t)

Solution: The system's natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500\,\frac{\text{N}}{\text{m}}}{30\,\text{kg}}} = 7.07\,\frac{\text{rad}}{\text{sec}}$$

The graphical breakdown of the solution is as shown below

The mathematical form of the excitation is

$$F(t) = 500[u(t) - u(t-1)] + (-1000t + 1500)[u(t-1) - u(t-1.5)]$$

The system response is given by

$$x(t) = x_1(t) - x_2(t) + x_3(t) - x_4(t)$$

where x_1, x_2, x_3, and x_4 are determined from Table 5.1 as follows

x_1: Step function, A = 500 N, $t_0 = 0$

$$x_1 = \frac{500\,\text{N}}{1500\,\frac{\text{N}}{\text{m}}}(1 - \cos 7.07\,t)u(t) = \frac{1}{3}(1 - \cos 7.07\,t)u(t)\,\text{m}$$

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x_2: Step function, A = 500 N, t_0 = 1 sec

$$x_2(t) = \frac{500\,\text{N}}{1500\,\frac{\text{N}}{\text{m}}}\left[1-\cos 7.07(t-1)\right]u(t-1) = \frac{1}{3}\left[1-\cos(7.07t-7.07)\right]u(t-1)\text{m}$$

x_3: Ramp function, A = -1000 N/sec, B = 1500 N, t_0 = 1 sec

$$x_3(t) = -\frac{1000\,\frac{\text{N}}{\text{sec}}}{1500\,\frac{\text{N}}{\text{m}}}\left[t-1.5+0.5\cos 7.07(t-1)-\frac{1}{7.07}\sin 7.70(t-1)\right]u(t-1)$$

$$= \frac{2}{3}\left[t-1.5+0.5\cos(7.07t-7.07)-0.141\sin(7.07t-7.07)\right]u(t-1)\text{m}$$

x_4: Ramp function, A = -1000 N/sec, B = 1500 N, t_0 = 1.5 sec

$$x_4(t) = \frac{-1000\,\frac{\text{N}}{\text{sec}}}{1500\,\frac{\text{N}}{\text{m}}}\left[t-1.5-\frac{1}{7.07}\sin 7.07(t-1.5)\right]u(t-1.5)$$

$$= -\frac{2}{3}\left[t-1.5-0.141\sin(7.07t-10.61)\right]u(t-1.5)\text{m}$$

Problem 5.14 illustrates (a) the graphical breakdown of an excitation whose form changes with time, and (b) the use of Table 5.1 to and the superposition principle to determine the response of an undamped system.

5.15 A machine tool with a mass of 30 kg is mounted on an undamped foundation of stiffness 1500 N/m. During operation, it is subject to the machining forces shown in Figure P5.15. Use the principle of superposition and the convolution integral to determine the response of the system to each force.

FIGURE P5.15

Given: m = 30 kg, k = 1500 N/m, F(t)

Find: x(t)

Solution: The natural frequency of the system is

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$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500\,\frac{\text{N}}{\text{m}}}{30\,\text{kg}}} = 7.07\,\frac{\text{rad}}{\text{sec}}$$

The graphical breakdown of F(t) into functions whose response are available from Table 5.1 is shown below.

The mathematical form of F(t) is

$$F(t) = 10000t[u(t) - u(t-0.1)] + 1000[u(t-0.1) - u(t-0.3)]$$
$$+ (-10000t + 40000)[u(t-0.3) - u(t-0.5)] + (10000t - 6000)[u(t-0.5) - u(t-0.6)]$$

Superposition is used to write

$$x(t) = x_1(t) - x_2(t) + x_3(t) - x_4(t) + x_5(t) - x_6(t) + x_7(t) - x_8(t)$$

where $x_i(t)$, i = 1,…,8 are determined from Table 5.1 as shown below

$x_1(t)$: Ramp function, A = 10000 N/sec, B = 0, $t_0 = 0$

$$x_1(t) = \frac{10000\,\text{N}}{1500\,\frac{\text{N}}{\text{m}}}\left(t - \frac{1}{7.07}\sin 7.07t\right)u(t)$$
$$= 6.67\,(t - 0.141\sin 7.07t)\,u(t)\,\text{m}$$

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$x_2(t)$: Ramp function, A = 10000 N/sec, B = 0, $t_0 = 0.1$ sec

$$x_2(t) = \frac{10000\,\text{N}}{1500\,\frac{\text{N}}{\text{m}}}\left[t - 0.1\cos 7.07(t-0.1) - 0.141\sin 7.07(t-0.1)\right]u(t-0.1)$$
$$= 6.67\left[t - 0.1\cos(7.07t - 0.707) - 0.141\sin(7.07t - 0.707)\right]u(t-0.1)\text{m}$$

$x_3(t)$: Step function, A = 1000 N, $t_0 = 0.1$ sec

$$x_3(t) = \frac{1000\,\text{N}}{1500\,\frac{\text{N}}{\text{m}}}\left[1 - \cos 7.07(t-0.1)\right]u(t-0.1)$$
$$= 0.667\left[1 - \cos(7.07t - 0.707)\right]u(t-0.1)\text{m}$$

$x_4(t)$: Step function, A = 1000 N, $t_0 = 0.3$ sec

$$x_4(t) = \frac{1000\,\text{N}}{1500\,\frac{\text{N}}{\text{m}}}\left[1 - \cos 7.07(t-0.3)\right]u(t-0.3)$$
$$= 0.667\left[1 - \cos(7.07t - 2.12)\right]u(t-2.12)\text{m}$$

$x_5(t)$: Ramp function, A = -10000 N/sec, B = 4000 N, $t_0 = 0.3$ sec

$$x_5(t) = -6.67\left[t - 0.4 + 0.1\cos(7.07t - 2.12) - 0.141\sin(7.07t - 2.12)\right]u(t-0.3)\text{m}$$

$x_6(t)$: Ramp function, A = -10000 N/sec, B = 4000 N, $t_0 = 0.5$ sec

$$x_6(t) = -6.67\left[t - 0.4 - 0.1\cos(7.07t - 3.54) - 0.141\sin(7.07t - 3.54)\right]u(t-0.5)\text{m}$$

$x_7(t)$: Ramp function, A = 10000 N/sec, B = -6000 N, $t_0 = 0.5$ sec

$$x_7(t) = 6.67\left[t - 0.6 + 0.1\cos(7.07t - 3.54) - 0.141\sin(7.07t - 3.54)\right]u(t-0.5)\text{m}$$

$x_8(t)$: Ramp function, A = 10000 N/sec, B = -6000 N, $t_0 = 0.6$ sec

$$x_8(t) = 6.67\left[t - 0.6 - 0.141\sin(7.07t - 4.24)\right]u(t-0.6)\text{m}$$

Problem 5.15 illustrates (a) graphical breakdown of an excitation that changes form at discrete times to functions whose responses can be determined from Table 5.1 and (b) use of superposition and Table 5.1 to determine the response of an undamped system due to an excitation that changes form at discrete times.

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5.16 A machine tool with a mass of 30 kg is mounted on an undamped foundation of stiffness 1500 N/m. During operation, it is subject to the machining forces shown in Figure P5.16. Use the principle of superposition and the convolution integral to determine the response of the system to each force.

FIGURE P5.16

Given: F(t), m = 30 kg, k = 1500 N/m,

Find: x(t)

Solution: the natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30\,\text{kg}}{1500\,\frac{\text{N}}{\text{m}}}} = 7.07\,\frac{\text{rad}}{\text{sec}}$$

The graphical breakdown of F(t) is shown below.

The mathematical form of F(t) is

$$F(t) = -1000t[u(t) - u(t-0.5)]$$
$$+(1000t - 1000)[u(t-0.5) - u(t-1.5)] + 500\,u(t-1.5)$$

The response of the system is

$$x(t) = x_1(t) - x_2(t) + x_3(t) - x_4(t) + x_5(t)$$

where x_1 , x_2 , x_3 , x_4 , and x_5 are determined using Table 5.1 as show below

$x_1(t)$: Ramp function, A = -1000 N/sec, B = 0, $t_0 = 0$

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$$x_1(t) = -\frac{1000\,\frac{\text{N}}{\text{sec}}}{1500\,\frac{\text{N}}{\text{m}}}(t - 0.141\sin 7.07t)u(t)$$

x_2: Ramp function, A = -1000 N/sec, B = 0, t_0 = 0.5 sec

$$x_2(t) = -0.667[t - 0.5\cos(7.07t - 3.54) - 0.141\sin(7.07t - 3.54)]u(t - 0.5)$$

x_3: Ramp function, A= 1000 N/sec, B=-1000 N, t_0=0.5 sec

$$x_3(t) = -0.667[t - 1 + 0.5\cos(7.07t - 3.54) - 0.141\sin(7.07t - 3.54)]u(t - 0.5)$$

x_4: Ramp function, A = 1000 N/sec, B = -1000 N, t_0 = 1.5 sec

$$x_4(t) = 0.667[t - 1 - 0.5\cos(7.07t - 10.61) - 0.141\sin(7.07t - 10.61)]u(t - 1.5)$$

x_5: Step function, A = 500 N, t_0 = 1.5 sec

$$x_5(t) = 0.333[1 - \cos(7.07t - 10.61)]u(t - 1.5)$$

Problem 5.16 illustrates (a) graphical breakdown of an excitation whose form changes with time and (b) the use of Table 5.1 and superposition to determine the response of a one-degree-of-freedom system due to an excitation whose form changes at discrete values of time.

5.17 A machine tool with a mass of 30 kg is mounted on an undamped foundation of stiffness 1500 N/m. During operation, it is subject to the machining forces shown in Figure P5.17. Use the principle of superposition and the convolution integral to determine the response of the system to each force.

Given: F(t), m = 30 kg, k = 1500 N/m

Find: x(t)

Solution: the natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30\,\text{kg}}{1500\,\frac{\text{N}}{\text{m}}}} = 7.07\,\frac{\text{rad}}{\text{sec}}$$

FIGURE P5.17

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The graphical breakdown of F(t) is shown below.

The mathematical form of F(t) is

$$F(t) = 300t[u(t) - u(t-2)] + 600e^{-0.2(t-2)}u(t-2)$$

The response of the system is

$$x(t) = x_1(t) - x_2(t) + x_3(t)$$

where x_1, x_2, and x_3 are determined using Table 5.1 as shown below

x_1: Ramp function, A = 300 N/sec, B = 0, $t_0 = 0$

$$x_1(t) = -\frac{300\,\frac{\text{N}}{\text{sec}}}{1500\,\frac{\text{N}}{\text{m}}}(t - 0.141\sin 7.07t)u(t)$$

x_2: Ramp function, A = 300 N/sec, B = 0, $t_0 = 2$ sec

$$x_2(t) = 0.2[t - 2\cos(7.07t - 14.14) - 0.141\sin(7.07t - 14.14)]u(t-2)$$

x_3: Exponential function, A = 600 N, $\alpha = 0.2\ \text{sec}^{-1}$, $t_0 = 2$ sec

$$x_3(t) = 0.4\left[e^{-0.2(t-2)} + \frac{0.2}{7.07}\sin(7.07t - 14.14) - \cos(7.07t - 14.14)\right]\frac{1}{1 + \left(\frac{0.2}{7.07}\right)^2}u(t-2)$$

$$= 0.3997\left[e^{-0.2(1-2)} + 0.0283\sin(7.07t - 14.14) - \cos(7.07t - 14.14)\right]u(t-2)$$

Problem 5.17 illustrates (a) graphical breakdown of an excitation whose form changes with time and (b) the use of Table 5.1 and superposition to determine the response of a one-degree-of-freedom system due to an excitation whose form changes at discrete values of time.

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5.18 A machine tool with a mass of 30 kg is mounted on an undamped foundation of stiffness 1500 N/m. During operation, it is subject to the machining forces shown in Figure P5.18. Use the principle of superposition and the convolution integral to determine the response of the system to each force.

Given: F(t), m = 30 kg, k = 1500 N/m

Find: x(t)

FIGURE P5.18

Solution: the natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30\,\text{kg}}{1500\,\frac{\text{N}}{\text{m}}}} = 7.07\,\frac{\text{rad}}{\text{sec}}$$

The graphical breakdown of F(t) is shown below.

The mathematical form of F(t) is

$$F(t) = 100t[u(t) - u(t-1)] + 100[u(t-1) - u(t-4)] + 450\delta(t-6)$$

The response of the system is

$$x(t) = x_1(t) - x_2(t) + x_3(t) - x_4(t) + x_5(t)$$

where x_1, x_2, x_3, x_4, and x_5 are determined using Table 5.1 as show below

x_1: Ramp function, A = 100 N/sec, B = 0, $t_0 = 0$

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$$x_1(t) = \frac{100\,\frac{\text{N}}{\text{sec}}}{1500\,\frac{\text{M}}{\text{m}}}(t - 0.141\sin 7.07t)u(t)$$

x_2: Ramp function, A = 100 N/sec, B = 0, t_0 = 1 sec

$$x_2(t) = 0.0333[t - \cos(7.07t - 7.07) - 0.141\sin(7.07t - 7.07)]u(t-1)$$

x_3: Step function, A = 100 N, t_0 = 1 sec

$$x_3(t) = 0.0333[1 - \cos(7.07t - 7.07)]u(t-1)$$

x_4: Step function, A = 100 N, t_0 = 4 sec

$$x_4(t) = 0.0333[1 - \cos(7.07t - 28.28)]u(t-4)$$

x_5: Impulse function, A = 450 N-sec, t_0 = 6 sec

$$x_5(t) = \frac{(450\,\text{N}-\text{sec})\left(7.07\,\frac{\text{rad}}{\text{sec}}\right)}{1500\,\frac{\text{N}}{\text{m}}}\sin(7.07t - 42.42)u(t-6)$$
$$= 2.12\sin(7.07t - 42.42)u(t-6)$$

Problem 5.18 illustrates (a) graphical breakdown of an excitation whose form changes with time and (b) the use of Table 5.1 and superposition to determine the response of a one-degree-of-freedom system due to an excitation whose form changes at discrete values of time.

5.19 The force applied to the 120 kg anvil of a forge hammer during operation is approximated as a rectangular pulse of magnitude 2000 N for a duration of 0.3 s. The anvil is mounted on a foundation of stiffness 2000 N/m and damping ratio 0.4. What is the maximum displacement of the anvil?

Given: m = 120 kg, F = 2000 N, t_0 = 0.3 s, k = 2000 N/m, ζ = 0.4

Find: x_{max}

Solution: The natural frequency of the anvil is

$$\omega_n = \sqrt{\frac{k}{m}} = 4.08 \text{ rad/s}$$

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The damped natural frequency is

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 3.74 \text{ rad/s}$$

The convolution integral is used to determine the response of the system for $t < t_0$ as

$$x(t) = \frac{1}{m\omega_d} \int_0^t F_0 e^{-\zeta\omega\ (t-\tau)} \sin(\omega_d (t-\tau))\, \mathrm{d}\tau$$

which is evaluated to

$$x(t) = F_0 \left[5.88\times10^{-2} - 5.88\times10^{-3} e^{-4t} \cos(3.74t) - 6.29\times10^{-3} \sin(3.74)\right]$$

A plot of the above expression reveals that x(t) does not reach a maximum in the interval from t = 0 to t = 0.3 The appropriate expression for x(t) for t > 0.3 s is

$$x(t) = \frac{1}{m\omega_d} \int_0^{t_0} F_0 e^{-\zeta\omega\ (t-\tau)} \sin(\omega_d (t-\tau))\, \mathrm{d}\tau$$

which is evaluated to

$$x(t) = 0.396e^{-4t+1.2} \cos(3.74t) + 1.115e^{-4t+1.2} \sin(3.74t) - 1.177e^{-4t} \cos(3.74t) - 0.126e^{-4t} \sin(3.74t)$$

The maximum value of x(t) is evaluated as $x_{max} = 0.909$ m.

Note: MATLAB was used to symbolically integrate the convolution integral and to determine the maximum of x(t).

Problem 5.19 illustrates the use of the convolution integral to determine the maximum response of a system.

5.20 A one-story frame structure houses a chemical laboratory. Figure P5.20 shows the results of a model test to predict the transient force to which the structure would be subject if an explosion would occur. The equivalent mass of the structure is 2000 kg and its equivalent stiffness is 5 × 10^6 N/m. Approximate the maximum displacement of the structure due to this blast.

FIGURE P5.20

Given: m = 2000 kg, k = 5 × 10^6 N/m, F(t)

Find: x_{max}

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Solution: The blast force is approximated by the force shown. The natural frequency of the structure is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5\times10^6\ \frac{\text{N}}{\text{m}}}{2000\,\text{kg}}} = 50\,\frac{\text{rad}}{\text{sec}}$$

The force for t < 0.2 sec is approximated by a ramp,

$$F(t) = 2.5\times10^4 t\ \text{N}, 0 < t < 0.2\,\text{sec}$$

The response during this time is determined using Table 5.1 for a ramp function with A = 2.5 × 10^4 N/sec, B = 0, t_0 = 0,

$$x(t) = \frac{2.5\times10^4\ \text{N}}{5\times10^6\ \frac{\text{N}}{\text{sec}}}(t - 0.02\sin 50t), 0 < t < 0.2\,\text{sec}$$

The extrema occur during this time interval at values of t such that

$$1 - \cos\omega_n t = 0, \omega_n t = 0, \frac{3}{2}\pi, 7\frac{\pi}{2}, \ldots$$

It is also possible for the absolute maximum during the interval to occur at t = 0.2 sec. The values are checked and it is found that a maximum displacement of 1.055 × 10^{-3} m occurs at t = 0.2 sec.

During the time interval between 0.2 sec and 1.0 sec, the excitation is approximated as a constant force of 5000 N. The response during this period is obtained using the principle of superposition and Table 5.1 The response is the response of the ramp function previously obtained minus the response due to the same ramp function but starting at 0.2 sec plus the response due to a step excitation starting at 0.2 sec. Using Table 5.1

Ramp function, A = 2.5 × 10^4 N/sec, B = 0, t_0 = 0.2 sec

$$x_b(t) = 0.005[t - 0.2\cos 50(t - 0.2) - 0.02\sin 50(t - 0.2)]$$

Step function, A = 5000 N, t_0 = 0.2 sec

$$x_c(t) = 0.001[1 - \cos 50(t - 0.2)]$$

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Combining the excitations

$$x(t) = X_a(t) - X_b(t) + X_c(t) = 0.001 - 1x10^{-4}\sin 50t + 1\times 10^{-4}\sin 50(t-0.2)$$

The times at which the maxima occur are obtained by taking the derivative and setting it to zero. This leads to the equation

$$\cos 50t = \cos 50(t-0.2)$$

whose solutions are

$$\omega_n t = \tan^{-1}(3.380)$$

This leads to a maximum of

$$x_{\max} = 1.16\times 10^{-3}\text{ m}$$

Problem 5.20 illustrates (a) the approximation of an excitation by a combination of functions whose responses are catalogued, (b) The use of Table 5.1, and (c) determination of the maximum response.

5.21 A 20 kg radio set is mounted in a ship on an undamped foundation of stiffness 1000 N/m. The ship is loosely tied to a dock. During a storm, the ship experiences the displacement of Figure P5.21. Determine the maximum acceleration of the radio.

FIGURE P5.21

Given: m = 20 kg, k = 1000 N/m

Find: a_{max}

Solution: The natural frequency of the radio is

$$\omega_n = \sqrt{\frac{k}{m}} = 7.07\text{ rad/s}$$

The radio is subject to the base motion of the ship. The mathematical representation of the base motion is

$$y(t) = 0.15\{1.67t[u(t)-u(t-0.6)] + (7-10t)[u(t-0.6)-u(t-0.7)]\}$$

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The convolution integral solution is used

$$x(t) = m\omega_n^2 \int_0^t y(\tau) \frac{1}{m\omega_n} \sin[\omega_n (t-\tau)]\, d\tau$$

Since the system is undamped, Table 5.1and the principle of linear superposition can be used to determine the system response,

$$x(t) = x_a(t) - x_b(t) + x_c(t) - x_d(t)$$

where each x(t) is the response due to a ramp function. To this end, note that $k = m\omega_n^2 = 1000$ and 0.15k = 150. The responses to each of the ramp inputs are determined using Table 5.1 as follows

$x_a(t)$: Ramp , A = 150(1.67) = 250, B = 0, $t_0 = 0$

$$x_a(t) = 0.25\left(t - \frac{1}{\omega_n}\sin\omega_n t\right)u(t)$$

$x_b(t)$: Ramp, A = 250, B = 0, $t_0 = 0.6$

$$x_b(t) = 0.25\left[t - 0.6\cos\omega_n(t-0.6) - \frac{1}{\omega_n}\sin\omega_n(t-0.6)\right]u(t-0.6)$$

$x_c(t)$: Ramp, A = 150(-10) = -1500, B = 150(7) = 1050, $t_0 = 0.6$

$$x_c(t) = -1.5\left[t - 0.7 + 0.1\cos\omega_n(t-0.6) - \frac{1}{\omega_n}\sin\omega_n(t-0.6)\right]u(t-0.6)$$

$x_d(t)$: Ramp, A = -1500, B = 1050, $t_0 = 0.7$

$$x_d(t) = -1.5\left[t - 0.7 - \frac{1}{\omega_n}\sin\omega_n(t-0.7)\right]u(t-0.7)$$

Note that the differential equation leads to

$$\ddot{x} + \omega_n^2 x = \omega_n^2 y$$
$$\ddot{x} = \omega_n^2 (y - x)$$

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To this end the acceleration is determined as

0 s < t < 0.6 s

$$\ddot{x} = \omega_n^2 \left[\frac{0.25}{\omega_n} \sin \omega_n t \right]$$

0.6 s < t < 0.7 s

$$\ddot{x} = \omega_n^2 \left[\frac{1}{\omega_n} \left(0.25 \sin \omega_n t - 1.25 \sin \omega_n (t - 0.6) \right) \right]$$

t > 0.7 s

$$\ddot{x} = \omega_n^2 \left[\frac{1}{\omega_n} \left(0.25 \sin \omega_n t - 1.25 \sin \omega_n (t - 0.6) + 1.5 \sin \omega_n (t - 0.7) \right) \right]$$

The maximum acceleration is 0.1722 m/s^2.

Problem 5.21 illustrates the maximum acceleration from a shock type input.

5.22 A personal computer of mass *m* is packed inside a box such that the stiffness and damping coefficient of the packing material are *k* and *c*, respectively. The package is accidentally dropped from a height *h* and lands on a hard surface without rebound. Set up the convolution integral whose evaluation leads to displacement of the computer relative to the package.

Given: m, k, c, h

Find: z_{max}

Solution: The velocity of the package is

$$v(t) = -gt[1 - u(t - t_0)], t_0 = \sqrt{\frac{2h}{g}}$$

The velocity of the package acts as a base excitation to the computer. The displacement of the computer relative to the package is

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$$z(t) = -\frac{1}{\sqrt{1-\zeta^2}}\int_0^t v(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau-\chi)d\tau$$

$$= \frac{g}{\sqrt{1-\zeta^2}}\int_0^t \tau[1-u(\tau-t_0)]e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau-\chi)d\tau$$

$$= \frac{g}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left\{\int_0^t \tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau-\chi)d\tau - u(t-t_0)\int_{t_0}^t \tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau-\chi)d\tau\right\}$$

Note that

$$\int \tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau-\chi)d\tau = -\frac{\tau e^{\zeta\omega_n\tau}}{\omega_n}\left[\zeta\sin\omega_d(t-\tau-\chi)-\sqrt{1-\zeta^2}\cos(t-\tau-\chi)\right]$$
$$+\frac{e^{\zeta\omega_n\tau}}{\omega_n^2}\left[(2\zeta^2-1)\sin\omega_d(t-\tau-\chi)-2\zeta\sqrt{1-\zeta^2}\cos\omega_d(t-\tau-\chi)\right]$$

Thus the system response is

$$x(t) = \frac{g}{\sqrt{1-\zeta^2}}\left\{\frac{t}{\omega_n}\left(-\zeta\sin\omega_d\chi-\sqrt{1-\zeta^2}\cos\omega_d\chi\right)\right.$$
$$+\frac{1}{\omega_n^2}\left[(2\zeta^2-1)\sin\omega_d\chi+2\zeta\sqrt{1-\zeta^2}\cos\omega_d\chi\right]$$
$$\left.+\frac{e^{-\zeta\omega_n\tau}}{\omega_n^2}\left[(2\zeta^2-1)\sin\omega_d(t-\chi)-2\zeta\sqrt{1-\zeta^2}\cos\omega_d(t-\chi)\right]\right\}$$

$$-u(t-t_0)\frac{g}{\sqrt{1-\zeta^2}}\left\{-\frac{t}{\omega_n}\left(\zeta\omega_n\sin\omega_d\chi+\sqrt{1-\zeta^2}\cos\omega_d\chi\right)\right.$$
$$+\frac{1}{\omega_n^2}\left[(2\zeta^2-1)\sin\omega_d\chi+2\zeta\sqrt{1-\zeta^2}\cos\omega_d\chi\right]$$
$$-e^{-\zeta\omega_n(t-t_0)}\frac{t_0}{\omega_n}\left[\zeta\sin\omega_d(t-t_0-\chi)-\sqrt{1-\zeta^2}\cos\omega_d(t-t_0-\chi)\right]$$
$$\left.+\frac{e^{-\zeta\omega_n(t-t_0)}}{\omega_n^2}\left[(2\zeta^2-1)\sin\omega_d(t-t_0-\chi)-2\zeta\sqrt{1-\zeta^2}\cos\omega_d(t-t_0-\chi)\right]\right\}$$

The maximum response will probably occur for $t > t_0$. However the algebra is too complicated to obtain an analytical solution. A numerical solution must be used.

Problem 5.22 illustrates the use of the convolution integral to determine the time dependent displacement of a one-degree-of-freedom system with viscous damping.

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5.23 Use the Laplace transform method to determine the response of a system at rest in equilibrium when subject to

$$F(t) = F_0 \cos\omega t[1 - u(t - t_0)]$$

for (a) $\zeta = 0$, (b) $0 < \zeta < 1$, (c) $\zeta = 1$, (d) $\zeta > 1$.

Given: F(t), ζ , ω_n

Find: x(t)

Solution: The Laplace transform of a system at rest in equilibrium when subject to an excitation F(t) is

$$X(s) = \frac{1}{m}\frac{F(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where, for this problem

$$F(s) = F_0 \frac{s}{s^2 + \omega^2}\left[1 - e^{-st_0}\right]$$

Hence

$$X(s) = \frac{F_0}{m}\frac{s}{(s^2 + \omega^2)(s^2 + 2\zeta\omega_n s + \omega_n^2)}\left[1 - e^{-st_0}\right]$$

(a) $\zeta = 0$.The system is undamped. Hence

$$X(s) = \frac{F_0 s(1 - e^{-st_0})}{(s^2 + \omega^2)(s^2 + \omega_n^2)}$$

A partial fraction decomposition yields

$$G(s) = \frac{F_0 s}{(s^2 + \omega^2)(s^2 + \omega_n^2)} = \frac{As + B}{(s^2 + \omega^2)} + \frac{Cs + D}{(s^2 + \omega_n^2)}$$

Setting up equations to solve for A, B, C and D lead to

$$A + C = 0$$

$$B + D = 0$$

$$A\omega_n^2 + C\omega^2 = F_0$$

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$$B\omega_n^2 + D\omega^2 = 0$$

The solutions of the above equations are

$$A = \frac{F_0}{\omega_n^2 - \omega^2}, B = 0, C = -\frac{F_0}{\omega_n^2 - \omega^2}, D = 0$$

which leads to

$$G(s) = \frac{F_0}{\omega_n^2 - \omega^2}\left(\frac{s}{(s^2 + \omega^2)} - \frac{s}{(s^2 + \omega_n^2)}\right)$$

Then

$$X(s) = \frac{F_0}{\omega_n^2 - \omega^2}\left(\frac{s}{(s^2 + \omega^2)} - \frac{s}{(s^2 + \omega_n^2)}\right)(1 - e^{-st_0})$$

Using the second shifting theorem the inversion of this transform is

$$x(t) = \frac{F_0}{\omega_n^2 - \omega^2}\{\cos\omega t - \cos\omega_n t - [\cos\omega(t - t_0) - \cos\omega_n(t - t_0)]u(t - t_0]\}$$

(b) $\zeta < 1$, The quadratic term in the denominator has no real roots. Thus it is left as a quadratic factor and partial faction decomposition yields

$$G(s) = \frac{s}{(s^2 + \omega^2) + (s^2 + 2\zeta\omega_n s + \omega_n^2)}$$
$$= \frac{1}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_n^2}\left[\frac{(\omega_n^2 - \omega^2)s + 2\zeta\omega_n\omega^2}{s^{2+}\omega^2} - \frac{(\omega_n^2 - \omega^2)s + 2\zeta\omega_n^3}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right]$$

Note that

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \zeta\omega_n)^2 + \omega_d^2$$

Hence after some algebra

$$g(t) = L^{-1}\{G(s)\} = \frac{1}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_n^2}\left[(\omega_n^2 - \omega^2)\cos\omega t + 2\zeta\omega\omega_n \sin\omega t - (\omega_n^2 - \omega^2)e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}(\omega^2 + \omega_n^2)e^{-\zeta\omega_n t}\sin\omega_d t\right]$$

Note that

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$$X(s)=\frac{F_0}{m}G(s)\left[1-e^{-st_0}\right]$$

Thus using linearity and the second shifting theorem

$$x(t)=\frac{F_0}{m}\left[g(t)-g(t-t_0)u(t-t_0)\right]$$

(c) $\zeta = 1$, The denominator has equal roots. Then

$$G(s)=\frac{As+B}{s^2+\omega^2}+\frac{C}{s+\omega_n}+\frac{D}{(s+\omega_n)^2}$$

where

$$A=\frac{\omega_n^2-\omega^2}{\omega_n^2\left(\omega_n^2+\omega^2\right)+\omega^2\left(\omega^2+\omega\omega_n\right)}$$

$$B=\frac{1}{2}\frac{\omega_n\omega^2+\omega^3+2\omega^2\omega_n}{\omega_n^2\left(\omega^2+\omega_n^2\right)+\omega^2\left(\omega^2+\omega\omega_n\right)}$$

$$C=-A$$

$$D=\frac{\omega_n\left(2\omega^2+\omega_n^2+\omega\omega_n\right)}{\omega_n^2\left(\omega_n^2+\omega^2\right)+\omega^2\left(\omega^2+\omega\omega_n\right)}$$

then

$$g(t)=A\cos\omega t+\frac{B}{\omega}\sin\omega t+Ce^{-\omega_n t}+Dte^{-\omega_n t}$$

and

$$x(t)=\frac{F_0}{m}\left[g(t)-g(t-t_0)u(t-t_0)\right]$$

(d) $\zeta > 1$,The denominator has two real roots

$$s_1=\omega_n\left(-\zeta-\sqrt{\zeta^2-1}\right)$$
$$s_2=\omega_n\left(-\zeta+\sqrt{\zeta^2-1}\right)$$

In this case

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$$G(s)=\frac{As+B}{s^2+\omega^2}+\frac{C}{s-s_1}+\frac{D}{s-s_2}$$

where A,B,C, and D solve

$$A+C+D=0$$
$$2\zeta\omega_n A+B-s_2C-s_1D=0$$
$$\omega_n^2A+2\zeta\omega_n B+\omega^2C+\omega^2D=1$$
$$A+\omega_n^2B-\omega^2s_2C-\omega^2s_1D=0$$

Then

$$g(t)=A\cos\omega t+\frac{B}{\omega}\sin\omega t+Ce^{s_1t}+De^{s_2t}$$

Problem 5.23 illustrates the use of the Laplace transform method to determine the transient response of (a) an undamped system (b) an underdamped system, (c) a critically damped system, and (d) an overdamped system.

5.24 Use the Laplace transform method to determine the response of an undamped SDOF system initially at rest in equilibrium when subject to a symmetric triangular pulse of magnitude F_0 and total duration t_0.

Given: m, k, F(t), x(0) = 0, x(0) = 0

Find: x(t)

Solution: The graphical breakdown of the triangular pulse is shown below

The mathematical form of the excitation is

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$$F(t)=\frac{2F_0 t}{t_0}\left[u(t)-u\left(t-\frac{t_0}{2}\right)\right]+\left(\frac{-2F_0}{t_0}t+2F_0\right)\left[u\left(t-\frac{t_0}{2}\right)-u(t-t_0)\right]$$
$$=\frac{2F_0}{t_0}tu(t)-\frac{4F_0}{t_0}\left(t-\frac{t_0}{2}\right)u\left(t-\frac{t_0}{2}\right)+\frac{2F_0}{t_0}(t-t_0)u(t-t_0)$$

The Laplace transform of F(t) is obtained using the second shifting theorem and transform pair 2 of Table B.1,

$$F(s)=\frac{2F_0}{t_0 s^2}\left[1-2e^{-s\frac{t_0}{2}}+e^{-st_0}\right]$$

The transform of the system response is

$$X(s)=\frac{F(s)}{m\left(s^2+\omega_n^2\right)}$$

Partial fraction decomposition yields

$$\frac{1}{s^2\left(s^2+\omega_n^2\right)}=\frac{1}{\omega_n^2}\left(\frac{1}{s^2}-\frac{1}{\left(s^2+\omega_n^2\right)}\right)$$

Hence

$$X(s)=\frac{2F_0}{mt_0\omega_n^2}\left(\frac{1}{s^2}-\frac{1}{s^2+\omega_n^2}\right)\left(1-2e^{-s\frac{t_0}{2}}+e^{-st_0}\right)$$

Application of the first shifting theorem and transform pairs 2 and 4 of table B1 are used to invert the transform and obtain the system response

$$x(t)=\frac{2F_0}{mt_0\omega_n^2}\left\{\left(t-\frac{1}{\omega_n}\sin\omega_n t\right)u(t)\right.$$
$$-2\left[t-\frac{t_0}{2}-\frac{1}{\omega_n}\sin\omega_n\left(t-\frac{t_0}{2}\right)\right]u\left(t-\frac{t_0}{2}\right)$$
$$\left.+\left[t-t_0-\frac{1}{\omega_n}\sin\omega_n(t-t_0)\right]u(t-t_0)\right\}$$

Problem 5.24 illustrates application of the Laplace transform to determine the response of an undamped one-degree-of-freedom system subject to an excitation whose form changes with time.

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5.25 Use the Laplace transform method to determine the response of an underdamped SDOF system to a rectangular pulse of magnitude F_0 and time t_0.

Given: $F(t) = F_0[u(t) - u(t - t_0)]$

Find: x(t) using Laplace transform method

Solution: The Laplace transform of F(t) is obtained using the second shifting theorem

$$F(s) = \frac{F_0}{s}\left(1 - e^{-st_0}\right)$$

Assume the initial conditions are both zero then

$$X(s) = \frac{F_0 / m(1 - e^{-st_0})}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Partial fraction decomposition leads to

$$X(s) = \frac{F_0}{m\omega_n^2}\left[\frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right](1 - e^{-st_0})$$

For an unerdamped system the quadratic denominator can be rewritten as $(s + \zeta\omega_n)^2 + \omega_d^2$. Then application f both the First Shifting Theorem and the Second Shifting Theorem leads to

$$x(t) = \frac{F_0}{m\omega_n^2}\left[1 - e^{-\zeta\omega_n t}(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]u(t) -$$
$$\frac{F_0}{m\omega_n^2}\left[1 - e^{-\zeta\omega_n (t - t_0)}(\cos\omega_d (t - t_0) + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d (t - t_0)\right]u(t - t_0)$$

Problem 5.25 illustrates the application of the Laplace transform method to derive the response of an underdamped one-degree-of-freedom system to a rectangular pulse. The solution requires application of the shifting theorems.

5.26 Use the Laplace transform method to derive the response of a SDOF system initially at rest in equilibrium when subject to a harmonic force F_0 sin ωt, when (a) $\omega \neq \omega_n$ and (b) $\omega = \omega_n$.

Given: ω, ω_n , F_0

Find: x(t) when (a) $\omega \neq \omega_n$, (b) $\omega = \omega_n$

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Solution: The transform of the response of an undamped one-degree-of-freedom system is attained using eq. (5. 25) with the initial conditions taken to be zero as

$$X(s)=\frac{1}{\tilde{m}}\frac{F(s)}{s^2+\omega_n^2}$$

(a) For $F(t) = F_0 \sin\omega t$, $\omega \neq \omega_n$,

$$F(s)=\frac{F_0\omega}{s^2+\omega^2}$$

Hence

$$X(s)=\frac{F_0\omega}{\tilde{m}}\frac{1}{\left(s^2+\omega^2\right)\left(s^2+\omega_n^2\right)}$$

Partial fraction decomposition is used leading to

$$X(s)=\frac{F_0\omega}{\tilde{m}\left(\omega^2-\omega_n^2\right)}\left(\frac{1}{s^2+\omega^2}-\frac{1}{s^2+\omega_n^2}\right)$$

Linearity and Table B.1 is used to invert the transform and give

$$x(t)=\frac{F_0\omega}{\tilde{m}\left(\omega^2-\omega_n^2\right)}\left(\frac{1}{\omega}\sin\omega t-\frac{1}{\omega_n}\sin\omega_n t\right)$$

(b) When $\omega = \omega_n$,

$$F(s)=\frac{F_0\omega_n}{s^2+\omega_n^2}$$

and

$$X(s)=\frac{F_0\omega_n}{\tilde{m}}\frac{1}{\left(s^2+\omega_n^2\right)^2}$$

A table of transforms more extensive than those listed in Table B.1 or use of properties not listed in Table B.2 used in conjunction with transforms listed in Table B.1 are required to invert the preceding transform. The result is

$$x(t)=\frac{F_0}{2\tilde{m}\omega_n^2}\left(\sin\omega_n t-\omega_n t\cos\omega_n t\right)$$

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Problem 5.26 illustrates the use of the Laplace transform method to derive the response of a one-degree-of-freedom undamped system.

5.27 Determine the transfer function for the relative displacement of a SDOF system with base motion defined as $G(s) = \frac{Z(s)}{Y(s)}$ where $Z(s)$ is the Laplace transform of the relative displacement and $Y(s)$ is the Laplace transform of the motion of the base.

Given: y(t)

Determine: $G(s) = \frac{Z(s)}{Y(s)}$

Solution: The differential equation governing the relative displacement is

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y}$$

Taking the Laplace transform of the equation assuming all initial conditions are zero leads to

$$s^2Z(x) + 2\zeta\omega_s sZ(s) + \omega_n^2 Z(s) = -s^2Y(s)$$

Solving for Z(s) leads to

$$G(s) = \frac{Z(s)}{Y(s)} = \frac{-s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Problem 5.27 illustrates the transfer function of a system with base motion.

5.28 Determine the transfer function for the force transmitted to the foundation for a SDOF system. The transfer function is defined as $G(s) = \frac{F_t(s)}{F(s)}$ where $F_t(s)$ is the Laplace transform of the transmitted force and $F(s)$ is the Laplace transform of the applied force.

Given: F(s)

Find: $G(s) = \frac{F_t(s)}{F(s)}$

Solution: The force transmitted to the foundation with a machine mounted on a spring and viscous damper is

$$F_t(t) = c\dot{x} + kx$$

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Taking the Laplace transform of this equation leads to

$$F_t(s) = csX(s) + kX(s)$$

The transform of the system output is related to the transform of the system input by

$$X(s) = \frac{F(s)}{ms^2 + cs + k}$$

Substituting for X(s) in the equation for $F_t(s)$leads to

$$F_t(s) = \frac{(cs + k)F(s)}{ms^2 + cs + k}$$

Solving for the transfer function yields

$$G(s) = \frac{F_t(s)}{F(s)} = \frac{cs + k}{ms^2 + cs + k}$$

which can also be written as

$$G(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Problem 5.28 illustrates the transfer function for the transmitted force.

5.29 Use the transfer function to determine the response of an SDOF system excited by motion of its base with m = 3 kg and k = 18,000 N/m where the base motion is shown in Figure P5.29.

FIGURE P5.29

Given: m = 3 kg, k = 18,000 N/m

Find: $x(t)$

Solution: The equation for the displacement of the undamped system due to motion of its base is

$$\ddot{x} + \omega_n^2 x = \omega_n^2 y$$

The transfer function is obtained as

$$G(s) = \frac{X(s)}{Y(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

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The natural frequency of the system is

$$\omega_n = \sqrt{\frac{18000 \text{ N/m}}{3 \text{ kg}}} = 77.5 \text{ rad/s}$$

Thus

$$G(s) = \frac{6000}{s^2 + 6000}$$

The input motion is given by

$$y(t) = 0.05t[u(t) - u(t-0.2)] + (0.02 - 0.05t)[u(t-0.2) - u(t-0.5)] + 0.005u(t-0.5)$$

$$= 0.05tu(t) + (0.02 - 0.01t)u(t-0.2) + (0.025 - 0.05t)u(t-0.5)$$

$$= 0.05tu(t) - 0.01(t-2)u(t-2) + 0.05(t-0.5)u(t-0.5)$$

The second shifting theorem is employed to take the Laplace transform of y(t) resulting in

$$Y(s) = \frac{0.05}{s^2} - \frac{0.01}{s^2}e^{-0.2s} + \frac{0.05}{s^2}e^{-0.5s}$$

Substituting Y(s) into the transfer function yields

$$X(s) = \frac{6000}{s^2(s^2 + 6000)}(0.05 - 0.01e^{-0.2s} + 0.05e^{-0.5s})$$

A partial fraction decomposition leads to

$$\frac{6000}{s^2(s^2 + 6000)} = \frac{1}{s^2} - \frac{1}{s^2 + 6000}$$

Thus

$$X(s) = \left(\frac{1}{s^2} - \frac{1}{s^2 + 6000}\right)(0.05 - 0.01e^{-0.2s} + 0.05e^{-0.5s})$$

Noting that $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+6000}\right\} = \frac{1}{77.5}\sin 77.5t = 0.0129 \sin 77.5t$ and using the second shifting theorem the inverse of X(s) is

$$x(t) = 0.05(t - 0.0129 \sin 77.5t) - 0.01[t - 0.2 + 0.0129 \sin 77.5(t-0.2)]u(t-0.2)$$
$$+ 0.05[t - 0.5 + 0.0129 \sin 77.5(t-0.5)]u(t-0.5)$$

Problem 5.29 illustrates the application of the Laplace transform method using transfer functions.

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5.30 Use the transfer function to determine the response of a SDOF system with m =1 kg, k = 100 N/m, and c = 6 N · s/m when the system is subject to motion of its base shown in Figure P5.30.

FIGURE P5.30

Given: m = 1 kg, k = 100 N/m, c = 6 N · s/m

Find: x(t)

Solution: The natural frequency for the system is 10 rad/s and its damping ratio is 0.3. The transfer function for the system is

$$G(s) = \frac{6s + 100}{s^2 + 6s + 100}$$

The motion of the base is given by

$$y(t) = 0.1t[u(t) - u(t-1)] + 0.1u(t-1)$$
$$= 0.1tu(t) - 0.1(t-1)u(t-1)$$

The Laplace transform of the base motion is

$$Y(s) = \frac{0.1}{s^2} - \frac{0.1}{s^2}e^{-s}$$

Then using the transfer function

$$X(s) = \left(\frac{6s + 100}{s^2 + 6s + 100}\right)\left(\frac{0.1}{s^2} - \frac{0.1}{s^2}e^{-s}\right) = \frac{6s + 100}{s^2(s^2 + 6s + 100)}(0.1 - 0.1e^{-s})$$

A partial fraction decomposition leads to

$$\frac{6s + 100}{s^2(s^2 + 6s + 100)} = \frac{1}{s^2} - \frac{1}{s^2 + 6s + 100}$$

Hence

$$X(s) = \left(\frac{1}{s^2} - \frac{1}{s^2 + 6s + 100}\right)(0.1 - 0.1e^{-s})$$

It is noted that

$$\frac{1}{s^2 + 6s + 100} = \frac{1}{(s+3)^2 + 91}$$

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and

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+100}\right\} = \frac{1}{\sqrt{91}}e^{-3t}\sin\sqrt{91}\,t = 0.105e^{-3t}\sin 9.54t$$

The second shifting theorem is used to invert the transform yielding

$$x(t) = 0.1(t - 0.105e^{-3t}\sin 9.54t) - 0.1\left[t - 1 - 0.105\,e^{-3(t-1)}\sin 9.54(t-1)\right]u(t-1)$$

Problem 5.30 illustrates application of the transfer function to determine the response of a damped SDOF system due to base motion.

5.31 Repeat Chapter Problem 5.30 if the system parameters are m = 1 kg, k = 200 N/m, and c = 30 N · s/m.

Given: m = 1 kg, k = 200 N/m, c = 30 N · s/m

Find: x(t)

Solution: The natural frequency for the system is 14.14rad/s and its damping ratio is 1.061 The transfer function for the system is

$$G(s) = \frac{30s+200}{s^2+30s+200}$$

The motion of the base is given by

$$y(t) = 0.1t[u(t) - u(t-1)] + 0.1u(t-1)$$

$$= 0.1tu(t) - 0.1(t-1)u(t-1)$$

The Laplace transform of the base motion is

$$Y(s) = \frac{0.1}{s^2} - \frac{0.1}{s^2}e^{-s}$$

Then using the transfer function

$$X(s) = \left(\frac{30s+200}{s^2+30s+200}\right)\left(\frac{0.1}{s^2} - \frac{0.1}{s^2}e^{-s}\right) = \frac{30s+200}{s^2(s^2+30s+200)}(0.1 - 0.1e^{-s})$$

A partial fraction decomposition leads to

$$\frac{30s+200}{s^2(s^2+30s+200)} = \frac{1}{s^2} - \frac{1}{s^2+30s+200}$$

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Hence

$$X(s) = \left(\frac{1}{s^2} - \frac{1}{s^2 + 30s + 200}\right)(0.1 - 0.1e^{-s})$$

It is noted that

$$\frac{1}{s^2 + 30s + 200} = \frac{1}{(s+10)(s+20)} = \frac{0.1}{s+10} - \frac{0.1}{s+20}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 30s + 200}\right\} = 0.1e^{-10t} - 0.1e^{-20t}$$

The second shifting theorem is used to invert the transform yielding

$$x(t) = 0.1(t - 0.1e^{-10t} + 0.1e^{-20t})u(t) - 0.1\left[t - 1 - 0.1e^{-10(t-1)} + 0.1e^{-20(t-1)}\right]u(t-1)$$

Problem 5.31 illustrates application of the transfer function to determine the response of a damped SDOF system due to base motion.

5.32 For the system of Figure P5.32(a), complete the following. (a) Determine its transfer function defined as $G(s) = \frac{X(s)}{Y(s)}$. (b) Use the transfer function to find the response of the system due to $y(t)$ as shown in Figure P5.32(b). Use m = 1 kg, k = 100 N/m, and c = 30 N · s/m.

Given: (b) m = 1 kg, k = 100 N/m and
c = 30 N · s/m, F(t) as shown

Find: $G(s), x(t)$

FIGURE P5.32

Solution: (a) The differential equation governing the motion of the block is

$$m\ddot{x} + c\dot{x} + 3kx = c\dot{y} + ky$$

Taking the Laplace transform of both sides of the equation setting all initial condition to zero leads to

$$ms^2X(s) + csX(s) + 3kX(s) = csY(s) + kY(s)$$

Solving for the transfer function leads to

$$G(s) = \frac{cs + k}{ms^2 + cs + 3k}$$

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(b) Substituting numbers into the transfer function leads to

$$G(s) = \frac{30s + 100}{s^2 + 30s + 300}$$

The base motion is given by

$$y(t) = \frac{0.001}{0.05} t[u(t) - u(t - 0.05)] + 0.001u(t - 0.05)$$
$$= 0.02tu(t) - 0.02(t - 0.05)u(t - 0.05)$$

The Laplace transform of the base motion is

$$Y(s) = \frac{0.02}{s^2} - \frac{0.02}{s^2} e^{-0.05s}$$

which when substituted into the transfer function yields

$$X(s) = \frac{0.02(30s + 100)}{s^2(s^2 + 30s + 300)}(1 - e^{-0.05s})$$

A partial fraction decomposition leads to

$$\frac{0.02(30s + 100)}{s^2(s^2 + 30s + 300)} = 0.02\left(\frac{1}{s^2} - \frac{1}{s^2 + 30s + 300}\right) = 0.02\left[\frac{1}{s^2} - \frac{1}{(s + 15)^2 + 75}\right]$$

The inverse Laplace transform of X(s) is x(t),

$$x(t) = 0.02\left(t - \frac{1}{\sqrt{75}} e^{-15t} \sin\sqrt{75}\, t\right)$$
$$- 0.02\left[t - 0.05 - \frac{1}{\sqrt{75}} e^{-15(t-0.05)} \sin\sqrt{75}(t - 0.05)\right] u(t - 0.05)$$

Problem 5.32 illustrates the use of the transfer function is solving base motion problems.

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5.33 For the system of Figure P5.33(a), complete the following.

FIGURE P5.33

(a) Determine its transfer function defined as $G(s) = \frac{\theta(s)}{Y(s)}$ where $\theta(s)$ is the Laplace transform of the angular displacement of the bar.

(b) Use the transfer function to determine $\theta(t)$ due to $y(t)$, as showing in Figure P5.33(b).

Given: $k_1 = 1000\frac{\text{N}}{\text{m}}, k_2 = 2000\frac{\text{N}}{\text{m}}, m = 12\text{ kg}, a = 20\text{ cm}, b = 80\text{ cm}, F(t)$

Find: $(s) = \frac{\theta(s)}{Y(s)}, \theta(t)$

Solution: FBDs of the system at an arbitrary instant are shown. Summing moments about the point of support using $(\sum M_o)_{ext} = (\sum M_o)_{eff}$ leads to

$$-1000(y + 0.2\theta)(0.2) - 2000(0.8\theta)(0.8) = 12(0.3\ddot{\theta})(0.3) + \frac{1}{12}(12)(1)^2\ddot{\theta}$$

which becomes upon simplification

$$2.08\ddot{\theta} + 155.2 = -200y(t)$$

Taking the Laplace transform of both sides of the above equation leads to

$$G(s) = \frac{\theta(s)}{Y(s)} = \frac{-200}{2.08s^2 + 155.2}$$

(b) The excitation is given by

$$y(t) = 0.1t[u(t) - u(t - 0.1)] + (0.002 - 0.1t)[u(t - 0.1) - u(t - 0.2)]$$
$$= 0.1tu(t) - 0.2(t - 0.1)u(t - 0.1) + 0.1(t - 2)u(t - 0.2)$$

The Laplace transform of y(t) is

$$Y(s) = \frac{0.1}{s^2}(1 - 2e^{-0.1s} + e^{-0.2s})$$

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Substitution into the transfer function yields

$$\theta(s) = \frac{-20}{s^2(2.08s^2 + 155.2)}(1 - 2e^{-0.1s} + e^{-0.2s})$$

A partial fraction decomposition leads to

$$\theta(s) = -\frac{20}{(2.08)(74.61)}\left(\frac{1}{s^2} - \frac{1}{s^2 + 74.61}\right)(1 - 2e^{-0.1s} + e^{-0.2s})$$

Inversion leads to

$$\theta(s) = -0.129\{t - 0.116\sin 8.67t - 2[t - 0.1 - 0.116\sin 8.67(t - 0.1)]u(t - 0.1) \\ + [t - 0.2 - 0.116\sin 8.67(t - 0.2)]u(t - 0.2)\}$$

Problem 5.33 illustrates application of the transfer function to base motion problems.

5.34 During its normal operation, a 144-kg machine tool is subject to a 15,000 N · s impulse. Design an efficient isolator such that the maximum force transmitted through the isolator is 2500 N and the maximum displacement is minimized.

Given: I = 25000 N · s, m = 144 kg, $F_{T,max.}$ = 2500 N

Find: k and c such that $x_{max.}$ is a minimum

Solution: For a given $F_{T,max.}$, $x_{max.}$ is minimized by selecting ζ such that $S(\zeta)$ is a minimum. This requires setting

$$\zeta = 0.4$$

The natural frequency is calculated from

$$Q(\zeta) = \frac{F_{T_{max.}}}{mv\omega_n}$$

$$\omega_n = \frac{F_{T_{max.}}}{mvQ(0.4)}$$

where from impulse-momentum

$$mv = I = 15000 \text{ N} \cdot \text{sec}$$

and from eq. (5.110)

$$Q(0.4) = 0.88$$

Thus

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$$\omega_n = \frac{2500\ \text{N}}{(15000\ \text{N}\cdot\text{sec})(0.88)} = 0.189\ \frac{\text{rad}}{\text{sec}}$$

The isolator stiffness is

$$k = m\omega_n^2 = (144\ \text{kg})\left(0.189\ \frac{\text{rad}}{\text{sec}}\right)^2 = 1.14\ \frac{\text{N}}{\text{m}}$$

The isolator damping coefficient is

$$c = 2\zeta m\omega_n = 2(0.4)(144\ \text{kg})\left(0.189\ \frac{\text{rad}}{\text{sec}}\right) = 43.55\ \frac{\text{N}\cdot\text{sec}}{\text{m}}$$

Problem 5.34 illustrates design of an isolator to protect a foundation from impulsive loading with minimum displacement.

5.35 A 110 kg pump is mounted on an isolator of stiffness 4×10^5 N/m and a damping ratio of 0.15. The pump is given a sudden velocity of 30 m/s. What is the maximum force transmitted through the isolator and what is the maximum displacement of the pump?

Given: m = 110 kg, k = 4 × 105 N/m, ζ = 0.15, v = 30 m/sec

Find: $F_{T,max}$, $x_{max.}$

Solution: The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times 10^5\ \frac{\text{N}}{\text{m}}}{110\ \text{kg}}} = 60.3\ \frac{\text{rad}}{\text{sec}}$$

The maximum transmitted force is calculated from

$$\frac{F_{T_{max.}}}{mv\omega_n} = Q(0.15)$$

where Q(0.15) = 0.844 as calculated using eq.(5.110). Thus

$$F_{T_{max.}} - 0.844(110\ \text{kg})\left(30\ \frac{\text{m}}{\text{sec}}\right)\left(60.3\ \frac{\text{rad}}{\text{sec}}\right) = 1.68\times 10^5\text{N}$$

The maximum displacement is obtained from

$$\frac{x_{max.}F_{T_{max,}}}{\frac{1}{2}mv^2} = S(0.15)$$

where S(0.15) = 1.823 is obtained from eq.(5.112). Then

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$$x_{max.} = \frac{1.823(110\text{ kg})\left(30\ \frac{\text{m}}{\text{sec}}\right)^2}{2\left(1.68\times10^5\ \frac{\text{N}}{\text{m}}\right)} = 0.537\text{ m}$$

Problem 5.35 illustrates the use of Q(ζ) and S(ζ).

5.36 During operation, a 50-kg machine tool is subject to the short-duration pulse of Figure P5.36. Design an isolator that minimizes the maximum displacement and reduces the maximum transmitted force to 5000 N. What is the maximum displacement of the machine tool when this isolator is used?

FIGURE P5.36

Given: m = 50 kg, F_{max} =5000

Find: isolator design, x_{max}

Solution: The total impulse applied to the machine due to the impulsive excitation is

$$I = \int_0^{t_0} F(t)\,dt$$

$$I = (30{,}000\text{ N})\left[(0.005\text{ s}) + \frac{1}{2}(0.005\text{ s})\right]$$

$$I = 225\text{ N}\cdot\text{s}$$

The initial velocity imparted to the machine due to the impulse is

$$v = \frac{I}{m} = \frac{225\text{ N}\cdot\text{s}}{50\text{ kg}} = 4.5\text{ m/s}$$

The maximum displacement is minimized by designing an isolator of damping ratio ζ = 0.4. Then if the maximum transmitted force is limited to 5,000 N, the maximum displacement is

$$x_{\max} = \frac{\frac{1}{2}mv^2}{F_{\max}}S(0.4) = \frac{\frac{1}{2}(50\text{ kg})(4.5\text{ m/s})^2}{5000\text{ N}}1.04 = 0.105\text{ m}$$

The natural frequency of the isolator is calculated by using Eq.(5.110)

$$\omega_n = \frac{F_{\max}}{mvQ(0.4)} = \frac{5000\text{ N}}{(50\text{ kg})(4.5\text{ m/s})(0.88)} = 25.3\text{ rad/s}$$

The maximum isolator stiffness is

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$$k = m\omega_n^2 = (50\text{ kg})(25.3\text{ rad/s})^2 = 3.19\times10^4\text{ N/m}$$

Problem 5.36 illustrates design of an isolator to protect against a short duration pulse.

5.37 Repeat Chapter Problem 5.36 for the short-duration pulse of Figure P5.37.

Given: m = 50 kg, F_{max} = 5000 N

Find: isolator design, x_{max}

FIGURE P5.37

Solution: The total impulse applied to the machine due to the impulsive excitation is

$$I = \int_0^{t_0} F(t)\,dt$$

$$I = \frac{1}{2}(20{,}000\text{ N})(0.01\text{ s})$$

$$I = 100\text{ N}\cdot\text{s}$$

The initial velocity imparted to the machine due to the impulse is

$$v = \frac{I}{m} = \frac{100\text{ N}\cdot\text{s}}{50\text{ kg}} = 2\text{ m/s}$$

The maximum displacement is minimized by designing an isolator of damping ratio ζ = 0.4. Then if the maximum transmitted force is limited to 5,000 N, the maximum displacement is

$$x_{max} = \frac{\frac{1}{2}mv^2}{F_{max}}S(0.4) = \frac{\frac{1}{2}(50\text{ kg})(2\text{ m/s})^2}{5000\text{ N}}1.04 = 0.0208\text{ m}$$

The natural frequency of the isolator is calculated by using Eq.(5.110)

$$\omega_n = \frac{F_{max}}{mvQ(0.4)} = \frac{5000\text{ N}}{(50\text{ kg})(2\text{ m/s})(0.88)} = 56.8\text{ rad/s}$$

The maximum isolator stiffness is

$$k = m\omega_n^2 = (50\text{ kg})(56.8\text{ rad/s})^2 = 1.61\times10^5\text{ N/m}$$

Problem 5.37 illustrates design of an isolator to protect against a short duration pulse.

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5.38 A ship is moored at a dock in rough seas and frequently impacts the dock. The maximum velocity change caused by the impact is 15 m/s. Design an isolator to protect a sensitive 80-kg navigational control system such that its maximum acceleration is 30 $\mathrm{m/s^2}$.

Given: m = 80 kg, $\Delta v = 15$ m/s, $a_{max} = 30\ \mathrm{m/s^2}$

Find: k,c

Solution: Since minimizing the transmitted accelerations is the only consideration, the damping ratio is chosen as $\zeta = 0.25$ such that $Q(\zeta)$is a minimum with a value of Q(0.25) = 0.81. Then the maximum transmitted acceleration is

$$\frac{a_{max}}{v\omega_n} = Q(0.25) = 0.81$$

which leads to

$$\omega_n = \frac{a_{max}}{0.81v} = \frac{30\ \mathrm{m/s^2}}{0.81(15\ \mathrm{m/s})} = 2.47\ \mathrm{rad/s}$$

The maximum stiffness is calculated as

$$k = m\omega_n^2 = (80\ \mathrm{kg})(2.47\ \mathrm{rad/s})^2 = 487.7\ \mathrm{N/m}$$

The isolator damping coefficient is

$$c = 2\zeta m\omega_n = 2(0.25)(80\ \mathrm{kg})\left(2.47\frac{\mathrm{rad}}{\mathrm{s}}\right) = 98.8\ \mathrm{N\cdot s/m}$$

Problem 5.38 illustrates shock isolation theory applied to a transmitted acceleration problem.

5.39 A one-story frame structure with an equivalent mass 12,000 kg and stiffness 1.8×10^6 N/m is subject to a blast whose force is given in Figure P5.39. What is the maximum deflection of the structure?

FIGURE P5.39

Given: $m_{eq} = 12{,}000$ kg, $k_{eq} = 1.8 \times 10^6\ \frac{\mathrm{N}}{\mathrm{m}}$, $F_0 = 35000$ N, $t_0 = 0.6$ s

Find: x_{max}

Solution: Calculations show that

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$$\frac{t_0}{T} = \frac{\omega_n t_0}{2\pi} = \frac{t_0}{2\pi}\sqrt{\frac{k}{m}} = \frac{0.6\text{ s}}{2\pi}\sqrt{\frac{1.8\text{x} \times 10^6\text{ N/m}}{12000\text{ kg}}} = 1.17$$

From the displacement spectrum for an undamped system subject to a triangular pulse

$$\frac{kx_{max}}{F_0} = 1.5$$

The maximum displacement is calculated as

$$x_{max} = \frac{1.5(35000\text{ N})}{1.8 \times 10^6\text{ N/m}} = 2.92\text{ cm}$$

Problem 5.39 illustrates use of a displacement spectrum to determine the maximum displacement of a structure.

5.40 A 20 kg machine is on a foundation that is subject to an acceleration that is modeled as a versed sine pulse of magnitude of 20 m/s^2 and duration of 0.4 s. Design an undamped isolator such that the maximum acceleration felt by the machine is 15 m/s^2. What is the maximum displacement of the machine tool relative to its foundation when the isolator is used?

Given: m = 20 kg, $a_0 = 20\text{ m/s}^2$, a_{max} = 15 m/s^2, $t_0 = 0.4$ s, $\zeta = 0$

Find: k, z_{max}

Solution: The ratio of the acceleration of the machine to the acceleration of its foundation is

$$\frac{a_{max}}{a_0} = 0.75$$

The force spectrum for a versed sine pulse is used to determine that for the acceleration ratio to be less than 0.76,

$$\frac{\omega_n t_0}{2\pi} < 0.29 \Longrightarrow \omega_n < \frac{(0.29)2\pi}{0.4\text{ s}} = 4.5\text{ rad/s}$$

The isolator stiffness is given by

$$k = m\omega_n^2 = (20\text{ kg})\left(4.5\frac{\text{rad}}{\text{s}}\right)^2 = 400\text{ N/m}$$

The maximum relative displacement is determined using the displacement spectrum. However since the isolator is undamped the displacement spectrum is the same as the force spectrum. For transmitted acceleration problems the vertical scale of the displacement spectrum becomes

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$$\frac{z_{max}\omega_n^2}{a_o} = 0.75 \Longrightarrow z_{max} = \frac{(0.75)(20\ \text{m/s}^2)}{(4.5\ \text{rad/s})^2} = 0.74\ \text{m}$$

Problem 5.40 illustrates the use of force and displacement spectra to design an isolator to protect a machine from a foundation acceleration.

5.41 During operation, a 100 kg machine tool is exposed to a force that is modeled as a sinusoidal pulse of magnitude of 3100 N and duration of 0.05 s. Design an isolator with a damping ratio of 0.1 such that the maximum force transmitted through the isolator is 2000 N and the maximum displacement of the machine tool is 3 cm.

Given: m = 100 kg, $F_0 = 3100$ N, $t_0 = 0.05$ s, $F_{max} = 2000$ N, x_{max} = 3 cm, $\zeta = 0.1$

Find: k, c

Solution: The maximum value of $\frac{F_T}{F_0}$ is $\frac{2100\ \text{N}}{3000\ \text{N}} = 0.645$. The force spectrum for a sinusoidal pulse yields $\frac{\omega_n t_0}{2\pi} = 0.25$ or $\omega_n < \frac{2\pi(0.25)}{0.05\ \text{s}} = 31.4\ \text{rad/s}$. The displacement spectrum for a sinusoidal pulse corresponding to a value of 0.25 on the horizontal scale yields $\frac{kx_{max}}{F_0} = 0.67 \Longrightarrow x_{max} = 0.21$ m which is less than the 3 cm required. The stiffness and damping coefficient of such an isolator are

$$k = m\omega_n^2 = (100\ \text{kg})(31.4\ \text{rad/s})^2 = 98600\ \text{N/m}$$

$$c = 2\zeta m\omega_n = 2(0.1)(100\ \text{kg})\left(31.4\frac{\text{rad}}{\text{s}}\right) = 528.0\ \text{N}\cdot\text{s/m}$$

Problem 5.41 illustrates use of the force and displacement spectra for sinusoidal pulse excitation.

5.42 During operation a 80 kg machine is subject to a triangular pulse with a magnitude of 30,000 N and duration of 0.15 s. What is the range of undamped isolator stiffness such that the maximum transmitted force is 15,000 N and the maximum displacement is 5 cm?

Given: m = 80 kg, F_0 = 30,000 N, $t_0 = 0.15$ s, $x_{max} = 5$ cm, $F_{max} = 15{,}000$N

Find: range of k

Solution: Limiting the maximum force to 15000 N implies that $\frac{F_{max}}{F_0} = 0.5$ which, from the force spectrum for an undamped system subject to a triangular pulse, leads to $\frac{t_0}{T} = 0.3$ which implies

$$T = \frac{0.15\ \text{s}}{3} = 0.05\ \text{s}.$$

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This leads to

$$\omega_n = \frac{2\pi}{0.05\ \text{s}} = 125.7\ \text{rad/s}$$

which leads to

$$k = (80\ \text{kg})\left(125.7\frac{\text{rad}}{\text{s}}\right)^2 = 1.26 \times 10^6\ \text{N/m}$$

This is an upper bound on the stiffness. Checking the maximum displacement with this stiffness

$$\frac{kx_{max}}{F_0} = 0.5 \Longrightarrow x_{max} = \frac{0.5(30000\ \text{N})}{1.26 \times 10^6\ \text{N/m}} = 1.19\ \text{cm}$$

The maximum value of $\frac{kx_{max}}{F_0}$ is 1.6. With this value $k = \frac{1.6(15000\ \text{N})}{0.05\ \text{m}} = 4.8 \times 10^5\ \text{N/m}$. For this value of k, $\frac{t_0}{T} = 1.89 \Longrightarrow \frac{kx_{max}}{F_0} = 1.1 \Longrightarrow k = 3.3 \times 10^5$. The iteration can continue, but this is close enough. Thus

$$3.3 \times 10^5 < k < 1.26 \times 10^6\ \text{N/m}$$

Problem 5.42 illustrates the use of the force and displacement spectra to determine a lower bound and an upper bound on the stiffness.

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CHAPTER 6: TWO DEGREE-OF-FREEDOM SYSTEMS

Short Answer Problems

6.1 True: A two-degree-of-freedom system has two natural frequencies. The normal mode solution is assumed and substituted into the differential equations. Two linearly independent solutions are obtained, one corresponding to each natural frequency.

6.2 False: The natural frequencies are determined by setting $|\omega^2\mathbf{M} - \mathbf{K}| = 0$.

6.3 False: The natural frequencies of a two degree-of-freedom system are independent of the choice of generalized coordinates used to model the system.

6.4 True: Even powers occur if the system is undamped. If the system has viscous damping odd powers occur as well.

6.5 False: The modal fraction represents the displacement of the second generalized coordinate compared to the displacement of the first.

6.6 False: The principal coordinates are the generalized coordinates for which the mass matrix and the stiffness matrix are diagonal matrices.

6.7 False: The free response of a damped two degree-of-freedom system has two modes of vibration, both of which may be underdamped.

6.8 True: A displacement of a node for a mode of a two degree-of-freedom system can serve as a principal coordinate. Since a node can only occur for the second mode, the node can serve as a principal coordinate for the first mode.

6.9 True: While the natural frequencies are independent of the choice of generalized coordinates, the modal fractions for a two degree-of-freedom system depend upon the choice of generalized coordinates.

6.10 True: The sinusoidal transfer function which is the transfer function evaluated at $i\omega$, where ω is the frequency of input, can be used to determine the steady-state response of a two-degree-of-freedom system. The steady-state response is $F_0\left|G_{ij}(i\omega)\right|$.

6.11 True: The addition of a vibration absorber adds another degree of freedom to the original system.

6.12 False: The undamped vibration absorber is tuned to the excitation frequency to eliminate steady-state vibrations of the primary systems.

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6.13 False: An optimally tuned damped vibration absorber is tuned such that the amplitude of vibration during start up is minimized as well as increasing the operating range.

6.14 False: Addition of a dynamic vibration absorber to a damped primary system will not eliminate the steady-state vibrations of the primary system at any speed.

6.15 True: A Houdaille vibration absorber is used on engine crankshafts and other devices.

6.16

6.17

6.18 The normal mode solution is $x(t) = Xe^{i\omega t}$. It is substituted into the differential equations for free vibration of a two-degree-of-freedom system to determine the natural frequencies and mode shapes. The mode shapes are the solutions to $|\omega^2\mathbf{M} - \mathbf{K}| = 0$ and the mode shapes are the corresponding non-trivial solutions of $\mathbf{KX}=\omega^2\mathbf{MX}$.

6.19 The assumed solution for an undamped system is $\mathbf{x}(t) = \mathbf{X}e^{i\omega t}$ while for a damped system $\mathbf{x}(t) = \mathbf{X}e^{\lambda t}$. The values of λ for a damped system may be complex while ω for an undamped system must real. The modal fraction for a damped system are also complex and occur in complex conjugate pairs.

6.20 A real solution for λ means that the mode associated with the value of λ is overdamped.

6.21 A complex solution for λ means that the mode associated with the value of λ is underdamped. Also the complex conjugate of the value is also a solution of the fourth order equation.

6.22 The transfer function $G_{1,2}(s)$is the transform of the response at the generalized coordinate described by x_1 due to a unit impulse applied at the generalized coordinate whose response is described by x_2.

6.23 The sinusoidal transfer function is $G(i\omega)$where ω is the frequency of excitation. It can be written as $|G(i\omega)|e^{i\phi}$ where $|G(i\omega)|F_0$ is the amplitude of the steady-state response and ϕ is the phase angle between the excitation and the response.

6.24 The differential equations defining the principal coordinates are

$$\ddot{p}_1 + \omega_1^2 p_1 = 0$$

$$\ddot{p}_2 + \omega_2^2 p_2 = 0$$

6.25 A modal fraction equal to zero implies that the second mode is uncoupled from the first mode.

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6.26 A modal fraction equal to one means that the system moves as a rigid body for that mode.

6.27 There are zero nodes corresponding to the lowest natural frequency of a tow degree of freedom system.

6.28 If the differential equations governing a two degree-of-freedom system are uncoupled when a certain set of generalized coordinates are used the coordinates must be principal coordinates for the system.

6.29 (b) the denominator D(s)

6.30 The appropriate convolution integral solution is

$$x_1(t) = \int_0^t f(\tau) h_{12}\,(t-\tau) d\tau$$

where $h_{12}(t) = \mathcal{L}^{-1}\{G_{12}(s)\}$.

6.31 The amplitudes and phases are determined by applying the initial conditions to

$$x_1(t) = A_1 \sin(\omega_1 t + \phi_1)$$

$$x_2(t) = A_2 \sin(\omega_2 t + \phi_2)$$

6.32 $G(s) = |G(i\omega)|e^{i\phi}$ where $|G(i\omega)| = \sqrt{Re^2[G(i\omega)] + Im^2[G(i\omega)]}$ and $\phi = \tan^{-1}\frac{Im[G(i\omega)]}{Re[G(i\omega)]}$

6.33 The vibration amplitude of the primary system when a dynamic vibration absorber tuned to the excitation frequency is added to the system is zero.

6.34 A dynamic vibration absorber works by adding one degree of freedom to a system whose natural frequency is ear the excitation frequency such that the two natural frequencies are away from the excitation frequency. One natural frequency is lower than the excitation frequency one is higher.

6.35 A vibration damper is used in situations where vibration control is necessary over a wide range of frequencies.

6.36 The two problems addressed by adding damping to a vibration absorber are the large amplitude during start up when the lowest natural frequency of the system is passed and the wide variation in amplitude of the primary system with operating speed.

6.37 The optimum damping ratio for a Houdaillle damper is $\zeta = \frac{1}{\sqrt{2(\mu+1)(\mu+2)}}$where μ is the ratio of the moment of inertia of the damper to the moment of inertia of the shaft.

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6.38 Given: $6\omega^4 - 27\omega^2 + 21 = 0$. The natural frequencies are obtained by applying the quadratic formula $\omega^2 = \frac{\left(27 \pm \sqrt{27^2 - 4(6)(21)}\right)}{[2(6)]} = 3.5, 1$ leading to $\omega = 1.87, 1$.

6.39 Given: $\begin{bmatrix} -\omega^2 + 3 & -2 \\ -2 & -\omega^2 + 2 \end{bmatrix} \begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (a) The equation is the frequency equation for a system such that $|\omega^2 \mathbf{M} - \mathbf{K}| = 0$. Thus the differential equations are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which comes from the system shown in Fig SP 6.39a. (b) The natural frequencies of the system are calculated from $\begin{bmatrix} -\omega^2 + 3 & -2 \\ -2 & -\omega^2 + 2 \end{bmatrix} = 0 = (-\omega^2 + 3)(-\omega^2 + 2) - 4 \Longrightarrow$ $\omega^4 - 5\omega^2 + 2 = 0$whose solutions are $\omega^2 = \frac{5 \pm \sqrt{17}}{2} \Longrightarrow \omega = 0.662, 2.14$ (c) The mode shapes are determined from the given equation. The first equation yields $\chi = \frac{-\omega^2 + 3}{2}$ which leads to $\chi_1 = 1.28$ and $\chi_2 = -0.781$ which lead to mode shapes of $\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.28 \end{bmatrix}$and $\mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.781 \end{bmatrix}$ (e) The mode shapes are illustrated in Figure SP 6.39b.

6.40 The mode shape corresponds to the higher mode.

6.41 Given: $G(s) = \frac{1}{s^4 + 3s^2 + 2}$ (a) $G(3i) = \frac{1}{(3i)^4 + 2(3i)^2 + 2} = \frac{1}{-79 + 18i} = -0.0120 - 0.00274i$ (b) The squares of the natural frequencies are the roots of $\omega^4 - 3\omega^2 + 2 = 0$ which are 1 and 2. Hence $\omega_1 = 1$ and $\omega_2 = \sqrt{2} = 1.41$

6.42 Given: $G(s) = \frac{1}{s^4 + 2s^3 + 4s^2 + 10s + 25}$, $F(t) = 2.5 \sin 4t$ The steady-state response is given by $x_1(t) = X \sin(4t - \emptyset)$ where
$X = F_0 |G(4i)| = 2.5 \left| \frac{1}{(4\mathrm{i})^4 + 2(4\mathrm{i})s^3 + 4(4\mathrm{i})^3 + 10(4\mathrm{i}) + 25} \right| = |10^{-13}(9.89 + 4.01\mathrm{i})| = 0.007$ and $\phi = \tan^{-1} \frac{Im\,[G(4i)]}{Re\,[G(4i)]} = 0.3853$. Thus $x_1(t) = 0.0107 \sin(4t - 0.3853)$.

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6.43 (a) $r_2 = 1$ (b) $r_1 = 1.05$ (c) $q = \frac{r_1}{r_2} = 1.05$

6.44 Given: $\omega_{11} = 100, \mu = 0.2$

$\omega_{1,2} = \frac{100}{\sqrt{2}}\sqrt{1 + 1.05^2(1+0.2) \pm \sqrt{1.05^4(1+0.2)^2 + 2(0.2-1)1.05^2 + 1}} =$
$81.5\frac{\text{rad}}{\text{s}}, 128.8\ \frac{\text{rad}}{\text{s}}$

6.45 Given: f = 30 Hz, $F_0 = 200$ N (a) The steady-state vibration of the machine are eliminated if the absorber is tuned to a frequency of 30 Hz = 188.5 rad/s (b) If $m_2 = 3$ kg the stiffness of the absorber is $k_2 = m_2\omega^2 = (3\text{ kg})(188.5\text{ rad/s})^2 = 1.07 \times 10^5$ N/m (c) $X_2 = \frac{F_0}{k_2} = \frac{200\ N}{1.07\times10^5\text{ N/m}} = 1.9$ mm (d) $\omega = 188.5$ rad/s

6.46 Given: $m_1 = 50$ kg, $m_2 = 10$ kg, $\omega_{11} = 100$ rad/s (a) $\mu = \frac{m_2}{m_1} = 0.2, q = \frac{1}{1+\mu} = \frac{1}{1+0.2} = 0.833$, $\omega_{22} = q\omega_{11} = 0.833(100) = 83.33$ rad/s (b) $\zeta_{opt} = \sqrt{\frac{3\mu}{8(1+\mu)}} = \sqrt{\frac{3(0.2)}{8(1.2)}} = 0.25$.

6.47 Given: $J_1 = 0.1\text{ kg}\cdot\text{m}^2, J_2 = 0.01\text{ kg}\cdot\text{m}^2$, $\frac{M_0}{k} = 0.002$ (a) $\mu = \frac{J_2}{J_1} = 0.1$,

(a) $\mu = \frac{J_2}{J_1} = 0.1$, $\zeta_{opt} = \sqrt{\frac{1}{2(1+\mu)(\mu+2)}} = \sqrt{\frac{1}{2(1.1)(2.1)}} = 0.465$ (b) $\Theta = \frac{M_0}{k}\frac{2+\mu}{\mu} = (0.002)\frac{2.1}{0.1} = 0.042$ rad

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Chapter Problems

6.1 Derive the differential equation governing the two degree-of-freedom system shown in Figure P6.1 using x_1 and x_2 as generalized coordinates.

FIGURE P6.1

Given: System shown

Find: Differential equations

Solution: Free-body diagrams of both masses are drawn at an arbitrary instant

Summation of forces on the FBD's lead to

$$-kx_1 + k(x_2 - x_1) = m\ddot{x}_1$$

and

$$-k(x_2 - x_1) - 2kx_2 = 2m\ddot{x}_2$$

The equations are rearranged to

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$2m\ddot{x}_2 - kx_1 + 3kx_2 = 0$$

Problem 6.1 illustrates the use of Newton's law to derive the differential equations for a 2DOF system.

6.2 Derive the differential equation governing the two degree-of-freedom system shown in Figure P6.2 using x and θ as generalized coordinates.

FIGURE P6.2

Given: System shown

Find: Differential equations

Solution: Free-body diagrams of the bar and the discrete mass, assuming small θ, are drawn at an arbitrary instant

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Summing moments on the bar using $(\sum M_O)_{ext} = (\sum M_O)_{eff}$ leads to

$$k\left(x-\frac{L}{2}\theta\right)\frac{L}{2} - kL\theta(L) + M_0\sin\omega t = m\frac{L}{2}\ddot{\theta}\left(\frac{L}{2}\right) + \frac{1}{12}mL^2\ddot{\theta} \qquad (a)$$

Summing forces on the mass using $\sum F_{ext} = \sum F_{eff}$ leads to

$$-k\left(x-\frac{L}{2}\theta\right) = m\ddot{x} \qquad (b)$$

Rearranging Eqs. (a) and (b) yields

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{3}{4}kL^2\theta - k\frac{L}{2}x = M_0\sin\omega t \qquad (c)$$

$$m\ddot{x} - k\frac{L}{2}\theta + kx = 0 \qquad (d)$$

Problem 6.2 illustrates the use of D'Alembert's principle to derive the differential equations governing a 2DOF system.

6.3 Derive the differential equations governing the two degree-of-freedom system shown in Figure P6.3 using θ_1 and θ_2 as generalized coordinates.

Given: System shown

Find: Differential equations

Solution: Free-body diagrams of the disks are drawn at an arbitrary instant

FIGURE P6.3

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Summing moments on disk A using $(\sum M_A)_{ext} = (\sum M_A)_{eff}$ leads to

$$-kr\theta_1(r) - k(r\theta_1 - r\theta_2)r = I_1\ddot{\theta}_1 + mr\ddot{\theta}_1(r) \qquad (a)$$

Summing moments on disk B using $(\sum M_B)_{ext} = (\sum M_B)_{eff}$ leads to

$$k(r\theta_1 - r\theta_2)r - (F_0 \sin\omega t)2r = I_2\ddot{\theta}_2 + m(2r\ddot{\theta}_2)(2r) \qquad (b)$$

Rearranging Eqs. (a) and (b) lead to

$$(I_1 + mr^2)\ddot{\theta}_1 + 2kr^2\theta_1 - kr^2\theta_2 = 0 \qquad (c)$$

and

$$(I_2 + 4mr^2)\ddot{\theta}_2 - kr^2\theta_1 + kr^2\theta_2 = -2F_0 r \sin\omega t \qquad (c)$$

Problem 6.3 illustrates the use of D'Alembert's principle to derive the differential equations governing a 2DOF system.

6.4 Derive the differential equations governing the two degree-of-freedom system shown in Figure P6.4 using θ_1 and θ_2 as generalized coordinates.

Given: System shown

Find: Differential equations

FIGURE P6.4

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Solution: Free-body diagrams of the disks are drawn at an arbitrary instant

Summing moments on disk A using $(\sum M_A)_{ext} = (\sum M_A)_{eff}$ leads to

$$-k_{t1}\theta_1 - k_{t2}(\theta_1 - \theta_2) = I_1\ddot{\theta}_1 \qquad (a)$$

Summing moments on disk B using $(\sum M_B)_{ext} = (\sum M_B)_{eff}$ leads to

$$k_{t2}(\theta_1 - \theta_2) = I_2\ddot{\theta}_2 \qquad (b)$$

Noting that

$$k_{t1} = \frac{J_1 G_1}{L} \qquad k_{t2} = \frac{J_2 G_2}{3L/2} \qquad (c)$$

Eqs. (a) and (b) are rearranged to

$$I_1\ddot{\theta}_1 + \left(\frac{J_1 G_1}{L} + \frac{J_2 G_2}{3L/2}\right)\theta_1 - \frac{J_2 G_2}{3L/2}\theta_2 = 0 \qquad (d)$$

and

$$I_2\ddot{\theta}_2 - \frac{J_2 G_2}{3L/2}\theta_1 + \frac{J_2 G_2}{3L/2}\theta_2 = 0 \qquad (e)$$

Problem 6.4 illustrates the use of D'Alembert's principle to derive the differential equations governing a 2DOF system.

6.5 A two degree-of-freedom model of an airfoil shown in Figure P6.5 is used for flutter analysis. Derive the governing differential equations using h and θ as generalized coordinates.

FIGURE P6.5

Given: h, θ

Find: Differential equations

Solutions: FBD's of the airfoil at an arbitrary instant are drawn.

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Summing Forces on the FBD using $\sum F_{ext} = \sum F_{eff}$ leads to

$$-k(h + \ell\theta) = m\ddot{h}$$

Summing moments according to $(\sum M_G)_{ext} = (\sum M_G)_{eff}$ leads to

$$-k_t\theta - k(h + \ell\theta)\ell = I\ddot{\theta}$$

Rearranging the equations leads to

$$m\ddot{h} + kh + k\ell\theta = 0$$

$$I\ddot{\theta} + k\ell h + (k_t + k\ell^2)\theta = 0$$

Problem 6.5 illustrates the derivation of differential equations governing the motion of a 2DOF system.

6.6 Derive the differential equations governing the damped two degree-of-freedom system shown using x_1 and x_2 as generalized coordinates.

FIGURE P6.6

Given: System shown

Find: Differential equations

Solution: Free-body diagrams of the blocks are drawn at an arbitrary instant

Summing forces on the first FBD leads to

$$-2c\dot{x}_1 - 2kx_1 + c(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) = 2m\ddot{x}_1 \qquad (a)$$

Summing forces on the second FBD leads to

$$-c(\dot{x}_2 - \dot{x}_1) - k(x_2 - x_1) + F_0 \sin\omega t = m\ddot{x}_2 \qquad (b)$$

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Equations (a) and (b) are rearranged to

$$2m\ddot{x}_1 + 3c\dot{x}_1 - c\dot{x}_2 + 3kx_1 - kx_2 = 0 \qquad (c)$$

and

$$m\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - kx_1 + kx_2 = F_0 \sin \omega t \qquad (d)$$

Problem 6.6 illustrates the use of Newton's law to derive the differential equations for a 2DOF system with viscous damping.

6.7 Derive the differential equations governing the damped two degree of freedom system shown in Figure P6.7 using x_1 and x_2 as generalized coordinates.

FIGURE P6.7

Given: System shown

Find: Differential equations

Solution: Free-body diagrams of the blocks are drawn at an arbitrary instant

Summing forces on the first FBD leads to

$$-c\dot{x}_1 - kx_1 + c(\dot{x}_2 - \dot{x}_1) = m\ddot{x}_1 \qquad (a)$$

Summing forces on the second FBD leads to

$$-c(\dot{x}_2 - \dot{x}_1) = 2m\ddot{x}_2 \qquad (b)$$

Equations (a) and (b) are rearranged to

$$m\ddot{x}_1 + 2c\dot{x}_1 - c\dot{x}_2 + kx_1 = 0 \qquad (c)$$

and

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$$2m\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 = 0 \qquad (d)$$

Problem 6.7 illustrates the use of Newton's law to derive the differential equations for a 2DOF system with viscous damping.

6.8 A two degree-of-freedom model of a machine tool is illustrated in Figure P6.8. Using x_1 and x_2 as generalized coordinates, derive the differential equations governing the motion of the system.

Given: System shown

Find: Differential equations

FIGURE P6.8

Solution: Free-body diagram of the bar is drawn at an arbitrary instant

Summing moments on the FBD using $(\sum M_A)_{ext} = (\sum M_A)_{eff}$ leads to

$$-(kx_1 + c\dot{x}_1)(a+b) = I\left(\frac{\ddot{x}_1 - \ddot{x}_2}{a+b}\right) + m\left(\frac{b\ddot{x}_1 + a\ddot{x}_2}{a+b}\right)(b) \qquad (a)$$

Summing moments on the FBD using $(\sum M_B)_{ext} = (\sum M_B)_{eff}$ leads to

$$(kx_2 + c\dot{x}_2)(a+b) = I\left(\frac{\ddot{x}_1 - \ddot{x}_2}{a+b}\right) - m\left(\frac{b\ddot{x}_1 + a\ddot{x}_2}{a+b}\right)(a) \qquad (b)$$

Rearranging Eqs. (a) and (b) leads to

$$\left(\frac{I + mb^2}{a+b}\right)\ddot{x}_1 + \left(\frac{mab - I}{a+b}\right)\ddot{x}_2 + c(a+b)\dot{x}_1 + k(a+b)x_1 = 0 \qquad (c)$$

and

$$\left(\frac{I - mab}{a+b}\right)\ddot{x}_1 - \left(\frac{I + ma^2}{a+b}\right)\ddot{x}_2 - c(a+b)\dot{x}_2 - k(a+b)x_2 = 0 \qquad (d)$$

Problem 6.8 illustrates the use of D'Alembert's principle to derive the differential equations governing a 2DOF system with viscous damping.

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6.9 Derive the differential equation of the two degree-of-freedom model of the machine tool of Chapter Problem 6.8 using x and θ as generalized coordinates.

FIGURE P6.8

Given: System shown

Find: Differential equations

Solution: Free-body diagram of the bar is drawn at an arbitrary instant

Summing moments on the FBD using $(\sum M_A)_{ext} = (\sum M_A)_{eff}$ leads to

$$-k(x + a\theta)(a + b) - c(\dot{x} + a\dot{\theta})(a + b) = I(\ddot{\theta}) + m\ddot{x}(b) \qquad (a)$$

Summing moments on the FBD using $(\sum M_B)_{ext} = (\sum M_B)_{eff}$ leads to

$$-k(x - b\theta)(a + b) - c(\dot{x} - b\dot{\theta})(a + b) = I(\ddot{\theta}) - m\ddot{x}(a) \qquad (b)$$

Rearranging Eqs. (a) and (b) leads to

$$I\ddot{\theta} + mb\ddot{x} + ca(a + b)\dot{\theta} + c(a + b)\dot{x} + ka(a + b)\theta + k(a + b)x = 0 \qquad (c)$$

and

$$I\ddot{\theta} - ma\ddot{x} + cb(a + b)\dot{\theta} - c\dot{x}(a + b) + kb(a + b)\theta - k(a + b)x = 0 \qquad (d)$$

Problem 6.9 illustrates the use of D'Alembert's principle to derive the differential equations governing a 2DOF system with viscous damping.

6.10 Determine the natural frequencies of the system of Figure P6.1 if m = 10 kg and k = 1 × 10^5 N/m. Determine and graphically illustrate the mode shapes. Identify any nodes.

FIGURE P6.1

Given: m = 10 kg, k = 1 × 10^5 N/m

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Find: $\omega_1, \omega_2, \chi_1, \chi_2$

Solution: The differential equations are derived in the solution of Chapter Problem 6.1. They are written in matrix form using the given values of k and m as

$$\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^5 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (a)$$

The natural frequencies are determined as the solutions of $|-\omega^2\mathbf{M} + \mathbf{K}| = 0$ or

$$\begin{vmatrix} -10\omega^2 + 2\phi & -\phi \\ -\phi & -20\omega^2 + 3\phi \end{vmatrix} = 0 = (-10\omega^2 + 2\phi)(-20\omega^2 + 3\phi) - \phi^2$$
$$= 200\omega^4 - 70\phi\omega^2 + 5\phi^2$$

where $\phi = 10^5$. The solutions of the equation are

$$\omega_1 = 100\frac{\text{rad}}{\text{s}} \qquad \omega_2 = 158.11\frac{\text{rad}}{\text{s}}$$

The modal fractions are determined by

$$\chi = \frac{-10\omega^2 + 2\phi}{\phi}$$

which when evaluated lead to $\chi_1 = 1$ and $\chi_2 = -0.5$. Hence the mode shapes are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

The mode shape vectors are illustrated below. There is a node for the second mode in the spring connecting the two masses.

Problem 6.10 illustrates the computation of natural frequencies and mode shapes for 2DOF systems.

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6.11 Determine the natural frequencies of the system of Figure P6.2 if m = 2 kg, L = 1 m and k = 1000 N/m. Determine the modal fractions for each mode.

FIGURE P6.2

Given: m = 2 kg, L = 1m, k = 1000 N/m

Find: $\omega_1, \omega_2, \chi_1, \chi_2$

Solution: The differential equations are derived in the solution of Chapter Problem 6.2 as

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{3}{4}kL^2\theta - k\frac{L}{2}x = 0 \qquad (a)$$

$$m\ddot{x} - k\frac{L}{2}\theta + kx = 0 \qquad (b)$$

Rewriting Eqs. (a) and (b) in matrix form using the given values

$$\begin{bmatrix} 0.667 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 750 & -500 \\ -500 & 1000 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (c)$$

The natural frequencies are determined as the solutions of $|-\omega^2\mathbf{M} + \mathbf{K}| = 0$ or

$$\begin{vmatrix} -0.667\omega^2 + 750 & -500 \\ -500 & -2\omega^2 + 1000 \end{vmatrix} = 0 = (-0.667\omega^2 + 750)(-2\omega^2 + 1000) - 500^2$$
$$= 1.33\omega^4 - 2166.7\omega^2 + 500000$$

The solutions of the equation are

$$\omega_1 = 16.69\frac{\text{rad}}{\text{s}} \qquad \omega_2 = 36.69\frac{\text{rad}}{\text{s}}$$

The modal fractions are determined by

$$\chi = \frac{-0.667\omega^2 + 750}{500}$$

which when evaluated lead to $\chi_1 = 1.128$ and $\chi_2 = -0.2949$. Hence the mode shapes are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.128 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.2949 \end{bmatrix}$$

Problem 6.11 illustrates the computation of natural frequencies and mode shapes for 2DOF systems.

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6.12 Determine the natural frequencies of the system of Figure P6.3 if m = 30 g, I_1 = 8 × 10^{-6} kg · m^2, I_2 = 2 × 10^{-5} kg · m^2, r = 5 mm, and k = 10 N/m. Determine the modal fraction for each mode.

FIGURE P6.3

Given: m = 30 g, $I_1 = 8 \times 10^{-6}$ kg · m^2,

$I_2 = 2 \times 10^{-5}$ kg · m^2, r = 5 mm, k = 10 N/m

Given: m = 2 kg, L = 1 m, k = 1000 N/m

Find: $\omega_1, \omega_2, \chi_1, \chi_2$

Solution: The differential equations are derived in the solution of Chapter Problem 6.3 as

$$(I_1 + mr^2)\ddot{\theta}_1 + 2kr^2\theta_1 - kr^2\theta_2 = 0 \qquad (a)$$

$$(I_2 + 4mr^2)\ddot{\theta}_2 - kr^2\theta_1 + kr^2\theta_2 = -2F_0 r \sin \omega t \qquad (b)$$

Rewriting Eqs. (a) and (b) in matrix form using the given values

$$10^{-5}\begin{bmatrix} 0.875 & 0 \\ 0 & 2.3 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + 10^{-4}\begin{bmatrix} 5 & -2.5 \\ -2.5 & 2.5 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (c)$$

The natural frequencies are determined as the solutions of $|-\omega^2\mathbf{M} + \mathbf{K}| = 0$ or

$$\begin{vmatrix} -0.875\psi\omega^2 + 5\phi & -2.5\phi \\ -2.5\phi & -2.3\psi\omega^2 + 2.5\phi \end{vmatrix} = 0$$
$$= (-0.875\psi\omega^2 + 5\phi)(-2.3\psi\omega^2 + 2.5\phi) - (2.5\phi)^2$$

where $\psi = 10^{-5}$ and $\phi = 10^{-4}$ The solutions of the equation are

$$\omega_1 = 2.219\frac{\text{rad}}{\text{s}} \qquad \omega_2 = 7.943\frac{\text{rad}}{\text{s}}$$

The modal fractions are determined by

$$\chi = \frac{-0.0875\omega^2 + 5}{2.5}$$

which when evaluated lead to $\chi_1 = 1.827$ and $\chi_2 = -0.2082$. Hence the mode shapes are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.827 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.2082 \end{bmatrix}$$

Problem 6.12 illustrates the computation of natural frequencies and mode shapes for 2DOF systems.

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6.13 Determine the natural frequencies of the system of Figure P6.4 if $I_1 = 0.3\ \text{kg}\cdot\text{m}^2$, $I_1 = 0.4\ \text{kg}\cdot\text{m}^2$, $J_1 = J_2 = 1.6\times10^{-8}\ \text{m}^4$, $G_1 = G_2 = 80\times10^9\ \frac{\text{N}}{\text{m}^2}$, and L = 30 cm. Determine the modal fractions for each mode. Identify any nodes.

FIGURE P6.4

Given: $I_1 = 0.3\ \text{kg}\cdot\text{m}^2$, $I_1 = 0.4\ \text{kg}\cdot\text{m}^2$, $J_1 = J_2 = 1.6\times10^{-8}\ \text{m}^4$, $G_1 = G_2 = 80\times10^9\ \frac{\text{N}}{\text{m}^2}$, and L = 30 cm.

Find: $\omega_1, \omega_2, \chi_1, \chi_2$

Solution: The differential equations are derived in the solution of Chapter Problem 6.4 as

$$I_1\ddot{\theta}_1 + \left(\frac{J_1G_1}{L} + \frac{J_2G_2}{3L/2}\right)\theta_1 - \frac{J_2G_2}{3L/2}\theta_2 = 0 \qquad (a)$$

and

$$I_2\ddot{\theta}_2 - \frac{J_2G_2}{3L/2}\theta_1 + \frac{J_2G_2}{3L/2}\theta_2 = 0 \qquad (b)$$

Rewriting Eqs. (a) and (b) in matrix form using the given values

$$\begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + 10^3\begin{bmatrix} 7.111 & -2.844 \\ -2.844 & 2.844 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (c)$$

The natural frequencies are determined as the solutions of $|-\omega^2\mathbf{M} + \mathbf{K}| = 0$ or

$$\begin{vmatrix} -0.3\omega^2 + 7.111\phi & -2.844\phi \\ -2.844\phi & -0.4\omega^2 + 2.844\phi \end{vmatrix} = 0$$
$$= (-0.3\omega^2 + 7.111\phi)(-0.4\omega^2 + 2.844\phi) - (2.844\phi)^2$$

where $\phi = 10^3$. The solutions of the equation are

$$\omega_1 = 61.11\frac{\text{rad}}{\text{s}} \qquad \omega_2 = 164.6\frac{\text{rad}}{\text{s}}$$

The modal fractions are determined by

$$\chi = \frac{-0.3\omega^2 + 7.111\phi}{2.844\phi}$$

which when evaluated lead to $\chi_1 = 2.106$ and $\chi_2 = -0.358$. Hence the mode shapes are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 2.106 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.358 \end{bmatrix}$$

A node exists for the second mode in the shaft that connects the two disks.

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Problem 6.13 illustrates the computation of natural frequencies and mode shapes for 2DOF systems.

6.14 An overhead crane is modeled as a two degree-of-freedom system as shown in Figure P6.14. The crane is modeled as a mass of 1000 kg on a steel (E = 200 × 10^9 N/m^2) fixed-fixed beam with a moment of inertia of 4.2 × 10^{-3} m^4 and length of 12 m. The crane has an elastic rope of diameter 20 cm. At a specific instant, the length of the rope is 10 m and is carrying a 300 kg load. What are the two natural frequencies of the system?

FIGURE P6.14

Given: m_1 = 1000 kg, E = $200 \times 10^9 \ \frac{\text{N}}{\text{m}^2}$, I = 4.2×10^{-3} m^4, L = 12 m, r = 10 cm, $L_r = 10$ m, $m_2 = 300$ kg

Find: $\omega_1, \omega_2,$

Solution: The stiffness of the beam is

$$k_1 = \frac{192EI}{L^3} = \frac{192\left(200 \times 10^9 \ \frac{\text{N}}{\text{m}^2}\right)(4.2 \times 10^{-3} \ \text{m}^4)}{(12 \ \text{m})^3} = 9.33 \times 10^7 \ \frac{\text{N}}{\text{m}}$$

The stiffness of the cable is

$$k_2 = \frac{EA}{L_2} = \frac{\left(200 \times 10^9 \ \frac{\text{N}}{\text{m}^2}\right)\pi(0.1 \ \text{m})^2}{10 \ \text{m}} = 5.24 \times 10^8 \ \frac{\text{N}}{\text{m}}$$

The differential equations for a model of the system as two suspended masses is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1000 & 0 \\ 0 & 300 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^8\begin{bmatrix} 6.16 & -5.24 \\ -5.24 & 5.24 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the solutions of $|-\omega^2\mathbf{M} + \mathbf{K}| = 0$ which are

$$\omega_1 = 266.6 \frac{\text{rad}}{\text{s}} \qquad \omega_2 = 1513.7 \ \frac{\text{rad}}{\text{s}}$$

Problem 6.14 illustrates the modeling and natural frequency computation of a 2DOF system.

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6.15 A seismometer of mass 30 g and stiffness 40 N/m is used to measure the vibrations of a SDOF system of mass 60 g and natural frequency 150 rad/s. It is feared that the mass of the seismometer may affect the vibrations that are to be measured. Check this out by calculating the natural frequencies of the two degree-of-freedom with the seismometer attached.

Given: $m_1 = 60\text{ g}, \omega_{n1} = 150\frac{\text{rad}}{\text{s}}, m_1 = 30\text{ g}, k_2 = 40\text{ N/m}$

Find: $\omega_1, \omega_{2,}$

Solution: The stiffness of the original system is

$$k_1 = m_1\omega_{n1}^2 = (0.06\text{ kg})(150\text{ rad/s})^2 = 1350\text{ N/m}$$

The 2DOF model of the seismometer attached to the machine is that of a mass suspended by a spring suspended from a mass-spring system. The governing differential equations for the model are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0.06 & 0 \\ 0 & 0.03 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 1390 & -40 \\ -40 & 40 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the solutions of $|-\omega^2\mathbf{M} + \mathbf{K}| = 0$ which are

$$\omega_1 = 35.95\ \frac{\text{rad}}{\text{s}} \qquad \omega_2 = 152.3\ \frac{\text{rad}}{\text{s}}$$

Problem 6.15 illustrates the modeling and natural frequency computation of a 2DOF system.

6.16 Calculate the natural frequencies and modal fractions for the system of Figure P6.16.

FIGURE P6.16

Given: System shown

Find: $\omega_1, \omega_2, X_1, X_2$

Solution: FBD's of the system at an arbitrary instant are shown below

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Summing forces on the FBD's using $\sum F_{ext} = \sum F_{eff}$ leads to

$$3\ddot{x}_1 + 6000x_1 - 4000x_2 = 0$$

$$4\ddot{x}_2 - 4000x_1 + 4000x_2 = 0$$

Assuming a normal mode solution to the differential equations $x_1(t) = X_1 e^{i\omega t}$ and $x_2(t) = X_2 e^{i\omega t}$ yields

$$-3\omega^2 X_1 + 6000X_1 - 4000X_2 = 0$$

$$-4\omega^2 X_2 - 4000X_1 + 4000X_2 = 0$$

These equations have a non-trivial solution if and only if

$$\begin{vmatrix} -3\omega^2 + 6000 & -4000 \\ -4000 & -4\omega^2 + 4000 \end{vmatrix} = 0$$

Evaluation of the determinant leads to

$$(-3\omega^2 + 6000)(-4\omega^2 + 4000) - (-4000)(-4000) = 0$$

or

$$12\omega^4 - 38000\omega^2 + 1.8x10^6 = 0$$

The solutions are

$$\omega_1 = 15.54\frac{\text{rad}}{\text{s}} \qquad \omega_2 = 52.52 \text{ rad/s}$$

The modal fractions are determined from

$$\chi = \frac{-3\omega^2 + 6000}{4000}$$

which leads to

$$\mathbf{X_1} = \begin{bmatrix} 1 \\ 1.388 \end{bmatrix} \qquad \mathbf{X_2} = \begin{bmatrix} 1 \\ -0.569 \end{bmatrix}$$

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Problem 6.16 illustrates natural frequencies and mode shapes of a 2DOF system.

6.17 Determine the forced response to the system of Figure P6.1 and Chapter Problems 6.1 and 6.10 if the left-hand mass is given an initial displacement of 0.001 m while the right-hand mass is held in equilibrium and the system is released from rest.

FIGURE P6.1

Given: $\omega_1 = 100\frac{\text{rad}}{\text{s}}$, $\omega_2 = 158.11\frac{\text{rad}}{\text{s}}$, $\mathbf{X}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{X}_2 = \begin{bmatrix}1\\-0.5\end{bmatrix}$, $x_1(0) = 0.001$ m, $x_2(0) = 0, \dot{x}_1(0) = 0, \dot{x}_2(0) = 0$.

Find: $x_1(t), x_2(t)$

Solution: The general solution of the differential equations is

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = C_1\begin{bmatrix}1\\2\end{bmatrix}\cos 100t + C_2\begin{bmatrix}1\\2\end{bmatrix}\sin 100t + C_3\begin{bmatrix}1\\-0.5\end{bmatrix}\cos 158.11t + C_4\begin{bmatrix}1\\-0.5\end{bmatrix}\sin 158.11t \qquad (a)$$

Application of initial conditions to Eq.(a) leads to

$$C_1 + C_3 = 0.001 \qquad (b)$$

$$2C_1 - 0.5C_3 = 0 \qquad (c)$$

$$100C_2 + 151.1C_4 = 0 \qquad (d)$$

$$2(100)C_2 - 0.5(158.1)C_4 = 0 \qquad (e)$$

The solution of Eqs. (c) through (e) is $C_1 = 0.0002, C_2 = 0, C_3 = 0.0008, C_4 = 0$ which leads to

$$x_1(t) = 0.0002\cos 100t + 0.0008\cos 158.1t \qquad (f)$$

$$x_2(t) = 0.0004\cos 100t - 0.0004\cos 158.1t \qquad (g)$$

Problem 6.17 illustrates the free response of a 2DOF system.

6.18 Determine the response of the system of Figure P6.2 and Chapter Problems 6.2 and 6.11 if the particle is given an initial velocity of 2 m/s when the system is in equilibrium.

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Given: $\omega_1 = 16.69\,\frac{\text{rad}}{\text{s}}$, $\omega_2 = 36.69\,\frac{\text{rad}}{\text{s}}$,

$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.128 \end{bmatrix}, \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.2949 \end{bmatrix}$,

$\theta(0) = 0, x(0) = 0, \dot{\theta}(0) = 0, \dot{x}(0) = 2\ \text{m/s}$

Find: $\theta(t), x(t)$

FIGURE P6.2

Solution: The general solution of the differential equations is

$$\begin{bmatrix} \theta(t) \\ x(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1.128 \end{bmatrix} \cos 16.69t + C_2 \begin{bmatrix} 1 \\ 1.128 \end{bmatrix} \sin 16.69t + C_3 \begin{bmatrix} 1 \\ -0.2949 \end{bmatrix} \cos 36.69t$$

$$+ C_4 \begin{bmatrix} 1 \\ -0.2949 \end{bmatrix} \sin 36.69t \qquad (a)$$

Application of initial conditions to Eq.(a) leads to

$$C_1 + C_3 = 0 \qquad (b)$$

$$1.128C_1 - 0.2949C_3 = 0 \qquad (c)$$

$$16.69C_2 + 36.69C_4 = 0 \qquad (d)$$

$$1.128(16.69)C_2 - 0.2949(36.69)C_4 = 0 \qquad (e)$$

The solution of Eqs. (c) through (e) is $C_1 = 0, C_2 = 0.0842, C_3 = 0, C_4 = -0.0383$ which leads to

$$\theta(t) = 0.0842 \cos 16.69t - 0.0383 \cos 36.69t \qquad (f)$$

$$x(t) = 0.0950 \cos 16.69t + 0.0113 \cos 36.69t \qquad (g)$$

Problem 6.18 illustrates the free response of a 2DOF system.

6.19 Determine the response of the system of Figure P6.4 and Chapter Problems 6.4 and 6.13 if the right-hand disk is given an angular displacement of 2° clockwise from equilibrium and the left-hand disk is given an angular displacement of 2° counterclockwise.

FIGURE P6.4

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Given: $\omega_1 = 64.1\frac{\text{rad}}{\text{s}}$, $\omega_2 = 164.6\frac{\text{rad}}{\text{s}}$, $\mathbf{X}_1 = \begin{bmatrix} 1 \\ 2.106 \end{bmatrix}$, $\mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.358 \end{bmatrix}$,
$\theta_1(0) = 2°, \theta_2(0) = 0, \dot{\theta}_1(0) = -2°, \dot{\theta}_2(0) = 0.$

Find: $\theta_1(t), \theta_2(t)$

Solution: The general solution of the differential equations is

$$\begin{bmatrix} \theta_1(t) \\ \theta_1(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2.106 \end{bmatrix} \cos 64.1t + C_2 \begin{bmatrix} 1 \\ 2.106 \end{bmatrix} \sin 64.1t + C_3 \begin{bmatrix} 1 \\ -0.358 \end{bmatrix} \cos 164.6t$$

$$+ C_4 \begin{bmatrix} 1 \\ -0.358 \end{bmatrix} \sin 164.6t \qquad (a)$$

Application of initial conditions to Eq.(a) leads to

$$C_1 + C_3 = 2 \qquad (b)$$

$$2.106C_1 - 0.358C_3 = -2 \qquad (c)$$

$$64.1C_2 + 164.1C_4 = 0 \qquad (d)$$

$$2.106(64.1)C_2 - 0.358(164.6)C_4 = 0 \qquad (e)$$

The solution of Eqs. (c) through (e) is $C_1 = -0.5207, C_2 = 0, C_3 = 2.5207\ C_4 = 0$ which leads to

$$\theta_1(t) = -0.5207 \cos 64.1t + 2.5207 \cos 164.6t \qquad (f)$$

$$\theta_2(t) = -1.0976 \cos 64.1t - 0.0924 \cos 36.69t \qquad (g)$$

Problem 6.19 illustrates the free response of a 2DOF system.

6.20 Determine the response of the system of Chapter Problem 6.14 if the crane is disturbed resulting in an initial velocity of 10 m/s downward.

FIGURE P6.14

Given: $x_1(0) = 0, x_2(0) = 0, \dot{x}_1(0) = 0, \dot{x}_2(0) = 10$ m/s

Find: $x_1(t), x_2(t)$

Solution: The natural frequencies determined in the solution of Chapter Problem 6.14 are

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$$\omega_1 = 266.6\frac{\text{rad}}{\text{s}} \qquad \omega_2 = 1513.7\ \frac{\text{rad}}{\text{s}}$$

The modal fractions are determined from

$$\begin{bmatrix} -1000\omega^2 + 6.16\times 10^8 & -5.24\times 10^8 \\ -5.24\times 10^8 & -300\omega^2 + 5.24\times 10^8 \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the first equation gives

$$\chi = \frac{-1000\omega^2 + 6.16\times 10^8}{5.24\times 10^8}$$

leading to

$$\mathbf{X_1} = \begin{bmatrix} 1 \\ 1.0309 \end{bmatrix} \qquad \mathbf{X_2} = \begin{bmatrix} 1 \\ -3.1971 \end{bmatrix}$$

The general solution of the differential equations is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 1.0399 \end{bmatrix}\cos 266.6t + C_2\begin{bmatrix} 1 \\ 1.0309 \end{bmatrix}\sin 266.6t + C_3\begin{bmatrix} 1 \\ -3.171 \end{bmatrix}\cos 1513t + C_4\begin{bmatrix} 1 \\ -3.171 \end{bmatrix}\sin 1513t$$

Application of the initial conditions gives

$$x_1(0) = 0 \Longrightarrow C_1 + C_3 = 0$$

$$x_2(0) = 0 \Longrightarrow 1.0399C_1 - 3.171C_3 = 0$$

$$\dot{x}_1(0) = 0 \Longrightarrow 266.6C_2 + 1513C_4 = 0$$

$$\dot{x}_2(0) = 10 \Longrightarrow 266.6(1.0399)C_2 + 1513(-3.171)C_4 = 10$$

The solution of the equations is $C_1 = 0, C_2 = 8.91\times 10^{-3}, C_3 = 0, C_4 = -1.57\times 10^{-3}$ resulting in

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 8.91\times 10^{-3}\begin{bmatrix} 1 \\ 1.0309 \end{bmatrix}\sin 266.6t - 1.57\times 10^{-3}\begin{bmatrix} 1 \\ -3.171 \end{bmatrix}\sin 1513t$$

Problem 6.20 illustrates application of initial conditions to the response of a 2DOF system.

6.21 Determine the output from the seismometer of Chapter Problem 6.15 if the 60 g mass is given an initial velocity of 15 m/s. Use a two degree-of-freedom system, remembering that the seismometer records the relative displacement between the seismic mass and the body whose vibrations are to be measured.

Given: $x_1(0) = 0, x_2(0) = 0, \dot{x}_1(0) = 15, \dot{x}_2(0) = 0$

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Find: $x_2 - x_1$

Solution: The natural frequencies are determined in the solution of Chapter Problem 6.15 as $\omega_1 = 35.95\ \frac{\text{rad}}{\text{s}}$, $\omega_2 = 152.3\ \frac{\text{rad}}{\text{s}}$. The modal fractions are determined from the equations

$$\begin{bmatrix} -0.06\omega^2 + 1390 & -40 \\ -40 & -0.03\omega^2 + 40 \end{bmatrix} \begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (a)$$

which leads to

$$\chi = \frac{-0.06\omega^2 + 1390}{40} \qquad (b)$$

Equation (b) is evaluated for $\omega_1 = 35.95\ \frac{\text{rad}}{\text{s}}$, leading to $\chi_1 = 32.81$ and for $\omega_2 = 152.3\ \frac{\text{rad}}{\text{s}}$ leading to $\chi_2 = -0.0429$. Thus the mode shape vectors are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 32.81 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.0429 \end{bmatrix} \qquad (c)$$

The general solution of the differential equations is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 32.81 \end{bmatrix} \cos 35.95t + C_2 \begin{bmatrix} 1 \\ 32.81 \end{bmatrix} \sin 35.95t + C_3 \begin{bmatrix} 1 \\ -0.0429 \end{bmatrix} \cos 152.3t + C_4 \begin{bmatrix} 1 \\ -0.0429 \end{bmatrix} \sin 152.3t \qquad (d)$$

Application of the initial conditions to Eq. (d) leads to

$$C_1 + C_3 = 0 \qquad (e)$$

$$32.81C_1 - 0.0429C_3 = 0 \qquad (f)$$

$$35.95C_2 + 152.3C_4 = 15 \qquad (g)$$

$$32.81(35.95)C_2 - 0.0429(152.3)C_4 = 0 \qquad (h)$$

The solution of Eqs. (e) through (h) are $C_1 = 0, C_2 = 0.000545, C_3 = 0, C_4 = 0.0984$. The response of the seismometer is

$$x_2(t) - x_1(t) = 32.81(0.000545)\sin 35.95t - 0.0429(0.0984)\sin 152.3t - 0.000545\sin 35.95t - 0.0984\sin 152.3t$$

$$= 0.0173\sin 35.85t - 0.1026\sin 153.2t \qquad (i)$$

Problem 6.21 illustrates (1) the application of initial conditions to the solution for a 2DOF system and (2) the effect of a seismometer on measuring the response of a system.

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6.22 Determine the free response of the system of Figure P6.6 if the left-hand mass is given an initial displacement of 0.001 m while the right-hand mass is held in equilibrium and the system is released from rest. Use m = 1 kg, k = 10,000 N/m and c = 100 N · s/m.

FIGURE P6.6

Given: m = 1 kg, k = 10,000 N/m, c = 100 N · s/m, $x_1(0) = 0.001, x_2(0) = 0$, $\dot{x}_1(0) = 01$, $\dot{x}_2(0) = 0$

Find: $x_1(t), x_2(t)$

Solution: The differential equations derived in the solution to Chapter Problem 6.6 are

$$2m\ddot{x}_1 + 3c\dot{x}_1 - c\dot{x}_2 + 3kx_1 - kx_2 = 0$$

and

$$m\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - kx_1 + kx_2 = 0$$

Substituting given values into these equations leads to

$$2\ddot{x}_1 + 300\dot{x}_1 - 100\dot{x}_2 + 30000x_1 - 10000x_2 = 0$$

$$\ddot{x}_2 - 100\dot{x}_1 + 100\dot{x}_2 - 10000x_1 + 10000x_2 = 0$$

Assuming a solution of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \chi \end{bmatrix} e^{\lambda t}$$

and substituting into the equations leads to

$$2\lambda^2 + 300\lambda - 100\lambda\chi + 30000 - 10000\chi = 0$$

$$\chi\lambda^2 - 100\lambda + 100\lambda\chi - 10000 + 10000\chi = 0$$

A non-trivial solution of the above equations exists if and only if

$$\begin{vmatrix} 2\lambda^2 + 300\lambda + 30000 & -100\lambda - 10000 \\ -100\lambda - 10000 & \lambda^2 + 100\lambda + 10000 \end{vmatrix} = 0$$

The solutions for λ are

$$\lambda = -27.437 + 135.174i, -27.437 - 135.174i, -32.291, -162.762$$

The value of χ corresponding to each λ is given by

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$$\chi = \frac{2\lambda^2 + 300\lambda + 30000}{100\lambda + 10000}$$

Substituting calculated values of λ into the equation for χ and recognizing that the first component of the mode shape vectors is equal to 1 leads to

$$\lambda = -27.437 + 135.174i \qquad \mathbf{X}_1 = \begin{bmatrix} 1 \\ 0.4153 + 0.7035i \end{bmatrix}$$

$$\lambda = -27.437 - 135.174i \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ 0.4153 - 0.7035i \end{bmatrix}$$

$$\lambda = -32.291 \qquad \mathbf{X}_3 = \begin{bmatrix} 1 \\ 0.3542 \end{bmatrix}$$

$$\lambda = -162.762 \qquad \mathbf{X}_4 = \begin{bmatrix} 1 \\ -2.2552 \end{bmatrix}$$

The general solution of the differential equations is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-27.437t}\left(C_1 \begin{bmatrix} 1 \\ 0.4153 + 0.7035i \end{bmatrix} e^{i135.17t} + C_2 \begin{bmatrix} 1 \\ 0.4153 - 0.7035i \end{bmatrix} e^{-i135.17t}\right) + C_3 \begin{bmatrix} 1 \\ 0.3542 \end{bmatrix} e^{-32.291t} + C_4 \begin{bmatrix} 1 \\ -2.552 \end{bmatrix} e^{-162.762t}$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-27.437t}\left\{A_1\left(\begin{bmatrix} 1 \\ 0.4153 \end{bmatrix}\cos 135.17t - \begin{bmatrix} 0 \\ 0.7035 \end{bmatrix}\sin 135.17\right) + A_2\left(\begin{bmatrix} 1 \\ 0.4153 \end{bmatrix}\sin 135.17t + \begin{bmatrix} 0 \\ 0.7035 \end{bmatrix}\cos 135.17\right)\right\} + A_3 \begin{bmatrix} 1 \\ 0.3542 \end{bmatrix} e^{-32.291t} + A_4 \begin{bmatrix} 1 \\ -2.552 \end{bmatrix} e^{-162.762t}$$

Application of the initial conditions leads to

$$x_1(0) = 0.001 = A_1 + A_3 + A_4$$

$$x_2(0) = 0 = 0.4153A_1 + 0.7035A_2 + 0.3542A_3 - 2.2552A_4$$

$$\dot{x}_1(0) = 0 = -27.437A_1 + 135.17A_2 - 32.291A_3 - 162.762A_4$$

$$\dot{x}_2(0) = 0 = [(-27.437)(0.4153) - (135.17)(0.7035)]A_1 + [(135.17)(0.4153) + (-27.437)(0.7035)]A_2 - (32.291)(0.3542)A_3 - (162.762)(-2.552)A_4$$

Solution of these equations leads to $A_1 = 1.119 \times 10^{-3}, A_2 = 4.41 \times 10^{-4}$, $A_3 = -4.43 \times 10^{-4}, A_4 = 2.54 \times 10^{-4}$. The response of the system is

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$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 10^{-4}\left[e^{-27.437t}\left\{11.19\left(\begin{bmatrix} 1 \\ 0.4153 \end{bmatrix}\cos 135.17t - \begin{bmatrix} 0 \\ 0.7035 \end{bmatrix}\sin 135.17\right)\right.\right.$$
$$\left. + 4.41\left(\begin{bmatrix} 1 \\ 0.4153 \end{bmatrix}\sin 135.17t + \begin{bmatrix} 0 \\ 0.7035 \end{bmatrix}\cos 135.17\right)\right\}$$
$$\left. - 4.43\begin{bmatrix} 1 \\ 0.3542 \end{bmatrix}e^{-32.291t} + 2.54\begin{bmatrix} 1 \\ -2.552 \end{bmatrix}e^{-162.762t}\right]$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 10^{-4}\left[e^{-27.437t}\left\{\left(\begin{bmatrix} 11.19 \\ 7.7496 \end{bmatrix}\cos 135.17t + \begin{bmatrix} 4.41 \\ -6.0407 \end{bmatrix}\sin 135.17\right)\right\}\right.$$
$$\left. - \begin{bmatrix} 4.43 \\ 1.57 \end{bmatrix}e^{-32.291t} + A_4\begin{bmatrix} 2.54 \\ -6.48 \end{bmatrix}e^{-162.762t}\right]$$

Problem 6.22 illustrates the damped response of a 2DOF system.

6.23 Determine the response of the system of Figure P6.7 if the lower mass is given a displacement from equilibrium of 0.004 m and the upper mass is held in its equilibrium position and the system is released. Use m = 5 kg, k = 4000 N/m and c = 30 N · s/m.

FIGURE P6.7

Given: m = 5 kg, k = 4000 N/m, c = 30 N · s/m, $x_1(0) = 0, x_2(0) = 0.004, \dot{x}_1(0) = 0, \dot{x}_2(0) = 0$

Find: $x_1(t), x_2(t)$

Solution: The differential equations derived in the solution of Chapter Problem 6.7 are

$$m\ddot{x}_1 + 2c\dot{x}_1 - c\dot{x}_2 + kx_1 = 0$$

and

$$2m\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 = 0$$

Substituting given values into these equations leads to

$$5\ddot{x}_1 + 60\dot{x}_1 - 30\dot{x}_2 + 4000x_1 = 0$$

$$10\ddot{x}_2 - 30\dot{x}_1 + 30\dot{x}_2 = 0$$

Assuming a solution of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \chi \end{bmatrix}e^{\lambda t}$$

and substituting into the equations leads to

$$5\lambda^2 + 60\lambda - 30\lambda\chi + 4000 = 0$$

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$$10\chi\lambda^2 - 30\lambda + 30\lambda\chi = 0$$

A non-trivial solution of the above equations exists if and only if

$$\begin{vmatrix} 5\lambda^2 + 60\lambda + 4000 & -30\lambda \\ -30\lambda & \lambda^2 + 30\lambda \end{vmatrix} = 0$$

The solutions for λ are

$$\lambda = 0, -33.9547, -4.0227 + 26.2801i, -4.0227 - 26.2801i$$

The value of χ corresponding to each λ except 0 is given by

$$\chi = \frac{5\lambda^2 + 60\lambda + 4000}{30\lambda}$$

The mode shape vector for the first mode is

$$\mathbf{X}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Substituting calculated values of λ into the equation for χ and recognizing that the first component of the mode shape vectors is equal to 1 leads to

$$\lambda = -4.0227 + 26.28014i \qquad \mathbf{X}_1 = \begin{bmatrix} 1 \\ 0.5707 + 0.5774i \end{bmatrix}$$

$$\lambda = -4.0227 - 26.28014i \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ 0.5707 - 0.5774i \end{bmatrix}$$

$$\lambda = -33.29547 \qquad \mathbf{X}_4 = \begin{bmatrix} 1 \\ -7.5859 \end{bmatrix}$$

The general solution of the differential equations is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-4.0227t}\left(C_1 \begin{bmatrix} 1 \\ 0.5707 + 0.5774i \end{bmatrix} e^{i26.2801t} + C_2 \begin{bmatrix} 1 \\ 0.5707 - 0.5774i \end{bmatrix} e^{-i26.2801t}\right) + C_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_4 \begin{bmatrix} 1 \\ -7.5859 \end{bmatrix} e^{-33.9547t}$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-4.0227t}\left\{A_1\left(\begin{bmatrix} 1 \\ 0.5707 \end{bmatrix}\cos 26.2801t - \begin{bmatrix} 0 \\ 0.5774 \end{bmatrix}\sin 26.2801t\right) + A_2\left(\begin{bmatrix} 1 \\ 0.5707 \end{bmatrix}\sin 26.2801t + \begin{bmatrix} 0 \\ 0.5774 \end{bmatrix}\cos 26.2801t\right)\right\} + A_3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + A_4\begin{bmatrix} 1 \\ -7.5859 \end{bmatrix} e^{-33.954t}$$

Application of the initial conditions leads to

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$$x_1(0) = 0 = A_1 + A_4$$

$$x_2(0) = 0.004 = 0.5707A_1 + 0.5774A_2 + A_3 - 7.5859A_4$$

$$\dot{x}_1(0) = 0 = -4.0227A_1 + 26.2801A_2 - 33.954A_4$$

$$\begin{aligned}\dot{x}_2(0) = 0 = {} & [(-4.0227)(0.5707) - (26.2801)(0.5774)]A_1 \\ & + [(26.2801)(0.5707) + (-4.0227)(0.5774)]A_2 \\ & - (33.954)(-7.5859)A_4\end{aligned}$$

Solution of these equations leads to $A_1 = 3.785 \times 10^{-3}, A_2 = 8.57 \times 10^{-4}, A_3 = -1.027 \times 10^{-3}, A_4 = 2.154 \times 10^{-4}$. The response of the system is

$$\begin{aligned}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 10^{-4} \Bigg[& e^{-4.0227t} \left\{ 37.85 \left(\begin{bmatrix} 1 \\ 0.5707 \end{bmatrix} \cos 26.2801t - \begin{bmatrix} 0 \\ 0.5774 \end{bmatrix} \sin 26.2801t \right) \right. \\ & \left. + 8.57 \left(\begin{bmatrix} 1 \\ 0.5707 \end{bmatrix} \sin 26.2801t + \begin{bmatrix} 0 \\ 0.5774 \end{bmatrix} \cos 26.2801t \right) \right\} - 10.27 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & + 2.154 \begin{bmatrix} 1 \\ -7.5859 \end{bmatrix} e^{-33.954t} \Bigg]\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 10^{-4} \Bigg[& e^{-4.0227t} \left\{ \left(\begin{bmatrix} 37.85 \\ 16.65 \end{bmatrix} \cos 26.2801t + \begin{bmatrix} 8.57 \\ -16.96 \end{bmatrix} \sin 26.2801t \right) \right\} \\ & - 10.27 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -16.34 \end{bmatrix} e^{-33.954t} \Bigg]\end{aligned}$$

Problem 6.23 illustrates the damped response of a 2DOF system.

6.24 Determine the free response of the system of Figure P6.8 if the machine tool has initial velocities of $\dot{x}(0) = 0.8$ m/s and $\dot{\theta}(0) = 5$ rad/s if $I = 0.03$ kg · m^2, $c = 100$ N · m, $m = 3$ kg, $a = 0.3$ m, $b = 0.4$ m, and $k = 3000$ N/m.

FIGURE P6.8

Given: $\dot{x}(0) = 0.8$ m/s , $\dot{\theta}(0) = 5$ rad/s, $I = 0.03$ kg · m^2, $c = 100$ N · m, $m = 3$ kg, $a = 0.3$ m, $b = 0.4$ m, $k = 3000$ N/m.

Find: $x(t), \theta(t)$

Solution: The differential equations are derived in the solution of Chapter Problem 6.9 as

$$I\ddot{\theta} + mb\ddot{x} + ca(a+b)\dot{\theta} + c(a+b)\dot{x} + ka(a+b)\theta + k(a+b)x = 0 \qquad (c)$$

$$I\ddot{\theta} - ma\ddot{x} + cb(a+b)\dot{\theta} - c\dot{x}(a+b) + kb(a+b)\theta - k(a+b)x = 0$$

Substituting in the given values leads to

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$$\begin{bmatrix} 0.03 & 1.2 \\ 0.03 & -0.9 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 21 & 70 \\ 28 & -70 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 630 & 2100 \\ 840 & -2100 \end{bmatrix}\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free solution is obtained by assuming

$$\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ \chi \end{bmatrix} e^{\lambda t}$$

which when substituted into the governing equations leads to

$$\begin{bmatrix} 0.03\lambda^2 + 21\lambda + 630 & 1.2\lambda^2 + 70\lambda + 2100 \\ 0.03\lambda^2 + 28\lambda + 840 & -0.09\lambda^2 - 70\lambda - 2100 \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Setting the determinant to zero leads to

$$-0.063\lambda^4 - 56.7\lambda^3 - 5131\lambda^2 + 2.058 \times 10^5\lambda + 3.087 \times 10^6 = 0$$

whose roots are

$$\lambda = -803.17, -31.62, -32.26 \pm 29.43i$$

The values of χ for each value of λ are determined from

$$\chi = -\frac{0.03\lambda^2 + 21\lambda + 630}{1.2\lambda^2 + 70\lambda + 2100}$$

leading to

$$\lambda = -803.17 \qquad \chi = -0.004328$$

$$\lambda = -31.62 \qquad \chi = 0.003763$$

$$\lambda = -32.26 \pm 29.43i \qquad \chi = 3.6828 \pm 0.7223i$$

The general solution is

$$\begin{aligned}\begin{bmatrix} \theta \\ x \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -0.004328 \end{bmatrix} e^{-803.17t} + C_2 \begin{bmatrix} 1 \\ 0.003763 \end{bmatrix} e^{-31.62t} \\ + e^{-32.26t}\left[C_3\left(\begin{bmatrix} 1 \\ 3.6828 \end{bmatrix}\cos 29.43t - \begin{bmatrix} 0 \\ 0.7223 \end{bmatrix}\sin 29.43t\right)\right. \\ \left. + C_4\left(\begin{bmatrix} 1 \\ 3.6828 \end{bmatrix}\sin 29.43t + \begin{bmatrix} 0 \\ 0.7223 \end{bmatrix}\cos 29.43t\right)\right]\end{aligned}$$

Application of initial conditions leads to

$$\theta(0) = 0 \Longrightarrow C_1 + C_2 + C_3 = 0$$

$$x(0) = 0 \Longrightarrow -0.004328C_1 + 0.003763C_2 + 3.6828C_3 + 0.7223C_4 = 0$$

$$\dot{\theta}(0) = 5 \Longrightarrow -803.17C_1 - 31.62C_2 - 32.26C_3 + 29.43C_4 = 5$$

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$$\dot{x}(0) = 0.8 \Longrightarrow 3.4761C_1 - 0.1190C_2 - 140.43C_3 + 85.08C_4 = 0.8$$

The solutions are

$$C_1 = -6.203 \times 10^{-3} \quad C_2 = 7.638 \times 10^{-3} \quad C_1 = -1.435 \times 10^{-3}$$

$$C_1 = -7.237 \times 10^{-3}$$

which leads to

$$\begin{bmatrix}\theta\\ x\end{bmatrix} = 10^{-3}\Big(-6.203\begin{bmatrix}1\\-0.004328\end{bmatrix}e^{-803.17t} + 7.638\begin{bmatrix}1\\0.003763\end{bmatrix}e^{-31.62t}$$
$$+ e^{-32.26t}\Big[-1.435\Big(\begin{bmatrix}1\\3.6828\end{bmatrix}\cos 29.43t - \begin{bmatrix}0\\0.7223\end{bmatrix}\sin 29.43t\Big)$$
$$- 7.237\Big(\begin{bmatrix}1\\3.6828\end{bmatrix}\sin 29.43t + \begin{bmatrix}0\\0.7223\end{bmatrix}\cos 29.43t\Big)\Big]\Big)$$

$$= 10^{-3}\Big(\begin{bmatrix}-6.203\\0.0268\end{bmatrix}e^{-803.17t} + \begin{bmatrix}7.368\\0.0275\end{bmatrix}e^{-31.62t}$$
$$+ e^{-32.26t}\Big[\Big(\begin{bmatrix}-1.435\\-5.2848\end{bmatrix}\cos 29.43t - \begin{bmatrix}0\\-1.0365\end{bmatrix}\sin 29.43t\Big)$$
$$+ 7.237\Big(\begin{bmatrix}-7.237\\-26.652\end{bmatrix}\sin 29.43t + \begin{bmatrix}0\\-5.3865\end{bmatrix}\cos 29.43t\Big)\Big]\Big)$$

Problem 6.24 illustrates the solution for the free response of a 2DOF system.

6.25 Determine the principal coordinates for the system of Figure P6.1 and Chapter Problem 6.10. Write the differential equations which the principal coordinates satisfy.

FIGURE P6.1

Given: System shown

Find: $p_1(t), p_2(t)$

Solution: The natural frequencies and mode shape vectors are determined in the solution to Chapter Problem 6.10 as $\omega_1 = 100\frac{\text{rad}}{\text{s}}$, $\omega_2 = 158.11\frac{\text{rad}}{\text{s}}$, $\mathbf{X}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\mathbf{X}_2 = \begin{bmatrix}1\\-0.5\end{bmatrix}$. The principal coordinates are

$$p_1 = \frac{1}{\chi_2 - \chi_1}(\chi_2 x_1 - x_2) = \frac{1}{-0.5-2}(-0.5x_1 - x_2) = 0.2x_1 + 0.4x_2$$

$$p_2 = \frac{1}{\chi_2 - \chi_1}(x_2 - \chi_1 x_1) = \frac{1}{-0.5-2}(x_2 - 2x_1) = -0.4x_2 + 0.8x_1$$

The principal coordinates satisfy the differential equations

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$$\ddot{p}_1 + 10{,}000p_1 = 0$$

$$\ddot{p}_2 + 25{,}000p_2 = 0$$

Problem 6.25 illustrates the calculation of the principal coordinates for a 2DOF system.

6.26 Determine the principal coordinates for the system of Figure P6.2 and Chapter Problem 6.11. Write the differential equations which the principal coordinates satisfy.

FIGURE P6.2

Given: system shown

Find: $p_1(t), p_2(t)$

Solution: The natural frequencies and mode shape vectors are determined in the solution to Chapter Problem 6.11 as

$\omega_1 = 16.69\frac{\text{rad}}{\text{s}}, \omega_2 = 36.69\frac{\text{rad}}{\text{s}}, \mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.128 \end{bmatrix}, \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.2949 \end{bmatrix}$. The principal coordinates are

$$p_1 = \frac{1}{\chi_2 - \chi_1}(\chi_2 x_1 - x_2) = \frac{1}{-0.2949 - 1.128}(-0.2949x_1 - x_2)$$
$$= 0.2072x_1 + 0.7027x_2$$

$$p_2 = \frac{1}{\chi_2 - \chi_1}(x_2 - \chi_1 x_1) = \frac{1}{-0.2949 - 1.128}(x_2 - 1.128x_1)$$
$$= -0.7027\, x_2 + 1.605x_1$$

The principal coordinates satisfy the differential equations

$$\ddot{p}_1 + 278.55p_1 = 0$$

$$\ddot{p}_2 + 1346.15p_2 = 0$$

Problem 6.26 illustrates the calculation of the principal coordinates for a 2DOF system.

6.27 Determine the principal coordinates for the system of Figure P6.3 and Chapter Problem 6.12. Write the differential equations which the principal coordinates satisfy.

FIGURE P6.3

Given: system shown

Find: $p_1(t), p_2(t)$

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Solution: The natural frequencies and mode shape vectors are determined in the solution to Chapter Problem 6.12 as

$\omega_1 = 2.219\,\frac{\text{rad}}{\text{s}}, \omega_2 = 7.943\,\frac{\text{rad}}{\text{s}}, \mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.827 \end{bmatrix}, \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.2082 \end{bmatrix}$. The principal coordinates are

$$p_1 = \frac{1}{\chi_2 - \chi_1}(\chi_2 x_1 - x_2) = \frac{1}{-0.2082 - 1.827}(-0.2082x_1 - x_2)$$
$$= 0.1023x_1 + 0.4914x_2$$

$$p_2 = \frac{1}{\chi_2 - \chi_1}(x_2 - \chi_1 x_1) = \frac{1}{-0.2082 - 1.827}(x_2 - 1.827x_1)$$
$$= -0.4914\,x_2 + 0.8977x_1$$

The principal coordinates satisfy the differential equations

$$\ddot{p}_1 + 4.924p_1 = 0$$

$$\ddot{p}_2 + 63.09p_2 = 0$$

Problem 6.27 illustrates the calculation of the principal coordinates for a 2DOF system.

6.28 Determine the principal coordinates for the system of Figure P6.4 and Chapter Problem 6.13. Write the differential equations which the principal coordinates satisfy.

FIGURE P6.4

Given: system shown

Find: $p_1(t), p_2(t)$

Solution: The natural frequencies and mode shape vectors are determined in the solution to Chapter Problem 6.13 as $\omega_1 = 61.11\,\frac{\text{rad}}{\text{s}}$, $\omega_2 = 164.6\,\frac{\text{rad}}{\text{s}}, \mathbf{X}_1 = \begin{bmatrix} 1 \\ 2.106 \end{bmatrix}, \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.358 \end{bmatrix}$.

The principal coordinates are

$$p_1 = \frac{1}{\chi_2 - \chi_1}(\chi_2 x_1 - x_2) = \frac{1}{-0.358 - 2.106}(-0.358x_1 - x_2) = 0.1453x_1 + 0.4058x_2$$

$$p_2 = \frac{1}{\chi_2 - \chi_1}(x_2 - \chi_1 x_1) = \frac{1}{-0.358 - 2.106}(x_2 - 2.106x_1)$$
$$= -0.4058\,x_2 + 0.8456x_1$$

The principal coordinates satisfy the differential equations

$$\ddot{p}_1 + 3734.4p_1 = 0$$

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$$\ddot{p}_2 + 27093p_2 = 0$$

Problem 6.28 illustrates the calculation of the principal coordinates for a 2DOF system.

6.29 Determine the principal coordinates for the system of Figure P6.8 if it had no damping. Write the differential equations which the principal coordinates satisfy. Use $I = 0.03\ \text{kg}\cdot\text{m}^2$, $m = 3\ \text{kg}$, $a = 0.3\ \text{m}$, $b = 0.4\ \text{m}$, $k = 3000\ \text{N/m}$

FIGURE P6.8

Given: $I = 0.03\ \text{kg}\cdot\text{m}^2$, $m = 3\ \text{kg}$, $a = 0.3\ \text{m}$, $b = 0.4\ \text{m}$, $k = 3000\ \text{N/m}$

Find: $p_1(t), p_2(t)$

Solution: The differential equations obtained in the solution of Chapter Problem 6.8 assuming no damping are

$$\left(\frac{I + mb^2}{a+b}\right)\ddot{x}_1 + \left(\frac{mab - I}{a+b}\right)\ddot{x}_2 + k(a+b)x_1 = 0$$

$$\left(\frac{I - mab}{a+b}\right)\ddot{x}_1 - \left(\frac{I + ma^2}{a+b}\right)\ddot{x}_2 + k(a+b)x_2 = 0$$

Substituting given values and writing the equations in matrix form leads to

$$\begin{bmatrix} 0.0729 & 0.0471 \\ -0.0471 & -0.0428 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2100 & 0 \\ 0 & -2100 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequencies are calculated from

$$\begin{vmatrix} -0.729\omega^2 + 2100 & -0.471\omega^2 \\ 0.471\omega^2 & 0.428\omega^2 - 2100 \end{vmatrix} = -0.08984\omega^4 + 2430\omega^2 - 4.41 \times 10^6$$

and are equal to

$$\omega_1 = 44.23\ \frac{\text{rad}}{\text{s}} \qquad \omega_1 = 158.5\ \frac{\text{rad}}{\text{s}}$$

The modal fractions are obtained by

$$\begin{bmatrix} -0.729\omega^2 + 2100 & -0.471\omega^2 \\ 0.471\omega^2 & 0.428\omega^2 - 2100 \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix}$$

which leads to

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$$\chi = -\frac{-0.729\omega^2 + 2100}{0.471\omega^2}$$

and

$$\chi_1 = -0.5888 \qquad \chi_2 = -1.3701$$

The principal coordinates satisfy

$$p_1 = \frac{1}{\chi_2 - \chi_1}(\chi_2 x_1 - x_2) = \frac{1}{-0.588 + 1.370}(-1.370x_1 - x_2) = -1.882x_1 - 1.376x_2$$

$$p_2 = \frac{1}{\chi_2 - \chi_1}(x_2 - \chi_1 x_1) = \frac{1}{-0.5888 + 1.3701}(x_2 + 0.5888x_1)$$
$$= 1.376\,x_2 + 0.7529x_1$$

The principal coordinates satisfy the differential equations

$$\ddot{p}_1 + 1956p_1 = 0$$

$$\ddot{p}_2 + 25092p_2 = 0$$

Problem 6.29 illustrates principal coordinates for a 2DOF system.

6.30 Determine the principal coordinates for the system of Chapter Problem 6.9. Write the differential equations which the principal coordinates satisfy if $I = 0.03$ kg · m^2, $c = 0$ N · s/m, $m = 3$ kg, $a = 0.3$ m, $b = 0.4$ m, and $k = 3000$ N/m.

FIGURE P6.8

Given: $I = 0.03\ \text{kg}\cdot\text{m}^2, m = 3\ \text{kg}, a = 0.3\ \text{m},\ b = 0.4\ \text{m}, k = 3000\ \text{N/m}$

Find: $p_1(t), p_2(t)$

Solution: The differential equations obtained in the solution of Chapter Problem 6.9 assuming no damping are

$$\begin{bmatrix} I & mb \\ -ma & I \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} ka(a+b) & k(a+b) \\ kb(a+b) & k(a+b) \end{bmatrix}\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Substitution of given values leads to

$$\begin{bmatrix} 0.03 & 1.2 \\ -0.9 & 0.03 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 630 & 2100 \\ 840 & 2100 \end{bmatrix}\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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The natural frequencies and mode shape vectors are determined in the solution to Chapter Problem 6.14 as $\omega_1 = 266.6\frac{\text{rad}}{\text{s}}$, $\omega_2 = 1513.7\ \frac{\text{rad}}{\text{s}}$. The modal fractions are determined from

$$\begin{bmatrix} -1000\omega^2 + 6.16\times 10^8 & -5.24\times 10^8 \\ -5.24\times 10^8 0 & -300\omega^2 + 5.24\times 10^8 \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which leads to

$$\chi = \frac{-1000\omega^2 + 6.16\times 10^8}{-5.24\times 10^8}$$

For $\omega_1 = 266.6\frac{\text{rad}}{\text{s}}$, $\chi_1 = 1.04$. For $\omega_2 = 1513.7\ \frac{\text{rad}}{\text{s}}$, $\chi_1 = -3.20$. The principal coordinates are

$$p_1 = \frac{1}{\chi_2 - \chi_1}(\chi_2 x_1 - x_2) = \frac{1}{-3.20 - 1.04}(-3.20x_1 - x_2) = 0.928x_1 + 0.290x_2$$

$$p_2 = \frac{1}{\chi_2 - \chi_1}(x_2 - \chi_1 x_1) = \frac{1}{-3.20 - 1.04}(x_2 - 1.04x_1) = -0.290\ x_2 + 0.301x_1$$

The principal coordinates satisfy the differential equations

$$\ddot{p}_1 + 71076p_1 = 0$$

$$\ddot{p}_2 + 2.913x10^6 p_2 = 0$$

Problem 6.30 illustrates the calculation of the principal coordinates for a 2DOF system.

6.31 Determine the response of the system of Figure P6.1 and Chapter Problem 6.10 due to a sinusoidal force $200\sin 110t$ N applied to the block whose displacement is x_1 using the method of undetermined coefficients.

FIGURE P6.1

Given: $F_0 = 200$ N, $\omega = 110$ rad/s

Find: $x_1(t), x_2(t)$

Solution: The differential equations governing the system are derived in the solution of Chapter Problem 6.1 and modified to take the applied force into account. They are

$$\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^5\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200\sin 110\ t \\ 0 \end{bmatrix} \qquad (a)$$

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The method of undetermined coefficients is used and a solution assumed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin 110t \qquad (b)$$

Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} -(110)^2(10) + 2 \times 10^5 & -1 \times 10^5 \\ -1 \times 10^5 & -(110)^2(20) + 3 \times 10^5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 200 \\ 0 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 10^{-3} \begin{bmatrix} -1.16 \\ -2.00 \end{bmatrix} \qquad (d)$$

Problem 6.31 illustrates application of the method of undetermined coefficients applied to determine the steady-state response of a 2DOF system.

6.32 Determine the response of the system of Figure P6.1 and Chapter Problem 6.10 due to a sinusoidal force $200 \sin 80t$ applied to the block whose displacement is x_2 using the Laplace transform method.

FIGURE P6.1

Given: $F(t) = 200 \sin 80t$

Find: Response of system using Laplace transforms

Solution: The differential equations governing the system are derived in the solution of Chapter Problem 6.1 and modified to take the applied force into account. They are

$$\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^5 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 200 \sin 80t \end{bmatrix} \qquad (a)$$

Taking the Laplace transform of Eq. (a) assuming all initial conditions are zero leads to

$$\begin{bmatrix} 10s^2 + 200000 & -100000 \\ -100000 & 20s^2 + 300000 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{(200)80}{s^2 + 64000} \end{bmatrix} \qquad (b)$$

where $X_i(s) = \mathcal{L}\{x_i(t)\}$. Solving Eq. (b) leads to

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{(200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9)(s^2 + 64000)} \begin{bmatrix} 1.6 \times 10^9 \\ 1.6 \times 10^5 s^2 + 3.2 \times 10^9 \end{bmatrix} \qquad (c)$$

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Partial fraction decompositions yield $X_1(s)$ and $X_2(s)$ which can be inverted to determine $x_1(t)$ and $x_2(t)$.

Problem 6.32 illustrates the use of the Laplace transform method to a two-degree-of-freedom system.

6.33 Determine the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a sinusoidal force $100 \sin 70t$ N applied to the particle using the method of undetermined coefficients.

FIGURE P6.2

Given: $F(t) = 100 \sin 70t$

Find: $x(t), \theta(t)$

Solution: The differential equations are derived in the solution of Chapter Problem 6.2 and put in a matrix form in the solution of Chapter Problem 6.11. They are modified to take the applied force into account as

$$\begin{bmatrix} 0.667 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 750 & -500 \\ -500 & 1000 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 100 \sin 70t \\ 0 \end{bmatrix} \qquad (a)$$

A steady-state solution is assumed as

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} X \\ \Theta \end{bmatrix} \sin 70t \qquad (b)$$

Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} -0.667(70)^2 + 750 & -500 \\ -500 & -2(70)^2 + 1000 \end{bmatrix} \begin{bmatrix} X \\ \Theta \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \qquad (c)$$

Equation (c) is solved leading to $X = -0.0402, \Theta = 0.0023$.

Problem 6.33 illustrates the use of the method of undetermined coefficients on 2DOF systems.

6.34 Determine the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a sinusoidal moment $50 \sin 90t$ N· m applied to the bar using the method of undetermined coefficients.

FIGURE P6.2

Given: $M(t) = 50 \sin 90t$

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Find: $x(t), \theta(t)$

Solution: The differential equations are derived in the solution of Chapter Problem 6.2 and put in a matrix form in the solution of Chapter Problem 6.11. They are modified to take the applied force into account as

$$\begin{bmatrix} 0.667 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 750 & -500 \\ -500 & 1000 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \sin 90t \end{bmatrix} \qquad (a)$$

A steady-state solution is assumed as

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} X \\ \Theta \end{bmatrix} \sin 90t \qquad (b)$$

Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} -0.667(90)^2 + 750 & -500 \\ -500 & -2(90)^2 + 1000 \end{bmatrix} \begin{bmatrix} X \\ \Theta \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \end{bmatrix} \qquad (c)$$

Equation (c) is solved leading to $X = 3.55 \times 10^{-4}, \Theta = -3.30 \times 10^{-3}$.

Problem 6.34 illustrates the use of the method of undetermined coefficients on 2DOF systems.

6.35 Determine the response of the system of Figure P6.1 and Chapter Problem 6.10 due to a (a) a unit impulse applied to the block whose displacement is x_1, and (b) a unit impulse applied to the block whose displacement is x_2.

FIGURE P6.1

Given: $F(t) = \delta(t)$ applied first to block 1 and then to block 2

Find: $x_1(t), x_2(t)$

Solution: The differential equations governing the motion of the system are derived in the solution of Chapter Problem 6.1 as

$$\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^5 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix} \qquad (a)$$

Taking the Laplace transform of the equations assuming all initial conditions are zero leads to

$$\begin{bmatrix} 10s^2 + 200000 & -100000 \\ -100000 & 20s^2 + 300000 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix} \qquad (b)$$

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where $X_i(s) = \mathcal{L}\{x_i(t)\}$ and $F_i(s) = \mathcal{L}\{F_i(t)\}$ Solving Eq. (b) leads to

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9} \begin{bmatrix} 1 \times 10^5 F_2(s) + (20s^2 + 300000)F_1(s) \\ 1 \times 10^5 F_1(s) + (10s^2 + 200000)F_2(s) \end{bmatrix} \qquad (c)$$

(a) Setting $F_2(s) = 0$ leads to the transfer functions

$$G_{1,1}(s) = \frac{20s^2 + 300000}{200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9} \qquad (d)$$

$$G_{2,1}(s) = \frac{1 \times 10^5}{200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9} \qquad (e)$$

The response due to a unit impulse applied to the particle whose displacement is x_1 is the inverse of the transfer functions $G_{1,1}(s)$ and $G_{2,1}(s)$. Partial fraction decompositions yield

$$G_{1,1}(s) = \frac{0.03811}{s^2 + 23596} + \frac{0.0619}{s^2 + 1043}$$

$$G_{2,1}(s) = -\frac{0.0011275}{s^2 + 23596} + \frac{0.0011275}{s^2 + 1043}$$

The impulsive responses are

$$x_1(t) = 0.248 \sin 153.61t + 1.917 \sin 32.296t \text{ mm}$$

$$x_2(t) = -7.314 \sin 153.61t + 34.9 \sin 32.296t \ \mu\text{m}$$

(b) Setting $F_1(t) = 0$ in the transfer functions leads to

$$G_{1,2}(s) = \frac{1 \times 10^5}{200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9}$$

$$G_{2,2}(s) = \frac{10s^2 + 200000}{200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9}$$

A partial fraction decomposition yields

$$G_{1,1}(s) = \frac{0.0079}{s^2 + 23596} + \frac{0.0402}{s^2 + 1043}$$

Inverting the transfer functions leads to the impulsive responses

$$x_1(t) = -7.314 \sin 153.61t + 34.9 \sin 32.296t \ \mu\text{m}$$

$$x_2(t) = 7.9 \sin 153.61t + 40.2 \sin 32.296t \text{ mm}$$

Problem 6.35 illustrates the transfer functions for a 2DOF system and the system's impulsive responses.

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6.36 Determine the response of the system of Figure P6.1 and Chapter Problem 6.10 due to the force of Figure P6.36 applied to the block whose displacement is x_1.

FIGURE P6.36

Given: force shown applied to block whose displacement is x_1

Find: $x_1(t), x_2(t)$

Solution: The differential equations governing the motion of the system are derived in the solution of Chapter Problem 6.1 as

$$\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^5 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F(t) \\ 0 \end{bmatrix} \qquad (a)$$

Taking the Laplace transform of the equations assuming all initial conditions are zero leads to

$$\begin{bmatrix} 10s^2 + 200000 & -100000 \\ -100000 & 20s^2 + 300000 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F_1(s) \\ 0 \end{bmatrix} \qquad (b)$$

where $X_i(s) = \mathcal{L}\{x_i(t)\}$ and $\mathrm{F}(s) = \mathcal{L}\{F(t)\}$ Solving Eq. (b) leads to

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9} \begin{bmatrix} (20s^2 + 300000)F(s) \\ 1 \times 10^5 F(s) \end{bmatrix} \qquad (c)$$

This leads to the transfer functions

$$G_{1,1}(s) = \frac{20s^2 + 300000}{200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9} \qquad (d)$$

$$G_{2,1}(s) = \frac{1 \times 10^5}{200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9} \qquad (e)$$

The force is

$$F(t) = 100[u(t) - u(t - 0.01)] + (200 - 10000t)[u(t - 0.01) - u(t - 0.02)]$$
$$= 100u(t) - 10000(t - 0.01)u(t - 0.01) + 100000(t - 0.02)u(t - 0.02)$$

The Laplace transform is obtained by the second shifting theorem

$$F(s) = \frac{100}{s} - \frac{10000}{s^2} e^{-0.01s} + \frac{10000}{s^2} e^{-0.02s}$$

The Laplace transforms of the responses are obtained by multiplying the transfer functions by F(s)

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$$X_1(s) = F(s)G_{1,1}(s)$$
$$= \frac{100(20s^2 + 300000)}{s(200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9)} + \frac{10000(20s^2 + 300000)}{s^2(200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9)}(-e^{-0.01s} + e^{-0.02s})$$

$$= \frac{-1.0482}{s} - \frac{0.1386s}{s^2 + (153.61)^2} + \frac{1.1868s}{s^2 + (32.73)^2} + 100\left(\frac{-1.0482}{s^2} - \frac{0.1386}{s^2 + (153.61)^2} + \frac{1.1868}{s^2 + (32.73)^2}\right)(-e^{-0.01s} + e^{-0.02s})$$

$$X_2(s) = F(s)G_{2,1}(s)$$
$$= \frac{1 \times 10^7}{s(200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9)} + \frac{1 \times 10^9}{s^2(200s^4 + 5 \times 10^6 s^2 + 5 \times 10^9)}(-e^{-0.01s} + e^{-0.02s})$$

$$= \frac{-0.377}{s} - \frac{0.0180s}{s^2 + (153.61)^2} + \frac{0.3956s}{s^2 + (32.73)^2} + 100\left(\frac{-0.377}{s^2} - \frac{0.0180}{s^2 + (153.61)^2} + \frac{0.3956}{s^2 + (32.73)^2}\right)(-e^{-0.01s} + e^{-0.02s})$$

Inverting the transforms using the second shifting theorem yields

$$\begin{aligned} x_1(t) = {} & -1.082 - 0.1386\cos 153.61t + 1.1868\cos 32.73t \\ & - 100[-1.0482(t - 0.01) - 9.02 \times 10^{-4}\sin 153.61(t - 0.01) \\ & + 0.0110\sin 32.73(t - 0.01)]u(t - 0.01) \\ & + 100[-1.0482(t - 0.02) - 9.02 \times 10^{-4}\sin 153.61(t - 0.02) \\ & + 0.0110\sin 32.73(t - 0.02)]u(t - 0.02) \end{aligned}$$

$$\begin{aligned} x_2(t) = {} & -0.377 - 0.0180\cos 153.61t + 0.3956\cos 32.73t \\ & - 100[-0.377(t - 0.01) - 1.149 \times 10^{-4}\sin 153.61(t - 0.01) \\ & + 0.0121\sin 32.73(t - 0.01)]u(t - 0.01) \\ & + 100[-0.377(t - 0.02) - 1.149 \times 10^{-4}\sin 153.61(t - 0.02) \\ & + 0.0121\sin 32.73(t - 0.02)]u(t - 0.02) \end{aligned}$$

Problem 6.36 illustrates the use of transfer functions to determine the transient response of a 2DOF system.

6.37 Determine the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a unit impulse applied to the particle.

Given: system shown

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Find: $x(t), \theta(t)$

FIGURE P6.2

Solution: The differential equations are derived in the solution of Chapter Problem 6.2 as

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{3}{4}kL^2\theta - k\frac{L}{2}x = 0$$

$$m\ddot{x} - k\frac{L}{2}\theta + kx = \delta(t)$$

Substituting the given values into the differential equations leads to

$$\frac{2}{3}\ddot{\theta} + 750\theta - 500x = 0$$

$$2\ddot{x} - 500\theta + 1000x = \delta(t)$$

The transfer functions are the Laplace transforms of the response due to a unit impulse applied to the system. Solving for the correct transfer functions leads to

$$\begin{bmatrix} \frac{2}{3}s^2 + 750 & -500 \\ -500 & 2s^2 + 1000 \end{bmatrix} \begin{bmatrix} G_{12}(s) \\ G_{22}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving for the transfer functions

$$G_{12}(s) = \frac{\begin{vmatrix} 0 & -500 \\ 1 & 2s^2 + 1000 \end{vmatrix}}{\begin{vmatrix} \frac{2}{3}s^2 + 750 & -500 \\ -500 & 2s^2 + 1000 \end{vmatrix}} = \frac{500}{1.67s^4 + 2167s^2 + 500000}$$

$$= \frac{500}{[s^2 + (16.68)^2][s^2 + (36.698)^2]}$$

$$G_{22}(s) = \frac{\begin{vmatrix} \frac{2}{3}s^2 + 750 & 0 \\ -500 & 1 \end{vmatrix}}{\begin{vmatrix} \frac{2}{3}s^2 + 750 & -500 \\ -500 & 2s^2 + 1000 \end{vmatrix}} = \frac{0.667s^2 + 750}{1.33s^4 + 2167s^2 + 500000}$$

$$= \frac{0.667s^2 + 750}{[s^2 + (16.68)^2][s^2 + (36.698)^2]}$$

Partial fraction decompositions lead to

$$G_{12}(s) = \frac{0.4682}{s^2 + (16.68)^2} - \frac{0.4682}{s^2 + (36.69)^2}$$

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$$G_{22}(s) = \frac{0.5285}{s^2 + (16.68)^2} + \frac{0.1385}{s^2 + (36.69)^2}$$

Taking the inverses of the Laplace transforms

$$\theta(t) = 0.0128 \sin 16.68t - 0.0281 \sin 36.69t$$

$$x(t) = 0.0317 \sin 16.68t + 0.00377 \sin 36.69t$$

Problem 6.37 illustrates the use of transfer functions to determine the transient response of a 2DOF system.

6.38 Determine the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a unit impulsive moment applied to the bar.

FIGURE P6.2

Given: system shown

Find: $x(t), \theta(t)$

Solution: The differential equations are derived in the solution of Chapter Problem 6.2 as

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{3}{4}kL^2\theta - k\frac{L}{2}x = \delta(t)$$

$$m\ddot{x} - k\frac{L}{2}\theta + kx = 0$$

Substituting the given values into the differential equations leads to

$$\frac{2}{3}\ddot{\theta} + 750\theta - 500x = \delta(t)$$

$$2\ddot{x} - 500\theta + 1000x = 0$$

The transfer functions are the Laplace transforms of the response due to a unit impulse applied to the system. Solving for the correct transfer functions leads to

$$\begin{bmatrix} \frac{2}{3}s^2 + 750 & -500 \\ -500 & 2s^2 + 1000 \end{bmatrix} \begin{bmatrix} G_{11}(s) \\ G_{21}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving for the transfer functions

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$$G_{11}(s) = \frac{\begin{vmatrix} 1 & -500 \\ 0 & 2s^2+1000 \end{vmatrix}}{\begin{vmatrix} \frac{2}{3}s^2+750 & -500 \\ -500 & 2s^2+1000 \end{vmatrix}} = \frac{2s^2+100}{1.67s^4+2167s^2+500000}$$

$$= \frac{2s^2+100}{[s^2+(16.68)^2][s^2+(36.698)^2]}$$

$$G_{21}(s) = \frac{\begin{vmatrix} \frac{2}{3}s^2+750 & 1 \\ -500 & 0 \end{vmatrix}}{\begin{vmatrix} \frac{2}{3}s^2+750 & -500 \\ -500 & 2s^2+1000 \end{vmatrix}} = \frac{500}{1.33s^4+2167s^2+500000}$$

$$= \frac{500}{[s^2+(16.68)^2][s^2+(36.698)^2]}$$

Partial fraction decompositions lead to

$$G_{11}(s) = \frac{-0.4274}{s^2+(16.68)^2} + \frac{2.4274}{s^2+(36.69)^2}$$

$$G_{21}(s) = \frac{0.4682}{s^2+(16.68)^2} - \frac{0.4682}{s^2+(36.69)^2}$$

Taking the inverses of the Laplace transforms

$$\theta(t) = 0.0256\sin 16.68t + 0.0662\sin 36.69t$$

$$x(t) = 0.0128\sin 16.68t - 0.0281\sin 36.69t$$

Problem 6.38 illustrates the use of transfer functions to determine the transient response of a 2DOF system.

6.39 Derive the response of Figure P6.2 and Chapter Problem 6.11 due to the force of Figure P6.39 applied downward to the end of the bar.

FIGURE P6.39

Given : F(t) as shown

Find: $x(t), \theta(t)$

Solution: The differential equations are derived in the solution of Chapter Problem 6.2 as

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{3}{4}kL^2\theta - k\frac{L}{2}x = LF(t)$$

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$$m\ddot{x} - k\frac{L}{2}\theta + kx = 0$$

Substituting the given values into the differential equations leads to

$$\frac{2}{3}\ddot{\theta} + 750\theta - 500x = LF(t)$$

$$2\ddot{x} - 500\theta + 1000x = 0$$

The transfer functions are the Laplace transforms of the response due to a unit impulse applied to the system. Solving for the correct transfer functions leads to

$$\begin{bmatrix} \frac{2}{3}s^2 + 750 & -500 \\ -500 & 2s^2 + 1000 \end{bmatrix} \begin{bmatrix} G_{11}(s) \\ G_{21}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving for the transfer functions

$$G_{11}(s) = \frac{\begin{vmatrix} 1 & -500 \\ 0 & 2s^2 + 1000 \end{vmatrix}}{\begin{vmatrix} \frac{2}{3}s^2 + 750 & -500 \\ -500 & 2s^2 + 1000 \end{vmatrix}} = \frac{2s^2 + 100}{1.67s^4 + 2167s^2 + 500000}$$
$$= \frac{2s^2 + 100}{[s^2 + (16.68)^2][s^2 + (36.698)^2]}$$

$$G_{21}(s) = \frac{\begin{vmatrix} \frac{2}{3}s^2 + 750 & 1 \\ -500 & 0 \end{vmatrix}}{\begin{vmatrix} \frac{2}{3}s^2 + 750 & -500 \\ -500 & 2s^2 + 1000 \end{vmatrix}} = \frac{500}{1.33s^4 + 2167s^2 + 500000}$$
$$= \frac{500}{[s^2 + (16.68)^2][s^2 + (36.698)^2]}$$

The moment is given by

$$M(t) = LF(t) = (1\ kg)\{200[u(t) - u(t - 0.3)] - 100[u(t - 0.3) - u(t - 0.5)]\ N\}$$
$$= 200u(t) - 300u(t - 0.3) + 100u(t - 0.5)$$

Taking the Laplace transform of this function yields

$$M(s) = \frac{100}{s}(2 - 3e^{-0.3s} + e^{-0.5s})$$

The Laplace transforms of the responses are

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$$\Theta(s) = M(s)G_{11}(s) = \frac{100(2s^2 + 100)(2 - 3e^{-0.3s} + e^{-0.5s})}{s[s^2 + (16.68)^2][s^2 + (36.698)^2]}$$

$$X(s) = M(s)G_{21}(s) = \frac{100(500)(2 - 3e^{-0.3s} + e^{-0.5s})}{s[s^2 + (16.68)^2][s^2 + (36.698)^2]}$$

Partial fraction decomposition leads to

$$\Theta(s) = 0.1\left(\frac{8.213}{s} - \frac{8.480s}{s^2 + (16.68)^2} + \frac{0.267s}{s^2 + (36.69)^2}\right)(2 - 3e^{-0.3s} + e^{-0.5s})$$

$$\mathrm{X}(s) = 0.1\left(\frac{5.134}{s} - \frac{6.459s}{s^2 + (16.68)^2} + \frac{1.335s}{s^2 + (36.69)^2}\right)(2 - 3e^{-0.3s} + e^{-0.5s})$$

Inverting the transforms leads to

$$\begin{aligned}\theta(t) = {} & 0.2(8.213 - 8.840\cos 16.68t + 0.267\cos 36.69t) \\ & - 0.3[8.213 - 8.840\cos 16.68(t - 0.3) + 0.267\cos 36.69(t - 0.3)]u(t - 0.3) \\ & + 0.1[8.213 - 8.840\cos 16.68(t - 0.5) + 0.267\cos 36.69(t - 0.5)]u(t - 0.5)\end{aligned}$$

$$\begin{aligned}x(t) = {} & 0.2(5.134 - 6.459\cos 16.68t + 1.335\cos 36.69t) \\ & - 0.3[5.134 - 6.459\cos 16.68(t - 0.3) + 1.335\cos 36.69(t - 0.3)]u(t - 0.3) \\ & + 0.1[5.134 - 6.459\cos 16.68(t - 0.5) + 1.335\cos 36.69(t - 0.5)]u(t - 0.5)\end{aligned}$$

Problem 6.39 illustrates the use of transfer functions to determine the transient response of a 2DOF system.

6.40 Derive the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a moment $M(t) = 10e^{-2t}$ N · m applied to the bar.

FIGURE P6.2

Given: $M(t) = 10e^{-2t}$ N· m, m = 2 kg, L = 1m, k = 1000 N/m

Find: $x(t), \theta(t)$

Solution: The differential equations are derived in the solution of Chapter Problem 6.2 as

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{3}{4}kL^2\theta - k\frac{L}{2}x = 10e^{-2t}$$

$$m\ddot{x} - k\frac{L}{2}\theta + kx = 0$$

Substituting the given values into the differential equations leads to

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$$\frac{2}{3}\ddot{\theta} + 750\theta - 500x = 10e^{-2t}$$

$$2\ddot{x} - 500\theta + 1000x = 0$$

The homogeneous solution is determined in Chapter Problem 6.11 as

$$\begin{bmatrix}\theta(t)\\ x(t)\end{bmatrix} = \begin{bmatrix}1\\ 1.128\end{bmatrix}(C_1\cos 16.69t + C_2\sin 16.69t) + \begin{bmatrix}1\\ -0.2949\end{bmatrix}(C_3\cos 36.69t + C_4\sin 36.69t)$$

The particular solution is assumed as

$$\begin{bmatrix}\theta(t)\\ x(t)\end{bmatrix} = \begin{bmatrix}A\\ B\end{bmatrix}e^{-2t}$$

Substitution into the differential equations leads to

$$-\frac{8}{3}A + 750A - 500B = 10$$

$$8B - 500A + 1000B = 0$$

Solution of the simultaneous equations leads to

$$\begin{bmatrix}\theta(t)\\ x(t)\end{bmatrix} = \begin{bmatrix}0.0202\\ 0.0102\end{bmatrix}e^{-2t}$$

The general solution is

$$\begin{bmatrix}\theta(t)\\ x(t)\end{bmatrix} = \begin{bmatrix}1\\ 1.128\end{bmatrix}(C_1\cos 16.69t + C_2\sin 16.69t) + \begin{bmatrix}1\\ -0.2949\end{bmatrix}(C_3\cos 36.69t + C_4\sin 36.69t) + \begin{bmatrix}0.0202\\ 0.0102\end{bmatrix}e^{-2t}$$

Assuming all initial conditions are zero leads to

$$\theta(0) = 0 = C_1 + C_3 + 0.0202$$

$$x(0) = 0 = 1.128C_1 - 0.2949C_3 + 0.0102$$

$$\dot{\theta}(0) = 0 = 16.69C_2 + 36.69C_4 - 0.0404$$

$$\dot{x}(0) = 0 = 1.128(16.69)C_2 - 0.2949(36.69)C_4 - 0.0204$$

Solving for the constants of integration and substitution into the solution leads to

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$$\begin{bmatrix}\theta(t)\\ x(t)\end{bmatrix} = 10^{-3}\left\{\begin{bmatrix}1\\ 1.128\end{bmatrix}(-11.355\cos 16.69t + 1.361\sin 16.69t) + \begin{bmatrix}1\\ -0.2949\end{bmatrix}(-8.842\cos 36.69t + 0.482\sin 36.69t)\right\} + \begin{bmatrix}0.0202\\ 0.0102\end{bmatrix}e^{-2t}$$

Problem 6.40 illustrates the particular solution of differential equations for 2DOF systems.

6.41 Determine the response of the system of Figure P6.6 due to a force $F(t) = 20\sin 20t$N applied to the block whose displacement is x_2 using the method of undetermined coefficients. Use m = 10 kg, k = 90,000 N/m, and c = 100 N·s/m.

FIGURE P6.6

Given: System shown, m = 10 kg, k = 90,000 N/m, and c = 100 N·s/m.

Find: $x_1(t), x_2(t)$

Solution: The differential equations governing the motion of the system are derived in the solution of Problem 6.6 as

$$2m\ddot{x}_1 + 3c\dot{x}_1 - c\dot{x}_2 + 3kx_1 - kx_2 = 0$$

$$m\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - kx_1 + kx_2 = F_0\sin\omega t$$

Substituting given numbers leads to

$$20\ddot{x}_1 + 300\dot{x}_1 - 100\dot{x}_2 + 27000x_1 - 9000x_2 = 0$$

$$10\ddot{x}_2 - 100\dot{x}_1 + 100\dot{x}_2 - 9000x_1 + 9000x_2 = 20\sin 20t$$

A steady-state solution is assumed as

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}u_1\\ u_2\end{bmatrix}\sin 20t + \begin{bmatrix}v_1\\ v_2\end{bmatrix}\cos 20t$$

Substitution into the differential equations leads to

$$\left(-400\begin{bmatrix}20 & 0\\ 0 & 10\end{bmatrix}\sin 20t + 20\begin{bmatrix}300 & -100\\ -100 & 100\end{bmatrix}\cos 20t + \begin{bmatrix}27000 & -9000\\ -9000 & 9000\end{bmatrix}\sin 20t\right)\begin{bmatrix}u_1\\ u_2\end{bmatrix} + \left(-400\begin{bmatrix}20 & 0\\ 0 & 20\end{bmatrix}\cos 20t - 20\begin{bmatrix}300 & -100\\ -100 & 100\end{bmatrix}\sin 20t + \begin{bmatrix}27000 & -9000\\ -9000 & 9000\end{bmatrix}\cos 20t\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 20\end{bmatrix}\sin 20t$$

or

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$$\begin{bmatrix} 19000 & -9000 & -6000 & 2000 \\ -9000 & 5000 & 2000 & -2000 \\ 6000 & -2000 & 19000 & -9000 \\ -2000 & 2000 & 9000 & 5000 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ 0 \\ 0 \end{bmatrix}$$

The solution of the above equations is

$$\begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = 10^{-2} \begin{bmatrix} 1.035 \\ 2.261 \\ -0.173 \\ -0.178 \end{bmatrix}$$

The solutions are

$$x_1(t) = 10^{-2}(1.035 \sin 20t \ - 0.173 \cos 20t)$$

$$x_2(t) = 10^{-2}(2.261 \sin 20t \ - 0.178 \cos 20t)$$

which are also written as

$$x_1(t) = 0.0149 \sin(20t + 0.165)$$

$$x_2(t) = 0.0227 \sin(20t + 0.079)$$

Problem 6.41 illustrates the use of the method of undetermined coefficients for damped 2DOF systems.

6.42 Determine the response of the system of Figure 6.7 due to a force $F(t) = 40 \sin 60t$ N applied to the block whose displacement is x_1 using the method of undetermined coefficients. Use m = 20 kg, k = 200,000 N/m, and c = 400 N·s/m.

FIGURE P6.7

Given: system shown, F(t) = 40 sin 60t N, m = 20 kg, k = 200,000 N/m, and c = 400 N · s/m.

Find: $x_1(t), x_2(t)$

Solution: The differential equations governing the motion of the system are derived in the solution of Problem 6.7 as

$$m\ddot{x}_1 + 2c\dot{x}_1 - c\dot{x}_2 + kx_1 = 40 \sin 60t$$

$$2m\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 = 0$$

Substituting given numbers leads to

$$20\ddot{x}_1 + 800\dot{x}_1 - 400\dot{x}_2 + 200000x_1 = 40 \sin 60t$$

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$$40\ddot{x}_2 - 400\dot{x}_1 + 400\dot{x}_2 = 0$$

A steady-state solution is assumed as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \sin 60t + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cos 60t$$

Substitution into the differential equations leads to

$$\left(-3600\begin{bmatrix} 20 & 0 \\ 0 & 40 \end{bmatrix} \sin 60t + 60\begin{bmatrix} 800 & -400 \\ -400 & 400 \end{bmatrix} \cos 60t + \begin{bmatrix} 200000 & 0 \\ 0 & 0 \end{bmatrix} \sin 60t\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$+\left(-3600\begin{bmatrix} 20 & 0 \\ 0 & 40 \end{bmatrix} \cos 60t - 60\begin{bmatrix} 800 & -400 \\ -400 & 400 \end{bmatrix} \sin 60t\right.$$
$$\left.+\begin{bmatrix} 200000 & 0 \\ 0 & 0 \end{bmatrix} \cos 60t\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \end{bmatrix} \sin 60t$$

or

$$\begin{bmatrix} 128000 & 0 & -48000 & 24000 \\ 0 & -144000 & 24000 & -24000 \\ 48000 & -24000 & 128000 & 0 \\ -24000 & 24000 & 0 & -144000 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution of the above equations is

$$\begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = 10^{-4}\begin{bmatrix} 2.807 \\ -0.269 \\ -1.103 \\ -0.513 \end{bmatrix}$$

The solutions are

$$x_1(t) = 10^{-4}(2.807 \sin 60t \ - 1.103 \cos 60t)$$

$$x_2(t) = -10^{-4}(0.269 \sin 20t \ + 0.513 \cos 20t)$$

which are also written as

$$x_1(t) = 2.819 \times 10^{-4} \sin(60t + 0.374)$$

$$x_2(t) = 5.788 \times 10^{-5} \sin(60t + 1.759)$$

Problem 6.42 illustrates the use of the method of undetermined coefficients for damped 2DOF systems.

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6.43 Determine the response of the system of Figure P6.8 due to a unit impulse applied at the mass center. Use m = 100 kg, I = 4.5 kg · m^2, k = 200,000 N/m, c = 500 N · s/m, b = 2 m, and a = 1 m.

FIGURE P6.8

Given: System shown with unit impulse applied at mass center, m = 100 kg, I = 4.5 kg · m^2, k = 200,000 N/m, c = 500 N · s/m, b = 2 m and a = 1 m

Find: $x(t), \theta(t)$

Solution: The differential equations derived for the problem in the solution of Chapter Problem 6.9 are

$$I\ddot{\theta} + mb\ddot{x} + ca(a+b)\dot{\theta} + c(a+b)\dot{x} + ka(a+b)\theta + k(a+b)x = \delta(t)b$$

$$I\ddot{\theta} - ma\ddot{x} + cb(a+b)\dot{\theta} - c\dot{x}(a+b) + kb(a+b)\theta - k(a+b)x = \delta(t)a$$

Substituting given values leads to

$$4.5\ddot{\theta} + 200\ddot{x} + 1500\dot{\theta} + 1500\dot{x} + 600000\theta + 600000x = 2\delta(t)$$

$$4.5\ddot{\theta} - 100\ddot{x} + 3000\dot{\theta} - 3000\dot{x} + 1200000\theta - 600000x = \delta(t)$$

Taking the Laplace transform of the differential equations assuming all initial conditions are zero leads to

$$4.5s^2\Theta(s) + 200s^2X(s) + 1500s\Theta(s) + 1500sX(s) + 600000\Theta(s) + 600000X(s) = 2$$

$$4.5s^2\Theta(s) - 100s^2X(s) + 3000s\Theta(s) - 3000sX(s) + 1200000\Theta(s) - 600000X(s) = 1$$

Writing the above equations in a matrix form

$$\begin{bmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{bmatrix} \begin{bmatrix} \Theta(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving by Cramer's rule

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$$\Theta(s) = \frac{\begin{vmatrix} 2 & 200s^2 + 1500s + 600000 \\ 1 & -100s^2 - 1500s - 600000 \end{vmatrix}}{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{vmatrix}}$$

$$= \frac{-400s^2 - 4500s - 1800000}{1350s^4 + 7.635 \times 10^5 s^3 + 3.122 \times 10^8 s^2 + 5.4 \times 10^9 s + 1.8 \times 10^{10}}$$

$$= \frac{-400s^2 - 4500s - 1800000}{1350(s^2 + 526.8s + 2.22 \times 10^5)(s + 2.31)(s + 15.58)}$$

$$= \frac{-0.00453}{s + 2.31} - \frac{0.00476}{s + 15.58} - \frac{0.00023s + 0.3203}{s^2 + 526.8s + 2.22 \times 10^5}$$

$$\mathrm{X}(s) = \frac{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 2 \\ 4.5s^2 + 3000s + 1200000 & 1 \end{vmatrix}}{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{vmatrix}}$$

$$= \frac{-4.5s^2 - 4500s - 1800000}{1350s^4 + 7.635 \times 10^5 s^3 + 3.122 \times 10^8 s^2 + 5.4 \times 10^9 s + 1.8 \times 10^{10}}$$

$$= \frac{-4.5s^2 - 4500s - 1800000}{1350(s^2 + 526.8s + 2.22 \times 10^5)(s + 2.31)(s + 15.58)} =$$

$$= \frac{-0.000452}{s + 2.31} + \frac{0.000452}{s + 15.58} + \frac{0.00001s + 0.003192}{s^2 + 526.8s + 2.22 \times 10^5}$$

Inverting the transforms leads to

$$\theta(t) = -0.00453e^{-2.31t} - 0.00476e^{-15.58t} - e^{-263.4t}(0.00023\cos 390.6t + 0.000649\sin 390.t)$$

$$x(t) = -0.00452e^{-2.31t} + 0.000452e^{-15.58t} + e^{-263.4t}(0.000001\cos 390.6t + 0.00000727\sin 390.t)$$

Problem 6.43 illustrates the use of Laplace transforms to solve 2DOF forced vibrations problems.

6.44 Determine the response of the system of Figure P6.8 and Chapter Problem 6.43 to a unit impulse applied t to the right end or the machine tool using x and θ as generalized coordinates.

FIGURE P6.8

Given: System shown with unit impulse applied at mass center, m = 100 kg, I = 4.5 kg · m^2, k = 200,000 N/m, c = 500 N · s/m, b = 2 m and a = 1 m

Find: $x(t), \theta(t)$

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Solution: The differential equations derived for the problem in the solution of Chapter Problem 6.9 are

$$I\ddot{\theta} + mb\ddot{x} + ca(a+b)\dot{\theta} + c(a+b)\dot{x} + ka(a+b)\theta + k(a+b)x = \delta(t)(a+b)$$

$$I\ddot{\theta} - ma\ddot{x} + cb(a+b)\dot{\theta} - c\dot{x}(a+b) + kb(a+b)\theta - k(a+b)x = 0$$

Substituting given values leads to

$$4.5\ddot{\theta} + 200\ddot{x} + 1500\dot{\theta} + 1500\dot{x} + 600000\theta + 600000x = 3\delta(t)$$

$$4.5\ddot{\theta} - 100\ddot{x} + 3000\dot{\theta} - 3000\dot{x} + 1200000\theta - 600000x = 0$$

Taking the Laplace transform of the differential equations assuming all initial conditions are zero leads to

$$4.5s^2\Theta(s) + 200s^2X(s) + 1500s\Theta(s) + 1500sX(s) + 600000\Theta(s) + 600000X(s) = 3$$

$$4.5s^2\Theta(s) - 100s^2X(s) + 3000s\Theta(s) - 3000sX(s) + 1200000\Theta(s) - 600000X(s) = 0$$

Writing the above equations in a matrix form

$$\begin{bmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{bmatrix} \begin{bmatrix} \Theta(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Solving by Cramer's rule

$$\Theta(s) = \frac{\begin{vmatrix} 3 & 200s^2 + 1500s + 600000 \\ 0 & -100s^2 - 1500s - 600000 \end{vmatrix}}{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{vmatrix}}$$

$$= \frac{-300s^2 - 4500s - 1800000}{1350s^4 + 7.635 \times 10^5 s^3 + 3.122 \times 10^8 s^2 + 5.4 \times 10^9 s + 1.8 \times 10^{10}}$$

$$= \frac{-300s^2 - 4500s - 1800000}{1350(s^2 + 526.8s + 2.22 \times 10^5)(s + 2.31)(s + 15.58)}$$

$$= \frac{-0.000453}{s + 2.31} + \frac{0.000470}{s + 15.58} - \frac{0.000017s + 0.2251}{s^2 + 526.8s + 2.22 \times 10^5}$$

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$$X(s) = \frac{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 3 \\ 4.5s^2 + 3000s + 1200000 & 0 \end{vmatrix}}{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{vmatrix}}$$

$$= \frac{-13.5s^2 - 9000s - 3600000}{1350s^4 + 7.635 \times 10^5 s^3 + 3.122 \times 10^8 s^2 + 5.4 \times 10^9 s + 1.8 \times 10^{10}}$$

$$= \frac{-13.5s^2 - 9000s - 3600000}{1350(s^2 + 526.8s + 2.22 \times 10^5)(s + 2.31)(s + 15.58)}$$

$$= \frac{-0.000905}{s + 2.31} + \frac{0.000903}{s + 15.58} + \frac{0.000002 + 0.002908}{s^2 + 526.8s + 2.22 \times 10^5}$$

Inverting the transforms leads to

$$\theta(t) = -0.000453e^{-2.31t} - 0.000470e^{-15.58t} - e^{-263.4t}(0.000017\cos 390.6t + 0.00056549\sin 390.6t)$$

$$x(t) = -0.000905e^{-2.31t} + 0.0009030e^{-15.58t} + e^{-263.4t}(0.000002\cos 390.6t + 0.00000610\sin 390.6t)$$

Problem 6.44 illustrates the use of Laplace transforms to find the transient response of 2DOF systems.

6.45 Determine the response of the system of Figure P6.8 and Chapter Problem 6.43 to the force shown in Figure P6.45 applied at the right end of the machine tool.

FIGURE P6.45

Given: System shown with unit impulse applied at mass center, m = 100 kg, I = 4.5 kg · m^2, k = 200,000 N/m, c = 500 N · s/m, b = 2 m and a = 1 m, triangular pulse input at right end of bar.

Find: $x(t), \theta(t)$

Solution: The differential equations derived for the problem in the solution of Chapter Problem 6.9 are

$$I\ddot{\theta} + mb\ddot{x} + ca(a+b)\dot{\theta} + c(a+b)\dot{x} + ka(a+b)\theta + k(a+b)x = 3F(t)$$

$$I\ddot{\theta} - ma\ddot{x} + cb(a+b)\dot{\theta} - c\dot{x}(a+b) + kb(a+b)\theta - k(a+b)x = 0$$

where

$$F(t) = 2000t[u(t) - u(t-0.05)] + (200 - 2000t)[u(t-0.05) - u(t-0.1)]$$

$$= 2000tu(t) - 4000(t-0.05)u(t-0.05) + 2000(t-0.1)u(t-0.1)$$

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Substituting given values leads to

$$4.5\ddot{\theta} + 200\ddot{x} + 1500\dot{\theta} + 1500\dot{x} + 600000\theta + 600000x = 3F(t)$$

$$4.5\ddot{\theta} - 100\ddot{x} + 3000\dot{\theta} - 3000\dot{x} + 1200000\theta - 600000x = 0$$

Taking the Laplace transform of the differential equations assuming all initial conditions are zero leads to

$$4.5s^2\Theta(s) + 200s^2X(s) + 1500s\Theta(s) + 1500sX(s) + 600000\Theta(s) + 600000X(s) = 3F(s)$$

$$4.5s^2\Theta(s) - 100s^2X(s) + 3000s\Theta(s) - 3000sX(s) + 1200000\Theta(s) - 600000X(s) = 0$$

where

$$F(s) = \frac{2000}{s^2}(1 - 2e^{-0.05s} + e^{-0.1s})$$

Writing the above equations in a matrix form

$$\begin{bmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{bmatrix} \begin{bmatrix} \Theta(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} 3F(s) \\ 0 \end{bmatrix}$$

Solving by Cramer's rule

$$\Theta(s) = \frac{\begin{vmatrix} 3 & 200s^2 + 1500s + 600000 \\ 0 & -100s^2 - 1500s - 600000 \end{vmatrix}}{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{vmatrix}}$$

$$= \frac{(-300s^2 - 4500s - 1800000)F(s)}{1350s^4 + 7.635 \times 10^5 s^3 + 3.122 \times 10^8 s^2 + 5.4 \times 10^9 s + 1.8 \times 10^{10}}$$

$$= \frac{(-300s^2 - 4500s - 1800000)F(s)}{1350(s^2 + 526.8s + 2.22 \times 10^5)(s + 2.31)(s + 15.58)}$$

$$\mathrm{X}(s) = \frac{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 3 \\ 4.5s^2 + 3000s + 1200000 & 0 \end{vmatrix}}{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{vmatrix}}$$

$$= \frac{(-13.5s^2 - 9000s - 3600000)F(s)}{1350s^4 + 7.635 \times 10^5 s^3 + 3.122 \times 10^8 s^2 + 5.4 \times 10^9 s + 1.8 \times 10^{10}}$$

$$= \frac{(-13.5s^2 - 9000s - 3600000)F(s)}{1350(s^2 + 526.8s + 2.22 \times 10^5)(s + 2.31)(s + 15.58)}$$

It is noted that the response due to a force F(s) is the convolution of the response due to a unit impulse with the force. To this end the response due to a unit impulse is found in Chapter Problem 6.44 as

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$$h_\theta(t) = -0.000453e^{-2.31t} - 0.000470e^{-15.58t} - e^{-263.4t}(0.000017\cos 390.6t + 0.00056549\sin 390.6t)$$

$$h_x(t) = -0.000905e^{-2.31t} + 0.0009030e^{-15.58t} + e^{-263.4t}(0.000002\cos 390.6t + 0.00000610\sin 390.6t)$$

The convolution solution to the problem is

$$\theta(t) = \int_0^t F(t-\tau)h_\theta(\tau)dt$$

$$x(t) = \int_0^t F(t-\tau)h_x(\tau)dt$$

Problem 6.45 illustrates the use of the convolution property of Laplace transforms.

6.46 A schematic of part of a power transmission system is shown in Figure P6.46. A motor of moment of inertia I = 100 kg · m^2 is mounted on a shaft of shear modulus $G = 80 \times 10^9$ N/m^2, polar moment of inertia $J = 2.3 \times 10^{-4}$ m^4, and length 10 cm. Gear A, of moment of inertia 50 kg · m^2 with 40 teeth is at the end of the shaft which meshes with a gear, gear B, of moment of inertia 25 kg · m^2 with 20 teeth. Gear B is on a shaft of elastic modulus $G = 80 \times 10^9$ N/m^2, polar moment of inertia $J = 1.2 \times 10^{-5}$m^4, and length 60 cm. At the end of the shaft is a large industrial fan of moment of inertia 300 kg · m^2. Determine the natural frequencies of the system and the modal fractions.

FIGURE P6.46

Given: System of Figure 6.46

Find: Natural frequencies

Solution: The system is actually an unrestained 3DOF system. The system is modeled by three disks attached by torsional springs. The stiffnesses of the torsional springs are

$$k_{t1} = \frac{J_1 G_1}{L_1} = \frac{(2.3 \times 10^{-4}\ \mathrm{m}^4)\left(80 \times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}\right)}{0.1\ \mathrm{m}} = 1.84 \times 10^8\ \frac{\mathrm{N \cdot m}}{\mathrm{rad}}$$

$$k_{t2} = \frac{J_2 G_2}{L_2} = \frac{(1.2 \times 10^{-5}\ \mathrm{m}^4)\left(80 \times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}\right)}{0.6\ \mathrm{m}} = 1.6 \times 10^6\ \frac{\mathrm{N \cdot m}}{\mathrm{rad}}$$

The gears mesh, thus

$$40\omega_A = 20\omega_B \Longrightarrow \omega_B = 2\omega_A$$

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The equivalent moment of inertia of the gears is calculated from

$$T = \frac{1}{2}(50)\omega_A^2 + \frac{1}{2}(25)(2\omega_A)^2 = \frac{1}{2}(150\ \mathrm{kg}\cdot\mathrm{m}^2)\omega_A^2 \Longrightarrow I_{eq} = 150\ \mathrm{kg}\cdot\mathrm{m}^2$$

The differential equations governing the disks are obtained by summing moments on the FBD's of the disks

$$00\ddot{\theta}_1 + 1.84\times10^8\theta_1 - 1.84\times10^8\theta_2 = 0$$

$$150\ddot{\theta}_2 - 1.84\times10^8\theta_1 + 1.872\times10^8\theta_2 - 3.2\times10^6\theta_3 = 0$$

$$300\ddot{\theta}_3 - 3.2\times10^6\theta_2 + -3.2\times10^6\theta_3 = 0$$

The natural frequencies of the system are calculated as

$$\omega_1 = 0 \qquad \omega_2 = 153\ \frac{\mathrm{rad}}{\mathrm{s}} \qquad \omega_3 = 1736\ \frac{\mathrm{rad}}{\mathrm{s}}$$

Problem 6.46 illustrates the natural frequencies of an unrestrained system.

6.47 Determine the natural frequencies and modal fractions for the two degree-of-freedom system sof Figure P6.47.

FIGURE P6.47

Given: m, k, r

Find: $\omega_1, \omega_2, \chi_1, \chi_2$

Solution: The Lagrangian for the system is

$$L = T - V = \frac{1}{2}2m\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-2x)^2$$

Application of Lagrange's equations leads to

$$\left(2m + \frac{I}{r^2}\right)\ddot{x} + 4x - 2y = 0$$

$$m\ddot{y} - 2x + y = 0$$

A normal mode solution is assumed as

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$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ \chi \end{bmatrix} e^{i\omega t}$$

and leads to

$$\begin{vmatrix} \left(m + \frac{I}{r^2}\right)\omega^2 + 4k & -2k \\ -2k & m\omega^2 + k \end{vmatrix} = 0$$

The determinant is expanded to

$$\left[\left(m + \frac{I}{r^2}\right)\omega^2 + 4k\right][m\omega^2 + k] - (-2k)(-2k) = 0$$

$$m\left(m + \frac{I}{r^2}\right)\omega^4 + k\left(5m + \frac{I}{r^2}\right)\omega^2 = 0$$

$$\omega_1 = 0, \omega_2 = \sqrt{\frac{k(5mr^2 + I)}{m(mr^2 + I)}}$$

The modal fractions are obtained from

$$\begin{bmatrix} \left(m + \frac{I}{r^2}\right)\omega^2 + 4k & -2k \\ -2k & m\omega^2 + k \end{bmatrix} \begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first equation leads to

$$\chi = \frac{\left(m + \frac{I}{r^2}\right)\omega^2 + 4k}{2k}$$

Evaluating for the lowest natural frequency

$$\chi_1 = 2$$

For the higher natural frequency

$$\chi_2 = \frac{9}{2} + \frac{I}{2m}$$

Problem 6.47 illustrates determination of natural frequencies and modal fractions for an unrestrained 2DOF system.

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6.48 Determine the frequency response of the system of Figure P6.1 and Chapter Problem 6.10 due to a sinusoidal force $F_0 \sin \omega t$ applied to the block whose displacement is x_1.

FIGURE P6.1

Given: system shown

Find: Frequency response

Solution: The differential equations of motion are derived in the solution of Chapter Problem 6.1 and put in matrix form in the solution of Chapter Problem 6.10. They are modified to take the force into account as

$$\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^5 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix} \qquad (a)$$

Using the method of undetermined coefficients a solution is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin \omega t \qquad (b)$$

Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} -10\omega^2 + 200000 & -100000 \\ -100000 & -20\omega^2 + 300000 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is obtained using Cramer's rule

$$X_1 = \frac{\begin{vmatrix} F_0 & -100000 \\ 0 & -20\omega^2 + 300000 \end{vmatrix}}{\begin{vmatrix} -10\omega^2 + 200000 & -100000 \\ -100000 & -20\omega^2 + 300000 \end{vmatrix}} = \frac{F_0(-20\omega^2 + 300000)}{200\omega^4 - 7 \times 10^6 \omega^2 + 5 \times 10^{10}} \qquad (d)$$

$$X_2 = \frac{\begin{vmatrix} -10\omega^2 + 200000 & F_0 \\ -100000 & 0 \end{vmatrix}}{\begin{vmatrix} -10\omega^2 + 200000 & -100000 \\ -100000 & -20\omega^2 + 300000 \end{vmatrix}} = \frac{100000 F_0}{200\omega^4 - 7 \times 10^6 \omega^2 + 5 \times 10^{10}} \qquad (e)$$

Plots of X_1/F_0 and X_2/F_0 are presented

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Problem 6.48 illustrates determination of the frequency response of a 2DOF system.

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6.49 Determine the frequency response of the system of Figure P6.1 and Chapter Problem 6.10 due to a sinusoidal force $F_0 \sin \omega t$ applied to the block whose displacement is x_2.

FIGURE P6.1

Given: system shown

Find: Frequency response

Solution: The differential equations of motion are derived in the solution of Chapter Problem 6.1 and put in matrix form in the solution of Chapter Problem 6.10. They are modified to take the force into account as

$$\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^5 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_0 \sin \omega t \end{bmatrix} \qquad (a)$$

Using the method of undetermined coefficients a solution is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin \omega t \qquad (b)$$

Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} -10\omega^2 + 200000 & -100000 \\ -100000 & -20\omega^2 + 300000 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_0 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is obtained using Cramer's rule

$$X_1 = \frac{\begin{vmatrix} 0 & -100000 \\ F_0 & -20\omega^2 + 300000 \end{vmatrix}}{\begin{vmatrix} -10\omega^2 + 200000 & -100000 \\ -100000 & -20\omega^2 + 300000 \end{vmatrix}} = \frac{100000 F_0}{200\omega^4 - 7 \times 10^6 \omega^2 + 5 \times 10^{10}} \qquad (d)$$

$$X_2 = \frac{\begin{vmatrix} -10\omega^2 + 200000 & 0 \\ -100000 & F_0 \end{vmatrix}}{\begin{vmatrix} -10\omega^2 + 200000 & -100000 \\ -100000 & -20\omega^2 + 300000 \end{vmatrix}} = \frac{F_0(-10\omega^2 + 200000)}{200\omega^4 - 7 \times 10^6 \omega^2 + 5 \times 10^{10}} \qquad (e)$$

Plots of X_1/F_0 and X_2/F_0 are presented

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Problem 6.49 illustrates the frequency response of a 2DOF system.

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6.50 Determine the frequency response of the system of Figure P6.2 and Chapter Problem 6.11 due to a sinusoidal force $F_0 \sin \omega t$ applied to the particle.

FIGURE P6.2

Given: system shown

Find: Frequency response

Solution: The differential equations of motion are derived in the solution of Chapter Problem 6.2 and put in matrix form in the solution of Chapter Problem 6.11. They are modified to take the force into account as

$$\begin{bmatrix} 0.667 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 750 & -500 \\ -500 & 1000 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix} \qquad (a)$$

Using the method of undetermined coefficients a solution is assumed as

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} X \\ \Theta \end{bmatrix} \sin \omega t \qquad (b)$$

Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} -0.667\omega^2 + 750 & -500 \\ -500 & -2\omega^2 + 1000 \end{bmatrix} \begin{bmatrix} X \\ \Theta \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is obtained using Cramer's rule

$$X_1 = \frac{\begin{vmatrix} F_0 & -500 \\ 0 & -2\omega^2 + 1000 \end{vmatrix}}{\begin{vmatrix} -0.667\omega^2 + 750 & -500 \\ -500 & -2\omega^2 + 1000 \end{vmatrix}} = \frac{F_0(-2\omega^2 + 1000)}{1.333\omega^4 - 2167\omega^2 + 500000} \qquad (d)$$

$$X_2 = \frac{\begin{vmatrix} -0.667\omega^2 + 750 & F_0 \\ -500 & 0 \end{vmatrix}}{\begin{vmatrix} -0.667\omega^2 + 750 & -500 \\ -500 & -2\omega^2 + 1000 \end{vmatrix}} = \frac{500F_0}{1.333\omega^4 - 2167\omega^2 + 500000} \qquad (e)$$

Plots of X_1/F_0 and X_2/F_0 are presented

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Problem 6.50 illustrates the frequency response for a 2DOF system.

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6.51 Determine the frequency response of the system of the system of Figure P6.7 and Chapter Problem 6.42 due to a sinusoidal force $F_0 \sin \omega t$ applied to the block whose displacement is x_1.

FIGURE P6.7

Given: system shown, F(t) = 40 sin 60t N, m = 20 kg, k = 200,000 N/m, and c = 400 N · s/m.

Find: X_1, X_2

Solution: The differential equations governing the motion of the system are derived in the solution of Problem 6.7 as

$$m\ddot{x}_1 + 2c\dot{x}_1 - c\dot{x}_2 + kx_1 = F_0 \sin \omega t$$

$$2m\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 = 0$$

Substituting given numbers leads to

$$20\ddot{x}_1 + 800\dot{x}_1 - 400\dot{x}_2 + 200000x_1 = F_0 \sin \omega t$$

$$40\ddot{x}_2 - 400\dot{x}_1 + 400\dot{x}_2 = 0$$

A steady-state solution is assumed as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \sin 60t + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cos 60t$$

Substitution into the differential equations leads to

$$\left(-\omega^2 \begin{bmatrix} 20 & 0 \\ 0 & 40 \end{bmatrix} \sin \omega t + \omega \begin{bmatrix} 800 & -400 \\ -400 & 400 \end{bmatrix} \cos \omega t + \begin{bmatrix} 200000 & 0 \\ 0 & 0 \end{bmatrix} \sin \omega t\right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$+ \left(-\omega^2 \begin{bmatrix} 20 & 0 \\ 0 & 40 \end{bmatrix} \cos \omega t - \omega \begin{bmatrix} 800 & -400 \\ -400 & 400 \end{bmatrix} \sin \omega t$$
$$+ \begin{bmatrix} 200000 & 0 \\ 0 & 0 \end{bmatrix} \cos \omega t\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t$$

or

$$\begin{bmatrix} -20\omega^2 + 200000 & 0 & -800\omega & 400\omega \\ 0 & -40\omega^2 & 400\omega & -400\omega \\ 800\omega & -400\omega & -20\omega^2 + 200000 & 0 \\ -400\omega & 400\omega & 0 & -40\omega^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution of the above equations leads to

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$$\begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = \frac{F_0}{\omega^6 - 1.79 \times 10^4 \omega^4 + 4.94 \times 10^7 \omega^2 + 1 \times 10^{10}} \begin{bmatrix} -0.05\omega^4 + 485\omega^2 + 5 \times 10^4 \\ -25\omega^2 + 5 \times 10^4 \\ -2\omega^3 - 50\omega \\ \frac{1}{2}\omega^3 - 5.01 \times 10^3 \end{bmatrix}$$

Then

$$U = \sqrt{u_1^2 + u_2^2}$$

$$V = \sqrt{v_1^2 + v_2^2}$$

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Problem 6.51 illustrates frequency response of a 2DOF system.

6.52 Determine the frequency response of the system of Figure P6.8 and Chapter Problem 6.43 due to a sinusoidal force $F_0 \sin \omega t$ applied to the mass center of the machine tool.

Given: System of Figure P6.8 and Problem 6.43, Sinusoidal force applied at the middle of the bar

Find: frequency response

Solution: The differential equations governing the motion of the system with a force applied at the middle of the bar are derived in the solution of Chapter Problem 6.8 and the numbers substituted in the solution of Chapter Problem 6.43. The Laplace transforms are taken in the solution of Chapter Problem 6.43. The results are

$$4.5s^2\Theta(s) + 200s^2X(s) + 1500s\Theta(s) + 1500sX(s) + 600000\Theta(s) + 600000X(s) = 2F(s)$$

$$4.5s^2\Theta(s) - 100s^2X(s) + 3000s\Theta(s) - 3000sX(s) + 1200000\Theta(s) - 600000X(s) = F(s)$$

The transfer functions are calculated from

$$\begin{bmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{bmatrix} \begin{bmatrix} \Theta(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} 2F(s) \\ F(s) \end{bmatrix}$$

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which lead to

$$G_1(s) = \frac{\Theta(s)}{F(s)} = \frac{\begin{vmatrix} 2 & 200s^2 + 1500s + 600000 \\ 1 & -100s^2 - 1500s - 600000 \end{vmatrix}}{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{vmatrix}}$$

$$= \frac{-400s^2 - 4500s - 1800000}{1350s^4 + 7.635 \times 10^5 s^3 + 3.122 \times 10^8 s^2 + 5.4 \times 10^9 s + 1.8 \times 10^{10}}$$

and

$$G_2(s) = \frac{X(s)}{F(s)} = \frac{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 2 \\ 4.5s^2 + 3000s + 1200000 & 1 \end{vmatrix}}{\begin{vmatrix} 4.5s^2 + 1500s + 600000 & 200s^2 + 1500s + 600000 \\ 4.5s^2 + 3000s + 1200000 & -100s^2 - 1500s - 600000 \end{vmatrix}}$$

$$= \frac{-4.5s^2 - 4500s - 1800000}{1350s^4 + 7.635 \times 10^5 s^3 + 3.122 \times 10^8 s^2 + 5.4 \times 10^9 s + 1.8 \times 10^{10}}$$

The sinusoidal transfer function is used to determine the frequency response

$$G_1(i\omega) = \frac{-400(i\omega)^2 - 4500(i\omega) - 1800000}{1350(i\omega)^4 + 7.635 \times 10^5 (i\omega)^3 + 3.122 \times 10^8 (i\omega)^2 + 5.4 \times 10^9 (i\omega) + 1.8 \times 10^{10}}$$

$$= \frac{-4500(i\omega) - 1800000 + 400\omega^2}{1.8 \times 10^{10} + 1350\omega^4 - 3.122 \times 10^8 \omega^2 + i(5.4 \times 10^9 \omega - 7.635 \times 10^5 \omega^3)}$$

The frequency response is $|G_1(i\omega)|$ plotted on the vertical scale versus ω.

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$$G_2(i\omega) = \frac{-4.5(i\omega)^2 - 4500i\omega - 1800000}{1350(i\omega)^4 + 7.635\times 10^5(i\omega)^3 + 3.122\times 10^8(i\omega)^2 + 5.4\times 10^9(i\omega) + 1.8\times 10^{10}}$$

$$= \frac{-4500(i\omega) - 1800000 + 4.5\omega^2}{1.8\times 10^{10} + 1350\omega^4 - 3.122\times 10^8\omega^2 + i(5.4\times 10^9\omega - 7.635\times 10^5\omega^3)}$$

Problem 6.52 illustrates frequency response

6.53 Determine the frequency response of the system of Figure P6.8 and Chapter Problem 6.43 due to a sinusoidal force F_0 sin ωt applied to the right end of the machine tool.

Given: System of Figure P6.8 and Problem 6.43, Sinusoidal force applied at end of the bar

Find: frequency response

Solution: The differential equations governing the motion of the system with a force applied at the end of the bar are modified form those derived in the solution of Chapter Problem 6.8 and the numbers substituted in the solution of Chapter Problem 6.43. The Laplace transforms are taken in the solution of Chapter Problem 6.43. The results are

$$4.5s^2\Theta(s) + 200s^2X(s) + 1500s\Theta(s) + 1500sX(s) + 600000\Theta(s) + 600000X(s) = 3F(s)$$

$$4.5s^2\Theta(s) - 100s^2X(s) + 3000s\Theta(s) - 3000sX(s) + 1200000\Theta(s) - 600000X(s) = 0$$

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The transfer functions are calculated from

$$\begin{bmatrix} 4.5s^2+1500s+600000 & 200s^2+1500s+600000 \\ 4.5s^2+3000s+1200000 & -100s^2-1500s-600000 \end{bmatrix}\begin{bmatrix}\Theta(s)\\ X(s)\end{bmatrix}=\begin{bmatrix}3F(s)\\ 0\end{bmatrix}$$

which lead to

$$G_1(s)=\frac{\Theta(s)}{F(s)}=\frac{\begin{vmatrix}3 & 200s^2+1500s+600000\\ 0 & -100s^2-1500s-600000\end{vmatrix}}{\begin{vmatrix}4.5s^2+1500s+600000 & 200s^2+1500s+600000\\ 4.5s^2+3000s+1200000 & -100s^2-1500s-600000\end{vmatrix}}$$

$$=\frac{-300s^2-4500s-1800000}{1350s^4+7.635\times10^5s^3+3.122\times10^8s^2+5.4\times10^9s+1.8\times10^{10}}$$

and

$$G_2(s)=\frac{X(s)}{F(s)}=\frac{\begin{vmatrix}4.5s^2+1500s+600000 & 3\\ 4.5s^2+3000s+1200000 & 0\end{vmatrix}}{\begin{vmatrix}4.5s^2+1500s+600000 & 200s^2+1500s+600000\\ 4.5s^2+3000s+1200000 & -100s^2-1500s-600000\end{vmatrix}}$$

$$=\frac{-13.5s^2-4500s-3600000}{1350s^4+7.635\times10^5s^3+3.122\times10^8s^2+5.4\times10^9s+1.8\times10^{10}}$$

The sinusoidal transfer function is used to determine the frequency response

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Problem 6.53 illustrates frequency response.

6.54 A 50 kg lathe mounted on an elastic foundation of stiffness 4×10^5 N/m has a vibration amplitude of 35 cm when the motor speed is 95 rad/sec. Design an undamped dynamic vibration absorber such that the steady–state vibrations are completely eliminated at 95 rad/sec and the maximum displacement of the absorber mass at this speed is 5 cm.

Given: $m_1 = 50$ kg, $k_1 = 4 \times 10^5$ N/m, $X_1 = 35$ cm, $\omega = 95$ rad/sec, $X_2 = 5$ cm

Find: k_2, m_2

Solution: The natural frequency of the original system is

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{4 \times 10^5 \ \frac{\text{N}}{\text{m}}}{50 \text{ kg}}} = 89.44 \ \frac{\text{rad}}{\text{sec}}$$

When the original system operates at 95 rad/sec, its frequency ratio is

$$r_1 = \frac{\omega}{\omega_{11}} = \frac{95 \ \frac{\text{rad}}{\text{sec}}}{89.44 \ \frac{\text{rad}}{\text{sec}}} = 1.062$$

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The excitation amplitude is calculated from

$$\frac{m\omega_{11}^2 X}{F_0} = M(1.062,0) = \frac{1}{(1.062)^2 - 1} = 7.805$$

$$F_0 = \frac{(50\text{ kg})\left(89.44\ \frac{\text{rad}}{\text{sec}}\right)^2 (0.35\text{ m})}{7.822} = 17900\text{ N}$$

In order for the steady-state amplitude of the absorber mass be limited to 5 cm,

$$X_2 = \frac{F_0}{k_2} = 5\text{ cm}, \quad k_2 = \frac{17900\text{ N}}{0.05\text{ m}} = 3.58 \times 10^5\ \frac{\text{N}}{\text{m}}$$

Then, if steady-state vibrations of the original system are eliminated at 95 rad/sec

$$\omega_{22} = \sqrt{\frac{k_2}{m_2}} = 95\ \frac{\text{rad}}{\text{sec}}$$

$$m_2 = \frac{k_2}{\omega_{22}^2} = \frac{3.58 \times 10^5\ \frac{\text{N}}{\text{m}}}{\left(95\ \frac{\text{rad}}{\text{sec}}\right)^2} = 39.67\text{ kg}$$

Problem 6.54 illustrates undamped absorber design.

6.55 What is the required stiffness of an undamped dynamic vibration absorber whose mass is 5 kg to eliminate vibrations of a 25 kg machine of natural frequency 125 rad/sec when the machine operates at 110 rad/s?

Given: $m_1 = 25$ kg, $m_2 = 5$ kg, $\omega_n = 125$ rad/sec, $\omega = 110$ rad/sec

Find: k_2

Solution: In order for steady-state vibrations of the machine to be completely eliminated when the machine operates at 100 rad/sec, the natural frequency of the absorber must be 110 rad/sec,

$$\omega_{22} = \sqrt{\frac{k_2}{m_2}} = 110\ \frac{\text{rad}}{\text{sec}}$$

Hence

$$k_2 = m_2\omega_{22}^2 = (5\text{ kg})\left(110\ \frac{\text{rad}}{\text{sec}}\right)^2 = 6.05 \times 10^4\ \frac{\text{N}}{\text{m}}$$

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Problem 6.55 illustrates design of an undamped vibration absorber to eliminate steady-state vibrations of a one-degree-of-freedom system at a single frequency.

6.56 A 35 kg machine is attached to the end of a cantilever beam of length 2 m, elastic modulus 210×10^9 N/m^2, and moment of inertia 1.3×10^{-7} m^4. The machine operates at 180 rpm and has a rotating unbalance of 0.3 kg · m.

(a) What is the required stiffness of an undamped absorber of mass 5 kg such that steady–state vibrations are eliminated at 180 rpm?

(b) With the absorber in place, what are the natural frequencies of the system?

(c) For what range of operating speeds will the steady–state amplitude of the machine be less than 8 mm?

Given: $m_1 = 35$ kg, $L = 2$ m, $E = 210 \times 10^9$ N/m^2, $I = 1.3 \times 10^{-7}$ m^4, $\omega = 180$ rpm $= 18.85$ rad/sec, $m_0e = 0.3$ kg · m, $m_2 = 5$ kg

Find: (a) k2 (b) ω_1, ω_2 (c) range of ω such that $X_1 < 8$ cm

Solution: (a) The stiffness of the cantilever beam is

$$k_1 = \frac{3EI}{L^3} = \frac{3\left(210 \times 10^9 \ \frac{\text{N}}{\text{m}^2}\right)(1.3 \times 10^{-7} \ \text{m}^4)}{(2 \ \text{m})^3} = 10{,}240 \ \frac{\text{N}}{\text{m}}$$

The natural frequency of the original system is

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{10{,}240 \ \frac{\text{N}}{\text{m}}}{35 \ \text{kg}}} = 17.1 \ \frac{\text{rad}}{\text{sec}}$$

The original system is modeled as a one-degree-of-freedom mass-spring system. Inertia effects of the beam are ignored. Steady-state vibrations of the original system are eliminated when the machine operates at 180 rpm if the natural frequency of the absorber is 180 rpm,

$$\omega_{22} = \sqrt{\frac{k_2}{m_2}} = 180 \ \text{rpm} = 18.85 \ \frac{\text{rad}}{\text{sec}}$$

$$k_2 = m_2\omega_{22}^2 = (5 \ \text{kg})\left(18.85 \ \frac{\text{rad}}{\text{sec}}\right)^2 = 1777 \ \frac{\text{N}}{\text{m}}$$

(b) For the absorber designed

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$$q = \frac{\omega_{22}}{\omega_{11}} = \frac{18.85\ \frac{\text{rad}}{\text{sec}}}{17.1\ \frac{\text{rad}}{\text{sec}}} = 1.102, \quad \mu = \frac{m_2}{m_1} = \frac{5\ \text{kg}}{35\ \text{kg}} = 0.143$$

The natural frequencies of the resulting two degree-of-freedom system are calculated from eq.(6.46)

$$\omega_{1,2} = \frac{17.1\ \frac{\text{rad}}{\text{sec}}}{\sqrt{2}}\Big[1 + (1.102)^2(1.143) \pm \sqrt{(1.102)^4(1.143)^2 + 2(-0.857)(1.102)^2 + 1}\Big]^{\frac{1}{2}}$$

or

$$\omega_1 = 14.65\ \frac{\text{rad}}{\text{sec}}, \quad \omega_2 = 21.99\ \frac{\text{rad}}{\text{sec}}$$

(c) In order for the amplitude of the original system to be less than 8 cm at a given operating speed,

$$0.008\ m > X_1 = \frac{m_0 e\omega^2}{k_1}\left|\frac{1 - r_2^2}{q^2 r_2^4 - r_2^2 - (1+\mu)q^2 r_2^2 + 1}\right| = \frac{m_0 e\omega_{22}^2}{k_1} r_2^2 \left|\frac{1 - r_2^2}{q^2 r_2^4 - r_2^2 - (1+\mu)q^2 r_2^2 + 1}\right|$$

Hence

$$r_2^2\left|\frac{1 - r_2^2}{q^2 r_2^4 - r_2^2 - (1+\mu)q^2 r_2^2 + 1}\right| > \left|\frac{\left(10240\ \frac{\text{N}}{\text{m}}\right)(0.008\ \text{m})}{(0.3\ \text{kg} - \text{m})\left(18.85\ \frac{\text{rad}}{\text{sec}}\right)^2}\right| = 0.769$$

Using the positive sign of the absolute value leads to the upper bound on the frequency range. Rearranging leads to

$$r_2^4 - 1.465 r_2^2 + 0.3975 = 0$$

whose appropriate solution is $r_2 = 1.108$ which leads to $\omega < 19.85$ rad/sec. Use of the negative sign leads to the lower bound on the frequency range. Rearranging leads to

$$r_2^4 + 12.26 r_2^2 - 11.29 = 0$$

whose appropriate solution is $r_2 = 0.928$ which leads to $\omega > 17.49$ rad/sec.

$$17.49\ \frac{\text{rad}}{\text{sec}} < \omega < 19.85\ \frac{\text{rad}}{\text{sec}}$$

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Problem 6.56 illustrates (a) design of an undamped absorber, (b) calculation of the natural frequencies with the absorber in place, and (c) the effective operating range with the absorber in place.

6.57 A 150 kg pump experiences large amplitude vibrations when operating at 1500 rpm. Assuming this is the natural frequency of a SDOF system, design a dynamic vibration absorber such that the lowest natural frequency of the two degree-of-freedom system is less than 1300 rpm and the higher natural frequency is greater than 1700 rpm.

Given: $m_1 = 50$ kg, $\omega_{11} = \omega = 1500$ rpm = 157.1 rad/sec, $\omega_1 < 1300$ rpm = 136.1 rad/sec, $\omega_2 > 1700$ rpm = 178.0 rad/sec

Find: k_2, m_2

Solution: The absorber is designed such that

$$q = \frac{\omega_{22}}{\omega_{11}} = 1$$

Then from eq.(6.46) with q = 1,

$$\omega_2^2 = \frac{\omega_{11}^2}{2}\left(2 + \mu + \sqrt{\mu^2 + 4\mu}\right)$$

$$\omega_1^2 = \frac{\omega_{11}^2}{2}\left(2 + \mu - \sqrt{\mu^2 + 4\mu}\right)$$

Hence

$$\omega_1^2 + \omega_2^2 = \omega_{11}^2(2 + \mu)$$

or

$$\mu = \frac{\omega_1^2 + \omega_2^2}{\omega_{11}^2} - 2 = \frac{\left(136.1\ \frac{\text{rad}}{\text{sec}}\right)^2 + \left(178.0\ \frac{\text{rad}}{\text{sec}}\right)^2}{\left(157.1\ \frac{\text{rad}}{\text{sec}}\right)^2} - 2 = 0.0356$$

The absorber mass must be greater than

$$m_2 = \mu m_1 = (0.0356)(150\text{ kg}) = 5.34\text{ kg}$$

The absorber stiffness is

$$k_2 = m_2\omega_{22}^2 = m_2\omega_{11}^2 = (5.34\text{ kg})\left(157.1\ \frac{\text{rad}}{\text{sec}}\right)^2 = 1.32 \times 10^5\ \frac{\text{N}}{\text{m}}$$

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Problem 6.57 illustrates the difference in natural frequencies for the system when the absorber is added.

6.58 A solid disk of diameter 30 cm and mass 10 kg is attached to the end of a solid 3-cm-diameter, 1-m-long steel ($G = 80 \times 10^9$ N/m^2) shaft. A torsional vibration absorber consists of a disk attached to a shaft that is then attached to the primary system. If the absorber disk has a mass of 3 kg and a diameter of 10 cm, what is the required diameter of a 50-cm-long absorber shaft to eliminate steady–state vibrations of the original system when excited at 500 rad/sec?

Given: $r_D = 30$ cm, $m_D = 10$ kg, $L_1 = 1$ m, $r_s = 3$ cm, $G = 80 \times 10^9$ N/m^2, $m_{D2} = 3$ kg, $r_{D2} = 10$ cm, $L_2 = 50$ cm, $\omega = 500$ rad/sec

Find: r_{s2}

Solution: The addition of the shaft and disk to the original system acts as a dynamic vibration absorber for the torsional oscillations. Steady-state torsional oscillations of the original system are eliminated if the natural frequency of the absorber coincides with the excitation frequency. That is

$$\omega_{22} = \sqrt{\frac{k_{t_2}}{I_2}} = 500\ \frac{\text{rad}}{\text{sec}}$$

The moment of inertia of the absorber disk is

$$I_2 = \frac{1}{2} m_{D_2} r_{D_2^2} = \frac{1}{2}(3\ \text{kg})(0.05\ \text{m})^2 = 0.00375\ \text{kg}\cdot\text{m}^2$$

Thus

$$k_t = I_{D_2}\omega_{22}^2 = (0.00375\ \text{kg}\cdot\text{m}^2)\left(500\ \frac{\text{rad}}{\text{sec}}\right)^2 = 937.5\ \frac{\text{N}\cdot\text{m}}{\text{rad}}$$

The radius of the shaft is calculated from

$$k_t = \frac{\pi r_{s_2^4} G}{2L_{s_2}}$$

$$r_{s_2} = \left(\frac{2\left(937.5\ \frac{\text{N}\cdot\text{m}}{\text{rad}}\right)(0.5\ \text{m})}{\pi\left(80\times 10^9\ \frac{\text{N}}{\text{m}}\right)}\right)^{\frac{1}{4}} = 7.8\ \text{mm}$$

Hence the required shaft diameter is

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$$d = 15.6 \text{ mm}$$

Problem 6.58 illustrates design of a vibration absorber for a torsional system.

6.59 A 200 kg machine is placed on a massless simply supported beam shown in Figure P6.59. The machine has a rotating unbalance of 1.41 kg · m and operates at 3000 rpm. The steady-state vibrations of the machine are to be absorbed by hanging a mass attached to a 40 cm steel cable from the location on the beam where the mass is attached. What is the required diameter of the cable such that machine vibrations are eliminated at 3000 rpm and the amplitude of the absorber mass is less than 50 mm?

FIGURE P6.59

Given: m = 200 kg, m_0e = 1.41 kg · m, ω = 3000 rpm, L = 40 cm, E = 210 × 10^9 N/m^2, $X_{2,max}$ = 50

Find: d

Solution: The steady-state vibrations of the location on the beam where the absorber is attached are absorbed if the absorber is tuned to the excitation frequency,

$$\sqrt{\frac{k_2}{m_2}} = \omega = 3000 \text{ rpm}\left(2\pi \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right) = 314.6 \frac{\text{rad}}{\text{sec}}$$

The steady-state amplitude of the absorber is

$$X_2 = \frac{F_0}{k_2}$$

Requiring the amplitude to be less than 50 mm leads to

$$0.05 \text{ m} > \frac{(1.41 \text{ kg} \cdot \text{m})\left(314.6 \frac{\text{rad}}{\text{sec}}\right)^2}{\text{k}_2}$$

$$k_2 < 2.78 \times 10^6 \frac{\text{N}}{\text{m}}$$

This leads to

$$k_2 = \frac{AE}{L} = \frac{\pi d^2 E}{4L}$$

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$$d = \sqrt{\frac{4k_2L}{\pi E}} = 0.00259 \text{ m}$$

Problem 6.59 illustrates the use of a mass attached to a flexible cable as a vibration absorber.

6.60 The disk in Figure P6.60 rolls without slip and the pulley is massless. What is the mass of the block that should be hung from the cable such that steady-state vibrations of the cylinder are eliminated when ω = 120 rad/sec?

FIGURE P6.60

Given: $\omega = 120$ rad/sec, $k_1 = 5 \times 10^5$ N/m, $m_1 = 25$ kg, $r_D = 40$ cm, $r_1 = 20$ cm, $r_2 = 40$ cm, $k_2 = 3 \times 10^6$ N/m

Find: m_2 such that $X_1 = 0$

Solution: The block of mass m_2 acts as a vibration absorber. When an absorber is added to a system, steady-state vibrations of the point to which the absorber is attached vanish when the absorber frequency is equal to the excitation frequency. If θ is the angular displacement of the pulley, then the displacement of the center of the disk is

$$X_1 = (0.2 \text{ m})\theta$$

Thus if, pulley oscillations vanish, so do oscillations of the cylinder.

Hence, in order to eliminate steady-state vibrations of the cylinder at 120 rad/sec,

$$\omega_{22} = \sqrt{\frac{k_2}{m_2}} = \omega = 120 \ \frac{\text{rad}}{\text{sec}}$$

$$m_2 = \frac{k_2}{\omega^2} = \frac{3 \times 10^6 \ \frac{\text{N}}{\text{m}}}{\left(120 \ \frac{\text{rad}}{\text{sec}}\right)^2} = 208.3 \text{ kg}$$

Problem 6.60 illustrates design of a dynamic vibration absorber to eliminate steady-state vibrations of a one-degree-of-freedom system.

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6.61 Vibration absorbers are used in boxcars to protect sensitive cargo from large accelerations due to periodic excitations provided by rail joints. For a particular railway, joints are spaced 5 m apart. The boxcar, when empty, has a mass of 25,000 kg. Two absorbers, each of mass 12,000 kg, are used. Absorbers for a particular boxcar are designed to eliminate vibrations of the main mass when the boxcar is loaded with a 12,000 kg cargo and travels at 100 m/s. The natural frequency of the unloaded boxcar is 165 rad/sec.

(a) At what speeds will resonance occur for the boxcar with a 12,000 kg cargo?

(b) What is the best speed for the boxcar when it is loaded with a 25,000 kg cargo?

Given: d = 5 m, m_0 = 25,000 kg, m_a = 12,000 kg, m_c = 12,000 kg, v = 100 m/s, ω_n(unloaded) = 165 rad/sec

Find: (a) ω_1, ω_2, (b) v for m_c=25,000 kg

Solution: (a) The vertical oscillations of the boxcar by itself are modeled using a one-degree-of-freedom system. The mass of the system is the mass of the boxcar plus the mass of its cargo. When the boxcar is unloaded, it has a mass of 25000 kg and a natural frequency of 165 rad/sec. Hence the equivalent stiffness for one-degree-of-freedom model is

$$k_{eq} = m\omega_n^2 = (25000\text{ kg})\left(165\ \frac{\text{rad}}{\text{sec}}\right)^2 = 6.906 \times 10^8\ \frac{\text{N}}{\text{m}}$$

The absorbers are assumed to be placed such that they are equidistant from the center of the boxcar. Thus the vibrations of the boxcar with the absorbers are modeled by the system shown below.

The differential equations of motion governing the three-degree-of-freedom system are

$$(m_0 + m_c)\ddot{x}_1 + (k_{eq} + 2k_2)x_1 - k_2x_2 - k_2x_3 = F_0 sin\omega t$$

$$m_a\ddot{x}_2 + k_2x_1 - k_2x_2 = 0$$

$$m_a\ddot{x}_3 + k_2x_1 - k_2x_3 = 0$$

Note that x_2 and x_3 are interchangeable in the above equations. Hence $x_2 = x_3$ and the above equations become

$$(m_0 + m_c)\ddot{x}_1 + (k_{eq} + 2k_2)x_1 - 2k_2x_2 = F_0 sin\omega t$$

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$$m_a\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

Note that when the second equation is multiplied by 2 the two equations are identical to the equations describing the motion of a one-degree-of-freedom with a single vibration absorber added of mass $2m_a$ and stiffness $2k_2$.

The period of excitation is the time it takes for the boxcar to travel between joints,

$$t = \frac{d}{v} = \frac{5\text{ m}}{100\ \frac{\text{m}}{\text{sec}}} = 0.05\text{ sec}$$

Thus the period of excitation is

$$\omega = \frac{2\pi}{t} = 125.6\ \frac{\text{rad}}{\text{sec}}$$

The natural frequency of the boxcar with a 12,000 kg cargo is

$$\omega_n = \sqrt{\frac{m_0 + m_c}{k_{eq}}} = \sqrt{\frac{37000\text{ kg}}{6.906 \times 10^8\ \frac{\text{N}}{\text{m}}}} = 135.6\ \frac{\text{rad}}{\text{sec}}$$

The absorbers have been designed to eliminate vibrations at 125.6 rad/sec. Thus the natural frequencies of the resulting two degree-of-freedom system can be obtained using eq.(6.46) with

$$q = \frac{125.6\ \frac{\text{rad}}{\text{sec}}}{135.6\ \frac{\text{rad}}{\text{sec}}} = 0.926, \quad \mu = \frac{24{,}000\text{ kg}}{37{,}000\text{ kg}} = 0.649$$

ABSORB can also be used. The output from ABSORB follows.

UNDAMPED ABSORBER DESIGN USING ABSORB.BAS

Primary system parameters
- Mass = 3.700E+04 kg
- Stiffness = 6.906E+08 N/m
- Excitation frequency = 1.256E+02 rad/sec
- Excitation amplitude = 1.000E+02 N

Design specifications
- Absorber mass = 2.400E+04 kg

The results
- Absorber stiffness = 3.786E+08 N/m
- Steady-state absorber amplitude = 2.641E-07 m
- Lower natural frequency = 8.965E+01 rad/sec

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Higher natural frequency = 1.914E+02 rad/sec

ABSORB reveals that the natural frequencies of the resulting system are

$$\omega_1 = 89.7\ \frac{\text{rad}}{\text{sec}}, \quad \omega_2 = 191.4\ \frac{\text{rad}}{\text{sec}}$$

The boxcar speeds to induce these frequencies are

$$v = \frac{\omega d}{2\pi} = 71.3\ \frac{\text{rad}}{\text{sec}}, \quad 152.3\ \frac{\text{rad}}{\text{sec}}$$

(b) Steady-state vibrations of the primary system are eliminated only at the frequency of the absorber,

$$\omega = \sqrt{\frac{k_2}{m_a}} = 125.6\ \frac{\text{rad}}{\text{sec}}$$

If an absorber is already in place, the addition of mass to the primary system does not alter the absorber frequency and hence the frequencies at which steady-state vibrations are eliminated. The addition of mass to the primary system does not affect the steady-state amplitudes at other speeds, the steady-state amplitudes of the absorbers, and the resulting natural frequencies.

Thus, in light of the above, the best speed of the boxcar with any loading is 100 m/sec, as long as the same absorber is in place and the railway joints are 5 m apart.

Problem 6.61 illustrates (a) the use of multiple identical vibration absorbers is modeled as if adding a single vibration absorber with a multiple mass and multiple stiffness, (b) the determination of the natural frequencies of a system with a vibration absorber, and (c) steady-state oscillations are eliminated only at the speed to which the absorber is tuned.

6.62 A 500 kg reciprocating machine is mounted on a foundation of equivalent stiffness 5×10^6 N/m. When operating at 800 rpm, the machine produces an unbalanced harmonic force of magnitude 50,000 N. Two cantilever beams with end masses are added to the machine to act as absorbers. The beams are made of steel ($E = 210 \times 10^9$ N/m^2) and have a moment of inertia of 4×10^{-6} m^4. A 10 kg mass is attached to each beam. The absorbers are adjustable in that the location of the mass on the absorber can be varied.

(a) How far away from the support should the masses be located when the machine is operating at 800 rpm? What is the amplitude of the absorber mass?

(b) If the compressor operates at 1000 rpm and produces a harmonic force of amplitude 100,000 N, where should the absorber masses be placed and what is their vibration amplitude?

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Given: $m_1 = 500$ kg, $k_1 = 500 \times 10^6$ N/m, $E = 210 \times 10^9$ N/m^2, $I = 4 \times 10^{-6}$ m^4, $m_2 = 10$ kg (a) $\omega = 800$ rpm = 83.77 rad/sec, $F_0 = 50{,}000$ N, (b) $\omega = 1000$ rpm = 104.7 rad/sec, $F_0 = 50{,}000$ N

Find: (a) and (b) L, x_2

Solution: It is shown in the solution of problem 6.67 that the addition of two identical absorbers to a one-degree-of-freedom system is equivalent to adding a single absorber of twice the mass and twice the stiffness of each of the absorbers. The cantilever beam is assumed to be negligible mass and acts as an absorber of stiffness

$$k_2 = \frac{3EI}{L^3}$$

and natural frequency

$$\omega_{22} = \sqrt{\frac{3EI}{mL^3}}$$

where L is the distance from the support to the absorber mass.

Steady-state vibrations of the primary system are eliminated if the absorber frequency ω_{22} is equal to the excitation frequency.

(a) For $\omega = 83.77$ rad/sec, with m = 20 kg

$$k_{2_{eq}} = m\omega_{22}^2 = (20\text{ kg})\left(83.77\ \frac{\text{rad}}{\text{sec}}\right)^2 = 1.4 \times 10^5\ \frac{\text{N}}{\text{m}}$$

Since two absorbers are used, k_{eq} is twice the stiffness of a single absorber. Hence the stiffness of a single absorber is

$$k_{eq} = 7 \times 10^4\ \frac{\text{N}}{\text{m}}$$

which leads to

$$L = \left(\frac{3EI}{k_{eq}}\right)^{\frac{1}{3}} = 3.30\text{ m}$$

The steady-state amplitude of the absorber is

$$X_2 = \frac{F_0}{2k_2} = \frac{50000\ \frac{\text{N}}{\text{m}}}{1.4 \times 10^5\ \frac{\text{N}}{\text{m}}} = 0.357\text{ m}$$

(b) Repeating the calculations of (a) with $\omega = 104.7$ rad/sec and $F_0 = 100{,}000$ N leads to

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$$k_{2_{eq}} = (20\text{ kg})\left(104.7\ \frac{\text{rad}}{\text{sec}}\right)^2 = 2.19 \times 10^5\ \frac{\text{N}}{\text{m}}$$

$$L = \left(\frac{3\left(210 \times 10^9\ \frac{\text{N}}{\text{m}}\right)(4 \times 10^{-6}\text{ m}^4)}{0.5\left(2.19 \times 10^5\ \frac{\text{N}}{\text{m}}\right)}\right)^{\frac{1}{3}} = 2.84\text{ m}$$

$$X_2 = \frac{F_0}{2k_2} = \frac{100000\text{ N}}{2.19 \times 10^5\ \frac{\text{N}}{\text{m}}} = 0.456\text{ m}$$

Problem 6.62 illustrates (a) the use of identical multiple absorbers (b) the use of cantilever beams of vibration absorbers, and (c) calculation of the steady-state absorber amplitude.

6.63 A 100 kg machine is placed at the midspan of a 2-m-long cantilever beam ($E = 210 \times 10^9$ N/m^2, $I = 2.3 \times 10^{-6}$ m^4). The machine produces a harmonic force of amplitude 60,000 N. Design a damped vibration absorber of mass 30 kg such that when hung from the beam at midspan, the steady-state amplitude of the machine is less than 8 mm at all speeds between 1300 and 2000 rpm.

Given: $m_1 = 100$ kg, $E = 210 \times 10^9$ N/m^2, $I = 2.3 \times 10^{-6}$ m^4, $L = 2$ m, $m_2 = 30$ kg, $X_{1,max} = 8$ mm, 136.14 rad/sec < ω < 209.44 rad/sec

Find: k_2, c_2

Solution: The equivalent stiffness of the beam at the location where the machine is attached is

$$k = \frac{3EI}{\left(\frac{L}{2}\right)^3} = \frac{3\left(210 \times 10^6\ \frac{\text{N}}{\text{m}^2}\right)(2.3 \times 10^{-6}\text{ m}^4)}{(1\text{ m})^3} = 1.49 \times 10^6\ \frac{\text{N}}{\text{m}}$$

The natural frequency of the primary system is

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{1.49 \times 10^6\ \frac{\text{N}}{\text{m}}}{100\text{ kg}}} = 122.1\ \frac{\text{rad}}{\text{sec}}$$

If the maximum amplitude of the primary system is required to be less than 8 mm, then the magnification factor for the primary system has an upper bound of

$$M_1 < \frac{k_1 X_{1_{max}}}{F_0} = \frac{\left(1.49 \times 10^6\ \frac{N}{m}\right)(0.008\ m)}{60000\ N} = 0.199$$

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First consider the addition of a damped absorber with the optimum damping ration and the optimum tuning. Use of ABSORB.BAS shows that the steady-state amplitudes at the boundaries of the operating range are

$$X_1(136.14 \text{ rad/sec}) = 0.104 \text{ m}$$

$$X_1(309.44 \text{ rad/sec}) = 0.0213 \text{ m}$$

both of which exceed the maximum allowable.

Next consider the design of an undamped absorber with steady-state vibrations eliminated at the lowest operating speed. The range of frequencies for which the steady-state amplitude is less than 8 mm is 128 rad/sec < ω < 156.8 rad/sec.

An undamped absorber is generally not suitable for use over such a wide operating range. Indeed, id steady-state vibrations are eliminated at one operating speed; there is a very small range around that speed such that the steady-state amplitude is less than 8 mm. In fact, as the speed is further away from the tuning speed, the steady-state amplitude gets very large. This leads to a situation where the absorber works over a small range. Outside of this range, the steady-state amplitude with the absorber attached is much larger than the steady-state amplitude of the system without the absorber.

One possible solution is to increase the absorber mass. However calculations show that the absorber mass would have to be increased to over 400 kg, in order for the absorber to work.

Thus a damped absorber is more useful over a wide range of operating speeds. However, since M_1 is required to be less than 0.199 over the enter operating range, it is not possible to design an appropriate absorber with an absorber mass of 30 kg. ABSORB.BAS is used to test several absorber designs. The output follows. The final design selected uses a frequency ratio of 1.21 and a damping ratio of 0.15. The steady-state amplitudes at the ends of the operating range are

$$X_1\ (136.14 \text{ rad/sec}) = 0.0286 \text{ m}, \quad X_1\ (209.44 \text{ rad/sec}) = 0.0273 \text{ m}$$

The plot from ABSORB follows showing that the addition of the absorber leads to a decrease in amplitude over most of the operating range.

DAMPED VIBRATION ABSORBER ANALYSIS

PRIMARY SYSTEM PARAMETERS

Mass = 1.000E+02 kg
Stiffness = 1.229E+06 N/m
Excitation frequency = 1.361E+02 rad/sec
Excitation amplitude = 6.000E+04

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ABSORBER PARAMETERS (DIMENSIONAL)
Mass = 3.000E+01 kg
Stiffness = 2.572E+05 N/m
Damping coefficient = 1.634E+03 N-sec/m

ABSORBER PARAMETERS (NONDIMENSIONAL)
Mass = 3.000E-01
Frequency ratio = 7.692E-01
Damping ratio = 2.942E-01

STEADY-STATE AMPLITUDES AT SPECIFIED FREQUENCIES
For omega = 1.361E+02 rad/sec, X = 1.040E-01 m
For omega = 2.094E+02 rad/sec, X = 2.132E-02 m

MAXIMUM START-UP AMPLITUDE = 1.150E-01 m at OMEGA = 8.599E+01 rad/sec

UNDAMPED ABSORBER DESIGN USING ABSORB.BAS

PRIMARY SYSTEM PARAMETERS
Mass = 1.000E+02 kg
Stiffness = 4.490E+06 N/m
Excitation frequency = 1.361E+02 rad/sec
Excitation amplitude = 6.000E+04

Design specifications
Absorber mass = 3.000E+01 kg
Maximum steady-state amplitude = 8.000E-03 m

The results
Absorber stiffness = 5.560E+05 N/m
Lowest operating speed = 1.281E+02 rad/sec
Highest operating speed = 1.568E+02 rad/sec
Lower natural frequency = 1.248E+02 rad/sec
Higher natural frequency = 2.311E+02 rad/sec

UNDAMPED ABSORBER DESIGN USING ABSORB.BAS

PRIMARY SYSTEM PARAMETERS
Mass = 1.000E+02 kg
Stiffness = 1.490E+06 N/m
Excitation frequency = 1.361E+02 rad/sec
Excitation amplitude = 8.000E-03

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Design specifications
Absorber mass = 3.000E+01 kg
Maximum steady-state amplitude = 8.000E-03 m

The results
Absorber stiffness = 5.560E+05 N/m
Lowest operating speed = 9.920E+01 rad/sec
Highest operating speed = 1.675E+02 rad/sec
Lower natural frequency = 9.644E+01 rad/sec
Higher natural frequency = 1.723E+02 rad/sec

DAMPED VIBRATION ABSORBER ANALYSIS

PRIMARY SYSTEM PARAMETERS
Mass = 1.000E+02 kg
Stiffness = 1.490E+00 N/m
Excitation frequency = 1.000E+02 rad/sec
Excitation amplitude = 6.000E+04

ABSORBER PARAMETERS (DIMENSIONAL)
Mass = 3.000E+01 kg
Stiffness = 4.470E-01 N/m
Damping coefficient = 1.099E+00 N-sec/m

ABSORBER PARAMETERS (NONDIMENSIONAL)
Mass = 3.000E-01
Frequency ratio = 1.000E+00
Damping ratio = 1.500E-01

STEADY-STATE AMPLITUDES AT SPECIFIED FREQUENCIES
For omega = 1.361E+02 rad/sec, X = 3.237E-02 m
For omega = 2.094E+02 rad/sec, X = 1.368E-02 m

MAXIMUM START-UP AMPLITUDE = 3.459E+05 m at OMEGA = 9.411E-02 rad/sec

DAMPED VIBRATION ABSORBER ANALYSIS

PRIMARY SYSTEM PARAMETERS
Mass = 1.000E+02 kg
Stiffness = 1.490E+06 N/m
Excitation frequency = 1.361E+02 rad/sec

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Excitation amplitude = 6.000E+04

ABSORBER PARAMETERS (DIMENSIONAL)
Mass = 3.000E+01 kg
Stiffness = 6.545E+05 N/m
Damping coefficient = 1.329E+03 N-sec/m

ABSORBER PARAMETERS (NONDIMENSIONAL)
Mass = 3.000E-01
Frequency ratio = 1.210E+00
Damping ratio = 1.500E-01

STEADY-STATE AMPLITUDES AT SPECIFIED FREQUENCIES
For omega = 1.361E+02 rad/sec, X = 2.855E-02 m
For omega = 2.094E+02 rad/sec, X = 2.730E-02 m

MAXIMUM START-UP AMPLITUDE = 7.754E-01 m at OMEGA = 9.900E+01 rad/sec

Problem 6.63 illustrates (a) the design of a damped vibration absorber (b) the increase in operating range achieved by using a damped absorber, and (c) not all problems with strict specifications (e.g. small steady-state amplitude and small mass ratio) have solutions.

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6.64 Repeat Chapter Problem 6.63 if the excitation is due to a rotating unbalance of magnitude 0.33 kg · m.

Given: $m_1 = 100$ kg, $E = 210 \times 10^9$ N/m^2, $I = 2.3 \times 10^{-6}$ m^4, $L = 2$ m, $m_2 = 30$ kg, $X_{1,max} = 8$ mm, 136.14 rad/sec < ω < 209.44 rad/sec, $m_0e = 0.33$ kg · m

Find: k_2, c_2

Solution: The equivalent stiffness of the beam at the location where the machine is attached is

$$k = \frac{3EI}{\left(\frac{L}{2}\right)^3} = \frac{3\left(210 \times 10^6 \ \frac{\text{N}}{\text{m}^2}\right)(2.3 \times 10^{-6} \ \text{m}^4)}{(1 \ \text{m})^3} = 1.49 \times 10^6 \ \frac{\text{N}}{\text{m}}$$

The natural frequency of the primary system is

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{1.49 \times 10^6 \ \frac{\text{N}}{\text{m}}}{100 \ \text{kg}}} = 122.1 \ \frac{\text{rad}}{\text{sec}}$$

In order for the absorber to work over such a wide range of frequencies, the absorber must be tuned such that the second peak in the frequency response curve is much smaller than the first. ABSORB.BAS is used to help design such an absorber. ABSORB.BAS is used, trying a variety of absorber designs. The excitation force at the lowest operating speed (6100 N) is used for input, and the steady-state amplitude at five operating speeds are printed, assuming an excitation of 6100 N. An absorber with q = 1.21 and ζ = 0.20 is chosen. The output from ABSORB for this absorber design follows, as well as the frequency response curves plotted from ABSORB.

Since the excitation is actually a frequency squared excitation the amplitudes at the upper operating speeds are incorrect. Since the excitation force at the lowest operating speed is used, the true amplitude at another speed is obtained by multiplying the printed amplitude by the square of the ratio of the excitation frequency to 136.14 rad/sec. This results in the following steady-state amplitudes

$$X_1(150 \ \text{rad/sec}) = 0.00345 \ \text{m}, \quad X_1(175 \ \text{rad/sec}) = 0.00516 \ \text{m}$$

$$X_1(200 \ \text{rad/sec}) = 0.00595 \ \text{m}, \quad X_1(209.44 \ \text{rad/sec}) = 0.00581 \ \text{m}$$

Hence the design is acceptable.

DAMPED VIBRATION ABSORBER ANALYSIS

PRIMARY SYSTEM PARAMETERS
Mass = 1.000E+02 kg
Stiffness = 1.490E+06 N/m

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Excitation frequency = 1.360E+02 rad/sec
Excitation amplitude = 6.100E+03

ABSORBER PARAMETERS (DIMENSIONAL)
Mass = 3.000E+01 kg
Stiffness = 6.545E+05 N/m
Damping coefficient = 1.772E+03 N-sec/m

ABSORBER PARAMETERS (NONDIMENSIONAL)
Mass = 3.000E-01
Frequency ratio = 1.210E+00
Damping ratio = 2.000E-01

STEADY-STATE AMPLITUDES AT SPECIFIED FREQUENCIES
For omega = 1.361E+02 rad/sec, X = 3.478E-03 m
For omega = 1.500E+02 rad/sec, X = 2.844E-03 m
For omega = 1.750E+02 rad/sec, X = 3.128E-03 m
For omega = 2.000E+02 rad/sec, X = 2.758E-03 m
For omega = 2.094E+02 rad/sec, X = 2.453E-03 m

MAXIMUM START-UP AMPLITUDE = 6.293E-02 m at OMEGA = 9.900E+01 rad/sec

Problem 6.64 illustrates (a) the design of a damped vibration absorber for a system with a frequency squared excitations, and (b) the use of ABSORB.BAS.

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6.65 For the absorber designed in Chapter Problem 6.63, what is the minimum steady-state amplitude of the machine and at what speed does it occur?

Given: system of Problem 6.63

Find: $X_{1,min}$, ω

Solution: Please refer to the solution of Problem 6.63 for the details of the absorber design. The plot from ABSORB.BAS is shown below. The minimum magnification factor appears to be 0.55 at a frequency ratio of 1.2. Using the values determined in Problem 6.63, this yields

$$X_1 = \frac{F_0 M}{k_1} = \frac{(60000\ N)(0.55)}{1.49 \times 10^6\ \frac{N}{m}} = 0.0221\ m$$

$$\omega = r\omega_{11} = (1.2)(122.1\ \text{rad/sec}) = 146.5\ \text{rad/sec}$$

Problem 6.65 illustrates the use of ABSORB.BAS to determine the minimum steady-state amplitude for a given absorber design.

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6.66 Determine values of k and c such that the steady-state amplitude of the center of the cylinder in Figure P6.66 is less than 4 mm for 60 rad/sec < ω < 110 rad/sec.

FIGURE P6.66

Given: $m_1 = 40$ kg, $k_1 = 5 \times 10^5$ N/m, r = 40 cm, $M_0 = 200$ N-m, 60 rad/sec < ω < 110 rad/sec, $r_1 = 20$ cm, $r_2 = 40$ cm, $m_2 = 8$ kg, X_1 < 4mm

Find: c, k

Solution: Let x_1 be the displacement of the mass center of the disk and x_2 be the displacement of the block, both measured from the system's equilibrium position. Assuming no slip between the disk and the surface, the kinetic energy of the system at an arbitrary time is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}\frac{1}{2}mr^2\left(\frac{\dot{x}_1}{r}\right)^2 + \frac{1}{2}m_2\dot{x}_2^2$$

The potential energy at an arbitrary time is

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left(x_2 - \frac{r_2}{r_1}x_1\right)^2$$

The work done by the nonconservative forces as the system moves through variations δx_1 and δx_2 is

$$\delta Q = -c\left(\dot{x}_2 - \frac{r_2}{r_1}\dot{x}_1\right)\delta\left(x_2 - \frac{r_2}{r_1}x_1\right) + \frac{M_0}{r}sin\omega t\delta x_1$$

Thus using Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = Q_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = Q_2$$

to derive the governing differential equations leads to

$$\frac{3}{2}m_1\ddot{x}_1 + \left[k_1 + \left(\frac{r_2}{r_1}\right)^2 k_2\right]x_1 - \frac{r_2}{r_1}x_2 = \frac{M_0}{r}sin\omega t$$

$$m_2\ddot{x}_2 + \frac{r_2}{r_1}k_2x_1 + k_2x_2 - c\frac{r_2}{r_1}\dot{x}_1 + c\dot{x}_2 = 0$$

Let

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$$z = \frac{r_2}{r_1} x_1$$

Then using z as a dependent variable instead of x_1 leads to

$$\frac{3}{2}\frac{r_1}{r_2} m_1 \ddot{z} + \left[k_1 + \left(\frac{r_2}{r_1}\right)^2 k_2\right]\frac{r_1}{r_2} z - \frac{r_2}{r_1} k_2 x_2 = \frac{M_0}{r} sin\omega t$$

$$m_2 \ddot{x}_2 - k_2 z + k_2 x_2 - c\dot{z} + c\dot{x}_2 = 0$$

Multiplying the first equation by r_2/r_1 and the second equation by $(r_2/r_1)^2$ leads to

$$\frac{3}{2} m_1 \ddot{z} + \left[k_1 + \left(\frac{r_2}{r_1}\right)^2 k_2\right] z - \left(\frac{r_2}{r_1}\right)^2 k_2 x_2 = \frac{M_0}{r}\frac{r_2}{r_1} sin\omega t$$

$$m_2 \left(\frac{r_2}{r_1}\right)^2 \ddot{x}_2 - k_2 \left(\frac{r_2}{r_1}\right)^2 z + k_2 \left(\frac{r_2}{r_1}\right)^2 x_2 - c \left(\frac{r_2}{r_1}\right)^2 \dot{z} + c \left(\frac{r_2}{r_1}\right)^2 \dot{x}_2 = 0$$

Now define

$$m_1^* = \frac{3}{2} m_1 = 60\ kg, \quad k_1^* = k_1 = 5 \times 10^5\ \frac{N}{m}, \quad F_0^* = \frac{M_0}{r}\frac{r_2}{r_1} = 1000\ N$$

$$m_2^* = m_2 \left(\frac{r_2}{r_1}\right)^2 = 32\ kg, \quad k_2^* = k_2 \left(\frac{r_2}{r_1}\right)^* = 4k, \quad c^* = \left(\frac{r_2}{r_1}\right)^2 c = 4c$$

which leads to

$$m_1^* \ddot{z} + (k_1^* + k_2^*) z - k_2^* x_2 = F_0^* sin\omega t$$

$$m_2^* \ddot{x}_2 - k_2^* z + k_2^* z - c^* \dot{z} + c^* z = 0$$

which are identical to the differential equation governing the motion of the system with a damped vibration absorber. Hence the 8 kg block acts as a vibration absorber.

The program ABSORB.BAS is used to determine the parameters of the optimum damped vibration absorber using the information given. The steady-state amplitudes for z are determined at 60 rad/sec and 110 rad/sec. Both are less than 8 mm, so the design is acceptable. The output from ABSORB follows.

DAMPED VIBRATION ABSORBER ANALYSIS

PRIMARY SYSTEM PARAMETERS

Mass = 6.000E+01 kg
Stiffness = 5.000E+05 N/m
Excitation frequency = 9.000E+01 rad/sec
Excitation amplitude = 1.000E+03

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ABSORBER PARAMETERS (DIMENSIONAL)
Mass = 3.200E+01 kg
Stiffness = 1.134E+05 N/m
Damping coefficient = 1.376E+03 N-sec/m

ABSORBER PARAMETERS (NONDIMENSIONAL)
Mass = 5.333E-01
Frequency ratio = 6.522E-01
Damping ratio = 3.612E-01

STEADY-STATE AMPLITUDES AT SPECIFIED FREQUENCIES
For omega = 6.000E+01 rad/sec, X = 4.245E-03 m
For omega = 1.100E+02 rad/sec, X = 3.309E-03 m

MAXIMUM START-UP AMPLITUDE = 4.377E-03 m at OMEGA = 5.578E+01 rad/sec

The required stiffness and damping coefficients are calculated as

$$k_2 = \left(\frac{r_1}{r_2}\right)^2 k_2^* = \frac{1}{4}\left(1.134 \times 10^5 \ \frac{\mathrm{N}}{\mathrm{m}}\right) = 2.835 \times 10^4 \ \frac{\mathrm{N}}{\mathrm{m}}$$

$$c = \left(\frac{r_1}{r_2}\right)^2 c^* = \frac{1}{4}(1.37610^3 \ \mathrm{N-sec/m}) = 342 \ \mathrm{N-sec/m}$$

Problem 6.66 illustrates (a) that the differential equations for many two-degree-of-freedom systems can be put into the form of the equations derived governing the motion of the system with a damped vibration absorber and (b) the design of a damped vibration absorber.

6.67 Use the Laplace transform method to analyze the situation of an undamped absorber attached to a viscously damped system, as shown in Figure P6.67.

FIGURE P6.67

(a) Determine the steady-state amplitude of the mass m_1.

(b) Use the results of part (a) to design an absorber for a 123 kg machine of natural frequency 87 rad/sec and damping ration 0.13. Use an absorber mass of 35 kg.

Given: (a) m_1, k_1, c, m_2, k_2, F_0, ω

(b) m_1 = 123 kg, ω_{11} = 87 rad/sec, $\zeta = 0.13, m_2 = 35$ kg

Solution: (a) The differential equations governing the motion of the system are

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$$m_1\ddot{x}_1 + c\dot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_0\sin\omega t$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0 \tag{1}$$

Define

$$\bar{x}_1(s) = \mathcal{L}\{x_1(t)\}, \quad \bar{x}_2(s) = \mathcal{L}\{x_2(t)\} \tag{2}$$

Taking the Laplace transforms of eq.(1), using the definitions in eq.(2), properties of the transform, and known transform pairs leads to

$$m_1s^2X_1(s) + csX_1(s) + (k_1 + k_2)X_1(s) - k_2X_2(s) = \frac{F_0\omega}{s^2 + \omega^2}$$

$$m_2s^2X_2(s) - k_2X_1(s) + k_2X_2(s) = 0$$

or

$$(m_1s^2 + cs + k_1 + k_2)X_1(s) - k_2X_2(s) = \frac{F_0\omega}{s^2 + \omega^2}$$

$$-k_2X_1(s) + (m_2s^2 + k_2)X_2(s) = 0$$

Application of Cramer's rule leads to

$$X_1(s) = \frac{1}{D(s)}\det\begin{bmatrix} \dfrac{F_0\omega}{s^2 + \omega^2} & -k_2 \\ 0 & m_2s^2 + k_2 \end{bmatrix} = \frac{F_0\omega(m_2s^2 + k_2)}{(s^2 + \omega^2)D(s)}$$

where

$$D(s) = \det\begin{bmatrix} m_1s^2 + cs + k_1 + k_2 & -k_2 \\ -k_2 & m_2s^2 + k_2 \end{bmatrix}$$
$$= (m_1s^2 + cs + k_1 + k_2)(m_2s^2 + k_2) - k_2^2$$

Note that

$$X(s) = \frac{Us + V}{s^2 + \omega^2} + \frac{Q(s)}{D(s)}$$

$$F_0\omega(m_2s^2 + k_2) = (Us + V)D(s) + (s^2 + \omega^2)Q(s)$$

Setting s = iω in the above equation leads to

$$F_0\omega(k_2 - m_2\omega^2) = (i\omega U + V)D(i\omega)$$

$$i\omega U + V = \frac{F_0\omega(k_2 - m_2\omega^2)\bar{D}(i\omega)}{D(i\omega)\bar{D}(i\omega)}$$

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$$U = \frac{1}{\omega} Im\left[\frac{F_0\omega(k_2 - m_2\omega^2)\bar{D}(i\omega)}{D(i\omega)\bar{D}(i\omega)}\right]$$

$$V = Re\left[\frac{F_0\omega(k_2 - m_2\omega^2)\bar{D}(i\omega)}{D(i\omega)\bar{D}(i\omega)}\right]$$

The steady-state solution is obtained by inverting

$$X(s) = \frac{Us + V}{s^2 + \omega^2}$$

which leads to

$$x(t) = U\cos\omega t + \frac{V}{\omega}\sin\omega t$$

$$= A\sin(\omega t - \phi)$$

where

$$A = \sqrt{U^2 + \left(\frac{V}{\omega}\right)^2} = \frac{F_0(k_2 - m_2\omega^2)}{D(i\omega)\bar{D}(i\omega)}\sqrt{[Re(\bar{D}(i\omega))]^2 + [Im(\bar{D}(i\omega))]^2} = \frac{F_0(k_2 - m_2\omega^2)}{\sqrt{D(i\omega)\bar{D}(i\omega)}}$$

(b) Note that the steady-state amplitude of the primary system is zero

$$= \frac{F_0(k_2 - m_2\omega^2)}{\sqrt{\omega^2c^2(k_2 - m_2\omega^2)^2 + [(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2]^2}}$$

when the absorber is tuned to the primary system's excitation frequency. Assume that the primary system is being excited at a frequency near its natural frequency. Then choose

$$k_2 - m_2\omega^2 = 0$$

$$k_2 = m_2\omega^2 = (35\text{ kg})\left(87\ \frac{\text{rad}}{\text{sec}}\right)^2 = 2.65 \times 10^5\ \frac{\text{N}}{\text{m}}$$

Problem 6.67 illustrates that the steady-state vibrations of a primary system with viscous damping can be eliminated by addition of an undamped vibration absorber.

6.68 Design an undamped absorber such that the steady-state motion of the 25 kg machine component in Figure P6.68 ceases when the absorber is added. What is the steady-state amplitude of the 31 kg component?

Given: $m_1 = 25$ kg, $m_2 = 31$ kg, $k_1 = 5 \times 10^4$ N/m, $k_2 = 4 \times 10^4$ N/m, $F_0 = 200$ N, $\omega = 67$ rad/sec, $m_3 = 5$ kg

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Find: k_3, X_2

FIGURE P6.68

Solution: Steady-state vibrations of the 25 kg mass will be eliminated if the absorber is added to the 25 kg mass and tuned to the excitation frequency. The resulting three-degree-of-freedom system is shown. The absorber stiffness is calculated by

$$k_3 = m_3\omega^2 = (5\text{ kg})\left(67\ \frac{\text{rad}}{\text{sec}}\right)^2 = 2.25 \times 10^4\ \frac{\text{N}}{\text{m}}$$

The differential equations governing the motion of the three-degree-of-freedom system are

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 + k_3 & -k_2 & -k_3 \\ -k_2 & k_2 & 0 \\ -k_3 & 0 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_0 \sin\omega t \\ 0 \end{bmatrix}$$

The steady-state response is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \sin\omega t$$

Substitution of the preceding into the differential equations leads to

$$\begin{bmatrix} k_1 + k_2 + k_3 - m_1\omega^2 & -k_2 & -k_3 \\ -k_2 & k_2 - m_2\omega^2 & 0 \\ -k_3 & 0 & k_3 - m_3\omega^2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_0 \\ 0 \end{bmatrix}$$

The third of the above equations leads to $U_1 = 0$, as expected. The second equation then gives

$$U_2 = \frac{F_0}{k_2 - m_2\omega^2} = \frac{200\text{N}}{4 \times 10^4\ \frac{\text{N}}{\text{M}} - (31\text{ kg})\left(67\ \frac{\text{rad}}{\text{sec}}\right)^2} = 2.017\text{ mm}$$

Problem 6.68 illustrates (a) the use of an absorber in a two-degree-of-freedom system, (b) the use of undetermined coefficients to determine steady-state amplitudes for a two-degree-of-freedom system.

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6.69 A 300 kg compressor is placed at the end of a cantilever beam of length 1.8 m, elastic modulus 200×10^9 N/m^2 and moment of inertia 1.8×10^{-5} m^4. When the compressor operates at 1000 rpm, it has a steady-state amplitude of 1.2 mm. What is the compressor's steady-state amplitude when a 30 kg absorber of damping coefficient 500 N · s/m and stiffness 1.3×10^5 N/m is added to the end of the beam?

Given: Beam: L = 1.8 m, E = 200×10^9 N/m^2, I = 1.8×10^{-5} m^4; Compressor: m = 300 kg, $\omega = 1000$ rpm, X = 1.2 mm; Absorber: m = 30 kg, c = 500 N · s/m, k = 1.3×10^5 N/m

Find: X_1

Solution: The stiffness of the beam is

$$k = \frac{3EI}{L^3} = 1.85 \times 10^6 \text{ N/m}$$

The natural frequency of the mass attached to the beam is

$$\omega_n = \sqrt{\frac{k}{m}} = 78.57 \text{ rad/s}$$

which leads to a frequency ratio of

$$r_1 = \frac{\omega}{\omega_n} = \frac{(1000 \text{ rev/m})(2\pi \text{ r/rev})\left(\frac{1 \text{ min}}{60 \text{ s}}\right)}{78.57 \text{ rad/s}} = 1.33$$

The steady-state amplitude without the absorber is 1.2 mm. Modeling it as an undamped mass spring system the amplitude is

$$X = \frac{F_0}{k}\frac{1}{|1 - r_1^2|} \Longrightarrow F_0 = (1.85 \times 10^6 \text{ N/m})(0.0012 \text{ m})|1 - 1.33^2| = 1.73 \times 10^3 \text{ N}$$

The absorber properties are

$$\omega_{22} = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{1.3 \times 10^5 \text{ N/m}}{30 \text{ kg}}} = 65.83 \text{ rad/s}$$

$$\zeta = \frac{c}{2\sqrt{k_2 m_2}} = 0.1266$$

$$\mu = \frac{m_2}{m_1} = \frac{30 \text{ kg}}{300 \text{ kg}} = 0.1$$

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$$r_2 = \frac{\omega}{\omega_{22}} = 1.59$$

$$q = \frac{\omega_{22}}{\omega_{11}} = \frac{65.83 \text{ rad/s}}{78.57 \text{ rad/s}} = 0.8379$$

The amplitude with the absorber in place is given by Eq. (6.100) and is given by

$$\frac{k_1 X_1}{F_0} = \sqrt{\frac{(2\zeta r_1 q)^2 + (r_1^2 - q^2)^2}{\{r_1^4 - [1 + (1+\mu)q^2 r_1^2] + q^2\}^2 + (2\zeta r_1 q)^2[1 - r_1^2(1+\mu)]^2}}$$

which implies that

$$X_1 = \frac{F_0}{k_1}\sqrt{\frac{(2\zeta r_1 q)^2 + (r_1^2 - q^2)^2}{\{r_1^4 - [1 + (1+\mu)q^2 r_1^2] + q^2\}^2 + (2\zeta r_1 q)^2[1 - r_1^2(1+\mu)]^2}}$$

$$= \frac{1.73 \times 10^3 \text{ N}}{1.85 \times 10^6 \frac{\text{N}}{\text{m}}}\sqrt{\frac{[2(0.1266)(1.33)(0.8379)]^2 + [(1.33)^2 - (0.8379)^2]^2}{\{A + [2(0.1266)(1.33)(0.8379)]^2[1 - (1.33)^2](1 + 0.1)^2\}}}$$

where

$$A = [(1.33)^4 - [1 + [1 + (1 + 0.1)(0.8379)^2(1.33)^2)] + (0.8379)^2]^2$$

Then

$$X_1 = 0.69 \text{ mm}$$

Problem 6.69 illustrates the use of a damped vibration absorber.

6.70 An engine has a moment of inertia of 7.5 kg · m^2 and a natural frequency of 125 Hz. Design a Houdaille damper such that the engine's maximum magnification factor is 4.8. During operation, the engine is subject to a harmonic torque of magnitude 150 N · m at a frequency of 120 Hz. What is the engine's steady-state amplitude when the absorber is used?

Given: J_1 = 7.5 kg· m^2, f_n = 125 Hz, M_{all} = 4.8, M_0 = 150 N · m , f = 120 Hz

Find: c, J_2, X

Solution: The problem is to choose μ and ζ such that

$$4.8 < \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2}}$$

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for all r. For $\mu = 0.1$ and $\zeta = 0.1$ the magnification factor is plotted versus r

The maximum value of M is 1.8. This is an acceptable design. The moment of inertia of the Houdaille damper is

$$\mu = \frac{J_2}{J_1} \Longrightarrow J_2 = \mu J_1 = 0.1(7.5\ \mathrm{kg \cdot m^2}) = 0.75\ \mathrm{kg \cdot m^2}$$

The damping coefficient is

$$\zeta = \frac{c}{2J_2\sqrt{\frac{k}{J_1}}} = \frac{c}{2J_2\omega_{11}} \Longrightarrow c = 2\zeta J_2\omega_{11} = 2(0.1)(0.75\ \mathrm{kg \cdot m^2})\left(125\ \frac{\mathrm{cycles}}{\mathrm{s}}\right)\left(\frac{2\pi\ \mathrm{rad}}{\mathrm{cycle}}\right)$$
$$= 117.8\ \mathrm{N \cdot s \cdot m}$$

The steady-state amplitude at 120 Hz is

$$\Theta_1 = \frac{\mathrm{M_0}}{\mathrm{J_1}\omega_{11}^2}\sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2}} = \left(\frac{150\ \mathrm{N \cdot m}}{(7.5\ \mathrm{kg \cdot m^2})\left(120 \cdot 2\pi\ \frac{\mathrm{r}}{\mathrm{s}}\right)^2}\right)$$

$$\sqrt{\frac{4(0.1)^2 + \left(\frac{120}{125}\right)^2}{4(0.1)^2\left[\left(\frac{120}{125}\right)^2 + 0.1\left(\frac{120}{125}\right)^2 - 1\right]^2 + \left[\left(\frac{120}{125}\right)^2 - 1\right]^2\left(\frac{120}{125}\right)^2}}$$

$$= 7.95 \times 10^{-4}\ \mathrm{rad}$$

Problem 6.70 illustrates a Houdaille damper.

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6.71 A 200 kg machine is subjected to an excitation of magnitude 1500 N. The machine is mounted on a foundation of stiffness 2.8×10^6 N/m. What are the mass and damping ~~ratio~~ coefficients of an optimally designed vibration damper such that the maximum amplitude is 3 mm?

Given: $F_0 = 1500$ N, $k_1 = 2.8 \times 10^6$ N/m, $X_{max} = 3$ mm

Find: m_2, ζ

Solution: Choosing $\mu = 0.2$ lead to the following plot of steady-state amplitude versus frequency ratio

The maximum amplitude is 2.4 mm, less than the 3 mm allowed. Thus

$$m_2 = \mu m_1 = 0.2(200) = 40 \text{ kg}$$

and

$$\zeta = \sqrt{\frac{3\mu}{8(1+\mu)}} = \sqrt{\frac{3(0.2)}{8(1+0.2)}} = 0.25$$

Problem 6.71 illustrates the design of a damped vibration absorber.

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CHAPTER 7: MODELING OF MDOF SYSTEMS

Short Answer Questions

7.1 True: The differential equations for a linear MDOF system have the form $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}$.

7.2 False: Lagrange's equations can be used to derive the differential equations governing the motion of linear and nonlinear systems.

7.3 True: Lagrange's equation for a non-conservative system are $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_i$ for i-1,2,…,n where L is the Lagrangian and $Q_i$are the generalized forces.

7.4 False: The FBD method, when applied to a MDOF linear system, occasionally leads to symmetric mass, stiffness and damping matrices.

7.5 True: Lagrange's equations are guaranteed to give symmetric stiffness, damping and mass matrices. For a linear system, for example, the potential energy has a quadratic form $V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}\, x_i x_j$. The coefficient k_{ij} is indistinguishable from k_{ji} and therefore the stiffness matrix is symmetric (a formal proof uses Maxwell's reciprocity theorem).

7.6 True: Quadratic forms of potential and kinetic energies are used to determine the stiffness and mass matrices for linear MDOF systems.

7.7 False: A system is dynamically coupled if the mass matrix for the system is not diagonal.

7.8 True: A system may be dynamically coupled when one set of generalized coordinates is used, but not dynamically couple when another set is used.

7.9 False: The flexibility matrix is the inverse of the stiffness matrix.

7.10 False: A diagonal stiffness matrix means that $k_{ij} = 0$ for all $i \neq j$.

7.11 True: When one generalized coordinate represents a linear displacement and one generalized coordinate represents an angular displacement the elements of the mass matrix will have different dimensions.

7.12 True: The stiffness matrix is determined from potential energy and the potential energy is a function of position only.

7.13 True: Flexibility influence coefficients a_{ij} are calculated by applying a unit load at x_i and calculating the displacements at x_j.

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7.14 True: The flexibility matrix does not exist for an unrestrained system.

7.15 True: The flexibility matrix is the inverse of the stiffness matrix. If the flexibility matrix is a diagonal matrix so is the stiffness matrix.

7.16 False: Influence coefficients are used to derive the differential equations.

7.17 The equations for an undamped MDOF system are $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}$

7.18 Lagrange's equations for a conservative system are are $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$, i = 1.2…,n

7.19 A linear system is dynamically coupled with respect to a set of generalized coordinates if the mass matrix is not a diagonal matrix.

7.20 Rayleigh's dissipation function is used to generate a quadratic form for the non-conservative work done by the viscous damping forces. It is the negative of the power dissipated by the viscous dampers. It is then used in Lagrange's equations to generate the viscous damping terms.

7.21 A variation is a change in the dependent variable from its actual path to a varied path.

7.22 The method of virtual work is applied to calculate the work done by non-conservative forces.

7.23 Maxwell's reciprocity relation is that the stiffness influence coefficient k_{ij} is equal to k_{ji}. It is used to show that the stiffness matrix is symmetric.

7.24 The form of the differential equations are $\boldsymbol{A}\mathbf{M}\ddot{\mathbf{x}} + \mathbf{A}\mathbf{C}\dot{\mathbf{x}} + \mathbf{x} = \mathbf{A}\mathbf{F}$.

7.25 The system is given displacements $x_1 = 1, x_2 = 0$ and $\theta = 0$. Then forces are applied to keep the system in equilibrium at the particle whose displacement is x_1, the particle whose displacement is $x_2$and a moment is applied where θ is defined. The moment applied where θ is defined is k_{13}

7.26 A unit moment is defined acting where $\theta$is defined. No other forces or moments are applied to the system. The displacement of the particle described by x_1 is a_{13}.

7.27 A unit velocity is applied to the particle whose displacement is x_1 and no velocity applied to the particle whose displacement is x_2 or no angular velocity applied to the rotational coordinate and the system of impulses to cause this velocity calculated. The impulse that must be applied to the particle whose displacement is x_2 is m_{12}.

7.28 A unit angular velocity is applied for the rotational coordinate θ and the velocities for the particles whose displacements are x_1 and $x_2$set to zero. The system of impulses required to cause this is calculated. The impulse applied to the particle whose displacement is x_1 is m_{31}.

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7.29 Given: System of Figure SP7.29. The kinetic energy at an arbitrary instant is $T = \frac{1}{2}(10)\dot{x}_1^2 + \frac{1}{2}(4)\dot{x}_2^2 + \frac{1}{2}(8)\dot{x}_3^2$

FIGURE SP7.29

7.30 Given: System of Figure 7.29 The potential energy of the system at an arbitrary instant is $V = \frac{1}{2}(3000)x_1^2 + \frac{1}{2}(2000)(x_2 - x_1)^2 + \frac{1}{2}(600)(x_3 - x_2)^2$

7.31 Given: System of Figure SP7.29. Rayleigh's dissipation function is $\Im = -\frac{1}{2}(120)\dot{x}_1^2 - \frac{1}{2}(300)(\dot{x}_2 - \dot{x}_1)^2 - \frac{1}{2}(100)(\dot{x}_3 - \dot{x}_2)^2 - \frac{1}{2}(150)\dot{x}_3^2$

7.32 Given: $\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(2\dot{x} - \dot{y})^2\right]$ Results of the calculation yield $\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(2\dot{x} - \dot{y})^2\right] = \frac{d}{dt}(2(2\dot{x} - 3\dot{y})(2)) = 8\ddot{x} - 12\ddot{y}$.

7.33 Given: Figure of SP7.33 The virtual work done by the external forces is $\delta W = F(t)\delta x + \frac{M(t)}{r}\delta y$.

FIGURE SP7.33

7.34 Given: System of Figure SP7.34. The virtual work done by the external forces is $\delta w + F_1(t)\delta + \frac{L}{2}F_2(t)\delta\theta$. Thus $Q_1 = F_1(t)$ and $Q_2 = \frac{L}{2}F_2(t)$.

FIGURE SP7.34

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7.35 Given: $V = 5x_1^2 + 4x_1x_2 + 2x_1x_3 + 8x_2^2 + 3x_2x_2 + 6x_3^2$. The stiffness matrix is determined form the quadratic form of potential energy as

$$\mathbf{K} = \begin{bmatrix} 10 & 2 & 1 \\ 2 & 8 & 3/2 \\ 1 & 2 & 6 \end{bmatrix}$$

7.36 Given: $T = 3\left(\dot{x}_2 - \frac{1}{2}\dot{x}_1\right)^2 + 12\left(\dot{x}_2 + \frac{1}{3}\dot{x}_1\right)^2 + 4\dot{x}_3^2$. The mass matrix of the system is calculated from the quadratic form of kinetic energy as

$$M = \begin{bmatrix} 4.83 & 5 & 0 \\ 5 & 30 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

7.37 Given: Load of 50 N applied to 250 kg mass, $x_1 = 3$ mm, $x_2 = 5$ mm, $x_3 = 2.5$ mm. The flexibility influence coefficients are $a_{12} = a_{21} = \frac{0.003\text{ m}}{50\text{ N}} = 6 \times 10^{-5}\ \frac{\text{m}}{\text{N}}$, $a_{22} = \frac{0.005\text{ m}}{50\text{ N}} = 1 \times 10^{-4}\ \frac{\text{m}}{\text{N}}$, $a_{32} = a_{23} = \frac{0.0025\text{ m}}{50\text{ N}} = 5 \times 10^{-5}\ \frac{\text{m}}{\text{N}}$.

FIGURE SP7.37

7.38 Given: System of Figure SP7.38. The stiffness influence coefficients are $k_{13} = k_{31} = 0$, $k_{23} = k_{32} = \frac{100\text{ N}}{0.003\text{ m}} = 3.33 \times 10^4\ \frac{\text{N}}{\text{m}}$, $k_{33} = \frac{300\text{ N}}{0.003\text{ m}} = 1 \times 10^5\ \frac{\text{N}}{\text{m}}$.

FIGURE SP7.38

7.39 Given: System of Figure SP7.39. The system is unrestrained, thus $\det \mathbf{K} = 0$.

FIGURE SP7.39

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7.40 Given: Block A is given an impulse of 3 N · s which induces a velocity of 15 m/s and all other velocities are zero. Then $m_{11} = \frac{3\ \text{N·s}}{15\ \text{m/s}} = 0.2$ kg, $m_{12} = m_{21} = 0$ and $m_{13} = m_{13} = 0$.

FIGURE SP7.40

7.41 Given: Bar of SP7.41. Impulse and momentum diagrams are shown. Using the principle of linear impulse and momentum $m\bar{v} = -6$. Since the angular velocity is zero the inertia influence coefficient is, $m_{11} = 2$. Using angular impulse and angular momentum , $10\, m_{12} + (6\ \text{N} \cdot \text{s}) = 0 \Longrightarrow \text{m}_{12} = -0.6 \Longrightarrow \text{m}_{12} = \text{m}_{21} = -0.6$.

FIGURE SP7.41

7.42 (a)-(iv); (b)-(xv); (c)-(x); (d)-(i); (e)-(i); (f)-(v); (g)-(viii); (h)-(ii); (i)-(vi); (j)-(ix); (k)-(xi); (l)-(iii); (m)-(xiii); (n)-(vii)

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Chapter Problems

7.1 Use the free-body diagram method to derive the differential equations governing the motion of the system shown in Figure P7.1 using the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form.

FIGURE P7.1
(Problems 7.1, 7.8, 7.23, 7.30, 7.36, 7.51, 7.66)

Given: generalized coordinates x_1, x_2, x_3

Find: matrix form of differential equations

Solution: Free body diagrams of the blocks at an arbitrary time are shown below

The following conservation law is applied to each free body diagram

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$

Block A:

$$-kx_1 + 2k(x_2 - x_1) = m\ddot{x}_1$$

Block B:

$$-2k(x_2 - x_1) + k(x_3 - x_2) = m\ddot{x}_2$$

Block C:

$$-k(x_3 - x_2) - kx_3 = m\ddot{x}_3$$

The above equations are rearranged and summarized in matrix form as

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$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}+\begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.1 illustrates application of conservation laws to derive the governing differential equations for a linear three-degree-of-freedom system and the matrix formulation of the differential equations.

7.2 Use the free-body diagram to derive the differential equations governing the motion of the system shown in Figure P7.2 using the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form.

FIGURE P7.2
(Problems 7.2, 7.9, 7.31, 7.37, 7.52, 7.67)

Given: θ, x_1, x_2 as generalized coordinates

Find: matrix form of differential equations

Solution: Free body diagrams of the system at an arbitrary time are shown below where θ is assumed small.

Summing moments acting on the bar about its support,

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$$\left(\sum M_o\right)_{ext.} = \left(\sum M_o\right)_{eff.}$$

$$-kL\theta(L) + k\left(x_1 - \frac{2}{3}L\theta\right)\left(\frac{2}{3}L\right) = \frac{L}{2}m\ddot{\theta}\left(\frac{L}{2}\right) + \frac{1}{12}mL^2\ddot{\theta}$$

$$m\frac{L^2}{3}\ddot{\theta} + \frac{13}{9}kL^2\theta - \frac{2}{3}kLx_1 = 0$$

Summing forces on the first block

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$

$$-k\left(x_1 - \frac{2}{3}L\theta\right) + 2k(x_2 - x_1) = m\ddot{x}_1$$

$$m\ddot{x}_1 - \frac{2}{3}kL\theta + 3kx_1 - 2kx_2 = 0$$

Summing forces on the second block

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$

$$-2k(x_2 - x_1) = 2m\ddot{x}_2$$

$$2m\ddot{x}_2 - 2kx_1 + 2kx_2 = 0$$

The matrix formulation of the differential equations is

$$\begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{13}{9}kL^2 & -\frac{2}{3}kL & 0 \\ -\frac{2}{3}kL & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix}\begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.2 illustrates application of conservation laws to derive the differential equation for a three-degree-of-freedom system and the matrix formulation of the resulting linear differential equations.

7.3 Use the free-body diagram method to derive the differential equations governing the motion of the system shown in Figure P7.3 using the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form.

FIGURE P7.3
(Problems 7.3, 7.10, 7.24, 7.38, 7.53, 7.68)

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Given: θ, x_1, and x_2 as generalized coordinates

Find: matrix form of differential equations

Solution: Assume small θ. Free body diagrams at an arbitrary instant are shown below.

Summing moments acting on the bar about its point of support

$$\left(\sum M_o\right)_{ext.} = \left(\sum M_o\right)_{eff.}$$

$$-\frac{3}{4}kL\theta\left(\frac{3}{4}L\right) + 2k\left(x_2 - L\theta\right)\left(L\right) - k\left(x_1 - \frac{L}{2}\theta\right)\left(\frac{L}{2}\right) = m\frac{L}{2}\ddot{\theta}\left(\frac{L}{2}\right) + \frac{1}{12}mL^2\ddot{\theta}$$

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{45}{16}kL^2\theta - \frac{1}{2}kLx_1 - 2kLx_2 = 0$$

Summing forces on the first block

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$

$$-k\left(x_1 - \frac{1}{2}L\theta\right) = m\ddot{x}_1$$

$$m\ddot{x}_1 - \frac{1}{2}kL\theta + kx_1 = 0$$

Summing forces on the second block

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$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$
$$-2k\left(x_2 - L\theta\right) = 2m\ddot{x}_2$$
$$2m\ddot{x}_2 - 2kL\theta + 2kx_2 = 0$$

The matrix form of the differential equations is

$$\begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{45}{16}kL^2 & -\frac{1}{2}kL & -2k \\ -\frac{1}{2}kL & k & 0 \\ -2k & 0 & 2k \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.3 illustrates application of basic conservation laws to derive the differential equations governing the motion of a three-degree-of-freedoms system and their matrix formulation.

7.4 Use the free-body diagram method to derive the differential equation governing the motion of the system shown in Figure P7.4 using the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form.

FIGURE P7.4
(Problems 7.4, 7.11, 7.25, 7.39, 7.54, 7.69)

Given: x_1, x_2, and x_3 as generalized coordinates

Find: differential equations

Solution: Assume small displacements. Free body diagrams of the components of the system at an arbitrary instant are shown below.

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The governing differential equations are derived by applying the basic conservation laws. Summing forces acting on the bar,

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$

$$-kx_1 - k\left(x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_2\right) - kx_2 = \frac{1}{2}m\left(\ddot{x}_1 + \ddot{x}_2\right)$$

$$\frac{1}{2}m\ddot{x}_1 + \frac{1}{2}m\ddot{x}_2 + \frac{3}{2}kx_1 + \frac{3}{2}kx_2 - kx_3 = 0$$

Summing moments about the mass center of the bar,

$$\left(\sum M_G\right)_{ext.} = \left(\sum M_G\right)_{eff.}$$

$$kx_1\left(\frac{L}{2}\right) - kx_2\left(\frac{L}{2}\right) = \frac{1}{12}mL\left(\ddot{x}_2 - \ddot{x}_1\right)$$

$$-\frac{1}{12}mL\ddot{x}_1 + \frac{1}{12}mL\ddot{x}_2 - \frac{1}{2}kLx_1 + \frac{1}{2}kLx_2 = 0$$

Summing forces acting on the block,

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$

$$-k\left(x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_2\right) = m\ddot{x}_3$$

$$m\ddot{x}_3 - \frac{1}{2}kx_1 - \frac{1}{2}kx_2 + kx_3 = 0$$

The matrix formulation of the differential equations is

$$\begin{bmatrix} \frac{1}{2}m & \frac{1}{2}m & 0 \\ -\frac{1}{12}mL & \frac{1}{12}mL & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} \frac{3}{2}k & \frac{3}{2}k & -k \\ -\frac{1}{2}kL & \frac{1}{2}kL & 0 \\ -\frac{k}{2} & -\frac{k}{2} & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that neither the mass or stiffness matrix is symmetric when this formulation is used.

Problem 7.4 illustrates application of basic conservation laws to derive the differential equations governing the motion of a three-degree-of-freedom system.

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7.5 Use the free-body diagram method to derive the differential equations governing the motion of the system shown in Figure P7.5 using the generalized coordinates. Make linearizing assumptions and write the resulting differential equations in matrix form.

FIGURE P7.5
(Problems 7.5, 7.12, 7.26, 7.40, 7.55, 7.70)

Given: system shown
Find: differential equations

Solution: The gravity forces balance with the forces in the springs when the system is in equilibrium. Thus neither is shown on the free body diagrams or used in writing the differential equations. Free body diagrams of the bar and the block are shown below at an arbitrary time.

Summing forces on the free body diagrams of the bar

$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$

$$-2k(x_1 + 0.4L\theta) - k(x_1 - 0.4L\theta) - 2k(x_1 + 0.1L\theta - x_2) = m_1\ddot{x}_1$$

$$m_1\ddot{x}_1 + 5kx_1 + 0.6kL\theta - 2kx_2 = 0$$

Summing moments about the mass center of the bar

$$\left(\sum M_G\right)_{ext.} = \left(\sum M_G\right)_{eff.}$$

$$-2k(x_1 + 0.4L\theta)(0.4L) + k(x_1 - 0.4L\theta)(0.4L) - 2k(x_1 + 0.1L\theta - x_2)(0.1L) = \bar{I}\ddot{\theta}$$

$$\bar{I}\ddot{\theta} + 0.6kLx_1 + 0.5kL^2\theta - 0.2kx_2 = 0$$

Summing forces on the free body diagrams of the block

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$$\left(\sum F\right)_{ext.} = \left(\sum F\right)_{eff.}$$
$$2k\left(x_1 + 0.1L\theta - x_2\right) = m_2\ddot{x}_2$$
$$m_2\ddot{x}_2 - 2kx_1 - 0.2kL\theta + 2kx_2 = 0$$

The differential equations can be summarized in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & \bar{I} & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 5k & 0.6kL & -2k \\ 0.6kL & 0.5kL^2 & -0.2kL \\ -2k & -0.2kL & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.5 illustrates (a) derivation of governing differential equations for a three degree-of-freedom system by applying conservation laws to appropriate free body diagrams and (b) development of the matrix form of the differential equations for a linear system.

7.6 Use the free-body diagram method to derive the differential equations governing the motion of the system shown in Figure P7.6 using the generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form.

FIGURE P7.6
(Problems 7.6, 7.13, 7.41, 7.56, 7.71)

Solution: Summing forces on the FBDs below lead to

$$m\ddot{x}_1 + 3c\dot{x}_1 - 2c\dot{x}_2 + 3kx_1 - 2kx_2 = 0$$

$$3m\ddot{x}_2 - 2c\dot{x}_1 + 3c\dot{x}_2 - c\dot{x}_3 - 2kx_1 + 6kx_2 - 2kx_3 = 0$$

$$2m\ddot{x}_3 - c\dot{x}_2 + c\dot{x}_3 - 2kx_2 + 2kx_3 = 0$$

Problem 7.6 illustrates the FBD method applied to a 3DOF system.

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7.7 Use the free-body diagram method to derive the differential equations governing the motion of the system shown in Figure P7.7 using the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form.

FIGURE P7.7
(Problems 7.7, 7.14, 7.42, 7.57, 7.72)

Given: system shown

Find: differential equations in matrix form

Solution: The small angle assumption is used. Free-body diagrams of the system at an arbitrary instant are shown below.

Summing moments about A

$$\left(\sum M_A\right)_{ext} = \left(\sum M_A\right)_{eff}$$

$$k\left(x_1 - \frac{2L}{3}\theta\right)\left(\frac{2L}{3}\right) - kL\theta(L) - cL\dot{\theta}(L) = \frac{1}{12}mL^2\ddot{\theta} + m\frac{L}{2}\ddot{\theta}\left(\frac{L}{2}\right)$$

$$\frac{1}{3}mL^2\ddot{\theta} - cL^2\dot{\theta} + \frac{13}{9}kL^2\theta - \frac{2L}{3}kx_1 = 0$$

Summing forces acting on the upper block

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$

$$-k\left(x_1 - \frac{2L}{3}\theta\right) + 2k(x_2 - x_1) + 2c(\dot{x}_2 - \dot{x}_1) = m\ddot{x}_1$$

$$m\ddot{x}_1 + 2c\dot{x}_1 - 2c\dot{x}_2 - \frac{2L}{3}k\theta + 3kx_1 - 2kx_2 = 0$$

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Summing forces acting on the lower block

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$
$$-2k(x_2 - x_1) - 2c(\dot{x}_2 - \dot{x}_1) + F(t) = 2m\ddot{x}_2$$
$$2m\ddot{x}_2 - 2c\dot{x}_1 + 2c\dot{x}_2 - 2kx_1 + 2kx_2 = F(t)$$

The matrix formulation of the differential equations is

$$\begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} cL^2 & 0 & 0 \\ 0 & 2c & -2c \\ 0 & -2c & 2c \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{13}{9}kL^2 & -\frac{2L}{3}k & 0 \\ -\frac{2L}{3}k & 3k & -2k \\ 0 & -2k & k \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix}$$

Problem 7.7 illustrates the use of the free-body diagram method to derive the differential equations governing the motion of a three-degree-of-freedom system and the formulation of the differential equations in matrix form.

7.8 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.1. Use the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.1
(Problems 7.1, 7.8, 7.23, 7.30, 7.36, 7.51, 7.66)

Given: x_1, x_2, and x_3 as generalized coordinates

Find: Differential equations using Lagrange's equations, nature of coupling

Solution: Using the indicated generalized coordinates, the system's kinetic energy at an arbitrary time is

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$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$$

The potential energy at an arbitrary time is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}2k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 + \frac{1}{2}kx_3^2$$

The Lagrangian is L = T - V. Application of Lagrange's equations to this system yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}(m\dot{x}_1) + kx_1 + 2k(x_2 - x_1)(-1) = 0$$

$$m\ddot{x}_1 + 3kx_1 - 2kx_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{d}{dt}(m\dot{x}_2) + 2k(x_2 - x_1) + k(x_3 - x_2)(-1) = 0$$

$$m\ddot{x}_2 - 2kx_1 + 3kx_2 - kx_3 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} = 0$$

$$\frac{d}{dt}(m\dot{x}_3) + k(x_3 - x_2) + kx_3 = 0$$

$$m\ddot{x}_3 - kx_2 + 2kx_3 = 0$$

The matrix form of the differential equations is

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the mass matrix is a diagonal matrix the system, using these generalized coordinates, is not dynamically coupled. Since the stiffness matrix is not a diagonal matrix the system, using these generalized coordinates is statically coupled.

Problem 7.8 illustrates the use of Lagrange's equations to derive the differential equation for a three-degree-of-freedom system, the formulation of these equations in matrix form, and the nature of their coupling.

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7.9 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.2. Use the indicated generalized coordinates. Make linearizing assumptions, and write the resulting differential equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.2
(Problems 7.2, 7.9, 7.31, 7.37, 7.52, 7.67)

Given: θ, x_1 and x_2 as generalized coordinates

Find: Matrix form of differential equations, nature of coupling

Solution: Assuming small θ, the kinetic energy of the system is

$$T = \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}2m\dot{x}_2^2$$

The potential energy of the system is

$$V = \frac{1}{2}k(L\theta)^2 + \frac{1}{2}k\left(x_1 - \frac{2}{3}L\theta\right)^2 + \frac{1}{2}2k(x_2 - x_1)^2$$

The Lagrangian is L = T - V. Applying Lagrange's equations to this problem yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{1}{3}mL^2\dot{\theta}\right) + kL^2\theta + k\left(x_1 - \frac{2}{3}L\theta\right)\left(-\frac{2}{3}L\right) = 0$$

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{13}{9}kL^2\theta - \frac{2}{3}kLx_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}(m\dot{x}_1) + k\left(x_1 - \frac{2}{3}L\theta\right) + 2k(x_2 - x_1)(-1) = 0$$

$$m\ddot{x}_1 - \frac{2}{3}kL\theta + 3kx_1 - 2kx_2 = 0$$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=0$$

$$\frac{d}{dt}\left(2m\dot{x}_2\right)-2k\left(x_2-x_1\right)=0$$

$$2m\ddot{x}_2-2kx_1+2kx_2=0$$

The matrix formulation of the differential equations is

$$\begin{bmatrix}\frac{1}{3}mL^2 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & 2m\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\begin{bmatrix}\frac{13}{9}kL^2 & -\frac{2}{3}kL & 0\\ -\frac{2}{3}kL & 3k & -2k\\ 0 & -2k & 2k\end{bmatrix}\begin{bmatrix}\theta\\ x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Since the mass matrix is a diagonal matrix the system is not dynamically coupled. Since the stiffness matrix is not a diagonal matrix, the system is statically coupled.

Problem 7.9 illustrates application of Lagrange's equations to derive the differential equations for a three-degree-of-freedom system, their expression in matrix form, and the nature of their coupling.

7.10 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.3. Use the indicated generalized coordinates. Make linearizing assumptions, and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.3
(Problems 7.3, 7.10, 7.24, 7.38, 7.53, 7.68)

Given: θ, x_1, and x_2 as generalized coordinates

Find: matrix form of differential equations, nature of coupling

Solution: Assuming small θ, the kinetic energy of the system is

$$T=\frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2+\frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2+\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}2m\dot{x}_2^2$$

The system's potential energy is

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$$V = \frac{1}{2}k\left(\frac{3}{4}L\theta\right)^2 + \frac{1}{2}k\left(x_1 - \frac{1}{2}L\theta\right)^2 + \frac{1}{2}2k\left(x_2 - L\theta\right)^2$$

The Lagrangian is L = T - V. Application of Lagrange's equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{1}{3}mL^2\dot{\theta}\right) + \frac{9}{16}kL^2\theta + k\left(x_1 - \frac{1}{2}L\theta\right)\left(-\frac{L}{2}\right) + 2k\left(x_2 - L\theta\right)\left(-L\right) = 0$$

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{45}{16}kL^2\theta - \frac{1}{2}kLx_1 - 2kLx_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}\left(m\dot{x}_1\right) + k\left(x_1 - \frac{1}{2}L\theta\right) = 0$$

$$m\ddot{x}_1 - \frac{1}{2}kL\theta + kx_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{d}{dt}\left(m\dot{x}_2\right) + 2k\left(x_2 - L\theta\right) = 0$$

$$2m\ddot{x}_2 - 2kL\theta + 2kx_2 = 0$$

The matrix form of the differential equations is

$$\begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{45}{16}kL^2 & -\frac{1}{2}kL & -2k \\ -\frac{1}{2}kL & k & 0 \\ -2k & 0 & 2k \end{bmatrix}\begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the mass matrix is a diagonal matrix, the system is not dynamically coupled. Since the stiffness matrix is not a diagonal matrix, the system is statically coupled.

Problem 7.10 illustrates application of Lagrange's equations to derive the governing differential equations for a linear three-degree-of-freedom system and the nature of their coupling.

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7.11 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.4. Use the indicated generalized coordinates. Make linearizing assumptions, and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.4
(Problems 7.4, 7.11, 7.25, 7.39, 7.54, 7.69)

Given: x_1, x_2, and x_3 as generalized coordinates

Find: differential equations, nature of coupling

Solution: Assuming small displacements, the kinetic energy of the system is

$$T = \frac{1}{2}m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)^2 + \frac{1}{2}m\dot{x}_3^2$$

The potential energy of the system is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k\left(x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_2\right)^2$$

The Lagrangian is L = T - V. Application of Lagrange's equations gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}\left[\frac{m}{4}(\dot{x}_1 + \dot{x}_2) + \frac{m}{12}(\dot{x}_2 - \dot{x}_1)(-1)\right] + kx_1 + k\left(x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_2\right)\left(-\frac{1}{2}\right) = 0$$

$$\frac{1}{3}m\ddot{x}_1 + \frac{1}{6}m\ddot{x}_2 + \frac{5}{4}kx_1 + \frac{1}{4}kx_2 - \frac{1}{2}kx_3 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{d}{dt}\left[\frac{1}{4}m(\dot{x}_1 + \dot{x}_2) + \frac{1}{12}m(\dot{x}_2 - \dot{x}_1)\right] + kx_2 + k\left(x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_2\right)\left(-\frac{1}{2}\right) = 0$$

$$\frac{1}{6}m\ddot{x}_1 + \frac{1}{3}m\ddot{x}_2 + \frac{1}{4}kx_1 + \frac{5}{4}kx_2 - \frac{1}{2}kx_3 = 0$$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right)-\frac{\partial L}{\partial x_2}=0$$

$$\frac{d}{dt}(m\dot{x}_3)+k\left(x_3-\frac{1}{2}kx_1-\frac{1}{2}kx_2\right)=0$$

$$m\ddot{x}_3-\frac{1}{2}kx_1-\frac{1}{2}kx_2+kx_3=0$$

The matrix formulation of the system of equations is

$$\begin{bmatrix} \frac{1}{3}m & \frac{1}{6}m & 0 \\ \frac{1}{6}m & \frac{1}{3}m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}+\begin{bmatrix} \frac{5}{4}k & \frac{1}{4}k & -\frac{1}{2}k \\ \frac{1}{4}k & \frac{5}{4}k & -\frac{1}{2}k \\ -\frac{1}{2}k & -\frac{1}{2}k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the mass matrix is not a diagonal matrix the system is dynamically coupled. Since the stiffness matrix is not a diagonal matrix the system is statically coupled.

Problem 7.11 illustrates application of Lagrange's equations to derive the governing equations for a statically and dynamically coupled three-degree-of-freedom system.

7.12 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.5. Use the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.5
(Problems 7.5, 7.12, 7.26, 7.40, 7.55, 7.70)

Given: x_1, θ, and x_2 as generalized coordinates

Find: differential equations, nature of coupling

Solution: The kinetic energy of the system at an arbitrary time is

$$T=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}\bar{I}\dot{\theta}^2+\frac{1}{2}m_2\dot{x}_2^2$$

The potential energy of the system at an arbitrary time is

$$V=\frac{1}{2}k(x_1-0.4L\theta)^2+\frac{1}{2}(2k)(x_2-x_1-0.1L\theta)^2+\frac{1}{2}(2k)(x_1+0.4L\theta)^2$$

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The Lagrangian is

$$L = T - V = \frac{1}{2}\Big[m_1\dot{x}_1^2 + \bar{I}\dot{\theta}^2 + m_2\dot{x}_2^2 - k\left(x_1 - 0.4L\theta\right)^2$$
$$- 2k\left(x_2 - x_1 - 0.1L\theta\right)^2 - 2k\left(x_1 + 0.4kL\theta\right)^2\Big]$$

Applying Lagrange's equations for each of the generalized coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$0 = \frac{d}{dt}\left(m_1\dot{x}_1\right) + k\left(x_1 - 0.4L\theta\right) + 2k\left(x_2 - x_1 - 0.1L\theta\right)\left(-1\right) + 2k\left(x_1 + 0.4L\theta\right)$$

$$m_1\ddot{x}_1 + 5kx_1 + 0.6kL\theta - 2kx_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$0 = \frac{d}{dt}\left(\bar{I}\dot{\theta}\right) + k\left(x_1 - 0.4L\theta\right)\left(-0.4L\right)$$
$$+ 2k\left(x_2 - x_1 - 0.1L\theta\right)\left(-0.1L\right) + 2k\left(x_1 + 0.4L\theta\right)\left(0.4L\right)$$

$$\bar{I}\ddot{\theta} + 0.6kLx_1 + 0.5kL^2\theta - 0.2kLx_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$0 = \frac{d}{dt}\left(m_2\dot{x}_2\right) + 2k\left(x_2 - x_1 - 0.1L\theta\right)$$

$$m_2\ddot{x}_2 - 2kx_1 - 0.2kL\theta + 2kx_2 = 0$$

The matrix form of the differential equations is

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & \bar{I} & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 5k & 0.6kL & -2k \\ 0.6kL & 0.5kL^2 & -0.2kL \\ -2k & -0.2kL & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the mass matrix is diagonal, the differential equations are not dynamically coupled. Since the stiffness matrix is not diagonal, the differential equations are statically coupled.

Problem 7.12 illustrates application of Lagrange's equations to derive the differential equations governing the motion of a three-degree-of-freedom system.

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7.13 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.6. Use the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.6
(Problems 7.6, 7.13, 7.41, 7.56, 7.71)

Given: System shown with x_1, x_2 andx_3 as generalized coordinates

Find: Differential equations using Lagrange's equations

Solution: The kinetic energy of the system at an arbitrary instant in terms of the chosen generalized coordinates is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}3m\dot{x}_2^2 + \frac{1}{2}2m\dot{x}_3^2$$

The potential energy of the system at an arbitrary instant in terms of the chosen generalized coordinates is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}2k(x_2 - x_1)^2 + \frac{1}{2}2kx_2^2 + \frac{1}{2}2k(x_3 - x_2)^2$$

Rayleigh's dissipation functions for the problem is

$$\Im = -\frac{1}{2}c\dot{x}_1^2 - \frac{1}{2}2c(\dot{x}_2 - \dot{x}_1)^2 - \frac{1}{2}c(\dot{x}_3 - \dot{x}_2)^2$$

Application of Lagrange's equations to this system with $L = T - V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} - \frac{\partial \Im}{\partial \dot{x}_1} = 0$$

$$m\ddot{x}_1 + 3c\dot{x}_1 - 2c\dot{x}_2 + 3kx_1 - 2kx_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} - \frac{\partial \Im}{\partial \dot{x}_2} = 0$$

$$3m\ddot{x}_2 - 2c\dot{x}_1 + 3c\dot{x}_2 - c\dot{x}_3 - 2kx_1 + 6kx_2 - 2kx_3 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} - \frac{\partial \Im}{\partial \dot{x}_3} = 0$$

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$$2m\ddot{x}_3 - c\dot{x}_2 + c\dot{x}_3 - 2kx_2 + 2kx_3 = 0$$

The system is statically coupled.

Problem 7.13 illustrates use of Lagrange's equations for a damped 3DOF system.

7.14 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.7. Use the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.7
(Problems 7.7, 7.14, 7.42, 7.57, 7.72)

Given: System shown with θ, x_1 and x_2 as generalized coordinates

Find: Differential equations using Lagrange's equations

Solution: The kinetic energy of the system at an arbitrary instant using the indicated generalized coordinates is

$$T = \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}2m\dot{x}_2^2$$

The potential energy of the system at an arbitrary instant using the indicated generalized coordinates is

$$V = \frac{1}{2}k(L\theta)^2 + \frac{1}{2}k\left(x_1 - \frac{2L}{3}\theta\right)^2 + \frac{1}{2}2k(x_2 - x_1)^2$$

Rayleigh's dissipation function is

$$\Im = -\frac{1}{2}c\left(L\dot{\theta}\right)^2 - \frac{1}{2}c\left(\dot{x}_1 - \frac{2L}{3}\dot{\theta}\right)^2 - \frac{1}{2}2k(\dot{x}_2 - \dot{x}_1)^2$$

The virtual work done by external forces is

$$\delta W = F(t)\delta x_2$$

which implies

$$Q_1 = 0, Q_2 = 0, Q_3 = F(t)$$

Application of Lagrange's equations to this system with $L = T - V$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}-\frac{\partial \Im}{\partial \dot{\theta}}=Q_1$$

$$\frac{1}{3}mL^2\ddot{\theta}-cL^2\dot{\theta}+\frac{13}{9}kL^2\theta-\frac{2L}{3}kx_1=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}-\frac{\partial \Im}{\partial \dot{x}_1}=Q_2$$

$$m\ddot{x}_1+2c\dot{x}_1-2c\dot{x}_2-\frac{2L}{3}k\theta+3kx_1-2kx_2=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}-\frac{\partial \Im}{\partial \dot{x}_2}=Q_3$$

$$2m\ddot{x}_2-2c\dot{x}_1+2c\dot{x}_2-2kx_1+2kx_2=F(t)$$

The differential equations written in matrix form are

$$\begin{bmatrix}\frac{1}{3}mL^2 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & 2m\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\begin{bmatrix}cL^2 & 0 & 0\\ 0 & 2c & -2c\\ 0 & -2c & 2c\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \dot{x}_1\\ \dot{x}_2\end{bmatrix}+\begin{bmatrix}\frac{13}{9}kL^2 & -\frac{2L}{3}k & 0\\ -\frac{2L}{3}k & 3k & -2k\\ 0 & -2k & k\end{bmatrix}\begin{bmatrix}\theta\\ x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ F(t)\end{bmatrix}$$

The system is statically coupled.

Problem 7.14 illustrates application of Lagrange's equations to a damped 3DOF system.

7.15 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.15. Use the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

Thin disk of mass m and radius r rolls without slip relative to center of mass $2m$. x_D is absolute displacement of mass center of disk.

FIGURE P7.15
(Problems 7.15, 7.27, 7.32, 7.43, 7.58, 7.73)

Given: x_C and x_D as generalized coordinates

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Find: differential equations, nature of coupling

Solution: Let x_C be the displacement of the cart, measured from its equilibrium position. Let x_D be the absolute displacement of the center of the disk, measured from its equilibrium position. Assume the disk rolls without slip relative to the cart. Then the angular velocity of the disk is

$$\omega_D = \frac{\dot{x}_D - \dot{x}_C}{r}$$

The kinetic energy of the system is

$$T = \frac{1}{2}2m\dot{x}_C^2 + \frac{1}{2}m\dot{x}_D^2 + \frac{1}{2}\frac{1}{2}mr^2\left(\frac{\dot{x}_D - \dot{x}_C}{r}\right)^2$$

The potential energy of the system is

$$V = \frac{1}{2}kx_C^2 + \frac{1}{2}2k(x_D - x_C)^2 + \frac{1}{2}k(x_D - x_C)^2$$
$$= \frac{1}{2}kx_C^2 + \frac{1}{2}3k(x_D - x_C)^2$$

The Lagrangian is L = T - V. Applying Lagrange's equations to the system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_C}\right) - \frac{\partial L}{\partial x_C}$$
$$= \frac{d}{dt}\left[2m\dot{x}_C + \frac{1}{2}m(\dot{x}_D - \dot{x}_C)(-1)\right] + kx_C + 3k(x_D - x_C)(-1)$$
$$= \frac{5}{2}m\ddot{x}_C - \frac{1}{2}m\ddot{x}_C + 4kx_C - 3kx_D$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_D}\right) - \frac{\partial L}{\partial x_D}$$
$$= \frac{d}{dt}\left[m\dot{x}_D + \frac{1}{2}m(\dot{x}_D - \dot{x}_C)\right] + 3k(x_D - x_C)$$
$$= -\frac{1}{2}m\ddot{x}_C + \frac{3}{2}m\ddot{x}_D - 3kx_C - 3kx_D$$

Let δx_C and δx_D be virtual displacements. The work done as the system moves through these virtual displacements is

$$\delta W = F(t)\delta x_C$$

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Hence

$$F_1(t) = F(t), \quad F_2(t) = 0$$

The matrix form of the differential equations is

$$\begin{bmatrix} \frac{5}{2}m & -\frac{1}{2}m \\ -\frac{1}{2}m & \frac{3}{2}m \end{bmatrix} \begin{bmatrix} \ddot{x}_C \\ \ddot{x}_D \end{bmatrix} + \begin{bmatrix} 4k & -3k \\ -3k & 3k \end{bmatrix} \begin{bmatrix} x_C \\ x_D \end{bmatrix} = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}$$

Since the mass matrix is not a diagonal matrix the system is dynamically coupled. Since the stiffness matrix is not a diagonal matrix the system is statically coupled.

Problem 7.15 illustrates (a) application of Lagrange's equations to derive the differential equations governing the motion of a two-degree-of-freedom system, (b) their matrix form, and (c) the nature of their coupling.

7.16 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.16. Use the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.16
(Problems 7.16, 7.33, 7.44, 7.59, 7.74)

Given: x_C and x_D as generalized coordinates

Find: differential equations and nature of coupling

Solution: Note that x_C is the absolute displacement of the cart, measured from the system's equilibrium position, and x_D is the absolute displacement of the center of the disk from the system's equilibrium position. Assume the disk rolls without slip relative to the cart, and there is no friction between the cart and the floor.

The system has a spring, both ends of which are connected to the disk. One end of the spring is connected to the center of the disk, while its other end is connected to a point A, which in equilibrium is at the top of the disk. As the system moves, the end attached to the center of the disk has its displacement, x_D-x_C. If the disk were translating, but not rotating point A would also have a displacement of x_D-x_C. However, since the disk is rotating, point

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A moves relative to point. Since the disk rolls without slip relative to the cart, the angular displacement of point A is

$$x_A = x_D - x_C + r\frac{x_D - x_C}{r} = 2x_D - 2x_C$$

The total change in length of the spring is

$$x_S = 3x_D - 3x_C$$

The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}2m\dot{x}_C^2 + \frac{1}{2}m\dot{x}_D^2 + \frac{1}{2}\frac{1}{2}mr^2\left(\frac{\dot{x}_D - \dot{x}_C}{r}\right)^2$$

The potential energy of the system at an arbitrary time is

$$V = \frac{1}{2}kx_C^2 + \frac{1}{2}kx_C^2 + \frac{1}{2}k(x_D - x_C)^2 + \frac{1}{2}2k(3x_D - 3x_C)^2$$

The Lagrangian is

$$L = T - V = \frac{1}{2}\left[2m\dot{x}_C^2 + m\dot{x}_D^2 + \frac{1}{2}m(\dot{x}_D - \dot{x}_C)^2 - 2kx_C^2 - k(x_D - x_C)^2 - 2k(3x_D - 3x_C)^2\right]$$

Application of Lagrange's equations for each of the generalized coordinates leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_C}\right) - \frac{\partial L}{\partial x_C} = 0$$

$$0 = \frac{d}{dt}\left[2m\dot{x}_C + \frac{1}{2}m(\dot{x}_D - \dot{x}_C)(-1)\right] + 2kx_C + k(x_D - x_C)(-1) + 2k(3x_C - 3x_D)(-3)$$

$$\frac{5}{2}m\ddot{x}_C - \frac{1}{2}m\ddot{x}_D + 21kx_C - 19kx_D = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_D}\right) - \frac{\partial L}{\partial x_D} = 0$$

$$0 = \frac{d}{dt}\left[m\dot{x}_D + \frac{1}{2}m(\dot{x}_D - \dot{x}_C)\right] + k(x_D - x_C) + 2k(3x_D - 3x_C)(3)$$

$$-\frac{1}{2}\ddot{x}_C + \frac{3}{2}\ddot{x}_D - 19kx_C + 19kx_C = 0$$

The matrix form of the differential equations is

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$$\begin{bmatrix} \frac{5}{2}m & -\frac{1}{2}m \\ -\frac{1}{2}m & \frac{3}{2}m \end{bmatrix}\begin{bmatrix} \ddot{x}_C \\ \ddot{x}_C \end{bmatrix}+\begin{bmatrix} 21k & -19k \\ -19k & 19k \end{bmatrix}\begin{bmatrix} x_C \\ x_D \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the mass matrix is not a diagonal matrix, the differential equations are dynamically coupled. Since the stiffness matrix is not a diagonal matrix, the differential equations are statically coupled.

Problem 7.16 illustrates the application of Lagrange's equations to derive the differential equations governing the motion of a two-degree-of-freedom system that is both statically and dynamically coupled.

7.17 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.17. Use the indicated generalized coordinates. Make linearizing assumptions, and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.17
(Problems 7.17, 7.45, 7.60, 7.75)

Given: x_1, θ, and x_2 as generalized coordinates

Find: differential equations, nature of coupling

Solution: The kinetic energy of the system at an arbitrary instant of time is

$$T=\frac{1}{2}2m\dot{x}_1^2+\frac{1}{2}I\dot{\theta}^2+\frac{1}{2}(2m)\dot{x}_2^2+\frac{1}{2}(2I)\left(\frac{2}{r}\dot{x}_2\right)^2$$
$$=\frac{1}{2}\left[m\dot{x}_1^2+I\dot{\theta}^2+\left(2m+\frac{8}{r^2}I\right)\dot{x}_2^2\right]$$

The potential energy of the system at an arbitrary instant of time is

$$V=\frac{1}{2}k(x_1-r\theta)^2+\frac{1}{2}(2k)(2x_2+r\theta)^2$$

The Lagrangian is

$$L=T-V=\frac{1}{2}\left[m\dot{x}_1^2+I\dot{\theta}^2+\left(2m+\frac{8}{r^2}I\right)\dot{x}_2^2\right.$$
$$\left.-k(x_1-r\theta)^2-2k(2x_2+r\theta)^2\right]$$

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Applying Lagrange's equations for each of the generalized coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=0$$

$$\frac{d}{dt}\left(m\dot{x}_1\right)+k\left(x_1-r\theta\right)$$

$$m\ddot{x}_1+kx_1-kr\theta=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$

$$0=\frac{d}{dt}\left(I\dot{\theta}\right)+k\left(x_1-r\theta\right)\left(-r\right)+2k\left(2x_2+r\theta\right)\left(r\right)$$

$$I\ddot{\theta}-krx_1+3kr^2\theta+4krx_2=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=0$$

$$0=\frac{d}{dt}\left[\left(2m+\frac{8}{r^2}I\right)\dot{x}_2\right]+2k\left(2x_2+r\theta\right)\left(2\right)$$

$$\left(2m+\frac{8}{r^2}I\right)\ddot{x}_2+4kr\theta+8kx_2=0$$

The matrix form of the differential equations is

$$\begin{bmatrix} m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 2m+\frac{8}{r^2}I \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k & -kr & 0 \\ -kr & 3kr^2 & 4kr \\ 0 & 4kr & 8k \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the mass matrix is diagonal the differential equations are not dynamically coupled. Since the stiffness matrix is not diagonal, the system is statically coupled.

Problem 7.17 illustrates application of Lagrange's equations to derive the differential equations governing the motion of a conservative three-degree-of-freedom system.

7.18 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.18. Use the indicated generalized coordinates. Make linearizing assumptions, and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

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Given: x_1, θ, and x_2 as generalized coordinates

Find: differential equations, nature of coupling

Solution: The kinetic energy of the system at an arbitrary time is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{x}_2^2$$

The potential energy of the system at an arbitrary time is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_1 - r\theta)^2 + \frac{1}{2}k(x_2 - 2r\theta)^2$$

FIGURE P7.18
(Problems 7.18, 7.46, 7.61, 7.76)

The Lagrangian is

$$L = T - V = \frac{1}{2}\left[m\dot{x}_1^2 + I\dot{\theta}^2 + m\dot{x}_2^2 - kx_1^2 - k(x_1 - r\theta)^2 - k(x_2 - 2r\theta)^2\right]$$

Applying Lagrange's equations for each of the generalized coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$0 = \frac{d}{dt}(m\dot{x}_1) + kx_1 + k(x_1 - r\theta)$$

$$m\ddot{x}_1 + 2kx_1 - kr\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$0 = \frac{d}{dt}(I\dot{\theta}) + k(x_1 - r\theta)(-r) + k(x_2 - 2r\theta)(-2r)$$

$$I\ddot{\theta} - krx_1 + 5kr^2\theta - 2krx_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$0 = \frac{d}{dt}(m\dot{x}_2) + k(x_2 - 2r\theta)$$

$$m\ddot{x}_2 - 2kr\theta + kx_2 = 0$$

The matrix form of the differential equations is

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$$\begin{bmatrix} m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} 2k & -kr & 0 \\ -kr & 5kr^2 & -2kr \\ 0 & -2kr & k \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the mass matrix is not diagonal, the system is not dynamically coupled. Since the stiffness matrix is diagonal, the system is statically coupled.

Problem 7.18 illustrates application of Lagrange's equations to derive the differential equations governing the motion of a three-degree-of-freedom system.

7.19 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.19. Use the indicated generalized coordinates. Make linearizing assumptions, and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.19
(Problems 7.19, 7.28, 7.34, 7.47, 7.62, 7.77)

Given: x_1, θ, and x_2 as generalized coordinates

Find: differential equations, nature of coupling

Solution: The kinetic energy of the system at an arbitrary instant is

$$T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2+\frac{1}{2}\frac{1}{12}mL^2\left(\frac{2}{L}\dot{x}\right)^2$$

The potential energy of the system at an arbitrary instant is

$$V=\frac{1}{2}k\left(x_1-\frac{L}{2}\theta\right)^2+\frac{1}{2}k\left(x_1-\frac{L}{2}\theta\right)^2+\frac{1}{2}k\left(\frac{x_2}{2}-x_1-\frac{L}{2}\theta\right)^2$$

The Lagrangian is

$$L=T-V=\frac{1}{2}\left[m\dot{x}_1^2+\frac{1}{12}mL^2\dot{\theta}^2+\frac{m}{3}\dot{x}_2^2-k\left(x_1-\frac{L}{2}\theta\right)^2\right.$$
$$\left.-k\left(x_1+\frac{L}{2}\theta\right)^2-k\left(\frac{x_2}{2}-x_1-\frac{L}{2}\theta\right)^2\right]$$

Applying Lagrange's equations for each of the generalized coordinates yields

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=0$$

$$0=\frac{d}{dt}\left(m\dot{x}_1\right)+k\left(x_1-\frac{L}{2}\theta\right)+k\left(x_1+\frac{L}{2}\theta\right)+k\left(\frac{x_2}{2}-x_1-\frac{L}{2}\theta\right)(-1)$$

$$m\ddot{x}_1+3kx_1+k\frac{L}{2}\theta-\frac{k}{2}x_2=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$

$$0=\frac{d}{dt}\left(\frac{1}{12}mL^2\dot{\theta}\right)+k\left(x_1-\frac{L}{2}\theta\right)\left(-\frac{L}{2}\right)+k\left(x_1+\frac{L}{2}\theta\right)\left(\frac{L}{2}\right)+k\left(\frac{x_2}{2}-x_1-\frac{L}{2}\theta\right)\left(-\frac{L}{2}\right)$$

$$\frac{1}{12}mL^2\ddot{\theta}+k\frac{L}{2}x_1+3k\frac{L^2}{4}\theta-k\frac{L}{4}x_2=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=0$$

$$0=\frac{d}{dt}\left(\frac{m}{3}\dot{x}_2\right)+k\left(\frac{x_2}{2}-x_1-\frac{L}{2}\theta\right)\left(\frac{1}{2}\right)$$

$$\frac{m}{3}\ddot{x}_2-\frac{k}{2}x_1-k\frac{L}{4}\theta+\frac{k}{4}x_2=0$$

The matrix form of the differential equations is

$$\begin{bmatrix} m & 0 & 0 \\ 0 & \frac{1}{12}mL^2 & 0 \\ 0 & 0 & \frac{m}{3} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} 3k & k\frac{L}{2} & -\frac{k}{2} \\ k\frac{L}{2} & 3k\frac{L^2}{4} & -k\frac{L}{4} \\ -\frac{k}{2} & -k\frac{L}{4} & \frac{k}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the mass matrix is diagonal, the differential equations are not dynamically coupled. Since the stiffness matrix is diagonal, the differential equations are statically coupled.

Problem 7.19 illustrates application of Lagrange's equations to derive the differential equations governing the motion of a three-degree-of- freedom system.

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7.20 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.20. Use the indicated generalized coordinates. Make linearizing assumptions, and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.20
(Problems 7.20, 7.35, 7.48, 7.63, 7.78)

Given: θ_1, θ_2 and x as generalized coordinates

Find: differential equations, nature of coupling

Solution: The kinetic energy at an arbitrary instant in terms if the indicated generalized coordinates is

$$T = \frac{1}{2}m\frac{L}{3}\dot{\theta}_1^2 + \frac{1}{2}m\frac{L}{3}\dot{\theta}_2^2 + \frac{1}{2}m\dot{x}^2$$

The potential energy of the system at an arbitrary instant assuming small angular displacement is

$$V = \frac{1}{2}k\left(x - \frac{2L}{3}\theta_1\right)^2 + \frac{1}{2}k\left(\frac{2L}{3}\theta_2 - x\right)^2 + mg\frac{L}{2}(1-\cos\theta_1) + mg\frac{L}{2}(1-\cos\theta_2)$$

Assuming small angles the potential energy becomes

$$V = \frac{1}{2}k\left(x - \frac{2L}{3}\theta_1\right)^2 + \frac{1}{2}k\left(\frac{2L}{3}\theta_2 - x\right)^2 + mg\frac{L}{2}\theta_1^2 + mg\frac{L}{2}\theta_2^2$$

Application of Lagrange's equations with $L = T - V$ gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$m\frac{L^2}{3}\ddot{\theta}_1 + \left(k\frac{4L^2}{9} + mg\frac{L}{2}\right)\theta_1 - k\frac{2L}{3}x_1 + mg\frac{L}{2}\theta_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$m\frac{L^2}{3}\ddot{\theta}_2 + \left(k\frac{4L^2}{9} + mg\frac{L}{2}\right)\theta_2 - k\frac{2L}{3}x_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

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$$m\ddot{x} + 2kx - \frac{2L}{3}\theta_1 - \frac{2L}{3}\theta_2 = 0$$

The matrix form of the differential equations is

$$\begin{bmatrix} m\frac{L^2}{3} & 0 & 0 \\ 0 & m\frac{L^2}{3} & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} \frac{4kL^2}{9} + mg\frac{L}{2} & 0 & -\frac{2kL}{3} \\ 0 & \frac{4kL^2}{9} + mg\frac{L}{2} & -\frac{2kL}{3} \\ -\frac{2kL}{3} & -\frac{2kL}{3} & 2k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system is statically coupled

Problem 7.20 illustrates application of Lagrange's equations to a 3DOF system.

7.21 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.21. Use the indicated generalized coordinates. Make linearizing assumptions, and write the equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.21
(Problems 7.21, 7.49, 7.64, 7.79)

Given: θ_1, θ_2, and θ_3 as generalized coordinates

Find: differential equations, nature of coupling

Solution: The kinetic energy at an arbitrary instant in terms if the indicated generalized coordinates is

$$T = \frac{1}{2}J\dot{\theta}_1^2 + \frac{1}{2}J\dot{\theta}_2^2 + \frac{1}{2}J\dot{\theta}_3^2$$

The potential energy of the system at an arbitrary instant assuming small angular displacement is

$$V = \frac{1}{2}k_1 r^2\theta_1^2 + \frac{1}{2}k_2(r\theta_1 + r\theta_2)^2 \frac{1}{2}k_3(r\theta_2 + r\theta_3)^2 + \frac{1}{2}k_4 r^2\theta_3^2$$

Application of Lagrange's equations with $L = T - V$ gives

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)-\frac{\partial L}{\partial \theta_1}=0$$

$$J\ddot{\theta}_1+r^2(k_1+k_2)\theta_1+k_2r^2\theta_2=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\frac{\partial L}{\partial \theta_2}=0$$

$$J\ddot{\theta}_2+k_2r^2\theta_1+(k_2+k_3)r^2\theta_2+k_3r^2\theta_3=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right)-\frac{\partial L}{\partial \theta_3}=0$$

$$J\ddot{\theta}_3+k_3r^2\theta_2+(k_3+k_4)r^2\theta_3=0$$

The differential equations written in matrix form are

$$\begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix}+\begin{bmatrix} r^2k_1+r^2k_2 & r^2k_2 & 0 \\ r^2k_2 & r^2k_2+r^2k_3 & r^2k_3 \\ 0 & r^2k_3 & r^2k_3+r^2k_4 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system is statically coupled

Problem 7.21 illustrates application of Lagrange's equations to a 3DOF system.

7.22 Use Lagrange's equations to derive the differential equations governing the motion of the system shown in Figure P7.22. Use the indicated generalized coordinates. Make linearizing assumptions, and write the equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

FIGURE P7.22
(Problems 7.22, 7.29, 7.50, 7.65, 7.80)

Given: x_1, x_2 and x_3 as generalized coordinates

Find: differential equations, nature of coupling

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Solution: The kinetic energy at an arbitrary instant in terms if the indicated generalized coordinates is

$$T = \frac{1}{2}2m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2}\frac{1}{12}2mL^2\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)^2 + \frac{1}{2}m\dot{x}_3^2$$

The potential energy of the system at an arbitrary instant assuming small angular displacement is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k\left(\frac{3x_2 + x_1}{4}\right)^2 + \frac{1}{2}k(x_3 - x_2)^2 + \frac{1}{2}kx_3^2$$

Rayleigh's dissipation function is

$$\Im = -\frac{1}{2}c\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 - \frac{1}{2}c(\dot{x}_3 - \dot{x}_2)^2$$

The virtual work done by external forces is

$$\delta W = M\delta\left(\frac{x_2 - x_1}{L}\right) + F\delta x_3$$

which implies

$$Q_1 = -\frac{M}{L}, \quad Q_2 = \frac{M}{L}, \quad Q_3 = F$$

Application of Lagrange's equations with $L = T - V$ gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} - \frac{\partial \Im}{\partial \dot{x}_1} = Q_1$$

$$\frac{2m}{3}\ddot{x}_1 + \frac{m}{3}\ddot{x}_2 + \frac{1}{4}c\dot{x}_1 + \frac{1}{4}c\dot{x}_2 + \frac{17}{16}kx_1 + \frac{3}{4}kx_2 = -\frac{M}{L}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} - \frac{\partial \Im}{\partial \dot{x}_2} = Q_2$$

$$\frac{m}{3}\ddot{x}_1 + \frac{2m}{3}\ddot{x}_2 + \frac{1}{4}c\dot{x}_1 + \frac{5}{4}c\dot{x}_2 - c\dot{x}_3 + \frac{3}{4}kx_1 + \frac{25}{16}kx_2 - kx_3 = \frac{M}{L}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} - \frac{\partial \Im}{\partial \dot{x}_3} = Q_2$$

$$m\ddot{x}_3 - c\dot{x}_2 + c\dot{x}_3 - kx_2 + 2kx_3 = F$$

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The matrix form of the differential equations are

$$\begin{bmatrix} \frac{2m}{3} & \frac{m}{3} & 0 \\ \frac{m}{3} & \frac{2m}{3} & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{4}c & \frac{1}{4}c & 0 \\ \frac{1}{4}c & \frac{5}{4}c & -c \\ 0 & -c & c \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} \frac{17}{16}k & \frac{3}{4}k & 0 \\ \frac{3}{4}k & \frac{25}{16}k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{M}{L} \\ \frac{M}{L} \\ F \end{bmatrix}$$

The system is both statically coupled and dynamically coupled

Problem 7.22 illustrates the application of Lagrange's equations for a damped 3DOF system wiht external loads.

7.23 Determine the kinetic energy of the system shown at an arbitrary instant for the system of Figure P7.1. Put the kinetic energy in quadratic form. Use the quadratic form to determine the mass matrix for the system.

FIGURE P7.1
(Problems 7.1, 7.8, 7.23, 7.30, 7.36, 7.51, 7.66)

Given: system shown

Find: T, differential equations

Solution: The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$$

The kinetic energy is already in the quadratic form of Eq.(5.7). Since the kinetic energy does not contain coupling between the generalized coordinates the mass matrix id a diagonal matrix and the system is not dynamically coupled. The mass matrix is

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

Problem 7.23 illustrates the determination of the mass matrix from the quadratic form of the kinetic energy.

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7.24 Determine the kinetic energy of the system shown at an arbitrary instant for the system of Figure P7.3. Put the kinetic energy in a quadratic form. Use the quadratic form to determine the mass matrix for the system.

FIGURE P7.3
(Problems 7.3, 7.10, 7.24, 7.38, 7.53, 7.68)

Given: system shown

Find: T, differential equations

Solution: The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}\frac{mL^2}{3}\dot{\theta}^2 + \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}2m\dot{x}_2^2$$

The system's mass matrix is

$$\mathbf{M} = \begin{bmatrix} \frac{mL^2}{3} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

Problem 7.24 illustrates the use of the quadratic form of kinetic energy to determine the mass matrix of a system.

7.25 Determine the kinetic energy at an arbitrary instant for the system of Figure P7.4. Put the kinetic energy in its quadratic form. Use the quadratic form of the kinetic energy to determine the mass matrix for the system.

FIGURE P7.4
(Problems 7.4, 7.11, 7.25, 7.39, 7.54, 7.69)

Given: system shown

Find: T, differential equations

Solution: Using the small angle assumption, kinematics is used to obtain the displacement of the mass center of the bar and the angular rotation of the bar as

$$\bar{x} = \frac{1}{2}(x_1 + x_2)$$

$$\theta = \frac{1}{L}(x_2 - x_1)$$

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The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2} m\dot{\bar{x}}^2 + \frac{1}{2}\frac{1}{12} mL^2\dot{\theta}^2 + \frac{1}{2} m\dot{x}_3^2$$

$$T = \frac{1}{2}\frac{m}{4}(\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2}\frac{1}{12} m(\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2} m\dot{x}_3^2$$

The quadratic form of the kinetic energy is

$$T = \frac{1}{2}\left(\frac{m}{3}\dot{x}_1^2 + \frac{1}{3}\dot{x}_1\dot{x}_2 + \frac{m}{3}\dot{x}_2^2 + m\dot{x}_3^2\right)$$

The mass matrix is

$$\mathbf{M} = \begin{bmatrix} \frac{m}{3} & \frac{m}{6} & 0 \\ \frac{m}{6} & \frac{m}{3} & 0 \\ 0 & 0 & m \end{bmatrix}$$

Problem 7.25 illustrates the determination of the mass matrix from the quadratic form of the kinetic energy.

7.26 Determine the kinetic energy of the system at an arbitrary instant for the system of Figure P7.5. Put the kinetic energy in a quadratic form. Use the quadratic form to determine the mass matrix for the system.

FIGURE P7.5
(Problems 7.5, 7.12, 7.26, 7.40, 7.55, 7.70)

Given: system shown

Find: T, differential equations

Solution: The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2} m_1\dot{x}_1^2 + \frac{1}{2} I\dot{\theta}_2^2 + \frac{1}{2} m_2\dot{x}_2^2$$

The kinetic energy is already in the quadratic form of Eq.(5.7). Since the kinetic energy does not contain coupling between the generalized coordinates the mass matrix is a diagonal matrix and the system is not dynamically coupled. The mass matrix is

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$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & m_2 \end{bmatrix}$$

Problem 7.26 illustrates the determination of the mass matrix from the quadratic form of the kinetic energy.

7.27 Determine the kinetic energy of the system at an arbitrary instant for the system of Figure P7.15 Put the kinetic energy in a quadratic form. Use the quadratic form to determine the mass matrix for the system.

Thin disk of mass m and radius r rolls without slip relative to center of mass $2m$. x_D is absolute displacement of mass center of disk.

FIGURE P7.15
(Problems 7.15, 7.27, 7.32, 7.43, 7.58, 7.73)

Given: system shown

Find: T, differential equations

Solution: Assuming no slip between the disk and the cart, the displacement of the center of mass of the cart and the angular rotation of the disk are

$$\bar{x} = x_D$$
$$\theta = \frac{1}{r}(x_D - x_C)$$

The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\dot{x}_D^2 + \frac{1}{2}\frac{1}{2}mr^2\left(\frac{\dot{x}_D - \dot{x}_C}{r}\right)^2 + \frac{1}{2}2m\dot{x}_C^2$$

The quadratic form of the kinetic energy is

$$T = \frac{1}{2}\left(\frac{5}{2}m\dot{x}_C^2 - m\dot{x}_C\dot{x}_D + \frac{3}{2}m\dot{x}_D^2\right)$$

The mass matrix is

$$\mathbf{M} = \begin{bmatrix} \frac{5}{2}m & -\frac{1}{2}m \\ -\frac{1}{2}m & \frac{3}{2}m \end{bmatrix}$$

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Problem 7.27 illustrates the determination of the mass matrix from the quadratic form of the kinetic energy.

7.28 Determine the kinetic energy of the system shown at an arbitrary instant for the system of Figure P7.19. Put the kinetic energy in a quadratic form. Use the quadratic form to determine the mass matrix for the system.

FIGURE P7.19
(Problems 7.19, 7.28, 7.34, 7.47, 7.62, 7.77)

Given: system shown

Find: T, differential equations

Solution: The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} I\dot{\theta}_2^2 + \frac{1}{2} I\left(\frac{2}{L}\dot{x}_2\right)^2$$

The kinetic energy is already in the quadratic form. Since the kinetic energy does not contain coupling between the generalized coordinates the mass matrix is a diagonal matrix and the system is not dynamically coupled. The mass matrix is

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{4}{L^2} I \end{bmatrix}$$

Problem 7.28 illustrates the determination of the mass matrix from the quadratic form of the kinetic energy.

7.29 Determine the kinetic energy of the system at an arbitrary instant for the systems of Figures P7.22. Put the kinetic energy in a quadratic form. Use the quadratic form to determine the mass matrix for the system.

FIGURE P7.22
(Problems 7.22, 7.29, 7.50, 7.65, 7.80)

Given: system shown

Find: T, differential equations

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Solution: The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}2m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2}\frac{1}{12}2mL^2\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)^2 + \frac{1}{2}m\dot{x}_3^2$$

$$= \frac{1}{2}\frac{2m}{3}\dot{x}_1^2 + \frac{1}{2}\frac{2m}{3}\dot{x}_1\dot{x}_2 + \frac{1}{2}\frac{2m}{3}\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$$

The mass matrix is determined from the quadratic form of potential energy as

$$\mathbf{M} = \begin{bmatrix} \frac{2m}{3} & \frac{m}{3} & 0 \\ \frac{m}{3} & \frac{2m}{3} & 0 \\ 0 & 0 & m \end{bmatrix}$$

Problem 7.29 illustrates the use of the quadratic form of potential energy to determine the mass matrix for a system.

7.30 Determine the potential energy of the system at an arbitrary instant for the system of Figure P7.1. Put the potential energy in a quadratic form. Use the quadratic form to determine the stiffness matrix for the system.

FIGURE P7.1
(Problems 7.1, 7.8, 7.23, 7.30, 7.36, 7.51, 7.66)

Given: system shown

Find: V, stiffness matrix

Solution: The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}2k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 + \frac{1}{2}kx_3^2$$

The quadratic form of the potential energy is

$$V = \frac{1}{2}\left(3kx_1^2 - 4kx_1x_2 + 3kx_2^2 - 2kx_2x_3 + 2kx_3^2\right)$$

The stiffness matrix is determined from the quadratic form of the potential energy as

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$$\mathbf{K} = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 2k \end{bmatrix}$$

Problem 7.30 illustrates the use of the quadratic form of the potential energy to determine the stiffness matrix of a multi-degree-of-freedom system.

7.31 Determine the potential energy of the system at an arbitrary instant for the system of Figure P7.2. Put the potential energy in a quadratic form. Use the quadratic form to determine the stiffness matrix for the system.

FIGURE P7.2
(Problems 7.2, 7.9, 7.31, 7.37, 7.52, 7.67)

Given: system shown

Find: V, stiffness matrix

Solution: The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}kL\theta^2 + \frac{1}{2}k\left(x_1 - \frac{2L}{3}\theta\right)^2 + \frac{1}{2}2k(x_2 - x_1)^2$$

The quadratic form of the potential energy is

$$V = \frac{1}{2}\left(\frac{13}{9}kL^2\theta^2 - \frac{4L}{3}k\theta x_1 + 3kx_1^2 - 4kx_1x_2 + 2kx_2^2\right)$$

The stiffness matrix is determined from the quadratic form of the potential energy as

$$\mathbf{K} = \begin{bmatrix} \frac{13}{9}kL^2 & -\frac{2L}{3}k & 0 \\ -\frac{2L}{3}k & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix}$$

Problem 7.31 illustrates the use of the quadratic form of the potential energy to determine the stiffness matrix of a multi-degree-of-freedom system.

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7.32 Determine the potential energy of the system at an arbitrary instant for the system of Figure P7.15. Put the potential energy in a quadratic form. Use the quadratic form to determine the stiffness matrix for the system.

FIGURE P7.15
(Problems 7.15, 7.27, 7.32, 7.43, 7.58, 7.73)

Given: system shown

Find: V, stiffness matrix

Solution: The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}kx_C^2 + \frac{1}{2}2k(x_D - x_C)^2 + \frac{1}{2}k(x_D - x_C)^2$$

The quadratic form of the potential energy is

$$V = \frac{1}{2}\left(4kc_C^2 - 6kx_C x_D + 3kx_D^2\right)$$

The stiffness matrix is determined from the quadratic form of the potential energy as

$$\mathbf{K} = \begin{bmatrix} 4k & -3k \\ -3k & 3k \end{bmatrix}$$

Problem 7.32 illustrates the use of the quadratic form of the potential energy to determine the stiffness matrix of a multi-degree-of-freedom system.

7.33 Determine the potential energy of the system at an arbitrary for the systems of Figure P7.16. Put the potential energy in a quadratic form. Use the quadratic form to determine the stiffness matrix for the system.

FIGURE P7.16
(Problems 7.16, 7.33, 7.44, 7.59, 7.74)

Given: system shown

Find: V, stiffness matrix

Solution: The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}2kx_C^2 + \frac{1}{2}k(x_D - x_C)^2 + \frac{1}{2}2k\left[3(x_D - x_C)\right]^2$$

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The quadratic form of the potential energy is

$$V = \frac{1}{2}\left(21kc_C^2 - 38kx_C x_D + 19kx_D^2\right)$$

The stiffness matrix is determined from the quadratic form of the potential energy as

$$\mathbf{K} = \begin{bmatrix} 21k & -19k \\ -19k & 19k \end{bmatrix}$$

Problem 7.33 illustrates the use of the quadratic form of the potential energy to determine the stiffness matrix of a multi-degree-of-freedom system.

7.34 Determine the potential energy of the system at an arbitrary instant for the system of Figure P7.19. Put the potential energy in a quadratic form. Use the quadratic form to determine the stiffness matrix for the system.

FIGURE P7.19
(Problems 7.19, 7.28, 7.34, 7.47, 7.62, 7.77)

Given: system shown

Find: V, stiffness matrix

Solution: The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}k\left(x_1 - \frac{L}{2}\theta\right)^2 + \frac{1}{2}k\left(x_1 + \frac{L}{2}\theta\right)^2 + \frac{1}{2}k\left(\frac{1}{2}x_2 - x_1 - \frac{L}{2}\theta\right)^2$$

The quadratic form of the potential energy is

$$V = \frac{1}{2}\left(3kx_1^2 - kLx_1\theta - kx_1x_2 + \frac{3}{4}kL^2\theta^2 - \frac{1}{2}kL\theta x_2 + \frac{1}{4}kx_2^2\right)$$

The stiffness matrix is determined from the quadratic form of the potential energy as

$$\mathbf{K} = \begin{bmatrix} 3k & -\frac{1}{2}kL & -\frac{1}{2}k \\ -\frac{1}{2}kL & \frac{3}{4}kL^2 & -\frac{1}{4}kL \\ -\frac{1}{2}k & -\frac{1}{4}kL & \frac{1}{4}k \end{bmatrix}$$

Problem 7.34 illustrates the use of the quadratic form of the potential energy to determine the stiffness matrix of a multi-degree-of-freedom system.

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7.35 Determine the potential energy of the system at an arbitrary instant for the system of Figure P7.20. Put the potential energy in its quadratic form. Use the quadratic form to determine the stiffness matrix for the system.

FIGURE P7.20
(Problems 7.20, 7.35, 7.48, 7.63, 7.78)

Given: system shown

Find: V, stiffness matrix

Solution: The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}k\left(x - \frac{2L}{3}\theta_1\right)^2 + \frac{1}{2}k\left(\frac{2L}{3}\theta_2 - x\right)^2$$

$$= \frac{1}{2}\frac{4kL^2}{9}\theta_1^2 + \frac{1}{2}\frac{4kL^2}{9}\theta_2^2 + \frac{1}{2}2kx^2 - \frac{1}{2}\frac{4kL}{3}\theta_1 x - \frac{1}{2}\frac{4kL}{3}\theta_2 x$$

The stiffness matrix is determined as

$$\mathbf{K} = \begin{bmatrix} \frac{4kL^2}{9} & 0 & -\frac{2kL}{3} \\ 0 & \frac{4kL^2}{9} & -\frac{2kL}{3} \\ -\frac{2kL}{3} & -\frac{2kL}{3} & 2k \end{bmatrix}$$

Problem 7.35 illustrates the use of the quadratic form of potential energy to determine the stiffness matrix of a system.

7.36 Derive the stiffness matrix for the system of Figure P7.1 using the indicated generalized coordinates and stiffness influence coefficients.

FIGURE P7.1
(Problems 7.1, 7.8, 7.23, 7.30, 7.36, 7.51, 7.66)

Given: x_1, x_2, and x_3 as generalized coordinates.

Find: **K**

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Solution: The first column of the stiffness matrix is obtained by setting $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$ and finding the forces required to maintain this as an equilibrium position.

Summing forces to zero on each of the above diagrams leads to

$$k_{11} = 3k, \quad k_{21} = -2k, \quad k_{31} = 0$$

The second column matrix is obtained by setting $x_1 = 0$, $x_2 = 1$, and $x_3 = 0$.

Summing forces to zero on the above diagrams leads to

$$k_{12} = -2k, \quad k_{22} = 3k, \quad k_{32} = -k$$

The third column is obtained by setting $x_1 = 0$, $x_2 = 0$, and $x_3 = 1$

Summing forces to zero on the above diagrams leads to

$$k_{13} = 0, \quad k_{23} = -k, \quad k_{33} = 2k$$

Hence the stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 2k \end{bmatrix}$$

Problem 7.36 illustrates the use of stiffness influence coefficients to determine the elements of the stiffness matrix for a three-degree-of-freedom system.

FIGURE P7.2
(Problems 7.2, 7.9, 7.31, 7.37, 7.52, 7.67)

7.37 Derive the stiffness matrix for the system of Figure P7.2 using the indicated generalized coordinates and stiffness influence coefficients.

Given: θ, x_1, and x_2 as generalized coordinates

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Find: **K**

Solution: The first column of the stiffness matrix is obtained by setting $\theta = 1$, $x_1 = 0$, and $x_2 = 0$. The small angle assumption is used. The first column is the set of moments and forces necessary to maintain this as an equilibrium position.

Summing moments acting on the bar about its point of support

$$\sum M_0 = 0 = k_{11} - kL(L) - \frac{2}{3}kL\left(\frac{2}{3}L\right) \Rightarrow k_{11} = \frac{13}{9}kL^2$$

Summing forces to zero on the blocks

$$k_{21} = -\frac{2}{3}kL, \quad k_{31} = 0$$

The second column is obtained by setting $\theta = 0$, $x_1 = 1$, and $x_2 = 0$.

Summing moments acting on the bar about its point of support and summing forces on the blocks lead to

$$k_{12} = -\frac{2}{3}kL, \quad k_{22} = 3k, \quad k_{32} = -2k$$

The third column is obtained by setting $\theta = 0$, $x_1 = 0$, and $x_2 = 1$.

Application of equations of statics leads to

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$$k_{13} = 0, \quad k_{23} = -2k, \quad k_{33} = 2k$$

Thus the stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} \frac{13}{9}kL^2 & -\frac{2}{3}kL & 0 \\ -\frac{2}{3}kL & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix}$$

Problem 7.37 illustrates the use of stiffness influence coefficients to derive the stiffness matrix for a three-degree-of-freedom system where one generalized coordinate represents an angular displacement.

7.38 Derive the stiffness matrix for the system of Figure P7.3 using the indicated generalized coordinates and stiffness influence coefficients.

FIGURE P7.3
(Problems 7.3, 7.10, 7.24, 7.38, 7.53, 7.68)

Given: θ, x_1, and x_2 as generalized coordinates

Find: **K**

Solution: The first column of stiffness the stiffness matrix is obtained by setting $\theta = 1$, $x_1 = 0$, and $x_2 = 0$ and finding the system of moments and forces necessary to maintain the system in this equilibrium position.

Summing moments acting on the bar about its point of support

$$\sum M_0 = 0 = k_{11} - k\frac{L}{2}\left(\frac{L}{2}\right) - \frac{3}{4}kL\left(\frac{3}{4}L\right) - 2kL(L) \Rightarrow k_{11} = \frac{45}{16}kL^2$$

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Summing forces acting on each of the blocks to zero yields

$$k_{21} = -\frac{1}{2}kL, \quad k_{31} = -2kL$$

The second column is obtained by setting $\theta = 0$, $x_1 = 1$, and $x_2 = 0$.

Application of the laws of statics leads to

$$k_{12} = -\frac{1}{2}kL, \quad k_{22} = k, \quad k_{32} = 0$$

The third column is obtained by setting $\theta = 0$, $x_1 = 0$, and $x_2 = 1$.

Application of the laws of statics leads to

$$k_{13} = -2kL, \quad k_{23} = 0, \quad k_{33} = 2k$$

Thus the stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} \frac{45}{16}kL^2 & -\frac{1}{2}kL & -2kL \\ -\frac{1}{2}kL & k & 0 \\ -2kL & 0 & 2k \end{bmatrix}$$

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Problem 7.38 illustrates application of stiffness influence coefficients to determine the stiffness matrix for a three-degree-of-freedom linear system.

7.39 Derive the stiffness matrix for the system of Figure P7.4 using the indicated generalized coordinates and stiffness influence coefficients.

FIGURE P7.4
(Problems 7.4, 7.11, 7.25, 7.39, 7.54, 7.69)

Given: x_1, x_2, and x_3 as generalized coordinates

Find: **K**

Solution: The first column of the stiffness matrix is obtained by setting $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$. The column of stiffness influence coefficients is the system of forces necessary to maintain the system in equilibrium in this position.

Summing moments on the bar

$$\sum M_B = 0 = k_{11}L - kL - \frac{1}{2}k\left(\frac{1}{2}L\right) \Rightarrow k_{11} = \frac{5}{4}k$$

$$\sum M_A = 0 = k_{21}L - \frac{1}{2}k\left(\frac{1}{2}L\right) \Rightarrow k_{12} = \frac{1}{4}k$$

Summing forces on the block

$$\sum F = 0 = k_{31} + \frac{1}{2}k \Rightarrow k_{31} = -\frac{1}{2}k$$

The second column is obtained by setting $x_1 = 0$, $x_2 = 1$, and $x_3 = 0$.

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Summing moments on the bar

$$\sum M_B = 0 = k_{12}L - \frac{1}{2}k\left(\frac{1}{2}L\right) \Rightarrow k_{12} = \frac{1}{4}k$$

$$\sum M_A = 0 = k_{22}L - kL - \frac{1}{2}k\left(\frac{1}{2}L\right) \Rightarrow k_{22} = \frac{5}{4}k$$

Summing forces on the block

$$\sum F = 0 = k_{32} + \frac{1}{2}k \Rightarrow k_{32} = -\frac{1}{2}k$$

The third column is obtained by setting $x_1 = 0$, $x_2 = 0$, and $x_3 = 1$.

Summing moments on the bar

$$\sum M_B = 0 = k_{13}L + k\left(\frac{L}{2}\right) \Rightarrow k_{13} = -\frac{1}{2}k$$

$$\sum M_A = 0 = k_{23}L + k\left(\frac{L}{2}\right) \Rightarrow k_{23} = -\frac{1}{2}k$$

Summing forces on the block

$$\sum F = 0 = k_{33} - k \Rightarrow k_{33} = k$$

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Thus the stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} \frac{5}{4}k & \frac{1}{4}k & -\frac{1}{2}k \\ \frac{1}{4}k & \frac{5}{4}k & -\frac{1}{2}k \\ -\frac{1}{2}k & -\frac{1}{2}k & k \end{bmatrix}$$

Problem 7.39 illustrates the use of stiffness influence coefficients to determine the stiffness matrix for a three-degree-of- freedom system.

7.40 Derive the stiffness matrix for the system of Figure P7.5 using the indicated generalized coordinates and stiffness influence coefficients.

FIGURE P7.5
(Problems 7.5, 7.12, 7.26, 7.40, 7.55, 7.70)

Given: System shown

Find: **K**

Solution: The first column of the stiffness matrix is obtained by setting $x_1 = 1$, $\theta = 0$, and $x_2 = 0$ and finding the system of forces and moments necessary to maintain this configuration in equilibrium. Consider the following free body diagrams

Equilibrium equations are applied to determine the stiffness influence coefficients

$$rod: \sum F = 0 = k + 2k + 2k - k_{11},\ k_{11} = 5k$$

$$\sum M_G = 0 = k_{21} - 2k(0.4L) - 2k(0.1L) + k(0.4L),\ k_{21} = 0.6kL$$

$$block: \sum F = 0 = 2k + k_{31},\ k_{31} = -2k$$

The second column is obtained by setting $x_1 = 0$, $\theta = 1$, $x_2 = 0$.

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Application of equilibrium equations

$$rod: \sum F = 0 = 0.4kL - 0.2kL - 0.8kL + k_{12}, \quad k_{12} = 0.6kL$$
$$\sum M_G = 0 = k_{22} - 0.8kL(0.4L) - 0.2kL(0.1L) - 0.4kL(0.4L), \quad k_{22} = 0.5kL^2$$
$$block: \sum F = 0 = 0.2kL + k_{32}, \; k_{32} = -0.2kL$$

The third column is obtained by setting $x_1 = 0$, $\theta = 0$, $x_2 = 1$.

Application of

equilibrium equations

$$rod: \sum F = 0 = k_{13} + 2k, \; k_{13} = -2k$$
$$\sum M_G = 0 = k_{23} + 2k(0.1L), \; k_{23} = -0.2kL$$
$$block: \sum F = 0 = -2k + k_{33}, \; k_{33} = 2k$$

The stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} 5k & 0.6kL & -2k \\ 0.6kL & 0.5kL^2 & -0.2kL \\ -2k & -0.2kL & 2k \end{bmatrix}$$

Problem 7.40 illustrates use of stiffness influence coefficients to develop the stiffness matrix for a three-degree-of-freedom system including an angular coordinate as a generalized coordinate.

7.41 Derive the stiffness matrix for the system of Figure P7.6 using the indicated generalized coordinates and stiffness influence coefficients.

FIGURE P7.6
(Problems 7.6, 7.13, 7.41, 7.56, 7.71)

Given: x_1, x_2, and x_3 as generalized coordinates.

Find: **K**

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Solution: The first column of the stiffness matrix is obtained by setting $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$ and finding the forces required to maintain this as an equilibrium position.

Summing forces to zero on each of the above diagrams leads to

$$k_{11} = 3k, \quad k_{21} = -2k, \quad k_{31} = 0$$

The second column matrix is obtained by setting $x_1 = 0$, $x_2 = 1$, and $x_3 = 0$.

Summing forces to zero on the above diagrams leads to

$$k_{12} = -2k, \quad k_{22} = 6k, \quad k_{32} = -2k$$

The third column is obtained by setting $x_1 = 0$, $x_2 = 0$, and $x_3 = 1$

Summing forces to zero on the above diagrams leads to

$$k_{13} = 0, \quad k_{23} = -2k, \quad k_{33} = 2k$$

Hence the stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 6k & -2k \\ 0 & -2k & 2k \end{bmatrix}$$

Problem 7.41 illustrates use of stiffness influence coefficients to develop the stiffness matrix for a three-degree-of-freedom system.

7.42 Derive the stiffness matrix for the system of Figure P7.7 using the indicated generalized coordinates and stiffness influence coefficients.

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FIGURE P7.7
(Problems 7.7, 7.14, 7.42, 7.57, 7.72)

Given: system shown

Find: **K**

Solution: To determine the first column of the stiffness matrix, set $\theta_1 = 1$, $x_1 = 0$, and $x_2 = 0$. Free-body diagrams of the system in this equilibrium position are shown below.

Summing moments about the pin support leads to

$$\sum M_A = 0$$

$$-kL(L) - \frac{2L}{3}k\left(\frac{2L}{3}\right) + k_{11} = 0$$

$$k_{11} = \frac{13}{9}kL^2$$

Summing forces acting on the blocks to zero leads to

$$k_{21} = -\frac{2L}{3}k, \quad k_{31} = 0$$

The second column of the stiffness matrix is obtained by setting θ_1=0, x_1=1, and x_2=0. Free-body diagrams are shown. Summing moments about the pin support to zero and summing forces acting on the blocks to zero leads to

$$k_{12} = -\frac{2L}{3}k, \quad k_{22} = 3k, \quad k_{32} = -k$$

The third column is obtained by setting θ_1=0, x_1=0, and x_2=1. Free-body diagrams for this position are shown. Summing moments about the pin support to zero and summing forces acting on the blocks to zero leads to

$$k_{13} = 0, \quad k_{23} = -2k, \quad k_{33} = 2k$$

Hence the stiffness matrix for the system is

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$$\mathbf{K} = \begin{bmatrix} \frac{4}{9}kL^2 & -\frac{2L}{3}k & 0 \\ -\frac{2L}{3} & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix}$$

Problem 7.42 illustrates the determination of the stiffness matrix of a multi-degree-of-freedom.

7.43 Derive the stiffness matrix for the system of Figure P7.15 using the indicated generalized coordinates and stiffness influence coefficients.

Thin disk of mass m and radius r rolls without slip relative to center of mass $2m$. x_D is absolute displacement of mass center of disk.

FIGURE P7.15
(Problems 7.15, 7.27, 7.32, 7.43, 7.58, 7.73)

Given: x_C and x_D as generalized coordinates

Find: **K**

Solution: The first column of the stiffness matrix is determined by setting $x_C = 1$ and $x_D = 0$. The elements of the first column of stiffness influence coefficients are the forces required to hold the system in this position.

Summing moments about the contact point on the disk

$$\sum M_o = 0 = 2kr + kr + k_{21}r \Rightarrow k_{21} = -3k$$

Summing forces on the cart

$$\sum F = 0 = k_{11} + k_{21} - k \Rightarrow k_{11} = k + 3k = 4k$$

The second column is obtained by setting $x_C = 0$ and $x_D = 1$

Applying the equations of equilibrium

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$$\sum M_o = 0 = k_{22}r - 2kr - kr \Rightarrow k_{22} = 3k$$
$$\sum F = 0 = k_{12} + k_{22} \Rightarrow k_{12} = -3k$$

Hence the stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} 4k & -3k \\ -3k & 3k \end{bmatrix}$$

Problem 7.43 illustrates development of the stiffness matrix for a two-degree-of-freedom system from stiffness influence coefficients.

7.44 Derive the stiffness matrix for the system of Figure P7.16 using the indicated generalized coordinates and stiffness influence coefficients.

FIGURE P7.16
(Problems 7.16, 7.33, 7.44, 7.59, 7.74)

Given: x_C and x_D as generalized coordinates

Find: **K**

Solution: Note that x_C is the absolute displacement of the cart, measured from the system's equilibrium position, and x_D is the absolute displacement of the center of the disk from the system's equilibrium position. Assume the disk rolls without slip relative to the cart, and there is no friction between the cart and the floor.

The system has a spring, both ends of which are connected to the disk. One end of the spring is connected to the center of the disk, while its other end is connected to a point A, which in equilibrium is at the top of the disk. As the system moves, the end attached to the center of the disk has its displacement, x_D-x_C. If the disk were translating, but not rotating point A would also have a displacement of x_D-x_C. However, since the disk is rotating, point A moves relative to point. Since the disk rolls without slip relative to the cart, the angular displacement of point A is

$$x_A = x_D - x_C + r\frac{x_D - x_C}{r} = 2x_D - 2x_C$$

The total change in length of the spring is

$$x_s = 3x_D - 3x_C$$

The first column of the stiffness matrix is obtained by setting x_C = 1 and x_D = 0 and determining the forces applied to the cart and disk necessary to maintain the system in this position in equilibrium. Free-body diagrams of this equilibrium position are shown below.

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Summing moments acting on the disk about the point of contact between the disk and cart leads to

$$k_{21}r + 6k(2r) + 6k(r) + k(r) = 0, \quad k_{21} = -19k$$

Summing forces acting on the cart and disk assembly leads to

$$k_{21} + k_{11} = 2k, \quad k_{11} = 21k$$

The second column of the stiffness matrix is obtained by setting $x_C = 0$ and $x_D = 1$ and determining the forces applied to the cart and disk necessary to maintain the system in this position in equilibrium.

Summing moments acting on the disk about the point of contact between the disk and the cart yields

$$k_{22}(r) - 6k(2r) - 6k(r) - k(r) = 0, \quad k_{22} = 19k$$

Summing forces acting on the cart and disk assembly leads to

$$k_{12} + k_{22} = 0, k_{22} = -19k$$

Thus

$$\boldsymbol{K} = \begin{bmatrix} 21k & -19k \\ -19k & 19k \end{bmatrix}$$

Problem 7.44 illustrates the use of stiffness influence coefficients to determine the stiffness matrix of a two-degree-of-freedom statically coupled system.

7.45 Derive the stiffness matrix for the system of Figure P7.17 using the indicated generalized coordinates and stiffness influence coefficients.

Given: x_1, θ, x_2 as generalized coordinates

FIGURE P7.17
(Problems 7.17, 7.45, 7.60, 7.75)

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Find: **K**

Solution: The first column of the stiffness matrix is obtained by setting $x_1 = 1$, $\theta = 0$, $x_2 = 0$ and finding the applied system of forces and moments necessary to maintain this position in equilibrium, as illustrated by the free-body diagrams below.

Applying the equations of equilibrium to the free-body diagrams

$$block\ 1 \sum F = 0 = k_{11} - k, \quad k_{11} = k$$

$$pulley\,1 \sum M_O = 0 = k_{21} + kr, \quad k_{21} = -kr$$

$$pulley\,2 \sum M_G = 0 = k_{31}\frac{r}{2}, \quad k_{31} = 0$$

The second column of the stiffness matrix is obtained by setting $x_1 = 0$, $\theta = 1$, and $x_2 = 0$.

Applying the equations of equilibrium to the free-body diagrams yields

$$block\ 1 \sum F = 0 = k_{12} + kr, \quad k_{12} = -kr$$

$$pulley\,1 \sum M_O = 0 = k_{22} - kr(r) - 2kr(r), \quad k_{22} = 3kr^2$$

$$pulley\,2 \sum M_G = 0 = -2kr(r) + k_{32}\frac{r}{2}, \quad k_{32} = 4kr$$

The third column of the stiffness matrix is obtained by setting $x_1 = 0$, $\theta = 0$, and $x_2 = 1$.

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Applying the equations of equilibrium to the free-body diagrams gives

$$block\,1 \sum F = 0 = k_{13}, \quad k_{13} = 0$$

$$pulley\,1 \sum M_O = 0 = k_{23} - 4k(r), \quad k_{23} = 4kr$$

$$pulley\,2 \sum M_G = 0 = k_{33}\frac{r}{2} - 4kr, \quad k_{33} = 8k$$

Hence the stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} k & -kr & 0 \\ -kr & 3kr^2 & 4kr \\ 0 & 4kr & 8k \end{bmatrix}$$

Problem 7.45 illustrates the use of stiffness influence coefficients to derive the stiffness matrix for a three-degree-of-freedom system.

7.46 Derive the stiffness matrix for the system of Figure P7.18 using the indicated generalized coordinates and stiffness influence coefficients.

FIGURE P7.18
(Problems 7.18, 7.46, 7.61, 7.76)

Given: x_1, x_2, θ as generalized coordinates

Find: **K**

Solution: The first column is obtained by setting $x_1 = 1$, $x_2 = 0$, and $\theta = 0$ and finding the system of loads necessary to maintain this as an equilibrium configuration. Applying the laws of statics to the free body diagrams:

Block A:

$$\sum F = 0 \Rightarrow k_{11} = 2k$$

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Pulley:

$$\sum M_o = 0 \Rightarrow k_{31} = -rk$$

Block B:

$$\sum F = 0 \Rightarrow k_{21} = 0$$

The second column is obtained by setting $x_1 = 0$, $x_2 = 1$, and $\theta = 0$

Applying the laws of statics to the free body diagrams:

Block A:

$$\sum F = 0 \Rightarrow k_{12} = 0$$

Pulley:

$$\sum M_o = 0 \Rightarrow k_{32} = -2rk$$

Block B:

$$\sum F = 0 \Rightarrow k_{22} = k$$

The third column is obtained by setting $x_1 = 0$, $x_2 = 1$, and $\theta = 1$. Applying the laws of statics to the free body diagrams:

Block A:

$$\sum F = 0 \Rightarrow k_{13} = -rk$$

Pulley:

$$\sum M_o = 0 \Rightarrow k_{33} = 2kr(2r) + kr(r) = 5kr^2$$

Block B:

$$\sum F = 0 \Rightarrow k_{23} = -2rk$$

The stiffness matrix is

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$$\boldsymbol{K} = \begin{bmatrix} 2k & 0 & -rk \\ 0 & k & -2rk \\ -rk & -2rk & 5kr^2 \end{bmatrix}$$

Problem 7.46 illustrates development of the stiffness matrix using stiffness influence coefficients for a three-degree-of-freedom system where two generalized coordinates represent linear displacements and one generalized coordinate represents an angular displacement.

7.47 Derive the stiffness matrix for the system of Figure P7.19 using the indicated generalized coordinates and stiffness influence coefficients.

FIGURE P7.19
(Problems 7.19, 7.28, 7.34, 7.47, 7.62, 7.77)

Given: x_1, θ, and x_2 as generalized coordinates.

Find: **K**

Solution: The first column of the stiffness matrix is obtained by setting $x_1 = 1$, $\theta = 0$, and $x_2 = 0$ and determining the forces necessary to maintain the system in this position in equilibrium.

Applying the equations of static equilibrium to the system yields

$$upper \sum F = 0 = k_{11} - 3k, \quad k_{11} = 3k$$

$$upper \sum M_G = 0 = k_{21} - k\frac{L}{2} - k\frac{L}{2} + k\frac{L}{2}, \quad k_{21} = k\frac{L}{2}$$

$$lower \sum M_G = 0 = k_{31}\frac{L}{2} + k\frac{L}{4}, \quad k_{31} = -\frac{k}{2}$$

The second column is obtained by setting $x_1 = 0$, $\theta = 1$, and $x_2 = 0$

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Applying the equations of static equilibrium to the system yields

$$upper \sum F = 0 = k_{12} - k\frac{L}{2} - k\frac{L}{2} + k\frac{L}{2}, \qquad k_{12} = k\frac{L}{2}$$

$$upper \sum M_G = 0 = k_{22} - k\frac{L^2}{4} - k\frac{L^2}{4} - k\frac{L^2}{4}, \qquad k_{22} = 3k\frac{L^2}{4}$$

$$lower \sum M_G = 0 = k_{32}\frac{L}{2} + \frac{kL^2}{8}, \qquad k_{32} = -k\frac{L}{4}$$

The third column is obtained by setting $x_1 = 0$, $\theta = 0$, and $x_2 = 1$

Applying the equations of equilibrium to the free-body diagrams yields

$$upper \sum F = 0 = k_{13} + \frac{k}{2}, \qquad k_{13} = -\frac{k}{2}$$

$$upper \sum M_G = 0 = k_{23} + \frac{k}{2}\frac{L}{2}, \qquad k_{23} = -k\frac{L}{4}$$

$$lower \sum M_G = 0 = k_{33}\frac{L}{2} - \frac{k}{2}\left(\frac{L}{4}\right), \qquad k_{33} = \frac{k}{4}$$

Thus the stiffness matrix is

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$$\boldsymbol{K} = \begin{bmatrix} 3k & k\frac{L}{2} & -\frac{k}{2} \\ k\frac{L}{2} & 3k\frac{L^2}{4} & -k\frac{L}{4} \\ -\frac{k}{2} & -k\frac{L}{4} & k\frac{L}{4} \end{bmatrix}$$

Problem 7.47 illustrates application of stiffness influence coefficients to determine the stiffness matrix for a three-degree-of-freedom system.

7.48 Derive the stiffness matrix for the system of Figure P7.20 using the indicated generalized coordinates and stiffness influence coefficients.

FIGURE P7.20
(Problems 7.20, 7.35, 7.48, 7.63, 7.78)

Given: θ_1, θ_2, and x as generalized coordinates.

Find: **K**

Solution: The first column of the stiffness matrix is obtained by setting $\theta_1 = 1, \theta_2 = 0$ and $x = 0$.

Summing moments about the pin supports of the bars and summing forces on the block

$$-k\frac{2L}{3}\left(\frac{2L}{3}\right) - mg\frac{L}{2} + k_{11} = 0 \Longrightarrow k_{11} = k\frac{4L^2}{9} + \frac{mgL}{2}$$

$$k_{21} = 0$$

$$k\frac{2L}{3} + k_{31} = 0 \Longrightarrow k_{31} - k\frac{2L}{3}$$

The second column of the stiffness matrix is obtained by setting $\theta_1 = 0, \theta_2 = 1$ and $x = 0$. Summing moments about the pin supports of the bars and summing forces on the block

$$k_{12} = 0$$

$$-k\frac{2L}{3}\left(\frac{2L}{3}\right) - \frac{mgL}{2} + k_{22} = 0 \Longrightarrow k_{22} = k\frac{4L^2}{9} + \frac{mgL}{2}$$

$$k\frac{2L}{3} + k_{32} = 0 \Longrightarrow k_{32} - k\frac{2L}{3}$$

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The third column of the stiffness matrix is obtained by setting $\theta_1 = 0, \theta_2 = 0$ and $x = 1$. Summing moments about the pin supports of the bars and summing forces on the block

$$k_{13} + k\left(\frac{2L}{3}\right) = 0 \Longrightarrow k_{13} = -k\frac{2L}{3}$$

$$k\frac{2L}{3} + k_{23} = 0 \Longrightarrow k_{232} - k\frac{2L}{3}$$

$$k_{33} - k - k = 0 \Longrightarrow k_{33} = 2k$$

The stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} \frac{4kL^2}{9} + mg\frac{L}{2} & 0 & -\frac{2kL}{3} \\ 0 & \frac{4kL^2}{9} + mg\frac{L}{2} & -\frac{2kL}{3} \\ -\frac{2kL}{3} & -\frac{2kL}{3} & 2k \end{bmatrix}$$

Problem 7.48 illustrates the use of stiffness influence coefficients to calculate the stiffness matrix.

7.49 Derive the stiffness matrix for the system of Figure P7.21 using the indicated generalized coordinates and stiffness influence coefficients.

Given: θ_1, θ_2, and θ_3 as generalized coordinates.

Find: **K**

FIGURE P7.21
(Problems 7.21, 7.49, 7.64, 7.79)

Solution: The first column of the stiffness matrix is obtained by setting $\theta_1 = 1, \theta_2 = 0$ and $\theta_3 = 0$.

Summing moments about the pin supports of each disk leads to

$$k_{11} - k_1 r(r) - k_2 r(r) \Longrightarrow k_{11} = r^2 k_1 + r^2 k_2$$

$$k_{21} - k_2 r(r) \Longrightarrow k_{21} = r^2 k_2$$

$$k_{31} = 0$$

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The second column of the stiffness matrix is obtained by setting $\theta_1 = 0, \theta_2 = 1$ and $\theta_3 = 0$. Summing moments about the pin supports of each disk leads to

$$k_{12} - k_2 r(r) \Longrightarrow k_{12} = r^2 k_2$$

$$k_{22} - k_2 r(r) - k_3 r(r) \Longrightarrow k_{22} = r^2 k_2 + r^2 k_3$$

$$k_{32} - k_3 r(r) \Longrightarrow k_{32} = r^2 k_3$$

The third column of the stiffness matrix is obtained by setting $\theta_1 = 0, \theta_2 = 0$ and $\theta_3 = 1$. Summing moments about the pin supports of each disk leads to

$$k_{13} = 0$$

$$k_{23} - k_3 r(r) \Longrightarrow k_{23} = r^2 k_3$$

$$k_{33} - k_3 r(r) - k_4 r(r) \Longrightarrow k_{33} = r^2 k_3 + r^2 k_4$$

The stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} r^2 k_1 + r^2 k_2 & r^2 k_2 & 0 \\ r^2 k_2 & r^2 k_2 + r^2 k_3 & r^2 k_3 \\ 0 & r^2 k_3 & r^2 k_3 + r^2 k_4 \end{bmatrix}$$

Problem 7.49 illustrates the use of stiffness influence coefficients to calculate the stiffness matrix.

7.50 Derive the stiffness matrix for the system of Figure P7.22 using the indicated generalized coordinates and stiffness influence coefficients.

Given: x_1, x_2, and x_3 as generalized coordinates.

Find: **K**

FIGURE P7.22
(Problems 7.22, 7.29, 7.50, 7.65, 7.80)

Solution: The first column of the stiffness matrix is obtained by setting $x_1 = 1, x_2 = 0$ and $x_3 = 0$.

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Summing moments about both ends of the bar and summing forces on the block lead to

$$k_{11}L - kL - \frac{k}{4}\left(\frac{L}{4}\right) = 0 \Longrightarrow k_{11} = \frac{17}{16}k$$

$$k_{21}L - \frac{k}{4}\left(\frac{3L}{4}\right) = 0 \Longrightarrow k_{21} = \frac{3}{16}k$$

$$k_{31} = 0$$

The second column of the stiffness matrix is obtained by setting $x_1 = 0, x_2 = 1$ and $x_3 = 0$. Summing moments about both ends of the bar and summing forces on the block lead to

$$k_{12}L - \frac{3k}{4}\left(\frac{L}{4}\right) = 0 \Longrightarrow k_{12} = \frac{3}{4}k$$

$$k_{22}L - \frac{3k}{4}\left(\frac{3L}{4}\right) - kL = 0 \Longrightarrow k_{22} = \frac{25}{16}k$$

$$k_{32} + k = 0 \Longrightarrow k_{32} = -k$$

The third column of the stiffness matrix is obtained by setting $x_1 = 0, x_2 = 0$ and $x_3 = 1$. Summing moments about both ends of the bar and summing forces on the block lead to

$$k_{13}L = 0 \Longrightarrow k_{13} = 0$$

$$k_{23}L + kL = 0 \Longrightarrow k_{23} = -k$$

$$k_{33} - k - k = 0 \Longrightarrow k_{33} = 2k$$

The stiffness matrix is

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$$\mathbf{K} = \begin{bmatrix} \frac{17}{16}k & \frac{3}{4}k & 0 \\ \frac{3}{4}k & \frac{25}{16}k & -k \\ 0 & -k & 2k \end{bmatrix}$$

Problem 7.50 illustrates the application of stiffness influence coefficients to calculate a stiffness matrix.

7.51 Determine the flexibility matrix for the system of Figure P7.1 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.1
(Problems 7.1, 7.8, 7.23, 7.30, 7.36, 7.51, 7.66)

Given: system shown

Find: **A**

Solution: The first column of the flexibility matrix is determined by applying a unit load on the leftmost block and determining the resulting equilibrium position. Free-body diagrams of this position are shown below

Summing forces acting on the blocks leads to

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$$\sum F = 0$$
$$1 - ka_{11} + 2k(a_{21} - a_{11}) = 0$$
$$-2k(a_{21} - a_{11}) + k(a_{31} - a_{31}) = 0$$
$$-k(a_{31} - a_{21}) - ka_{31} = 0$$

These equations can be rewritten as

$$3ka_{11} - 2ka_{21} = 1$$
$$-2ka_{11} + 3ka_{21} - ka_{31} = 0$$
$$-ka_{21} + 2ka_{31} = 0$$

The solution of the simultaneous equations is

$$a_{11} = \frac{5}{7k}, \quad a_{21} = \frac{4}{7k}, \quad a_{31} = \frac{2}{7k}$$

The second column of the flexibility matrix is obtained by applying a unit load to the middle block. Free-body diagrams for this equilibrium position are shown. Summing forces on the blocks to zero leads to

$$3ka_{12} - 2ka_{22} = 0$$
$$-2ka_{12} + 3ka_{22} - ka_{32} = 1$$
$$-ka_{22} + 2ka_{32} = 0$$

The solution of the simultaneous equations is

$$a_{12} = \frac{4}{7k}, \quad a_{22} = \frac{6}{7k}, \quad a_{32} = \frac{3}{7k}$$

The third column of the flexibility matrix is obtained by applying a unit load to the rightmost block. Free-body diagrams illustrating this equilibrium position are shown. Summing forces to zero on these free-body diagrams leads to

$$3ka_{13} - 2ka_{23} = 0$$
$$-2ka_{13} + 3ka_{23} - ka_{33} = 0$$
$$-ka_{23} + 2ka_{33} = 1$$

The solution of these simultaneous equations is

$$a_{13} = \frac{2}{7k}, \quad a_{23} = \frac{3}{7k}, \quad a_{33} = \frac{5}{7k}$$

The flexibility matrix is

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$$\mathbf{A} = \frac{1}{7k}\begin{bmatrix} 5 & 4 & 2 \\ 4 & 6 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Problem 7.51 illustrates determination of the flexibility matrix for a three-degree-of-freedom system using flexibility influence coefficients.

7.52 Determine the flexibility matrix for the system of Figure P7.2 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.2
(Problems 7.2, 7.9, 7.31, 7.37, 7.52, 7.67)

Given: x_1, x_2, and θ as generalized coordinates

Find: **A**

Solution: The flexibility matrix is obtained by applying unit forces or moments to locations whose displacement or rotation is described by generalized coordinates. The flexibility influence coefficients are the resulting static displacements. The first column of the flexibility matrix is obtained by applying a clockwise unit moment of the bar. Free-body diagrams of the resulting equilibrium positions are shown below.

Application of the principles of static equilibrium leads to

$$a_{11} = \theta = \frac{16}{9kL^2}, \quad a_{21} = x_1 = \frac{8}{9kL}, \quad a_{31} = x_2 = \frac{16}{9kL}$$

The second column is obtained by applying a downward unit force to the block whose displacement is described by x_1. Free-body diagrams of the resulting equilibrium position are shown below.

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Application of the principles of static equilibrium leads to

$$a_{12} = \theta = \frac{8}{9kL}, \quad a_{22} = x_1 = \frac{13}{9k}, \quad a_{32} = x_2 = \frac{8}{9k}$$

The third column is obtained by applying a downward unit force to the block whose displacement is described by x_2.

Application of the principles of static equilibrium leads to

$$a_{13} = \theta = \frac{16}{9kL}, \quad a_{23} = x_1 = \frac{8}{9k}, \quad a_{33} = x_2 = \frac{41}{18k}$$

Hence

$$\boldsymbol{A} = \begin{bmatrix} \frac{16}{kL^2} & \frac{8}{9kL} & \frac{16}{9kL} \\ \frac{8}{9kL} & \frac{13}{9k} & \frac{8}{9k} \\ \frac{16}{9kL} & \frac{8}{9k} & \frac{41}{18k} \end{bmatrix}$$

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Problem 7.52 illustrates the use of flexibility influence coefficients to derive the flexibility matrix of a three-degree-of-freedom system in which one of the generalized coordinates is an angular displacement.

7.53 Determine the flexibility matrix for the system of Figure P7.3 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.3
(Problems 7.3, 7.10, 7.24, 7.38, 7.53, 7.68)

Given: x_1, θ, and x_2 as generalized coordinates

Find: **A**

Solution: Consider the static equilibrium position of the system when it is subject to a force f_1 applied at the mass center of the bar, a moment m applied clockwise to the bar, and a force f_2 applied to the block.

Summing forces on the bar

$$\sum F = 0 = f_1 + 2k(x_2 - x_1 - 0.1L\theta) - k(x_1 - 0.4L\theta) - 2k(x_1 + 0.4L\theta)$$
$$5kx_1 - 2kx_2 + 0.6kL\theta = f_1$$

Summing moments on the bar

$$\sum M_G = 0 = m - 2k(x_1 + 0.4L\theta)(0.4L) + 2k(x_2 - x_1 - 0.1L\theta)(0.1L) + k(x_1 - 0.4L\theta)(0.4L)$$
$$0.6kLx_1 - 0.2kLx_2 + 0.5kL^2\theta = m$$

Summing forces on the block

$$\sum F = 0 = f_2 - 2k(x_2 - x_1 - 0.1L\theta)$$
$$-2kx_1 - 0.2kL\theta + 2kx_2 = f_2$$

The first column of the flexibility matrix is found by setting $f_1 = 1$. $m = 0$, and $f_2 = 0$. The solutions correspond to $a_{11} = x_1$, $a_{21} = x_2$, $a_{13} = \theta$. Simultaneous solution of these equations yield

$$a_{11} = \frac{0.375}{k}, \quad a_{21} = \frac{0.344}{k}, \quad a_{31} = -\frac{0.313}{kL}$$

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The second column of the flexibility matrix is found by setting $f_1 = 0$, $m = 0$, and $f_2 = 1$. The solutions correspond to $a_{12} = x_1$, $a_{22} = x_2$, $a_{32} = \theta$. Simultaneous solution of these equations yield

$$a_{12} = \frac{0.343}{k}, \quad a_{22} = \frac{0.836}{k}, \quad a_{32} = -\frac{0.078}{kL}$$

The third column of the flexibility matrix is found by setting $f_1 = 0$, $m = 1$ and $f_2 = 0$. The solutions correspond to $a_{13} = x_1$, $a_{23} = x_2$, and $a_{33} = \theta$. Simultaneous solution these equations yield

$$a_{13} = \frac{0.625}{kL}, \quad a_{23} = \frac{-0.078}{kL}, \quad a_{33} = \frac{2.34}{kL^2}$$

Thus the flexibility matrix is

$$\boldsymbol{A} = \frac{1}{kL^2}\begin{bmatrix} 0.375L^2 & 0.344L^2 & -0.313L \\ 0.344L^2 & 0.836L^2 & -0.078L \\ -0.313L & -0.078L & 2.34 \end{bmatrix}$$

Problem 7.53 illustrates calculation of the flexibility matrix for a three-degree-of-freedom mechanical system using flexibility influence coefficients.

7.54 Determine the flexibility matrix for the system of Figure P7.4 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.4
(Problems 7.4, 7.11, 7.25, 7.39, 7.54, 7.69)

Given: x_1, x_2 and x_3 as generalized coordinates

Find: **A**

Solution: Consider an arbitrary deflected shape when the system is subject to static loads f_1 applied at the left end of the bar, f_2 applied at right left end of the bar and f_3 applied to the block.

Summing moments about each end of the bar and summing forces on the block leads to

$$\sum M_B = 0 = f_1 L - kx_1 L + k\left(x_3 - \frac{x_1 + x_2}{2}\right)\frac{L}{2} = 0$$

$$\sum M_A = 0 = f_2 L - kx_2 L + k\left(x_3 - \frac{x_1 + x_2}{2}\right)\frac{L}{2} = 0$$

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$$\sum F = 0 = f_3 - k\left(x_3 - \frac{x_1 + x_2}{2}\right)$$

Set $f_1 = 1, f_2 = 0, f_3 = 0$. Then $x_1 = a_{11}, x_2 = a_{21}, x_3 = a_{31}$. The simultaneous equations are solved yielding $a_{11} = \frac{1}{k}, a_{21} = 0, a_{31} = \frac{1}{2k}$

Set $f_1 = 0, f_2 = 1, f_3 = 0$. Then $x_1 = a_{12}, x_2 = a_{22}, x_3 = a_{32}$. The simultaneous equations are solved yielding $a_{12} = 0, a_{22} = \frac{1}{k}, a_{33} = \frac{1}{2k}$

Set $f_1 = 0, f_2 = 0, f_3 = 1$. Then $x_1 = a_{13}, x_2 = a_{23}, x_3 = a_{33}$. The simultaneous equations are solved yielding $a_{13} = \frac{1}{2k}, a_{23} = \frac{1}{2k}, a_{33} = \frac{3}{2k}$

The flexibility matrix is

$$\mathbf{A} = \frac{1}{k}\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Problem 7.54 illustrates the use of flexibility influence coefficients.

7.55 Determine the flexibility matrix for the system of Figure P7.5 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.5
(Problems 7.5, 7.12, 7.26, 7.40, 7.55, 7.70)

Given: x_1, θ and x_2 as generalized coordinates

Find: **A**

Solution: Consider an arbitrary deflected shape when the system is subject to static loads f_1 applied at the mass center of the bar, a moment m applied clockwise to the bar and f_2 applied to the block.

Summing moments about the mass center of the bar summing forces on the bar and the block

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$$\sum M_G = 0 = m - 2k(x_1 + 0.3L\theta)(0.3L) + k(x_1 - 0.4L\theta)(0.4L) + 2k[x_2 - (x_1 + 0.1L\theta)](0.1L) = 0$$

$$\sum F = 0 = f_1 - k(x_1 - 0.4L\theta) \quad + 2k[x_2 - (x_1 + 0.1L\theta)] - 2k(x_1 + 0.3L\theta) = 0$$

$$\sum F = 0 = f_3 - 2k[x_2 - (x_1 + 0.1L\theta)]$$

Set $f_1 = 1, m = 0, f_2 = 0$. Then $x_1 = a_{11}, \theta = a_{21}, x_2 = a_{31}$. The simultaneous equations are solved yielding $a_{11} = \frac{0.3367}{k}, a_{21} = \frac{0.102}{Lk}, a_{31} = \frac{0.3464}{k}$

Set $f_1 = 0, m = 1, f_2 = 0$. Then $x_1 = a_{12}, \theta = a_{22}, x_2 = a_{32}$. The simultaneous equations are solved yielding $a_{12} = \frac{0.102}{Lk}, a_{22} = \frac{3.061}{L^2k}0, a_{32} = \frac{0.4082}{Lk}$

Set $f_1 = 0, m = 0, f_2 = 1$. Then $x_1 = a_{13}, \theta = a_{23}, x_2 = a_{33}$. The simultaneous equations are solved yielding $a_{13} = \frac{0.3464}{k}, a_{23} = \frac{0.4082}{Lk}0, a_{33} = \frac{0.8878}{k}$

The flexibility matrix is

$$\mathbf{A} = \frac{1}{k}\begin{bmatrix} 0.3367 & \frac{0.102}{L} & 0.3464 \\ \frac{0.102}{L} & \frac{3.061}{L^2} & \frac{0.4082}{L} \\ 0.3464 & \frac{0.4082}{L} & 0.8878 \end{bmatrix}$$

Problem 7.55 illustrates the use of flexibility influence coefficients.

7.56 Determine the flexibility matrix for the system of Figure P7.6 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.6
(Problems 7.6, 7.13, 7.41, 7.56, 7.71)

Given: x_1, x_2 and x_3 as generalized coordinates

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Find: **A**

Solution: Consider an arbitrary deflected shape when the system is subject to static loads f_1 applied to the block whose displacement is x_1, f_2 applied to the block whose displacement is x_2 and f_3 applied to the block whose displacement is x_3.

Summing forces on each block leads to

$$\sum F = 0 = f_1 - kx_1 + 2k(x_2 - x_1) = 0$$

$$\sum F = 0 = f_2 - 2kx_2 - 2k(x_2 - x_1) + 2k(x_3 - x_2) = 0$$

$$\sum F = 0 = f_3 - 2k(x_3 - x_2)$$

Set $f_1 = 1, f_2 = 0, f_3 = 0$. Then $x_1 = a_{11}, x_2 = a_{21}, x_3 = a_{31}$. The simultaneous equations are solved yielding $a_{11} = \frac{1}{2k}, a_{21} = \frac{1}{4k}, a_{31} = \frac{1}{4k}$

Set $f_1 = 0, f_2 = 1, f_3 = 0$. Then $x_1 = a_{12}, x_2 = a_{22}, x_3 = a_{32}$. The simultaneous equations are solved yielding $a_{12} = \frac{1}{4k}, a_{22} = \frac{3}{8k}, a_{32} = \frac{3}{8k}$

Set $f_1 = 0, f_2 = 0, f_3 = 1$. Then $x_1 = a_{13}, x_2 = a_{23}, x_3 = a_{33}$. The simultaneous equations are solved yielding $a_{13} = \frac{1}{4k}, a_{21} = \frac{3}{8k}, a_{31} = \frac{7}{8k}$

The flexibility matrix is

$$\mathbf{A} = \frac{1}{k}\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{3}{8} & \frac{7}{8} \end{bmatrix}$$

Problem 7.56 illustrates the use of flexibility influence coefficients.

FIGURE P7.7
(Problems 7.7, 7.14, 7.42, 7.57, 7.72)

7.57 Determine the flexibility matrix for the system of Figure P7.7 using the indicated generalized coordinates and flexibility influence coefficients.

Given: x_1, x_2 and x_3 as generalized coordinates

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Find: **A**

Solution: Consider an arbitrary deflected shape when the system is subject to static loads m applied clockwise to the bar, f_1 applied to the block whose displacement is x_1 and f_2 applied to the block whose displacement is x_2.

Summing moments about the pin support of the bar and summing forces on each block leads to

$$\sum M_O = 0 = m - kL\theta(L) + k\left(x_1 - \frac{2L}{3}\theta\right)\frac{2L}{3} = 0$$

$$\sum F = 0 = f_1 - k\left(x_1 - \frac{2L}{3}\theta\right) + 2k(x_2 - x_1) = 0$$

$$\sum F = 0 = f_2 - 2k(x_2 - x_1)$$

Set m= $1, f_1 = 0, f_2 = 0$. Then $\theta = a_{11}, x_1 = a_{21}, x_2 = a_{31}$. The simultaneous equations are solved yielding $a_{11} = \frac{1}{L^2 k}, a_{21} = \frac{2}{3Lk}, a_{31} = \frac{2}{3Lk}$

Set m= $0, f_1 = 1, f_2 = 0$. Then $\theta = a_{12}, x_1 = a_{22}, x_2 = a_{32}$. The simultaneous equations are solved yielding $a_{12} = \frac{2}{3Lk}, a_{22} = \frac{13}{9k}, a_{32} = \frac{13}{9k}$

Set m= $0, f_1 = 0, f_2 = 1$. Then $\theta = a_{13}, x_1 = a_{23}, x_2 = a_{33}$. The simultaneous equations are solved yielding $a_{13} = \frac{2}{3Lk}, a_{23} = \frac{13}{9k}, a_{33} = \frac{35}{18k}$

The flexibility matrix is

$$\mathbf{A} = \frac{1}{k}\begin{bmatrix} \frac{1}{L^2} & \frac{2}{3L} & \frac{2}{3L} \\ \frac{2}{3L} & \frac{13}{9} & \frac{13}{9} \\ \frac{2}{3L} & \frac{13}{9} & \frac{35}{18} \end{bmatrix}$$

Problem 7.57 illustrates the use of flexibility influence coefficients.

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7.58 Determine the flexibility matrix for the system of Figure P7.15 using the indicated generalized coordinates and flexibility influence coefficients.

Thin disk of mass m and radius r rolls without slip relative to center of mass $2m$. x_D is absolute displacement of mass center of disk.

FIGURE P7.15
(Problems 7.15, 7.27, 7.32, 7.43, 7.58, 7.73)

Given: x_1, x_2 and x_3 as generalized coordinates

Find: **A**

Solution: Consider an arbitrary deflected shape when the system is subject to static loads f_1 applied to the cart and f_2 applied to the disk. Since there is no slip between the cart and the disk, the forces developed in the spring attached to the cart and the disk are $x_D - x_C$.

Summing forces acting on the cart to zero yields

$$f_1 - kx_C + 2k(x_D - x_C) + k(x_D - x_C) = 0$$

Summing forces on the disk leads to

$$f_2 - 2k(x_D - x_C) - k(x_D - x_C) = 0$$

The first column of the flexibility matrix is obtained by setting $f_1 = 1, f_2 = 0$ and $x_C = a_{11}, x_D = a_{21}$The equations are solved to yield $a_{11} = \frac{1}{k}, a_{21} = \frac{1}{k}$. The second column of the flexibility matrix is obtained by setting $f_1 = 0, f_2 = 1$ and $x_C = a_{12}, x_D = a_{22}$The equations are solved to yield $a_{12} = \frac{1}{k}, a_{22} = \frac{41}{3k}$. Thus the flexibility matrix is

$$\mathbf{A} = \frac{1}{k}\begin{bmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{bmatrix}$$

Problem 7.58 illustrates the use of flexibility influence coefficients.

7.59 Determine the flexibility matrix for the system of Figure P7.16 using the indicated generalized coordinates and flexibility influence coefficients.

Thin disk of mass m and radius r rolls without slip relative to center of cart. x_D is absolute displacement of mass center of disk.

FIGURE P7.16
(Problems 7.16, 7.33, 7.44, 7.59, 7.74)

Given: x_1, x_2 and x_3 as generalized coordinates

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Find: **A**

Solution: The system has a spring, both ends of which are connected to the disk. One end of the spring is connected to the center of the disk, while its other end is connected to a point A, which in equilibrium is at the top of the disk. As the system moves, the end attached to the center of the disk has its displacement, x_D-x_C. If the disk were translating, but not rotating point A would also have a displacement of x_D-x_C. However, since the disk is rotating, point A moves relative to point. Since the disk rolls without slip relative to the cart, the angular displacement of point A is

$$x_A = x_D - x_C + r\frac{x_D - x_C}{r} = 2x_D - 2x_C$$

Consider an arbitrary deflected shape when the system is subject to static loads f_1 applied to the cart and f_2 applied to the disk. Free body diagrams of the static position are shown.

Summing forces on the cart gives

$$f_1 + f_2 - kx_C - kx_C = 0$$

Summing moments about the point of contact between the cart and the disk leads to

$$f_2 r + 6k(x_C - x_D)(2r) + 6k(x_C - x_D)r - k(x_D - x_C)r = 0$$

Set $f_1 = 1, f_2 = 0$. Then $x_C = a_{11}, x_D = a_{21}$. The simultaneous equations are solved yielding $a_{11} = \frac{1}{2k}, a_{21} = \frac{1}{2k}$

Set $f_1 = 0, f_2 = 1,$. Then $x_C = a_{12}, x_D = a_{22}$. The simultaneous equations are solved yielding $a_{12} = \frac{1}{2k}, a_{22} = \frac{21}{38k}$

The flexibility matrix is

$$\mathbf{A} = \frac{1}{2k}\begin{bmatrix} 1 & 1 \\ 1 & \frac{21}{19} \end{bmatrix}$$

Problem 7.59 illustrates the use of flexibility influence coefficients.

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7.60 Determine the flexibility matrix for the system of Figure P7.17 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.17
(Problems 7.17, 7.45, 7.60, 7.75)

Given: x_1, θ and x_2 as generalized coordinates

Find: **A**

Solution: Consider an arbitrary deflected shape when the system is subject to static loads f_1 applied to the block whose displacement is x_1, m is a counterclockwise moment applied to the leftmost pulley and, f_2 applied to the block whose displacement is x_2.

Summing moments about the center of each pulley and summing forces on the block lead to

$$\sum F = 0 = f_1 - k(x_1 - r\theta) = 0$$

$$\sum M_A = 0 = m + k(x_1 - r\theta)r - 2k(2x_2 + r\theta)r = 0$$

$$\sum M_B = 0 = f_2\frac{r}{2} - 2k(2x_2 + r\theta)r$$

Set $f_1 = 1, m = 0, f_2 = 0$. Then $x_1 = a_{11}, \theta = a_{21}, x_2 = a_{31}$. The simultaneous equations are solved yielding $a_{11} = \frac{3}{2k}, a_{21} = \frac{1}{2kr}, a_{31} = -\frac{1}{4k}$

Set $f_1 = 0, m = 1, f_3 = 0$. Then $x_1 = a_{12}, \theta = a_{22}, x_2 = a_{32}$. The simultaneous equations are solved yielding $a_{12} = \frac{1}{2kr}, a_{22} = \frac{1}{2kr^2}, a_{32} = -\frac{1}{4kr}$

Set $f_1 = 0, m = 0, f_3 = 1$. Then $x_1 = a_{13}, \theta = a_{23}, x_2 = a_{33}$. The simultaneous equations are solved yielding $a_{13} = -\frac{1}{4k}, a_{23} = -\frac{1}{4kr}, a_{33} = \frac{1}{4k}$

The flexibility matrix is

$$\mathbf{A} = \frac{1}{k}\begin{bmatrix} \frac{3}{2} & \frac{1}{2r} & -\frac{1}{4} \\ \frac{1}{2r} & \frac{1}{2r^2} & -\frac{1}{4r} \\ -\frac{1}{4} & -\frac{1}{4r} & \frac{1}{4} \end{bmatrix}$$

Problem 7.60 illustrates the use of flexibility influence coefficients.

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7.61 Determine the flexibility matrix for the system of Figure P7.18 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.18
(Problems 7.18, 7.46, 7.61, 7.76)

Given: x_1, x_2, and θ as generalized coordinates

Find: **A**

Solution: The flexibility matrix is obtained by applying unit loads to the locations whose displacements are described by the generalized coordinates and determining the resulting static displacements. Consider the free-body diagrams of the system components when a force f_1 is applied to block 1, a force f_2 is applied to block 2 and a moment m is applied to the pulley. These forces and moments are applied to be consistent with the chosen positive directions of the generalized coordinates.

The equations of static equilibrium are applied to the free-body diagrams yielding

$$\begin{aligned} 2kx_1 - kr\theta &= f_1 \\ kx_2 - 2kr\theta &= f_2 \\ -krx_1 - 2krx_2 + 5kr^2\theta &= m \end{aligned} \qquad (1)$$

The first column of the flexibility matrix is obtained by setting $f_1 = 1$, $f_2 = 0$, and $m = 0$. The flexibility influence coefficients are $x_1 = a_{11}$, $x_2 = a_{21}$, $\theta = a_{31}$. Equations (1) are solved yielding

$$a_{11} = \frac{1}{k}, \quad a_{21} = \frac{2}{k}, \quad a_{32} = \frac{1}{kr}$$

The second column of the flexibility matrix is obtained by setting $f_1 = 0$, $f_2 = 1$, and $m = 0$. The flexibility influence coefficients are $x_1 = a_{12}$, $x_2 = a_{22}$, $\theta = a_{32}$. Equations (1) are solved yielding

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$$a_{12} = \frac{2}{k}, \quad a_{22} = \frac{9}{k}, \quad a_{32} = \frac{4}{kr}$$

The third column of the flexibility matrix is obtained by setting $f_1 = 0$, $f_2 = 0$, and $m = 1$. The flexibility influence coefficients are $x_1 = a_{13}$, $x_2 = a_{23}$, $\theta = a_{33}$. Equations (1) are solved yielding

$$a_{13} = \frac{1}{kr}, \quad a_{23} = \frac{4}{kr}, \quad a_{33} = \frac{2}{kr^2}$$

The flexibility matrix is

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{k} & \frac{2}{k} & \frac{1}{kr} \\ \frac{2}{k} & \frac{9}{k} & \frac{4}{kr} \\ \frac{1}{kr} & \frac{4}{kr} & \frac{2}{kr^2} \end{bmatrix}$$

Problem 7.61 illustrates the use of flexibility influence coefficients to determine the flexibility matrix for a three-degree-of-freedom system.

7.62 Determine the flexibility matrix for the system of Figure P7.19 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.19
(Problems 7.19, 7.28, 7.34, 7.47, 7.62, 7.77)

Given: x_1, x_2, and θ as generalized coordinates

Find: **A**

Solution: The flexibility matrix is obtained by applying unit loads to the locations whose displacements are described by the generalized coordinates and determining the resulting static displacements. Consider the free-body diagrams of the system components when a force f_1 is applied to the mass center of the upper bar, a force f_2 is applied to the mass center of the lower bar 2 and a moment m is applied to the upper bar. These forces and moments are applied to be consistent with the chosen positive directions of the generalized coordinates.

The equations of static equilibrium are applied to the

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free-body diagrams yielding

$$
\begin{aligned}
3kx_1 + k\frac{L}{2}\theta - \frac{k}{2}x_2 &= f_1 \\
k\frac{L}{2}x_1 + 3k\frac{L^2}{4}\theta - k\frac{L}{4}x_2 &= m \\
-\frac{k}{2}x_1 - k\frac{L}{4}\theta + \frac{k}{4}x_2 &= f_2
\end{aligned}
\tag{1}
$$

The first column of the flexibility matrix is obtained by setting $f_1 = 1$, $f_2 = 0$, and $m = 0$. The flexibility influence coefficients are $x_1 = a_{11}$, $x_2 = a_{31}$, $\theta = a_{21}$. Equations (1) are solved yielding

$$a_{11} = \frac{1}{k}, \quad a_{21} = 0, \quad a_{32} = \frac{2}{k}$$

The second column of the flexibility matrix is obtained by setting $f_1 = 0$, $f_2 = 0$, and $m = 1$. The flexibility influence coefficients are $x_1 = a_{12}$, $x_2 = a_{32}$, $\theta = a_{22}$. Equations (1) are solved yielding

$$a_{12} = 0, \quad a_{22} = \frac{4}{kL^2}, \quad a_{32} = \frac{4}{kL}$$

The third column of the flexibility matrix is obtained by setting $f_1 = 0$, $f_2 = 1$, and $m = 0$. The flexibility influence coefficients are $x_1 = a_{13}$, $x_2 = a_{33}$, $\theta = a_{23}$. Equations (1) are solved yielding

$$a_{13} = \frac{2}{k}, \quad a_{23} = \frac{4}{kL}, \quad a_{33} = \frac{16}{k}$$

The flexibility matrix is

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{k} & 0 & \frac{2}{k} \\ 0 & \frac{4}{kL^2} & \frac{4}{kL} \\ \frac{2}{k} & \frac{4}{kL} & \frac{16}{k} \end{bmatrix}$$

Problem 7.62 illustrates the use of flexibility influence coefficients to determine the flexibility matrix for a three-degree-of-freedom system.

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7.63 Determine the flexibility matrix for the system of Figure P7.20 using the indicated generalized coordinates and flexibility influence coefficients.

FIGURE P7.20
(Problems 7.20, 7.35, 7.48, 7.63, 7.78)

Given: θ_1, θ_2 and x as generalized coordinates

Find: **A**

Solution: Consider an arbitrary deflected shape when the system is subject to static moments m_1 applied counterclockwise to the leftmost bar, m_2 counterclockwise moment applied to the rightmost bar and a static force f applied to the block whose displacement is x.

Summing moments about the pin supports of the bars and summing forces on the block lead to

$$\sum M_A = 0 = m_1 + k\left(x - \frac{2L}{3}\theta_1\right)\frac{2L}{3} - mg\frac{L}{2}\theta_1 = 0$$

$$\sum M_B = 0 = m_2 - k\left(\frac{2L}{3}\theta_2 - x\right)\frac{2L}{3} - mg\frac{L}{2}\theta_2$$

$$\sum F = 0 = f - k\left(x - \frac{2L}{3}\theta_1\right) + k\left(\frac{2L}{3}\theta_2 - x\right) = 0$$

Set $m_1 = 1, m_2 = 0, f = 0$. Then $\theta_1 = a_{11}, \theta_2 = a_{21}, x = a_{31}$. The simultaneous equations are solved yielding $a_{11} = \frac{8Lk+18mg}{mgL(8Lk+9mg)}, a_{21} = \frac{8Lk}{mgL(8Lk+9mg)}, a_{31} = \frac{2}{3mg}$

Set $m_1 = 0, m_2 = 1, f = 0$. Then $\theta_1 = a_{12}, \theta_2 = a_{22}, x = a_{32}$. The simultaneous equations are solved yielding $a_{12} = \frac{8Lk}{mgL(8Lk+9mg)}, a_{22} = \frac{8Lk+18mg}{mgL(8Lk+9mg)}, a_{32} = \frac{2}{3mg}$

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Set $m_1 = 0, m_2 = 0, f = 1$. Then $\theta_1 = a_{13}, \theta_2 = a_{23}, f = a_{33}$. The simultaneous equations are solved yielding $a_{13} = \frac{2}{3mg}, a_{23} = \frac{2}{3mg}, a_{33} = \frac{\frac{4kL}{9}+mg/2}{mgk}$

Problem 7.63 illustrates the use of flexibility influence coefficients.

7.64 Determine the flexibility matrix for the system of Figure P7.21 using the indicated generalized coordinates and flexibility influence coefficients.

Given: θ_1, θ_2 and θ_3 as generalized coordinates

Find: **A**

FIGURE P7.21
(Problems 7.21, 7.49, 7.64, 7.79)

Solution: Consider an arbitrary deflected shape when the system is subject to static moments m_1 applied to the pulley whose angular displacement is θ_1, m_2 is a moment applied to the pulley whose angular displacement is $\theta_2$and, m_3 applied to the pulley whose angular displacement is θ_3.

Summing moments about the center of each pulley leads to

$$\sum M_A = 0 = m_1 - k_1(r\theta_1)r - k_2(r\theta_2 + r\theta_1)r = 0$$

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$$\sum M_B = 0 = m_2 - k_2(r\theta_2 + r\theta_1)r - k_3(r\theta_2 + r\theta_3)r$$

$$\sum M_c = 0 = m_3 - k_4(r\theta_3)r - k_3(r\theta_2 + r\theta_3)r = 0$$

Set $m_1 = 1, m_2 = 0, m_3 = 0$. Then $\theta_1 = a_{11}, \theta_2 = a_{21}, \theta_3 = a_{31}$. The simultaneous equations are solved yielding $a_{11} = (3k_3 + 3k_4 + k_2k_3 + k_2k_4 - k_3^2)/D, a_{21} = k_2(k_3 + k_4)/D, a_{31} = k_2k_3/D$ where $D = r^2(2k_1k_3 + 3k_1k_4 + 2k_2k_3 + 3k_2k_4 +$ *k1k2k3+k1k2k4*

Set $m_1 = 0, m_2 = 1, m_3 = 0$. Then $\theta_1 = a_{12}, \theta_2 = a_{22}, \theta_3 = a_{32}$. The simultaneous equations are solved yielding $a_{12} = k_2(k_3 + k_4)/D, a_{22} = (k_1 + k_2)(k_3 + k_4)/D, a_{32} = k_3(k_1 + k_2)/D$

Set $m_1 = 0, m_2 = 0, m_3 = 1$. Then $\theta_1 = a_{13}, \theta_2 = a_{23}, \theta_3 = a_{33}$. The simultaneous equations are solved yielding $a_{13} = k_2k_3/D, a_{23} = k_3(k_1 + k_2)/D, a_{33} = (3k_1 + 3k_2 +$ *k1k2/D*

Problem 7.64 illustrates the use of flexibility influence coefficients.

7.65 Determine the flexibility matrix for the system of Figure P7.22 using the indicated generalized coordinates and flexibility influence coefficients.

Given: x_1, x_2 and x_3 as generalized coordinates

Find: **A**

FIGURE P7.22
(Problems 7.22, 7.29, 7.50, 7.65, 7.80)

Solution: Consider an arbitrary deflected shape when the system is subject to static loads f_1 applied to the left end of the bar, f_2 applied to the right end of the bar and f_3 is applied to the block.

Summing moments about each end of the bar and summing forces on the block lead to

$$\sum M_B = 0 = f_1L - kx_1L - k\left(\frac{x_1 + 3x_2}{4}\right)\frac{L}{4} = 0$$

$$\sum M_A = 0 = f_2L - k\left(\frac{x_1 + 3x_2}{4}\right)\frac{3L}{4} + k(x_3 - x_2)L$$

$$\sum F = 0 = f_3 - kx_3 - k(x_3 - x_2) = 0$$

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Set $f_1 = 1, f_2 = 0, f_3 = 0$. Then $x_1 = a_{11}, x_2 = a_{21}, x_3 = a_{31}$. The simultaneous equations are solved yielding $a_{11} = \frac{34}{35k}, a_{21} = -\frac{6}{35k}, a_{31} = -\frac{3}{35k}$

Set $f_1 = 0, f_2 = 1, f_3 = 0$. Then $x_1 = a_{12}, x_2 = a_{22}, x_3 = a_{32}$. The simultaneous equations are solved yielding $a_{12} = -\frac{6}{35k}, a_{22} = \frac{34}{35k}, a_{32} = \frac{17}{35k}$

Set $f_1 = 0, f_2 = 0, f_3 = 1$. Then $x_1 = a_{13}, x_2 = a_{23}, x_3 = a_{33}$. The simultaneous equations are solved yielding $a_{13} = -\frac{3}{35k}, a_{23} = \frac{17}{35k}, a_{33} = \frac{26}{35k}$

The flexibility matrix is

$$\mathbf{A} = \frac{1}{35k}\begin{bmatrix} 34 & -6 & -3 \\ -6 & 34 & 27 \\ -3 & 17 & 26 \end{bmatrix}$$

Problem 7.65 illustrates the use of flexibility influence coefficients.

7.66 Determine the mass matrix for the system of Figure P7.1 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.1
(Problems 7.1, 7.8, 7.23, 7.30, 7.36, 7.51, 7.66)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_1 = 1, \dot{x}_2 = 0, \dot{x}_3 = 0$ and calculating the impulses needed to cause them.

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Applying the principle of impulse and momentum to each block

$$m_{11} = m, m_{21} = 0, m_{31} = 0$$

The second column is calculated by setting $\dot{x}_1 = 0, \dot{x}_2 = 1, \dot{x}_3 = 0$. Applying the principle of impulse and momentum to each block

$$m_{12} = 0, m_{22} = m, m_{32} = 0$$

The third column is calculated by setting $\dot{x}_1 = 0, \dot{x}_2 = 0, \dot{x}_3 = 1$. Applying the principle of impulse and momentum to each block

$$m_{13} = 0, m_{23} = m, m_{33} = m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

Problem 7.66 illustrates the use of inertia influence coefficients.

7.67 Determine the mass matrix for the system of Figure P7.2 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.2
(Problems 7.2, 7.9, 7.31, 7.37, 7.52, 7.67)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{\theta} = 1, \dot{x}_1 = 0, \dot{x}_2 = 0$ and calculating the impulses needed to cause them.

Applying the principle of angular impulse and momentum about the pin support to the bar principle of impulse and momentum to each block

$$m_{11} = \frac{1}{12}mL^2 + m\frac{L}{2}\left(\frac{L}{2}\right) \Rightarrow m_{11} = m\frac{L^2}{3}$$

$$m_{21} = 0,$$

$$m_{31} = 0$$

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The second column is calculated by setting $\dot{\theta} = 0, \dot{x}_1 = 1, \dot{x}_2 = 0$. Applying the principle of angular impulse and momentum about the pin support to the bar principle of impulse and momentum to each block

$$m_{12} = 0, m_{22} = m, m_{32} = 0$$

The third column is calculated by setting $\dot{\theta} = 0, \dot{x}_1 =, \dot{x}_2 = 1$. Applying the principle of angular impulse and momentum about the pin support to the bar principle of impulse and momentum to each block

$$m_{13} = 0, m_{23} = m, m_{33} = 2m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} m\dfrac{L^2}{3} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

Problem 7.67 illustrates the use of inertia influence coefficients.

7.68 Determine the mass matrix for the system of Figure P7.3 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.3
(Problems 7.3, 7.10, 7.24, 7.38, 7.53, 7.68)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{\theta} = 1, \dot{x}_1 = 0, \dot{x}_2 = 0$ and calculating the impulses needed to cause them.

Applying the principle of angular impulse and momentum about the pin support to the bar principle of impulse and momentum to each block

$$m_{11} = \frac{1}{12}mL^2 + m\frac{L}{2}\left(\frac{L}{2}\right) \Rightarrow m_{11} = m\frac{L^2}{3}$$

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$$m_{21} = 0,$$

$$m_{31} = 0$$

The second column is calculated by setting $\dot{\theta} = 0, \dot{x}_1 = 1, \dot{x}_2 = 0$. Applying the principle of angular impulse and momentum about the pin support to the bar principle of impulse and momentum to each block

$$m_{12} = 0, m_{22} = m, m_{32} = 0$$

The third column is calculated by setting $\dot{\theta} = 0, \dot{x}_1 =, \dot{x}_2 = 1$. Applying the principle of angular impulse and momentum about the pin support to the bar principle of impulse and momentum to each block

$$m_{13} = 0, m_{23} = 0, m_{33} = 2m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} m\dfrac{L^2}{3} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

Problem 7.68 illustrates the use of inertia influence coefficients.

7.69 Determine the mass matrix for the system of Figure P7.4 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.4
(Problems 7.4, 7.11, 7.25, 7.39, 7.54, 7.69)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_1 = 1, \dot{x}_2 = 0, \dot{x}_3 = 0$ and calculating the impulses needed to cause them.

Applying the principle of angular impulse and momentum to the bar about each end and the principle of linear impulse and momentum to the block leads to

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$$m_{11}L = \frac{m}{2}\left(\frac{L}{2}\right) + \frac{1}{12}mL^2\left(\frac{1}{L}\right) \Longrightarrow m_{11} = \frac{m}{3}$$

$$m_{12}L = \frac{m}{2}\left(\frac{L}{2}\right) - \frac{1}{12}mL^2\left(\frac{1}{L}\right) \Longrightarrow m_{12} = \frac{m}{6}$$

$$m_{31} = 0$$

The second column is calculated by setting $\dot{x}_1 = 0, \dot{x}_2 = 1, \dot{x}_3 = 0$. Applying the principle of angular impulse and momentum to the bar about each end and the principle of linear impulse and momentum to the block leads to

$$m_{12}L = \frac{m}{2}\left(\frac{L}{2}\right) - \frac{1}{12}mL^2\left(\frac{1}{L}\right) \Longrightarrow m_{12} = \frac{m}{6}$$

$$m_{22}L = \frac{m}{2}\left(\frac{L}{2}\right) - \frac{1}{12}mL^2\left(\frac{1}{L}\right) \Longrightarrow m_{22} = \frac{m}{3}$$

$$m_{32} = 0$$

The third column is calculated by setting $\dot{x}_1 = 0, \dot{x}_2 = 0, \dot{x}_3 = 1$. Applying the principle of angular impulse and momentum to the bar about each end and the principle of linear impulse and momentum to the block leads to

$$m_{13} = 0$$

$$m_{23} = 0$$

$$m_{33} = m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} \frac{m}{3} & \frac{m}{6} & 0 \\ \frac{m}{6} & \frac{m}{3} & 0 \\ 0 & 0 & m \end{bmatrix}$$

Problem 7.69 illustrates the use of inertia influence coefficients.

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7.70 Determine the mass matrix for the system of Figure P7.5 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.5
(Problems 7.5, 7.12, 7.26, 7.40, 7.55, 7.70)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_1 = 1, \dot{\theta} = 0, \dot{x}_2 = 0$ and calculating the impulses needed to cause them.

Applying the principle of angular impulse and momentum about the mass center and the principle of linear impulse and momentum to the bar and the block leads to

$$m_{11} = m_1$$

$$m_{21} = 0$$

$$m_{31} = 0$$

The second column is calculated by setting $\dot{x}_1 = 0, \dot{\theta} = 1, \dot{x}_2 = 0$. Applying the principle of angular impulse and momentum about the mass center and the principle of linear impulse and momentum to the bar and the block leads to

$$m_{12} = 0$$

$$m_{22} = I$$

$$m_{32} = 0$$

The third column is calculated by setting $\dot{x}_1 = 0, \dot{\theta} = 0, \dot{x}_2 = 1$. Applying the principle of angular impulse and momentum about the mass center and the principle of linear impulse and momentum to the bar and the block leads to

$$m_{13} = 0$$

$$m_{23} = 0$$

$$m_{33} = m_2$$

Thus the mass matrix for this problem is

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$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & m_2 \end{bmatrix}$$

Problem 7.70 illustrates the use of inertia influence coefficients.

7.71 Determine the mass matrix for the system of Figure P7.6 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.6
(Problems 7.6, 7.13, 7.41, 7.56, 7.71)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_1 = 1, \dot{x}_2 = 0, \dot{x}_3 = 0$ and calculating the impulses needed to cause them.

Applying the principle of impulse and momentum to each block

$$m_{11} = m, m_{21} = 0, m_{31} = 0$$

The second column is calculated by setting $\dot{x}_1 = 0, \dot{x}_2 = 1, \dot{x}_3 = 0$. Applying the principle of impulse and momentum to each block

$$m_{12} = 0, m_{22} = 3m, m_{32} = 0$$

The third column is calculated by setting $\dot{x}_1 = 0, \dot{x}_2 = 0, \dot{x}_3 = 1$. Applying the principle of impulse and momentum to each block

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$$m_{13} = 0, m_{23} = 0, m_{33} = 2m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

Problem 7.71 illustrates the use of inertia influence coefficients.

FIGURE P7.7
(Problems 7.7, 7.14, 7.42, 7.57, 7.72)

7.72 Determine the mass matrix for the system of Figure P7.7 using the indicated generalized coordinates and inertia influence coefficients.

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{\theta} = 1, \dot{x}_1 = 0, \dot{x}_2 = 0$ and calculating the impulses needed to cause them.

Applying the principle of angular impulse and momentum about the pin support to the bar principle of impulse and momentum to each block

$$m_{11} = \frac{1}{12}mL^2 + m\frac{L}{2}\left(\frac{L}{2}\right) \Longrightarrow m_{11} = m\frac{L^2}{3}$$

$$m_{21} = 0,$$

$$m_{31} = 0$$

The second column is calculated by setting $\dot{\theta} = 0, \dot{x}_1 = 1, \dot{x}_2 = 0$. Applying the principle of angular impulse and momentum about the pin support to the bar principle of impulse and momentum to each block

$$m_{12} = 0, m_{22} = m, m_{32} = 0$$

The third column is calculated by setting $\dot{\theta} = 0, \dot{x}_1 = 0, \dot{x}_2 = 1$. Applying the principle of angular impulse and momentum about the pin support to the

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bar principle of impulse and momentum to each block

$$m_{13} = 0, m_{23} = m, m_{33} = 2m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} m\frac{L^2}{3} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

Problem 7.72 illustrates the use of inertia influence coefficients.

7.73 Determine the mass matrix for the system of Figure P7.3 using the indicated generalized coordinates and inertia influence coefficients.

Given: System shown

Find: **M**

Thin disk of mass m and radius r rolls without slip relative to center of mass $2m$. x_D is absolute displacement of mass center of disk.

FIGURE P7.15
(Problems 7.15, 7.27, 7.32, 7.43, 7.58, 7.73)

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_c = 1,\ \dot{x}_D = 0$ and calculating the impulses needed to cause them. The disk rolls without slip relative to the cart. Thus the angular velocity of the disk is -1/r.

Applying angular impulse and momentum about the mass center to the disk

$$r\int Fdt = \frac{1}{2}mr$$

Applying the principle of impulse and momentum to the cart yields

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$$m_{11} - \int F dt = 2m \Longrightarrow m_{11} = \frac{5}{2}m$$

Applying the principle of impulse and momentum to the disk

$$m_{21} + \int F dt = 0 \Longrightarrow m_{21} = -\frac{1}{2}m$$

The second column of the mass matrix is obtained by applying $\dot{x}_c =$, $\dot{x}_D = 1$. Due to the no slip condition the angular velocity of the disk is 1/r. Applying angular impulse and momentum about the mass center to the disk

$$r\int F dt = -\frac{1}{2}mr$$

Applying the principle of impulse and momentum to the cart yields

$$m_{12} - \int F dt = 0 \Longrightarrow m_{11} = -\frac{1}{2}m$$

Applying the principle of impulse and momentum to the disk

$$m_{22} + \int F dt = 0 \Longrightarrow m_{21} = \frac{1}{2}m$$

The mass matrix for the system is

$$\mathbf{M} = \begin{bmatrix} \frac{5}{2}m & -\frac{1}{2}m \\ -\frac{1}{2}m & \frac{1}{2}m \end{bmatrix}$$

Problem 7.73 illustrates inertia influence coefficients.

7.74 Determine the mass matrix for the system of Figure P7.16 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.16
(Problems 7.16, 7.33, 7.44, 7.59, 7.74)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_c = 1$, $\dot{x}_D = 0$ and calculating the impulses needed to cause them. The disk rolls without slip relative to the cart. Thus the angular velocity of the disk is -1/r.

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Applying angular impulse and momentum about the mass center to the disk

$$r\int Fdt = \frac{1}{2}mr$$

Applying the principle of impulse and momentum to the cart yields

$$m_{11} - \int Fdt = m \Longrightarrow m_{11} = \frac{3}{2}m$$

Applying the principle of impulse and momentum to the disk

$$m_{21} + \int Fdt = 0 \Longrightarrow m_{21} = -\frac{1}{2}m$$

The second column of the mass matrix is obtained by applying $\dot{x}_c =, \ \dot{x}_D = 1$. Due to the no slip condition the angular velocity of the disk is 1/r. Applying angular impulse and momentum about the mass center to the disk

$$r\int Fdt = -\frac{1}{2}mr$$

Applying the principle of impulse and momentum to the cart yields

$$m_{12} - \int Fdt = 0 \Longrightarrow m_{11} = -\frac{1}{2}m$$

Applying the principle of impulse and momentum to the disk

$$m_{22} + \int Fdt = 0 \Longrightarrow m_{21} = \frac{1}{2}m$$

The mass matrix for the system is

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$$\mathbf{M} = \begin{bmatrix} \frac{3}{2}m & -\frac{1}{2}m \\ -\frac{1}{2}m & \frac{1}{2}m \end{bmatrix}$$

Problem 7.74 illustrates inertia influence coefficients.

7.75 Determine the mass matrix for the system of Figure P7.17 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.17
(Problems 7.17, 7.45, 7.60, 7.75)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_1 = 1, \dot{\theta} = 0, \dot{x}_2 = 0$ and calculating the impulses needed to cause them.

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Applying the principle of impulse and momentum to the leftmost block and the principle of linear impulse and momentum to each pulley yields

$$m_{11} = m$$

$$m_{21} = 0$$

$$m_{31} = 0$$

The second column is calculated by setting $\dot{x}_1 = 0, \dot{\theta} = 1, \dot{x}_2 = 0$. Applying the principle of impulse and momentum to the leftmost block and the principle of linear impulse and momentum to each pulley yields

$$m_{12} = 0$$

$$m_{22} = I$$

$$m_{32} = 0$$

The third column is calculated by setting $\dot{x}_1 = 0, \dot{\theta} = 0, \dot{x}_2 = 1$. Applying the principle of impulse and momentum to the leftmost block and the principle of linear impulse and momentum to each pulley yields

$$m_{13} = 0,$$

$$m_{23} = 0$$

$$m_{33}\frac{r}{2} = 2m\frac{r}{2} + \frac{2I}{r} \Longrightarrow m_{33} = 2m + \frac{4I}{r^2}$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 2m + \frac{4I}{r^2} \end{bmatrix}$$

Problem 7.75 illustrates the use of inertia influence coefficients.

7.76 Determine the mass matrix for the system of Figure P7.18 using the indicated generalized coordinates and inertia influence coefficients.

Given: System shown

Find: **M**

FIGURE P7.18
(Problems 7.18, 7.46, 7.61, 7.76)

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Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_1 = 1, \dot{\theta} = 0, \dot{x}_2 = 0$ and calculating the impulses needed to cause them.

Applying the principle of impulse and momentum to each block and the principle of angular impulse and momentum to the pulley

$$m_{11} = m$$

$$m_{21} = 0$$

$$m_{31} = 0$$

The second column is calculated by setting $\dot{x}_1 = 0, \dot{\theta} = 1, \dot{x}_2 = 0$. Applying the principle of impulse and momentum to each block and the principle of angular impulse and momentum to the pulley

$$m_{12} = 0,$$

$$m_{22} = I$$

$m_{32} = 0$

The third column is calculated by setting $\dot{x}_1 = 0, \dot{\theta} = 1, \dot{x}_2 = 0$. Applying the principle of impulse and momentum to each block and the principle of angular impulse and momentum to the pulley

$$m_{13} = 0$$

$$m_{23} = 0$$

$$m_{33} = m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & m \end{bmatrix}$$

Problem 7.76 illustrates the use of inertia influence coefficients.

7.77 Determine the mass matrix for the system of Figure P7.19 using the indicated generalized coordinates and inertia influence coefficients.

Given: System shown

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Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_1 = 1, \dot{\theta} = 0, \dot{x}_2 = 0$ and calculating the impulses needed to cause them.

FIGURE P7.19
(Problems 7.19, 7.28, 7.34, 7.47, 7.62, 7.77)

Applying the principle of angular impulse and momentum to each bar and the principle of linear impulse and momentum to the upper bar gives

$$m_{11} = m$$

$$m_{21} = 0$$

$$m_{31} = 0$$

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The second column is calculated by setting $\dot{x}_1 = 0, \dot{\theta} = 1, \dot{x}_2 = 0$. Applying the principle of angular impulse and momentum to each bar and the principle of linear impulse and momentum to the upper bar gives

$$m_{12} = 0$$

$$m_{22} = \frac{1}{12} mL^2$$

$$m_{32} = 0$$

The third column is calculated by setting $\dot{x}_1 = 0, \dot{\theta} = 0, \dot{x}_2 = 1$. Applying the principle of angular impulse and momentum to each bar and the principle of linear impulse and momentum to the upper bar gives

$$m_{13} = 0$$

$$m_{23} = m$$

$$m_{33} \frac{L}{2} = \frac{1}{12} mL^2 \left(\frac{2}{L}\right) \Longrightarrow m_{33} = \frac{1}{3} m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & \frac{1}{12} mL^2 & 0 \\ 0 & 0 & \frac{1}{3} m \end{bmatrix}$$

Problem 7.77 illustrates the use of inertia influence coefficients.

7.78 Determine the mass matrix for the system of Figure P7.20 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.20
(Problems 7.20, 7.35, 7.48, 7.63, 7.78)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{\theta}_1 = 1, \dot{\theta}_1 = 0, \dot{x} = 0$ and calculating the impulses needed to cause them.

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Applying the principle of angular impulse and momentum about the pin support of each bar and the principle of linear impulse and momentum to the block yields

$$m_{11} = \frac{1}{12} mL^2$$

$$m_{21} = 0$$

$$m_{31} = 0$$

The second column is calculated by setting $\dot{\theta}_1 = 0, \dot{\theta}_1 = 1, \dot{x} = 0$. Applying the principle of angular impulse and momentum about the pin support of each bar and the principle of linear impulse and momentum to the block yields

$$m_{12} = 0$$

$$m_{22} = \frac{1}{12} mL^2$$

$$m_{32} = 0$$

The third column is calculated by setting $\dot{\theta}_1 = 0, \dot{\theta}_1 = 0, \dot{x} = 1$. Applying the principle of angular impulse and momentum about the pin support of each bar and the principle of linear impulse and momentum to the block yields

$$m_{13} = 0$$

$$m_{23} = m,$$

$$m_{33} = m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} \frac{1}{12} mL^2 & 0 & 0 \\ 0 & \frac{1}{12} mL^2 & 0 \\ 0 & 0 & m \end{bmatrix}$$

Problem 7.78 illustrates the use of inertia influence coefficients.

7.79 Determine the mass matrix for the system of Figure P7.21 using the indicated generalized coordinates and inertia influence coefficients.

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FIGURE P7.21
(Problems 7.21, 7.49, 7.64, 7.79)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{\theta}_1 = 1, \dot{\theta}_2 = 0, \dot{\theta}_3 = 0$ and calculating the impulses needed to cause them.

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Applying the principle of angular impulse and momentum to each disk

$$m_{11} = J, m_{21} = 0, m_{31} = 0$$

The second column is calculated by setting $\dot{\theta}_1 = 0, \dot{\theta}_2 = 1, \dot{\theta}_3 = 0$. Applying the principle of angular impulse and momentum to each disk

$$m_{12} = 0, m_{22} = J, m_{32} = 0$$

The third column is calculated by setting $\dot{\theta}_1 = 0, \dot{\theta}_2 = 0, \dot{\theta}_3 = 1$. Applying the principle of angular impulse and momentum to each disk

$$m_{13} = 0, m_{23} = 0, m_{33} = J$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}$$

Problem 7.79 illustrates the use of inertia influence coefficients.

7.80 Determine the mass matrix for the system of Figure P7.22 using the indicated generalized coordinates and inertia influence coefficients.

FIGURE P7.22
(Problems 7.22, 7.29, 7.50, 7.65, 7.80)

Given: System shown

Find: **M**

Solution: Using inertia influence coefficients to calculate the first column of the mass matrix imagine a set of velocities imparted to the system by impulses such that $\dot{x}_1 = 1, \dot{x}_2 = 0, \dot{x}_3 = 0$ and calculating the impulses needed to cause them.

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Applying the principle of angular impulse and momentum about each end of the bar and the principle of linear impulse and momentum on the block yields

$$m_{11}L = \frac{m}{2}\left(\frac{L}{2}\right) + \frac{1}{12}mL^2\left(\frac{1}{L}\right) \Longrightarrow m_{11} = \frac{m}{3}$$

$$m_{12}L = \frac{m}{2}\left(\frac{L}{2}\right) - \frac{1}{12}mL^2\left(\frac{1}{L}\right) \Longrightarrow m_{12} = \frac{m}{6}$$

$$m_{31} = 0$$

The second column is calculated by setting $\dot{x}_1 = 0, \dot{x}_2 = 1, \dot{x}_3 = 0$. Applying the principle of angular impulse and momentum to the bar about each end and the principle of linear impulse and momentum to the block leads to

$$m_{12}L = \frac{m}{2}\left(\frac{L}{2}\right) - \frac{1}{12}mL^2\left(\frac{1}{L}\right) \Longrightarrow m_{12} = \frac{m}{6}$$

$$m_{22}L = \frac{m}{2}\left(\frac{L}{2}\right) - \frac{1}{12}mL^2\left(\frac{1}{L}\right) \Longrightarrow m_{22} = \frac{m}{3}$$

$$m_{32} = 0$$

The third column is calculated by setting $\dot{x}_1 = 0, \dot{x}_2 = 0, \dot{x}_3 = 1$. Applying the principle of angular impulse and momentum to the bar about each end and the principle of linear impulse and momentum to the block leads to

$$m_{13} = 0$$

$$m_{23} = 0$$

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$$m_{33} = m$$

Thus the mass matrix for this problem is

$$\mathbf{M} = \begin{bmatrix} \frac{m}{3} & \frac{m}{6} & 0 \\ \frac{m}{6} & \frac{m}{3} & 0 \\ 0 & 0 & m \end{bmatrix}$$

Problem 7.80 illustrates the use of inertia influence coefficients.

7.81 Derive the differential equations governing the torsional oscillations of the turbomotor of Figure P7.81. The motor operates at 800 rpm and the turbine shafts turns at 3200 rpm.

Given: $I_{Motor} = 1800$ kg · m^2,
$I_{Turbine} = 600$ kg · m^2, $I_A = 400$ kg · m^2,
$I_B = 80$ kg · m^2
Turbine shaft: G = 80 × 10^9 N/m^2, L = 2.1 m, d = 180 mm
Motor shaft: G = 80 × 10^9 N/m^2, L = 1.4 m, d = 305 mm

Find: differential equations

FIGURE P7.81

Solution: The torsional stiffness of the motor shaft is

$$k_m = \frac{J_m G_m}{L_m} = \frac{\pi(0.305\,\text{m})^4(80\times10^9\ \text{N/m}^2)}{32(1.4\,\text{m})} = 4.85\times10^7\ \text{N-m/rad}$$

The torsional stiffness of the turbine shaft is

$$k_t = \frac{J_t G_t}{L_t} = \frac{\pi(0.180\,\text{m})^4(80\times10^9\ \text{N/m}^2)}{32(2.1\,\text{m})} = 3.93\times10^6\ \text{N-m/rad}$$

It is noted that since θ_2 represents the rotation of gear B, the angular rotation of gear A is $\theta_2/4$. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}I_m\dot{\theta}_1^2 + \frac{1}{2}I_A\left(\frac{\dot{\theta}_2}{4}\right)^2 + \frac{1}{2}I_B\dot{\theta}_2^2 + \frac{1}{2}I_t\dot{\theta}_3^2$$

The mass matrix for the system is determined from the kinetic energy as

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$$\mathbf{M} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & \frac{I_A}{16} + I_B & 0 \\ 0 & 0 & I_t \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 1800 & 0 & 0 \\ 0 & 105 & 0 \\ 0 & 0 & 600 \end{bmatrix} \text{kg - m}^2$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2} k_m \left(\frac{\theta_2}{4} - \theta_1 \right)^2 + \frac{1}{2} k_t (\theta_3 - \theta_2)^2$$

$$V = \frac{1}{2} \left[k_m \theta_1^2 - \frac{1}{2} k_m \theta_1 \theta_2 + \left(\frac{1}{16} k_m + k_t \right) \theta_2^2 - 2k_t \theta_2 \theta_3 + k_t \theta_3^2 \right]$$

The stiffness matrix is determined from the potential energy as

$$\mathbf{K} = \begin{bmatrix} k_m & -\frac{1}{4} k_m & 0 \\ -\frac{1}{4} k_m & \frac{1}{16} k_m + k_t & -k_t \\ 0 & -k_t & k_t \end{bmatrix}$$

$$\mathbf{K} = 10^7 \begin{bmatrix} 4.85 & 1.21 & 0 \\ -1.21 & 0.696 & -0.393 \\ 0 & -0.393 & 0.393 \end{bmatrix} \text{N - m/rad}$$

Thus the differential equations governing the motion of the system are

$$\begin{bmatrix} 1800 & 0 & 0 \\ 0 & 105 & 0 \\ 0 & 0 & 60 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + 10^7 \begin{bmatrix} 4.85 & -1.21 & 0 \\ -1.21 & 0.696 & -0.393 \\ 0 & -0.393 & 0.393 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.81 illustrates the derivation of differential equations for a three-degree-of-freedom system.

7.82 Derive the differential equations governing the torsional oscillations of the system of Figure P7.82.

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FIGURE P7.82

Given: system shown, J, G, L

Find: differential equations

Solution: The torsional stiffness of a shaft of length L, polar moment of inertia J, and shear modulus G is

$$k_t = \frac{JG}{L}$$

When the length of the shaft is doubled the torsional stiffness is doubled. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} I_3 \dot{\theta}_3^2 + \frac{1}{2} I_1 \dot{\theta}_4^2 + \frac{1}{2} I_5 \dot{\theta}_5^2$$

A diagonal mass matrix is determined from the kinetic energy as

$$\mathbf{M} = \begin{bmatrix} I_1 & 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 \\ 0 & 0 & 0 & I_4 & 0 \\ 0 & 0 & 0 & 0 & I_5 \end{bmatrix}$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2} k_t (\theta_2 - \theta_1)^2 + \frac{1}{2} 2k_y (\theta_3 - \theta_2)^2 + \frac{1}{2} k_t (\theta_4 - \theta_3)^2 + \frac{1}{2} k_t (\theta_5 - \theta_4)^2$$

The stiffness matrix is determined from the potential energy as

$$\mathbf{K} = \begin{bmatrix} k_t & -k_t & 0 & 0 & 0 \\ -k_t & 3k_t & -2k_t & 0 & 0 \\ 0 & -2k_t & 3k_t & -k_t & 0 \\ 0 & 0 & -k_t & 2k_t & -k_t \\ 0 & 0 & 0 & -k_t & k_t \end{bmatrix}$$

Thus the differential equations are

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$$\begin{bmatrix} I_1 & 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 \\ 0 & 0 & 0 & I_4 & 0 \\ 0 & 0 & 0 & 0 & I_5 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \end{bmatrix} + \begin{bmatrix} k_t & -k_t & 0 & 0 & 0 \\ -k_t & 3k_t & -2k_t & 0 & 0 \\ 0 & -2k_t & 3k_t & -k_t & 0 \\ 0 & 0 & -k_t & 2k_t & -k_t \\ 0 & 0 & 0 & -k_t & k_t \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.82 illustrates the derivation of the differential equations governing the motion of a torsional system.

7.83 A rotor of mass m is mounted on an elastic shaft with journal bearings at both ends. A three degree-of-freedom model of the system is shown in Figure P7.83. Each journal bearing is modeled as a spring in parallel with a viscous damper. Derive the differential equations governing the transverse motion of the system.

FIGURE P7.83

Given: system shown

Find: differential equations.

Solution: The differential equations governing the motion of the 3DOF system are

$$\mathbf{AM}\ddot{\mathbf{x}} + \mathbf{AC}\dot{\mathbf{x}} + \mathbf{x} = \mathbf{0}$$

where $x = [x_1 \quad x_2 \quad x_3]^T$. The mass matrix is given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix}$$

The damping matrix is given by

$$\mathbf{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}$$

The flexibility matrix is calculated using flexibility influence coefficients. The system is modeled as a beam with a spring at both ends, otherwise the ends are free. The differential equation governing the displacement of the beam due to a concentrated unit load at z=a is solved leading to

$$EIw(x) = C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3 z + C_4 + \frac{1}{6}(z-a)^3 u(z-a)$$

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The beam satisfies the following boundary conditions

$$\frac{d^2w}{dx^2}(0) = 0 \quad EI\frac{d^3w}{dx^3}(0) = -kw(0)$$

$$\frac{d^2w}{dx^2}(L) = 0 \quad EI\frac{d^3w}{dx^3}(L) = kw(L)$$

Application of the boundary conditions leads to

$$C_1 = -\left(1 - \frac{a}{L}\right)$$

$$C_2 = 0$$

$$C_3 = \frac{EI}{kL}\left[\left(1 - \frac{a}{L}\right)\left(\frac{1}{6}\frac{kL^3}{EI} - 2\right) - \frac{1}{6}(L-a)^3\right]$$

$$C_4 = \frac{EI}{k}\left(1 - \frac{a}{L}\right)$$

Set $a = L$ and calculate $a_{13} = w(0), a_{23} = w\left(\frac{L}{2}\right), a_{33} = w(L)$. Set $a = \frac{L}{2}$ and calculate $a_{12} = w(0), a_{22} = w\left(\frac{L}{2}\right), a_{32} = w(L)$.Then use symmetry to calculate $a_{11} = a_{33}$.

Problem 7.83 illustrates derivation of the differential equation involving a beam attached to springs.

7.84 A three degree-of-freedom model of a railroad bridge is shown in Figure P7.84. The bridge is composed of three rigid spans. Each span is pinned at its base. Using the angular displacements of the spans as generalized coordinates, derive the differential equations governing the motion of the bridge.

FIGURE P7.84

Given: three degree-of-freedom model of railroad bridge

Find: differential equations

Solution: The kinetic energy of the system at an arbitrary instant is

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$$T=\frac{1}{2}I\dot{\theta}_1^2+\frac{1}{2}m\left(\ell\dot{\theta}_1\right)^2+\frac{1}{2}I\dot{\theta}_2^2+\frac{1}{2}m\left(\ell\dot{\theta}_2\right)^2+\frac{1}{2}I\dot{\theta}_3^2+\frac{1}{2}m\left(\ell\dot{\theta}_3\right)^2+\frac{1}{2}I\dot{\theta}_4^2+\frac{1}{2}m\left(\ell\dot{\theta}_4\right)^2+$$
$$\frac{1}{2}I\dot{\theta}_5^2+\frac{1}{2}m\left(\ell\dot{\theta}_5\right)^2$$
$$T=\frac{1}{2}(I+m\ell^2)\dot{\theta}_1^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_2^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_3^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_4^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_5^2$$

The potential energy of the system at an arbitrary instant is

$$V=\frac{1}{2}k_1(h\theta_1)^2+\frac{1}{2}k_2(h\theta_2-h\theta_1)^2+\frac{1}{2}k_2(h\theta_3-h\theta_1)^2+\frac{1}{2}k_1(h\theta_3)^2$$

The Lagrangian is

$$L=T-V=\frac{1}{2}(I+m\ell^2)\dot{\theta}_1^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_2^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_3^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_4^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_5^2-$$
$$\left[\frac{1}{2}k_1(h\theta_1)^2+\frac{1}{2}k_2(h\theta_2-h\theta_1)^2+\frac{1}{2}k_2(h\theta_3-h\theta_1)^2+\frac{1}{2}k_1(h\theta_3)^2\right]$$

Application of Lagrange's equations leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)-\frac{\partial L}{\partial\theta_1}=0$$
$$\frac{d}{dt}\left[(I+m\ell^2)\dot{\theta}_1\right]+k_1h^2\theta_1+k_2(h\theta_2-h\theta_1)(-h)=0$$
$$(I+m\ell^2)\ddot{\theta}_1+(k_1+k_2)h^2\theta_1-k_2h^2\theta_2=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\frac{\partial L}{\partial\theta_2}=0$$
$$\frac{d}{dt}\left[(I+m\ell^2)\dot{\theta}_2\right]++k_2(h\theta_2-h\theta_1)(h)+k(h\theta_3-h\theta_2)(-h)=0$$
$$(I+m\ell^2)\ddot{\theta}_2-k_2h^2\theta_1+2k_2h^2\theta_2-k_2h^2\theta_3=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_3}\right)-\frac{\partial L}{\partial\theta_3}=0$$
$$\frac{d}{dt}\left[(I+m\ell^2)\dot{\theta}_3\right]+k_1h^2\theta_3+k_2(h\theta_3-h\theta_2)(h)=0$$
$$(I+m\ell^2)\ddot{\theta}_3-k_2h^2\theta_2+(k_1+k_2)h^2\theta_3=0$$

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Matrix formulation of the differential equations is

$$\begin{bmatrix} I+m\ell^2 & 0 & 0 \\ 0 & I+m\ell^2 & 0 \\ 0 & 0 & I+m\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix}+\begin{bmatrix} (k_1+k_2)h^2 & -k_2h^2 & 0 \\ -k_2h^2 & 2k_2h^2 & -k_2h^2 \\ 0 & -k_2h^2 & (k_1+k_2)h^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.84 illustrates application of Lagrange's equations to derive the differential equations governing the motion of a three-degree-of-freedom system.

7.85 A five degree-of-freedom model of a railroad bridge is shown in Figure P7.85. The bridge is composed of five rigid spans. The connection between each span and its base is modeled as a torsional spring. Using the angular displacements of the spans as generalized coordinates, derive the differential equations governing the motion of the bridge.

FIGURE P7.85

Given: five-degree-of-freedom model of railroad bridge

Find: differential equations

Solution: The kinetic energy of the system at an arbitrary instant is

$$T=\frac{1}{2}(I+m\ell^2)\dot{\theta}_1^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_2^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_3^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_4^2+\frac{1}{2}(I+m\ell^2)\dot{\theta}_5^2$$

The potential energy of the system at an arbitrary instant is

$$V=\frac{1}{2}k_1(h\theta_1)^2+\frac{1}{2}k_2(h\theta_2-h\theta_1)^2+\frac{1}{2}k_2(h\theta_3-h\theta_3)^2+$$
$$\frac{1}{2}k_2(h\theta_4-h\theta_3)^2+\frac{1}{2}k_2(h\theta_5-h\theta_4)^2+\frac{1}{2}k_t\theta_1^2+\frac{1}{2}k_t\theta_2^2+\frac{1}{2}k_t\theta_3^2+\frac{1}{2}k_t\theta_4^2+\frac{1}{2}k_t\theta_5^2$$

The Lagrangian is

$$L=T-V$$

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Application of Lagrange's equations leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)-\frac{\partial L}{\partial \theta_1}=0$$

$$\frac{d}{dt}\left[(I+m\ell^2)\dot{\theta}_1\right]+\left[k_1h^2\theta_1+k_2(h\theta_2-h\theta_1)(-h)+k_t\theta_1\right]=0$$

$$(I+m\ell^2)\ddot{\theta}_1+\left[(k_1+k_2)h^2+k_t\right]\theta_1-k_2h^2\theta_2=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\frac{\partial L}{\partial \theta_2}=0$$

$$\frac{d}{dt}\left[(I+m\ell^2)\dot{\theta}_2\right]+\left[k_2(h\theta_2-h\theta_1)(h)+k_2(h\theta_3-h\theta_2)(-h)+k_t\theta_1\right]=0$$

$$(I+m\ell^2)\ddot{\theta}_1-k_2h^2\theta_1+\left[k_2h^2+k_t\right]\theta_1-k_2h^2\theta_3=0$$

Lagrange's equations are applied in the same fashion for the remaining generalized coordinates. The resulting differential equations can be written in matrix form as

$$\begin{bmatrix} I+m\ell^2 & 0 & 0 & 0 & 0 \\ 0 & I+m\ell^2 & 0 & 0 & 0 \\ 0 & 0 & I+m\ell^2 & 0 & 0 \\ 0 & 0 & 0 & I+m\ell^2 & 0 \\ 0 & 0 & 0 & 0 & I+m\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \end{bmatrix}+$$

$$\begin{bmatrix} (k_1+k_2)h^2+k_t & -k_2h^2 & 0 & 0 & 0 \\ -k_2h^2 & k_2h^2+k_t & -k_2h^2 & 0 & 0 \\ 0 & -k_2h^2 & k_2h^2+k_t & -k_2h^2 & 0 \\ 0 & 0 & -k_2h^2 & k_2h^2+k_t & -k_2h^2 \\ 0 & 0 & 0 & -k_2h^2 & (k_1+k_2)h^2+k_t \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.85 illustrates application of Lagrange's equations to derive the differential equations governing the motion of a five-degree-of-freedom model of a railroad bridge.

7.86 A four degree-of-freedom model of an aircraft wing is shown in Figure P7.86. Derive the flexibility matrix for the model.

Given: Aircraft wing shown

FIGURE P7.86

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Find: Flexibility matrix

Solution: The flexibility matrix is that of a stepped cantilever beam. The displacement function for a cantilever beam is calculated from Appendix D as

$$y(z) = \frac{1}{EI}\left[\frac{1}{6}(z-a)^3 - \frac{z^3}{6} + a\frac{z^2}{2}\right]$$

The displacement function for a stepped cantilever beam is calculated section by section. The first section is fixed at x = 0 and attached to the second section. It has boundary conditions and matching conditions of

$$y_1(0) = 0$$

$$\frac{dy_1}{dz}(0) = 0$$

$$y_1(z_1) = y_2(z_1)$$

$$\frac{dy_1}{dz}(z_1) = \frac{dy_2}{dz}(z_1)$$

Subsequent conditions are matching conditions except for the last element where is has

$$y_3(z_3) = y_4(z_3)$$

$$\frac{dy_3}{dz}(z_3) = \frac{dy_4}{dz}(z_3)$$

$$\frac{d^2y_4}{dz^2}(L) = 0$$

$$\frac{d^3y_4}{dz^3}(L) = 0$$

The flexibility matrix is calculated by applying a unit load at z_i and calculating the deflection at z_j.

Problem 7.86 illustrates calculation of the flexibility matrix.

7.87 Figure P7.87 illustrates a three degree-of-freedom model of an aircraft. A rigid fuselage is attached to two thin flexible wings. An engine is attached to each wing, but the wings themselves are of negligible mass. Derive the differential equations governing the motion of the system.

FIGURE P7.87

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Given: 3DOF model of aircraft as shown

Find: Differential equations governing the model.

Solution: Since the mass is concentrated at the locations described by the generalized coordinates the kinetic energy is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$$

from which the mass matrix is determined as

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}$$

The stiffness matrix is determined by modeling the wings as cantilever beams of stiffness $k = \frac{3EI}{L^3}$. Stiffness influence coefficients are used to determine the stiffness matrix. For example assume $x_1 = 1, x_2 = 0\,, x_3 = 0$. The masses are assumed to be connected by springs of stiffness k as shown below. The stiffness matrix for such a model is

$$\mathbf{K} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

Thus the differential equations governing the mode are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \frac{3EI}{L^3}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.87 illustrates a 3DOF model of an airplane.

7.88 An airplane is modeled as two flexible wings attached to a rigid fuselage (Figure P7.88). Use two degrees-of-freedom to model each wing and derive the differential equations governing the motion of the five degrees-of-freedom system.

FIGURE P7.88

Given: 5DOF model of airplane shown

Find: differential equations

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Solution: Since the mass is concentrated at the locations described by the generalized coordinates the kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}M\dot{x}_3^2 + \frac{1}{2}m_2\dot{x}_4^2 + \frac{1}{2}m_1\dot{x}_5^2$$

from which the mass matrix is determined as

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & m_1 \end{bmatrix}$$

The wings are modeled as cantilever beams of length L, but the support at the center of the plane can move. The flexibility matrix does not exist; hence the stiffness matrix is necessary. The FBDs for calculation of the stiffness matrix are given below.

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Use the equation for the deflection of a beam that is fixed at free at x = L and x = –L but fixed at x = 0 to determine the stiffness matrix. The additional constraint is that the beam has no slope at x = 0. The differential equations are written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$

Problem 7.88 illustrate calculation of the stiffness matrix for a structure.

7.89 A drum of mass *m* is being hoisted by an overhead crane as illustrated in Figure P7.89. The crane is modeled as a simply supported beam with a winch at its midspan. The cable connecting the crane to the drum is of stiffness *k*. Derive the differential equations governing the motion of the system using three degrees-of-freedom for the beam and one for the displacement of the load.

FIGURE P7.89

Given: system shown

Find: differential equations

Solution: The inertia of the beam is modeled by placing three particles of mass $m_b/4$ along the span of the beam. The mass of the winch, m_w, is added to the mass of the particle at the midspan. In terms of the generalized coordinates as specified in the above figure, the mass matrix for the system is

$$\mathbf{M} = \begin{bmatrix} \frac{m_b}{4} & 0 & 0 & 0 \\ 0 & \frac{m_b}{4} + m_w & 0 & 0 \\ 0 & 0 & \frac{m_b}{4} & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$$

The flexibility matrix for this system will be specified. The first three rows and columns of the flexibility matrix are the rows and columns of a three-degree-of-freedom model of a pinned-pinned beam. These flexibility influence coefficients are determined using Table D.2. The results are

$$\mathbf{A}_1 = \frac{L^3}{768EI}\begin{bmatrix} 7 & 11 & 9 \\ 11 & 16 & 11 \\ 9 & 11 & 7 \end{bmatrix}$$

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In order to determine the fourth row and fourth column of the flexibility matrix for the system at hand, consider a unit force applied to the mass. Free-body diagrams of the mass and the beam are shown below.

Application of a unit force to the block leads to a unit force applied at the midspan of the beam. Hence the first three elements of the fourth row of the flexibility matrix are the elements of the second column of the flexibility matrix for the three-degree-of-freedom model of the beam. From the free-body diagram of the machine

$$a_{44} = a_{22} + \frac{1}{k}$$

Hence the flexibility matrix for the four-degree-of-freedom system is

$$\mathbf{A} = \frac{L^3}{768EI}\begin{bmatrix} 7 & 11 & 9 & 11 \\ 11 & 16 & 11 & 16 \\ 9 & 11 & 7 & 11 \\ 11 & 16 & 11 & 16 + \dfrac{768EI}{kL^3} \end{bmatrix}$$

The differential equations are then written as

$$\frac{L^3}{768EI}\begin{bmatrix} 7 & 11 & 9 & 11 \\ 11 & 16 & 11 & 16 \\ 9 & 11 & 7 & 11 \\ 11 & 16 & 11 & 16 + \dfrac{768EI}{kL^3} \end{bmatrix}\begin{bmatrix} m_b/4 & 0 & 0 & 0 \\ 0 & m_b/4 + m_w & 0 & 0 \\ 0 & 0 & m_b/4 & 0 \\ 0 & 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7.89 illustrates the lumped mass modeling of a continuous system.

7.90 The beam shown in Figure P7.90 is made of an elastic material of elastic modulus 210×10^9 N/m^2 and has a cross-sectional moment of inertia 1.3×10^{-5} m^4. Determine the flexibility matrix when a three degrees-of-freedom model is used to analyze the beam's vibrations. Use the displacements of the particles shown as generalized coordinates. Use Table D.2 for deflection calculations.

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FIGURE P7.90

Given: system shown

Find: **A**, E = 210 × 10^9 N/m^2, I = 1.3 × 10^{-5} m^4

Solution: The first column of the flexibility matrix is obtained by determining the deflections at the three nodal points due to a unit concentrated load applied at the location of the first nodal point. Using Table D.2, for a concentrated load applied at a, the deflection of a point at x ≤ a is

$$y = \frac{1}{EI}\left(C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2}\right)$$

where

$$C_1 = \frac{1}{2}\left(1 - \frac{a}{L}\right)\left[\left(\frac{a}{L}\right)^2 - 2\frac{a}{L} - 2\right]$$

$$C_2 = \frac{1}{2}a\left(1 - \frac{a}{L}\right)\left(2 - \frac{a}{L}\right)$$

For a concentrated load applied at a = 0.4 m, a/L = 0.25 and

$$C_1 = -0.914, \quad C_2 = 0.2625$$

and

$$a_{11} = y(x = 0.4\,m) = 4.12 \times 10^{-9}\,\frac{m}{N}$$

For a concentrated load applied at x = 0.8 m, a/L = 0.5 and

$$C_1 = -0.6875, \quad C_2 = 0.3$$

and

$$a_{12} = y(x = 0.4\,m) = 6.11 \times 10^{-9}\,\frac{m}{N}$$

$$a_{22} = y(x = 0.8\,m) = 1.368 \times 10^{-8}\,\frac{m}{N}$$

For a concentrated load applied at x = 1.2 m, a/L = 0.75 and

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$$C_1 = -0.3672, \quad C_2 = 0.7875$$

and

$$a_{13} = y(x = 0.4\,m) = 4.06 \times 10^{-9}\,\frac{m}{N}$$

$$a_{23} = y(x = 0.8\,m) = 1.05 \times 10^{-8}\,\frac{m}{N}$$

$$a_{33} = y(x = 1.2\,m) = 1.071 \times 10^{-8}\,\frac{m}{N}$$

Hence

$$\boldsymbol{A} = 1 \times 10^{-9} \begin{bmatrix} 4.12 & 6.11 & 4.06 \\ 6.11 & 13.68 & 10.50 \\ 4.06 & 10.50 & 10.71 \end{bmatrix} \frac{m}{N}$$

Problem 7.90 illustrates (1) the use of Table D.2 and (2) determination of flexibility influence coefficients when a finite number of degrees-of-freedom are used to model a continuous system.

7.91 The beam shown in Figure P7.91 is made of an elastic material of elastic modulus 210×10^9 N/m^2 and has a cross-sectional moment of inertia 1.3×10^{-5} m^4. Determine the flexibility matrix when a three degrees-of-freedom model is used to analyze the beam's vibrations. Use the displacements of the particles shown as generalized coordinates. Use Table D.2 for deflection calculations.

FIGURE P7.91

Given: system shown

Find: **A**, E = 210×10^9 N/m^2, I = 1.3×10^{-5} m^4

Solution: The first column of the flexibility matrix is obtained by determining the deflections at the three nodal points due to a unit concentrated load applied at the location of the first nodal point. Using Table D.2, for a concentrated load applied at a, the deflection of a point at $x \le a$ is

$$y = \frac{1}{EI}\left(C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2}\right)$$

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where

$$C_1 = -\left(1-\frac{a}{L}\right)^2\left(1+2\frac{a}{L}\right)$$

$$C_2 = a\left(1-\frac{a}{L}\right)^2$$

For a concentrated load applied at a = 0.8 m, a/L = 0.25 and

$$C_1 = -0.844, \quad C_2 = 0.45$$

and

$$a_{11} = y(x = 0.8\,m) = 2.64\times10^{-8}\,\frac{m}{N}$$

For a concentrated load applied at x = 1.6 m, a/L = 0.5 and

$$C_1 = -0.5, \quad C_2 = 0.4$$

and

$$a_{12} = y(x = 0.8\,m) = 3.13\times10^{-8}\,\frac{m}{N}$$

$$a_{22} = y(x = 1.6\,m) = 6.25\times10^{-8}\,\frac{m}{N}$$

For a concentrated load applied at x = 2.4 m, a/L = 0.75 and

$$C_1 = -0.1563, \quad C_2 = 0.15$$

and

$$a_{13} = y(x = 0.8\,m) = 1.27\times10^{-8}\,\frac{m}{N}$$

$$a_{23} = y(x = 1.6\,m) = 3.13\times10^{-8}\,\frac{m}{N}$$

$$a_{33} = y(x = 2.4\,m) = 2.64\times10^{-8}\,\frac{m}{N}$$

Hence

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$$\boldsymbol{A} = 1\times10^{-8}\begin{bmatrix} 2.64 & 3.13 & 1.27 \\ 3.13 & 6.25 & 3.13 \\ 1.27 & 3.13 & 2.64 \end{bmatrix}\frac{m}{N}$$

Problem 7.91 illustrates (1) the use of Table D.2 and (2) determination of flexibility influence coefficients when a finite number of degrees-of-freedom are used to model a continuous system.

7.92 The beam shown in Figure P7.92 is made of an elastic material of elastic modulus 210×10^9 N/m^2 and has a cross-sectional moment of inertia of 1.3×10^{-5} m^4. Determine the flexibility matrix when a three degree-of-freedom model is used to analyze the beam's vibrations. Use the displacements of the particles shown as generalized coordinates. Use Table D.2 for deflection calculations.

FIGURE P7.92

Given: E = 210×10^9 N/m^2, I = 1.3×10^{-5} m^4, x_1, x_2, and x_3 as generalized coordinates

Find: **A**

Solution: The general equation for the deflection of a cantilever beam with an overhang 1.8 m from the fixed end due to a unit concentrated load a distance a from the fixed end is

$$y = \frac{1}{EI}\left[\frac{1}{6}(x-a)^3 u(x-a) + \frac{R}{6}(x-1.8)^3 u(x-1.8) + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4\right]$$

where R is the reaction at the simple support.

The quickest determination of the flexibility matrix is to use reciprocity and the procedure illustrated below. The first column is calculated by applying a unit load to the particle whose displacement is x_1. Then a = 0.6 m and the constants of integration are evaluated using Table D.2 as

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$$\frac{a}{x_1} = \frac{0.6\,m}{1.8\,m} = \frac{1}{3}$$

$$c_1 = -\frac{3}{2} + \frac{3}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right)^3 = -\frac{23}{27}$$

$$c_2 = \frac{1.8}{2}\left(\frac{2}{3}\right)\left(\frac{5}{9}\right) = \frac{1}{3}$$

$$C_3 = C_4 = 0$$

Then a_{11} is calculated as the deflection at x = 0.6 m,

$$a_{11} = \frac{1}{EI}\left[-\frac{23}{27}\frac{(0.6)^3}{6} + \frac{1}{3}\frac{(0.6)^2}{2}\right] = \frac{0.0293}{EI}$$

The second column is calculated by applying a concentrated load at the location whose displacement is x_2. Then a = 1.2 m and the constants of integration are evaluated from table D.2 as

$$\frac{a}{x_1} = \frac{1.2\,m}{1.8\,m} = \frac{2}{3}$$

$$C_1 = -\frac{3}{2} + \frac{3}{2}\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right)^3 = -\frac{13}{27}$$

$$C_2 = \frac{1.8}{2}\left(\frac{1}{3}\right)\left(\frac{8}{9}\right) = \frac{4}{15}$$

$$C_3 = C_4 = 0$$

Then $a_{12} = a_{21}$ is the deflection at x = 0.6 m,

$$a_{12} = \frac{1}{EI}\left[-\frac{13}{27}\frac{(0.6)^3}{6} + \frac{4}{15}\frac{(0.6)^2}{2}\right] = \frac{0.0307}{EI}$$

and a_{22} is the deflection at x = 1.2 m,

$$a_{22} = \frac{1}{EI}\left[-\frac{13}{27}\frac{(1.2)^3}{6} + \frac{4}{15}\frac{(1.2)^2}{2}\right] = \frac{0.0533}{EI}$$

The third column is calculated by applying a unit concentrated load to the particle whose displacement is x_3. Then a = 2.6 m and the constants of integration are evaluated from table D.2 as

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$$\frac{a}{x_1} = \frac{2.6\,m}{1.8\,m} = \frac{13}{9}$$

$$C_1 = -\frac{3}{2} + \frac{3}{2}\left(\frac{13}{9}\right) = \frac{2}{3}$$

$$C_2 = \frac{1.8}{2}\left(-\frac{4}{9}\right) = -\frac{2}{5}$$

$$C_3 = C_4 = 0$$

Then $a_{13} = a_{31}$ is the deflection at x = 0.6 m,

$$a_{13} = \frac{1}{EI}\left[\frac{2}{3}\frac{(0.6)^3}{6} - \frac{2}{5}\frac{(0.6)^2}{2}\right] = -\frac{0.048}{EI}$$

$a_{23} = a_{32}$ is the deflection at x = 1.2 m,

$$a_{23} = \frac{1}{EI}\left[\frac{2}{3}\frac{(1.2)^3}{6} - \frac{2}{5}\frac{(1.2)^2}{2}\right] = \frac{-0.096}{EI}$$

and a_{33} is the deflection at x = 2.6 m

$$a_{33} = \frac{1}{EI}\left[\frac{2}{3}\frac{(2.6)^3}{6} - \frac{2}{5}\frac{(2.6)^2}{2}\right] = \frac{0.600}{EI}$$

Note that

$$\frac{1}{EI} = 3.66 \times 10^{-7}\ N - m^2$$

The flexibility matrix for this three-degree-of-freedom model is

$$\boldsymbol{A} = \frac{1}{EI}\begin{bmatrix} 0.0293 & 0.0307 & -0.048 \\ 0.0307 & 0.0533 & -0.096 \\ -0.048 & -0.096 & 0.600 \end{bmatrix}$$

Problem 7.92 illustrates the use of Table D.2 to calculate the flexibility matrix for a three-degree-of-freedom model of a cantilever beam with an overhang.

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7.93 The beam shown in Figure P7.93 is made of an elastic material of elastic modulus 210×10^9 N/m^2 and has a cross-sectional moment of inertia 1.3×10^{-5} m^4. Determine the flexibility matrix when a three degrees-of-freedom model is used to analyze the beam's vibrations. Use the displacements of the particles shown as generalized coordinates. Use Table D.2 for deflection calculations.

FIGURE P7.93

Given: system shown

Find: **A**, E = 210×10^4 N/m^2, I = 1.3×10^{-5} m^2

Solution: The first column of the flexibility matrix is obtained by determining the deflections at the three nodal points due to a unit concentrated load applied at the location of the first nodal point. Using Table D.2, for a concentrated load applied at a, the deflection of a point at x ≤ a is

$$y = \frac{1}{EI}\left(C_1 \frac{x^3}{6} + C_3 x + \frac{1}{6} R(x - x_1)^3 u(x - x_1)\right)$$

where the intermediate support is at x = x_1 and

$$C_1 = \frac{a}{x_1} - 1, \quad C_3 = -\left(1 - \frac{a}{x_1}\right)\frac{x_1^{\,2}}{6}\left[\left(1 - \frac{a}{x_1}\right)^2 u(x_1 - a) - 1\right]$$

$$R = -\frac{a}{x_1}$$

Noting that L = 2 m, for a concentrated load applied at a = 0.4 m, a/x_1 = 0.333 and

$$C_1 = -0.667, \quad C_3 = 0.00987, \quad R_1 = -0.333$$

and

$$a_{11} = y(x = 0.4\,m) = 1.042 \times 10^{-8}\,\frac{m}{N}$$

For a concentrated load applied at x = 0.8 m, a/x_1 = 0.667 and

$$C_1 = -0.333, \quad C_3 = 0.0316, \quad R = -0.667$$

and

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$$a_{12} = y(x = 0.4\,m) = 9.18 \times 10^{-9}\ \frac{m}{N}$$

$$a_{22} = y(x = 0.8\,m) = 1.04 \times 10^{-8}\ \frac{m}{N}$$

For a concentrated load applied at x = 2.0 m, a/x_1L = 1.67 and

$$C_1 = 0.667, \quad C_3 = 0. - 0.2469, \quad R = -1.67$$

and

$$a_{13} = y(x = 0.4m) = -2.08 \times 10^{-8}\ \frac{m}{N}$$

$$a_{23} = y(x = 0.8m) = -2.61 \times 10^{-8}\ \frac{m}{N}$$

$$a_{33} = y(x = 2.0m) = -1.56 \times 10^{-7}\ \frac{m}{N}$$

Hence

$$\boldsymbol{A} = 1 \times 10^{-8} \begin{bmatrix} 1.04 & 0.912 & -2.084 \\ 0.912 & 1.042 & -2.61 \\ -2.084 & -2.61 & 15.63 \end{bmatrix} \frac{m}{N}$$

Problem 7.93 illustrates (1) the use of Table D.2 and (2) determination of flexibility influence coefficients when a finite number of degrees-of-freedom are used to model a continuous system.

7.94 Determine the stiffness matrix for the three degree-of-freedom model of the free-free beam of Figure P7.94.

FIGURE P7.94

Given: x_1, x_2, and x_3 as generalized coordinates, E, I, L

Find: **K**

Solution: The stiffness matrix is determined from stiffness influence coefficients by determining the system of forces required at the nodal points to impose unit deflections. Consider the deflection of the free-free

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beam due to concentrated loads applied at the nodal points as shown.

$$EIy(x)=\frac{1}{6}F_1\left(x-\frac{L}{4}\right)^3 u\left(x-\frac{L}{4}\right)+\frac{1}{6}F_2\left(x-\frac{L}{2}\right)^3 u\left(x-\frac{L}{2}\right)+\frac{1}{6}F_3\left(x-\frac{3}{4}L\right)^3 u\left(x-\frac{3}{4}L\right)+$$
$$C_1\frac{x^3}{6}+C_2\frac{x^2}{2}+C_3x+C_4$$

Requiring the shear and moment to be zero at each end leads to $C_1 = C_2 = 0$. Since the system is unrestrained, a general application of loads leads to rigid body motion of the beam. In order for the beam to be in static equilibrium under the application of these loads,

$$\sum F=0=F_1+F_2+F_3$$
$$\sum M_2=0=F_1\frac{L}{4}-F_3\frac{L}{4}$$

Thus from the above

$$F_1=F_3=f,\quad F_2=-2f \tag{1}$$

The deflection equation for the free-free beam in static equilibrium becomes

$$EIy(x)=\frac{1}{6}f\left(x-\frac{L}{4}\right)^3 u\left(x-\frac{L}{4}\right)-\frac{1}{3}\left(x-\frac{L}{2}\right)^3 u\left(x-\frac{L}{2}\right)$$
$$+\frac{1}{6}\left(x-\frac{3}{4}L\right)^3 u\left(x-\frac{3}{4}L\right)+C_3x+C_4$$

The constants C_3 and C_4 cannot be determined from application of statics or boundary conditions. The deflections at the nodal points are evaluated as

$$x_1=y\left(\frac{L}{4}\right)=\frac{1}{EI}\left(C_3\frac{L}{4}+C_4\right)$$
$$x_2=y\left(\frac{L}{2}\right)=\frac{1}{EI}\left(\frac{1}{384}fL^3+C_3\frac{L}{2}+C_4\right) \tag{2}$$
$$x_3=y\left(\frac{3}{4}L\right)=\frac{1}{EI}\left(-\frac{1}{192}fL^3+\frac{3}{4}LC_3+C_4\right)$$

The first column of the stiffness matrix is determined by setting $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$. Equations (2) are solved simultaneously to yield

$$k_{11}=f=\frac{384EI}{7L^3}$$

Then from eq.(1)

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$$k_{21} = -2f = -\frac{768EI}{7L^3},\ k_{31} = f = \frac{384EI}{7L^3}$$

The second and third columns of the stiffness matrix could be obtained in a similar manner. However, using the required symmetry of the stiffness matrix and eq.(1) leads to

$$\boldsymbol{K} = \frac{384EI}{7L^3}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Note that the first and third columns of the stiffness matrix are identical and the second column is -2 times either of the other columns. This indeed implies the stiffness matrix is singular and that the flexibility matrix does not exist. Also note that modeling of a free-free beam requires at least three degrees of freedom. If only one or two degrees of freedom are used, the system would not have a static equilibrium position when nodal forces are applied.

Problem 7.94 illustrates the modeling of the vibrations of a free-free beam with a finite number of degrees of freedom. The stiffness matrix is determined from the beam's deflection equation, statics, and symmetry.

7.95 Using a two degree-of-freedom model, derive the differential equations governing the forced vibration of the system of Figure P7.95.

FIGURE P7.95

Given: E, I, ρ, A, F_0, L, 2DOF

Find: differential equations

Solution: The inertia of the beam is modeled by placing two particles along the span of the beam. The particles are equidistant from themselves and the supports. Let x_1 and x_2 be the generalized coordinates, representing the displacements of the particles. The differential equations governing forced vibration of the system are written as

$$\boldsymbol{AM\ddot{x}} + \boldsymbol{x} = \boldsymbol{AF}$$

where **A** is the flexibility matrix, **M** is the mass matrix, and **F** is the force vector. Each particle represents the mass of a certain portion of the beam. Since the supports have zero displacement, the kinetic energy of the particles near the supports is small and is ignored in

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the modeling. Thus, the mass of each of the particles is one third of the total mass of the beam. The diagonal mass matrix is

$$\boldsymbol{M} = \frac{1}{3}\rho AL\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The effect of the distributed loading is replaced by concentrated loads at each nodal point such that the resultant of the distributed load is the sum of the concentrated forces and the resultant moment of the distributed loading about any point is the sum of the moments of the concentrated loads about that point. Due to symmetry, it is easy to see

$$F_1 = F_0\frac{L}{6}, \quad F_2 = F_0\frac{L}{6}$$

The flexibility matrix is determined using Table D.2. The first column is obtained by applying a unit load to the first particle and determining the resulting deflections. Table D.2 is used with a = L/3.Then

$$a_{11} = y\left(x = \frac{L}{3}\right) = 5.03\times 10^{-3}\frac{L^3}{EI}$$

The second column of the flexibility matrix is obtained by applying a unit load to the second particle and determining the resulting deflections. Table D.2 is used with a = 2L/3, resulting in

$$a_{12} = a_{21} = y\left(x = \frac{L}{3}\right) = 5.258\times 10^{-3}\frac{L^3}{EI}$$

$$a_{22} = y\left(x = \frac{2}{3}L\right) = 9.145\times 10^{-3}\frac{L^3}{EI}$$

Hence the governing differential equations are

$$\frac{\rho AL^4}{3EI}\left(1\times 10^{-3}\right)\begin{bmatrix} 5.03 & 5.26 \\ 5.26 & 9.15 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{F_0 L^4}{6EI}\left(1\times 10^{-3}\right)\begin{bmatrix} 5.03 & 5.26 \\ 5.26 & 9.15 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= F_0\frac{L^4}{6EI}\left(1\times 10^{-3}\right)\begin{bmatrix} 10.29 \\ 14.41 \end{bmatrix}$$

Problem 7.95 illustrates the modeling of the forced vibration of a fixed-pinned beam using two degrees of freedom.

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7.96 Use a two-degree-of-freedom model to derive the differential equations governing the motion of the system of Figure P7.96. A thin disk of mass moment of inertia I_D is attached to the end of the fixed-free beam. Use x, the vertical displacement of the disk, and θ, the slope of the beam, as generalized coordinates.

FIGURE P7.96

Given: x and θ as generalized coordinates, L, E, I_b, m, I

Find: differential equations

Solution: Define

$$\boldsymbol{x} = \begin{bmatrix} x \\ \theta \end{bmatrix}$$

The differential equations governing forced vibration can be written as

$$\boldsymbol{AM}\ddot{\boldsymbol{x}} + \boldsymbol{x} = \boldsymbol{AF}$$

where **A** is the flexibility matrix, **M** is the mass matrix, and **F** is the force vector. Inertia effects of the beam are ignored. Thus the mass matrix is simply a diagonal matrix,

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}$$

The first column of the flexibility matrix is determined by applying a unit load at the end of the beam and determining the deflection at the end (a_{11}) and the slope at the end (a_{21}). Using Table D.2, the deflection of a cantilever beam due to a concentrated load at its end is

$$y(x) = \frac{1}{EI_b}\left(-\frac{x^3}{6} + L\frac{x^2}{2}\right)$$

Then

$$a_{11} = y(L) = \frac{L^3}{3EI_b}, \quad a_{21} = y'(L) = \frac{L^2}{2EI_b}$$

The second column of the flexibility matrix is obtained by applying a clockwise unit moment at the end of the beam and determining the deflection at the end (a_{12}) and the slope at the end (a_{22}). The equation for the deflection of the beam due to a concentrated moment at its end is

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$$y(x) = \frac{x^2}{2EI_b}$$

Hence

$$a_{12} = y(L) = \frac{L^2}{2EI_b}, \quad a_{22} = y'(L) = \frac{L}{EI_b}$$

Hence the flexibility matrix is

$$\boldsymbol{A} = \frac{1}{EI_b}\begin{bmatrix} \frac{L^3}{3} & \frac{L^2}{2} \\ \frac{L^2}{2} & L \end{bmatrix}$$

The distributed triangular loading is statically equivalent to a force F, equal to the resultant of the distributed loading, and moment M, equal to the moment of the distributed loading about the end of the beam, applied at the end of the beam. For the triangular loading

$$F = \frac{1}{2}F_0 L \sin\omega t$$

$$M = \frac{L}{3}F = F_0\frac{L^2}{6}\sin\omega t$$

Hence the differential equations become

$$\frac{1}{EI_b}\begin{bmatrix} \frac{L^3}{3} & \frac{L^2}{2} \\ \frac{L^2}{2} & L \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} x \\ \theta \end{bmatrix} = \frac{1}{EI_b}\begin{bmatrix} \frac{L^3}{3} & \frac{L^2}{2} \\ \frac{L^2}{2} & L \end{bmatrix}\begin{bmatrix} \frac{1}{2}F_0 L \\ \frac{1}{6}F_0 L^2 \end{bmatrix}\sin\omega t$$

Problem 7.96 illustrates the modeling of the forced vibrations of a beam using a finite number of degrees of freedom. One of the generalized coordinates is the slope of the beam at its end. The modeling includes determination of the flexibility matrix and the consistent force vector.

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CHAPTER 8: FREE VIBRATIONS OF MDOF SYSTEMS

Short Answer Questions

8.1 False: The natural frequencies of a MDOF system are square roots of the eigenvalues of $\mathbf{M^{-1}K}$.

8.2 False: An *n* degree of freedom system has *n* natural frequencies.

8.3 True: The natural frequencies are either the square roots of the eigenvalues of $\mathbf{M^{-1}K}$ or the reciprocals of the square roots of the eigenvalues of $\mathbf{AM}$. In either case the mode shapes are the eigenvcectors. Hence $\left(\mathbf{AM} - \frac{\mathbf{1}}{\boldsymbol{\omega}^{\mathbf{2}}}\mathbf{I}\right)\mathbf{X} = \mathbf{0}$.

8.4 True: A node is a particle that has zero displacement.

8.5 False, The mode shape vectors are orthogonal with respect to the kinetic energy (or potential energy) inner product. That is $\mathbf{X_j^T M X_i} = 0$ (or $\mathbf{X_j^T K X_i} = 0$).

8.6 False: The mode-shape vector corresponding to a natural frequency $\omega$for a MDOF system is unique only to a multiplicative constant.

8.7 True: The eigenvectors are the mode shape vectors and they are normalized by requiring that the kinetic energy inner product of a mode shape vector with itself is one.

8.8 False: The modal matrix is matrix whose columns are the normalized mode shape vectors.

8.9 True: Proportional damping occurs when the damping matrix is a linear combination of the stiffness matrix and the damping matrix. In this case the coefficient multiplying the stiffness matrix is zero.

8.10 True: To determine the natural frequencies of a MDOF system a polynomial of order 2n is derived forω. However, the polynomial only has even powers and can be reduced to a polynomial of order n.

8.11 True: The modal matrix is defined as the matrix whose columns are the normalized eigenvectors. The elements of $\mathbf{P^T M P}$ are the kinetic energy inner products of the mode shape vectors $\left(\mathbf{X_i}, \mathbf{X_j}\right)_M$which by orthogonality are zero unless i = j and by normalization are 1 if i = j. Thus the matrix is the identity matrix.

8.12 True: The result is due to mode shape normalization.

8.13 True: $\det \mathbf{K} = 0$ implies that the system is unrestrained.

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8.14 True: The flexibility matrix is the inverse of the stiffness matrix and $\det \mathbf{K} = 0$ for an unrestrained system.

8.15 False: Rayleigh's quotient can be applied to obtain an upper bound on the lowest natural frequency.

8.16 False: The damping ratio for a proportionally damped system where the proportional damping is proportional to the stiffness matrix is proportional to the natural frequency.

8.17 True: Matrix iteration uses fixed-point iteration to determine the natural frequencies one at a time.

8.18 False: If $[1 \quad 2]^T$ is a mode shape vector corresponding to a natural frequency of 100 rad/s for a two DOF system then $[2 \quad 4]^T$ is also a mode shape vector corresponding to 100 rad/s. (Mode shape vectors corresponding to the same mode are multiples on one another).

8.19 The normal mode solution is a solution where the response of the system is assumed to be synchronous, $\mathbf{x}(t) = \mathbf{X}e^{i\omega t}$. It is called the normal mode solution because of an orthogonality condition between the mode shapes, the possible values of **X.**

8.20 The dynamical matrix is **AM**, the product of the flexibility matrix and the mass matrix.

8.21 The natural frequencies of an *n*DOF system are the reciprocal of the square roots of the eigenvalues of **AM**.

8.22 The free response is a linear combination of the mode shape vectors times a $\sin \omega_i t$ and $\cos \omega_i t$,

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{X}_i (C_{1i} \sin \omega_i t + C_{2i} \cos \omega_i t)$$

Initial conditions are applied to determine the constants of integration.

8.23 A rigid-body mode corresponds to z natural frequency of zero.

8.24 Two linearly independent mode shape vectors correspond to a natural frequency that is a double root of the characteristic equation.

8.25 The potential energy scalar product of two vectors **x** and **y** is $(\mathbf{x}, \mathbf{y})_V = \mathbf{y}^{\mathbf{T}}\mathbf{K}\mathbf{x}$**.**

8.26 When the kinetic energy scalar product is taken between a mode shape vector and itself it is proportional to twice the kinetic energy associated with that mode.

8.27 The property of commutivity of scalar products is satisfied for the kinetic energy scalar product because the mass matrix is symmetric.

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8.28 Mode shape orthogonality refers to the condition where $\mathbf{X}_i$ and $\mathbf{X}_j$ are mode shape vectors corresponding to $\left(\mathbf{X}_i, \mathbf{X}_j\right)_V = \mathbf{X}_j{}^{\mathbf{T}}\mathbf{K}\,\mathbf{X}_i = 0$ and $\left(\mathbf{X}_i, \mathbf{X}_j\right)_M = \mathbf{X}_j{}^{T}\mathbf{M}\,\mathbf{X}_i = 0$.

8.29 A normalized mode shape vector satisfies $(\mathbf{X}_i, \mathbf{X}_i)_M = \mathbf{X}_j{}^{T}\mathbf{M}\,\mathbf{X}_i = 1$

8.30 Rayleigh's quotient for an arbitrary *n*-dimensional vector **x** is defined as $R(\mathbf{x}) = \frac{(\mathbf{x},\mathbf{x})_V}{(\mathbf{x},\mathbf{x})_M}$.

8.31 Rayleigh's quotient is stationary when it is evaluated for a mode shape vector.

8.32 The modal matrix is non-singular because its columns are the normalized eigenvectors of a MDOF system. The eigenvectors are guaranteed to be linearly independent since they are orthogonal.

8.33 The expansion theorem for a MDOF system with mode normalized mode shape vectors $\mathbf{X_1}, \mathbf{X_2} \ldots, \mathbf{X_n}$ is for any arbitrary n dimensional vector **y**

$$\mathbf{y} = \sum_{i=1}^{n} (\mathbf{y}, \mathbf{X_i})_M \mathbf{X}_i$$

8.34 The principal coordinates are the coordinates where the differential equations for a MDOF system are uncoupled. When the principal coordinates are used as generalized coordinates the stiffness matrix and the mass matrix are diagonal matrices.

8.35 Matrix iteration is used with the dynamic matrix **D=AM**. An initial guess is made and a sequence is initiated $\mathbf{x_{i+1}} = \mathbf{AMx_i}$. The iteration eventually converges such that the ratio of the mth element of $\mathbf{x_{i+1}}$ and $\mathbf{x_i}$ converges to ω_1^2.

8.36 The modal damping ratio is defined for systems with proportional damping. The differential equation for the principal coordinate p_i of a MDOF system with proportional damping is

$$\ddot{p}_i + 2\zeta_i\omega_i\dot{p}_i + \omega_i^2 p_i = g_i$$

The modal damping ratio for the *i*th mode is ζ_i.

8.37 A system with proportional damping has a damping matrix given by $\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M}$. The transformation between the original generalized coordinates and the principal coordinates is $\mathbf{x} = \mathbf{Pp}$. If the transformation is used in the governing equation and multiplied by $\boldsymbol{P}^T$, the term $\mathbf{P^TCP}$ becomes $\alpha\mathbf{P^TKP} + \boldsymbol{\beta}\mathbf{M^TCP} = \alpha\boldsymbol{\Omega} + \beta\mathbf{I}$ which is a diagonal matrix.

8.38 The determinant of $\mathbf{M^{-1}K}$ is zero if the lowest natural frequency is zero.

8.39 There should be two nodes corresponding to the third mode.

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8.40 Given: Eigenvalues of $\mathbf{M^{-1}K}$ are 20, 50 and 100. The eigenvalues of **AM** are the reciprocals of the eigenvalues of $\mathbf{M^{-1}K}$. They are 0.05, 0.02, and 0.01.

8.41 Given: Eigenvalues of $\mathbf{M^{-1}K}$ are 16, 49, 100 and 225. The natural frequencies are the square roots of the eigenvalues $\omega = 4,7,10,15$.

8.42 Given: system of Fig. SP8.42, $\mathbf{x} = [3\ 2 - 1]^T$, $\mathbf{y} = [1 \quad -2 \quad 3]^T$. The stiffness matrix is $K = \begin{bmatrix} 8000 & -3000 & 0 \\ -3000 & 4000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix}$.

$$(\mathbf{x},\mathbf{y})_K = (\mathbf{Kx},\mathbf{y}) = \mathbf{y^T Kx} = [1 \quad -2 \quad 3] \begin{bmatrix} 8000 & -3000 & 0 \\ -3000 & 4000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} =$$

$$[1 \quad -2 \quad 3]^T \begin{bmatrix} 18000 \\ 0 \\ -3000 \end{bmatrix} = 9000$$

FIGURE SP 8.42

8.43 Given: system of Fig. SP8.42, $\mathbf{x} = [3\ 2 - 1]^T$. Rayleigh's quotient is $R(\mathbf{x}) = \frac{(\mathbf{x},\mathbf{x})_\mathbf{K}}{(\mathbf{x},\mathbf{x})_\mathbf{M}}$ where $(\mathbf{x},\mathbf{x})_\mathbf{K} = [3 \quad 2 \quad -1] \begin{bmatrix} 8000 & -3000 & 0 \\ -3000 & 4000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 57000$ and $(\mathbf{x},\mathbf{x})_\mathbf{M} = [3 \quad 2 \quad -1] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 26$. Hence $R(\mathbf{x}) = \frac{57000}{26} = 2192.3$

8.44 Given: $\mathbf{X}_1 = [1 \quad 2]^T$, $\boldsymbol{M} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. The second mode shape vector $\mathbf{X}_2$ must be ortohogonal to the first mode shape vector $\mathbf{X_2^T M X_1} = 0$. Assuming $\mathbf{X}_2 = [a \quad b]^T$, $\mathbf{X_2^T M X_1} = [a \quad b]^T \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2a + 6b = 0 \Longrightarrow a = -3b$. Thus the mode shape vector for the second mode is $\mathbf{X}_2 = [-3 \quad 1]^T$

8.45 Given: $\mathbf{X}_1 = [1 \quad 2]^T$. The mode shape vector is for the first mode corresponding to the lower natural frequency because it does not allow for any nodes.

8.46 Given: $\mathbf{X_1} = [1 \quad 2]^T$, $\boldsymbol{M} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. The normalized mode shape vector is such that $\mathbf{X_1^T M X_1} = 1$. Assume that $\mathbf{X_1} = C[1 \quad 2]^T$. Then $\mathbf{X_1^T M X_1} = C[1 \quad 2]^T \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} C \begin{bmatrix} 1 \\ 2 \end{bmatrix} = C^2 14 \Longrightarrow C = \frac{1}{\sqrt{14}} = 0.267$. The normalized mode shape vector is: $\mathbf{X_1} = [0.267 \quad 0.534]^T$

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8.47 Given: $\mathbf{X}_1 = [0.1 \quad 0.3]^T$, $\boldsymbol{K} = \begin{bmatrix} 200 & -100 \\ -100 & 300 \end{bmatrix}$. For a normalized mode shape vector $\mathbf{X_1^T K X_1} = \omega_1^2$. To this end $[0.1 \quad 0.3]^T \begin{bmatrix} 200 & -100 \\ -100 & 300 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} = 23$. Thus $\omega_1 = \sqrt{23} = 4.80$

8.48 Given: $\mathbf{X}_1 = [1 \quad 2 \quad 2.5]^T$ and $\mathbf{X}_2 = [1 \quad 2 \quad -2]^T$ $M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$. If the vectors are mode shape vectors they satisfy an orthogonality relation $\mathbf{X_2^T M X_1} = 0$. To this end $[1 \quad 2 \quad -2]^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2.5 \end{bmatrix} = m(1 + 4 - 5) = 0$. Yes they can be mode shape vectors.

8.49 Given: ω_1 =10 rad/s, ω_2 =25 rad/s and ω_3 =50 rad/s . The differential equations satisfied by the principal coordinates are

$$\ddot{p}_1 + 100p_1 = 0$$

$$\ddot{p}_2 + 625p_2 = 0$$

$$\ddot{p}_3 + 250p_3 = 0$$

8.50 The coefficient of proportional damping is $\alpha = \frac{2\zeta_1}{\omega_1} = \frac{2(0.1)}{10} = 0.02$. The higher modal damping ratios are $\zeta_2 = \frac{\alpha\omega_2}{2} = \frac{0.02(25)}{2} = 0.25$ and $\zeta_3 = \frac{\alpha\omega_3}{2} = \frac{0.02(50)}{2} = 0.5$.

8.51 The equations are

$$\begin{bmatrix} 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 5 & 0 & 0 & 3 & -1 & 0 \\ 0 & 3 & 0 & -1 & 4 & -3 \\ 0 & 0 & 2 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \\ \dot{y}_6 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 50 & -20 & 0 \\ 0 & 0 & 0 & -20 & 100 & -80 \\ 0 & 0 & 0 & 0 & -80 & 120 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

8.52 (a) rad/s (b) dimensionless (c) s (d) $1/\text{kg}^{0.5}$ (e) $\text{m}/\text{kg}^{0.5}$ (f) $\text{kg}^{0.5}$ (g) $\text{m}/\text{kg}^{0.5}$ (h) $1/\text{kg}^{0.5}$ (i) s^2/m^2

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Chapter Problems

8.1 Calculate the natural frequencies and mode shapes for the system shown in Figures P8.1 by calculating the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. Graphically illustrate the mode shapes. Identify any nodes.

FIGURE P8.1

Given: m,k

Find: ω_1, ω_2, $\mathbf{X_1}$, $\mathbf{X_2}$

The differential equations governing the motion of the system are

$$\begin{bmatrix} m & 0 \\ 0 & 3m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (a)$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M^{-1}K}$

$$\mathbf{M^{-1}K} = \frac{\mathrm{k}}{\mathrm{m}}\begin{bmatrix} 3 & -2 \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \qquad (b)$$

which are determined from

$$\begin{vmatrix} 3\phi - \lambda & -2\phi \\ -\frac{2}{3}\phi & \frac{2}{3}\phi - \lambda \end{vmatrix} = (3\phi - \lambda)\left(\frac{2}{3}\phi - \lambda\right) - (-2\phi)\left(-\frac{2}{3}\phi\right) = \lambda^2 - \frac{11}{3}\phi\lambda + \frac{2}{3}\phi^2 = 0 \qquad (c)$$

where $\phi = \frac{k}{m}$. The solutions of Eq. (c) are

$$\lambda = 0.1919\phi, 3.478\phi \qquad (d)$$

The natural frequencies are the square roots of the eigenvalues

$$\omega_1 = 0.438\sqrt{\frac{k}{m}} \qquad \omega_2 = 1.846\sqrt{\frac{k}{m}} \qquad (e)$$

The mode shape vectors are obtained by solving $(\mathbf{M^{-1}K} - \boldsymbol{\lambda}\mathbf{I})\mathbf{X} = \mathbf{0}$. The solution of this equation for $\lambda = \lambda_1 = 0.1919\phi$ is

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.40 \end{bmatrix}$$

and for $\lambda = \lambda_2 = 3.4789\phi$ is

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$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.237 \end{bmatrix}$$

The mode shape vectors are illustrated below. There is a node for the second mode in the spring connecting the masses.

Problem 8.1 illustrates determination of the natural frequencies of a 2DOF system.

8.2 Calculate the natural frequencies and mode shapes for the system shown in Figures P8.2 by calculating the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. Graphically illustrate the mode shapes. Identify any nodes.

FIGURE P8.2

Given: m, k

Find: ω_1, ω_2, $\mathbf{X_1}$, $\mathbf{X_2}$

The differential equations governing the motion of the system are

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 4k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (a)$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M^{-1}K}$

$$\mathbf{M^{-1}K} = \frac{\mathrm{k}}{\mathrm{m}} \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix} \qquad (b)$$

which are determined from

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$$\begin{vmatrix} 3\phi - \lambda & -2\phi \\ -2\phi & 4\phi - \lambda \end{vmatrix} = (3\phi - \lambda)(4\phi - \lambda) - (-2\phi)(-2\phi) = \lambda^2 - 7\phi\lambda + 8\phi^2 = 0 \qquad (c)$$

where $\phi = \frac{k}{m}$. The solutions of Eq. (c) are

$$\lambda = 1.4384\phi, 5.5616\phi \qquad (d)$$

The natural frequencies are the square roots of the eigenvalues

$$\omega_1 = 1.994\sqrt{\frac{k}{m}} \qquad \omega_2 = 2.2353\sqrt{\frac{k}{m}} \qquad (e)$$

The mode shape vectors are obtained by solving $(\mathbf{M}^{-1}\mathbf{K} - \boldsymbol{\lambda}\mathbf{I})\mathbf{X} = \mathbf{0}$. The solution of this equation for $\lambda = \lambda_1 = 1.4384\phi$ is

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 0.781 \end{bmatrix}$$

and for $\lambda = \lambda_2 = 5.561\phi$ is

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ -1.281 \end{bmatrix}$$

The mode shape vectors are illustrated below. There is a node for the second mode in the spring connecting the masses.

Problem 8.2 illustrates determination of the natural frequencies of a 2DOF system.

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8.3 Calculate the natural frequencies and mode shapes for the system shown in Figures P8.3 by calculating the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. Graphically illustrate the mode shapes. Identify any nodes.

FIGURE P8.3
(Problems 8.3, 8.22.)

Given: k, L, m

Find: ω_1, ω_2, $\mathbf{X_1}$, $\mathbf{X_2}$

Solution: Let θ be the clockwise angular displacement of the bar, measured from the system's equilibrium position, and let x be the downward displacement of the block, measured from the system's equilibrium position. The kinetic energy of the system at an arbitrary instant, expressed in terms of these generalized coordinates is

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}k(x - a\theta)^2 + \frac{1}{2}k(x + b\theta)^2$$

The Lagrangian is

$$L = T - V$$

Application of Lagrange's equations leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(I\dot{\theta}\right) + \left[k\left((x - a\theta)(-a) + k(x + b\theta)(b)\right]\right. = 0$$

$$I\ddot{\theta} + k(a^2 + b^2)\theta - k(a - b)x = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(m\dot{x}) + \left[k(x - a\theta) + k(x + b\theta)\right] = 0$$

$$m\ddot{x} - k(a - b)\theta + 2kx = 0$$

The matrix formulation of the differential equations is

$$\begin{bmatrix} I & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} k(a^2 + b^2) & -k(a - b) \\ -k(a - b) & 2k \end{bmatrix}\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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or

$$\begin{bmatrix} 0.6 & 0 \\ 0 & 1.5 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + 10^6 \begin{bmatrix} 3.46 & -2.2 \\ -2.2 & 4 \end{bmatrix}\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 1.67 & 0 \\ 0 & 0.67 \end{bmatrix} 10^6 \begin{bmatrix} 3.46 & -2.2 \\ -2.2 & 4 \end{bmatrix} = 10^6 \begin{bmatrix} 5.757 & -3.667 \\ -1.467 & 2.667 \end{bmatrix}$$

which are determined from

$$\begin{vmatrix} 5.767\phi - \lambda & -3.667\phi \\ -1.467\phi & 2.667\phi - \lambda \end{vmatrix} = (5.767\phi - \lambda)(2.667\phi - \lambda) - 1.467\phi(-3.667\phi)$$
$$= \lambda^2 - 8.433\phi\lambda + 10\phi^2 = 0 \qquad (c)$$

where $\phi = 10^6$. The solutions of Eq. (c) are

$$\lambda = 1.427\phi, 7.006\phi \qquad (d)$$

The natural frequencies are the square roots of the eigenvalues

$$\omega_1 = 1194.7 \qquad \omega_2 = 2646.9 \qquad (e)$$

The mode shape vectors are obtained by solving $(\mathbf{M}^{-1}\mathbf{K} - \boldsymbol{\lambda}\mathbf{I})\mathbf{X} = \mathbf{0}$. The solution of this equation for $\lambda = \lambda_1 = 1.427\phi$ is

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.183 \end{bmatrix}$$

and for $\lambda = \lambda_2 = 7.006\phi$ is

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.338 \end{bmatrix}$$

The mode shape vectors are illustrated below. There is a node for the second mode on the bar.

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Problem 8.3 illustrates determination of the natural frequencies of a 2DOF system.

8.4 Calculate the natural frequencies and mode shapes for the system shown in Figures P8.4 by calculating the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. Graphically illustrate the mode shapes. Identify any nodes.

FIGURE P8.4

Given: k, L, m

Find: ω_1, ω_2, $\mathbf{X_1}$, $\mathbf{X_2}$

Solution: Let θ be the clockwise angular displacement of the bar, measured from the system's equilibrium position, and let x be the downward displacement of the block, measured from the system's equilibrium position. The kinetic energy of the system at an arbitrary instant, expressed in terms of these generalized coordinates is

$$T = \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2 + \frac{1}{2}2m\dot{x}^2$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}k\left(x - \frac{L}{2}\theta\right)^2 + \frac{1}{2}kx^2$$

The Lagrangian is

$$L = T - V$$

Application of Lagrange's equations leads to

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$

$$\frac{d}{dt}\left(\frac{1}{12}mL^2\dot{\theta}\right)+\left[k\left((x-\frac{L}{2}\theta\right)\left(-\frac{L}{2}\right)\right]=0$$

$$\frac{1}{12}mL^2\ddot{\theta}+k\frac{L^2}{4}\theta-k\frac{L}{2}x=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

$$\frac{d}{dt}(2m\dot{x})+\left[k\left(x-\frac{L}{2}\theta\right)+kx\right]=0$$

$$2m\ddot{x}-k\frac{L}{2}\theta+2kx=0$$

The matrix formulation of the differential equations is

$$\begin{bmatrix}\frac{1}{12}mL^2 & 0\\ 0 & 2m\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{x}\end{bmatrix}+\begin{bmatrix}k\frac{L^2}{4} & -k\frac{L}{2}\\ -k\frac{L}{2} & 2k\end{bmatrix}\begin{bmatrix}\theta\\ x\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end

$$\mathbf{M}^{-1}\mathbf{K}=\begin{bmatrix}\frac{12}{mL^2} & 0\\ 0 & \frac{1}{2m}\end{bmatrix}\begin{bmatrix}\frac{kL^2}{4} & -k\frac{L}{2}\\ -k\frac{L}{2} & 2k\end{bmatrix}=\frac{k}{m}\begin{bmatrix}3 & -\frac{6}{L}\\ -\frac{L}{4} & 1\end{bmatrix}$$

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$$\left|\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right| = 0$$

$$\begin{vmatrix} 3\phi - \lambda & -\frac{6}{L}\phi \\ -\frac{L}{4}\phi & \phi - \lambda \end{vmatrix} = 0 \quad \phi = \frac{k}{m}$$

$$(3\phi - \lambda)(\phi - \lambda) - \frac{3}{2}\phi^2 = 0$$

$$\lambda^2 - 4\phi\lambda + \frac{3}{2}\phi^2 = 0$$

$$\lambda_1 = \frac{1}{2}(4 - \sqrt{10})\phi = 0.419\phi$$

$$\lambda_2 = \frac{1}{2}(4 + \sqrt{10}) = 3.58\phi$$

$$\omega_1 = \sqrt{\lambda_1} = 0.647\sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\lambda_2} = 1.89\sqrt{\frac{k}{m}}$$

The mode shapes are determined from

$$(\mathbf{M}^{-1}\mathbf{K} - \lambda_1)\mathbf{X_1} = \mathbf{0}$$

$$\begin{bmatrix} 3\phi - 0.419\phi & -\frac{6}{L}\phi \\ -\frac{L}{4}\phi & \phi - 0.419\phi \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2.581\phi X_{11} - \frac{6}{L}\phi X_{12} = 0$$

$$X_{12} = 0.430 L X_{11}$$

$$(\mathbf{M}^{-1}\mathbf{K} - \lambda_2\mathbf{I})\mathbf{X_2} = \mathbf{0}$$

$$\begin{bmatrix} 3\phi - 3.58\phi & -\frac{6}{L}\phi \\ -\frac{L}{4}\phi & \phi - 3.58\phi \end{bmatrix} \begin{bmatrix} X_{21} \\ X_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-0.58\phi X_{21} - \frac{6}{L}\phi X_{22} = 0$$

$$X_{22} = -0.0967 L X_{21}$$

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The mode shape vectors are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 0.430L \end{bmatrix}$$

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.0947L \end{bmatrix}$$

The mode shapes are graphically illustrated below.

Problem 8.4 illustrates the determination of natural frequencies and mode shapes for a two-degree-of-freedom system.

8.5 Calculate the natural frequencies and mode shapes for the system shown in Figures P8.5 by calculating the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. Graphically illustrate the mode shapes. Identify any nodes.

FIGURE P8.5

Given: m, k

Find: ω_1, ω_2, ω_3, mode shapes

Solution: The differential equations governing the motion of the three-degree-of-freedom system are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end

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$$\mathbf{M}^{-1}\mathbf{K} = \frac{k}{m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} 2 & -1 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\left|\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right| = 0$$

$$\begin{vmatrix} 2\phi - \lambda & -\phi & 0 \\ -\frac{1}{2}\phi & \phi - \lambda & -\frac{1}{2}\phi \\ 0 & -\frac{1}{2}\phi & \phi - \lambda \end{vmatrix} = 0 \qquad \phi = \frac{k}{m}$$

$$\beta^3 - 4\beta^2 + 4.5\beta - 1.25 = 0 \qquad \beta = \frac{\lambda}{\phi}$$

$$\beta = 0.419, 1.344,\ 2.240$$

$$\omega_1 = \sqrt{\lambda_1} = 0.647\sqrt{\frac{k}{m}} \quad \omega_2 = \sqrt{\lambda_2} = 1.159\sqrt{\frac{k}{m}} \quad \omega_3 = \sqrt{\lambda_3} = 1.497\sqrt{\frac{k}{m}}$$

The first mode shape is determined from

$$\left|\mathbf{M}^{-1}\mathbf{K} - \lambda_1\mathbf{I}\right| = 0$$

$$\begin{vmatrix} 2\phi - 0.419\phi & -\phi & 0 \\ -0.5\phi & \phi - 0.419\phi & -0.5\phi \\ 0 & -0.5\phi & \phi - 0.419\phi \end{vmatrix}\begin{bmatrix} X_{11} \\ X_{12} \\ X_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first and third equations are used to give

$$X_{11} = 0.632X_{12}$$
$$X_{13} = 0.861X_{12}$$

leading to the first mode shape of

$$\mathbf{X_1} = \begin{bmatrix} 0.632 \\ 1 \\ 0.861 \end{bmatrix}$$

The second and third mode shapes are obtained in a similar manner as

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$$\mathbf{X}_2 = \begin{bmatrix} 0.763 \\ 1 \\ -1.452 \end{bmatrix} \quad \mathbf{X}_3 = \begin{bmatrix} -2.054 \\ 1 \\ -0.4028 \end{bmatrix}$$

The mode shapes are graphically illustrated below. The second mode has one node. The third mode has two nodes.

Problem 8.5 illustrates the determination of natural frequencies and mode shapes for a three-degree-of-freedom system.

8.6 Calculate the natural frequencies and mode shapes for the system shown in Figures P8.6 by calculating the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. Graphically illustrate the mode shapes. Identify any nodes.

FIGURE P8.6

Given: m, k

Find: ω_1, ω_2, ω_3, mode shapes

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Solution: The differential equations governing the motion of the three-degree-of-freedom system are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}+\begin{bmatrix} 2k & -k & 0 \\ -k & 4k & -k \\ 0 & -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end

$$\mathbf{M}^{-1}\mathbf{K}=\frac{k}{m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}=\frac{k}{m}\begin{bmatrix} 2 & -1 & 0 \\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\left|\mathbf{M}^{-1}\mathbf{K}-\lambda\mathbf{I}\right|=0$$

$$\begin{vmatrix} 2\phi-\lambda & -\phi & 0 \\ -\frac{1}{3}\phi & \frac{4}{3}\phi-\lambda & -\frac{1}{3}\phi \\ 0 & -\frac{1}{2}\phi & \phi-\lambda \end{vmatrix}=0 \qquad \phi=\frac{k}{m}$$

$$\beta^3-\frac{13}{3}\beta^2+\frac{11}{2}\beta-2=0 \qquad \beta=\frac{\lambda}{\phi}$$

$$\beta=0.634, 1.333,\ 2.366$$

$$\omega_1=\sqrt{\lambda_1}=0.796\sqrt{\frac{k}{m}} \quad \omega_2=\sqrt{\lambda_2}=1.155\sqrt{\frac{k}{m}} \quad \omega_3=\sqrt{\lambda_3}=1.538\sqrt{\frac{k}{m}}$$

The first mode shape is determined from

$$\left|\mathbf{M}^{-1}\mathbf{K}-\lambda_1\mathbf{I}\right|=0$$

$$\begin{vmatrix} 2\phi-0.634\phi & -\phi & 0 \\ -0.333\phi & 1.333\phi-0.634\phi & -0.333\phi \\ 0 & -0.5\phi & \phi-0.634\phi \end{vmatrix}\begin{bmatrix} X_{11} \\ X_{12} \\ X_{13} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first and third equations are used to give

$$X_{11}=0.732X_{12}$$
$$X_{13}=1.366X_{12}$$

leading to the first mode shape of

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$$\mathbf{X}_1 = \begin{bmatrix} 0.732 \\ 1 \\ 1.366 \end{bmatrix}$$

The second and third mode shapes are obtained in a similar manner as

$$\mathbf{X}_2 = \begin{bmatrix} 1.5 \\ 1 \\ -1.5 \end{bmatrix} \quad \mathbf{X}_3 = \begin{bmatrix} -2.73 \\ 1 \\ -0.366 \end{bmatrix}$$

The mode shapes are graphically illustrated below. The second mode has one node. The third mode has two nodes.

Problem 8.6 illustrates the determination of natural frequencies and mode shapes for a three-degree-of-freedom system.

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8.7 Calculate the natural frequencies and mode shapes for the system shown in Figures P8.7 by calculating the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. Graphically illustrate the mode shapes. Identify any nodes.

FIGURE P8.7

Given: m, k, L

Find: ω_1, ω_2, ω_3, mode shapes

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{5}{4}kL^2 & -k\frac{L}{2} & 0 \\ -k\frac{L}{2} & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} \frac{3}{mL^2} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{2m} \end{bmatrix} \begin{bmatrix} \frac{5}{4}kL^2 & -k\frac{L}{2} & 0 \\ -k\frac{L}{2} & 2k & -k \\ 0 & -k & k \end{bmatrix} = \frac{k}{m} \begin{bmatrix} \frac{15}{4} & -\frac{3}{2L} & 0 \\ -\frac{L}{2} & 2 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are determined from

$$\left|\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right| = 0 = \begin{bmatrix} \frac{15}{4}\phi - \lambda & -\frac{3}{2L}\phi & 0 \\ -\frac{L}{2}\phi & 2\phi - \lambda & -\phi \\ 0 & -\frac{1}{2}\phi & \frac{1}{2}\phi - \lambda \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = 0.1878\phi, \lambda_1 = 1.9351\phi, \lambda_1 = 4.127\phi$$

The natural frequencies are

$$\omega_1 = 0.4334\sqrt{\frac{k}{m}}, \omega_2 = 1.3911\sqrt{\frac{k}{m}}, \omega_3 = 2.032\sqrt{\frac{k}{m}}$$

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The mode shapes are determined from the eigenvalue problem. For an eigenvalue λ the first equation implies

$$X_1 = \frac{\frac{3}{2L}\phi}{\frac{15}{4}\phi - \lambda} X_2$$

and the third equation leads to

$$X_3 = \frac{\frac{1}{2}\phi}{\frac{1}{2}\phi - \lambda}$$

Choosing $X_2 = 1$ these lead to

$$X_1 = \begin{bmatrix} \frac{0.421}{L} \\ 1 \\ 1.601 \end{bmatrix} \qquad X_2 = \begin{bmatrix} \frac{0.8265}{L} \\ 1 \\ -0.3480 \end{bmatrix} \qquad X_2 = \begin{bmatrix} -\frac{3.978}{L} \\ 1 \\ -0.1379 \end{bmatrix}$$

Problem 8.7 illustrates calculation of natural frequencies and mode shapes for a 3DOF system.

8.8 Two machines are placed on a massless fixed-pinned beam of Figure P8.8. Determine the natural frequencies for the system.

FIGURE P8.8

Given: $m_1 = 20$ kg, $m_2 = 30$ kg, $E = 210 \times 10^9$ N/m^2, $I = 5.6 \times 10^{-4}$ m^4

Find: natural frequencies

Solution: The natural frequencies are the reciprocals of the square roots of the eigenvalues of **AM** where

$$\mathbf{M} = \begin{bmatrix} 20 & 0 \\ 0 & 30 \end{bmatrix}$$

and the flexibility matrix **A** is determined using flexibility influence coefficients as

$$\mathbf{A} = 10^{-10} \begin{bmatrix} 9.1837 & 6.0091 \\ 6.0091 & 7.2562 \end{bmatrix}$$

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The natural frequencies are the reciprocals of the square roots of the eigenvalues of **AM**. To this end letting $\phi = 10^{-7}$

$$|\mathbf{AM} - \lambda\mathbf{I}| = 0 = \begin{vmatrix} 1.837\phi - \lambda & 1.803\phi \\ 1.202\phi & 2.177\phi - \lambda \end{vmatrix} = \lambda^2 - 4.013 \times 10^{-7} + 1.832 \times 10^{-14}$$

The eigenvalues are

$$\lambda_1 = 5.25 \times 10^{-8} \quad \lambda_1 = 3.489 \times 10^{-7}$$

and the natural frequencies are

$$\omega_1 = 1.663 \times 10^3 \ \frac{\text{rad}}{\text{s}} \qquad \omega_2 = 4.364 \times 10^3 \ \frac{\text{rad}}{\text{s}}$$

Problem 8.8 illustrates the calculation of natural frequencies using the flexibility matrix.

8.9 Determine the natural frequencies and mode shapes for the system of Figure P7.2 if $k = 3.4 \times 10^5$ N/m, $L = 1.5$ m and $m = 4.6$ kg.

FIGURE P7.2
(Problems 7.2, 7.9, 7.31, 7.37, 7.52, 7.67)

Given: $k = 3.4 \times 10^5 \ \frac{\text{N}}{\text{m}}, L = 1.5 \text{ m}, m = 4.6 \text{ kg}$

Find: natural frequencies and mode shapes

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{13}{9}kL^2 & -\frac{2}{3}kL & 0 \\ -\frac{2}{3}kL & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting in given values leads to

$$\begin{bmatrix} 3.45 & 0 & 0 \\ 0 & 4.6 & 0 \\ 0 & 0 & 9.2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 1.11\times10^6 & -3.4\times10^5 & 0 \\ -3.4\times10^5 & 1.02\times10^6 & -6.8\times10^5 \\ 0 & -6.8\times10^5 & 6.8\times10^5 \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end letting $\phi = 10^5$

$$\left|\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right| = 0 = \begin{vmatrix} 3.82\phi - \lambda & -0.986\phi & 0 \\ -0.740\phi & 2.218\phi - \lambda & -1.478\phi \\ 0 & -0.740\phi & 0.740\phi - \lambda \end{vmatrix}$$
$$= -\lambda^3 + 6.17 \times 10^5\lambda^2 - 9.33 \times 10^{10}\lambda + 1.22 \times 10^{15}$$

The solutions of the cubic equation are

$$\lambda_1 = 1.441 \times 10^4, \lambda_2 = 2.222 \times 10^5, \lambda_3 = 3.806 \times 10^5$$

from which the natural frequencies are obtained as

$$\omega_1 = 120.04\ \frac{\text{rad}}{\text{s}}, \qquad \omega_2 = 471.4\ \frac{\text{rad}}{\text{s}}, \quad \omega_3 = 617.06\ \frac{\text{rad}}{\text{s}}$$

The mode shape vectors are solutions of $\mathbf{M}^{-1}\mathbf{K}\mathbf{X} = \lambda\mathbf{X}$ which are

$$\begin{bmatrix} 3.217\phi & -0.986\phi & 0 \\ -0.740\phi & 2.218\phi & -1.478\phi \\ 0 & -0.740\phi & 0.740\phi \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \lambda \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

The first equation gives

$$X_1 = \frac{0.986\phi}{3.217\phi - \lambda} X_2$$

The third equation gives

$$X_3 = \frac{0.740\phi}{0.740\phi - \lambda} X_2$$

Setting $X_2 = 1$ this leads to

$$\mathbf{X}_1 = \begin{bmatrix} 0.311 \\ 1 \\ 1.242 \end{bmatrix}, \qquad \mathbf{X}_2 = \begin{bmatrix} 0.901 \\ 1 \\ -0.499 \end{bmatrix}, \qquad \mathbf{X}_3 = \begin{bmatrix} -2.01 \\ 1 \\ -0.241 \end{bmatrix}$$

Problem 8.9 illustrates calculation of natural frequencies and mode shapes for a 3DOF system.

8.10 Determine the natural frequencies of the system of Figure P7.5 if k = 2500 N/m, m_1 = 2.4 kg, m_2 = 1.6 kg, I = 0.65 kg · m^2, and L = 1 m.

FIGURE P7.5
(Problems 7.5, 7.12, 7.26, 7.40, 7.55, 7.70)

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Given: $k = 2500\ \frac{\text{N}}{\text{m}}, m_1 = 2.4\ \text{kg}, m_2 = 1.6\ \text{kg}, I = 0.65\ \text{kg}\cdot\text{m}^2, L = 1\ \text{m}$

Find: natural frequencies and mode shapes

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & \bar{I} & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 5k & 0.6kL & -2k \\ 0.6kL & 0.5kL^2 & -0.2kL \\ -2k & -0.2kL & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting given values leads to

$$\begin{bmatrix} 2.4 & 0 & 0 \\ 0 & 0.65 & 0 \\ 0 & 0 & 1.6 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 12500 & 1500 & -5000 \\ 1500 & 1250 & -500 \\ -5000 & -500 & 5000 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M^{-1}K}$. Letting $\phi = 10^3$

$$\left|\mathbf{M^{-1}K} - \lambda\mathbf{I}\right| = 0 = \begin{vmatrix} 5.208\phi - \lambda & 0.625\phi & -2.0833\phi \\ 2.307\phi & 1.923\phi - \lambda & -0.769\phi \\ -1.923\phi & -0.1923\phi & 1.923\phi - \lambda \end{vmatrix}$$
$$= -\lambda^3 + 9.054\times 10^3\lambda^2 - 1.813\times 10^7\lambda + 9.816\times 10^9$$

The solutions of the cubic equation are

$$\lambda_1 = 9.364\times 10^2, \lambda_2 = 1.621\times 10^3, \lambda_3 = 6.447\times 10^3$$

from which the natural frequencies are obtained as

$$\omega_1 = 30.60\ \frac{\text{rad}}{\text{s}}, \qquad \omega_2 = 40.26\ \frac{\text{rad}}{\text{s}}, \quad \omega_3 = 80.60\ \frac{\text{rad}}{\text{s}}$$

The mode shape vectors are solutions of $\mathbf{M^{-1}KX} = \lambda\mathbf{X}$ which are

$$\begin{bmatrix} 5.208\phi & 0.625\phi & -2.0833\phi \\ 2.307\phi & 1.923\phi & -0.769\phi \\ -1.923\phi & -0.1923\phi & 1.923\phi \end{bmatrix}\begin{bmatrix} X_1 \\ \Theta \\ X_2 \end{bmatrix} = \lambda\begin{bmatrix} X_1 \\ \Theta \\ X_2 \end{bmatrix}$$

The first equation gives

$$(5.208\phi - \lambda)X_1 + 0.625\phi\Theta - 2.0833\phi X_2 = 0$$

The third equation gives

$$-1.923\phi X_1 - 0.1923\phi\Theta + (1.923\phi - \lambda)\phi X_2 = 0$$

Setting $X_2 = 1$ this leads to

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$$\mathbf{X}_1 = \begin{bmatrix} 0.5688 \\ -0.5475 \\ 1 \end{bmatrix}, \qquad \mathbf{X}_2 = \begin{bmatrix} -0.4138 \\ 0.5708 \\ 1 \end{bmatrix}, \qquad \mathbf{X}_3 = \begin{bmatrix} -2.2482 \\ -1.3037 \\ 1 \end{bmatrix}$$

Problem 8.10 illustrates calculation of natural frequencies and mode shapes for a 3DOF system.

8.11 Determine the natural frequencies and mode shapes of the system of Figure P7.17 if $k = 10{,}000$ N/m, $m = 3$ kg, $I = 0.6$ kg · m^2, and $r = 80$ cm.

FIGURE P7.17
(Problems 7.17, 7.45, 7.60, 7.75)

Given: $k = 10000\ \frac{\text{N}}{\text{m}}, m = 3\ \text{kg}, I = 0.6\ \text{kg} \cdot \text{m}^2,$
$r = 80$ cm

Find: natural frequencies and mode shapes

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 2m + \frac{8}{r^2} I \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -kr & 0 \\ -kr & 3kr^2 & 4kr \\ 0 & 4kr & 8k \end{bmatrix} \begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting given values leads to

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 13.5 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 10000 & -8000 & 0 \\ -8000 & 19200 & 32000 \\ 0 & 32000 & 80000 \end{bmatrix} \begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end letting $\phi = 10^4$

$$\left|\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right| = 0 = \begin{vmatrix} 0.3333\phi - \lambda & -0.2667\phi & 0 \\ -1.333\phi & 3.2\phi - \lambda & 5.3337\phi \\ 0 & 0.2370\phi & 0.5926\phi - \lambda \end{vmatrix}$$
$$= -\lambda^3 + 4.126 \times 10^4 \lambda^2 - 1.541 \times 10^8 \lambda$$

The solutions of the cubic equation are

$$\lambda_1 = 0, \lambda_2 = 4.15210^3, \lambda_3 = 3.7107 \times 10^4$$

from which the natural frequencies are obtained as

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$$\omega_1 = 0\ \frac{\text{rad}}{\text{s}}, \qquad \omega_2 = 64.43\frac{\text{rad}}{\text{s}}, \qquad \omega_3 = 192.6\ \frac{\text{rad}}{\text{s}}$$

The mode shape vectors are solutions of $\mathbf{M}^{-1}\mathbf{K}\mathbf{X} = \lambda\mathbf{X}$ which are

$$\begin{bmatrix} 0.3333\phi & -0.2667\phi & 0 \\ -1.333\phi & 3.2\phi & 5.3337\phi \\ 0 & 0.2370\phi & 0.5926\phi \end{bmatrix} \begin{bmatrix} X_1 \\ \Theta \\ X_2 \end{bmatrix} = \lambda \begin{bmatrix} X_1 \\ \Theta \\ X_2 \end{bmatrix}$$

The first equation gives

$$X_1 = \frac{0.2667\phi}{0.3333\phi - \lambda}\Theta$$

The third equation gives

$$X_3 = \frac{0.237\phi}{0.5926\phi - \lambda}\Theta$$

Setting $\Theta = 1$ this leads to

$$\mathbf{X}_1 = \begin{bmatrix} 0.8002 \\ 1 \\ 0.3999 \end{bmatrix}, \qquad \mathbf{X}_2 = \begin{bmatrix} -3.2558 \\ 1 \\ 1.3361 \end{bmatrix}, \qquad \mathbf{X}_3 = \begin{bmatrix} -0.0790 \\ 1 \\ -0.0760 \end{bmatrix}$$

Problem 8.11 illustrates calculation of natural frequencies and mode shapes for a 3DOF system.

8.12 Determine the natural frequencies and mode shapes of the system of Figure P7.19 if $k = 12{,}000$ N/m and each bar is of mass 12 kg and length 4 m.

FIGURE P7.19
(Problems 7.19, 7.28, 7.34, 7.47, 7.62, 7.77)

Given: $k = 12000\ \frac{\text{N}}{\text{m}}, m = 12\ \text{kg}, L = 4\ \text{m}$

Find: natural frequencies and mode shapes

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & \frac{1}{12}mL^2 & 0 \\ 0 & 0 & \frac{m}{3} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & k\frac{L}{2} & -\frac{k}{2} \\ k\frac{L}{2} & 3k\frac{L^2}{4} & -k\frac{L}{4} \\ -\frac{k}{2} & -k\frac{L}{4} & \frac{k}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Substituting given values leads to

$$\begin{bmatrix} 12 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 36000 & 24000 & -6000 \\ 24000 & 144000 & -12000 \\ -6000 & -12000 & 3000 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end

$$\left|\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right| = 0 = \begin{vmatrix} 3000-\lambda & 2000 & -5000 \\ 1500 & 9000-\lambda & -750 \\ -1500 & -3000 & 750-\lambda \end{vmatrix}$$
$$= -\lambda^3 + 1.275\times 10^4\lambda^2 - 3\times 10^7\lambda + 9\times 10^9$$

The solutions of the cubic equation are

$$\lambda_1 = 3.509\times 10^2, \lambda_2 = 2.624\times 10^3, \lambda_3 = 9.775\times 10^3$$

from which the natural frequencies are obtained as

$$\omega_1 = 18.77\ \frac{\text{rad}}{\text{s}}, \qquad \omega_2 = 51.22\ \frac{\text{rad}}{\text{s}}, \quad \omega_3 = 98.87\ \frac{\text{rad}}{\text{s}}$$

The mode shape vectors are solutions of $\mathbf{M}^{-1}\mathbf{K}\mathbf{X} = \lambda\mathbf{X}$ which are

$$\begin{bmatrix} 3000 & 2000 & -5000 \\ 1500 & 9000 & -750 \\ -1500 & -3000 & 750 \end{bmatrix}\begin{bmatrix} X_1 \\ \Theta \\ X_2 \end{bmatrix} = \lambda\begin{bmatrix} X_1 \\ \Theta \\ X_2 \end{bmatrix}$$

The first equation gives

$$(3000-\lambda)X_1 + 2000\Theta - 5000X_2 = 0$$

The third equation gives

$$-1500X_1 - 3000\Theta + (750-\lambda)\phi X_2 = 0$$

Setting $X_2 = 1$ this leads to

$$\mathbf{X}_1 = \begin{bmatrix} 2.8706 \\ -1.3023 \\ 1 \end{bmatrix}, \qquad \mathbf{X}_2 = \begin{bmatrix} -10.0162 \\ 4.3834 \\ 1 \end{bmatrix}, \qquad \mathbf{X}_3 = \begin{bmatrix} -1.4169 \\ -2.2999 \\ 1 \end{bmatrix}$$

Problem 8.12 illustrates calculation of natural frequencies and mode shapes for a 3DOF system.

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8.13 A 400 kg machine is placed at the midspan of a 3-m long, 200-kg simply supported beam. The beam is made of a material of elastic modulus 200×10^9 N/m^2 and has a cross-sectional moment of inertia of 1.4×10^{-5} m^4. Use a three degree-of-freedom model to approximate the system's lowest natural frequency.

Given: m = 400 kg, L = 3 m, m_b = 200 kg, E = 200×10^9 N/m^2, I = 1.4×10^{-5} m^4

Find: ω_1

Solution: The inertia effects of the beam are approximated by lumping particles of mass $m_b/4$ at x = L/4, L/2, and 3L/4 along the span of the beam. The machine is placed at the misdpan of the beam. Hence the mass matrix for a three degree-of-freedom model is

$$\mathbf{M} = \begin{bmatrix} \frac{m_b}{4} & 0 & 0 \\ 0 & \frac{m_b}{4} + m & 0 \\ 0 & 0 & \frac{m_b}{4} \end{bmatrix} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 450 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

The flexibility matrix for the beam is determined. The deflection of a particle a distance z along the neutral axis of a simply supported beam, measured from the left support, due to a concentrated unit load applied a distance a from the left support is

$$y(z) = \frac{L^3}{6EI}\left(1 - \frac{a}{L}\right)\left[\frac{a}{L}\left(2 - \frac{a}{L}\right)\frac{z}{L} - \left(\frac{z}{L}\right)^3\right]$$

for a ≥ z. The elements of the third column of the flexibility matrix are the displacements induced by a unit concentrated load at a = 3L/4. Then

$$y(z) = \frac{L^3}{24EI}\left[\frac{15}{16}\frac{z}{L} - \left(\frac{z}{L}\right)^3\right]$$

and the flexibility influence coefficients are

$$a_{13} = y\left(\frac{L}{4}\right) = \frac{7L^3}{768EI} \quad a_{23} = y\left(\frac{L}{2}\right) = \frac{11L^3}{768EI} \quad a_{33} = y\left(\frac{3L}{4}\right) = \frac{3L^3}{256EI}$$

The second column of the flexibility matrix is determined by placing a unit concentrated load at a = L/2. Then

$$y(z) = \frac{L^3}{12EI}\left(\frac{3z}{4L} - \left(\frac{z}{L}\right)^3\right)$$

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Note that due to symmetry only a_{22} needs to be calculated. To this end

$$a_{22} = y\left(\frac{L}{2}\right) = \frac{L^3}{48EI}$$

Then from symmetry of the flexibility matrix $a_{32} = a_{23}$ and from symmetry of the beam $a_{12} = a_{32}$. Then from symmetry and reciprocity, $a_{21} = a_{12}$ and $a_{31} = a_{13}$. Thus the flexibility matrix is

$$\mathbf{A} = \frac{L^3}{EI}\begin{bmatrix} \frac{3}{256} & \frac{11}{768} & \frac{7}{768} \\ \frac{11}{768} & \frac{1}{48} & \frac{11}{768} \\ \frac{7}{768} & \frac{11}{768} & \frac{3}{256} \end{bmatrix} = 10^{-7}\begin{bmatrix} 1.130 & 1.381 & 0.879 \\ 1.381 & 2.009 & 1.381 \\ 0.879 & 1.381 & 1.130 \end{bmatrix}$$

The natural frequencies are the reciprocals of the square roots of the eigenvalues of **AM**. To this end

$$\mathbf{AM} = 10^{-7}\begin{bmatrix} 1.130 & 1.381 & 0.879 \\ 1.381 & 2.009 & 1.381 \\ 0.879 & 1.381 & 1.130 \end{bmatrix}\begin{bmatrix} 50 & 0 & 0 \\ 0 & 450 & 0 \\ 0 & 0 & 50 \end{bmatrix} = 10^{-4}\begin{bmatrix} 00565 & 0.6215 & 0.0439 \\ 0.0691 & 0.9040 & 0.0691 \\ 0.0439 & 0.6215 & 0.0565 \end{bmatrix}$$

$$|\mathbf{AM} - \lambda\mathbf{I}| = 0$$

$$10^{-4}\begin{vmatrix} 00565 - 10^4\lambda & 0.6215 & 0.0439 \\ 0.0691 & 0.9040 - 10^4\lambda & 0.0691 \\ 0.0439 & 0.6215 & 0.0565 - 10^4\lambda \end{vmatrix} = 0$$

$$10^{12}\lambda^3 - 1.017\times10^8\lambda^2 + 1.758\times10^2\lambda - 6.235\times10^{-5} = 0$$

$$\lambda = 5.00\times10^{-1},\ 1.26\times10^{-2},\ 9.995\times10^{-3}$$

$$\omega_1 = \frac{1}{\sqrt{\lambda_3}} = \frac{1}{\sqrt{9.995\times10^{-3}}} = 100.02 \text{ rad/s}$$

Problem 8.13 illustrates the determination of natural frequencies using the flexibility matrix.

8.14 A 500 kg machine is placed at the end of a 3.8-m long, 190-kg fixed-free beam. The beam is made of a material of elastic modulus 200×10^9 N/m^2 and has a cross-sectional moment of inertia of 1.4×10^{-5} m^4. Use a three degree-of-freedom model to approximate the two lowest natural frequencies of the system.

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Given: m = 500 kg, L = 3.8 m, m_b = 190 kg, E = 200 × 10^9 N/m^2, I = 1.4 × 10^{-5} m^4

Find: ω_1, ω_2

Solution: The inertia effects of the beam are modeled by placing particles along the span of the beam as shown below. The mass matrix for this model is

$$\mathbf{M} = \begin{bmatrix} \frac{m_b}{3} & 0 & 0 \\ 0 & \frac{m_b}{3} & 0 \\ 0 & 0 & \frac{m_b}{6} + m \end{bmatrix} = \begin{bmatrix} 63.33 & 0 & 0 \\ 0 & 63.33 & 0 \\ 0 & 0 & 531.67 \end{bmatrix}$$

The flexibility matrix is determined for the fixed-free beam. The third column is determined by placing a concentrated unit load at the end of the beam. Then if y(z) is the resulting deflected shape of the beam

$$a_{13} = y\left(\frac{L}{3}\right) \quad a_{23} = y\left(\frac{2L}{3}\right) \quad a_{33} = y(L)$$

The second column of the flexibility matrix is determined by placing a concentrated unit load at z = 2L/3. If y(z) is the resulting deflected shape of the beam then

$$a_{12} = y\left(\frac{L}{3}\right) \quad a_{22} = y\left(\frac{2L}{3}\right) \quad a_{32} = y(L)$$

The first column of the flexibility matrix is obtained by placing a unit concentrated load at z = L/3. From symmetry of the flexibility matrix only a_{11} = y(L/3) needs to be calculated. The resulting flexibility matrix is

$$\mathbf{A} = \frac{L^3}{EI}\begin{bmatrix} 0.0123 & 0.0309 & 0.0494 \\ 0.0309 & 0.0988 & 0.1728 \\ 0.0494 & 0.1728 & 0.3333 \end{bmatrix} = 10^{-5}\begin{bmatrix} 0.0242 & 0.0605 & 0.0968 \\ 0.0605 & 0.1936 & 0.3387 \\ 0.0968 & 0.3387 & 0.6532 \end{bmatrix}$$

The natural frequencies are the reciprocals of the square roots of the eigenvalues of **AM**. To this end

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$$\mathbf{AM} = 10^{-5}\begin{bmatrix} 0.0242 & 0.0605 & 0.0968 \\ 0.0605 & 0.1936 & 0.3387 \\ 0.0968 & 0.3387 & 0.6532 \end{bmatrix}\begin{bmatrix} 63.33 & 0 & 0 \\ 0 & 63.33 & 0 \\ 0 & 0 & 531.67 \end{bmatrix} = 10^{-5}\begin{bmatrix} 1.53 & 3.83 & 51.45 \\ 3.83 & 12.25 & 180.1 \\ 6.13 & 21.45 & 347.3 \end{bmatrix}$$

$$|\mathbf{AM} - \lambda\mathbf{I}| = 0$$

$$\begin{vmatrix} 1.53 - 10^5\lambda & 3.83 & 51.45 \\ 3.83 & 12.25 - 10^5\lambda & 180.1 \\ 6.13 & 21.45 & 347.3 - 10^5\lambda \end{vmatrix} = 0$$

$$1\times10^{15}\lambda^3 - 3.6\times10^{12}\lambda^2 6.15\times10^7\lambda - 9.816 = 0$$

$$\lambda = 1.78\times10^{-6},\ \ 1.533\times10^{-5},\ \ 3.594\times10^{-3}$$

$$\omega_1 = \frac{1}{\sqrt{\lambda_3}} = \frac{1}{\sqrt{3.594\times10^{-3}}} = 16.68\ \text{rad/s}$$

$$\omega_2 = \frac{1}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{1.533\times10^{-5}}} = 749.1\ \text{rad/s}$$

Problem 8.14 illustrates the determination of natural frequencies using the flexibility matrix.

8.15 Determine the two lowest natural frequencies of the railroad bridge of Chapter Problem 7.84 if $k_1 = 5.5 \times 10^7$ N/m, $k_2 = 1.2 \times 10^7$ N/m, $m = 15{,}000$ kg, $I = 1.6 \times 10^6$ kg · m^2, $l = 6.7$ m, and $h = 8.8$ m.

FIGURE P7.84

Given: $k_1 = 5.5 \times 10^7$ N/m, $k_2 = 1.2 \times 10^7$ N/m, m = 15,000 kg, I = 1.6×10^6 kg · m^2, l = 6.7 m, and h = 8.8 m

Find: ω_1, ω_2

Solution: The differential equations governing the motion of the system are derived in the solution of Problem 7.84. They are

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$$\begin{bmatrix} I+m\ell^2 & 0 & 0 \\ 0 & I+m\ell^2 & 0 \\ 0 & 0 & I+m\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix}+\begin{bmatrix} (k_1+k_2)h^2 & -k_2h^2 & 0 \\ -k_2h^2 & 2k_2h^2 & -k_2h^2 \\ 0 & -k_2h^2 & (k_1+k_2)h^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting given values the mass matrix becomes

$$\mathbf{M}=\begin{bmatrix} I+m\ell^2 & 0 & 0 \\ 0 & I+m\ell^2 & 0 \\ 0 & 0 & I+m\ell^2 \end{bmatrix}=10^6\begin{bmatrix} 2.273 & 0 & 0 \\ 0 & 2.273 & 0 \\ 0 & 0 & 2.273 \end{bmatrix}$$

Substituting given values the stiffness matrix becomes

$$\mathbf{K}=\begin{bmatrix} (k_1+k_2)h^2 & -k_2h^2 & 0 \\ -k_2h^2 & 2k_2h^2 & -k_2h^2 \\ 0 & -k_2h^2 & (k_1+k_2)h^2 \end{bmatrix}=10^9\begin{bmatrix} 4.288 & -0.768 & 0 \\ -0.768 & 1.536 & -0.768 \\ 0 & -0.768 & 4.288 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end

$$\mathbf{M}^{-1}\mathbf{K}=10^{-7}\begin{bmatrix} 4.40 & 0 & 0 \\ 0 & 4.40 & 0 \\ 0 & 0 & 4.40 \end{bmatrix}10^9\begin{bmatrix} 4.288 & -0.768 & 0 \\ -0.768 & 1.536 & -0.768 \\ 0 & -0.768 & 4.288 \end{bmatrix}=$$

$$10^3\begin{bmatrix} 1.89 & -0.338 & 0 \\ -0.338 & 0.678 & -0.338 \\ 0 & -0.338 & 1.886 \end{bmatrix}$$

$$\left|\mathbf{M}^{-1}\mathbf{K}-\lambda\mathbf{I}\right|=0$$

$$\begin{vmatrix} 1.89-10^{-3}\lambda & -0.338 & 0 \\ -0.338 & 0.678-10^{-3}\lambda & -0.338 \\ 0 & -0.338 & 1.886-10^{-3}\lambda \end{vmatrix}=0$$

$$\lambda^3-4.448\times10^3\lambda^2+5.878\times10^6\lambda-1.973\times10^9=0$$

$$\lambda=5.098\times10^2,\ 1.886\times10^3, 2.052\times10^3$$

$$\omega_1=\sqrt{\lambda_1}=\sqrt{5.098\times10^2}=22.58\ \text{rad/s}$$

$$\omega_2=\sqrt{\lambda_2}=\sqrt{1.886\times10^3}=43.43\ \text{rad/s}$$

Problem 8.15 illustrates the determination of natural frequencies for a three-degree-of-freedom system.

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8.16 Determine the natural frequencies of the system of Chapter Problem 7.89. The beam is of length 5 m, made of a material of elastic modulus 200×10^9 N/m^2, and has a cross-sectional moment of inertia of 1.4×10^{-5} m^4. The total mass of the beam is 320 kg. The mass of the winch is 115 kg. The winch cable is made of a material of elastic modulus 200×10^9 N/m^2 and a cross-sectional area of 3.4×10^{-2} m^2. The length of the cable is 5.5 m and the mass being lifted is 715 kg.

FIGURE P7.89

Given: $L_b = 5$ m, $E_b = 200 \times 10^9$ N/m^2, $I_b = 1.4 \times 10^{-5}$ m^4, $m_b = 320$ kg, $m_w = 115$ kg, $E_w = 200 \times 10^9$ N/m^2, $A = 3.4 \times 10^{-2}$ m^2, $L_c = 5.5$ m, $m = 715$ kg

Find: ω_1, ω_2, ω_3, ω_4

Solution: The stiffness of the cable is

$$k = \frac{AE_c}{L_c} = \frac{(3.4\times10^{-2}\ \text{m}^2)(200\times10^9\ \text{N/m}^2)}{5.5\ \text{m}} = 1.23\times10^9\ \text{N/m}$$

The differential equations governing the motion of the system are derived in the solution of Problem 5.77. They are

$$\frac{L^3}{768EI}\begin{bmatrix} 7 & 11 & 9 & 11 \\ 11 & 16 & 11 & 16 \\ 9 & 11 & 7 & 11 \\ 11 & 16 & 11 & 16+\dfrac{768EI}{kL^3} \end{bmatrix}\begin{bmatrix} m_b/4 & 0 & 0 & 0 \\ 0 & m_b/4+m_w & 0 & 0 \\ 0 & 0 & m_b/4 & 0 \\ 0 & 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the reciprocals of the square roots of the eigenvalues of **AM**. To this end

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$$\mathbf{AM} = \frac{L^3}{768EI}\begin{bmatrix} 7 & 11 & 9 & 11 \\ 11 & 16 & 11 & 16 \\ 9 & 11 & 7 & 11 \\ 11 & 16 & 11 & 16+\dfrac{768EI}{kL^3} \end{bmatrix}\begin{bmatrix} m_b/4 & 0 & 0 & 0 \\ 0 & m_b/4+m_w & 0 & 0 \\ 0 & 0 & m_b/4 & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$$

where

$$\frac{L^3}{768EI} = \frac{(5\text{ m})^3}{768(200\times10^9\text{ N/m}^2)(1.4\times10^{-5}\text{ m}^4)} = 2.71\times10^{-7}\text{ m/N}$$

$$\frac{768EI}{kL^3} = \frac{1}{(1.23\times10^{-9}\text{ N/m})(2.7\times10^{-7}\text{ m/N})} = 0.034$$

Thus

$$\mathbf{AM} = 2.71\times10^{-7}\begin{bmatrix} 7 & 11 & 9 & 11 \\ 11 & 16 & 11 & 16 \\ 9 & 11 & 7 & 11 \\ 11 & 16 & 11 & 16.034 \end{bmatrix}\begin{bmatrix} 80 & 0 & 0 & 0 \\ 0 & 195 & 0 & 0 \\ 0 & 0 & 80 & 0 \\ 0 & 0 & 0 & 715 \end{bmatrix}$$

$$\mathbf{AM} = 10^{-4}\begin{bmatrix} 1.516 & 5.809 & 1.948 & 21.298 \\ 2.383 & 8.449 & 2.383 & 30.979 \\ 1.950 & 5.809 & 1.516 & 21.298 \\ 2.383 & 8.449 & 2.383 & 31.045 \end{bmatrix}$$

$$|\mathbf{AM} - \lambda\mathbf{I}| = 0$$

$$\begin{vmatrix} 1.516-10^4\lambda & 5.809 & 1.948 & 21.298 \\ 2.383 & 8.449-10^4\lambda & 2.383 & 30.979 \\ 1.950 & 5.809 & 1.516-10^4\lambda & 21.298 \\ 2.383 & 8.449 & 2.383 & 31.045-10^4\lambda \end{vmatrix} = 0$$

$$10^{16}\lambda^4 - 4.3\times10^{13}\lambda^3 - 1.03\times10^9\lambda^2 - 1.37\times10^4\lambda - 4.57\times10^{-2} = 0$$

$$\lambda = 4.3\times10^{-2},\ -4.33\times10^{-5},\ 1.79\times10^{-5},\ 1.38\times10^{-6}$$

The negative sign on one of the eigenvalues indicates that the flexibility matrix is not positive definite. This occurs because of the large stiffness of the cable compared to the beam. The smallest natural frequency is

$$\omega_1 = \frac{1}{\sqrt{\lambda_4}} = 852.8\text{ rad/s}$$

Problem 8.16 illustrates determination of natural frequencies using the flexibility matrix.

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8.17 Determine the free vibration response of the railroad bridge of Chapter Problem 8.15 if a ground disturbance initially leads to $\theta_1 = 0.8^o$ with $\theta_2 = \theta_3 = 0$.

FIGURE P7.84

Given: $k_1 = 5.5 \times 10^7$ N/m, $k_2 = 1.2 \times 10^7$ N/m, m = 15,000 kg, $I = 1.6 \times 10^6$ kg · m^2, l = 6.7 m, h = 8.8 m, $\theta_1(0) = 0.8^o$, $\theta_2(0) = \theta_3(0) = 0$

Find: **x**(t)

Solution: The matrix $\mathbf{M}^{-1}\mathbf{K}$ for this system is determined in the solution of Problem 6.17 as

$$\mathbf{M}^{-1}\mathbf{K} = 10^3 \begin{bmatrix} 1.89 & -0.338 & 0 \\ -0.338 & 0.678 & -0.338 \\ 0 & -0.338 & 1.886 \end{bmatrix}$$

Its eigenvalues are determined as

$$\lambda = 5.098 \times 10^2,\ 1.886 \times 10^3, 2.052 \times 10^3$$

The natural frequencies are the square roots of the eigenvalues

$$\omega_1 = 22.58 \text{ rad/s}$$
$$\omega_2 = 43.43 \text{ rad/s}$$
$$\omega_3 = 45.30 \text{ rad/s}$$

The mode shape for the first mode is determined from

$$\left(\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right)\mathbf{X} = \mathbf{0}$$

$$10^3 \begin{bmatrix} 1.89 - 0.5098 & -0.338 & 0 \\ -0.338 & 0.678 - 0.5098 & -0.338 \\ 0 & -0.338 & 1.886 - 0.5098 \end{bmatrix} \begin{bmatrix} \Theta_{11} \\ \Theta_{12} \\ \Theta_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first and third equations lead to

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$$\Theta_{11} = \frac{0.338}{1.996 - 0.5098}\Theta_{12} = 0.246\Theta_{12}$$
$$\Theta_{13} = \frac{0.338}{1.996 - .5098}\Theta_{12} = 0.246\Theta_{12}$$

Thus the first mode shape vector

$$\Theta_1 = \begin{bmatrix} 0.246 \\ 1 \\ 0.46 \end{bmatrix}$$

The modes shape vectors for the second and third modes are similarly obtained as

$$\Theta_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \Theta_3 = \begin{bmatrix} -2.04 \\ 1 \\ -2.04 \end{bmatrix}$$

The general solution of the differential equations is

$$\mathbf{x}(t) = C_1\Theta_1 \sin(\omega_1 t + \phi_1) + C_2\Theta_2 \sin(\omega_2 t + \phi_1) + C_3\Theta_1 \sin(\omega_2 t + \phi_2)$$

The initial conditions are

$$\mathbf{x}(0) = \begin{bmatrix} 0.8° \\ 0 \\ 0 \end{bmatrix} \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Application of the initial velocity conditions leads to

$$\phi_1 = \phi_2 = \phi_3 = \frac{\pi}{2}$$

Application of the initial displacement conditions leads to

$$\begin{bmatrix} 0.8° \\ 0 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 0.246 \\ 1 \\ 0.246 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + C_3 \begin{bmatrix} -2.04 \\ 1 \\ -2.04 \end{bmatrix}$$

The solution of the above equations is

$$C_1 = 0.1752° \quad C_2 = 0.4005° \quad C_3 = -0.1752°$$

The response of the system is

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$$\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\\ \theta_3(t)\end{bmatrix} = .1752^\circ\begin{bmatrix}0.246\\ 1\\ 0.246\end{bmatrix}\sin\left(22.58t+\frac{\pi}{2}\right) + 0.4405^\circ\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}\sin\left(43.43t+\frac{\pi}{2}\right)$$
$$-0.1752^\circ\begin{bmatrix}-2.04\\ 1\\ -2.04\end{bmatrix}\sin\left(45.30t+\frac{\pi}{2}\right)$$

Problem 8.17 illustrates the application of initial conditions to determine the free-vibration response of a multi-degree-of-freedom system.

8.18 A robot arm is 60 cm long, made of a material of elastic modulus $200 \times 10^9 \mathrm{N/m^2}$ and has the cross section Figure P8.18. The total mass of the arm is 850 g. A tool of mass 1 kg is attached to the end of the arm. Assume one end of the arm is pinned and the other end is free. Use a three degree-of-freedom model to determine the arm's natural frequencies.

FIGURE P8.18

Given: E = $200 \times 10^9 \mathrm{N/m^2}$, L = 0.6 m, $m_b = 0.85$ kg, m = 1 kg

Find: natural frequencies from 3DOF model.

Solution: The mass of the beam is lumped at 3 locations along the axis of the beam. Since one end is a pinned support and the other end is free the masses of these particles is 0.85 kg = 0.142 kg. The total mass matrix is

$$\begin{bmatrix}0.142 & 0 & 0\\ 0 & 0.284 & 0\\ 0 & 0 & 1.142\end{bmatrix}$$

The stiffness matrix must be calculated as the beam is pinned-free and hence unrestrained. The natural frequencies are the square roots of the eigenvalues of $\mathbf{M^{-1}K}$.

Problem 8.18 illustrates the calculation of natural frequencies.

8.19 A 30,000 kg locomotive is coupled to a fully loaded 20,000 kg boxcar and moving at 6.5 m/s. The assembly is coupled to a stationary and empty 5,000 kg cattle car. The stiffness of each coupling is 5.7×10^5 N/m. (a) What are the natural frequencies of the three-car assembly? (b) Mathematically describe the motion of the cattle car after coupling.

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Given: $m_1 = 30{,}000$ kg, $m_2 = 20{,}000\ kg$, $m_3 = 5{,}000$ kg, $v_1 = v_2 = 6.5\ \frac{m}{s}$, $v_3 = 0, k = 5.7 \times 10^5$ N/m

Find: natural frequencies, $x_3(t)$

Solution: (a) Let x_1 represent the motion of the locomotive, x_2 the motion of the boxcar and x_3 the motion of the cattle car. The differential equations of motion of the assembly are

$$\begin{bmatrix} 30000 & 0 & 0 \\ 0 & 20000 & 0 \\ 0 & 0 & 5000 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + 5.7x10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M^{-1}K}$. To this end

$$\left|\mathbf{M^{-1}K} - \lambda\mathbf{I}\right| = 0 = \begin{vmatrix} 19-\lambda & -19 & 0 \\ -28.5 & 57-\lambda & -28.5 \\ 0 & -114 & 114-\lambda \end{vmatrix} = -\lambda^3 + 190\lambda^2 - 5965\lambda$$

The solutions of the cubic equation are

$$\lambda_1 = 0, \lambda_2 = 39.61, \lambda_3 = 150.39$$

from which the natural frequencies are obtained as

$$\omega_1 = 0\ \frac{\text{rad}}{\text{s}}, \qquad \omega_2 = 6.293\ \frac{\text{rad}}{\text{s}}, \quad \omega_3 = 12.264\ \frac{\text{rad}}{\text{s}}$$

(b) The mode shape vectors are solutions of $\mathbf{M^{-1}KX} = \lambda\mathbf{X}$ which are

$$\begin{bmatrix} 19 & -19 & 0 \\ -28.5 & 57 & -28.5 \\ 0 & -114 & 114 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \lambda \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

The first equation gives

$$X_1 = \frac{19}{19-\lambda} X_2$$

The third equation gives

$$X_3 = \frac{114}{114-\lambda} X_2$$

Setting $X_2 = 1$ this leads to

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{X}_2 = \begin{bmatrix} -0.9241 \\ 1 \\ 1.5324 \end{bmatrix}, \qquad \mathbf{X}_3 = \begin{bmatrix} -0.1440 \\ 1 \\ -3.1324 \end{bmatrix}$$

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The general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = (C_1 + C_2 t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.9241 \\ 1 \\ 1.5324 \end{bmatrix}(C_3 \cos 6.293t + C_4 \sin 6.293t)$$
$$+ \begin{bmatrix} -0.1440 \\ 1 \\ -3.1324 \end{bmatrix}(C_5 \cos 12.264t + C_6 \sin 12.264t)$$

The initial conditions are

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 6.5 \\ 6.5 \\ 0 \end{bmatrix}$$

Application of the initial conditions to the general solution leads to $C_1 = C_3 = C_5 = 0$ and $C_2 = 5.9076, C_4 = -0.1380, C_6 = 0.1191$. Then $x_3(t) = 5.9076t - 0.4809 \sin 6.293t - 0.3731 \sin 12.264t$

Problem 8.19 illustrates natural frequency calculations for an unrestrained system and application of initial conditions to determine constants of integration.

8.20 Determine the natural frequencies and mode shapes for the three degree-of-freedom model of the airplane of Chapter Problem 7.87. Assume $m = 3.5$ m.

FIGURE P7.87

Given: $m = 3.5$ m

Find: ω_1, ω_2, ω_3, and corresponding mode shapes

Solution: The differential equations governing the motion of the three-degree-of-freedom system are derived in the solution of Problem 7.87 as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \frac{3EI}{L^3}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end with $m = 3.5$m

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$$\mathbf{M}^{-1}\mathbf{K} = \frac{3EI}{mL^3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{3EI}{mL^3}\begin{bmatrix} 1 & -1 & 0 \\ -\frac{2}{7} & \frac{4}{7} & -\frac{2}{7} \\ 0 & -1 & 1 \end{bmatrix}$$

$$\left|\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right| = 0$$

$$\begin{vmatrix} \phi - \lambda & -\phi & 0 \\ -\frac{2}{7}\phi & \frac{4}{7}\phi - \lambda & -\frac{2}{7}\phi \\ 0 & -\phi & \phi - \lambda \end{vmatrix} \qquad \phi = \frac{3EI}{mL^3}$$

$$\beta^3 - 2.5714\beta^2 + 1.5713\beta = 0 \qquad \beta = \frac{\lambda}{\phi}$$

$$\beta = 0,\ 1,\ 1.5714$$

$$\omega_1 = \sqrt{\lambda_1} = 0$$

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{3EI}{mL^3}} = 1.7321\sqrt{\frac{EI}{mL^3}}$$

$$\omega_3 = \sqrt{\lambda_3} = \sqrt{1.5714\frac{3EI}{mL^3}} = 2.1712\sqrt{\frac{EI}{mL^3}}$$

Note that system is unrestrained and hence its lowest natural frequency is zero

The mode shape vector for the first mode is obtained by

$$\left(\mathbf{M}^{-1}\mathbf{K} - \lambda_1\mathbf{I}\right)\mathbf{X_1} = \mathbf{0}$$

$$\begin{bmatrix} 1-0 & -1 & 0 \\ -.2857 & .5714-0 & -.2857 \\ 0 & -1 & 1-0 \end{bmatrix}\begin{bmatrix} X_{11} \\ X_{12} \\ X_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_{11} = X_{12} \quad X_{13} = X_{12}$$

$$\mathbf{X_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The modes shape vectors for the second and third modes are obtained in a similar fashion leading to

$$\mathbf{X_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{X_3} = \begin{bmatrix} 1.750 \\ -1 \\ 1.750 \end{bmatrix}$$

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Problem 8.20 illustrates determination of natural frequencies and mode shapes for an unrestrained system.

8.21 Determine the natural frequencies and mode shapes of the torsional system of Problem 7.81.

FIGURE P7.81

Given: $I_{Motor} = 1800$ kg · m^2, $I_{Turbine} = 600$ kg · m^2, $I_{GearA} = 400$ kg · m^2, $I_{GearB} = 80$ kg · m^2, Turbine shaft: G = 80 × 10^9 N/m^2, L = 2.1 m, d = 180 mm, Motor shaft: G = 80 × 10^9 N/m^2, L = 1.4 m, d = 305 mm

Find: ω and mode shapes

Solution: The torsional stiffness of the motor shaft is

$$k_m = \frac{J_m G_m}{L_m} = \frac{\pi(0.305\text{ m})^4(80\times10^9\text{ N/m}^2)}{32(1.4\text{ m})} = 4.85\times10^7\text{ N - m/rad}$$

The torsional stiffness of the turbine shaft is

$$k_t = \frac{J_t G_t}{L_t} = \frac{\pi(0.180\text{ m})^4(80\times10^9\text{ N/m}^2)}{32(2.1\text{ m})} = 3.93\times10^6\text{ N - m/rad}$$

It is noted that since θ_2 represents the rotation of gear B, the angular rotation of gear A is $\theta_2/4$. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}I_m\dot{\theta}_1^2 + \frac{1}{2}I_A\left(\frac{\dot{\theta}_2}{4}\right)^2 + \frac{1}{2}I_B\dot{\theta}_2^2 + \frac{1}{2}I_t\dot{\theta}_3^2$$

The mass matrix for the system is determined from the kinetic energy as

$$\mathbf{M} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & \frac{I_A}{16} + I_B & 0 \\ 0 & 0 & I_t \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 1800 & 0 & 0 \\ 0 & 105 & 0 \\ 0 & 0 & 600 \end{bmatrix}\text{kg - m}^2$$

The potential energy of the system at an arbitrary instant is

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$$V = \frac{1}{2}k_m\left(\frac{\theta_2}{4} - \theta_1\right)^2 + \frac{1}{2}k_t(\theta_3 - \theta_2)^2$$

$$V = \frac{1}{2}\left[k_m\theta_1^2 - \frac{1}{2}k_m\theta_1\theta_2 + \left(\frac{1}{16}k_m + k_t\right)\theta_2^2 - 2k_t\theta_2\theta_3 + k_t\theta_3^2\right]$$

The stiffness matrix is determined from the potential energy as

$$\mathbf{K} = \begin{bmatrix} k_m & -\frac{1}{4}k_m & 0 \\ -\frac{1}{4}k_m & \frac{1}{16}k_m + k_t & -k_t \\ 0 & -k_t & k_t \end{bmatrix}$$

$$\mathbf{K} = 10^7\begin{bmatrix} 4.85 & 1.21 & 0 \\ -1.21 & 0.696 & -0.393 \\ 0 & -0.393 & 0.393 \end{bmatrix}\text{N - m/rad}$$

Thus the differential equations governing the motion of the system are

$$\begin{bmatrix} 1800 & 0 & 0 \\ 0 & 105 & 0 \\ 0 & 0 & 60 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + 10^7\begin{bmatrix} 4.85 & -1.21 & 0 \\ -1.21 & 0.696 & -0.393 \\ 0 & -0.393 & 0.393 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end

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$$\mathbf{M}^{-1}\mathbf{K} = 10^7 \begin{bmatrix} 5.56\times10^{-4} & 0 & 0 \\ 0 & 9.54\times10^{-3} & 0 \\ 0 & 0 & 1.67\times10^{-3} \end{bmatrix} \begin{bmatrix} 4.85 & -1.21 & 0 \\ -1.21 & 0.696 & -0.393 \\ 0 & -0.393 & 0.393 \end{bmatrix}$$

$$\mathbf{M}^{-1}\mathbf{K} = 10^5 \begin{bmatrix} 0.2694 & -0.0674 & 0 \\ -1.1548 & 0.6630 & -0.3743 \\ 0 & -0.0655 & 0.0655 \end{bmatrix}$$

$$\left|\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right| = 0$$

$$\begin{vmatrix} 0.2694-10^{-5}\lambda & -0.0674 & 0 \\ -1.1548 & 0.6630-10^{-5}\lambda & -0.3743 \\ 0 & -0.0655 & 0.06550-10^{-5}\lambda \end{vmatrix} = 0$$

$$10^{-15}\lambda^3 - 9.979\times10^{-11}\lambda^2 + 1.374\times10^{-6}\lambda = 0$$

$$\lambda = 0,\ 1.65\times10^4,\ \ 8.33\times10^5$$

$$\omega_1 = \sqrt{\lambda_1} = 0 \text{ rad/s},\quad \omega_2 = \sqrt{\lambda_2} = 128.4 \text{ rad/s}\quad \omega_3 = \sqrt{\lambda_3} = 288.6 \text{ rad/s}$$

The mode shape corresponding to the lowest natural frequency is obtained from

$$\left(\mathbf{M}^{-1}\mathbf{K} - \lambda_1\mathbf{I}\right)\mathbf{X}_1 = \mathbf{0}$$

The mode shape vectors are obtained as

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}\ \mathbf{X}_2 = \begin{bmatrix} -0.4741 \\ -0.7353 \\ 0.4843 \end{bmatrix}\ \mathbf{X}_3 = \begin{bmatrix} 0.1183 \\ -0.7894 \\ 0.0844 \end{bmatrix}$$

Problem 8.21 illustrates determination of natural frequencies and mode shapes for an unrestrained system.

8.22 Use a four degree-of-freedom model to approximate the two lowest nonzero natural frequencies of a free-free beam.

Given: four degree-of-freedom model of free-free beam

Find: natural frequencies

Solution: The four degree-of-freedom model of the beam is shown below.

The mass matrix for the model is

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$$\mathbf{M}=\begin{bmatrix}\frac{m}{4} & 0 & 0 & 0\\ 0 & \frac{m}{4} & 0 & 0\\ 0 & 0 & \frac{m}{4} & 0\\ 0 & 0 & 0 & \frac{m}{4}\end{bmatrix}$$

The stiffness matrix is

$$\mathbf{K}=\frac{EI}{L^3}10^3\begin{bmatrix}0.2 & -0.45 & 0.3 & -0.05\\ 0.45 & 1.2 & -1.05 & 0.3\\ 0.3 & -1.05 & 1.2 & -0.45\\ -.05 & 0.3 & -0.45 & 0.2\end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. To this end

$$\mathbf{M}^{-1}\mathbf{K}=\frac{4\times10^3 EI}{mL^3}\begin{bmatrix}0.2 & -0.45 & 0.3 & -0.05\\ 0.45 & 1.2 & -1.05 & 0.3\\ 0.3 & -1.05 & 1.2 & -0.45\\ -.05 & 0.3 & -0.45 & 0.2\end{bmatrix}$$

$$\left|\mathbf{M}^{-1}\mathbf{K}-\lambda\mathbf{I}\right|=0$$

$$\begin{vmatrix}0.2-\beta\lambda & -0.45 & 0.3 & -0.05\\ 0.45 & 1.2-\beta\lambda & -1.05 & 0.3\\ 0.3 & -1.05 & 1.2-\beta\lambda & -0.45\\ -.05 & 0.3 & -0.45 & 0.2-\beta\lambda\end{vmatrix}=0 \quad \beta=\frac{mL^3}{4000EI}$$

$$\mu^4-2.8\mu^3+0.75\mu=0 \quad \mu=\beta\lambda$$

$$\mu=0,0,0.3,2.5$$

$$\omega_3=\sqrt{\frac{\mu}{\beta}}=\sqrt{\frac{0.3(4000)EI}{mL^3}}=34.64\sqrt{\frac{EI}{mL^3}}$$

$$\omega_4=\sqrt{\frac{2.5(4000)EI}{mL^3}}=100\sqrt{\frac{EI}{mL^3}}$$

Problem 8.22 illustrates determination of natural frequencies of an unrestrained system.

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8.23 A pipe extends from a wall as showing in Figure P8.23. The pipe is supported at A to prevent transverse displacement, but not to prevent rotation. Under what conditions will the pipe's lowest natural frequency of transverse vibrations coincide with its frequency of free torsional vibrations?

FIGURE P8.23

Given: pipe extending from wall

Find: relation between parameter for which natural frequency of transverse motion is equal to the natural frequency of torsional motion.

Solution: The moment of inertia of the beam is

$$I = \frac{\pi}{4}[(r+t)^4 - r^4]$$

The polar moment of inertia is

$$J = \frac{\pi}{2}[(r+t)^4 - r^4]$$

A two degree of freedom model is used. For the transverse vibrations place a mass a distance L/3 from the fixed support and a mass at the end of the beam. If the total mass of the beam is m, the mass matrix becomes

$$\mathbf{M} = \begin{bmatrix} \frac{m}{3} & 0 \\ 0 & \frac{m}{6} \end{bmatrix}$$

The flexibility matrix is found using appendix D with a fixed-free beam with an overhang,

$$\mathbf{A} = \frac{L^3}{EI}\begin{bmatrix} 0.003906 & -0.05208 \\ -0.05208 & 0.0401 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of **AM**.

The torisonal model can be built using two degrees of freedom.

Problem 8.23 illustrates the natural frequencies using a flexibility matrix.

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8.24 Show that Rayleigh's quotient $R(\mathbf{X})$ is stationary if and only if $\mathbf{X}$ is a mode shape vector.

Given: R($\mathbf{X}$)

Show: R($\mathbf{X}$) is stationary if and only if $\mathbf{X}$ is a mode shape vector

Solution: For an arbitrary n dimensional vector $\mathbf{X}$ Rayleigh's quotient is defined as

$$R(\mathbf{X}) = \frac{(\mathbf{X},\mathbf{X})_{\mathbf{K}}}{(\mathbf{X},\mathbf{X})_{\mathbf{M}}}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the symmetric mass and stiffness matrices for a *n*-degree-of-freedom system. Let $\omega_1 \le \omega_2 \le \ldots \le \omega_n$ be the natural frequencies of the system with corresponding normalized mode shape vectors $\mathbf{X_1}$ $\mathbf{X_2}$ … $\mathbf{X_n}$. Then from the expansion theorem there exists scalar values c_i i = 1,2,…n such that

$$\mathbf{X} = \sum_{i=1}^{n} c_i \mathbf{X_i}$$

Substituting the expansion theorem into Rayleigh's quotient leads to

$$R(\mathbf{X}) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j (\mathbf{X_i},\mathbf{X_j})_{\mathbf{K}}}{\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j (\mathbf{X_i},\mathbf{X_j})_{M}}$$

Using mode shape orthogonality and mode shape normalization properties in the above leads to

$$R(\mathbf{X}) = \frac{\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{c_i^2}\omega_i^2}{\sum_{i=1}^{n} c_i^2}$$

Rayleigh's quotient is stationary if

$$\frac{\partial R}{\partial c_1} = \frac{\partial R}{\partial c_2} = \cdots = \frac{\partial R}{\partial c_n} = 0$$

To this end

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$$\frac{\partial R}{\partial c_k} = \frac{2c_k\omega_k^2\sum_{i=1}^{n}c_i^2 - 2c_k\sum_{i=1}^{n}c_i^2\omega_i^2}{\left(\sum_{i=1}^{n}c_i^2\right)^2} = 0$$

$$c_k\left(\omega_k^2\sum_{i=1}^{n}c_i^2 - \sum_{i=1}^{n}c_i^2\omega_i^2\right) = 0 \quad k = 1,2,\ldots,n$$

If **X** is the mode shape vector corresponding to the pth mode then, $c_k = 0$ for $k \neq p$. Thus the above equations are identically satisfied for $k \neq p$. When $k = p$ the equation reduces to

$$\omega_p^2 - \omega_p^2 = 0$$

Thus Rayleigh's quotient is stationary if **X** is a mode shape vector.

Now consider these equations to determine if there are any additional possibilities. Each equation is satisfied if $c_k = 0$ or $\omega_k^2\sum_{i=1}^{n}c_i^2 - \sum_{i=1}^{n}\omega_i^2c_i^2 = 0$. The only possible solution is for all $c_k = 0$ except for one, in which case **X** is a mode shape vector.

Problem 8.24 illustrates that Rayleigh's quotient is stationary when evaluated for a mode shape vector.

8.25 Use Rayleigh's quotient to determine an upper bound on the lowest natural frequency of the system of Figure P8.7. Use at least four trial vectors.

Given: m, k

Find: an upper bound on the lowest natural frequency

Solution: The mass and stiffness matrices for this system are

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 4k & -k \\ 0 & -k & 2k \end{bmatrix}$$

FIGURE P8.7

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Rayleigh's quotient will be used to approximate an upper bound on the lowest natural frequency. The minimum possible value of Rayleigh's quotient is the square of the lowest natural frequency. Thus if Rayleigh's quotient is evaluated for arbitrary vectors an upper bound can be established for the lowest natural frequency. Four trial vectors are used

$$\mathbf{W_1} = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix} \quad \mathbf{W_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{W_3} = \begin{bmatrix} 1 \\ 0.75 \\ 0.5 \end{bmatrix} \quad \mathbf{W_4} = \begin{bmatrix} 1 \\ 2 \\ 0.5 \end{bmatrix}$$

Applying Rayleigh's quotient to the first trial vector

$$R(\mathbf{W_1}) = \frac{\begin{bmatrix} 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2k & -k & 0 \\ -k & 4k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}} = \frac{3.5k}{3.75m} = 0.933\frac{k}{m}$$

$$\omega_1 < \sqrt{R(\mathbf{W_1})} = 0.9661\sqrt{\frac{k}{m}}$$

Application of Rayleigh's quotient to the other trial vectors leads to

$$R(\mathbf{W_2}) = 1.111\frac{k}{m} \quad \omega_1 < 1.054\sqrt{\frac{k}{m}}$$

$$R(\mathbf{W_3}) = 0.902\frac{k}{m} \quad \omega_1 < 0.950\sqrt{\frac{k}{m}}$$

$$R(\mathbf{W_4}) = \frac{k}{m} \qquad \omega_1 < \sqrt{\frac{k}{m}}$$

Hence the upper bound on the lowest natural frequency is $\omega_1 < 0.950\sqrt{k/m}$.

Problem 8.25 illustrates application of Rayleigh's quotient to determine an upper bound on the lowest natural frequency on a multi-degree-of-freedom system.

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8.26 An alternative method to derive the uncoupled equations governing the motion of the free vibrations of a *n*DOF system in terms of principal coordinates is to introduce a linear transformation between the generalized coordinates **x** and the principal coordinates **p** as **x** = **Pp**, where **P** is the modal matrix, the matrix whose columns are normalized mode shapes. Follow these steps to derive the equations governing the principal coordinates:

(a) Rewrite Eq.(8.3) using the principal coordinates as dependent variables by introducing the linear transformation in Eq.(8.3).

(b) Premultiply the resulting equation by $\mathbf{P}^T$.

(c) Use the orthonormality of mode shapes to show that $\mathbf{P}^T\mathbf{MP}$ and $\mathbf{P}^T\mathbf{KP}$ are diagonal matrices.

(d) Write the uncoupled equations for the principal coordinates.

Solution: The differential equations governing the motion of a linear nDOF system are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{Kx} = \mathbf{0}$$

Substituting for generalized coordinates in terms of principal coordinates leads to

$$\mathbf{MP}\ddot{\mathbf{p}} + \mathbf{KPp} = \mathbf{0}$$

Pre-multiplying the above equation by $\mathbf{P}^{\mathrm{T}}$ leads to

$$\mathbf{P}^{\mathrm{T}}\mathbf{MP}\ddot{\mathbf{p}} + \mathbf{P}^{\mathrm{T}}\mathbf{KPp} = \mathbf{0}$$

Consider

$$\mathbf{D} = \mathbf{P}^{\mathrm{T}}\mathbf{MP} = \mathbf{P}^{\mathrm{T}}\mathbf{B} \qquad \mathbf{B} = \mathbf{MP}$$

$$b_{ij} = \sum_{k=1}^{n} m_{ik}\, p_{kj}$$

$$d_{ij} = \sum_{\ell=1}^{n} p_{\ell_i}\, b_{\ell j}$$

$$d_{ij} = \sum_{\ell=1}^{n}\sum_{k=1}^{n} p_{\ell i}\, m_{ik}\, p_{kj}$$

Noting that the *j*th column of **P** is the normalized mode shape vector $\mathbf{X_j}$

$$d_{ij} = \mathbf{X}_j^T\mathbf{MX_j} = \delta_{ij}$$

Mode shape orthonormality is used to derive the above. Hence

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$$\mathbf{P}^{\mathbf{T}}\mathbf{MP} = \mathbf{I}$$

Using similar methods it can be shown that

$$\mathbf{P}^{\mathbf{T}}\mathbf{KP} = \Omega$$
$$\Omega_{ij} = \omega_j^2 \delta_{ij}$$

Hence $\mathbf{\Omega}$ is a diagonal matrix with the squares of the natural frequencies along the diagonal. Thus the equations become

$$\ddot{\mathbf{p}} + \Omega\mathbf{p} = \mathbf{0}$$

Differential equations for the individualized principal coordinates become

$$\ddot{p}_1 + \omega_1^2 p_1 = 0$$
$$\ddot{p}_2 + \omega_2^2 p_2 = 0$$
$$\vdots$$
$$\ddot{p}_n + p_n = 0$$

Problem 8.26 illustrates an alternative method to derive the uncoupled differential equations for the principal coordinates.

8.27 Use the method of Chapter Problem 8.26 to derive the uncoupled equations governing the principal coordinates for a system with proportional damping.

Solution: The differential equations governing the free vibrations of a system with proportional damping are

$$\mathbf{M}\ddot{\mathbf{x}} + (\alpha\mathbf{K} + \beta\mathbf{M})\dot{\mathbf{x}} + \mathbf{Kx} = \mathbf{0}$$

Introduction of principal coordinates leads to

$$\mathbf{MP}\ddot{\mathbf{p}} + \alpha\mathbf{KP}\dot{\mathbf{p}} + \beta\mathbf{MP}\dot{\mathbf{p}} + \mathbf{KPp} = \mathbf{0}$$

Pre-multiplying the above equation by $\mathbf{P}^{\mathbf{T}}$ leads to

$$\mathbf{P}^{\mathbf{T}}\mathbf{MP}\ddot{\mathbf{p}} + \alpha\mathbf{P}^{\mathbf{T}}\mathbf{KP}\dot{\mathbf{p}} + \beta\mathbf{P}^{\mathbf{T}}\mathbf{MP}\dot{\mathbf{p}} + \mathbf{P}^{\mathbf{T}}\mathbf{KPp} = \mathbf{0}$$

In the solution of Problem 8.26 it is shown that

$$\mathbf{P}^{\mathbf{T}}\mathbf{MP} = \mathbf{I}$$
$$\mathbf{P}^{\mathbf{T}}\mathbf{KP} = \Omega$$

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where $\mathbf{\Omega}$ is a diagonal matrix with the squares of the natural frequencies along the diagonal. Thus the differential equations become

$$\ddot{\mathbf{p}} + (\alpha\Omega + \beta)\dot{\mathbf{p}} + \Omega\mathbf{p} = \mathbf{0}$$
$$\ddot{p}_1 + (\alpha\omega_1^2 + \beta)\dot{p}_1 + \omega_1^2 p_1 = 0$$
$$\ddot{p}_2 + (\alpha\omega_2^2 + \beta)\dot{p}_2 + \omega_2^2 p_2 = 0$$
$$\vdots$$
$$\ddot{p}_n + (\alpha\omega_n^2 + \beta)\dot{p}_n + \omega_n^2 p_n = 0$$

Hence the differential equations governing the motion of the principal coordinates for a nDOF system with proportional damping are uncoupled.

Problem 8.27 illustrates the use of principal coordinates to uncouple the differential equations governing the motion of a system with proportional damping.

8.28 Determine the free vibration response of the system of Figure P8.28 if the system is released from rest after the 3 kg block is displaced 5 mm.

FIGURE P8.28

Given: $k_1 = 5 \times 10^5$ N/m, $k_2 = 1 \times 10^5$ N/m, $k_3 = 4 \times 10^5$ N/m, $c_1 = 2000$ N · s/m, $c_2 = 400$ N · s/m, $c_3 = 1600$ N · s/m, $m_1 = 5$ kg, $m_2 = 3$ kg, $x_2(0) = 5$ mm

Find: $\mathbf{x}(t)$

Solution: The differential equations governing the motion of the system can be derived using either the free-body diagram method or Lagrange's equations as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2400 & -400 \\ -400 & 2000 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + 10^5\begin{bmatrix} 6 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The damping matrix is proportional to the stiffness matrix with a constant of proportionality of

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$$\alpha = \frac{c_{11}}{k_{11}} = \frac{2400 \text{ N} \cdot \text{s/m}}{6 \times 10^5 \text{ N/m}} = 0.004 \text{ s}$$

The natural frequencies, normalized mode shapes, and modal matrix for the system are determined (using MATLAB) as

$$\omega_1 = 329.4 \text{ rad/s} \qquad \mathbf{X_1} = \begin{bmatrix} -0.4087 \\ -0.2343 \end{bmatrix}$$

$$\omega_2 = 422.1 \text{ rad/s} \qquad \mathbf{X_2} = \begin{bmatrix} 0.1815 \\ -0.5727 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -0.4087 & 0.1815 \\ -0.2343 & -0.5727 \end{bmatrix}$$

The modal damping ratios are

$$\zeta_1 = \frac{1}{2}\alpha\omega_1 = \frac{1}{2}(0.004 \text{ s})(329.4 \text{ rad/s}) = 0.659$$

$$\zeta_2 = \frac{1}{2}\alpha\omega_2 = \frac{1}{2}(0.004 \text{ s})(422.1 \text{ rad/s}) = 0.844$$

The damped natural frequencies become

$$\omega_{d1} = \omega_1\sqrt{1-\zeta_1^2} = 247.8 \text{ rad/s}$$

$$\omega_{d2} = \omega_2\sqrt{1-\zeta_2^2} = 226.3 \text{ rad/s}$$

The free-vibration response in terms of the principal coordinates is

$$p_1(t) = A_1 e^{-217.1t}\sin(247.8t - \phi_1)$$

$$p_2(t) = A_2 e^{-356.3t}\sin(226.3t - \phi_2)$$

The initial conditions for the principal coordinates are

$$\mathbf{p}(0) = \mathbf{P}^{-1}\mathbf{x}(0) = \begin{bmatrix} -0.0035 \\ -0.0079 \end{bmatrix}$$

$$\dot{\mathbf{p}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Application of initial conditions leads to

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$$0.0035 = A_1 \sin\phi_1$$
$$0.0079 = A_2 \sin\phi_2$$
$$0 = 217.1 \sin\phi_1 + 247.8 \cos\phi_1$$
$$0 = 356.3 \sin\phi_1 + 226.3 \cos\phi_1$$

whose solutions are

$$\phi_1 = -0.8513 \text{ rad} \quad \phi_2 = -0.5659 \text{ rad}$$
$$\mathrm{A}_1 = -0.0047 \qquad \mathrm{A}_2 = -0.0147$$

The solution for the principal coordinates is

$$p_1(t) = -.0047e^{-217.1t} \sin(2478t + 0.8513)$$
$$p_2(t) = -0.0147e^{-356.3t} \sin(226.3t + 0.5659)$$

the solution for the original generalized coordinates is

$$\mathbf{x} = \mathbf{Pp} = \begin{bmatrix} -0.4087 & 0.1815 \\ -0.2343 & -0.5727 \end{bmatrix} \begin{bmatrix} -.0047e^{-217.1t} \sin(2478t + 0.8513) \\ -0.0147e^{-356.3t} \sin(226.3t + 0.5659) \end{bmatrix}$$
$$x_1(t) = 0.0019e^{-17.1t} \sin(247.8t + 0.8513) - 0.00085e^{-336.3t} \sin(226.3t + 0.5659)$$
$$x_2(t) = 0.0034e^{-17.1t} \sin(247.8t + 0.8513) + 0.0084e^{-336.3t} \sin(226.3t + 0.5659)$$

Problem 8.28 illustrates the free-vibration response of a system with proportional damping

8.29 If the modal damping ratio for the lowest mode of Chapter Problem 8.13 is 0.03, determine the modal damping ratio for the higher modes and determine the response of the system if the machine is displaced 2 mm and released.

Given: m_m = 400 kg, L= 3 m, m_b = 200 kg, E = 200 × 10^9 N/m^2, I = 1.4 × 10^{-5} m^4, ζ_1 = 0.03, $x_2(0)$ = 2 mm

Find: ζ_2, ζ_3, **x**(t)

Solution: The inertia effects of the beam are approximated by lumping particles of mass $m_b/4$ at x = L/4, L/2, and 3L/4 along the span of the beam. The machine is placed at the midspan of the beam. Hence the mass matrix for a three-degree-of-freedom model is

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$$\mathbf{M} = \begin{bmatrix} \frac{m_b}{4} & 0 & 0 \\ 0 & \frac{m_b}{4} + m & 0 \\ 0 & 0 & \frac{m_b}{4} \end{bmatrix} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 450 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

The flexibility matrix for the beam is determined. The deflection of a particle a distance z along the neutral axis of a simply supported beam, measured from the left support, due to a concentrated unit load applied a distance a from the left support is

$$y(z) = \frac{L^3}{6EI}\left(1 - \frac{a}{L}\right)\left[\frac{a}{L}\left(2 - \frac{a}{L}\right)\frac{z}{L} - \left(\frac{z}{L}\right)^3\right]$$

for a ≥ z. The elements of the third column of the flexibility matrix are the displacements induced by a unit concentrated load at a = 3L/4. Then

$$y(z) = \frac{L^3}{24EI}\left[\frac{15}{16}\frac{z}{L} - \left(\frac{z}{L}\right)^3\right]$$

and the flexibility influence coefficients are

$$a_{13} = y\left(\frac{L}{4}\right) = \frac{7L^3}{768EI} \quad a_{23} = y\left(\frac{L}{2}\right) = \frac{11L^3}{768EI} \quad a_{33} = y\left(\frac{3L}{4}\right) = \frac{3L^3}{256EI}$$

The second column of the flexibility matrix is determined by placing a unit concentrated load at a = L/2. Then

$$y(z) = \frac{L^3}{12EI}\left(\frac{3z}{4L} - \left(\frac{z}{L}\right)^3\right)$$

Note that due to symmetry only a_{22} needs to be calculated. To this end

$$a_{22} = y\left(\frac{L}{2}\right) = \frac{L^3}{48EI}$$

Then from symmetry of the flexibility matrix $a_{32} = a_{23}$ and from symmetry of the beam $a_{12} = a_{32}$. Then from symmetry and reciprocity, $a_{21} = a_{12}$ and $a_{31} = a_{13}$. Thus the flexibility matrix is

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$$\mathbf{A} = \frac{L^3}{EI}\begin{bmatrix} \frac{3}{256} & \frac{11}{768} & \frac{7}{768} \\ \frac{11}{768} & \frac{1}{48} & \frac{11}{768} \\ \frac{7}{768} & \frac{11}{768} & \frac{3}{256} \end{bmatrix} = 10^{-7}\begin{bmatrix} 1.130 & 1.381 & 0.879 \\ 1.381 & 2.009 & 1.381 \\ 0.879 & 1.381 & 1.130 \end{bmatrix}$$

The natural frequencies are the reciprocals of the square roots of the eigenvalues of **AM**. To this end

$$\mathbf{AM} = 10^{-7}\begin{bmatrix} 1.130 & 1.381 & 0.879 \\ 1.381 & 2.009 & 1.381 \\ 0.879 & 1.381 & 1.130 \end{bmatrix}\begin{bmatrix} 50 & 0 & 0 \\ 0 & 450 & 0 \\ 0 & 0 & 50 \end{bmatrix} = 10^{-4}\begin{bmatrix} 00565 & 0.6215 & 0.0439 \\ 0.0691 & 0.9040 & 0.0691 \\ 0.0439 & 0.6215 & 0.0565 \end{bmatrix}$$

$$|\mathbf{AM} - \lambda\mathbf{I}| = 0$$

$$10^{-4}\begin{vmatrix} 00565 - 10^4\lambda & 0.6215 & 0.0439 \\ 0.0691 & 0.9040 - 10^4\lambda & 0.0691 \\ 0.0439 & 0.6215 & 0.0565 - 10^4\lambda \end{vmatrix} = 0$$

$$10^{12}\lambda^3 - 1.017\times10^8\lambda^2 + 1.758\times10^2\lambda - 6.235\times10^{-5} = 0$$

$$\lambda = 5.00\times10^{-1},\ 1.26\times10^{-2},\ 9.995\times10^{-3}$$

$$\omega_1 = \frac{1}{\sqrt{\lambda_3}} = \frac{1}{\sqrt{9.995\times10^{-3}}} = 100.02 \text{ rad/s}$$

$$\omega_2 = \frac{1}{\sqrt{\lambda_2}} = 892.4 \text{ rad/s}$$

$$\omega_3 = \frac{1}{\sqrt{\lambda_1}} = 1.42\times10^3 \text{ rad/s}$$

The modal matrix is the matrix of normalized mode shapes. MATLAB is used to determine the modal matrix as

$$\mathbf{P} = \begin{bmatrix} 0.4393 & -0.7071 & 0.7030 \\ 0.7151 & 0 & -0.1080 \\ 0.4943 & 0.7071 & 0.7030 \end{bmatrix}$$

For proportional damping the damping matrix is assumed proportional to the stiffness matrix with a constant of proportionality of

$$\alpha = \frac{2\zeta_1}{\omega_1} = \frac{2(0.03)}{100.02 \text{ rad/s}} = 6.0\times10^{-4} \text{ s}$$

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The modal damping ratios for the higher modes are

$$\zeta_2 = \frac{\alpha}{2}\omega_2 = 0.2677$$

$$\zeta_3 = \frac{\alpha}{2}\omega_3 = 0.4255$$

The damped natural frequencies for the modes are

$$\begin{aligned}
\omega_{di} &= \omega_i\sqrt{1-\zeta_i^2} \\
\omega_{d1} &= 99.98 \text{ rad/s} \\
\omega_{d2} &= 859.8 \text{ rad/s} \\
\omega_{d3} &= 1.284\times 10^3 \text{ rad/s}
\end{aligned}$$

The solution for the principal coordinates is

$$\begin{aligned}
p_i(t) &= A_i e^{-\zeta_i\omega_i t}\sin(\omega_{di}t-\phi_i) \\
p_1(t) &= A_1 e^{-26.77t}\sin(99.98t-\phi_1) \\
p_2(t) &= A_2 e^{-238.9t}\sin(859.9t-\phi_2) \\
p_3(t) &= A_3 e^{-603.7t}\sin(1284t-\phi_4)
\end{aligned}$$

The initial conditions for the generalized coordinates are

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0.002 \\ 0 \end{bmatrix} \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The initial conditions for the principal coordinates are

$$\mathbf{p}(0) = \mathbf{P}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1.549 & 20.17 & 1.549 \\ -5 & 0 & 5 \\ 7.754 & -6.573 & 4.574 \end{bmatrix}\begin{bmatrix} 0 \\ 0.002 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0403 \\ 0 \\ -0.0132 \end{bmatrix}$$

$$\dot{\mathbf{p}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Application of the initial conditions leads to

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$$0.0403 = -A_1 \sin\phi_1$$
$$0 = -A_2 \sin\phi_2$$
$$-0.0132 = -A_3 \sin\phi_3$$
$$0 = 26.77 \sin\phi_1 + 99.98 \cos\phi_1$$
$$0 = 238.9 \sin\phi_2 + 859.9 \cos\phi_2$$
$$0 = 603.7 \sin\phi_3 + 1284 \cos\phi_3$$

The solution of the above set of equation is

$$\phi_1 = -1.31 \text{ rad} \quad \phi_2 = -1.30 \text{ rad} \quad \phi_3 = -1.13 \text{ rad}$$
$$A_1 = 0.0417 \quad A_2 = 0 \quad A_3 = -0.0146$$

Thus the solutions for the principal coordinates are

$$p_1(t) = 0.0417 e^{-26.77t} \sin(99.98t + 1.31)$$
$$p_2(t) = 0$$
$$p_3(t) = -0.0146 e^{-603.7t} \sin(1284t + 1.13)$$

The original generalized coordinates are obtained from

$$\mathbf{x} = \mathbf{Pp} = \begin{bmatrix} 0.49493 & -0.7071 & 0.7030 \\ 0.7151 & 0 & -0.1080 \\ 0.4943 & 0.7071 & 0.7030 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}$$

The free-vibration response of the machine is

$$x_2(t) = 0.7151 p_1(t) - 0.1080 p_3(t)$$
$$x_2(t) = 0.0298 e^{-26.77t} \sin(99.98t + 1.31) + 0.000268 e^{-603.7t} \sin(1284t + 1.13)$$

Problem 8.29 illustrates the free-vibration response of a system.

8.30 Determine the free-vibration response of the bar of Figure P8.30. The mass center is displaced 1 cm from equilibrium while the bar is held horizontal and the system is released from this position.

FIGURE P8.30

Given: a = 1.5 m, b = 0.5 m, k = 4 × 10^5 N/m, c = 500 N · s/m, m = 120 kg, x(0) = 1 cm, θ(0) = 0, I = 15 kg · m^2

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Find: **x**(t)

Solution: The mass center of the bar is assumed at G. The chosen generalized coordinates are x, the displacement of the mass center from equilibrium, and θ, the clockwise angular rotation of the bar from equilibrium. The differential equations governing the free vibrations of the system can be derived using Lagrange's equations. The results are

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2c & c(b-a) \\ c(b-a) & c(a^2+b^2) \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & k(b-a) \\ k(b-a) & k(a^2+b^2) \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 120 & 0 \\ 0 & 15 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 1000 & -500 \\ -500 & 1250 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + 10^5\begin{bmatrix} 8 & -4 \\ -4 & 10 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The damping matrix is proportional to the stiffness matrix with

$$\alpha = \frac{c}{k} = 0.0125$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The normalized mode shape vectors are the corresponding eigenvectors. The results are

$$\omega_1 = 72.25 \text{ rad/s} \quad \mathbf{X_1} = \begin{bmatrix} 0.09023 \\ 0.03916 \end{bmatrix}$$

$$\omega_2 = 261.0 \text{ rad/s} \quad \mathbf{X_2} = \begin{bmatrix} 0.01384 \\ -0.25521 \end{bmatrix}$$

The modal damping ratios are

$$\zeta_1 = \frac{1}{2}\alpha\omega_1 = .4516$$

$$\zeta_2 = \frac{1}{2}\alpha\omega_2 = 1.631$$

Thus the lowest mode is underdamped while the higher mode is overdamped. The initial conditions for the principal coordinates become

$$\begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix} = \mathbf{P}^{-1}\mathbf{x}(0) = \begin{bmatrix} 0.09023 & 0.01384 \\ 0.03916 & -0.25521 \end{bmatrix}^{-1}\begin{bmatrix} 0.01 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix} = \begin{bmatrix} 0.1083 \\ 0.0166 \end{bmatrix}$$

It is noted that since the bar is released from this initial position, all initial velocities are zero. The solution for $p_1(t)$ is determined using Eq.(2.45)

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$$p_1(t) = 0.1213e^{-32.62t}\sin(64.46t + 1.102)$$

the solution for $p_2(t)$ is determined using Eq.(2.63)

$$p_2(t) = 0.02536e^{-89.38t} - 0.008750e^{-762.0t}$$

The solution for the original generalized coordinates is obtained from

$$\mathbf{x} = \mathbf{Pp}$$

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} 0.09023 & 0.01384 \\ 0.03916 & -0.25521 \end{bmatrix} \begin{bmatrix} 0.1213e^{-32.62t}\sin(64.46t + 1.102) \\ 0.02536e^{-89.38t} - 0.008750e^{-762.0t} \end{bmatrix}$$

$$x(t) = 0.01095e^{-32.62t}\sin(64.46t + 1.102) + 0.000351e^{-89.38t} - 0.0001211e^{-762/0t}$$

$$\theta(t) = 0.00475e^{-32.62t}\sin(64.46t + 1.102) - 0.00647e^{-89.38t} + 0.00233e^{-762/0t}$$

Problem 8.30 illustrates the free-vibration response of a system with proportional damping.

8.31 Determine the free-vibration response of the system of Figure P8.31.

FIGURE P8.31

Given: $m_1 = 4$ kg, $m_2 = 6$ kg, $k_1 = 1 \times 10^6$ N/m, $k_2 = 2 \times 10^6$ N/m, $c = 400$ N · s/m

Find: $\mathbf{x}(t)$

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 400 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + 10^6 \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The damping matrix is not proportional to a linear combination f the mass and stiffness matrices. Thus the theory for a general damping matrix is used. The augmented mass and stiffness matrices become

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$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \\ 4 & 0 & 400 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 3\times10^6 & -2\times10^6 \\ 0 & 0 & -2\times10^6 & 2\times10^6 \end{bmatrix}$$

The eigenvalues and mode shapes of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$ are

$$\lambda_1 = 36.34 - 99.83i \quad \mathbf{X_1} = \begin{bmatrix} 0.2126 - 9.8685i \\ -0.0585 - 0.4438i \\ -0.000877 - 0.000181i \\ 0.000446 + 0.0000423i \end{bmatrix}$$

$$\lambda_2 = 36.34 + 99.83i \quad \mathbf{X_2} = \begin{bmatrix} 0.2126 + 9.8685i \\ -0.0585 + 0.4438i \\ -0.000877 + 0.000181i \\ 0.000446 - 0.0000423i \end{bmatrix}$$

$$\lambda_3 = 13.66 - 288.6i \quad \mathbf{X_3} = \begin{bmatrix} 0.1241 - 0.55i \\ 0.1898 - 0.7677i \\ -0.00205 - 0.000329i \\ -0.00272 - 0.000529i \end{bmatrix}$$

$$\lambda_4 = 13.66 + 288.6i \quad \mathbf{X_4} = \begin{bmatrix} 0.1241 + 0.55i \\ 0.1898 + 0.7677i \\ -0.00205 + 0.000329i \\ -0.00272 + 0.000529i \end{bmatrix}$$

The general solution is of the form

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$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ x_1\\ x_2\end{bmatrix} = e^{-36.34t}\left(C_1\begin{bmatrix}0.2126-0.8685i\\ -0.0585-0.4438i\\ -0.000877-0.000181i\\ 0.000446+0.0000423i\end{bmatrix}e^{i99.83t} + C_2\begin{bmatrix}0.2126+0.8685i\\ -0.0585+0.4438i\\ -0.000877+0.000181i\\ 0.000446-0.0000423i\end{bmatrix}e^{-i99.83t}\right)$$

$$+e^{-13.66t}\left(C_3\begin{bmatrix}0.1241-0.5875i\\ 0.1898-0.7677i\\ -0.00205-0.000329i\\ -0.00272-0.000529i\end{bmatrix}e^{i288.6t} + C_4\begin{bmatrix}0.1241+0.5875i\\ 0.1898+0.7677i\\ -0.00205+0.000329i\\ -0.00272+0.000529i\end{bmatrix}e^{-i288.6t}\right)$$

Initial conditions must be applied to determine the constants of integration. Euler's identity is used to rewrite the complex exponentials as trigonometric functions.

Problem 8.31 illustrates the free-vibration response of a system with general damping.

8.32 Determine the free-vibration response of the system of Figure P8.32.

FIGURE P8.32

Given: a = 1.5 m, b = 0.5 m, k = 4 × 10^5 N/m, c = 500 N · s/m, m = 120 kg, I = 15 kg · m^2

Find: **x**(t)

Solution: The mass center of the bar is assumed at G. The chosen generalized coordinates are x, the displacement of the mass center from equilibrium, and θ, the clockwise angular rotation of the bar from equilibrium. The differential equations governing the free vibrations of the system can be derived using Lagrange's equations. The results are

$$\begin{bmatrix}m & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} + \begin{bmatrix}c & -ca\\ -ca & ca^2\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}2k & k(b-a)\\ k(b-a) & k(a^2+b^2)\end{bmatrix}\begin{bmatrix}x\\ \theta\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}120 & 0\\ 0 & 15\end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} + \begin{bmatrix}500 & -750\\ -750 & 1125\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} + 10^5\begin{bmatrix}8 & -4\\ -4 & 10\end{bmatrix}\begin{bmatrix}x\\ \theta\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

The damping matrix is not proportional to a linear combination of the mass and stiffness matrices. Thus the theory for a general damping matrix is used. To this end, define

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$$\mathbf{y} = \begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ x \\ \theta \end{bmatrix}$$

$$\widetilde{\mathbf{M}} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 120 & 0 \\ 0 & 0 & 0 & 15 \\ 120 & 0 & 500 & -750 \\ 0 & 15 & -750 & 1125 \end{bmatrix}$$

$$\widetilde{\mathbf{K}} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} = \begin{bmatrix} -120 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & 0 & 8\times10^5 & -4\times10^5 \\ 0 & 0 & -4\times10^5 & 1\times10^6 \end{bmatrix}$$

The eigenvalues and eigenvectors of $\widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{K}}$ are

$$\lambda_1 = 39.34 - 257.9i \quad \mathbf{X_1} = \begin{bmatrix} 0.05487 + 0.01343i \\ -0.9955 - 0.07619i \\ 0.00001918 - 0.0002156i \\ 0.0002865 + 0.003815i \end{bmatrix}$$

$$\lambda_2 = 39.34 + 257.9i \quad \mathbf{X}_2 = \begin{bmatrix} 0.05487 - 0.01343i \\ -0.9955 + 0.07619i \\ 0.00001918 + 0.0002156i \\ 0.0002865 - 0.003815i \end{bmatrix}$$

$$\lambda_3 = 0.246 - 72.27i \quad \mathbf{X_3} = \begin{bmatrix} 0.9012 + 0.1670i \\ 0.3892 + 0.09136i \\ 0.002268 - 0.01248i \\ 0.001246 - 0.005390i \end{bmatrix}$$

$$\lambda_4 = 0.246 + 72.27i \quad \mathbf{X}_4 = \begin{bmatrix} 0.9012 - 0.1670i \\ 0.3892 - 0.09136i \\ 0.002268 + 0.01248i \\ 0.001246 + 0.005390i \end{bmatrix}$$

The free-vibration response is

$$\mathbf{y} = C_1\mathbf{X_1}e^{-\lambda_1 t} + C_2\mathbf{X_2}e^{-\lambda_2 t} + C_3\mathbf{X_3}e^{-\lambda_3 t} + C_4\mathbf{X_4}e^{-\lambda_4 t}$$

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leading to

$$x(t) = e^{-39.34t}\left[C_1\left(0.00001918 - 0.0002156i\right)e^{i257.9t} + C_2\left(0.00001918 + 0.0002156i\right)e^{-i257.9t}\right]$$
$$+ e^{-0.246t}\left[C_3(0.002268 - 0.01248i)e^{i72.7t} + C_4(0.002268 + 0.01248i)e^{-i72.7t}\right]$$

$$\theta(t) = e^{-39.34t}\left[C_1(0.0002865 + 0.003815i)e^{i257.9t} + C_2(0.0002865 - 0.003815i)e^{-i257.9t}\right]$$
$$+ e^{-0.246t}\left[C_3(0.001246 - 0.005390i)e^{i72.27t} + C_4(0.001246 + 0.005390i)e^{-i72.27t}\right]$$

Problem 8.32 illustrates the free-vibration response of a system with.

8.33 Determine the free-vibration response of the system of Chapter Problem 7.87 when $E = 200 \times 10^9$ N/m^2, $I = 1.5 \times 10^{-6}$ m^4, $L = 0.8$ m, $k = 1.5 \times 10^5$ N/m, $c = 250$ N · s/m, $m_1 = 4$ kg, m_2 6.1 kg.

FIGURE P7.83

Given: $E = 200 \times 10^9$ N/m^2, $I = 1.5 \times 10^{-6}$ m^4, $L = 0.8$ m, $k = 1.5 \times 10^5$ N/m, $c = 250$ N · s/m, $m_1 = 4$ kg, m_2 6.1 kg

Find: **x**(t)

Solution: The differential equations are of the form

$$\mathbf{AM}\ddot{\mathbf{x}} + \mathbf{AC}\dot{\mathbf{x}} + \mathbf{x} = \mathbf{0}$$

where the mass matrix is

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix}$$

and the damping matrix is

$$\mathbf{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}$$

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The flexibility matrix is determined using flexibility influence coefficients. If a unit load is applied to the shaft a distance a from the left end the moment and deflection equations are

$$EIy''(x) = (x-a)u(x-a) + C_1 x + C_2$$
$$EIy(x) = \frac{1}{6}(x-a)^3 u(x-a) + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4$$

The moment a both ends is zero,

$$y''(0) = 0 \quad \Rightarrow C_2 = 0$$
$$y''(L) = 0$$

Consider the third column of the flexibility matrix. A unit load is applied at the right end of the shaft.

Application of the equations of equilibrium leads to

$$y(0) = 0$$
$$y(L) = \frac{1}{k}$$

Application of the conditions to the deflection equations leads to

$$C_1 = 0$$
$$C_4 = 0$$
$$C_3 = \frac{EI}{kL}$$

Hence

$$y(x) = \frac{x}{kL}$$

and the flexibility coefficients are

$$a_{13} = y(0) = 0$$
$$a_{23} = y(L/2) = \frac{1}{2k}$$
$$a_{33} = y(L) = \frac{1}{k}$$

The first column is obtained using symmetry as

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$$a_{11} = \frac{1}{k}$$
$$a_{21} = \frac{1}{2k}$$
$$a_{31} = 0$$

The second column of the flexibility matrix is obtained by placing a unit load at the midspan of the rotor.

Application of equilibrium leads to

$$y(0) = \frac{1}{2k}$$
$$y(L) = \frac{1}{2k}$$

Application of these conditions to the deflection equation leads to

$$C_1 = -\frac{1}{2}$$
$$C_4 = \frac{EI}{2k}$$
$$C_3 = \frac{L^3}{16}$$
$$EIy(x) = \frac{1}{6}(x - L/2)^3 u(x - L/2) - \frac{1}{2}\frac{x^3}{6} + \frac{L^3}{16}x + \frac{EI}{2k}$$

Then

$$a_{12} = y(0) = \frac{1}{2k}$$
$$a_{22} = y(L/2) = \frac{1}{2k} + \frac{L^3}{48EI}$$
$$a_{32} = y(L) = \frac{1}{2k}$$

The differential equations are

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$$\begin{bmatrix} \frac{1}{k} & \frac{1}{2k} & 0 \\ \frac{1}{2k} & \frac{1}{2k}+\frac{L^3}{48EI} & \frac{1}{2k} \\ 0 & \frac{1}{2k} & \frac{1}{k} \end{bmatrix} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} +$$

$$\begin{bmatrix} \frac{1}{k} & \frac{1}{2k} & 0 \\ \frac{1}{2k} & \frac{1}{2k}+\frac{L^3}{48EI} & \frac{1}{2k} \\ 0 & \frac{1}{2k} & \frac{1}{k} \end{bmatrix} \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substitution of given values leads to

$$\mathbf{A} = 10^{-5} \begin{bmatrix} 0.6667 & 0.3333 & 0 \\ 0.3333 & 0.3369 & 0.3333 \\ 0 & 0.3333 & 0.6667 \end{bmatrix}$$

The stiffness matrix is the inverse of the flexibility matrix

$$\mathbf{K} = 10^{7} \begin{bmatrix} 0.7181 & -1.4062 & 0.7031 \\ -1.4062 & 2.8125 & -1.4062 \\ 0.7031 & -1.4062 & 0.7181 \end{bmatrix}$$

The mass and damping matrices are

$$\mathbf{M} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6.1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 250 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 250 \end{bmatrix}$$

Using the notation Chapter 8

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$$\widetilde{\mathbf{M}} = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 0 & 250 & 0 & 0 \\ 0 & 6.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 250 \end{bmatrix}$$

$$\widetilde{\mathbf{K}} = \begin{bmatrix} -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.181\times10^{6} & -1.4062\times10^{7} & 7.031\times10^{6} \\ 0 & 0 & 0 & -1.4062\times10^{7} & 2.8125\times10^{7} & -1.4062\times10^{6} \\ 0 & 0 & 0 & 7.031\times10^{6} & -1.4062\times10^{7} & 7.181\times10^{6} \end{bmatrix}$$

The eigenvalues and eigenvectors of $\widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{K}}$ are

$$\lambda_1 = 1.359\times10 - 2.853\times10^{3} i \quad \Phi_1 = \begin{bmatrix} 5.943\times10^{-3} + 5.195\times10^{-1} i \\ -2.206\times10^{-2} - 6.779\times10^{-1} i \\ 5.493\times10^{-3} + 5.195\times10^{-1} i \\ 1.820\times10^{-4} - 2.792\times10^{-6} i \\ -2.375\times10^{-4} + 8.864\times10^{-6} i \\ 1.820\times10^{-4} - 2.792\times10^{-6} i \end{bmatrix}$$

$$\lambda_2 = 1.359\times10 + 2.853\times10^{3} i \quad \Phi_1 = \begin{bmatrix} 5.943\times10^{-3} - 5.195\times10^{-1} i \\ -2.206\times10^{-2} + 6.779\times10^{-1} i \\ 5.493\times10^{-3} - 5.195\times10^{-1} i \\ 1.820\times10^{-4} + 2.792\times10^{-6} i \\ -2.375\times10^{-4} - 8.864\times10^{-6} i \\ 1.820\times10^{-4} + 2.792\times10^{-6} i \end{bmatrix}$$

$$\lambda_3 = 3.125\times10 - 1.911\times10^{2} \quad \Phi_3 = \begin{bmatrix} 3.566\times10^{-1} + 6.105\times10^{-1} i \\ 0 \\ -3.566\times10^{-1} - 6.105\times10^{-1} i \\ 2.812\times10^{-3}\times10^{-3} - 2.326\times10^{-3} i \\ 0 \\ -2.814\times10^{-3} + 2.326\times10^{-3} i \end{bmatrix}$$

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$$\lambda_4 = 3.125\times 10 + 1.911\times 10^2 \quad \Phi_4 = \begin{bmatrix} 3.566\times 10^{-1} - 6.105\times 10^{-1} i \\ 0 \\ -3.566\times 10^{-1} + 6.105\times 10^{-1} i \\ 2.812\times 10^{-3}\times 10^{-3} + 2.326\times 10^{-3} i \\ 0 \\ -2.814\times 10^{-3} - 2.326\times 10^{-3} i \end{bmatrix}$$

$$\lambda_5 = 1.766\times 10^1 - 1.446\times 10^2 i \quad \Phi_5 = \begin{bmatrix} 4.874\times 10^{-1} - 3.078\times 10^{-1} i \\ 4.899\times 10^{-1} - 3.086\times 10^{-1} i \\ 4.874\times 10^{-1} - 3.078\times 10^{-1} i \\ -2.502\times 10^{-3} - 3.063\times 10^{-3} i \\ -2.509\times 10^{-3} - 3.085\times 10^{-3} i \\ -2.502\times 10^{-3} - 3.063\times 10^{-3} i \end{bmatrix}$$

$$\lambda_6 = 1.766\times 10^1 + 1.446\times 10^2 i \quad \Phi_6 = \begin{bmatrix} 4.874\times 10^{-1} + 3.078\times 10^{-1} i \\ 4.899\times 10^{-1} + 3.086\times 10^{-1} i \\ 4.874\times 10^{-1} + 3.078\times 10^{-1} i \\ -2.502\times 10^{-3} + 3.063\times 10^{-3} i \\ -2.509\times 10^{-3} + 3.085\times 10^{-3} i \\ -2.502\times 10^{-3} + 3.063\times 10^{-3} i \end{bmatrix}$$

The general solution for the generalized coordinates is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = e^{-13.58t} \begin{bmatrix} 1.802 - .00279i \\ -2.375 + 0.00886i \\ 1.820 - 0.00279i \end{bmatrix} 10^{-4} \left(C_1 \cos 2853t + C_2 \sin 2853t\right)$$

$$+ e^{-31.25t} \begin{bmatrix} 2.814 - 2.326i \\ 0 \\ -2.814 + 2.326i \end{bmatrix} 10^{-3} \left(C_3 \cos 191.1t + C_4 \sin 191.1t\right)$$

$$+ e^{-17.66t} \begin{bmatrix} -2.502 - 3.063i \\ -2.509 - 3.805i \\ -2.502 - 3.063i \end{bmatrix} 10^{-3} \left(C_5 \cos 144.6t + C_6 \sin 144.6t\right)$$

Problem 8.33 illustrates the free-vibration response of a three-degree-of-freedom system with general damping.

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CHAPTER 9: FORCED VIBRATIONS OF MDOF SYSTEMS

Short Answer Questions

9.1 False: The Laplace transform method can be used to determine the response of a system with proportional damping.

9.2 True: Principal coordinates are used to uncouple both the free vibrations problem and the forced vibrations problem.

9.3 True: For proportional damping that is proportional to the stiffness matrix the modal damping ratios are $\zeta_i = \frac{\alpha}{2}\omega_i$, the higher modes are more highly damped.

9.4 False: The elements of the inverse of the impedance matrix are the transfer functions $G_{i,j}(s)$.

9.5 False: The principal coordinates are used to determine the transient response as well as the steady-state response of a system.

9.6 True: The modal matrix is the vector of normalized mode shapes. $\mathbf{P}^T\mathbf{F}$ is a vector whose individual components are $(\mathbf{X_i}, \mathbf{F})$.

9.7 False: The *k*th component of **G,** the vector on the right hand side of the equations defining the generalized coordinate is calculated by taking the standard scalar product of the force vector with the *k*th normalized mode shape.

9.8 The determinant of the impedance matrix of an *n* degree of freedom system is a polynomial of order 2n.

9.9 (c) $\ddot{p}_1 + 900\dot{p}_1 = g_1(t)$

9.10 To derive modal analysis, the <u>expansion theorem</u> is used to write the general solution as a linear combination of the principal coordinates.

9.11 The <u>standard</u> scalar product is taken with both sides of the equation after the linear combination is substituted into the differential equations.

9.12 The equations are <u>uncoupled</u> using mode shape <u>orthogonality</u> with respect to <u>the kinetic energy scalar product</u> and <u>the potential energy scalar product</u>.

9.13 The <u>convolution</u> integral can be used to solve the resulting non-homogenous differential equations.

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9.14 For systems with a general damping matrix the differential equations governing the nDOF system is written as 2n first-order differential equations.

9.15 The vector $\tilde{\mathbf{F}}$ is defined as the $2n \times 1$ vector force vector.

9.16 The modal matrix $\tilde{\mathbf{P}}$ is defined as the matrix whose columns are normalized by $\bar{\Phi}_i M \Phi_i = 1$.

9.17 The differential equations governing the principal coordinates of the system are $\dot{p}_i + \gamma_i p_i = 0$.

9.18 The differential equations have a solution, $\tilde{p}_i = \int_0^t \tilde{g}_i(t) e^{-\gamma_i(t-\tau)} d\tau$, called the convolution integral.

9.19 (a) Modal analysis uncovers the time scales in the equations, thus the stability of each equation can be determined. (b) Modal analysis leads to an uncoupled set of differential equations, which are easier to solve than a coupled set using numerical methods.

9.20 (a) Modal analysis uncovers the periods of the individual modes. The step size in numerical integration of the convolution integral is determined by the smaller of the period for that mode and the duration of the force. (b) Modal analysis leads to an uncoupled set of differential equations, which are easier to solve than a coupled set using numerical methods.

9.21 (a) The differential equation for the first mode shape is

$$\ddot{p}_1 + 400 p_1 = 2.9 \sin 54t$$

(b) The steady-state solution of the differential equation is

$$p_1 = \frac{2.9}{(54)^2 - 400} \sin 54t = 1.15 \times 10^{-3} \sin 54t$$

(c) The fourth mode has a natural frequency of 55 rad/s which is closest to the 54 rad/s. It will have the largest contribution to the response.

(d) The relation between the fifth generalized coordinate and the principal coordinates is

$$x_5 = 2.0 p_1 - 0.15 p_2 + 0.2 p_3 - 0.60 p_4 + 0.40 p_5$$

9.22 (a) The differential equation for the fourth mode shape is

$$\ddot{p}_4 + 259.7 \dot{p}_4 + 8649 p_1 = -0.3 \sin 54t$$

(b) The frequency ratio is $r = \frac{54}{93} = 0.581$. The steady-state solution of the differential equation is

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$$p_4 = \frac{0.3}{(93)^2} M(0.581,1.395)\sin(54t - \pi - \phi) = 1.98x10^{-5}\sin(54t - 4.32)$$

(c) Modes 1,2 and 3 are underdamped. Modes 4 and 5 are overdamped.

(d) Knowing that **C** is proportional to **K** the constant of proportionality is given by $\zeta_1 = \frac{\alpha}{2}\omega_1 \Longrightarrow \alpha = \frac{2(0.3)}{20} = 0.03$

9.23 The impedance matrix is given by

$$\mathbf{Z} = s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} = \begin{bmatrix} 2s^2 + s + 5 & -3 & 0 \\ -3 & 2s^2 + 7 & -4 \\ 0 & -4 & 3s^2 + 2s + 4 \end{bmatrix}$$

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Chapter Problems

9.1 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.1. Use the method of undetermined coefficients.

FIGURE P9.1

Given: System shown

Find: X_1

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3000 & -2000 & 0 \\ -2000 & 5000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \sin 20t \end{bmatrix} \qquad (a)$$

The system is undamped and is subject to a single frequency excitation at 20 rad/s. Thus a solution is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sin 20t \qquad (b)$$

Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} 2200 & -2000 & 0 \\ -2000 & 3600 & -3000 \\ 0 & -3000 & 600 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 10^{-3} \begin{bmatrix} -3.39 \\ -3.73 \\ -1.97 \end{bmatrix} \qquad (d)$$

Problem 9.1 illustrates the use of the method of undetermined coefficients to determine the forced response of a three-degree-of-freedom system.

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9.2 Determine the steady-state amplitude for the mass hanging from the end of the bar in the system in Figure P9.2 Use the method of undetermined coefficients.

FIGURE P9.2

Given: $m_1 = 30\ \text{kg}, m_2 = 20\ \text{kg}, k_1 = 4 \times 10^5\ \frac{\text{N}}{\text{m}}, k_2 = 3 \times 10^5\ \frac{\text{N}}{\text{m}}, I = 1.8\ \text{kg} \cdot \text{m}^2, L = 0.8\ \text{m},$ $M_0 = 20\ \text{N} \cdot \text{m}, \omega = 45\ \frac{\text{rad}}{\text{s}}$

Find: Y

Solution: The kinetic energy at an arbitrary instant is

$$T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 \qquad (a)$$

The potential energy at an arbitrary instant is

$$V = \frac{1}{2} k_1 \left(x - \frac{L}{2}\theta\right)^2 + \frac{1}{2} 2k_1 x^2 + \frac{1}{2} k_2 \left(y - x - \frac{L}{2}\theta\right)^2 \qquad (b)$$

The virtual work done by the external moment is

$$\delta W = M_0 \sin \omega t\ \delta\theta \qquad (c)$$

Lagrange's equations are applied to yield

$$\begin{bmatrix} I & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} (k_1 + k_2)\frac{L^2}{2} & (k_2 - k_1)\frac{L}{2} & -k_2\frac{L}{2} \\ (k_2 - k_1)\frac{L}{2} & 3k_1 + k_2 & -k_2 \\ -k_2\frac{L}{2} & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} M_0 \sin \omega t \\ 0 \\ 0 \end{bmatrix} \qquad (d)$$

Substituting given values into Eq. (d) leads to

$$\begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + 10^6 \begin{bmatrix} 0.224 & -0.04 & -0.12 \\ -0.04 & 1.5 & -0.3 \\ -0.12 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 20 \sin 45t \\ 0 \\ 0 \end{bmatrix} \qquad (e)$$

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The method of undetermined coefficients is used and a solution assumed as

$$\begin{bmatrix}\theta\\ x\\ y\end{bmatrix} = \begin{bmatrix}\Theta\\ X\\ Y\end{bmatrix}\sin 45t \qquad (f)$$

Substitution of Eq. (f) into Eq. (e) leads to

$$10^6\begin{bmatrix}0.2204 & -0.04 & -0.12\\ -0.04 & 1.432 & -0.3\\ -0.12 & -0.3 & 0.2529\end{bmatrix}\begin{bmatrix}\Theta\\ X\\ Y\end{bmatrix} = \begin{bmatrix}20\\ 0\\ 0\end{bmatrix} \qquad (g)$$

The solution of Eq. (g) is

$$\begin{bmatrix}\Theta\\ X\\ Y\end{bmatrix} = 10^{-3}\begin{bmatrix}1.435\\ 0.241\\ 0.961\end{bmatrix} \qquad (h)$$

Problem 9.2 illustrates the use of the method of undetermined coefficients for an undamped 3DOF system.

9.3 Determine the steady-state amplitude of vibration of the mass m_3 of the system in Figure P9.3. Use the method of undetermined coefficients.

FIGURE P9.3

Given: $m_1 = 18\text{ kg}, m_2 = 20\text{ kg}, m_3 = 45\text{ kg}, k_1 = 3\times10^4\ \frac{\text{N}}{\text{m}}, k_2 = 4.5\times10^4\ \frac{\text{N}}{\text{m}},\ k_3 = 1\times10^4\ \frac{\text{N}}{\text{m}},\ I = 1.4\text{ kg}\cdot\text{m}^2,\ r = 0.25\text{ m}, F_0 = 25\text{ N}, \omega = 35\ \frac{\text{rad}}{\text{s}}$

Find: Y

Solution: The kinetic energy at an arbitrary instant is

$$T = \frac{1}{2}(I + m_1 r^2)\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_3\dot{y}^2 \qquad (a)$$

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The potential energy at an arbitrary instant is

$$V = \frac{1}{2}k_1(r\theta)^2 + \frac{1}{2}k_2(x - 2r\theta)^2 + \frac{1}{2}k_3(y - x)^2 \qquad (b)$$

The virtual work done by the external moment is

$$\delta W = F_0 \sin \omega t \;\; \delta y \qquad (c)$$

Lagrange's equations are applied to yield

$$\begin{bmatrix} I + m_1 r^2 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} (k_1 + 4k_2)r^2 & -2rk_2 & 0 \\ -2rk_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_0 \sin \omega t \end{bmatrix} \qquad (d)$$

Substituting given values into Eq. (d) leads to

$$\begin{bmatrix} 2.525 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 45 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + 10^4 \begin{bmatrix} 1.312 & -2.25 & 0 \\ -2.25 & 5.5 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 25 \sin 35t \end{bmatrix} \qquad (e)$$

The method of undetermined coefficients is used and a solution assumed as

$$\begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} \Theta \\ X \\ Y \end{bmatrix} \sin 35t \qquad (f)$$

Substitution of Eq. (f) into Eq. (e) leads to

$$10^4 \begin{bmatrix} 1.0027 & -2.25 & 0 \\ -2.25 & 3.05 & -1 \\ 0 & -1 & -5.51 \end{bmatrix} \begin{bmatrix} \Theta \\ X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 25 \end{bmatrix} \qquad (g)$$

The solution of Eq. (g) is

$$\begin{bmatrix} \Theta \\ X \\ Y \end{bmatrix} = 10^{-4} \begin{bmatrix} 6.995 \\ 3.1117 \\ -6.231 \end{bmatrix} \qquad (h)$$

Problem 9.3 illustrates the use of the method of undetermined coefficients for an undamped 3DOF system.

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9.4 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.4. Use the method of undetermined coefficients.

FIGURE P9.4

Given: $k = 1 \times 10^4 \ \frac{\text{N}}{\text{m}}, m = 10 \text{ kg}, c = 100 \ \frac{\text{N·s}}{\text{m}}, F_0 = 20 \text{ N}, \omega = 15 \ \frac{\text{rad}}{\text{s}}$

Find: X_1, X_2, X_3

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} 100 & -100 & 0 \\ -100 & 100 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + 10^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \sin 15t \end{bmatrix} \qquad (a)$$

The method of undetermined coefficients is used and a solution assumed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} e^{15it} \qquad (b)$$

Only the imaginary part is used in the solution. Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} 7750 + 1500i & -1500i & 0 \\ -1500i & 15500 + 1500i & -20000 \\ 0 & -20000 & 17750 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 10^{-3} \begin{bmatrix} 0.0010 - 0.0573i \\ -0.2948 - 0.0626i \\ -0.2194 - 0.0706i \end{bmatrix} \qquad (d)$$

The solutions are

$$x(t) = -5.73 \times 10^{-5} \cos 15t + 1.0 \times 10^{-6} \sin 15t \qquad (e)$$

$$y(t) = -6.36 \times 10^{-5} \cos 15t - 2.948 \times 10^{-4} \sin 15t \qquad (f)$$

$$z(t) = -7.063 \times 10^{-5} \cos 15t - 2.194 \times 10^{-4} \sin 15t \qquad (e)$$

Problem 9.4 illustrates the application of the method of undetermined coefficients to a 3DOF system with damping.

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9.5 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.5. Use the method of undetermined coefficients.

FIGURE P9.5

Given: System shown

Find: X_1, X_2, X_3

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 50 & -20 & 0 \\ -20 & 50 & -30 \\ 0 & -30 & 30 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 300 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 100 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20\sin 15t \end{bmatrix} \qquad (a)$$

The system is viscously damped and is subject to a single frequency excitation at 15 rad/s. Thus a solution is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} e^{i15t} \qquad (b)$$

Only the imaginary part of Eq. (b) will be used. Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} -1050+750i & -100-300i & 0 \\ -100-300i & -700+750i & -100-450i \\ 0 & -100-450i & -800+450i \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 10^{-3}\begin{bmatrix} 0.7+1.90i \\ -6.45+4.89i \\ -17.6-6.90i \end{bmatrix} \qquad (d)$$

The steady-state amplitudes are the magnitudes of the U's

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 10^{-3}\begin{bmatrix} 2.0 \\ 8.1 \\ 18.9 \end{bmatrix}$$

Problem 9.5 illustrates the solution for the amplitudes of vibration of a three-degree-of-freedom system with viscous damping.

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9.6 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.6. Use the method of undetermined coefficients.

FIGURE P9.6

Given: System shown

Find: X_1, X_2, X_3

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 80 & -80 & 0 \\ -80 & 140 & -60 \\ 0 & -60 & 60 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 4000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 40\sin 30t \end{bmatrix} \qquad (a)$$

The system is viscously damped and is subject to a single frequency excitation at 30 rad/s. Thus a solution is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} e^{i30t} \qquad (b)$$

Only the imaginary part of Eq. (b) will be used. Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} -200+2400i & -1000-2400i & 0 \\ -1000-2400i & 2800+4200i & -3000-1800i \\ 0 & -3000-1800i & 2400+1800i \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 40 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 10^{-3}\begin{bmatrix} -7.64-0.80i \\ -5.94-3.92i \\ 3.82-3.38i \end{bmatrix} \qquad (d)$$

The steady-state amplitudes are the magnitudes of the U's

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 10^{-3}\begin{bmatrix} 7.69 \\ 7.12 \\ 5.05 \end{bmatrix}$$

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Problem 9.6 illustrates the solution for the amplitudes of vibration of a three-degree-of-freedom system with viscous damping.

9.7 Determine the steady-state responses of each of the masses of the system in Figure P9.7. Use the method of undetermined coefficients.

FIGURE P9.7

Given: System shown

Find: $x_1(t), x_2(t), x_3(t)$

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 40 & -20 & 0 \\ -20 & 50 & -30 \\ 0 & -30 & 30 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 600 & -100 & 0 \\ -100 & 300 & -200 \\ 0 & -200 & 200 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20\sin 50t \end{bmatrix} \qquad (a)$$

The system is viscously damped and is subject to a single frequency excitation at 50 rad/s. Thus a solution is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} e^{i50t} \qquad (b)$$

Only the imaginary part of Eq. (b) will be used. Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} -17400 + 2000i & -100 - 1000i & 0 \\ -100 - 1000i & -17200 + 2500i & -200 - 1500i \\ 0 & -200 - 1500i & -12300 + 1500i \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 10^{-5}\begin{bmatrix} 0.8 + 0.1i \\ -1.8 + 13.7i \\ -158 - 19.3i \end{bmatrix} \qquad (d)$$

The steady-state responses are

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$$x_1(t) = 1 \times 10^{-6} \cos 50t + 8 \times 10^{-6} \sin 50t \qquad (e)$$

$$x_2(t) = -1.8 \times 10^{-5} \cos 50t + 13.7 \times 10^{-5} \sin 50t \qquad (f)$$

$$x_3(t) = 15.8 \times 10^{-4} \cos 50t + 1.93 \times 10^{-4} \sin 50t \qquad (g)$$

Problem 9.7 illustrates the solution for the steady-state response vibration of a three-degree-of-freedom system with viscous damping.

9.8 Determine the steady-state responses of each of the masses of the system in P9.8. Use the method of undetermined coefficients.

FIGURE P9.8

Given: System shown

Find: $x_1(t), x_2(t), x_3(t)$

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 150 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 3000 & -2000 & 0 \\ -2000 & 2000 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 20 \sin(30t + \frac{\pi}{4}) \\ 20 \sin 30t \end{bmatrix} \qquad (a)$$

The system is viscously damped and is subject to a single frequency excitation at 30 rad/s. Thus a solution is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} e^{i30t} \qquad (b)$$

Only the imaginary part of Eq. (b) will be used. Substitution of Eq. (b) into Eq. (a) leads to

$$\begin{bmatrix} 1200 + 4500i & -2000 - 3000i & 0 \\ -2000 - 3000i & -700 + 6000i & -3000i \\ 0 & -3000i & -1800 + 3000i \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 20e^{i\pi/4} \\ 20 \end{bmatrix} \qquad (c)$$

The solution of Eq. (c) is

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$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 10^{-3} \begin{bmatrix} -4.57 - 1.48i \\ -4.98 - 3.71i \\ -8.24 - 5.44i \end{bmatrix} \qquad (d)$$

The steady-state responses are

$$x_1(t) = -1.48 \times 10^{-3} \cos 50t - 4.57 \times 10^{-3} \sin 50t \qquad (e)$$

$$x_2(t) = -3.71 \times 10^{-3} \cos 50t - 4.98 \times 10^{-3} \sin 50t \qquad (f)$$

$$x_3(t) = -5.44 \times 10^{-3} \cos 50t - 8.44 \times 10^{-3} \sin 50t \qquad (g)$$

Problem 9.8 illustrates the solution for the steady-state response vibration of a three-degree-of-freedom system with viscous damping.

9.9 Determine the steady-state response of the hanging mass in the system of Figure P9.9. Use the method of undetermined coefficients.

FIGURE P9.9

Given: $m_1 = 30$ kg, $m_2 = 20$ kg, $k_1 = 4 \times 10^5 \ \frac{\text{N}}{\text{m}}$, $k_2 = 3 \times 10^5 \ \frac{\text{N}}{\text{m}}$, $I = 1.8 \text{ kg} \cdot \text{m}^2$, $L = 0.8$ m, $M_0 = 20 \text{ N} \cdot \text{m}$, $\omega = 45 \ \frac{\text{rad}}{\text{s}}$, $\text{c} = 5000 \ \frac{\text{N·s}}{\text{m}}$

Find: $\theta(t), x(t), y(t)$

Solution: The kinetic energy at an arbitrary instant is

$$T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_1 \dot{y}^2 \qquad (a)$$

The potential energy at an arbitrary instant is

$$V = \frac{1}{2} k_1 \left(x - \frac{L}{2}\theta \right)^2 + \frac{1}{2} 2k_1 x^2 + \frac{1}{2} k_2 \left(y - x - \frac{L}{2}\theta \right)^2 \qquad (b)$$

Rayleigh's dissipation function for the system is

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$$\Im = -\frac{1}{2}c\left(\dot{x} - \frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}2c\dot{x}^2 \qquad (c)$$

The virtual work done by the external moment is

$$\delta W = M_0 \sin \omega t \;\; \delta\theta \qquad (d)$$

Lagrange's equations are applied to yield

$$\begin{bmatrix} I & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} c\frac{L^2}{2} & -c\frac{L}{2} & 0 \\ -c\frac{L}{2} & 3c & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} (k_1+k_2)\frac{L^2}{2} & (k_2-k_1)\frac{L}{2} & -k_2\frac{L}{2} \\ (k_2-k_1)\frac{L}{2} & 3k_1+k_2 & -k_2 \\ -k_2\frac{L}{2} & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} \theta \\ x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} M_0 \sin \omega t \\ 0 \\ 0 \end{bmatrix} \qquad (e)$$

Substituting given values into Eq. (d) leads to

$$\begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 20 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + 10^4\begin{bmatrix} 0.16 & -0.4 & 0 \\ -0.4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} + 10^6\begin{bmatrix} 0.224 & -0.04 & -0.12 \\ -0.12 & 1.5 & -0.3 \\ -0.12 & -0.3 & 0.3 \end{bmatrix}\begin{bmatrix} \theta \\ x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} 20 \sin 45t \\ 0 \\ 0 \end{bmatrix} \qquad (f)$$

The method of undetermined coefficients is used and a solution assumed as

$$\begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} \Theta \\ X \\ Y \end{bmatrix} \sin 45t \qquad (g)$$

Only the imaginary part of the solution is used. Substitution of Eq. (f) into Eq. (e) leads to

$$10^6\begin{bmatrix} 0.2204 + 0.0702i & -0.04 - 0.180i & -0.12 \\ -0.04 - 0.180i & 1.432 + 1.35i & -0.3 \\ -0.12 & -0.3 & 0.2529 \end{bmatrix}\begin{bmatrix} \Theta \\ X \\ Y \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix} \qquad (h)$$

The solution of Eq. (g) is

$$\begin{bmatrix} \Theta \\ X \\ Y \end{bmatrix} = 10^{-3}\begin{bmatrix} 0.1303 - 0.0464i \\ 0.0182 - 0.0087i \\ 0.0814 - 0.0315i \end{bmatrix} \qquad (i)$$

The steady-state response of the hanging mass is

$$y(t) = -3.15 \times 10^{-5} \cos 45t + 8.14 \times 10^{-5} \sin 45t \qquad (j)$$

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Problem 9.9 illustrates the use of the method of undetermined coefficients for an damped 3DOF system.

9.10 Determine the steady-state amplitudes of vibration of each of the masses of the system of Figure P9.1. Use the Laplace transform method.

FIGURE P9.1

Given: X_1, X_2, X_3

Find: steady-state response using Laplace transforms

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3000 & -2000 & 0 \\ -2000 & 5000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10\sin 20t \end{bmatrix}$$

The impedance matrix is

$$\mathbf{Z}(s) = \begin{bmatrix} 2s^2 + 3000 & -2000 & 0 \\ -2000 & 4s^2 + 5000 & -3000 \\ 0 & -3000 & 6s^2 + 3000 \end{bmatrix}$$

The transfer functions for the solutions are

$$\mathbf{G}(s) = \mathbf{Z}^{-1}(s) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{s^6 + 3250s^4 + 237500s^2 + 125000000}$$

$$\begin{bmatrix} \frac{s^4}{2} + 875s^2 + 12500 & 250s^2 + 12500 & 12500 \\ 250s^2 + 12500 & \frac{s^4}{4} + 500s^2 + 187500 & 125s^2 + 187500 \\ 12500 & 125s^2 + 187500 & \frac{s^4}{6} + \frac{1375s^2}{3} + \frac{687500}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$$

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$$\frac{1}{s^6+3250s^4+237{,}500s^2+125{,}000{,}000}\begin{bmatrix}12{,}500\\125s^2+187{,}500\\\frac{s^4}{6}+\frac{1375s^2}{3}+\frac{687{,}500}{3}\end{bmatrix}$$

The steady-state response is obtained using the sinusoidal transfer function. The amplitudes are $10|G(20i)|$. To the end

$$\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}=10^{-3}\begin{bmatrix}-3.39\\-3.73\\-1.97\end{bmatrix}$$

Problem 9.10 illustrates the use of the sinusoidal transfer function to determine the steady-state amplitude.

9.11 Determine the steady-state amplitudes of vibration of the hanging mass in the system of Figure P9.2. Use the Laplace transform method.

FIGURE P9.2

Given: system shown

Find: Y

Solution: The kinetic energy at an arbitrary instant is

$$T=\frac{1}{2}I\dot{\theta}^2+\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_2\dot{y}^2 \qquad (a)$$

The potential energy at an arbitrary instant is

$$V=\frac{1}{2}k_1\left(x-\frac{L}{2}\theta\right)^2+\frac{1}{2}2k_1x^2+\frac{1}{2}k_2\left(y-x-\frac{L}{2}\theta\right)^2 \qquad (b)$$

The virtual work done by the external moment is

$$\delta W=M_0\sin\omega t\ \delta\theta \qquad (c)$$

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Lagrange's equations are applied to yield

$$\begin{bmatrix} I & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} (k_1+k_2)\dfrac{L^2}{2} & (k_2-k_1)\dfrac{L}{2} & -k_2\dfrac{L}{2} \\ (k_2-k_1)\dfrac{L}{2} & 3k_1+k_2 & -k_2 \\ -k_2\dfrac{L}{2} & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} M_0 \sin \omega t \\ 0 \\ 0 \end{bmatrix} \qquad (d)$$

Substituting given values into Eq. (d) leads to

$$\begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 20 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + 10^6 \begin{bmatrix} 0.224 & -0.04 & -0.12 \\ -0.04 & 1.5 & -0.3 \\ -0.12 & -0.3 & 0.3 \end{bmatrix}\begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 20 \sin 45t \\ 0 \\ 0 \end{bmatrix}$$

The impedance matrix is

$$\mathbf{Z}(s) = \begin{bmatrix} 1.8s^2 + 0.224 \times 10^6 & -0.04 \times 10^6 & -0.12 \times 10^6 \\ -0.04 \times 10^6 & 30s^2 + 1.5 \times 10^6 & -0.3 \times 10^6 \\ -0.12 \times 10^6 & -0.3 \times 10^6 & 20s^2 + 0.3 \times 10^6 \end{bmatrix}$$

The sinusoidal transfer function required for solving this problem is

$$G_{13}(s) = \frac{\begin{vmatrix} 1 & -0.04 \times 10^6 & -0.12 \times 10^6 \\ 0 & 30s^2 + 1.5 \times 10^6 & -0.3 \times 10^6 \\ 0 & -0.3 \times 10^6 & 20s^2 + 0.3 \times 10^6 \end{vmatrix}}{\begin{vmatrix} 1.8s^2 + 0.224 \times 10^6 & -0.04 \times 10^6 & -0.12 \times 10^6 \\ -0.04 \times 10^6 & 30s^2 + 1.5 \times 10^6 & -0.3 \times 10^6 \\ -0.12 \times 10^6 & -0.3 \times 10^6 & 20s^2 + 0.3 \times 10^6 \end{vmatrix}}$$

$$= \frac{600s^4 + 3.9 \times 10^7 s^2 + 3.6 \times 10^{11}}{1080s^6 + 2.046 \times 10^{10} s^4 + 8.92 \times 10^{122} s^2 + 5.568 \times 10^{16}}$$

The solution is

$$y(t) = 20|G(i45)| \sin(45t + \phi)$$

$$= 1.47 \times 10^{-4} \sin\left(45t + \frac{\pi}{2}\right)$$

Problem 9.11 illustrates the use of the sinusoidal transfer function to determine the steady state response of a system.

9.12 Determine the steady-state amplitude of vibration of the mass m_3 of the system in Figure P9.3 using the Laplace transform method.

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FIGURE P9.3

Given: $m_1 = 18\text{ kg}, m_2 = 20\text{ kg}, m_3 = 45\text{ kg}, k_1 = 3 \times 10^4\ \frac{\text{N}}{\text{m}}, k_2 = 4.5 \times 10^4\ \frac{\text{N}}{\text{m}}, k_3 = 1 \times 10^4\ \frac{\text{N}}{\text{m}},\ I = 1.4\text{ kg} \cdot \text{m}^2,\ r = 0.25\text{ m}, F_0 = 25\text{ N}, \omega = 35\ \frac{\text{rad}}{\text{s}}$

Find: Y

Solution: The kinetic energy at an arbitrary instant is

$$T = \frac{1}{2}(I + m_1 r^2)\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_3\dot{y}^2 \qquad (a)$$

The potential energy at an arbitrary instant is

$$V = \frac{1}{2}k_1(r\theta)^2 + \frac{1}{2}k_2(x - 2r\theta)^2 + \frac{1}{2}k_3(y - x)^2 \qquad (b)$$

The virtual work done by the external moment is

$$\delta W = F_0 \sin\omega t\ \ \delta y \qquad (c)$$

Lagrange's equations are applied to yield

$$\begin{bmatrix} I + m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} (k_1 + 4k_2)r^2 & -2rk_2 & 0 \\ -2rk_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_0 \sin\omega t \end{bmatrix} \qquad (d)$$

Substituting given values into the differential equations leads to

$$\begin{bmatrix} 2.525 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 45 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + 10^4\begin{bmatrix} 1.312 & -2.25 & 0 \\ -2.25 & 5.5 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 25 \sin 35t \end{bmatrix}$$

Taking the Laplace transform of both sides and letting $X_i(s) = \mathcal{L}\{x_i(t)\}$ leads to

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$$\begin{bmatrix} 2.525s^2 + 1.312 \times 10^4 & -2.25 \times 10^4 & 0 \\ -2.25 & 20s^{2+}5.5 \times 10^4 & -1 \times 10^4 \\ 0 & -1 \times 10^4 & 45s^2 + 1 \times 10^4 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{875}{s^2 + 1225} \end{bmatrix}$$

Determining the transfer function for x_3

$$G_{33}(s) = \frac{\begin{vmatrix} 2.525s^2 + 1.312 \times 10^4 & -2.25 \times 10^4 & 0 \\ -2.25 \times 10^4 & 20s^2 + 5.5 \times 10^4 & 0 \\ 0 & -1 \times 10^4 & 1 \end{vmatrix}}{\begin{vmatrix} 2.525s^2 + 1.312 \times 10^4 & -2.25 \times 10^4 & 0 \\ -2.25 \times 10^4 & 20s^{2+}5.5 \times 10^4 & -1 \times 10^4 \\ 0 & -1 \times 10^4 & 45s^2 + 1 \times 10^4 \end{vmatrix}}$$

$$= \frac{55.5s^4 + 4.013 \times 10^5 s^2 + 2.154 \times 10^8}{2272.5s^6 + 1.856 \times 10^7 s^4 + 1.345 \times 10^{10} s^2 + 8.415 \times 10^{11}}$$

The response if given by the sinusoidal transfer function as

$$x_3(t) = 25|G(i35)| \sin(35t + \phi)$$

and is

$$x_3(t) = 6.23 \times 10^{-4} \sin\left(35t + \frac{\pi}{2}\right)$$

Problem 9.12 illustrates the use of the sinusoidal transfer function.

9.13 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.4. Use the Laplace transform method.

FIGURE P9.4

Given: $k = 1 \times 10^4 \ \frac{\text{N}}{\text{m}}, m = 10 \text{ kg}, c = 100 \ \frac{\text{N·s}}{\text{m}}, F_0 = 20 \text{ N}, \omega = 15 \ \frac{\text{rad}}{\text{s}}$

Find: X, Y, Z

Solution: The differential equations governing the motion of the system are

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$$\begin{bmatrix}10 & 0 & 0\\ 0 & 20 & 0\\ 0 & 0 & 10\end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}+\begin{bmatrix}100 & -100 & 0\\ -100 & 100 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}+10^4\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & -2\\ 0 & -2 & 2\end{bmatrix}\begin{bmatrix}\theta\\ x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 20\sin 15t\end{bmatrix}$$

The impedance matrix is

$$\mathbf{Z}(s)=\begin{bmatrix}10s^2+100s+1x10^4 & -100s & 0\\ -100s & 20s^2+100s+2\times 10^4 & -2\times 10^4\\ 0 & -2\times 10^4 & 10s^2+2\times 10^4\end{bmatrix}$$

The desired transfer functions are

$$\mathbf{G}_3(s)=Z^{-1}(s)\begin{bmatrix}0\\0\\1\end{bmatrix}=\frac{1}{s^6+4\times 10^3 s^5+4.5\times 10^4 s^3+3\times 10^6 s^2+1\times 10^7 s}$$

$$\begin{bmatrix}0.1s^4+0.5s^3+300s^2+100s & 0.5s^2+100s & 1000\\ 0.5s^2+100s & 0.05s^4+0.5s^3+150s^2+1000s+100000 & 100s^2+1000s+10000\\ 1000 & 100s^2+1000s+10000 & 0.1s^4+1.5s^3+200s^2+1500s+100000\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$=\begin{bmatrix}1000\\ 100s^2+1000s+10000\\ 0.1s^4+1.5s^3+200s^2+1500s+100000\end{bmatrix}$$

The steady state responses are obtained by evaluation of the sinusoidal transfer functions. The result is

$$x_1(t)=6\times 10^{-4}\sin(15t-1.55)$$

$$x_2(t)=3.3\times 10^{-3}\sin(15t-2.93)$$

$$x_3(t)=2.6\times 10^{-3}\sin(15t-2.89)$$

Problem 9.13 illustrates the use of the sinusoidal transfer function for systems with damping.

9.14 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.5. Use the Laplace transform method.

FIGURE P9.5

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Given: System shown

Find: X_1, X_2, X_3

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 50 & -20 & 0 \\ -20 & 50 & -30 \\ 0 & -30 & 30 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 300 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 100 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20\sin 15t \end{bmatrix}$$

The impedance matrix is

$$\mathbf{Z}(s) = \begin{bmatrix} 6s^2 + 50s + 300 & -20s - 100 & 0 \\ -20s - 100 & 4s^2 + 50s + 200 & -30s - 100 \\ 0 & -30s - 100 & 4s^2 + 30s + 100 \end{bmatrix}$$

The desired transfer functions are

$$\mathbf{G}_3(s) = Z^{-1}(s)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{12s^5 + 280s^4 + 2350s^3 + 11{,}750s^2 + 22{,}500s + 50{,}000}$$

$$\begin{bmatrix} 2s^3 + 30s^2 + 75s + 750 & 10s^2 + 75s + 250 & 75s + 250 \\ 10s^2 + 75s + 250 & 3s^4 + 47.5s^3 + 412.5s^2 + 1750s + 3750 & 22.5s^3 + 262.5s^2 + 1750s + 3750 \\ 75s + 250 & 22.5s^3 + 262.5s^2 + 1750s + 3750 & 3s^4 + 62.5s^3 + 562.5s^2 + 2625s + 6250 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 75s + 250 \\ 22.5s^3 + 262.5s^2 + 1750s + 3750 \\ 3s^4 + 62.5s^3 + 562.5s^2 + 2625s + 6250 \end{bmatrix}$$

The steady state amplitudes are $20|\mathbf{G}_3(15i)|$ which are

$$X_1 = 1.973 \times 10^{-3} \quad X_2 = 8.052 \times 10^{-3} \quad X_3 = 1.889 \times 10^{-2}$$

Problem 9.14 illustrates the use of the sinusoidal transfer function for damped problems.

9.15 Determine the response of the 2 kg mass of Figure P9.1 if the sinusoidal force is replaced by the triangular pulse of Figure P9.15. Use the Laplace transform method.

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FIGURE P9.15

FIGURE P9.1

Given: System shown

Find: X_1

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3000 & -2000 & 0 \\ -2000 & 5000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix}$$

where

$$F(t) = [200t - 200tu(t-0.1)] + (40 - 200t)[u(t-0.1) - u(t-0.2)]$$

$$= 200t - 400(t-0.1)u(t-0.1) + 200(t-0.2)u(t-0.2)$$

Using the second shifting theorem

$$F(s) = \frac{200}{s^2} - \frac{400}{s^2}e^{-0.1s} + \frac{200}{s^2}e^{-0.2s}$$

The impedance matrix is

$$\mathbf{Z}(s) = \begin{bmatrix} 2s^2 + 3000 & -2000 & 0 \\ -2000 & 4s^2 + 5000 & -3000 \\ 0 & -3000 & 6s^2 + 3000 \end{bmatrix}$$

The transfer functions for the solutions are

$$\mathbf{G}(s) = \mathbf{Z}^{-1}(s) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{12{,}500}{s^6 + 3250s^4 + 237{,}500s^2 + 125{,}000{,}000}$$

The solution for $X_1(s)$ is

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$$X_1(s) = G_{13}(s)F(s) = \frac{(12{,}500)200(1 - 2e^{-0.1s} + e^{-0.2s})}{s^2(s^6 + 3250s^4 + 237{,}500s^2 + 125{,}000{,}000)}$$

$$= 10^{-2}\left(\frac{2}{s^2} - \frac{2.175}{s^2 + 57} + \frac{0.222}{s^2 + 1000} - \frac{0.0447}{s^2 + 2193}\right)(1 - 2e^{-0.1s} + e^{-0.2s})$$

Taking the inverse Laplace transform of the above using the second shifting theorem leads to

$$\begin{aligned} x_1(t) = 10^{-2}\{&2t - 0.264\sin 7.55t + 0.00702\sin 31.63t - 0.000955\sin 46.83t \\ &- 2[2(t-0.1) - 0.264\sin 7.55(t-0.1) + 0.00702\sin 31.63(t-0.1) \\ &- 0.000955\sin 46.83(t-0.1)]u(t-0.1) \\ &+ [2(t-0.2) - 0.264\sin 7.55(t-0.2) + 0.00702\sin 31.63(t-0.2) \\ &- 0.000955\sin 46.83(t-0.2)]u(t-0.2)\} \end{aligned}$$

Problem 9.15 illustrates use of the Laplace transform method to find the transient response of a 3DOF system.

9.16 Determine the response of the 6 kg mass of Figure P9.1 if the sinusoidal force is replaced by the rectangular pulse of Figure P9.16. Use the Laplace transform method.

FIGURE P9.16

FIGURE P9.1

Given: System shown

Find: X_3

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3000 & -2000 & 0 \\ -2000 & 5000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix}$$

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where

$$F(t) = 20[u(t) - u(t - 0.5)]$$

Using the second shifting theorem

$$F(s) = \frac{20}{s}(1 - e^{-0.5s})$$

The impedance matrix is

$$\mathbf{Z}(s) = \begin{bmatrix} 2s^2 + 3000 & -2000 & 0 \\ -2000 & 4s^2 + 5000 & -3000 \\ 0 & -3000 & 6s^2 + 3000 \end{bmatrix}$$

The transfer functions for the solutions are

$$\mathbf{G}(s) = \mathbf{Z}^{-1}(s)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{s^6 + 3250s^4 + 237{,}500s^2 + 125{,}000{,}000}\begin{bmatrix} 12{,}500 \\ 125s^2 + 187{,}500 \\ \frac{s^4}{6} + \frac{1375s^2}{3} + \frac{687{,}500}{3} \end{bmatrix}$$

The solution for $X_3(s)$ is

$$X_3(s) = G_{33}(s)F(s) = \frac{20\left(\frac{s^4}{6} + \frac{1375s^2}{3} + \frac{687500}{3}\right)(1 - e^{-0.5s})}{s(s^6 + 3250s^4 + 237{,}500s^2 + 125{,}000{,}000)} =$$

$$= \frac{20\left(\frac{s^4}{6} + \frac{1375s^2}{3} + \frac{687{,}500}{3}\right)(1 - e^{-0.5s})}{s(s^2 + 57)(s^2 + 1000)(s^2 + 2193)}$$

$$= 3.33 \times 10^{-3}\left(\frac{11.0}{s} - \frac{10.6s}{s^2 + 57} - \frac{0.333s}{s^2 + 1000} - \frac{0.0275s}{s^2 + 2193}\right)(1 - e^{-0.5s})$$

Thus

$$\begin{aligned} x_3(t) = 3.33 \times 10^{-3}\{&11.0 - 10.6\cos 7.55t - 0.333\cos 31.62t - 0.0275\cos 46.83t \\ &- [11.0 - 10.6\cos 7.55(t - 0.5) - 0.333\cos 31.62(t - 0.5) \\ &- 0.0275\cos 46.83(t - 0.5)]u(t - 0.5)\} \end{aligned}$$

Problem 9.16 illustrates the use of the Laplace transform method to solve a transient MDOF vibrations problem.

9.17 Determine the response of the system of Figure P9.2 if the sinusoidal force is replaced by the force of Figure P9.17. Use the Laplace transform method.

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FIGURE P9.17

FIGURE P9.2

Given: system shown

Find: Y

Solution: The kinetic energy at an arbitrary instant is

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 \qquad (a)$$

The potential energy at an arbitrary instant is

$$V = \frac{1}{2}k_1\left(x - \frac{L}{2}\theta\right)^2 + \frac{1}{2}2k_1x^2 + \frac{1}{2}k_2\left(y - x - \frac{L}{2}\theta\right)^2 \qquad (b)$$

The virtual work done by the external moment is

$$\delta W = M_0 \sin\omega t \;\; \delta\theta \qquad (c)$$

Lagrange's equations are applied to yield

$$\begin{bmatrix} I & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} (k_1+k_2)\frac{L^2}{2} & (k_2-k_1)\frac{L}{2} & -k_2\frac{L}{2} \\ (k_2-k_1)\frac{L}{2} & 3k_1+k_2 & -k_2 \\ -k_2\frac{L}{2} & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} M_0 \sin\omega t \\ 0 \\ 0 \end{bmatrix} \qquad (d)$$

Substituting given values into Eq. (d) leads to

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$$\begin{bmatrix}1.8 & 0 & 0\\ 0 & 30 & 0\\ 0 & 0 & 20\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{x}\\ \ddot{y}\end{bmatrix}+10^6\begin{bmatrix}0.224 & -0.04 & -0.12\\ -0.04 & 1.5 & -0.3\\ -0.12 & -0.3 & 0.3\end{bmatrix}\begin{bmatrix}\theta\\ x\\ y\end{bmatrix}=\begin{bmatrix}20\sin 45t\\ 0\\ 0\end{bmatrix}$$

The impedance matrix is

$$\mathbf{Z}(s)=\begin{bmatrix}1.8s^2+0.224\times10^{-6} & -0.04\times10^{-6} & -0.12\times10^{-6}\\ -0.04\times10^{-6} & 30s^2+1.5\times10^{-6} & -0.3\times10^{-6}\\ -0.12\times10^{-6} & -0.3\times10^{-6} & 20s^2+0.3\times10^{-6}\end{bmatrix}$$

Invert the impedance matrix and calculate the inverse transform of Y(s).

Problem 917 illustrates the use of the Laplace transform method to solve a forced vibrations problem for a 3DOF system.

9.18 Repeat Chapter Problem 9.1 using modal analysis.

FIGURE P9.1

Given: System shown

Find: X_1

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix}2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 6\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{bmatrix}+\begin{bmatrix}3000 & -2000 & 0\\ -2000 & 5000 & -3000\\ 0 & -3000 & 3000\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 10\sin 20t\end{bmatrix} \qquad (a)$$

The natural frequencies are determined as the square roots of the eigenvalues of $\mathbf{M^{-1}K}$. They are

$$\omega_1=7.55\ \frac{\text{rad}}{\text{s}} \qquad \omega_2=31.63\ \frac{\text{rad}}{\text{s}} \qquad \omega_2=46.83\ \frac{\text{rad}}{\text{s}}$$

The modal matrix is the matrix whose columns are normalized eigenvectors of $\mathbf{M^{-1}K}$. It is

$$\mathbf{P}=\begin{bmatrix}0.1952 & -0.4714 & -0.4896\\ 0.2817 & -0.2357 & 0.3393\\ 0.3179 & 0.2357 & -0.1002\end{bmatrix}$$

The right-hand side vector is $\mathbf{P^TF}$ which is

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$$G = \begin{bmatrix} 0.3179 \\ 0.2357 \\ -0.1002 \end{bmatrix} 10 \sin 20t$$

The differential equations defining the principal coordinates are

$$\ddot{p}_1 + (7.55)^2 p_1 = 3.179 \sin 20t$$

$$\ddot{p}_2 + (32.63)^2 p_2 = 2.357 \sin 20t$$

$$\ddot{p}_3 + (46.83)^2 p_3 = -1.002 \sin 20t$$

The steady state solutions of the differential equations are

$$p_1(t) = \frac{3.179}{(7.55)^2 - (20)^2} \sin 20t = -9.27 \times 10^{-3} \sin 20t$$

$$p_2(t) = \frac{2.357}{(31.63)^2 - (20)^2} \sin 20t = 3.93 \times 10^{-3} \sin 20t$$

$$p_1(t) = \frac{-1.002}{(46.83)^2 - (20)^2} \sin 20t = -5.59 \times 10^{-4} \sin 20t$$

The steady-state solution for **x** is

$$\mathbf{x} = \mathbf{P}\mathbf{p} = \begin{bmatrix} 0.1952 & -0.4714 & -0.4896 \\ 0.2817 & -0.2357 & 0.3393 \\ 0.3179 & 0.2357 & -0.1002 \end{bmatrix} \begin{bmatrix} -9.27 \times 10^{-3} \\ 3.93 \times 10^{-4} \\ -5.59 \times 10^{-4} \end{bmatrix} \sin 20t = -10^{-3} \begin{bmatrix} 3.39 \\ 3.74 \\ 1.98 \end{bmatrix} \sin 20t$$

Problem 9.18 illustrates modal analysis for undamped systems.

9.19 Repeat Chapter Problem 9.2 using modal analysis.

FIGURE P9.2

Given: $m_1 = 30$ kg, $m_2 = 20$ kg, $k_1 = 4 \times 10^5 \ \frac{\text{N}}{\text{m}}$, $k_2 = 3 \times 10^5 \ \frac{\text{N}}{\text{m}}$, $I = 1.8 \text{ kg} \cdot \text{m}^2$, $L = 0.8$ m, $M_0 = 20 \text{ N} \cdot \text{m}$, $\omega = 45 \ \frac{\text{rad}}{\text{s}}$

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Find: Y

Solution: The kinetic energy at an arbitrary instant is

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 \qquad (a)$$

The potential energy at an arbitrary instant is

$$V = \frac{1}{2}k_1\left(x - \frac{L}{2}\theta\right)^2 + \frac{1}{2}2k_1x^2 + \frac{1}{2}k_2\left(y - x - \frac{L}{2}\theta\right)^2 \qquad (b)$$

The virtual work done by the external moment is

$$\delta W = M_0 \sin\omega t \;\; \delta\theta \qquad (c)$$

Lagrange's equations are applied to yield

$$\begin{bmatrix} I & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} (k_1+k_2)\frac{L^2}{2} & (k_2-k_1)\frac{L}{2} & -k_2\frac{L}{2} \\ (k_2-k_1)\frac{L}{2} & 3k_1+k_2 & -k_2 \\ -k_2\frac{L}{2} & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} M_0 \sin\omega t \\ 0 \\ 0 \end{bmatrix} \qquad (d)$$

Substituting given values into Eq. (d) leads to

$$\begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 20 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + 10^6\begin{bmatrix} 0.224 & -0.04 & -0.12 \\ -0.04 & 1.5 & -0.3 \\ -0.12 & -0.3 & 0.3 \end{bmatrix}\begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 20\sin 45t \\ 0 \\ 0 \end{bmatrix} \qquad (e)$$

The natural frequencies are determined as the square roots of the eigenvalues of $\mathbf{M^{-1}K}$. They are

$$\omega_1 = 86.44\frac{\text{rad}}{\text{s}} \qquad \omega_2 = 232.06\frac{\text{rad}}{\text{s}} \qquad \omega_3 = 357.94\frac{\text{rad}}{\text{s}}$$

The modal matrix is the matrix whose columns are normalized eigenvectors of $\mathbf{M^{-1}K}$. It is

$$\mathbf{P} = \begin{bmatrix} 0.1300 & -0.0076 & -0.7339 \\ 0.0535 & 0.1744 & 0.0077 \\ 0.2102 & -0.0602 & 0.0379 \end{bmatrix}$$

The right-hand side vector is $\mathbf{P^TF}$ which is

$$G = \begin{bmatrix} 0.1300 \\ -0.0076 \\ -0.73392 \end{bmatrix} 20\sin 45t$$

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The differential equations defining the principal coordinates are

$$\ddot{p}_1 + (86.44)^2 p_1 = 2.6 \sin 45t$$

$$\ddot{p}_2 + (232.06)^2 p_2 = -0.152 \sin 45t$$

$$\ddot{p}_3 + (357.94)^2 p_3 = -14.68 \sin 45t$$

The steady state solutions of the differential equations are

$$p_1(t) = \frac{2.6}{(86.44)^2 - (45)^2} \sin 45t = 4.77 \times 10^{-4} \sin 45t$$

$$p_2(t) = \frac{-0.152}{(232.06)^2 - (45)^2} \sin 45t = -2.93 \times 10^{-6} \sin 45$$

$$p_1(t) = \frac{-14.68}{(357.94)^2 - (45)^2} \sin 45t = -1.164 \times 10^{-4} \sin 45t$$

The steady-state solution for **x** is

$$\mathbf{x} = \mathbf{Pp} = \begin{bmatrix} 0.1300 & -0.0076 & -0.7339 \\ 0.0535 & 0.1744 & 0.0077 \\ 0.2102 & -0.0602 & 0.0379 \end{bmatrix} \begin{bmatrix} 4.77 \times 10^{-4} \\ -2.93 \times 10^{-6} \\ -1.14 \times 10^{-4} \end{bmatrix} \sin 45t = 10^{-4} \begin{bmatrix} 1.053 \\ 0.241 \\ 0.0919 \end{bmatrix} \sin 45t$$

Problem 9.19 illustrates modal analysis for undamped systems.

9.20 Repeat Chapter Problem 9.3 using modal analysis.

FIGURE P9.3

Given: $m_1 = 18$ kg, $m_2 = 20$ kg, $m_3 = 45$ kg, $k_1 = 3 \times 10^4 \ \frac{\text{N}}{\text{m}}$, $k_2 = 4.5 \times 10^4 \ \frac{\text{N}}{\text{m}}$, $k_3 = 1 \times 10^4 \ \frac{\text{N}}{\text{m}}$, $I = 1.4 \text{ kg} \cdot \text{m}^2$, $r = 0.25$ m, $F_0 = 25$ N, $\omega = 35 \ \frac{\text{rad}}{\text{s}}$

Find: Y

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Solution: The kinetic energy at an arbitrary instant is

$$T = \frac{1}{2}(I + m_1 r^2)\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_3\dot{y}^2 \qquad (a)$$

The potential energy at an arbitrary instant is

$$V = \frac{1}{2}k_1(r\theta)^2 + \frac{1}{2}k_2(x - 2r\theta)^2 + \frac{1}{2}k_3(y - x)^2 \qquad (b)$$

The virtual work done by the external moment is

$$\delta W = F_0 \sin \omega t \;\; \delta y \qquad (c)$$

Lagrange's equations are applied to yield

$$\begin{bmatrix} I + m_1 r^2 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} (k_1 + 4k_2)r^2 & -2rk_2 & 0 \\ -2rk_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_0 \sin \omega t \end{bmatrix} \qquad (d)$$

Substituting given values into Eq. (d) leads to

$$\begin{bmatrix} 2.525 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 45 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + 10^4 \begin{bmatrix} 1.312 & -2.25 & 0 \\ -2.25 & 5.5 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 25 \sin 35t \end{bmatrix} \qquad (e)$$

The natural frequencies are determined as the square roots of the eigenvalues of $\mathbf{M^{-1}K}$. They are

$$\omega_1 = 8.3124 \frac{\text{rad}}{\text{s}} \qquad \omega_2 = 26.966 \frac{\text{rad}}{\text{s}} \qquad \omega_3 = 85.86 \frac{\text{rad}}{\text{s}}$$

The modal matrix is the matrix whose columns are normalized eigenvectors of $\mathbf{M^{-1}K}$. It is

$$\mathbf{P} = \begin{bmatrix} 0.1574 & -0.3203 & -0.5184 \\ 0.0904 & -0.1606 & 0.1266 \\ 0.1312 & 0.0707 & -0.0039 \end{bmatrix}$$

The right-hand side vector is $\mathbf{P^T F}$ which is

$$G = \begin{bmatrix} 0.1312 \\ 0.0707 \\ -0.0039 \end{bmatrix} 25 \sin 35t$$

The differential equations defining the principal coordinates are

$$\ddot{p}_1 + (8.3122)^2 p_1 = 3.28 \sin 35t$$

$$\ddot{p}_2 + (26.96)^2 p_2 = 1.77 \sin 35t$$

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$$\ddot{p}_3 + (85.86)^2 p_3 = -0.0975 \sin 35t$$

The steady state solutions of the differential equations are

$$p_1(t) = \frac{3.28}{(8.312)^2 - (35)^2} \sin 35t = -2.83 \times 10^{-3} \sin 35t$$

$$p_2(t) = \frac{1.77}{(26.96)^2 - (35)^2} \sin 35t = -3.55 \times 10^{-3} \sin 35$$

$$p_1(t) = \frac{-0.0975}{(85.86)^2 - (35)^2} \sin 35t = -1.59 \times 10^{-5} \sin 35t$$

The steady-state solution for y is

$$y(t) = 0.1322 p_1(t) + 0.0707 p_2(t) - 0.0039 p_3(t) = -6.244 \times 10^{-4} \sin 35t$$

9.21 Repeat Chapter Problem 9.15 using modal analysis.

FIGURE P9.15

FIGURE P9.1

Given: System shown

Find: $x_1(t)$

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3000 & -2000 & 0 \\ -2000 & 5000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix}$$

The natural frequencies are determined as the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. They are

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$$\omega_1 = 7.55\ \frac{\text{rad}}{\text{s}} \qquad \omega_2 = 31.63\ \frac{\text{rad}}{\text{s}} \qquad \omega_2 = 46.83\ \frac{\text{rad}}{\text{s}}$$

The modal matrix is the matrix whose columns are normalized eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. It is

$$\mathbf{P} = \begin{bmatrix} 0.1952 & -0.4714 & -0.4896 \\ 0.2817 & -0.2357 & 0.3393 \\ 0.3179 & 0.2357 & -0.1002 \end{bmatrix}$$

The right-hand side vector is $\mathbf{P}^{\mathbf{T}}\mathbf{F}$ which is

$$G = \begin{bmatrix} 0.3179 \\ 0.2357 \\ -0.1002 \end{bmatrix} F(t)$$

where

$$F(t) = [200t - 200tu(t-0.1)] + (40 - 200t)[u(t-0.1) - u(t-0.2)]$$
$$= 200t - 400(t-0.1)u(t-0.1) + 200(t-0.2)u(t-0.2)$$

The differential equations defining the principal coordinates are

$$\ddot{p}_1 + (7.55)^2 p_1 = 3.179[200t - 400(t-0.1)u(t-0.1) + 200(t-0.2)u(t-0.2)]$$

$$\ddot{p}_2 + (32.63)^2 p_2 = 2.357[200t - 400(t-0.1)u(t-0.1) + 200(t-0.2)u(t-0.2)]$$

$$\ddot{p}_3 + (46.83)^2 p_3 = -1.002[200t - 400(t-0.1)u(t-0.1) + 200(t-0.2)u(t-0.2)]$$

The convolution integral is used to solve each equation. For example the solution to the first equation is

$$p_1(t) = \frac{1}{7.55}\int_0^t 3.179[200\tau - 400(\tau-0.1)u(\tau-0.1) + 200(\tau-0.2)u(\tau-0.2)]\sin 7.55(t-\tau)\,d\tau$$

Table 5.1 is used to facilitate the convolution integral. For example with delayed ramp functions, $m = 1, \omega_n = 7.55$ the convolution integral is used to obtain

$$p_1(t) = \frac{3.179(200)}{(7.55)^2}\left\{t - \frac{1}{7.55}\sin 7.55t - 2\left[t - 0.1 - \frac{1}{7.55}\sin 7.55(t-0.1)\right]u(t-0.1) + \left[t - 0.2 - \frac{1}{7.55}\sin 7.55(t-0.2)\right]u(t-0.2)\right\}$$

$$= 11.15\{t - 0.132\sin 7.55t - 2[t - 0.1 - 0.132\sin 7.55(t-0.1)]u(t-0.1) + [t - 0.2 - 0.132\sin 7.55(t-0.2)]u(t-0.2)\}$$

Similarly

$$p_2(t) = 0.443\{t - 0.0306\sin 32.63t - 2[t - 0.1 - 0.0.306\sin 32.63(t-0.1)]u(t-0.1) + [t - 0.2 - 0.0306\sin 32.63(t-0.2)]u(t-0.2)\}$$

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$$p_3(t) = -0.0914\{t - 0.0213\sin 46.83t - 2[t - 0.1 - 0.0.213\sin 46.83(t-0.1)]u(t-0.1) + [t - 0.2 - 0.0213\sin 46.83(t-0.2)]u(t-0.2)\}$$

The response of the system is obtained by **x = Pp**. Specifically for x_1

$$x_1(t) = 0.1952p_1(t) - 0.4714p_2(t) - 0.4896p_3(t)$$

Problem 9.21 illustrates the use of modal analysis to find the response of a 3DOF system subject to a transient excitation.

9.22 Repeat Chapter Problem 9.16 using modal analysis.

FIGURE P9.16

FIGURE P9.1

Given: System shown

Find: x_3

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3000 & -2000 & 0 \\ -2000 & 5000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix}$$

The natural frequencies are determined as the square roots of the eigenvalues of $\mathbf{M^{-1}K}$. They are

$$\omega_1 = 7.55\ \frac{\text{rad}}{\text{s}} \qquad \omega_2 = 31.63\ \frac{\text{rad}}{\text{s}} \qquad \omega_2 = 46.83\ \frac{\text{rad}}{\text{s}}$$

The modal matrix is the matrix whose columns are normalized eigenvectors of $\mathbf{M^{-1}K}$. It is

$$\mathbf{P} = \begin{bmatrix} 0.1952 & -0.4714 & -0.4896 \\ 0.2817 & -0.2357 & 0.3393 \\ 0.3179 & 0.2357 & -0.1002 \end{bmatrix}$$

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The right-hand side vector is $\mathbf{P}^{\mathbf{T}}\mathbf{F}$ which is

$$G = \begin{bmatrix} 0.3179 \\ 0.2357 \\ -0.1002 \end{bmatrix} F(t)$$

where

$$F(t) = 20[u(t) - u(t - 0.5)]$$

The differential equations defining the principal coordinates are

$$\ddot{p}_1 + (7.55)^2 p_1 = 3.179\{20[u(t) - u(t - 0.5)]\}$$

$$\ddot{p}_2 + (32.63)^2 p_2 = 2.357\{20[u(t) - u(t - 0.5)]\}$$

$$\ddot{p}_3 + (46.83)^2 p_3 = -1.002\{20[u(t) - u(t - 0.5)]\}$$

The convolution integral is used to solve each equation. For example the solution to the first equation is

$$p_1(t) = \frac{1}{7.55}\int_0^t 3.179\{20[u(\tau) - u(\tau - 0.5)]\} \sin 7.55(t - \tau)\, d\tau$$

Table 5.1 is used to facilitate the convolution integral. For example with delayed ramp functions, $m = 1, \omega_n = 7.55$ the convolution integral is used to obtain

$$p_1(t) = \frac{3.179(20)}{(7.55)^2}\{t - \cos 7.55t - [t - \cos\ 7.55(t - 0.5)]u(t - 0.5)\}$$

$$= 1.153\{t - \cos 7.55t - [t - \cos\ 7.55(t - 0.5)]u(t - 0.5)\}$$

Similarly

$$p_2(t) = 0.0443\{t - \cos 32.63t - [t - \cos\ 32.63(t - 0.5)]u(t - 0.5)\}$$

$$p_3(t) = -0.00913\{t - \cos 46.83t - [t - \cos\ 46.83(t - 0.5)]u(t - 0.5)\}$$

The response of the system is obtained by **x** = **Pp**. Specifically for x_3

$$x_3(t) = 0.3179p_1(t) + 0.2357p_2(t) - 0.1002p_3(t)$$

Problem 9.22 illustrates the use of modal analysis to find the response of a 3DOF system subject to a transient excitation.

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9.23 Repeat Chapter Problem 9.7 using modal analysis.

FIGURE P9.7

Given: System shown

Find: $x_1(t), x_2(t), x_3(t)$

Solution: The differential equations governing the motion of the system are

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 40 & -20 & 0 \\ -20 & 50 & -30 \\ 0 & -30 & 30 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 600 & -100 & 0 \\ -100 & 300 & -200 \\ 0 & -200 & 200 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20\sin 50t \end{bmatrix}$$

Rewriting the equations in the matrix form of Eq. (9.37). The matrices in the equation are

$$\breve{\mathbf{M}} = \begin{bmatrix} 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 5 & 0 & 0 & 40 & -20 & 0 \\ 0 & 7 & 0 & -20 & 50 & -30 \\ 0 & 0 & 5 & 0 & -30 & 30 \end{bmatrix}$$

$$\breve{\mathbf{K}} = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 600 & -100 & 0 \\ 0 & 0 & 0 & -100 & 300 & -200 \\ 0 & 0 & 0 & 0 & -200 & 200 \end{bmatrix}$$

$$\breve{\mathbf{F}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 20\sin 50t \end{bmatrix}$$

The eigenvalues and modal matrix for the system are

$$\lambda_{1,2} = 3.978 \mp i9.611, \quad \lambda_{3,4} = 6.049 \mp i6.657 \qquad \lambda_{5,62} = 0.5449 \mp i2.496$$

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$$\breve{\mathbf{P}} = \begin{bmatrix} 0.0112-0.0169i & 0.0112+0.0169i & 0.0048+0.0661i & 0.0048-0.00661i & -0.0274-0.1049i & -0.0274+0.0149i \\ -0.0446-0.0041i & -0.0446+0.0041i & 0.1377-0.1010i & 0.1377+0.1010i & -0.3378-0.4829i & -0.3378+0.4829i \\ -0.0009+0.0057i & -0.0008-0.0057i & -0.1361+0.0985i & -0.1361-0.0985i & -0.3395-0.5812i & -0.3395+0.5812i \\ -0.0019-0.0004i & -0.0019+0.0004i & 0.0051-0.0053i & 0.0051+0.0053i & -0.0378+0.0192i & -0.0378-0.0192i \\ -0.0002+0.0006i & -0.0002-0.0006i & -0.0186-0.0038i & -0.0186+0.0038i & -0.1564+0.1695i & -0.1564-0.1695i \\ 0.0005-0.0011i & 0.005+0.0011i & 0.0183+0.0038i & 0.0183-0.0038i & -0.1889+0.2012i & -0.1889-0.2012i \end{bmatrix}$$

The force vector is

$$\tilde{\mathbf{F}} = \begin{bmatrix} 0.0005-0.0001i \\ 0.0005+0.0001i \\ 0.0183+0.0038i \\ 0.0183-0.0038i \\ -0.1889+0.2012i \\ -0.1889+0.2012i \end{bmatrix} 20\sin 50t$$

The differential equations for the principal coordinates are

$$\dot{p}_1 + (3.9781-0.9610i)p_1 = (0.0005-0.0001)20\sin 50t$$

$$\dot{p}_2 + (3.9781+0.9610i)p_2 = (0.0005+0.0001)20\sin 50t$$

$$\dot{p}_3 + (6.0485-6.6566i)p_3 = (0.0183+0.0038i)20\sin 50t$$

$$\dot{p}_4 + (6.0485+6.6566i)p_5 = (0.0183-0.0038i)20\sin 50t$$

$$\dot{p}_5 + (0.5449-2.4964i)p_5 = (-0.1889+0.2012i)20\sin 50t$$

$$\dot{p}_6 + (0.5449+2.4964i)p_6 = (-0.1889-0.2012i)20\sin 50t$$

The differential equations for the principal coordinates are solved. The original coordinates are obtained by multiplying by the modal matrix.

Problem 9.23 illustrates the use of modal analysis for a damped system.

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9.24 Repeat Chapter Problem 9.9 using modal analysis.

FIGURE P9.9

Given: $m_1 = 30\ \text{kg}, m_2 = 20\ \text{kg}, k_1 = 4 \times 10^5\ \frac{\text{N}}{\text{m}}, k_2 = 3 \times 10^5\ \frac{\text{N}}{\text{m}}, I = 1.8\ \text{kg} \cdot \text{m}^2,$
$L = 0.8\ \text{m}, M_0 = 20\ \text{N} \cdot \text{m}, \omega = 45\ \frac{\text{rad}}{\text{s}}, \text{c} = 5000\ \frac{\text{N·s}}{\text{m}}$

Find: $\theta(t), x(t), y(t)$

Solution: The kinetic energy at an arbitrary instant is

$$T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_1 \dot{y}^2 \qquad (a)$$

The potential energy at an arbitrary instant is

$$V = \frac{1}{2} k_1 \left(x - \frac{L}{2}\theta \right)^2 + \frac{1}{2} 2k_1 x^2 + \frac{1}{2} k_2 \left(y - x - \frac{L}{2}\theta \right)^2 \qquad (b)$$

Rayleigh's dissipation function for the system is

$$\Im = -\frac{1}{2} c \left(\dot{x} - \frac{L}{2}\dot{\theta} \right)^2 + \frac{1}{2} 2c\dot{x}^2 \qquad (c)$$

The virtual work done by the external moment is

$$\delta W = M_0 \sin \omega t \ \ \delta\theta \qquad (d)$$

Lagrange's equations are applied to yield

$$\begin{bmatrix} I & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} c\frac{L^2}{2} & -c\frac{L}{2} & 0 \\ -c\frac{L}{2} & 3c & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} (k_1 + k_2)\frac{L^2}{2} & (k_2 - k_1)\frac{L}{2} & -k_2\frac{L}{2} \\ (k_2 - k_1)\frac{L}{2} & 3k_1 + k_2 & -k_2 \\ -k_2\frac{L}{2} & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} \theta \\ x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} M_0 \sin \omega t \\ 0 \\ 0 \end{bmatrix} \qquad (e)$$

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Substituting given values into Eq. (d) leads to

$$\begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + 10^4 \begin{bmatrix} 0.16 & -0.4 & 0 \\ -0.4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} + 10^6 \begin{bmatrix} 0.224 & -0.04 & -0.12 \\ -0.12 & 1.5 & -0.3 \\ -0.12 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} \theta \\ x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} 20 \sin 45t \\ 0 \\ 0 \end{bmatrix} \qquad (f)$$

A procedure similar to that of Problem 9.23 is followed.

Problem 9.24 illustrates the application of modal analysis to a damped system.

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CHAPTER 10: VIBRATIONS OF CONTINUOUS SYSTEMS

Short Answer Problems

10.1 True: Another name for a continuous system is a distributed parameter system.

10.2 True: The vibrations of a continuous system solve a partial differential equation. When the normal mode solution is used a differential eigenvalue problem is obtained which has an infinite, but countable, number of solutions.

10.3 False: The longitudinal vibrations of a bar are governed by the wave equations while the transverse vibrations of a beam are governed by a partial differential equation which is fourth-order in space and second-order in time (sometimes called the beam equation).

10.4 True: A free-free beam has a repeated natural frequency of zero.

10.5 False: Rayleigh's quotient defined for a system is stationary only for a mode shape of the system.

10.6 False: Four boundary conditions are necessary to determine the forced vibration response of a fixed-free beam.

10.7 True: The Rayleigh-Ritz method is an energy method that is used to approximate the free and forced responses of a continuous system by a finite expansion.

10.8 True: Mode shapes are orthogonal with respect to a potential energy scalar product and with respect to a kinetic energy scalar product.

10.9 True: A rigid-body mode is a mode shape corresponding to a natural frequency of zero. A pinned-free beam is an unrestrained system with its lowest natural frequency of zero.

10.10 True: This equation is used to approximate the bending moment in a beam.

10.11 The method of separation of variables is applied to solve for the free vibration response in a bar. The same method can be used for free vibrations of a beam.

10.12 The wave equation is second-order in the spatial coordinate. The beam equation is fourth-order in the spatial coordinate.

10.13 The process of introducing the independent variables $t^*$and x^* and the dependent variable w^* is called non-dimensionalization.

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10.14 (a) Four boundary conditions are required to determine the response of a beam undergoing transverse vibrations (b) Two boundary conditions are required to determine the response of a bar undergoing longitudinal vibrations. (c) Two boundary conditions are required to determine the response of a shaft undergoing torsional oscillations.

10.15 The boundary condition $\frac{\partial\theta}{\partial x}(L,t)$ means that the end at x = L is free, that is it is not subject to any torques or is not constrained. It is a representation that the shear stress at the end is equal to zero.

10.16 The boundary conditions are $\theta(0,t) = 0$ and $\frac{\partial\theta}{\partial x}(L,t) = 0$.

10.17 The boundary conditions are $\theta(0,t) = 0$ and $\frac{\partial\theta}{\partial x}(L,t) = -\frac{I}{\rho JL}\frac{\partial^2\theta}{\partial t^2}(L,t)$.

10.18 The boundary conditions are $\theta(0,t) = 0$ and $\frac{\partial\theta}{\partial x}(L,t) = -\frac{I}{\rho JL}\frac{\partial^2\theta}{\partial t^2}(L,t) - \frac{k_t L}{JG}\theta(L,t)$.

10.19 The boundary conditions are $w(0,t) = 0$ and $\frac{\partial w}{\partial x}(L,t) = -\frac{kL}{T}w(L,t)$.

10.20 The nondimensional natural frequency is $\omega^* = \sqrt{\frac{\rho L}{G}}\omega$ where ω is the dimensional natural frequency.

10.21 The normalization condition is $\int_0^L \rho A X_k^2(x)dx + mX_k^2(1) = 1$.

10.22 The normalization condition is $\int_0^L \rho A X_k^2(x)dx = 1$.

10.23 The term $EI\frac{\partial^4 w}{\partial x^4}$ represents the resultant normal stresses due to bending in the beam. The term $\rho A\frac{\partial^2 w}{\partial t^2}$ represents the inertia forces in the beam. The term $f(x,t)$ represents the external force/length to which the beam is subjected.

10.24 The characteristic equation for a fixed-free beam is $\cos\lambda^{1/4}\cosh\lambda^{1/4} = -1$ is an example of a <u>transcendental</u> equation to solve for λ.

10.25 The boundary conditions for the free vibrations of a fixed-free beam are $w(0,t) = 0, \frac{\partial w}{\partial x}(0,t) = 0, \frac{\partial^2 w}{\partial x^2}(L,t) = 0$ and $\frac{\partial^3 w}{\partial x^3}(L,t) = 0$.

10.26 The boundary conditions for a free-free beam are $\frac{\partial^2 w}{\partial x^2}(0,t) = 0$, $\frac{\partial^3 w}{\partial x^3}(0,t) = 0$, $\frac{\partial^2 w}{\partial x^2}(L,t) = 0$ and $\frac{\partial^3 w}{\partial x^3}(L,t) = 0$.

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10.27 The boundary conditions for a beam fixed at x = 0 and has a rigid mass attached at its free end are $w(0,t) = 0, \frac{\partial w}{\partial x}(0,t) = 0, \frac{\partial^2 w}{\partial x^2}(L,t) = 0$ and $\frac{\partial^3 w}{\partial x^3}(L,t) = \beta \frac{\partial^2 w}{\partial t^2}(L,t)$.

10.28 A pined-free beam is unrestrained thus admitting a natural frequency of zero. A fixed-pinned beam is restrained. When $\omega = 0$ is used in the mode shape of a fixed-pinned beam the resulting mode shape is the trivial response. This is not the case with the fixed-pinned beam.

10.29 (a) The potential energy of the bar at an instant of time is $\frac{1}{2}\int_0^L EA\left(X_k'\right)^2 dx$. (b) The kinetic energy of the bar is $\frac{\omega_k^2}{2}\int_0^L \rho A X_k^2\, dx$.

10.30 Rayleigh's quotient is $R(w) = \frac{\int_0^L EA(w')^2 dx}{\int_0^L \rho A w^2 dx}$

10.31 (a) The potential energy of this mode is is $\frac{1}{2}\int_0^L EI\left(X_k''\right)^2 dx$ (b) The kinetic energy of the mode is $\frac{\omega_k^2}{2}\int_0^L \rho A X_k^2\, dx$.

10.32 Rayleigh's quotient is $R(w) = \frac{\int_0^L EI(w'')^2 dx}{\int_0^L \rho A w^2 dx}$.

10.33 The wave speed is $c = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{80\times10^9 \frac{\mathrm{N}}{\mathrm{m}^2}}{7580 \frac{\mathrm{kg}}{\mathrm{m}^3}}} = 3.25 \times 10^3\ \frac{\mathrm{m}}{\mathrm{s}}$.

10.34 The wave speed is $c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{210\times10^9 \frac{\mathrm{N}}{\mathrm{m}^2}}{7580 \frac{\mathrm{kg}}{\mathrm{m}^3}}} = 5.26 \times 10^3\ \frac{\mathrm{m}}{\mathrm{s}}$

10.35 The relationship between the non-dimensional frequencies and the dimensional frequencies is $\omega_d = \omega_{nd}\sqrt{\frac{G}{\rho L^2}} = \omega_{nd}\sqrt{\frac{80\times10^9 \frac{\mathrm{N}}{\mathrm{m}^2}}{\left(7500 \frac{\mathrm{kg}}{\mathrm{m}^3}\right)(1.5)^2}} = 2.18 \times 10^3 \omega_{nd}$. The three lowest non-dimensional frequencies of a fixed-free bar are $\omega_{nd} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$. The three lowest dimensional natural frequencies for this shaft are $\omega_d = 3.43 \times 10^3\ \frac{\mathrm{rad}}{\mathrm{s}}$, $1.03 \times 10^4\ \frac{\mathrm{rad}}{\mathrm{s}}, 1.71 \times 10^4\ \frac{\mathrm{rad}}{\mathrm{s}}$.

10.36 The smallest solution of the equation is $\lambda = \left(\frac{\pi}{2}\right)^2$. The natural frequencies are the square roots of the eigenvalues, thus $\omega_1 = \frac{\pi}{2}$.

10.37 The three lowest values are $\lambda = 1.35, 4.112, 6.992$

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10.38 The three lowest values are $\lambda = 1.393, 4.659, 7.882$

10.39 For $u(x) = \sin 5\pi x$, the natural frequency is 5π. Thus $u(x,t) = \sin 5\pi x\ e^{i5\pi t}$. (a) The potential energy of the mode is $V = \frac{1}{2}\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{1}{2}\int_0^1 \left[5\pi \cos 5\pi x\ e^{i5\pi t}\right]^2 dx = \frac{25\pi^2}{4} e^{i10\pi t}$ (b) The kinetic energy is $T = \frac{1}{2}\int_0^1 \left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{1}{2}\int_0^1 \left[i5\pi \sin 5\pi x\ e^{i5\pi t}\right]^2 = -\frac{25\pi^2}{4} e^{i10\pi t}$. (c) The natural frequency associated with the mode is $\sqrt{\left|\frac{V}{T}\right|} = 1$

10.40 Given: $w(x) = \sqrt{2}\sin(3\pi x)$. (a) The potential energy of the mode is $V = \frac{1}{2}\int_0^1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = \frac{1}{2}\int_0^1 \left(-9\pi^2\sqrt{2}\sin 3\pi x\, e^{i9\pi^2 t}\right)^2 dx = 81\pi^4 e^{i18\pi^2 t}\int_0^1 (\sin 3\pi x)^2\, dx = \frac{81\pi^4}{2} e^{i18\pi^2 t}$ (b) The kinetic energy of the mode is
$T = \frac{1}{2}\int_0^1 \left(\frac{\partial w}{\partial t}\right)^2 dx = T = \frac{1}{2}\int_0^1 \left(\sqrt{2}\sin 3\pi x\, 9\pi^2 e^{i9\pi^2 t}\right)^2 dx = \frac{81\pi^2 e^{i18\pi^2 t}}{2}$ (c) The frequency is $9\pi^2$

10.41 Given: $L = 0.8\text{ m}, r = 0.03\text{ m}, E = 200\times 10^9\ \frac{\text{N}}{\text{m}^2}, \rho = 7600\ \frac{\text{kg}}{\text{m}^3}, \text{X(x)} = 33.91\cos 13.74\text{x}$. (a) The potential energy of the mode is $V = \frac{1}{2}\int_0^L EI\left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{1}{2}\int_0^{0.8}(200\times 10^9)\left[\frac{\pi}{4}(0.03)^4\right]\left[(33.91)(-13.74)\sin 13.74x\, e^{i13.74t}\right]^2 dx = 6.16\times 10^{12}\ e^{i27.58t}$

(b) The kinetic energy of the mode is
$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx =$
$\frac{1}{2}\int_0^{0.8}(7600)[\pi(0.03)^2]\left[33.91\cos 13.74x\,(i13.74)e^{i13.74t}\right]^2 dx = 1.87\times 10^6\ e^{i27.58t}$

(c) The natural frequency that corresponds to the mode is 13.74 rad/s.

10.42 Given: Carbon nanotube modeled as fixed-free beam $E = 1\text{ GPa}, \rho = 2.3\ \frac{\text{g}}{\text{cm}^3}$, L = 200 nm, r = 5 nm. The radius of a carbon atom is 0.34 nm. Thus the tube has an inner diameter of 4.66 nm and an outer diameter of 5.34 nm. Thus $A = \pi[(5.34\text{ nm})^2 - 4.66\text{ nm}2 = 21.36\text{ nm}2$ and $I = \pi 4 5.34\text{ nm}4 - 4.66\text{ nm}4 = 268.3\text{ nm}4$. The parameter $\sqrt{\frac{EI}{\rho AL^4}} = \sqrt{\frac{(1\text{ GPa})(268.3\text{ nm}^4)}{\left(2.3\ \frac{\text{g}}{\text{cm}^3}\right)(21.36\text{ nm}^2)(200\text{ nm})^4}} = 1.848\times 10^9\ \frac{\text{rad}}{\text{s}}$. The five lowest natural nondimensional frequencies of a fixed-fixed beam are 22.37, 61.66, 120.9, 199.9 and 298.6. Multiplying these by $1.848\times 10^9\ \frac{\text{rad}}{\text{s}}$ leads to $\omega_1 = 4.19\times 10^{10}\ \frac{\text{rad}}{\text{s}}$, $\omega_2 = 1.39\times 10^{11}\ \frac{\text{rad}}{\text{s}}$, $\omega_3 = 2.29\times 10^{11}\ \frac{\text{rad}}{\text{s}}$, $\omega_4 = 3.69\times 10^{11}\ \frac{\text{rad}}{\text{s}}$ and $\omega_5 = 5.52\times 10^{11}\ \frac{\text{rad}}{\text{s}}$.

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10.43 Given: $E = 210 \times 10^9 \frac{\text{N}}{\text{m}^2}, \rho = 7580 \frac{\text{kg}}{\text{m}^3}$ with $A = 1.2 \times 10^{-2}\,\text{m}^2, I = 4.0 \times 10^{-5}\,\text{m}^4$ and $L = 1.4$ m. The parameter $\sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{\left(210\times10^9 \frac{\text{N}}{\text{m}^2}\right)(4.0\times10^{-5}\,\text{m}^4)}{\left(7580 \frac{\text{kg}}{\text{m}^3}\right)(1.2\times10^{-2}\,\text{m}^2)(1.4\,\text{m})^4}} = 152.4$ s is multiplied by the nondimensional frequencies to obtain the dimensional natural frequencies. For the beam with an end spring the nondimensional parameter is $\beta = \frac{kL^3}{EI} = \frac{\left(3.5\times10^5 \frac{\text{N}}{\text{m}}\right)(1.4\ \text{m})^3}{\left(210\times10^9 \frac{\text{N}}{\text{m}^2}\right)(4.0\times10^{-5})} = 0.1143$. The three lowest solutions of the characteristic equation are $\omega_1 = 12.819, \omega_2 = 485.9, \omega_3 = 1307$. The dimensional frequencies are $\omega_1 = 1.953 \times 10^3 \frac{\text{rad}}{\text{s}}, \omega_2 = 7.404 \times 10^4 \frac{\text{rad}}{\text{s}}, \omega_3 = 1.99 \times 10^5 \frac{\text{rad}}{\text{s}}$

FIGURE SP10.43

10.44 Given: $E = 210 \times 10^9 \frac{\text{N}}{\text{m}^2}, \rho = 7580 \frac{\text{kg}}{\text{m}^3}$ with $A = 1.2 \times 10^{-2}\,\text{m}^2, I = 4.0 \times 10^{-5}\,\text{m}^4$ and $L = 1.4$ m. The parameter $\sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{\left(210\times10^9 \frac{\text{N}}{\text{m}^2}\right)(4.0\times10^{-5}\,\text{m}^4)}{\left(7580 \frac{\text{kg}}{\text{m}^3}\right)(1.2\times10^{-2}\,\text{m}^2)(1.4\,\text{m})^4}} = 152.4$ s is multiplied by the nondimensional frequencies to obtain the dimensional natural frequencies. For a pinned-pinned beam the three lowest non-dimensional natural frequencies are $\omega_1 = 9.87, \omega_2 = 38.48, \omega_3 = 88.83$. The dimensional frequencies are $\omega_1 = 1.507 \times 10^3 \frac{\text{rad}}{\text{s}}, \omega_2 = 4.334 \times 10^3 \frac{\text{rad}}{\text{s}}, \omega_3 = 1.353 \times 10^4 \frac{\text{rad}}{\text{s}}$

FIGURE SP10.44

10.45 Given: $E = 210 \times 10^9 \frac{\text{N}}{\text{m}^2}, \rho = 7580 \frac{\text{kg}}{\text{m}^3}$ with $A = 1.2 \times 10^{-2}\,\text{m}^2, I = 4.0 \times 10^{-5}\,\text{m}^4$ and $L = 1.4$ m. The parameter $\sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{\left(210\times10^9 \frac{\text{N}}{\text{m}^2}\right)(4.0\times10^{-5}\,\text{m}^4)}{\left(7580 \frac{\text{kg}}{\text{m}^3}\right)(1.2\times10^{-2}\,\text{m}^2)(1.4\,\text{m})^4}} = 152.4$ s is multiplied by the nondimensional frequencies to obtain the dimensional natural frequencies. For a pinned-free beam the three lowest non-dimensional natural frequencies are $\omega_1 = 0\ \omega_2 = 15.42, \omega_3 = 49.46$. The dimensional frequencies are $\omega_1 = 0 \frac{\text{rad}}{\text{s}}, \omega_2 = 2.349 \times 10^3 \frac{\text{rad}}{\text{s}}$, $\omega_3 = 7.535 \times 10^4 \frac{\text{rad}}{\text{s}}$

FIGURE SP10.45

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10.46 Given: $\frac{d^2X}{dx^2} + \lambda X = 0$ $X'(0) = 0$ $X'(1) = 0$. The solution to the differential equation is $X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$. Applying $X'(0) = 0 \Longrightarrow B = 0$. Then $X'(1) = 0 \Longrightarrow -A\sqrt{\lambda}\sin\sqrt{\lambda} = 0$. The solutions are $\lambda = (n\pi)^2$ for n = 0, 1, 2,….For n = 0 the solution is $X_0(x) = A$. Otherwise $X_n(x) = A\cos n\pi x$.

10.47 Given: $\frac{d^4X}{dx^4} - \lambda X = 0$, $x(0) = 0, X''^{(0)} = 0$, $X(1) = 0$, $X''(1) = 0$. This is the boundary value problem for a pinned-pinned beam. The eigenvalues are solutions of $\sin\lambda^{1/4} = 0$ which are $\lambda_n = (n\pi)^4$ for n = 1, 2, 3,…. The corresponding solution is $X_n(x) = C_n \sin n\pi x$.

10.48 (a) m/s (b) $\mathrm{N \cdot m^2}$ (c) rad/s (d) none (e) $\mathrm{rad^2/s^2}$ (f) N (g) $\mathrm{N \cdot m}$

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Chapter Problems

10.1 A 5000 N · m torque is statically applied to the free end of a solid 20-cm radius steel shaft ($G = 80 \times 10^9$ N/m^2, $\rho = 7500$ kg/m^3) with a length of 1.5 m that is fixed at one end and free at its other end. The torque is suddenly removed and torsional oscillations begin. Plot the time-dependent oscillations of the free end of the shaft.

Given: T = 5000 N · m, r = 20 cm, L = 1.5 m, ρ = 7500 kg/m^2, G = 80 × 10^9 N/m^2

Find: θ(L,t)

Solution: The mathematical problem governing the response of the system, in nondimensional variables, is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \tag{1}$$

subject to

$$\theta(0,t) = 0,\ \frac{\partial\theta}{\partial x}(1,t) = 0 \tag{2}$$

and

$$\theta(x,0) = \phi x \tag{3}$$

$$\frac{\partial\theta}{\partial t}(x,0) = 0 \tag{4}$$

where

$$\phi = \frac{TL}{JG} = \frac{(5000\ N-m)(1.5\ m)}{\frac{\pi}{2}(0.2\ m)^4\left(80\times10^9\ \frac{N}{m^2}\right)} = 3.73\times10^{-5}$$

Equation (3) is derived from a static analysis of the shaft's initial position. Equation (4) is a result of the shaft released from rest.

The natural frequencies and mode shapes for the fixed-free shaft are considered in Example 10.1. The natural frequencies are

$$\omega_n = (2k-1)\frac{\pi}{2}, \qquad k = 1,2,\ldots$$

The normalized mode shapes are

$$X_k(x) = \sqrt{2}sin(2k-1)\frac{\pi}{2}x$$

A product solution of the form

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$$\theta(x,t) = X(x)T(t)$$

where

$$T(t) = A_k cos\omega_k t + B_k sin\omega_k t \tag{5}$$

is assumed. Application of eq. (4) leads to

$$\frac{dT}{dt}(0) = 0$$

which in turns leads to

$$B_k = 0$$

Hence

$$\theta(x,t) = A_k\sqrt{2}sin(2k-1)\frac{\pi}{2}xcos(2k-1)\frac{\pi}{2}t \tag{6}$$

where the A_k remain arbitrary. Since equation (6) satisfies only homogeneous conditions, a general solution is a linear combination over all possible solutions

$$\theta(x,t) = \sum_{k=1}^{\infty} A_k\sqrt{2}sin(2k-1)\frac{\pi}{2}xcos(2k-1)\frac{\pi}{2}t$$

Application of eq.(3) leads to

$$\frac{8\phi}{\pi^2} = 3.02 \times 10^{-5}$$

A plot of the time dependent response follows.

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Problem 10.1 illustrates the free vibration response of a torsional shaft subject to an initial torque.

10.2 A 5000 N · m torque is statically applied to the midspan of a solid 20-cm radius steel shaft ($G = 80 \times 10^9$ N/m^2, $\rho = 7500$ kg/m^3) with a length of 1.5 m that is fixed at one end and free at its other end. The torque is suddenly removed and torsional oscillations begin. Determine an expression for the time-dependent angular displacement of the free end of the shaft.

Given: T = 5000 N · m, r = 20 cm, L = 1.5 m, ρ = 7500 kg/m^2, G = 80 × 10^9 N/m^2

Find: θ(L,t)

Solution: The mathematical problem governing the response of the system, in nondimensional variables, is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \tag{1}$$

subject to

$$\theta(0,t) = 0,\ \frac{\partial\theta}{\partial x}(1,t) = 0 \tag{2}$$

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and
$$\theta(x,0) = \begin{cases} \phi x & 0 \le x \le \frac{1}{2} \\ \frac{\phi}{2} & \frac{1}{2} \le x \le 1 \end{cases} \tag{3}$$

$$\frac{\partial \theta}{\partial t}(x,0) = 0 \tag{4}$$

where

$$\phi = \frac{TL}{JG} = \frac{(5000\ \mathrm{N \cdot m})(1.5\ \mathrm{m})}{\frac{\pi}{2}(0.2\ \mathrm{m})^4 \left(80 \times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}\right)} = 3.73 \times 10^{-5}$$

Equation (3) is derived from a static analysis of the shaft's initial position. Equation (4) is a result of the shaft released from rest.

The natural frequencies and mode shapes for the fixed-free shaft are considered in Example 10.1. The natural frequencies are

$$\omega_n = (2k-1)\frac{\pi}{2}, \qquad k = 1,2,\ldots$$

The normalized mode shapes are

$$X_k(x) = \sqrt{2} sin(2k-1)\frac{\pi}{2}x$$

A product solution of the form

$$\theta(x,t) = X(x)T(t)$$

where

$$T(t) = A_k cos\omega_k t + B_k sin\omega_k t \tag{5}$$

is assumed. Application of eq. (4) leads to

$$\frac{dT}{dt}(0) = 0$$

which in turns leads to

$$B_k = 0$$

Hence

$$\theta(x,t) = A_k\sqrt{2} sin(2k-1)\tfrac{\pi}{2}x cos(2k-1)\tfrac{\pi}{2}t \tag{6}$$

where the A_k remain arbitrary. Since equation (6) satisfies only homogeneous conditions, a general solution is a linear combination over all possible solutions

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$$\theta(x,t) = \sum_{k=1}^{\infty} A_k \sqrt{2} sin(2k-1)\frac{\pi}{2} x cos(2k-1)\frac{\pi}{2} t$$

Application of eq.(3) leads to

$$\theta(x,0) = \sum_{k=1}^{\infty} A_k \sqrt{2} \sin\,(2k-1)\frac{\pi}{2} \qquad (7)$$

Equation (7) is satisfied if

$$A_k = (\theta(x,0), X_k(x))$$

$$= \int_0^1 \sqrt{2}\theta(x,0) \sin\,(2k-1)\frac{\pi}{2} dx$$

$$= \sqrt{2}\phi \left[\int_0^{\frac{1}{2}} x sin(2k-1)\frac{\pi}{2} x dx + \int_{\frac{1}{2}}^1 \sin\,(2k-1)\frac{\pi}{2} x dx \right]$$

$$= \frac{\sqrt{2}\phi}{\left[(2k-1)\frac{\pi}{2}\right]^2} \sin\,(2k-1)\frac{\pi}{4}$$

Note that

$$\sin(2k-1)\frac{\pi}{4} = \begin{cases} \frac{\sqrt{2}}{2} & k = 1,2,5,6,8,19,\ldots \\ -\frac{\sqrt{2}}{2} & k = 3,4,7,8,11,12,\ldots \end{cases}$$

Hence

$$\theta(1,t) = 2\phi \sum_{k=1}^{\infty} \frac{1}{\left[(2k-1)\frac{\pi}{2}\right]^2} \sin\,(2k-1)\frac{\pi}{2} \sin\,(2k-1)\frac{\pi}{4} \cos\,(2k-1)\frac{\pi}{2} t$$

$$= 2\phi \left[\sum_{k=1,2,5,6,9,10}^{\infty} \frac{1}{\left[(2k-1)\frac{\pi}{2}\right]^2} \cos\,(2k-1)\frac{\pi}{2} t \right.$$

$$\left. - \sum_{k=3,4,7,8,11,12}^{\infty} \frac{1}{\left[(2k-1)\frac{\pi}{2}\right]^2} \cos\,(2k-1)\frac{\pi}{2} t \right]$$

Problem 10.2 illustrates application of initial conditions to determine the time dependent response of a torsional distributed parameter system.

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10.3 A steel shaft (ρ = 7850 kg/m^3, G = 85 × 10^9 N/m^2) with an inner radius of 30 mm, outer radius of 50 mm, and length of 1.0 m is fixed at both ends. Determine the three lowest natural frequencies of the shaft.

Given: fixed-fixed shaft, L = 1 m, r_i = 30 mm, r_o = 50 mm, ρ = 7850 kg/m^3, G = 85 × 10^9 N/m^2

Find: ω_1, ω_2, ω_3

Solution: The nondimensional natural frequencies for a fixed-fixed shaft are calculated in a manner similar to that for the fixed-free shaft of Example 10.1. The mode shape is

$$X(x) = C_1 cos\sqrt{\lambda}x + C_2 sin\sqrt{\lambda}x$$

Where C_1 and C_2 are integration constants and λ is the separation constant. The square roots of the separation constants are the natural frequencies. The boundary conditions for a fixed-fixed shaft are

$$\theta(0,t) = 0, \quad \theta(1,t) = 0$$

which lead to the following boundary conditions for the mode shape

$$X(0) = 0, \quad X(1) = 0$$

Application of the boundary condition at x-0 leads to $C_1 = 0$. Then

$$X(x) = C_2 sin\sqrt{\lambda}x$$

Application of the boundary condition at x = 1 leads to

$$C_2 sin\sqrt{\lambda} = 0$$

Choosing $C_2 = 0$ leads to the trivial solution. Thus in order to attain a non-trivial solution

$$sin\sqrt{\lambda} = 0, \quad \lambda = k^2\pi^2, \quad k = 1,2,\ldots$$

Hence the first three nondimensional natural frequencies are

$$\omega_1^* = \pi, \quad \omega_2^* = 2\pi, \quad \omega_3^* = 3\pi$$

From Eq.(10.13) it is obvious that the dimensional natural frequencies are

$$\omega_k = \frac{\omega_k^*}{L}\sqrt{\frac{G}{\rho}}$$

For the problem at hand

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$$\frac{1}{L}\sqrt{\frac{G}{\rho}} = \frac{1}{1\,m}\sqrt{\frac{85\times 10^9\,\frac{N}{m^2}}{7850\,\frac{kg}{m^3}}} = 3290\,\frac{\text{rad}}{\text{sec}}$$

Thus

$$\omega_k = 3290k\pi$$

Evaluation of the preceding equation for k = 1, 2, 3 leads to

$$\omega_1 = 10340\,\frac{\text{rad}}{\text{sec}},\quad \omega_2 = 20680\,\frac{\text{rad}}{\text{sec}},\quad \omega_3 = 31070\,\frac{\text{rad}}{\text{sec}}$$

Problem 10.3 illustrates application of the results of Example 10.1 to determine the natural frequencies of a fixed-free torsional shaft.

10.4 A 10,000-N · m torque is applied to the midspan of the shaft of Chapter Problem 10.3 and suddenly removed. Determine the time-dependent angular displacement of the midspan of the shaft.

Given: ρ =7850 kg/m^3, G = 85 × 10^9 N/m^2 , r_i = 30 mm, r_o = 50 mm, L = 1 m, T = 10,000 N · m

Find: θ(1/2,t)

Solution: The partial differential equation governing the angular oscillations of the shaft, written in nondimensional variables, is

$$\frac{\partial^2\theta}{\partial^2 x} = \frac{\partial^2\theta}{\partial^2 t} \tag{1}$$

The boundary conditions are

$$\theta(0,t) = 0,\quad \theta(1,t) = 0 \tag{2}$$

The initial displacement is determined from a static analysis of the initial position, leading to

$$\theta(x,0) = \begin{cases} \phi x, & 0 < x < \frac{1}{2} \\ \phi(1-x), & \frac{1}{2} < x < 1 \end{cases} \tag{3}$$

where

$$\phi = \frac{TL}{2JG} = \frac{(10000\,N-m)(1.0\,m)}{2\frac{\pi}{2}[(0.05\,m)^4-(0.03\,m)^4]\left(85\times 10^9\,\frac{N}{m^2}\right)} = 6.88\times 10^{-3}$$

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The initial velocity of the shaft is zero,

$$\frac{\partial \theta}{\partial t}(x,0) = 0 \tag{4}$$

A separation of variable solution of equation (1) is assumed as

$$\theta(x,t) = X(x)T(t)$$

The separation of variables solution is substituted into eq.(1). The usual separation argument is used resulting in ordinary differential equations for X(x) and T(t). The square roots of the separation constants are natural frequencies and the solutions for X(x) are the mode shapes. These are obtained in the solution of Problem 10.3 as

$$\omega_k = k\pi, \quad k = 1,2,\ldots, \quad X_k(x) = \sqrt{2}\sin k\pi x$$

The resulting solution for T(t) is

$$T_k(T) = A_k \cos k\pi t + B_k \sin k\pi t$$

(5)

Application of eq.(4) leads to $B_k = 0$. Then

$$\theta(x,t) = \sqrt{2}A_k \sin k\pi x \cos k\pi t$$

The most general solution is a linear combination over all possible solutions. Thus

$$\theta(x,t) = \sum_{k=1}^{\infty} B_k X_k(x)$$

Hence

$$B_k = \big(\theta(x,0), X_k(x)\big)$$

$$= \int_0^1 \theta(x,0)\sqrt{2}\sin k\pi x \; dx$$

$$= \sqrt{2}\phi\left[\int_0^{\frac{1}{2}} x \; sink\pi x \, dx + \int_{\frac{1}{2}}^{2} (1-x)\sin k\pi x \; dx\right]$$

$$= \sqrt{2}\phi \frac{2}{(k\pi)^2}\sin k\frac{\pi}{2}$$

Note that

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$$\sin k\frac{\pi}{2} = \begin{cases} 1, & k = 1,5,9,13,\ldots \\ 0, & k = 2,4,6,8,\ldots \\ -1, & k = 3,7,11,15,\ldots \end{cases}$$

Hence

$$\theta\left(\frac{1}{2}, t\right) = 4\frac{\phi}{\pi^2}\sum_{k=1,3,5}^{\infty}\frac{1}{k^2}\cos k\pi t$$

Problem 10.4 illustrates application of initial conditions to determine the free response of a torsional system.

10.5 A motor of mass moment of inertia 85 kg · m^2 is attached to the end of the shaft of Chapter Problem 10.1. Determine the three lowest frequencies of the shaft and motor assembly. Compare the lowest natural frequency to that obtained by making a one-degree-of-freedom model and approximating the inertia effects of the shaft.

Given: I = 85 kg · m^2, r = 20 cm, G = 80 × 10^9 N/m^2, ρ = 7500 kg/m^3, L = 1.5 m

Find: ω_1, ω_2, ω_3

Solution: The nondimensional partial differential equation governing the free torsional oscillations of the shaft is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \tag{1}$$

Since the shaft is fixed at x = 0

$$\theta(0,t) = 0 \tag{2}$$

From Table 10.1, the appropriate boundary condition at x = 1 is

$$\frac{\partial\theta}{\partial x} = -\beta\frac{\partial^2\theta}{\partial t^2} \tag{3}$$

where

$$\beta = \frac{I}{\rho JL} = \frac{85\ kg\text{-}m^2}{\left(7500\frac{kg}{m^3}\right)\frac{\pi}{2}(0.2\ m)^4(1.5\ m)} = 3.00 \tag{4}$$

A product solution is assumed of the form

$$\theta(x,t) = X(x)T(t) \tag{5}$$

Substituting eq.(5) into eq.(1) and using the usual separation argument leads to

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$$\frac{d^T}{dt^2} + \lambda T = 0 \tag{6}$$

where λ is the separation constant. Using eq.(5) in eq.(2) leads to

$$X(0) = 0 \tag{7}$$

Using eq.(5) in eq.(3) leads to

$$\frac{dX}{dx}(1)T(t) = -\beta X(1)\frac{d^2T}{dt^2} \tag{8}$$

Using eq.(6) in eq.(8) leads to

$$\frac{DX}{Dx}(1) = \beta\lambda X(1) \tag{9}$$

The solution for X(x) using eq.(6) is

$$X(x) = C_1 cos\sqrt{\lambda}x + C_2 sin\sqrt{\lambda}x \tag{10}$$

Application of eq.(7) to eq.(10) leads to C_1=0. Then application of eq.(9) to eq.(10) leads to

$$\sqrt{\lambda}cos\sqrt{\lambda} = \beta\lambda sin\sqrt{\lambda}$$

$$\beta tan\sqrt{\lambda} = \frac{1}{\sqrt{\lambda}} \tag{11}$$

Equation (11) is a transcendental equation which must be solved by trial and error. For β = 3.00, its lowest three roots are

$$\sqrt{\lambda} = 0.547, \qquad 3.244, \qquad 6.335$$

The square roots of the separation constants are the nondimensional natural frequencies. The dimensional natural frequencies are obtained from the nondimensional frequencies by

$$\omega = \omega^* \frac{1}{L}\sqrt{\frac{G}{\rho}}$$

For the problem at hand

$$\frac{1}{L}\sqrt{\frac{G}{\rho}} = \frac{1}{1.5\text{ m}}\sqrt{\frac{80 \times 10^9\ \frac{\text{N}}{\text{m}^2}}{7500\ \frac{\text{kg}}{\text{m}^3}}} = 2178\ \frac{\text{rad}}{\text{sec}}$$

Hence the dimensional natural frequencies are

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$$\omega_1 = 0.547(2178) = 1190\ \frac{\text{rad}}{\text{sec}}$$

$$\omega_2 = 3.244(2178) = 7065\ \frac{\text{rad}}{\text{sec}}$$

$$\omega_3 = 6.335(2178) = 13800\ \frac{\text{rad}}{\text{sec}}$$

When the system is modeled using one-degree-of-freedom it is modeled as a thin disk of mass moment of inertia I_{eq} attached to a torsional spring of stiffness k_{eq}. The inertia effects of the shaft are approximated by adding 1/3 of the mass moment of inertia of the shaft to the moment of inertia of the disk

$$I_{eq} = I + \frac{1}{3}\rho JL = 85\ \text{kg}\cdot\text{m}^2 + \frac{1}{3}\left(7500\ \frac{\text{kg}}{\text{m}^3}\right)\frac{\pi}{2}(0.2\ \text{m})^4(1.5\ \text{m}) = 94.42\ \text{kg}\cdot\text{m}^2$$

The equivalent stiffness of the shaft is

$$k = \frac{JG}{L} = \frac{\frac{\pi}{2}(0.2\ \text{m})^4\left(80\times10^9\ \frac{\text{N}}{\text{m}^2}\right)}{1.5\ \text{m}} = 1.33\times10^7\ \frac{\text{N}}{\text{m}}$$

The predicted natural frequency of the system is

$$\omega_n = \sqrt{\frac{K_{eq}}{I_{eq}}} = \sqrt{\frac{1.33\times10^7\ \frac{\text{N}}{\text{m}}}{94.42\ \text{kg}\cdot\text{m}^2}} = 375\ \frac{\text{rad}}{\text{sec}}$$

which is in significant error with the exact value of the lowest natural frequency.

Problem 10.5 illustrates (a) calculation of the natural frequencies of a shaft with an attached disk and (b) the poor comparison between the exact solution and a one-degree-of-freedom approximation.

10.6 Show the orthogonality of the two lowest mode shapes of the system in Chapter Problem 10.5.

Given: From solution of Problem 10.5, β = 3.0 and $\omega_1^* = 0.547, \omega_2^* = 3.244$

$$X_k(x) = A_k sin\omega_k^* x$$

Show: orthogonality

Solution: It is shown in Section 10.3 that the appropriate scalar product for a shaft with an attached disk is

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$$\left(X_j(x), X_k(x)\right) = \int_0^1 X_j(x)X_k(x)dx + \beta X_j(1)X_k(1) \qquad (1)$$

The first two mode shapes are

$$X_1(x) = A_1 sin0.547x, \quad X_2(x) = A_2 sin0.547x$$

For the system at hand, β = 3.00. Calculating the scalar product of the first two mode shapes

$$(X_1, X_2) = \int_0^1 X_1(x)X_2(x)\,dx + 3.00X_1(1)X_2(1)$$

$$= \int_0^1 A_1A_2 sin0.547x sin3.244x\,dx + 3.00 sin0.547 sin3.244$$

$$= \frac{sin(0.547 - 3.244)}{2(0.547 - 3.244)} - \frac{sin(0.547 + 3.244)}{2(0.547 + 3.244)} + 3.00 sin0.547 sin3.244$$

$$= 0.079722 + 0.0796 - 0.15947 \approx 0$$

Hence the first two mode shape are orthogonal with respect to the scalar product of eq.(1).

Problem 10.6 illustrates orthogonality of mode shapes for a torsional shaft with an attached disk.

10.7 Operation of the motor attached to the shaft of Chapter Problem 10.5 produces a harmonic torque of amplitude 2000 N · m at a frequency of 110 Hz. Determine the steady-state angular displacement of the end of the shaft.

Given: T = 2000 N · m, ω = 110 Hz, I = 85 kg ·m^2, r = 20 cm, ρ = 7500 kg/m^3, G = 80 × 10^9 N/m^2, L = 1.5 m

Find: θ(L,t)

Solution: The nondimensional form of the partial differential equation governing angular oscillations of the shaft is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \qquad (1)$$

The applied torque is

$$T = T_0 sin\tilde{\omega}t$$

where t is the nondimensional time and

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$$\tilde{\omega} = L\sqrt{\frac{\rho}{G}}\omega = (1.5\text{ m})\sqrt{\frac{7500\frac{\text{kg}}{\text{m}^3}}{80\times10^9\frac{\text{N}}{\text{m}^2}}}\left(110\ \frac{\text{cycles}}{\text{sec}}\right)\left(2\pi\ \frac{\text{rad}}{\text{cycles}}\right) = 0.317$$

The boundary condition at x = 0 is

$$\theta(0,t) = 0 \tag{2}$$

Summation of moments acting on the free body diagram of the disk leads to

$$\phi\ sin\tilde{\omega}t + \frac{\partial\theta}{\partial x}(1,t) = -\beta\frac{\partial^2\theta}{\partial^2 t}(1,t) \tag{3}$$

where

$$\phi = \frac{T_0 L}{JG} = \frac{(2000\text{ N}\cdot\text{m})(1.5\text{ m})}{\frac{\pi}{2}(0.2\text{ m})^4\left(80\times10^9\ \frac{\text{N}}{\text{m}^2}\right)} = 1.49\times10^{-5}\text{ rad}$$

The steady state solution is assumed as substitution of eq.(4) into eqs.(1)-(3) and canceling the sinωt terms from each equation leads to

$$\theta(xz,t) = u(x)sin\tilde{\omega}t \tag{4}$$

$$\frac{d^2u}{dx^2} + \tilde{\omega}^2 u = 0 \tag{5}$$

$$u(0) = 0 \tag{6}$$

$$\frac{du}{dx}(1) = \beta\omega^2 u(1) - \phi \tag{7}$$

The solution of eq.(5) is

$$u(x) = C_1\cos\omega x + C_2\sin\omega x \tag{8}$$

Application of eq.(6) to eq.(8) leads to $C_1 = 0$. Application of eq.(7) to eq.(8) leads to

$$C_2 = -\frac{\phi}{\tilde{\omega}cos\tilde{\omega} - \beta\tilde{\omega}^2 sin\tilde{\omega}}$$

Then the steady state amplitude at the end of the shaft is

$$u(1) = C_2 sin\tilde{\omega} = \frac{(1.49\times10^{-5})\text{sin}0.317}{0.317\text{cos}0.317 - (3.0)(0.317)^2\text{sin}0.317} = 2.25\times10^{-5}\text{rad}$$

Problem 10.7 illustrates the determination of the steady-state amplitude of angular oscillations of the end of shaft due to the application of a harmonic torque.

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10.8 A 20-cm diameter, 2-m-long steel shaft (ρ = 7600 kg/m^3, G = 80 × 10^9 N/m^2) has rotors of mass moment of inertia 110 kg · m^2 and 65 kg · m^2 attached to its ends. Determine the three lowest natural frequencies of the shaft. Compare the lowest non-zero natural frequency to that obtained by using two-degree-of-freedom model, ignoring the inertia of the shaft.

Given: I_1 = 110 kg · m^2, I_2 = 65 kg · m^2, r = 20 cm, L = 2 m, ρ = 7600 kg/m^3,
G = 80 × 10^9 N/m^2

Find: ω_1, ω_2, ω_3

Solution: The system is modeled as a torsional shaft with free ends, but disks attached to each of the free ends. Note that

$$\rho JL = \left(7600\ \frac{\text{kg}}{\text{m}^3}\right)\frac{\pi}{2}(0.2\ \text{m})^4(2\ \text{m}) = 38.15\ \text{kg}\cdot\text{m}^2$$

The partial differential equation governing the angular oscillations of the shaft, written using nondimensional variables, is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \tag{1}$$

Using Table 10.1 the boundary conditions at the ends of the shaft are

$$\frac{\partial\theta}{\partial x}(0,t) = \beta_1\frac{\partial^2\theta}{\partial t^2}(0,t) \tag{2}$$

$$\frac{\partial\theta}{\partial x}(1,t) = \beta_2\frac{\partial^2\theta}{\partial t^2}(1,t) \tag{3}$$

where

$$\beta_1 = \frac{I_1}{\rho JL} = 2.883, \quad \beta_2 = \frac{I_2}{\rho JL} = 1.704$$

A separation of variables solution is assumed as

$$\theta(x,t) = X(x)T(t) \tag{4}$$

Substitution of eq.(4) into eq.(1) leads to

$$\frac{d^2X}{dx^2} + \lambda X = 0$$

$$\frac{d^2T}{dt^2} + \lambda T = 0$$

where λ, the separation constant is the square of the natural frequencies. The solution for X(x) is

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$$X(x) = C_1 cos\sqrt{\lambda}x + C_2 sin\sqrt{\lambda}x \tag{6}$$

Application of eq.(2) to eq.(6) using the equation for T(t) leads to

$$\frac{dX}{dx}(0) = -\beta_1 \lambda X(0) \tag{7}$$

which when applied to eq.(7) leads to

$$C_1 = -\beta_1 \sqrt{\lambda} C_1 \tag{8}$$

Application of eq.(3) to eq.(6) using the equation for T(t) leads to

$$\frac{dX}{dx}(1) = -\beta_2 \lambda X(1) \tag{9}$$

which when applied to eq.(6) leads to

$$-C_1\sqrt{\lambda}sin\sqrt{\lambda} + C_2\sqrt{\lambda}cos\sqrt{\lambda} = \beta_2\left[C_1\sqrt{\lambda}cos\sqrt{\lambda} + C_2\sqrt{\lambda}sin\sqrt{\lambda}\right] \tag{10}$$

Using eq.(8) in eq.(10) and rearranging leads to

$$tan\sqrt{\lambda} = \frac{(\beta_1+\beta_2)\sqrt{\lambda}}{\beta_1\beta_2\lambda - 1} \tag{11}$$

Equation (11) is a transcendental equation used to solve for the system's natural frequencies. Note that λ = 0 is a solution of eq.(11). It also leads to a non-trivial mode shape. Hence λ = 0 is an eigenvalue and is the system's lowest natural frequency. This reflects the rigid body mode for the shaft. For the values given, eq,(11) becomes

$$tan\sqrt{\lambda} = \frac{4.587\sqrt{\lambda}}{4.912\lambda - 1}$$

The dimensional natural frequencies are calculated from the nondimensional natural frequencies by

$$\omega = \frac{\omega^*}{L}\sqrt{\frac{G}{\rho}} = 1622\omega^*$$

Hence

$$\omega_1 = 0, \quad \omega_2 = 1378\,\frac{rad}{sec}, \quad \omega_3 = 18870\,\frac{rad}{sec}$$

Now consider a two-degree-of-freedom model of the system, ignoring inertia effects of the shaft.

The differential equations governing the torsional oscillations of the two-degree-of-freedom model are

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$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where

$$k = \frac{JG}{L}$$

The characteristic equation for the system is

$$\lambda^2 - \left(\frac{k}{I_1} + \frac{k}{I_2}\right)\lambda = 0$$

which leads to natural frequencies of

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k}{I_1} + \frac{k}{I_2}} = 1570\ \frac{\text{rad}}{\text{sec}}$$

Problem 10.8 illustrates the determination of natural frequencies of a shaft with disks attached at both ends and the comparison of the exact natural frequencies with an approximation by considering a discrete system.

10.9 Determine an expression for the natural frequencies of the shaft of Figure P10.9.

FIGURE P10.9

Given: G, J, ρ, K, L

Solution: The partial differential equation governing torsional oscillations of the shaft is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \tag{1}$$

Since the shaft is fixed at x = 0

$$\theta(x, 0) = 0 \tag{2}$$

Using Table 10.1 the appropriate boundary condition for a shaft with an attached torsional spring at x = 1 is

$$\frac{\partial\theta}{\partial x}(1, t) = \beta\theta(1, t) \tag{3}$$

where

$$\beta = \frac{k_t L}{JG} \tag{4}$$

A product solution of eq.(1) is assumed as

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$$\theta(x,t) = X(x)T(t) \tag{5}$$

Substitution of eq.(5) into eq.(1) leads to

$$\frac{d^2T}{dt^2} + \lambda T = 0$$

$$\frac{d^2X}{dx^2} + \lambda X = 0 \tag{6}$$

where λ, the separation constant. The square roots of the separation constants are the natural frequencies. Application of eq.(5) to eqs.(2) and (3) lead to

$$X(0) = 0$$

$$\frac{dX}{dx}(1) = -\beta X(1) \tag{7}$$

The solution for X(x) from eq.(6) is

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x \tag{8}$$

Application of the boundary condition at x = 0 to eq.(8) leads to $C_1 = 0$. Application of the boundary condition at x = 1 to eq.(8) leads to

$$\sqrt{\lambda}\cos\sqrt{\lambda} = -\beta\sin\sqrt{\lambda}$$

$$\sqrt{\lambda} = -\beta\tan\sqrt{\lambda} \tag{9}$$

Equation (9) is a transcendental equation to solve for the separation constant. Noting that the square roots of the separation constants are the natural frequencies, eq.(9) can be rewritten as

$$\omega^* = -\beta\tan\omega^*$$

where ω* represents the nondimensional natural frequencies. The dimensional natural frequencies are

$$\omega = \frac{\omega^*}{L}\sqrt{\frac{G}{\rho}}$$

Problem 10.9 illustrates the development of the frequency equation for a shaft with an attached torsional spring.

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10.10 An oil well drilling tool is modeled as a bit attached to the end of long shaft, unrestrained from rotation at its fixed end.

(a) Determine the equation defining the natural frequencies of the drilling tool.
(b) For a particular operation, the shaft (ρ = 7500 kg/m^2, $G = 80 \times 10^9$ N/m^2) is 20 m long with a 20-cm diameter. The tool operates at a speed of 400 rad/sec. What are the limits on the moment of inertia of the drill bit such that the two lowest nonzero natural frequencies of the tool are not within 20% of the operating speed?

Given: ρ = 7500 kg/m^2, G = 80 × 10^9 N/m^2, L = 20 m, d = 20 cm, ω = 400 rad/sec

Find: I such that $\omega_1, \omega_2 < 320$ rad./sec and $\omega_1, \omega_2 > 480$ rad/sec

Solution: The drill bit attached to the shaft is modeled as a thin disk of moment of inertia I attached to the end of the shaft that is free to rotate at its opposite end. The partial differential equation governing torsional oscillations of the shaft is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \tag{1}$$

Since the end at x = 0 is free from rotation

$$\frac{\partial\theta}{\partial x}(0,t) = 0 \tag{2}$$

From Table 10.1 the appropriate boundary condition at x = 1 is

$$\frac{\partial\theta}{\partial x}(1,t) = -\beta\frac{\partial^2\theta}{\partial t^2}(1,t) \tag{3}$$

where

$$\beta = \frac{I}{\rho JL} \tag{4}$$

A product solution is assumed of the form

$$\theta(x,t) = X(x)T(t) \tag{5}$$

Substitution of eq.(5) into eq.(1) leads to the following ordinary differential equations

$$\frac{d^2T}{dt^2} + \lambda T = 0$$

$$\frac{d^2X}{dx^2} + \lambda X = 0 \tag{6}$$

where λ, the separation constant. The square roots of the separation constants are nondimensional natural frequencies. The solution for X(x) from eq.(6) is

$$X(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x \tag{7}$$

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Application of eq.(5) to eqs.(2) and (3) leads to

$$\frac{dX}{dx}(0) = 0 \tag{8}$$

$$dX(1)T(t) = -\beta X(1)\frac{D^2T}{dt^2}$$

Use of eq.(6) in the second of eqs.(8) leads to

$$\frac{dX}{dx}(1) = \beta\lambda X(1) \tag{9}$$

Application of the condition at x = 0 leads to $C_2 = 0$. Then

$$X(x) = C_1 \cos\sqrt{\lambda}x \tag{10}$$

Application of eq.(9) in eq.(10) leads to

$$-\sqrt{\lambda}\sin\sqrt{\lambda} = \beta\lambda\cos\sqrt{\lambda}$$

$$-\tan\sqrt{\lambda} = \beta\sqrt{\lambda} \tag{11}$$

Equation (11) is a transcendental equation to solve for the separation constants. Equation (11) is rewritten in terms of the nondimensional natural frequencies ω* as

$$-\tan\omega^* = \beta\omega^* \tag{12}$$

The dimensional natural frequencies are obtained from

$$\omega = \frac{\omega^*}{L}\sqrt{\frac{G}{\rho}} \tag{13}$$

For the particular system at hand, it is desired to design the bit such that the natural frequencies are out of the range from 320 rad/sec to 480 rad/sec. Note that

$$\frac{1}{L}\sqrt{\frac{G}{\rho}} = \frac{1}{20\text{ m}}\sqrt{\frac{80\times10^9\frac{\text{N}}{\text{m}^2}}{7500\frac{\text{kg}}{\text{m}^3}}} = 163.3$$

Hence in order for the natural frequency to be lower than 320 rad/sec

$$320 > \omega^*(163.3), \quad \omega^* < 1.959$$

However, from the frequency equation, it is only possible for the lowest nonzero natural frequency to be less than this value. Setting ω* = 1.959 leads to

$$\beta = -\frac{\tan\omega^*}{\omega^*} = -\frac{\tan(1.959)}{1.959} = 1.248$$

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In order for the natural frequency to be larger than 480 rad/sec

$$480 < 1.63\omega^*, \quad \omega^* > 2.939$$

Note, that is the smallest natural frequency is greater than 2.939, all natural frequencies are greater than 2.939. It is also not possible to have the second natural frequency in the range from 1.959 to 2.939. Setting ω* = 2.939 leads to

$$\beta = -\frac{\tan(2.939)}{2.939} = 0.0699$$

Hence, to avoid the frequency range from 320 rad/sec to 480 rad/sec, β < 0.0699 or β > 1.25. Noting that

$$\rho JL = \left(7500\ \frac{\text{kg}}{\text{m}^3}\right)\frac{\pi}{2}(0.2\ \text{m})^4(20\ \text{m}) = 377.0$$

leads to

$$I < 26.35\ \text{kg}\cdot\text{m}^2, \quad or\ I > 471.25\ \text{kg}\cdot\text{m}^2$$

Problem 10.10 illustrates (a) derivation of the natural frequency equation for a shaft free at one end and an attached disk at the other end and (b) modeling of a oil drilling operation.

10.11 The shaft of Chapter Problem 10.1 is at rest in equilibrium when the time-dependent moment of Figure P10.11 is applied to the end of the shaft. Determine the time-dependent form of the resulting torsional oscillations.

FIGURE P10.11

Given: M(t)

Find: θ(t)

Solution: The shaft is subject to an excitation per unit length of

$$T(t) = \frac{M_0}{L}\delta(x-1)[u(t) - u(t-t_0]$$

The nondimensional form of the governing partial differential equation is

$$\frac{\partial^2\theta}{\partial x^2} + \phi\delta(x-1)[u(t) - t(t-t_0)] = \frac{\partial^2\theta}{\partial t^2} \qquad (1)$$

where

$$\phi = \frac{M_0 L}{JG}$$

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The natural frequencies and mode shapes for the shaft are determined in the solution of Problem 10.4 as

$$\omega_k = (2k-1)\frac{\pi}{2}$$

$$x_k(x) = \sqrt{2}\sin\,(2k-1)\frac{\pi}{2}x$$

Modal analysis is used and the solution assumed as

$$\theta(x,t) = \sum_{k=1}^{\infty} x_k(x)p_x(t) \qquad (2)$$

Substitution of eq.(2) into eq.(1) leads to

$$\sum_{k=1}^{\infty}\frac{d^2X_k}{dx^2} + \phi\delta(x-1)[(u)t - u(t-t_0)] = \sum_{k=1}^{\infty}\ddot{p}_x X_k(x)$$

$$\sum_{k=1}^{\infty}(p_{\ddot{k}} + \omega_k^2 p_k)X_k = \phi\delta(x1)[u(t) - u(t-t_0)] \qquad (3)$$

Multiplying eq.(3) by $X_j(x)$ for an arbitrary x and integrating from 0 to 1 leads to

$$p_j + \omega_j^2 p_j = \phi[u(t) - u(t-t_0)](\delta\delta(x-1), X_j(x))$$

where

$$\big(\delta(x-1), X_j(x)\big) = \int_0^1 \delta\,(x-1)\sqrt{2}\,\sin(2k-1)\frac{\pi}{2}x dx = \sqrt{2}\sin\,(2k-1)\frac{\pi}{2}$$

Hence

$$p_j + \omega_j^2 p_j = \sqrt{2}\phi\,\sin(2k-1)\frac{\pi}{2}[u(t) - u(t-t_0)]$$

whose solution is

$$p_j(t) = \frac{\sqrt{2}\phi\sin\,(2k-1)\frac{\pi}{2}}{\omega_j^2}\{\big(1-\cos\omega_j t\big)u(t) - \big[1-\cos\omega_j(t-t_0)\big]u(t-t_0)\}$$

Problem 10.11 illustrates (1) the use of the unit impulse function to model a concentrated torque and (2) the application of modal analysis to determine the forced response of a torsional shaft.

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10.12 The shaft of Chapter Problem 10.1 is at rest in equilibrium when it is subject to the uniform time-dependent torque loading per unit length of Figure P10.12. Determine the time-dependent form of the resulting torsional oscillations.

FIGURE P10.12

Given: $\rho = 7500$ kg/m^3, L = 1.5 m, r = 20 cm, G = 80 × 10^9 N/m^2, M(t) shown

Find: θ(t)

Solution: The nondimensional form of the governing mathematical problem is

$$\frac{\partial^2\theta}{\partial^2 x} + \phi f(t) = \frac{\partial^2\theta}{\partial t^2} \tag{1}$$

where

$$\phi = \frac{M_0 L^2}{JG}$$

$$f(t) = [u(t) - u(t-t_0)] - [u(t-t_0) - u(t-2t_0)] = u(t) - 2u(t-t_0) + u(t-2t_0)$$

subject to

$$\theta(0,t) = 0, \quad \frac{\partial\theta}{\partial x}(1,t) = 0$$

$$\theta(x,0) = 0, \quad \frac{\partial\theta}{\partial x}(x,0) = 0$$

The natural frequencies and mode shapes for this shaft are determined in the solution of Chapter Problem 10.4 as

$$\omega_k = (2k-1)\frac{\pi}{2}$$

$$X_k(x) = \sqrt{2}\sin\,(2k-1)\frac{\pi}{2}x$$

A solution of (1) is assumed as

$$\theta(x,t) = \sum_{k=1}^{\infty} p_k(t) X_k(x) \tag{2}$$

Substitution of eq.(2) into eq.(1) leads to

$$\sum_{k=1}^{\infty} \frac{d^2 X_k}{dx^2} p_k + \phi f(t) = \sum_{k=1}^{\infty} X_k(x)\ddot{p}_k$$

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$$\sum_{k=1}^{\infty}\left(\ddot{p}_k X_k - \frac{d^2 X_k}{dx^2} p_k\right) = \phi f(t) \tag{3}$$

$$\sum_{k=1}^{\infty}(\ddot{p}_k + \omega_k^2) X_k = \phi f(t)$$

Multiply eq.(3) by $X_j(x)$ for an arbitrary j and integrating from 0 to 1 leads to

$$\sum_{k=1}^{\infty}(\ddot{p}_k + \omega_k^2 p_k)(X_j, X_k) = \phi f(t)(1, X_j)$$

$$\ddot{p}_j + \omega_j^2 p_j = \phi f(t)(1, X_j)$$

where

$$(1, X_j) = \int_0^1 \sqrt{2}\sin\,(2k-1)\frac{\pi}{2}x dx = \frac{2\sqrt{2}}{(2k-1)\pi}$$

Hence

$$p_j + \omega_j^2 p_j = \frac{2\phi\sqrt{2}}{(2k-1)\pi}[u(t) - 2u(t-t_0) + u(t-2t_0)]$$

Table 5.1 is used to determine the solution for each of the principal coordinates as

$$p_j = \frac{2\sqrt{2}}{\omega_j^2(2k-1)\pi}\{(1-\cos\omega_j t)u(t) - 2[1-\cos\omega_j(t-t_0)u(t-t_0)]\}$$

$$+[1-\cos\omega_j(t-2t_0)]u(t-2t_0)\}$$

Problem 10.12 illustrates the use of modal analysis to determine the torsional response of a circular shaft due to a time dependent moment applied along the length of the shaft.

10.13 The elastic bar of Figure P10.13 is undergoing longitudinal vibrations. Let $u(x, t)$ be the time-dependent displacement of a particle along the centroidal axis of the bar, initially a distance x from the left support.

(a) Draw free-body diagrams showing the external and effective forces acting on a differential element of thickness dx, a distance x from the left end of the bar at an arbitrary instant of time.
(b) Show that the governing partial differential equation is

FIGURE P10.13

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} \tag{1}$$

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(c) Introduce nondimensional variables to derive a nondimensional partial differential equation.

Given: E, ρ, A

Show: Eq. (1)

Solution: Consider free-body diagrams of a differential element of thickness dx, at an arbitrary instant. The cut of the differential element from the bar exposes internal axial forces acting in the cross section. The axial forces are the resultant of the normal stress distribution. The normal stress is assumed constant over a given cross-section. Thus the resultant normal force acting on a cross section of area A due to a normal stress is

$$F = \sigma(x,t)A$$

The axial stress varies over the length of the bar. At x + dx the axial stress is obtained using a Taylor series expansion about x,

$$\sigma(x+dx,t) = \sigma(x,t) + \frac{\partial\sigma}{\partial x}(x,t)dx$$

Summing forces acting on the free body diagrams

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$

leads to

$$-\sigma(x,t)Adx + \left[\sigma(x,t) + \frac{\partial\sigma}{\partial x}(x,t)dx\right]A = \rho A\frac{\partial^2 u}{\partial x^2}(x,t)$$

$$\frac{\partial\sigma}{\partial x} = \rho A\frac{\partial^2 u}{\partial t^2} \tag{2}$$

The material is assumed elastic

$$\sigma = E\epsilon$$

and the strain is related to the displacement by

$$\epsilon = \frac{\partial u}{\partial x}$$

leading to

$$\sigma = EA\frac{\partial u}{\partial x} \tag{3}$$

Substitution of eq.(3) into eq.(2) leads to

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$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

The following nondimensional variables are introduced

$$t^* = \frac{1}{L}\sqrt{\frac{E}{\rho}}t, \quad x^* = \frac{x}{L}, \quad u^* = \frac{u}{L} \tag{4}$$

When eqs.(4) are introduced in eq.(1), the following nondimensional equation results

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Problem 10.13 illustrates that the free vibrations of longitudinal waves in a bar governed by the wave equation.

10.14 Using the results of Chapter Problem 10.13, determine the natural frequencies of longitudinal vibrations of a bar fixed at one end and free at the other.

Given: E, ρ, L

Find: ω_1, ω_2, …

Solution: Using the results of Problem 10.13 the nondimensional partial differential equation governing the free longitudinal vibrations in a bar is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{1}$$

Since the bar is fixed at x = 0

$$u(x,t) = 0 \tag{2}$$

Since the end at x = 1 is free, it is subject to no external force, and no stress. Hence

$$\frac{\partial u}{\partial x}(1,t) = 0 \tag{3}$$

The problem specified by eqs.(1)-(3) is identical to the problem solved in Example 10.1 except that the dependent variable in the problem at hand is longitudinal displacement instead of angular displacement. Thus the nondimensional natural frequencies for a fixed-free bar are the same as the nondimensional natural frequencies for a fixed-free shaft. To this end

$$\omega_k^* = (2k-1)\frac{\pi}{2}, \quad k = 1,2,\ldots$$

The relationship between the dimensional time and the nondimensional time is

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$$t^* = \frac{1}{L}\sqrt{\frac{G}{\rho}}t$$

Hence the relationship between the nondimensional frequencies and dimensional frequencies is

$$\omega = \frac{\omega^*}{L}\sqrt{\frac{G}{\rho}}$$

Hence

$$\omega_k = (2k-1)\frac{\pi}{2L}\sqrt{\frac{G}{\rho}}, \quad k = 1,2,\ldots$$

Problem 10.14 illustrates free vibrations of a fixed-free bar.

10.15 Show the orthogonality of mode shapes of longitudinal vibrations of a bar fixed at one end and free at its other end.

Given: E, ρ, L

Find: orthogonality of mode shapes

Solution: The problem for free longitudinal vibrations in a fixed-free bar is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{1}$$

subject to

$$u(0,t) = 0 \tag{2}$$

and

$$\frac{\partial u}{\partial x}(1,t) = 0 \tag{3}$$

The problem given by eqs.(1)-(3) is identical to the problem solved in Example 10.1 except that the dependent variable in the problem is the longitudinal displacement instead of angular displacement. The nondimensional natural frequencies are given by

$$\omega_k^* = (2k-1)\frac{\pi}{2}, \quad k = 1,2,\ldots \tag{4}$$

The normalized mode shapes are

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$$X_k(x) = \sqrt{2}\sin\,(2k-1)\frac{\pi}{2}x$$

Testing the orthogonality of the mode shapes

$$\left(X_j(x), X_k(x)\right) = \int_0^1 X_j(x)X_k(x)dx$$

$$= \int_0^1 \sqrt{2}\sin\,(2j-1)\frac{\pi}{2}x\sqrt{2}\sin\,(2k-1)\frac{\pi}{2}xdx$$

$$\frac{\sin\,(2j-2k)\frac{\pi}{2}}{2j-2k} - \frac{\sin\,(2j+2k-2)\frac{\pi}{2}}{2j+2k-2}$$

Since j and k are both integers, the arguments of the sine functions are both integer multiples of π and thus the sine functions are zero. Thus the mode shapes are mutually orthogonal.

Problem 10.15 illustrates orthogonality of mode shapes with respect to the standard scalar product.

10.16 A large industrial piston operates at 1000 Hz. The piston head has a mass of 20 kg. The shaft is made of steel (ρ = 7500 kg/m^3, E = 210 × 10^9 N/m^2). For what shaft diameters will all natural frequencies be out of the range of 900 to 1100 Hz?

Given: ω = 1000 Hz, m = 20 kg, ρ = 7500 kg/m^3, L = 1 m, E = 210 × 10^9 N/m^2

Find: D such that natural frequencies are out of the range of 900 to 1100 Hz

Solution: The piston is modeled as a shaft which is free at one end and with a mass its other end as shown. The nondimensional wave equation governs longitudinal vibrations of the piston

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Since the end x=0 is free

$$\frac{\partial u}{\partial x}(0,t) = 0$$

Application of Newton's law to the piston mass leads to

$$\frac{\partial u}{\partial x}(1,t) = -\beta\frac{\partial^2 u}{\partial t^2}(1,t)$$

where

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$$\beta = \frac{m}{\rho AL}$$

A separation of variables solution u(x,t) = X(x)T(t) leads to the following problem for X(x)

$$X'' + \lambda X = 0$$
$$X'(0) = 0$$
$$X'(1) = \beta\lambda X(1)$$

where λ, the separation constant, is the square of the natural frequency. The solution for X(x) is as follows

$$X(x) = C_1 \cos\sqrt{\lambda x} + C_2 \sin\sqrt{\lambda x}$$
$$X'(0) = 0 \rightarrow C_2 = 0$$
$$X'(1) = \beta\lambda X(1) \rightarrow -\sqrt{\lambda}\sin\sqrt{\lambda} = \beta\lambda\cos\sqrt{\lambda}$$

The equation for the separation constant becomes

$$\tan\sqrt{\lambda} = -\beta\sqrt{\lambda}$$

It is desired for the piston to have no natural frequency within the range of 900 to 1100 Hz or 5650 to 6900 rad/s. If ω is a natural frequency of the piston in rad/ s the corresponding non-dimensional natural frequency for the piston is

$$\sqrt{\lambda} = L\sqrt{\frac{\rho}{E}}\omega = \frac{\omega}{5.29\times10^3 \text{ rad/s}}$$

Thus to avoid the piston having a natural frequency between 900 and 100 Hz,

$$\sqrt{\lambda} < \frac{5650}{5290} = 1.0679$$

or

$$\sqrt{\lambda} > \frac{6900}{5290} = 1.306$$

However for any value of β the first non-zero solution of the transcendental equation is greater than π/2. Thus for any value of shaft diameter the lowest non-zero natural frequency will be greater than 1100 Hz.

Problem 10.16 illustrates the natural frequency equation for longitudinal vibrations of a free-free shaft with an end mass.

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10.17 The free end of the piston of Chapter Problem 10.16 is subject to a force 1000 sin ωt N, where ω = 100 Hz. If the diameter of the shaft is 8 cm, determine the steady-state response of the piston.

Given: m = 20 kg, ρ = 7500 kg/m^3, L = 1 m, E = 210 × 10^9 N/m^2, d = 8 cm, F_0 = 1000 N, ω = 100 Hz

Find: X(L = 1 m)

Solution: The piston is modeled as a bar with a harmonic force applied at one end and a mass at its other end. The nondimensional differential equation and boundary conditions governing the problem are

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial u}{\partial x}(0,t) = \mu \sin \omega t$$

$$\frac{\partial u}{\partial x}(1,t) = -\beta \frac{\partial^2 u}{\partial t^2}(1,t)$$

where

$$\mu = \frac{F_0}{EA} = 2.37 \times 10^{-7}$$

$$\beta = \frac{m}{\rho A L} = 0.133$$

and the nondimensional frequency is

$$\omega = (100\,\text{Hz})(2\pi\ \text{rad/Hz})L\sqrt{\frac{\rho}{E}} = 0.1187$$

The steady-state solution is assumed as

$$u(x,t) = X(x)\sin \omega t$$

which when substituted into the original problem leads to the following problem for X(x)

$$X''(x) + \omega^2 X(x) = 0$$

$$X'(0) = \mu$$

$$X'(1) = \beta\omega^2 X(1)$$

The solution of the above problem proceeds as follows

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$$X(x) = C_1 \cos \omega x + C_2 \sin \omega x$$

$$X'(0) = \mu \rightarrow C_2 = \frac{\mu}{\omega}$$

$$X'(1) = \beta\omega^2 X(1) \rightarrow -\mu C_1 \sin \omega + \mu \cos \omega = \beta\omega^2 \left(C_1 \cos \omega + \frac{\mu}{\omega} \sin \omega \right)$$

$$C_1 = \frac{\mu \cos \omega - \beta\mu\omega \sin \omega}{\beta\omega^2 \cos \omega + \mu \sin \omega} = 1.26 \times 10^{-4}$$

Thus the steady-state response of the piston is

$$X(x) = 1.26 \times 10^{-4} \cos \omega x + 1.96 \times 10^{-6} \sin \omega x$$

The nondimensional steady-state amplitude of the piston mass is

$$X(1) = 1.256 \times 10^{-4}$$

The dimensional steady-state amplitude of the piston mass is

$$LX(1) = 1.256 \times 10^{-4} \text{ m}$$

Problem 10.17 illustrates the steady-state response of a bar subject to a harmonic excitation.

10.18 Determine the five lowest natural frequencies of the system of Figure P10.18.

FIGURE P10.18

Given: $\rho = 7500$ kg/m^3, $E = 200 \times 10^9$ N/m^2, $A = 1.5 \times 10^{-5}$ m^2, $L = 3$ m, $k_1 = 1 \times 10^6$ N/m, $k_2 = 1.5 \times 10^6$ N/m

Find: five lowest ω

Solution: The nondimensional partial differential equation and boundary conditions for the system are

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$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial u}{\partial x}(0,t) = \beta_1 u(0,t)$$
$$\frac{\partial u}{\partial x}(1,t) = \beta_2 u(1,t)$$

where

$$\beta_1 = \frac{k_1 L}{EA} = 1.0$$
$$\beta_2 = \frac{k_2 L}{EA} 1.5$$

A separation of variables solution u(x,t) = X(x)T(t) leads to the following problem for X(x)

$$X'' + \lambda X = 0$$
$$X'(0) = \beta_1 X(0)$$
$$X'(1) = -\beta_2 X(1)$$

The solution for X(x) is as follows

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{x}$$
$$X'(0) = \beta_1 X(0) \rightarrow C_2\sqrt{\lambda} = \beta_1 C_1 \rightarrow C_1 = \frac{\sqrt{\lambda}}{\beta_1} C_2$$
$$X'(1) = -\beta_2 X(1) \rightarrow -\frac{\lambda}{\beta_1}\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda} = -\beta_2\left(\frac{\sqrt{\lambda}}{\beta_1}\cos\sqrt{\lambda} + \sin\sqrt{\lambda}\right)$$

The following transcendental equation is obtained for the separation constant

$$\tan\sqrt{\lambda} = \frac{\sqrt{\lambda}\left(1 + \frac{\beta_2}{\beta_1}\right)}{\frac{\lambda}{\beta_1} - \beta_2}$$
$$\tan\sqrt{\lambda} = \frac{2.5\sqrt{\lambda}}{\lambda - 1.5}$$

The first five roots of the equation are

$$\sqrt{\lambda} = 1.423, 3.78, 6.50, 9.68, 12.76$$

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which are the nondimensional natural frequencies. The relation between the dimensional natural frequencies and the nondimensional natural frequencies is

$$\omega = \frac{1}{L}\sqrt{\frac{E}{\rho}}\sqrt{\lambda}$$

The first five dimensional natural frequencies are calculated as

$$\omega_1 = 2.45\times10^3 \text{ rad/s}$$
$$\omega_2 = 651\times10^3 \text{ rad/s}$$
$$\omega_3 = 1.12\times10^4 \text{ rad/s}$$
$$\omega_4 = 1.67\times10^4 \text{ rad/s}$$
$$\omega_5 = 2.20\times10^4 \text{ rad/s}$$

Problem 10.18 illustrates the natural frequencies for a bar with springs attached at both ends.

10.19 Determine the steady-state response of the system of Figure P10.19.

$\rho = 7500$ kg/m^3
$E = 200 \times 10^9$ N/m^2
$A = 4.5 \times 10^{-5}$ m^2
$k = 9 \times 10^5$ N/m
$F_0 = 800$ N
$\omega = 100$ rad/s
$L = 3.3$ m

FIGURE P10.19

Given: $\rho = 7500$ kg/m^3, $E = 200 \times 10^9$ N/m^2, $A = 4.5 \times 10^{-5}$ m^2, $k = 9 \times 10^5$ N/m, $F_0 = 800$ N, $\omega = 100$ rad/s, $L = 3.3$ m

Find: X(x)

Solution: The nondimensional partial differential equation and boundary conditions for the longitudinal displacement u(x,t) are

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial u}{\partial x}(0,t) = \beta u(0,t)$$
$$\frac{\partial u}{\partial x}(1,t) = \mu \sin \omega t$$

where

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$$\beta = \frac{kL}{EA} = 0.33$$
$$\mu = \frac{F_0}{EA} = 8.99 \times 10^{-5}$$

and the nondimesnional natural frequency is

$$\omega = (100\,\text{rad/s})L\sqrt{\frac{\rho}{E}} = 0.0639$$

The steady-state response is assumed as u(x,t) = X(x)sinωt leading to

$$X'' + \omega^2 X = 0$$
$$X'(0) = \beta X(0)$$
$$X'(1) = \mu$$

The solution for X(x) is

$$X(x) = C_1 \cos \omega x + C_2 \sin \omega x$$
$$X'(0) = \beta X(0) \rightarrow C_2 = \frac{\beta}{\omega} C_1 = 5.164 C_1$$
$$X'(1) = \mu \rightarrow C_1 = \frac{\mu}{\beta \cos \omega - \omega \sin \omega} = 2.73 \times 10^{-4}$$
$$C_2 = 1.41 \times 10^{-3}$$

The steady-state response is

$$X(x) = 2.73 \times 10^{-4} \cos \omega x + 1.41 \times 10^{-3} \sin \omega x$$

Problem 10.19 illustrates the steady-state response of a bar subject to a harmonic excitation.

10.20 Determine the steady-state response of the system shown.

$\rho = 7500$ kg/m^3
$E = 200 \times 10^9$ N/m^2
$A = 4.5 \times 10^{-5}$ m^2
$k = 9 \times 10^5$ N/m
$m = 2.5$ kg
$L = 3.5$ m
$F_0 = 600$ N
$\omega = 450$ rad/s

FIGURE P10.20

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Given: $\rho = 7500$ kg/m^3, $E = 200 \times 10^9$ N/m^2, $A = 4.5 \times 10^{-5}$ m^2, $k = 9 \times 10^5$ N/m, $m = 2.5$ kg, $L = 3.5$ m, $F_0 = 600$ N, $\omega = 450$ rad/s

Find: X(x)

Solution: The nondimensional partial differential equation and boundary conditions governing the longitudinal displacement u(x,t) are

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$
$$u(x,0) = 0$$
$$\frac{\partial u}{\partial x}(1,t) = -\delta \frac{\partial^2 u}{\partial t^2}(1,t) + \mu \sin \omega t - \beta u(1,t)$$

where

$$\delta = \frac{m}{\rho A L} = 2.12$$
$$\mu = \frac{F_0}{EA} = 6.67 \times 10^{-5}$$
$$\beta = \frac{kL}{EA} = 0.35$$
$$\omega = (450 \text{ rad/s}) L \sqrt{\frac{\rho}{E}} = 0.305$$

The steady-state solution is assumed as u(x,t) = X(x)sinωt, leading to

$$X'' + \omega^2 X = 0$$
$$X(0) = 0$$
$$X'(1) = (\delta\omega^2 - \beta) X(1) + \mu$$

The solution for X(x) is

$$X(x) = C_1 \cos \omega x + C_2 \sin \omega x$$
$$X(0) = 0 \rightarrow C_1 = 0$$
$$X'(1) = (\delta\omega^2 - \beta) X(1) + \mu \rightarrow C_2 = \frac{\mu}{\omega \cos \omega - (\delta\omega^2 - \beta) \sin \omega} = 1.978 \times 10^{-4}$$

Thus the nondimensional steady-state response is

$$X(x) = 1.978 \times 10^{-4} \sin(0.305x)$$

Problem 10.20 illustrates the steady-state response of a longitudinal bar with an end mass

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10.21 Draw frequency response curves for the response of the disk at the end of the shaft in Example 10.3. Plot the curves for $\beta = 0.5$, $\beta = 2.0$, and $\beta = 20.0$.

Given: system of Example 10.3, β = 0.5, β = 2.0, and β = 20.0

Find: frequency-response curves

Solution: The steady-state response of the system of Example 10.3 is calculated in Example 10.3 as

$$\theta(x,t) = \frac{T_0 L}{JG} \frac{\sin \tilde{\omega} x \sin \tilde{\omega} t}{\tilde{\omega} \cos \tilde{\omega} - \beta \tilde{\omega}^2 \sin \tilde{\omega}}$$

The nondimensional frequency response at the end of the shaft is

$$\frac{T_0 L}{JG} u(1) = \frac{\sin \tilde{\omega}}{\tilde{\omega} \cos \tilde{\omega} - \beta \tilde{\omega}^2 \sin \tilde{\omega}}$$

The frequency response curves for β = 0.5, β = 2.0, and β = 20.0 follow. The peaks really represent infinite amplitude responses smoothed over by a computer plotting routine. These occur at frequencies for which the denominator of the preceding equation vanishes, corresponding to the system's natural frequencies. Note that the range over which a large response occurs decreases as the frequencies increase.

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Problem 10.21 illustrates the frequency response curves for the steady-state response of a continuous system due to a harmonic excitation.

10.22 Determine the steady-state response of a circular shaft subject to a uniform torque per unit length of $T_0 \sin \omega t$ applied over its entire length.

Given: ρ, L, G, T_0, J, ω

Find: θ(x,t), steady-state

Solution: The nondimensional form of the governing partial differential equation is

$$\phi \sin \tilde{\omega} t + \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2} \tag{1}$$

where

$$\phi = \frac{L^2 T_0}{JG}, \quad \tilde{\omega} = \omega L \sqrt{\frac{\rho}{G}}$$

Since the shaft is fixed at x = 0,

$$\theta(0, t) = 0 \tag{2}$$

Since the shaft is free at x = 1

$$\frac{\partial \theta}{\partial x}(1, t) = 0 \tag{3}$$

Since the excitation is harmonic, the steady-state solution is assumed as

$$\theta(x, t) = f(x) \sin \tilde{\omega} t \tag{4}$$

Substitution of eq.(4) into eq.(1) leads to

$$\phi \sin \tilde{\omega} t + \frac{d^2 f}{dx^2} \sin \tilde{\omega} t = -\tilde{\omega}^2 f(x) \sin \tilde{\omega} t$$

$$\frac{d^2 f}{dx^2} + \tilde{\omega}^2 f = -\phi \tag{5}$$

The boundary conditions become

$$f(0) = 0 \tag{6}$$

and

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$$\frac{df}{dx}(1) = 0 \tag{7}$$

The general solution of eq.(5) is

$$f(x) = -\frac{\phi}{\tilde{\omega}^2} + C_1 \cos \tilde{\omega} t + C_2 \sin \tilde{\omega} t \tag{8}$$

Application of eq.(6) to eq.(8) leads to

$$C_1 = \frac{\phi}{\tilde{\omega}}$$

Application of eq.(7) to eq.(8) leads to

$$C_2 = \frac{\phi}{\tilde{\omega}^2} \cot \tilde{\omega}$$

Hence

$$\theta(x,t) = \frac{\phi}{\omega^2}(-\cos\tilde{\omega}x + \cot\tilde{\omega}\sin\tilde{\omega}x)\sin\tilde{\omega}t$$

Problem 10.22 illustrates determination of the steady-state response of a shaft subject to a harmonic torque per unit length.

10.23 Determine the steady-state response of the system of Figure P10.23.

FIGURE P10.23

Given: J, G, ρ, k_1, T_0, ω

Find: θ(x,t)

Solution: The nondimensional partial differential equation governing the forced vibrations of the system is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{1}$$

Since the shaft is fixed at x = 0

$$\theta(0,t) = 0 \tag{2}$$

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Summing moments about the center of the disk gives

$$\frac{\partial \theta}{\partial x}(1,t) = -\beta_1 \theta(1,t) + \phi \sin\tilde{\omega} t - \beta_2 \frac{\partial^2 \theta}{\partial t^2}(1,t) \tag{3}$$

where

$$\beta_1 = \frac{k_t L}{JG}, \quad \beta_2 = \frac{I}{\rho J L}$$

$$\phi = \frac{T_0 L}{JG}, \quad \tilde{\omega} = \omega L \sqrt{\frac{\rho}{G}}$$

Since the excitation is harmonic, the steady-state solution is assumed as

$$\theta(x,t) = f(x) \sin \tilde{\omega} t \tag{4}$$

Substitution of eq.(4) into eq.(1) leads to

$$\frac{d^2 f}{dx^2} + \tilde{\omega}^2 f = -\phi \tag{5}$$

Whose general solution is

$$f(x) = C_1 \cos\tilde{\omega} x + C_2 \sin\tilde{\omega} x \tag{6}$$

Application of eq.(4) to the boundary conditions, eqs.(2) and (3) leads to

$$f(0) = 0$$

$$\frac{df}{dx} = -\beta_1 f(1) + \phi + \beta_2 \omega^2 f(1) \tag{7}$$

Application of eq.(7) to eq.(6) leads to

$$C_1 = 0$$

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$$C_2 = \frac{\phi}{\tilde{\omega}\cos\tilde{\omega} + (\beta_1 - \beta_2\tilde{\omega}^2)\sin\tilde{\omega}}$$

Hence the steady-state response is

$$\theta(x,t) = \frac{\phi \sin\tilde{\omega}x \sin\tilde{\omega}t}{\tilde{\omega}\cos\tilde{\omega} + (\beta_1 - \beta_2\tilde{\omega}^2)\sin\tilde{\omega}}$$

Problem 10.23 illustrates determination of the steady-state response of a shaft subject to a harmonic torque at its end.

10.24 Propeller blades totaling 1200 kg with a total mass moment of inertia of 155 kg · m^2 are attached to a solid circular shaft (ρ = 5000 kg/m^3, G = 60 × 10^9 N/m^2, E = 140 × 10^9 N/m^2) of radius 40 cm and length 20 m. The other end of the shaft is fixed in an ocean liner. Determine (a) the lowest natural frequency for torsional oscillations of the propeller and (b) The lowest natural frequency of longitudinal motion of the propeller.

Given: m = 1200 kg, I = 155 kg · m^2, r = 40 cm, L = 20 m, $\rho = 5000\,\frac{\text{kg}}{\text{m}^3}, G = 60 \times 10^9\,\frac{\text{N}}{\text{m}^2}, E = 140 \times 10^9\,\frac{\text{N}}{\text{m}^2}$

Find: (a) ω_1 for torsional oscillations and (b) ω_1 for longitudinal oscillations

Solution: (a) The system is modeled as a disk of mass moment of inertia I attached to the end of a circular shaft fixed at its other end. The frequency equation for this system is determined in the solution of Example 10.2 as

$$\tan\sqrt{\lambda} = \frac{1}{\beta\sqrt{\lambda}}$$

where λ is a separation constant whose square roots are the natural frequencies and

$$\beta = \frac{I}{\rho JL} = \frac{155\ \text{kg}\cdot\text{m}^2}{\left(5000\,\frac{\text{kg}}{\text{m}^3}\right)\frac{\pi}{2}(0.4\ \text{m})^4(20\ \text{m})} = 0.0386$$

Hence the lowest nondimensional natural frequency is the smallest positive solution of

$$\tan\omega^* = \frac{1}{0.0386\omega^*}$$

which is $\omega^* = 1.512$. The dimensional natural frequency is obtained by

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$$\omega_1 = \frac{\omega^*}{L}\sqrt{\frac{G}{\rho}} = \frac{1.512}{20\ m}\sqrt{\frac{60 \times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}}{5000\frac{\mathrm{kg}}{\mathrm{m}^3}}} = 261.9\ \frac{\mathrm{rad}}{\mathrm{s}}$$

(b) The frequency equation for the nondimensional frequencies of free vibration for the longitudinal motion of a mass attached to the end of a shaft is

$$\tan\sqrt{\lambda} = \frac{1}{\beta\sqrt{\lambda}}$$

where

$$\beta = \frac{m}{\rho AL} = \frac{1200\ kg}{\left(5000\frac{\mathrm{kg}}{\mathrm{m}^3}\right)\pi(0.4\ \mathrm{m})^2(20\ \mathrm{m})} = 0.02387$$

Hence the frequency equation becomes

$$\tan\omega^* = \frac{1}{0.02387\omega^*}$$

The smallest positive solution of the above equation is $\omega^* = 1.534$. The dimensional natural frequency is

$$\omega_1 = \frac{\omega^*}{L}\sqrt{\frac{E}{\rho}} = \frac{1.534}{20\ \mathrm{m}}\sqrt{\frac{140 \times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}}{5000\frac{\mathrm{kg}}{\mathrm{m}^3}}} = 405.9\ \frac{\mathrm{rad}}{\mathrm{s}}$$

Problem 10.24 illustrates (a) the use of the frequency equation derived in Example 10.2 for the natural frequencies of a shaft fixed at one end and a disk at its other end and (b) the determination of the natural frequencies of longitudinal motion of a shaft fixed at one end and an attached mass at its other end.

10.25 A pipe used to convey fluid is cantilevered from a wall. The steel pipe (ρ = 7500 kg/m^3, $G = 80 \times 10^9$ N/m^2, $E = 200 \times 10^9$ N/m^2) has an inner radius of 20 cm, a thickness of 1 cm, and a length of 4.6 m. For an empty pipe determine (a) The five lowest natural frequencies for torsional oscillation, (b) the five lowest frequencies for longitudinal vibration, and (c) the five lowest natural frequencies for transverse vibration.

Given: L = 4.6 m, $r_i = 20$ cm, t = 1 cm, $\rho = 7500\frac{\mathrm{kg}}{\mathrm{m}^3}, E = 200 \times 10^9\frac{\mathrm{N}}{\mathrm{m}^2}, G = 80 \times 10^9\ \mathrm{N/m^2}$

Find: $\omega_1, \ldots, \omega_5$ for torsional oscillation, longitudinal vibration, transverse motion

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Solution: (a) For torsional oscillation the pipe is modeled as a fixed-free shaft. The nondimensional natural frequencies of a fixed-free shaft are calculated in the text as

$$\omega_k^* = (2k-1)\frac{\pi}{2}$$

The dimensional natural frequencies are calculated by

$$\omega_k = \omega_k^* \sqrt{\frac{G}{\rho L^2}}$$

where

$$\frac{1}{L}\sqrt{\frac{G}{\rho}} = \frac{1}{4.6\text{ m}}\sqrt{\frac{80\times 10^9\text{ N/m}^2}{7500\,\frac{\text{kg}}{\text{m}^3}}} = 710\text{ rad/s}$$

Hence the five lowest natural frequencies for torsional oscillation are

$$\omega_1 = 1115\,\frac{\text{rad}}{\text{s}},\qquad \omega_2 = 3346\,\frac{\text{rad}}{\text{s}},\qquad \omega_3 = 55765\,\frac{\text{rad}}{\text{s}},$$

$$\omega_4 = 10996\,\frac{\text{rad}}{\text{s}},\qquad \omega_5 = 14137\,\frac{\text{rad}}{\text{s}}$$

(b) For longitudinal motion the pipe is modeled as a fixed-free bar. It is shown in the text that the nondimensional natural frequencies for a fixed-free bar are

$$\omega_k^* = (2k-1)\frac{\pi}{2}$$

The dimensional natural frequencies are calculated by

$$\omega_k = \omega_k^* \sqrt{\frac{E}{\rho L^2}}$$

where

$$\frac{1}{L}\sqrt{\frac{E}{\rho}} = \frac{1}{4.6\text{ m}}\sqrt{\frac{200\times 10^9\text{ N/m}^2}{7500\,\frac{\text{kg}}{\text{m}^3}}} = 1123\text{ rad/s}$$

Hence the five lowest natural frequencies for torsional oscillation are

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$$\omega_1 = 1764\ \frac{\text{rad}}{\text{s}}, \qquad \omega_2 = 5292\ \frac{\text{rad}}{\text{s}}, \qquad \omega_3 = 8820\ \frac{\text{rad}}{\text{s}},$$

$$\omega_4 = 12350\ \frac{\text{rad}}{\text{s}}, \qquad \omega_5 = 158767\ \frac{\text{rad}}{\text{s}}$$

(c) For transverse motion the pipe is modeled by a fixed-free beam. From Table 10.4 the first five nondimensional frequencies for a fixed-free beam are

$$\omega_1^* = 3.516, \qquad \omega_1^* = 22.03, \qquad \omega_1^* = 61.70, \qquad \omega_1^* = 120.9, \qquad \omega_1^* = 199.9$$

The dimensional natural frequencies are calculated from

$$\omega_k = \omega_k^* \sqrt{\frac{EI}{\rho A L^4}}$$

where

$$A = \pi[(0.21\ \text{m})^2 - (0.20\ \text{m})^2] = 1.29 \times 10^{-2}\ \text{m}^2$$

$$I = \frac{\pi}{4}[(0.21\ \text{m})^4 - (0.20\ \text{m})^2] = 2.71 \times 10^{-4}\ \text{m}^4$$

$$\sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{\left(200 \times 10^9 \frac{\text{N}}{\text{m}^2}\right)(2.71 \times 10^{-4}\ \text{m}^4)}{\left(7500 \frac{\text{kg}}{\text{m}^3}\right)(1.29 \times 10^{-2}\ \text{m}^2)(4.6\ \text{m})^4}} = 35.4\ \frac{\text{rad}}{\text{s}}$$

Thus the five lowest natural frequencies of transverse vibration are

$$\omega_1 = 124.5\ \frac{\text{rad}}{\text{s}}, \quad \omega_2 = 779.8\ \frac{\text{rad}}{\text{s}}, \qquad \omega_3 = 2184\ \frac{\text{rad}}{\text{s}},$$

$$\omega_4 = 4280\ \frac{\text{rad}}{\text{s}}, \qquad \omega_5 = 70765\ \frac{\text{rad}}{\text{s}}$$

Problem 10.25 illustrates five modes of free vibration for a cantilevered pipe and their natural frequencies.

10.26 Verify the characteristic equation given in Table 10.4 for a pinned-free beam.

Given: pinned-free beam

Show: $\tan \lambda^{1/4} = \tanh \lambda^{1/4}$

Solution: The general form of the mode shape for a beam is

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$$y(x) = A\cos\lambda^{1/4}x + B\sin\lambda^{1/4}x + C\cosh\lambda^{1/4}x + D\sinh\lambda^{1/4}x$$

The boundary conditions for a pinned-free beam are

$$y(0) = 0 \quad y''(0) = 0 \quad y''(1) = 0 \quad y'''(1) = 0$$

Application of the boundary conditions to the mode shape yields

$$y(0) = 0 = A + C$$

$$y''(0) = 0 = -\lambda^{1/2}A + \lambda^{1/2}C$$

The preceding equations imply that A=C=0. Then application of the boundary conditions at x = 1 leads to

$$y''(1) = 0 = -\lambda^{1/2}B\sin\lambda^{1/4} + \lambda^{1/2}D\sinh\lambda^{1/4}$$

$$y'''(1) = 0 = -\lambda^{3/4}B\cos\lambda^{1/4} + \lambda^{3/4}D\sinh\lambda^{1/4}$$

Solving for B in terms of D leads to

$$B = \frac{\sinh\lambda^{1/4}}{\sin\lambda^{1/4}}D$$

which when substituted into the last equation becomes

$$0 = -\frac{\sinh\lambda^{1/4}}{\sin\lambda^{1/4}}\cos\lambda^{1/4} + \cosh\lambda^{1/4}$$

The above is rearranged to

$$\tan\lambda^{1/4} = \tanh\lambda^{1/4}$$

Problem 10.26 illustrates the derivation of the characteristic equation for a pinned-free beam.

10.27 Verify the characteristic equation given in Table 10.4 for a fixed-fixed beam.

Given: fixed-fixed beam

Show: $\cos\lambda^{1/4}\cosh\lambda^{1/4} = 1$

Solution: The general form of the mode shape for a beam is

$$y(x) = A\cos\lambda^{1/4}x + B\sin\lambda^{1/4}x + C\cosh\lambda^{1/4}x + D\sinh\lambda^{1/4}x$$

The boundary conditions for a pinned-free beam are

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$$y(0) = 0 \quad y'(0) = 0 \quad y(1) = 0 \quad y'(1) = 0$$

Application of the boundary conditions to the mode shape yields

$$y(0) = 0 = A + C$$

$$y'(0) = 0 = \lambda^{1/4}B + \lambda^{1/4}D$$

Application of the boundary conditions at x=1 leads to

$$y(1) = 0 = A\cos\lambda^{1/4} + B\sin\lambda^{1/4} + C\cosh\lambda^{1/4} + D\sinh\lambda^{1/4}$$

$$y'(1) = 0 = -\lambda^{1/4}\sin\lambda^{1/4} + \lambda^{1/4}\cos\lambda^{1/4} + C\,\lambda^{1/4}\sinh\lambda^{1/4} + D\cosh\lambda^{1/4}$$

Solving the first two equations for C = –A and D = –B and substituting in the last two equations leads to

$$0 = A\left(\cos\lambda^{\frac{1}{4}} - \cosh\lambda^{\frac{1}{4}}\right) + C(\sin\lambda^{\frac{1}{4}} - \sinh\lambda^{\frac{1}{4}})$$

$$0 = -A\left(\sin\lambda^{\frac{1}{4}} + \sinh\lambda^{\frac{1}{4}}\right) + C(\cos\lambda^{\frac{1}{4}} - \cosh\lambda^{\frac{1}{4}})$$

These two equations have a non-trivial solution only if the determinant of their coefficient matrix is zero

$$\begin{vmatrix} \cos\lambda^{\frac{1}{4}} - \cosh\lambda^{\frac{1}{4}} & \sin\lambda^{\frac{1}{4}} - \sinh\lambda^{\frac{1}{4}} \\ -\left(\sin\lambda^{\frac{1}{4}} + \sinh\lambda^{\frac{1}{4}}\right) & \cos\lambda^{\frac{1}{4}} - \cosh\lambda^{\frac{1}{4}} \end{vmatrix} = 0$$

or

$$\cos^2\lambda^{\frac{1}{4}} - 2\cos\lambda^{\frac{1}{4}}\cosh\lambda^{\frac{1}{4}} + \cosh^2\lambda^{\frac{1}{4}} + \sin^2\lambda^{\frac{1}{4}} - \sinh^2\lambda^{\frac{1}{4}} = 0$$

Application of trigonometric and hyperbolic trigonometric identities to the above leads to

$$\cos\lambda^{\frac{1}{4}}\cosh\lambda^{\frac{1}{4}} = 1$$

Problem 10.27 illustrates the development of the characteristic equation for a fixed-fixed beam.

10.28 Verify the orthogonality of the eigenfunctions given in Table 10.4 for a pinned-free beam.

Given: pinned-free beam

Show: orthogonality of eignefunctions

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Solution: The eigenfunctions given in Table 10.4 for a pinned-free beam are

$$X_k(x) = C_k\left(\sin\lambda_k^{1/4}x + \frac{\sin\lambda_k^{1/4}}{\sinh\lambda_k^{1/4}}\sinh\lambda_k^{1/4}x\right)$$

where

$$\tan\lambda_k^{1/4} = \tanh\lambda_k^{1/4}$$

The mode orthogonality condition is

$$(X_j, X_k) = \int_0^1 X_j(x)X_k(x)dx = 0$$

or

$$\int_0^1 C_j\,C_k\left(\sin\lambda_j^{1/4}x + \frac{\sin\lambda_j^{1/4}}{\sinh\lambda_j^{1/4}}\sinh\lambda_j^{1/4}x\right)\left(\sin\lambda_k^{1/4}x + \frac{\sin\lambda_k^{1/4}}{\sinh\lambda_k^{1/4}}\sinh\lambda_k^{1/4}x\right)dx$$

$$= \frac{\sin\left(\lambda_j^{1/4} - \lambda_k^{1/4}\right)}{2\left(\lambda_j^{1/4} - \lambda_k^{1/4}\right)} - \frac{\sin\left(\lambda_j^{1/4} + \lambda_k^{1/4}\right)}{2\left(\lambda_j^{1/4} + \lambda_k^{1/4}\right)}$$
$$+ \frac{\sin\lambda_j^{1/4}}{\left(\lambda_j^{1/2} + \lambda_k^{1/2}\right)}\left(\lambda_j^{1/4}\sin\lambda_k^{1/4}\cosh\lambda_j^{1/4} - \lambda_k^{1/4}\cos\lambda_k^{1/4}\sinh\lambda_j^{1/4}\right)$$
$$+ \frac{\sin\lambda_k^{1/4}}{\left(\lambda_j^{1/2} + \lambda_k^{1/2}\right)}\left(\lambda_k^{1/4}\sin\lambda_j^{1/4}\cosh\lambda_k^{1/4} - \lambda_j^{1/4}\cosh\lambda_j^{1/4}\sinh\lambda_k^{1/4}\right)$$
$$+ \frac{\sin\lambda_j^{1/4}\sin\lambda_k^{1/4}}{\sinh\lambda_j^{1/4}\sinh\lambda_k^{1/4}}\left[\frac{\sinh\left(\lambda_j^{1/4} + \lambda_k^{1/4}\right)}{2\left(\lambda_j^{1/4} + \lambda_k^{1/4}\right)} - \frac{\sinh\left(\lambda_j^{1/4} - \lambda_k^{1/4}\right)}{2\left(\lambda_j^{1/4} - \lambda_k^{1/4}\right)}\right]$$

The terms from the second and third lines of the above expression can be shown to be zero by use of the characteristic equation as can the terms from the first and fourth lines of the above expression. Thus the orthogonality condition is satisfied.

Problem 10.28 illustrates mode shape orthoghonality for a pinned-free beam.

10.29 Verify the orthogonality of the eigenfunctions given in Table 10.4 for a fixed-attached mass beam.

Given: fixed-attached mass beam

Show: mode shape orthogonality

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Solution: From Table 10.4 the mode shapes for a fixed-attached mass beam are

$$X_k(x) = C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x + \alpha_k\left(\sinh\lambda_k^{1/4}x - \sin\lambda_k^{1/4}x\right)\right]$$

where

$$\alpha_k = \frac{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} + \sinh\lambda_k^{1/4}}$$

The characteristic equation for the separation constants is

$$\lambda_k^{1/4}\left(\cos\lambda_k^{1/4}\cosh\lambda_k^{1/4} + 1\right) + \beta\left(\cos\lambda_k^{1/4}\sinh\lambda_k^{1/4} - \cosh\lambda_k^{1/4}\sin\lambda_k^{1/4}\right) = 0$$

where

$$\beta = \frac{m}{\rho AL}$$

The appropriate scalar product to test for orthogonality is

$$\left(X_j, X_k\right) = \int_0^1 X_j(x)X_k(x)dx + \beta X_j(1)X_k(1)$$

Let λ_j and λ_k be distinct values of the separation constant for a given value of β. Let $X_j(x)$ and $X_k(x)$ be their corresponding mode shapes. Showing orthogonality is equivalent to showing $\left(X_j, X_k\right) = 0$ or

$$\int_0^1 X_j(x)X_k(x)dx + \beta X_j(1)X_k(1) = 0$$

Details of the ensuing algebra are too voluminous to include in this solution. The integration defined by the scalar product is performed. Trigonometric identities may be used to help perform the integration and reduce the resulting expression. It is noted that both λ_j and λ_k satisfy the characteristic equation. Thus this expression is used to replace the terms involving β. After much, much, much algebra the mode shapes are proven to be orthogonal.

Problem 10.29 illustrates mode shape orthogonality for a beam with an attached mass where the appropriate scalar product includes a boundary term.

10.30 Determine the time-dependent displacement of the beam shown in Figure P10.30.

FIGURE P10.30

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Given: F_0, ω, ρ, E, I, A, L

Find: $w(x,t)$

Solution: The nondimensional partial differential equation governing the vibrations of the beam is

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \phi \sin \omega t$$

where

$$\phi = \frac{F_0 L^3}{EI} \qquad \omega = \omega \sqrt{\frac{\rho A L^4}{EI}}$$

The boundary conditions for the fixed-free beam are

$$w(0,t) = 0 \qquad \frac{\partial w}{\partial x}(0,t) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(1,t) = 0 \qquad \frac{\partial^3 w}{\partial x^3}(1,t) = 0$$

Since the excitation is harmonic and independent of x, a steady-state solution of the form

$$w(x,t) = u(x) \sin \omega t$$

is assumed. Substitution into the governing partial differential equation and boundary conditions leads to

$$\frac{d^4 u}{dx^4} - \omega^2 u = \phi$$

subject to

$$u(0) = 0 \qquad u'(0) = 0 \qquad u''(1) = 0 \qquad u'''(1) = 0$$

The general solution of the ordinary differential equation is

$$u(x) = A \cos \sqrt{\omega}\, x + B \sin \sqrt{\omega} x + C \cosh \sqrt{\omega} x + D \sinh \sqrt{\omega} x - \frac{\phi}{\omega^2}$$

Application of the boundary conditions leads to

$$A + C = \frac{\phi}{\omega^2}$$

$$B + D = 0$$

$$-\cos \sqrt{\omega}\, A - \sin \sqrt{\omega}\ B + \cosh \sqrt{\omega}\, C + \sinh \sqrt{\omega}\ D = 0$$

$$\sin \sqrt{\omega}\, A - \cos \sqrt{\omega}\ B + \sinh \sqrt{\omega}\, C + \cosh \sqrt{\omega}\ D = 0$$

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Simultaneous solution of the equations for the constants of integration leads to

$$A = \frac{\phi}{\omega^2}\left[\frac{1+\cos\sqrt{\omega}\cosh\sqrt{\omega}-\sin\sqrt{\omega}\sinh\sqrt{\omega}}{2(1+\cos\sqrt{\omega}\cosh\sqrt{\omega})}\right]$$

$$B = \frac{\phi}{\omega^2}\left[\frac{\cos\sqrt{\omega}\sinh\sqrt{\omega}+\sin\sqrt{\omega}\cosh\sqrt{\omega}}{2(1+\cos\sqrt{\omega}\cosh\sqrt{\omega})}\right]$$

$$C = \frac{\phi}{\omega^2}\left[\frac{1+\cos\sqrt{\omega}\cosh\sqrt{\omega}+\sin\sqrt{\omega}\sinh\sqrt{\omega}}{2(1+\cos\sqrt{\omega}\cosh\sqrt{\omega})}\right]$$

$$D = -\frac{\phi}{\omega^2}\left[\frac{\cos\sqrt{\omega}\sinh\sqrt{\omega}+\sin\sqrt{\omega}\cosh\sqrt{\omega}}{2(1+\cos\sqrt{\omega}\cosh\sqrt{\omega})}\right]$$

Problem 10.30 illustrates the determination of the steady-state response of a fixed-free beam subject to harmonic excitation.

10.31 Determine the time-dependent displacement for the beam shown in Figure P10.31.

FIGURE P10.31

Given: F_0, ω, ρ, E, I, A, L

Find: $w(x,t)$

Solution: The nondimensional partial differential equation governing the vibrations of the beam is

$$\frac{\partial^4 w}{\partial x^4}+\frac{\partial^2 w}{\partial t^2} = \phi\sin\omega t\left[u\left(x-\frac{1}{2}\right)-u(x-1)\right]$$

where

$$\phi = \frac{F_0 L^3}{EI} \qquad \omega = \omega\sqrt{\frac{\rho A L^4}{EI}}$$

The boundary conditions for the fixed-free beam are

$$w(0,t)=0 \qquad \frac{\partial w}{\partial x}(0,t)=0 \qquad \frac{\partial^2 w}{\partial x^2}(1,t)=0 \qquad \frac{\partial^3 w}{\partial x^3}(1,t)=0$$

Let $X_k(x)$ be the kth normalized mode shape for a fixed-free beam as specified in Table 10.4

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$$X_k(x) = C_k\left[\cosh\lambda_k^{1/4}x - \cos\lambda_k^{1/4}x - \alpha_k\left(\sinh\lambda_k^{1/4}x - \sin\lambda_k^{1/4}x\right)\right]$$

where

$$\alpha_k = \frac{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} + \sinh\lambda_k^{1/4}}$$

$$\cos\lambda_k^{1/4}\cos\lambda_k^{1/4} = -1$$

and C_k is chosen such that $\int_0^1 X_k^2(x)dx = 1$. Using the expansion theorem assume,

$$w(x,t) = \sum_{k=1}^{\infty} p_k(t)X_k(x)$$

Substituting the expansion into the governing partial differential equation, taking the scalar product with $x_j(x)$ for an arbitrary j and using mode shape orthogonality leads to

$$\ddot{p}_k + \omega_k^2 p_k = \beta_k \sin\omega t$$

where

$$\beta_k = \int_0^1 X_k(x)\phi\left[u\left(x-\frac{1}{2}\right) - u(x-1)\right]dx$$
$$= \phi\frac{C_k}{\lambda_k^{1/4}}\left[\sinh\lambda_k^{1/4} - \sin\lambda_k^{1/4} - \alpha_k\left(\cosh\lambda_k^{1/4} + \cos\lambda_k^{1/4}\right) - \sinh\frac{\lambda_k^{1/4}}{2}\right.$$
$$\left. + \sin\frac{\lambda_k^{1/4}}{2} + \alpha_k\left(\cosh\frac{\lambda_k^{1/4}}{2} + \cos\frac{\lambda_k^{1/4}}{2}\right)\right]$$

The solution of the ordinary differential equation subject to $p_k(0) = 0$ and $\dot{p}_k(0) = 0$ is

$$p_k(t) = \frac{\beta_k}{\lambda_k - \omega^2}\left(\sin\omega t - \frac{\omega}{\lambda_k^{1/2}}\sin\lambda_k^{1/2}t\right)$$

Problem 10.31 illustrates application of modal analysis to determine the forced response of a beam subject to a load applied only over a portion of the beam.

10.32 Determine the time-dependent displacement for the beam shown in Figure P10.32.

Given: F_0, ω, ρ, E, I, A, L

FIGURE P10.32

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Find: $w(x,t)$

Solution: The nondimensional partial differential equation governing the vibrations of the beam is

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \phi e^{-\alpha t}\delta\left(x - \frac{1}{2}\right)$$

where

$$\phi = \frac{F_0 L^3}{EI}$$

The boundary conditions for the fixed-free beam are

$$w(0,t) = 0 \qquad \frac{\partial w}{\partial x}(0,t) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(1,t) = 0 \qquad \frac{\partial^3 w}{\partial x^3}(1,t) = 0$$

Let $X_k(x)$ be the kth normalized mode shape for a fixed-free beam as specified in Table 10.4

$$X_k(x) = C_k\left[\cosh \lambda_k^{1/4} x - \cos \lambda_k^{1/4} x - \alpha_k\left(\sinh \lambda_k^{1/4} x - \sin \lambda_k^{1/4} x\right)\right]$$

where

$$\alpha_k = \frac{\cos \lambda_k^{1/4} + \cosh \lambda_k^{1/4}}{\sin \lambda_k^{1/4} + \sinh \lambda_k^{1/4}}$$

$$\cos \lambda_k^{1/4} \cos \lambda_k^{1/4} = -1$$

and $C_k$is chosen such that $\int_0^1 X_k^2(x)dx = 1$. Using the expansion theorem assume,

$$w(x,t) = \sum_{k=1}^{\infty} p_k(t) X_k(x)$$

Substituting the expansion into the governing partial differential equation, taking the scalar product with $x_j(x)$ for an arbitrary j and using mode shape orthogonality leads to

$$\ddot{p}_k + \omega_k^2 p_k = \beta_k e^{-\alpha t}$$

where

$$\beta_k = \int_0^1 X_k(x)\phi e^{-\alpha t}\delta\left(x - \frac{1}{2}\right)dx = \phi C_k\left[\cosh\frac{\lambda_k^{1/4}}{2} - \cos\frac{\lambda_k^{1/4}}{2} - \alpha_k\left(\sinh\frac{\lambda_k^{1/4}}{2} - \sin\frac{\lambda_k^{1/4}}{2}\right)\right]$$

The solution of the ordinary differential equation subject to $p_k(0) = 0$ and $\dot{p}_k(0) = 0$ is

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$$p_k(t) = \frac{\beta_k}{\lambda_k - \omega^2}\left(-\cos\lambda_k^{1/2}t + \frac{\alpha}{\lambda_k^{1/2}}\sin\lambda_k^{1/2}t + e^{-\alpha t}\right)$$

Problem 10.32 illustrates application of modal analysis to determine the forced response of a beam subject to a load applied only over a portion of the beam.

10.33 Determine the time-dependent displacement for the beam shown in Figure P10.33.

FIGURE P10.33

Given: F_0, ω, ρ, E, I, A, L, k

Find: $w(x,t)$

Solution: The nondimensional partial differential equation governing the vibrations of the beam is

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \phi\delta\left(x - \frac{1}{4}\right)\sin\omega t$$

where

$$\phi = \frac{F_0 L^3}{EI} \qquad \omega = \omega\sqrt{\frac{\rho A L^4}{EI}}$$

The boundary conditions for the fixed-free beam are

$$w(0,t) = 0 \qquad \frac{\partial w}{\partial x}(0,t) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(1,t) = 0 \qquad \frac{\partial^3 w}{\partial x^3}(1,t) + \beta w(1.t) = 0$$

where

$$\beta = \frac{kL^3}{EI}$$

Let $X_k(x)$ be the kth normalized mode shape for a fixed-attached spring beam as specified in Table 10.4

$$X_k(x) = C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x - \alpha_k\left(\sin\lambda_k^{1/4}x - \sinh\lambda_k^{1/4}x\right)\right]$$

where

$$\alpha_k = \frac{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} + \sinh\lambda_k^{1/4}}$$

$$\lambda_k^{3/4}\left(\cos\lambda_k^{1/4}\cos\lambda_k^{1/4} + 1\right) - \beta\left(\cos\lambda_k^{1/4}\sinh\lambda_k^{1/4} - \cosh\lambda_k^{1/4}\sin\lambda_k^{1/4}\right)$$

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and C_k is chosen such that $\int_0^1 X_k^2(x)dx = 1$. Using the expansion theorem assume,

$$w(x,t) = \sum_{k=1}^{\infty} p_k(t)X_k(x)$$

Substituting the expansion into the governing partial differential equation, taking the scalar product with $x_j(x)$ for an arbitrary j and using mode shape orthogonality leads to

$$\ddot{p}_k + \omega_k^2 p_k = \beta_k \sin \omega t$$

where

$$\beta_k = \int_0^1 X_k(x)\phi\delta\left(x - \frac{1}{4}\right)dx = X_j\left(\frac{1}{4}\right)$$

The solution of the ordinary differential equation subject to $p_k(0) = 0$ and $\dot{p}_k(0) = 0$ is

$$p_k(t) = \frac{\beta_k}{2\lambda_k}\left(\frac{1}{\frac{\omega}{\lambda_k^{1/2}} - 1}\left(\sin \omega t - \sin \lambda_k^{1/2} t\right) - \frac{1}{\frac{\omega}{\lambda_k^{1/2}} + 1}\left(\sin \omega t - \sin \lambda_k^{1/2} t\right)\right)$$

Problem 10.33 illustrates application of modal analysis to determine the forced response of a beam subject to a load applied only over a portion of the beam.

10.34 Determine the time-dependent displacement for the beam shown in Figure P10.34.

FIGURE P10.34

Given: β = 0.35, ω* = 1.2, F(x,t)

Find: w(x,t)

Solution: Since the excitation is harmonic, a program called CMODA.BAS can be used to determine the system response. CMODA.BAS is used with

$$\beta = 0.35, \quad \omega = 1.2$$

and all other input values set to 1. The required BASIC subroutine is written such that the magnitude of the excitation force is also 1. The nondimensional excitation is written as

$$F(x,t) = \left[u\left(x - \frac{1}{3}L\right) - u(x - \frac{2}{3}L)\right]\sin\omega t$$

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CMODA.BAS calculates the natural frequencies for the number of modes requested (5), and performs the modal analysis calculations. The result is the steady-state mode shape. The BASIC subroutine for the excitation, the natural frequency summary, numerical values for the mode shape, and a plot of the mode shape follow.

```
10000 F = 0
10010 IF X > .3333 AND X < .6667 THEN F = 1
10020 RETURN
```

Natural frequencies and mode shapes for a fixed-attached mass beam
with beta = 0.350

BEAM PROPERTIES
mass density = 1.000E+00 kg/m^3
elastic modulus = 1.000E+00 N/m^2
length = 1.000E+00 m
area = 1.000E+00 m^2
moment of inertia= 1.000E+00 m^4

Mode Number	Dimensionless Frequency	Natural frequency (rad/sec)	Normalization constant
1	2.88	2.88	0.752E+00
2	21.36	21.36	0.682E+00
3	61.02	61.02	0.665E+00
4	120.22	120.22	0.659E+00
5	199.17	199.17	0.658E+00

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x (m)	w (m)
0.000	0.000E+00
0.005	1.334E-06
0.010	5.323E-06
0.015	1.194E-05
0.020	2.117E-05
0.025	3.298E-05
0.030	4.735E-05
0.035	6.426E-05
0.040	8.369E-05
0.045	1.056E-04
0.050	1.300E-04
0.055	1.568E-04
0.060	1.861E-04
0.065	2.178E-04
0.070	2.518E-04
0.075	2.882E-04
0.080	3.269E-04
0.085	3.680E-04
0.090	4.113E-04
0.095	4.569E-04
0.100	5.047E-04
0.105	5.547E-04
0.110	6.070E-04
0.115	6.614E-04
0.120	7.179E-04
0.125	7.766E-04
0.130	8.374E-04
0.135	9.002E-04
0.140	9.651E-04
0.145	1.032E-03
0.150	1.101E-03
0.155	1.172E-03
0.160	1.245E-03
0.165	1.320E-03
0.170	1.396E-03
0.175	1.475E-03
0.180	1.555E-03
0.185	1.638E-03
0.190	1.722E-03
0.195	1.808E-03
0.200	1.896E-03
0.205	1.985E-03
0.210	2.076E-03
0.215	2.169E-03
0.220	2.264E-03
0.225	2.360E-03
0.230	2.458E-03
0.235	2.557E-03
0.240	2.658E-03
0.245	2.761E-03
0.250	2.865E-03
0.255	2.971E-03
0.260	3.078E-03
0.265	3.186E-03
0.270	3.296E-03
0.275	3.408E-03
0.280	3.521E-03
0.285	3.635E-03
0.290	3.750E-03
0.295	3.867E-03
0.300	3.985E-03
0.305	4.105E-03
0.310	4.226E-03
0.315	4.348E-03
0.320	4.471E-03
0.325	4.595E-03
0.330	4.721E-03
0.335	4.847E-03
0.340	4.975E-03
0.345	5.104E-03
0.350	5.234E-03
0.355	5.365E-03
0.360	5.497E-03
0.365	5.630E-03
0.370	5.764E-03
0.375	5.899E-03
0.380	6.035E-03
0.385	6.171E-03
0.390	6.309E-03
0.395	6.448E-03
0.400	6.587E-03
0.405	6.728E-03
0.410	6.869E-03
0.415	7.011E-03
0.420	7.154E-03
0.425	7.297E-03
0.430	7.442E-03
0.435	7.587E-03
0.440	7.732E-03
0.445	7.879E-03
0.450	8.026E-03
0.455	8.174E-03
0.460	8.322E-03
0.465	8.471E-03
0.470	8.621E-03
0.475	8.771E-03
0.480	8.922E-03
0.485	9.074E-03
0.490	9.226E-03
0.495	9.378E-03
0.500	9.531E-03
0.505	9.685E-03
0.510	9.839E-03
0.515	9.993E-03
0.520	1.015E-02
0.525	1.030E-02
0.530	1.046E-02
0.535	1.062E-02
0.540	1.077E-02
0.545	1.093E-02
0.550	1.109E-02
0.555	1.125E-02
0.560	1.140E-02
0.565	1.156E-02
0.570	1.172E-02
0.575	1.188E-02
0.580	1.204E-02
0.585	1.220E-02
0.590	1.236E-02
0.595	1.252E-02
0.600	1.268E-02
0.605	1.285E-02
0.610	1.301E-02
0.615	1.317E-02
0.620	1.333E-02
0.625	1.349E-02
0.630	1.366E-02
0.635	1.382E-02
0.640	1.398E-02
0.645	1.415E-02
0.650	1.431E-02
0.655	1.448E-02
0.660	1.464E-02
0.665	1.480E-02
0.670	1.497E-02
0.675	1.513E-02
0.680	1.530E-02
0.685	1.546E-02
0.690	1.563E-02
0.695	1.580E-02
0.700	1.596E-02
0.705	1.613E-02
0.710	1.629E-02
0.715	1.646E-02
0.720	1.663E-02
0.725	1.679E-02
0.730	1.696E-02
0.735	1.713E-02
0.740	1.730E-02
0.745	1.746E-02
0.750	1.763E-02
0.755	1.780E-02
0.760	1.797E-02
0.765	1.814E-02
0.770	1.830E-02
0.775	1.847E-02
0.780	1.864E-02
0.785	1.881E-02
0.790	1.898E-02
0.795	1.915E-02
0.800	1.932E-02
0.805	1.949E-02
0.810	1.966E-02
0.815	1.982E-02
0.820	1.999E-02
0.825	2.016E-02
0.830	2.033E-02
0.835	2.050E-02
0.840	2.067E-02
0.845	2.084E-02
0.850	2.101E-02
0.855	2.118E-02
0.860	2.136E-02
0.865	2.153E-02
0.870	2.170E-02
0.875	2.187E-02
0.880	2.204E-02
0.885	2.221E-02
0.890	2.238E-02
0.895	2.255E-02
0.900	2.272E-02
0.905	2.289E-02
0.910	2.306E-02
0.915	2.323E-02
0.920	2.341E-02
0.925	2.358E-02
0.930	2.375E-02
0.935	2.392E-02
0.940	2.409E-02
0.945	2.426E-02
0.950	2.443E-02
0.955	2.461E-02
0.960	2.478E-02
0.965	2.495E-02
0.970	2.512E-02
0.975	2.529E-02
0.980	2.546E-02
0.985	2.564E-02
0.990	2.581E-02
0.995	2.598E-02
1.000	2.615E-02

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10.35 A root manipulator is 60 cm long, made of steel (E = 210 × 10^9 N/m^2, ρ = 7500 kg/m^3) and has the cross section of Figure P10.35. One end of the manipulator is fixed and a 1-kg mass is attached to its opposite end. Determine the three lowest natural frequencies for transverse vibration of the manipulator.

FIGURE P10.35

Given: L = 60 cm, E = 210 × 10^9 N/m^2, $\rho = 7500$ kg/m^3, m = 1 kg

Find: $\omega_1, \omega_2, \omega_3$

Solution: The geometric properties of the robot's cross section are

$$A = (0.01\text{ m})(0.015\text{ m}) - (0.008\text{ m})(0.013\text{ m}) = 4.6 \times 10^{-5}\text{ m}^2$$

$$I = \frac{1}{12}[(0.01\text{ m})(0.015)^3 - (0.008\text{ m})(0.013\text{ m})^3] = 1.384 \times 10^{-9}\text{m}^4$$

The mass ratio is defined by

$$\beta = \frac{m}{\rho AL} = \frac{1\text{ kg}}{(7500\text{ kg/m}^3)(4.6 \times 10^{-5}\text{ m}^2)(0.6\text{ m})} = 4.83$$

The characteristic equation for a fixed-attached mass beam is

$$\lambda_k^{1/4}\left(\cos\lambda_k^{1/4}\cosh\lambda_k^{1/4} + 1\right) + 4.83\left(\cos\lambda_k^{1/4}\sinh\lambda_k^{1/4} - \cosh\lambda_k^{1/4}\sin\lambda_k^{1/4}\right) = 0$$

where $\lambda = \omega^2$. The three lowest nondimensional natural frequencies are

$$\omega_1 = 0.60\,, \omega_2 = 17.7, \omega_3 = 55.96$$

The dimensional natural frequencies are the nondimensional natural frequencies times $\sqrt{\frac{EI}{\rho AL^4}}$ which leads to

$$\omega_1 = 48.0\frac{\text{rad}}{\text{s}}, \omega_2 = 1408.3\frac{\text{rad}}{\text{s}}, \omega_3 = 4452\ \frac{\text{rad}}{\text{s}}$$

Problem 10.35 illustrates the frequency equation for a fixed-attached mass beam.

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10.36 The steam pipe of Figure P10.36 is suspended from the ceiling in an industrial plant. A heavy machine with a rotating unbalance is placed on the floor above the machine causing vibrations of the ceiling. If the frequency of the oscillations is 150 Hz and the amplitude of displacement of the pipe's left support is 0.5 mm and the amplitude of displacement of the pipe's right support is 0.8 mm, determine the maximum deflection of the center of the pipe. The pipe is modeled as a simply supported beam of length 5 m and has the cross section shown in Figure P10.36.

FIGURE P10.36

Given: $E = 210 \times 10^9$ N/m^2, $\rho = 7500$ kg/m^3, $r_0 = 10$ cm, $t = 1$ cm, $L = 5$ m, $x_l = 0.5$ mm, $x_r = 0.8$ mm, $\omega = 150$ Hz

Find: $y_{max}(L/2)$

Solution: The geometric properties of the pipe are

$$A = \pi(r_o^2 - r_i^2) = \pi[(0.1\ m)^2 - (0.09\ m)^2] = 5.97 \times 10^{-3}\ m^2$$

$$I = \frac{\pi}{4}(r_o^4 - r_i^4) = \frac{\pi}{4}[(0.1\ m)^2 - (0.09\ m)^2] = 2.7 \times 10^{-5}\ m^4$$

The nondimensional excitation frequency is

$$\tilde{\omega} = \omega\sqrt{\frac{\rho A L^4}{EI}} = \left(150\ \frac{cycles}{sec}\right)\left(2\pi\ \frac{rad}{cycle}\right)\sqrt{\frac{\left(7500\ \frac{kg}{m^3}\right)(5.97 \times 10^{-3} m^2)(5\ m)^4}{\left(210 \times 10^9\ \frac{N}{m^2}\right)(2.7 \times 10^{-5}\ m^4)}} = 68.33$$

The nondimensional partial differential equation governing the motion of the pipe is

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} = 0 \qquad (1)$$

The boundary conditions are for a pinned-pinned beam executing the harmonic motion described are

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$$y(0,t) = Y_1 \sin\tilde{\omega}t, \quad \frac{\partial^2 y}{\partial x^2}(0,t) = 0$$

$$y(1,t) = Y_2 \sin\tilde{\omega}t, \quad \frac{\partial^2 y}{\partial x^2}(1,t) = 0$$

For the remainder of the problem the ~ will be dropped from the nondimensional natural frequency. Since the excitation is harmonic, the steady-state vibrations of the pipe are assumed as

$$y(x,t) = u(x)\sin\omega t \tag{3}$$

Substitution of eq.(3) into eqs.(1) and (2) leads to the following problem for u(x)

$$\frac{d^4u}{dx^4} - \omega^2 u = 0 \tag{4}$$

subject to

$$u(0) = Y_1, \quad \frac{d^2u}{dx^2}(0) = 0 \tag{5}$$

$$u(1) = Y_2, \quad \frac{d^2u}{dx^2}(1) = 0$$

The general solution of eq,(4) is

$$u(x) = C_1\cos\sqrt{\omega}x + C_2\sin\sqrt{\omega}x + C_3\cosh\sqrt{\omega}x + C_4\sinh\sqrt{\omega}x \tag{6}$$

Application of eqs.(5) to eq.(6) leads to

$$C_1 + C_3 = Y_1$$

$$-C_1 + C_3 = 0$$

$$C_1 cos\sqrt{\omega} + C_2 sin\sqrt{\omega} + C_3 cosh\sqrt{\omega} + C_4 sinh\sqrt{\omega} = Y_2 \tag{7}$$

$$-C_1 cos\sqrt{\omega} - C_2 sin\sqrt{\omega} + C_3 cosh\sqrt{\omega} + C_4 sinh\sqrt{\omega} = 0$$

Equations (7) are solved simultaneously yielding

$$C_1 = \frac{Y_1}{2}$$

$$C_2 = \frac{1}{2sin\sqrt{\omega}}\left(Y_2 - Y_1 cos\sqrt{\omega}\right)$$

$$C_3 = \frac{Y_1}{2}$$

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$$C_4 = \frac{1}{2sin\sqrt{\omega}}\left(Y_2 - Y_1 cosh\sqrt{\omega}\right)$$

Substitution of given and calculated values leads to

$$u\left(\frac{1}{2}\right) = 7.082 \text{ mm}$$

Problem 10.36 illustrates (a) the modeling of a pipe as a pinned-pinned beam, (b) the boundary conditions for moveable beam supports, and (c) the steady-state response of a beam due to a harmonic excitation.

10.37 A simplified model of the rocket of Figure P10.37 is a free-free beam. (a) Calculate the five lowest natural frequencies for longitudinal vibration. (b) Calculate the five lowest natural frequencies for transverse vibration.

FIGURE P10.37

Given: E, I, A, J, L , G, ρ

Find: $\omega_1 - \omega_5$ for longitudinal motion and transverse vibrations

Solution: The nondimensional partial differential equation governing longitudinal vibrations of the free-free bar is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

The boundary conditions are $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(1,t) = 0$. A separation of variables solution is assumed as

$$u(x,t) = X(x)T(t)$$

and leads to

$$\frac{d^2X}{dx^2} + \lambda X = 0$$

where λ, the separation constant, is the square of the nondimensional natural frequency. The solution of the differential equation is

$$X(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x$$

The boundary conditions are applied to this solution as $X'(0) = 0 \Longrightarrow C_2 = 0$ and $X'(1) = 0 \Longrightarrow C_1\sqrt{\lambda}\sin\sqrt{\lambda} = 0$. For a non-trivial solution $\sqrt{\lambda} = n\pi$ for n = 0,1, 2, 3,…. Thus

$$\omega_n = n\pi, n = 0,1,2,3, \ldots$$

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The dimensional natural frequencies are related to the nondimensional natural frequencies by

$$\omega = \frac{\omega^*}{L}\sqrt{\frac{E}{\rho}}$$

Hence the dimensional frequencies are

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{E}{\rho}}$$

The first five frequencies are

$$\omega = 0, \frac{\pi}{L}\sqrt{\frac{E}{\rho}}, \frac{2\pi}{L}\sqrt{\frac{E}{\rho}}, \frac{3\pi}{L}\sqrt{\frac{E}{\rho}}, \frac{4\pi}{L}\sqrt{\frac{E}{\rho}}$$

(b) The first five frequencies for a free-free beam are given in Table 10.4 as

$$\omega_1^* = 0, \omega_2^* = 22.74, \omega_3^* = 61.66, \omega_4^* = 120.9, \omega_5^* = 199.9$$

The dimensional natural frequencies are related to the nondimensional frequencies by

$$\omega = \omega^*\sqrt{\frac{EI}{\rho A L^4}}$$

Problem 10.37 illustrates the development of the natural frequencies of a free-free bar and the natural frequencies of a free-free beam.

10.38 Longitudinal vibrations are initiated in the rocket of Figure P10.38 when thrust is developed. Determine the Laplace transform of the transient response U(x, s) when the thrust of Figure P10.38 is developed. Do not invert the transform.

FIGURE P10.38

Given: F(t)

Find: $\mathcal{L}\{u(x,t)\}$

Solution: The nondimensional partial differential equation governing longitudinal vibrations of the free-free bar is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

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The boundary conditions are $\frac{\partial u}{\partial x}(1,t) = 0$ and

$$\frac{\partial u}{\partial x}(0,t) = \alpha\left\{\frac{t}{t^*}[1 - u(t-t^*)] + u(t-t^*)\right\}$$

where

$$\alpha = \frac{F_0}{EA}, \qquad t^* = \frac{t_0}{L}\sqrt{\frac{G}{\rho}}$$

The initial conditions are $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = 0$. Define

$$U(x,s) = \mathcal{L}\{u(x,t)\} = \int_0^\infty u(x,t)e^{-st}dt$$

Taking the Laplace transform of the wave equation, interchanging the order of differentiation with respect to x and the transform, using linearity of the transforms and the properties of derivatives leads to

$$\frac{d^2U}{dx^2} - s^2U = 0$$

The general solution of the ordinary differential equation is

$$U(x,s) = C_1(s)\cosh sx + C_2(s)\sinh sx$$

Transforms of the boundary conditions are

$$\frac{dU}{dx}(0,s) = \alpha\left\{\frac{1}{s^2t^*}\left[1 - e^{-st^*}\right] + \frac{1}{s}e^{-st^*}\right\}$$

and

$$\frac{dU}{dx}(1,s) = 0$$

Application of the boundary condition at x-0 leads to

$$C_2(s) = \alpha\left\{\frac{1}{s^3t^*}\left[1 - e^{-st^*}\right] + \frac{1}{s^2}e^{-st^*}\right\}$$

Application of the boundary condition at x = 1 leads to

$$C_1(s) = -C_1(s)\coth s$$

Hence

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$$U(x,s) = \alpha\left\{\frac{1}{s^3 t^*}\left[1 - e^{-st^*}\right] + \frac{1}{s^2}e^{-st^*}\right\}(\sinh sx - \coth s \cosh sx)$$

Problem 10.38 illustrates the application of the Laplace transform method to derive the transform of the response of a continuous system due to time dependent boundary conditions.

10.39 Determine the response of a cantilever beam when the fixed support is subject to a displacement $f(t) = A \sin \omega t$. Use the Laplace transform method and determine the transform $W(x, s)$. Do not invert.

Given: cantilever beam with y(0,t) = A sin ωt

Find: $W(x, s)$

Solution: The nondimensional differential equation governing the motion of the system is

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 w}{\partial x^2} = 0 \tag{1}$$

The nondimensional boundary conditions are

$$y(0,t) = A^* sin\omega^* t, \quad \frac{\partial y}{\partial x}(0,t) = 0$$

$$\frac{\partial^2 y}{\partial^2 x}(1,t) = 0, \quad \frac{\partial^3 y}{\partial^3 x}(1,t) = 0 \tag{2}$$

where

$$A^* = \frac{A}{L}, \quad \omega^* = \omega\sqrt{\frac{\rho A L^4}{EI}}$$

The initial conditions are

$$y(x,0) = 0, \quad \frac{\partial y}{\partial t}(x,0) = 0$$

Define

$$\bar{y}(x,s) = \mathcal{L}\{y(x,t)\} = \int_0^\infty y(x,t)e^{-st}\,dt$$

Taking the Laplace transform of eq.(1), interchanging the order of differentiation with respect to x and the transform, using linearity and the property of the transforms of derivatives leads to

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$$\frac{d^4\bar{y}}{dx^4}(x,s) + s^2\bar{y}(x,s) = 0 \tag{3}$$

The general solution of eq.(3) is

$$\bar{y}(x,q) = C_1(q)cos\sqrt{q}x + C_2(q)sin\sqrt{q}x + C_3(q)cosh\sqrt{q}x + C_4(q)sinh\sqrt{q}x \tag{4}$$

where

$$q = is$$

Making the substitution for s in terms of q is a matter of convenience.

If the original transform variable s is used, the solution is written as

$$\bar{y}(x,s) = C_1(s)cos\sqrt{\frac{s}{2}}xcosh\sqrt{\frac{s}{2}}x + C_2(s)cos\sqrt{\frac{s}{2}}xsinh\sqrt{\frac{s}{2}}x$$

$$+C_3(s)sin\sqrt{\frac{s}{2}}xcosh\sqrt{\frac{s}{2}}x + C_4(s)sin\sqrt{\frac{s}{2}}xsinh\sqrt{\frac{s}{2}}x$$

Since the inverse of the transform is not required in this solution, the new transform variable s is used for the remainder of the problem,

Taking the transforms of the boundary conditions leads to

$$\bar{y}(0,s) = A^*\frac{\omega^*}{s^2+\omega^{*2}} = A^*\frac{\omega^*}{\omega^{*2}-q^2}, \quad \frac{d\bar{y}}{dx}(0,s)$$

$$\frac{d^2\bar{y}}{dx^2}(1,s) = 0, \quad \frac{d^3\bar{y}}{dx^3}(1,s) = 0$$

Application of the boundary conditions to eq.(4) leads to

$$A^*\frac{\omega^*}{\omega^{*2}-q^2} = C_1 + C_3$$

$$0 = C_2 + C_4$$

$$0 = -qC_1cos\sqrt{q} - qC_2sin\sqrt{q} + qC_3cosh\sqrt{q} + C_4sinh\sqrt{q}$$

$$0 = q^{\frac{3}{2}}C_1sin\sqrt{q} - q^{\frac{3}{2}}C_2cos\sqrt{q} + q^{\frac{3}{2}}C_3sinh\sqrt{q} + q^{\frac{3}{2}}C_4cosh\sqrt{q}$$

The solution of the above is obtained and substituted into eq.(4) leading to

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$$\bar{y}(x,s) = -A^* \frac{\omega^*}{2(\omega^{2^2} - q^2)(1 + cos\sqrt{q}cosh\sqrt{q})}[(1 - cos\sqrt{q}cosh\sqrt{q} + sin\sqrt{q}sinh\sqrt{q})cos\sqrt{q}x$$

$$-(cos\sqrt{q}sinh\sqrt{q} + cosh\sqrt{q}sin\sqrt{q}x)sin\sqrt{q}x$$

$$-(3 - cos\sqrt{q}cosh\sqrt{q} - sin\sqrt{q}sinh\sqrt{q})cosh\sqrt{q}x$$

$$+(cos\sqrt{q}sinh\sqrt{q} + cosh\sqrt{q}sin\sqrt{q})sinh\sqrt{q}x]$$

Problem 10.39 illustrates the use of the Laplace transform method to determine the response of a continuous system with time dependent boundary conditions.

10.40 The tail rotor blades of a helicopter have a rotating unbalance of magnitude 0.5 kg · m and operates at a speed of 1200 rpm. Modeling the tail section as a cantilever beam of length 3.5 m with A = 0.13 m^2 and $EI = 3.1 \times 10^6$ N · m^2, determine the steady-state response of the tail section.

Given: $m_0 e = 0.5\ \text{kg} \cdot \text{m},\ \omega = 1200\ \text{rpm} = 125.6\frac{\text{rad}}{\text{s}},\ L = 3.5\ \text{m},\ A = 0.13\ \text{m}^2,\ EI = 3.1 \times 10^6\ \text{N} \cdot \text{m}^2$

Find: $x(t)$

Solution: The nondimensional partial differential equation governing the transverse vibrations of the tail section is

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0$$

The appropriate boundary conditions are

$$w(0,t) = 0 \qquad \frac{\partial w}{\partial x}(0,t) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(1,t) = 0 \qquad \frac{\partial^3 w}{\partial x^3}(1,t) = \alpha \sin \omega t$$

where

$$\alpha = \frac{m_0 e \omega^2 L^2}{EI} = 0.0312$$

$$\omega = \omega\sqrt{\frac{\rho A L^4}{EI}} = 27.3$$

Since the excitation is harmonic a steady-state solution is assumed as

$$w(x,t) = u(x) \sin \omega t$$

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The assumed form of the steady-state solution is substituted into the differential equation and boundary conditions leading to

$$\frac{d^4u}{dx^4} - \omega^2 u = 0$$

$$u(0) = 0 \qquad u'(0) = 0 \qquad u''(1) = 0 \qquad u'''(1) = \alpha$$

The general solution of the ordinary differential equation is

$$y(x) = A\cos\sqrt{\omega}\,x + B\sin\sqrt{\omega}x + C\cosh\sqrt{\omega}x + D\sinh\sqrt{\omega}x$$

Application of the boundary conditions leads to

$$u(0) = 0 = A + C$$

$$u'(0) = 0 = B + D$$

$$u''(1) = 0 = -\cos\sqrt{\omega}\,A - \sin\sqrt{\omega}\;B + \cosh\sqrt{\omega}\;C + \sinh\sqrt{\omega}\;D$$

$$u'''(1) = \alpha = \omega^{3/2}\left(\sin\sqrt{\omega}\,A - \cos\sqrt{\omega}\,B + \sinh\sqrt{\omega}\,C + \cosh\sqrt{\omega}D\right)$$

The above equations are solved simultaneously and substituted into the solution leading to

$$\begin{aligned} w(x,t) = -\frac{\alpha}{2\omega^{3/2}\left(1+\cos\sqrt{\omega}\cosh\sqrt{\omega}\right)}\Big[&-\left(\sin\sqrt{\omega}+\sinh\sqrt{\omega}\right)\cos\sqrt{\omega}x \\ &+\left(\cos\sqrt{\omega}+\cosh\sqrt{\omega}\right)\sin\sqrt{\omega}x + \left(\sin\sqrt{\omega}+\sinh\sqrt{\omega}\right)\sinh\sqrt{\omega}x \\ &-\left(\cos\sqrt{\omega}+\cosh\sqrt{\omega}\right)\cosh\sqrt{\omega}x\Big]\sin\omega t \end{aligned}$$

Problem 10.40 illustrates the determination of the steady-state response of a continuous system due to a harmonic excitation.

10.41 Determine the steady-state amplitude of the engine of Figure P10.41.

FIGURE P10.41

Given: $\rho = 7800$ kg/m^3, $E = 200 \times 10^9$ N/m^2, $I = 4.5 \times 10^{-6}$ m^4, $A = 1.6 \times 10^{-3}$ m^2, $m = 55$ kg, $k = 5 \times 10^4$ N/m, $m_0e = 1.8$ kg · m, $\omega = 300$ rad/s, $L = 4.1$ m

Find: X

Solution: A steady-state solution is assumed as $w(x,t) = X(x)\sin\omega t$, leading to

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$$X^{IV} - \omega^2 X = 0$$
$$X(0) = 0$$
$$X'(0) = 0$$
$$X''(1) = 0$$
$$X'''(1) = \alpha + (\beta - \delta\omega^2)X(1)$$

The solution of the differential equation is

$$X(x) = C_1 \cos\sqrt{\omega}x + C_2 \sin\sqrt{\omega}x + C_3 \cosh\sqrt{\omega}x + C_4 \sinh\sqrt{\omega}x$$

Application of the boundary conditions leads to

$$X(0) = 0 \rightarrow C_1 + C_3 = 0$$
$$X'(0) = 0 \rightarrow C_2 + C_4 = 0$$

$$X''(1) = 0 \rightarrow -C_1 \cos\sqrt{\omega} - C_2 \sin\sqrt{\omega} + C_3 \cosh\sqrt{\omega} + C_4 \sinh\sqrt{\omega} = 0$$
$$X'''(1) = \alpha + (\beta - \delta\omega^2)X(1) \rightarrow \omega^{3/2}\left(C_1 \sin\sqrt{\omega} - C_2 \cos\sqrt{\omega} + C_3 \sinh\sqrt{\omega} + C_4 \cosh\sqrt{\omega}\right) =$$
$$\alpha + (\beta - \delta\omega^2)\left(C_1 \cos\sqrt{\omega} + C_2 \sin\sqrt{\omega} + C_3 \cosh\sqrt{\omega} + C_4 \sinh\sqrt{\omega}\right)$$

The above is used to solve for the constants of integration leading to

$$X(x) = -9.37\times10^{-6}\cos(1.402x) + 7.51\times10^{-6}\sin(1.402x)$$
$$+9.37\times10^{-6}\cosh(1.402x) - 7.51\times10^{-6}\sinh(1.402x)$$

The nondimensional amplitude of the engine is X(1) = 1.67 × 10^{-5}. The dimensional steady-state amplitude of the engine is

$$LX(1) = 4.79\times10^{-5}\ \text{m}$$

Problem 10.41 illustrates the continuous system analysis of a machine with a rotating unbalance fixed to the end of a beam.

10.42 Show that the differential equation governing free vibration of a uniform beam subject to a constant axial load, *P*, is

$$EI\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

Given: Partial differential equation

Find: nondimensional partial differential equation

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Solution: The chain rule is applied to give

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x^*}\frac{dx^*}{dx} = \frac{1}{L}\frac{\partial w}{\partial x^*}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial t^*}\frac{dt^*}{dt} = \sqrt{\frac{EI}{\rho A L^4}}\frac{\partial w}{\partial x^*}$$

Applying the chain rule to obtain higher-order derivatives and substituting into the partial differential equations yields

$$\frac{EI}{L^4}\frac{\partial^4 (Lw^*)}{\partial x^{*4}} - \frac{P}{L^2}\frac{\partial^2 (Lw^*)}{\partial x^{*2}} + \frac{\rho A E I}{\rho A L^4}\frac{\partial^2 (Lw^*)}{\partial t^{*2}} = 0$$

$$\frac{\partial^4 w}{\partial x^4} - \alpha\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} = 0$$

where

$$\alpha = \frac{PL^3}{EI}$$

and the *'s have been dropped from the nondimensional variables.

Problem 10.42 illustrates the nondimensionalization of the partial differential equations governing free transverse vibrations of a uniform beam with an axial load.

10.43 Determine the frequency equation for a simply supported beam subject to an axial load.

Given: simply supported beam with an axial load

Find: frequency equation

Solution: The nondimensional partial differential equation governing free vibrations of a beam with an axial load is

$$\frac{\partial^4 w}{\partial x^4} - \alpha\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} = 0$$

The boundary conditions for a simply supported beam are

$$w(0,t) = 0 \qquad \frac{\partial w}{\partial x}(0,t) = 0 \qquad w(1,t) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(1,t) = 0$$

A product solution is assumed as

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$$w(x,t) = X(x)T(t)$$

Substitution of the product solution into the partial differential equation and boundary conditions leads to

$$\frac{d^4X}{dx^4} - \alpha\frac{d^2X}{dx^2} - \lambda W = 0$$

$$X(0) = 0, \qquad X'(0) = 0, \qquad X(0) = 0, \qquad X''(1) = 0$$

where λ is the usual separtation constant. The general solution of the ordinary differential equation is assumed as

$$X(x) = Ae^{sx}$$

which upon substitution into the differential equation leads to

$$s^4 - \alpha s^2 - \lambda = 0$$

whose solutions are

$$s = i\mu = i\left[\frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + 4\lambda}\right)\right]^{1/2}$$

$$s = \nu = \left[\frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + 4\lambda}\right)\right]^{1/2}$$

Thus the general solution to the ordinary differential equation is

$$X(x) = A\cos\mu x + B\sin\mu x + C\cosh\upsilon x + D\sinh\nu x$$

Application of the boundary conditions at x = 0 leads to

$$X(0) = 0 = A + C$$

$$X'(0) = 0 = \mu B + \nu D$$

which leads to

$$A = -C, \qquad D = -\frac{\mu}{\nu}B$$

Application of the boundary conditions at x = 1 lead to

$$X(1) = 0 = \cos\mu A + \sin\mu B + \cosh\nu C + \sinh\nu D$$

$$X''(1) = 0 = -\mu^2\cos\mu A - \mu^2\sin\mu B + \nu^2\cosh\nu C + \nu^2\sinh\nu D$$

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A non-trivial solution of the above exists only if the determinant of the coefficient matrix is zero. The nondimensional natural frequencies are the square roots of the separation constants, λ. The dimensional natural frequencies are

$$\omega_n = \omega_n^* \sqrt{\frac{EI}{\rho A L^4}}$$

Problem 10.43 illustrates the natural frequencies of a fixed-free beam with an axial load.

10.44 Determine the frequency equation for a fixed-pinned beam subject to an axial load.

Given: fixed-pinned beam with an axial load

Find: frequency equation

Solution: The nondimensional partial differential equation governing free vibrations of a beam with an axial load is

$$\frac{\partial^4 w}{\partial x^4} - \alpha \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} = 0$$

The boundary conditions for a fixed-pinned beam are

$$w(0,t) = 0 \qquad \frac{\partial w}{\partial x}(0,t) = 0 \quad w(x,0) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(1,t) = 0$$

A product solution is assumed as

$$w(x,t) = X(x)T(t)$$

Substitution of the product solution into the partial differential equation and boundary conditions leads to

$$\frac{d^4 X}{dx^4} - \alpha \frac{d^2 X}{dx^2} - \lambda W = 0$$

$$X(0) = 0, \qquad X'(0) = 0, \qquad X(0) = 0, \qquad X''(1) = 0$$

where λ is the usual separtation constant. The general solution of the ordinary differential equation is assumed as

$$X(x) = Ae^{sx}$$

which upon substitution into the differential equation leads to

$$s^4 - \alpha s^2 - \lambda = 0$$

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whose solutions are

$$s = i\mu = i\left[\frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + 4\lambda}\right)\right]^{1/2}$$

$$s = \nu = \left[\frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + 4\lambda}\right)\right]^{1/2}$$

Thus the general solution to the ordinary differential equation is

$$X(x) = A\cos\mu x + B\sin\mu x + C\cosh\nu x + D\sinh\nu x$$

Application of the boundary conditions at x = 0 leads to

$$X(0) = 0 = A + C$$

$$X'(0) = 0 = \mu B + \nu D$$

which lead to

$$A = -C, \quad D = -\frac{\mu}{\nu}B$$

Application of the boundary conditions at x = 1 lead to

$$X(1) = 0 = \cos\mu\, A + \sin\mu\, B + \cosh\nu\, C + \sinh\nu\, D$$

$$X'(1) = 0 = -\mu\sin\mu\, A + cosB + \nu\sinh\nu + \nu\cosh\nu\, D$$

A non-trivial solution of the above exists only if the determinant of the coefficient matrix is zero. The nondimensional natural frequencies are the square roots of the separation constants, λ. The dimensional natural frequencies are

$$\omega_n = \omega_n^* \sqrt{\frac{EI}{\rho A L^4}}$$

Problem 10.44 illustrates the natural frequencies of a fixed-pinned beam with an axial load.

10.45 A fixed-fixed beam is made of a material with a coefficient of thermal expansion α. After installed, the temperature is decreased by ΔT. Determine the beam's frequency equation.

Given: $\alpha, \Delta T$, fixed-fixed beam

Find: frequency equation

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Solution: The nondimensional partial differential equation governing free vibrations of a beam with an axial load is

$$\frac{\partial^4 w}{\partial x^4} - \beta \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} = 0$$

where

$$\beta = \frac{\alpha A \Delta T L^2}{I}$$

The boundary conditions for a fixed-free beam are

$$w(0,t) = 0 \qquad \frac{\partial w}{\partial x}(0,t) = 0 \qquad w(1,t) = 0 \qquad \frac{\partial w}{\partial x}(1,t) = 0$$

A product solution is assumed as

$$w(x,t) = X(x)T(t)$$

Substitution of the product solution into the partial differential equation and boundary conditions leads to

$$\frac{d^4 X}{dx^4} - \alpha \frac{d^2 X}{dx^2} - \lambda W = 0$$

$$X(0) = 0, \qquad X'(0) = 0, \qquad X(1) = 0, \qquad X'(1) = 0$$

where λ is the usual separtation constant. The general solution of the ordinary differential equation is assumed as

$$X(x) = Ae^{sx}$$

which upon substitution into the differential equation leads to

$$s^4 - \alpha s^2 - \lambda = 0$$

whose solutions are

$$s = i\mu = i\left[\frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + 4\lambda}\right)\right]^{1/2}$$

$$s = \nu = \left[\frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + 4\lambda}\right)\right]^{1/2}$$

Thus the general solution to the ordinary differential equation is

$$X(x) = A\cos\mu x + B\sin\mu x + C\cosh\upsilon x + D\sinh\nu x$$

Application of the boundary conditions at x = 0 leads to

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$$X(0) = 0 = A + C$$

$$X'(0) = 0 = \mu B + \nu D$$

which lead to

$$A = -B, \quad D = -\frac{\mu}{\nu} B$$

Application of the boundary conditions at x = 1 leads to

$$X(1) = 0 = A\cos\mu + B\sin\mu + C\cosh\nu + D\sinh\nu$$

$$X'(1) = 0 = -A\mu\sin\mu + B\mu\cos\mu + C\nu\sinh\nu + D\,\nu\cosh\nu$$

A non-trivial solution exists when the determinant of the coefficient matrix is zero for the previous equations which results in

$$2\nu + (\nu^2 - \mu^2)\sin\mu\sin\nu - 2\nu\cos\mu\cosh\nu = 0$$

The above equation is the frequency equation when the forms of μ and υ are substituted in terms of λ where λ is the square of the natural frequency.

Problem 10.45 illustrates the natural frequencies of a fixed-fixed beam subject to an axial compressive load due to a temperature decrease.

10.46 Show orthogonality of the mode shapes for a simply supported beam subject to an axial load.

Given: simply supported beam with an axial load

Show: orthogonality of mode shapes

Solution: The free vibrations of a simply supported beam are studied in Chapter Problem 10.43. The general mode shape in terms of constants of integration is

$$X(x) = C_1\cos\mu x + C_2\sin\mu x + C_3\cosh\nu x + C_4\sinh\nu\mu x\, C_1\cos\mu x$$

Application of the boundary conditions leads to

$$C_1 = C_3 = C_4 = 0$$

and

$$\mu = n\pi, \qquad n = 1,2,3,\ldots$$

Hence the mode shape is

$$X_n(x) = C_2\sin n\pi x$$

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Consider the scalar product

$$(f,g)=\int_0^1 f(x)g(x)dx$$

and consider the scalar product of two mode shapes corresponding to different natural frequencies

$$\left(X_n(x),X_m(x)\right)=\int_0^1 C_nC_m\sin n\pi x\sin m\pi x\,dx=0$$

Problem 10.46 illustrates that the mode shapes for a simply supported beam subject to an axial load are mutually orthogonal with respect to the kinetic energy scalar product.

10.47 Use Rayleigh's quotient to approximate the lowest natural frequency of a torsional shaft fixed at both ends.

Given: fixed-fixed shaft, L,ρ,J,G

Find: approximation for ω_1

Solution: The time dependent displacement of a fixed-fixed shaft is approximated as

$$u(x,t)=u(x)\sin\omega t$$

where u(x) is chosen to satisfy the boundary conditions, u(0) = 0 and u(L) = 0. One such function that satisfies both boundary conditions is

$$u(x)=A\sin\left(\frac{\pi x}{L}\right)$$

The lowest natural frequency can be approximated using Rayleigh's quotient by

$$\omega_1^2=\frac{(u,u)_V}{(u,u)_T}$$

Substituting the assumed mode into the scalar products involved in Rayleigh's quotient leads to

$$(u,u)_V=\int_0^1 JG\left(\frac{du}{dx}\right)^2dx=\frac{JGA^2\pi^2}{L^2}\int_0^1\cos^2\left(\frac{\pi x}{L}\right)dx=\frac{JGA^2\pi^2}{2L}$$

$$(u,u)_T=\int_0^1\rho Ju^2dx=\rho JA^2\int_0^1\sin^2\left(\frac{\pi x}{L}\right)dx=\frac{\rho JA^2L}{2}$$

Thus the lowest natural frequency is approximated as

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$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{G}{\rho}}$$

Problem 10.47 illustrates use of Rayleigh's quotient to approximate the lowest natural frequency of a fixed-fixed torsional shaft.

10.48 Use Rayleigh's quotient to approximate the lowest natural frequency of a torsional shaft with a disk of mass moment of inertia I placed at its midspan. The shaft is fixed at both ends.

Given: fixed-fixed shaft, L, ρ, J, G, I

Find: approximation for ω_1

Solution: The time dependent displacement of a fixed-fixed shaft is approximated as

$$u(x,t) = u(x)\sin\omega t$$

where u(x) is chosen to satisfy the boundary conditions, u(0) = 0 and u(L) = 0. One such function that satisfies both boundary conditions is

$$u(x) = A\sin\left(\frac{\pi x}{L}\right)$$

The lowest natural frequency can be approximated using Rayleigh's quotient by

$$\omega_1^2 = \frac{(u,u)_V}{(u,u)_T}$$

Substituting the assumed mode into the scalar products involved in Rayleigh's quotient leads to

$$(u,u)_V = \int_0^1 JG\left(\frac{du}{dx}\right)^2 dx = \frac{JGA^2\pi^2}{L^2}\int_0^1 \cos^2\left(\frac{\pi x}{L}\right)dx = \frac{JGA^2\pi^2}{2L}$$

$$(u,u)_T = \int_0^1 \rho J u^2 dx + I u^2\left(\frac{L}{2}\right) = \rho J A^2\int_0^1 \sin^2\left(\frac{\pi x}{L}\right)dx + IA^2 = \frac{\rho J A^2 L}{2} + IA^2$$

Thus the lowest natural frequency is approximated as

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{GJ}{\rho J + \frac{2I}{L^2}}}$$

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Problem 10.48 illustrates use of Rayleigh's quotient to approximate the lowest natural frequency of a fixed-fixed torsional shaft with a concentrated inertia element at its midspan.

10.49 Use Rayleigh's quotient to approximate the lowest natural frequency of a fixed-fixed beam.

Given: fixed-fixed beam, L, ρ, I, A, E

Find: approximation for ω_1

Solution: The time dependent displacement of a fixed-fixed shaft is approximated as $w(x,t) = u(x)\sin\omega t$ where u(x) is chosen to satisfy the boundary conditions, w(0) = 0, $\frac{du}{dx}(0) = 0$, u(L) = 0, and $\frac{du}{dx}(1) = 0$. A fourth-order polynomial is of the form

$$u(x) = x^4 + Ax^3 + Bx^2 + Cx + D$$

In order to satisfy the boundary conditions at zero, C = D = 0. The boundary conditions at x = 1 are used to give $A = -4L, B = 6L^2$. The lowest natural frequency can be approximated using Rayleigh's quotient by

$$\omega_1^2 = \frac{(u,u)_V}{(u,u)_T}$$

Substituting the assumed mode into the scalar products involved in Rayleigh's quotient leads to

$$(u,u)_V = \int_0^1 EI\left(\frac{d^2u}{dx^2}\right)^2 dx = EI\int_0^1 (12x^2 - 24Lx + 12L^2)^2\,dx = \frac{144EI\,L^5}{5}$$

$$(u,u)_T = \int_0^1 \rho A u^2 dx = \rho A\int_0^1 (x^4 - 4Lx^3 + 6L^2x^2)^2\,dx = \frac{278\rho AL^9}{315}$$

Hence

$$\omega_1 = \sqrt{\frac{4536EI}{139\rho AL^4}} = 5.71\sqrt{\frac{EI}{\rho AL^4}}$$

Problem 10.49 illustrates use of Rayleigh's quotient to approximate the lowest natural frequency of a fixed-fixed beam.

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10.50 Use Rayleigh's quotient to approximate the lowest natural frequency of a simply supported beam with a mass m at its midspan. Use $w(x) = \sin(\frac{\pi x}{L})$ as the trial function.

Given: simply supported beam, L, ρ, I, A, E, m

Find: approximation for ω_1

Solution: The time dependent displacement of a pinned-pinned shaft is approximated as $w(x,t) = u(x)\sin\omega t$ where u(x) is chosen to satisfy the boundary conditions, w(0) = 0, $\frac{d^2u}{dx^2}$ (0) = 0, u(L) = 0, and $\frac{d^2u}{dx^2}$ $(1) = 0$. The lowest natural frequency can be approximated using Rayleigh's quotient by

$$\omega_1^2 = \frac{(u,u)_V}{(u,u)_T}$$

Substituting the assumed mode into the scalar products involved in Rayleigh's quotient leads to

$$(u,u)_V = \int_0^1 EI\left(\frac{d^2u}{dx^2}\right)^2 dx = EI\int_0^1 \left(\frac{\pi}{L}\cos\frac{\pi x}{L}\right)^2 dx = \frac{EI\,\pi^4}{2L^3}$$

$$(u,u)_T = \int_0^1 \rho A u^2 dx + mu^2\left(\frac{L}{2}\right) = \rho A\int_0^1 \left(\sin\frac{\pi x}{L}\right)^2 dx = \frac{\rho AL}{2} + m$$

Hence

$$\omega_1 = \frac{\pi^2}{L}\sqrt{\frac{EI}{L(\rho AL + m)}}$$

Problem 10.50 illustrates use of Rayleigh's quotient to approximate the lowest natural frequency of a pinned-pinned beam with a mass at its midspan.

10.51 Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies of a fixed-free beam.

Given: fixed-free beam

Find: ω_1, ω_2

Solution: The nondimensional natural frequencies are calculated. Thus the nondimensional forms of the scalar products are used in the Rayleigh-Ritz analysis and the solution developed using nondimensional frequencies. Since it is desired to approximate the two lowest natural frequencies two linearly independent trial functions must be used. Both

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functions satisfy the boundary conditions. If f(x) is a trial function then $f(0) = 0, \frac{df}{dx}(0) = 0, \frac{d^2f}{dx^2}(1) = 0 \ \frac{d^3f}{dx^3}(1) = 0$. The trail functions are generated from polynomials. In order for the boundary conditions to be satisfied the trial functions must be at least of order four. However two trial functions are required, at least one must be of order five. The boundary conditions are used to generate two functions as

$$f_1(x) = x^5 - 10x^3 + 20x^2$$

$$f_2(x) = x^5 - 2.5x^4 + 5x^2$$

The mode shape is approximated by

$$w(x) = C_1 f_1(x) + C_2 f_2(x)$$

The nondimensional kinetic energy scalar product is calculated as

$$(w,w)_T = \int_0^1 w^2(x)dx = \int_0^1 (C_1 f_1(x) + C_2 f_2(x))^2\, dx$$
$$= 30.49C_1^2 + 9.50(2C_1C_2) + 2.96C_2^2$$

The potential energy scalar product is calculated as

$$(w,w)_V = \int_0^1 w^2(x)dx = \int_0^1 (C_1 f''_1(x) + C_2 f''_2(x))^2\, dx$$
$$= 377.14C_1^2 + 117.14(2C_1C_2) + 37.14C_2^2$$

Rayleigh's quotient is a minimum when

$$R\frac{\partial(w,w)_T}{\partial C_1} - \frac{\partial(w,w)_V}{\partial C_1} = 0$$

$$R\frac{\partial(w,w)_T}{\partial C_2} - \frac{\partial(w,w)_V}{\partial C_2} = 0$$

which leads to

$$(30.49R - 377.14)C_1 + (9.50R - 117.14)C_2 = 0$$

$$(9.50R - 117.14)C_1 + (2.96R - 37.14)C_2 = 0$$

The preceding equations have a non-trivial solution only if the determinant of the coefficient matrix is zero. This leads to

$$0.194R^2 - 26.0R + 285.0 = 0$$

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whose solutions are

$$R = 12.04, 122.0$$

The natural frequency approximations are the square roots of the calculated values of R,

$$\omega_1 = 3.47, \omega_2 = 11.04$$

Problem 10.51 illustrates the use of the Rayleigh-Ritz method to approximate the lowest natural frequencies of a fixed-free beam. The lowest approximation is very close to the exact natural frequency of 3.52

10.52 Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies of the system of Figure P10.52.

FIGURE P10.52

Given: $\rho = 6000\,\frac{\text{kg}}{\text{m}^3}, E = 210 \times 10^9\,\frac{\text{N}}{\text{m}^2}, L = 2\text{ m}, k = 1 \times 10^6\,\frac{\text{N}}{\text{m}}, A = 7 \times 10^{-4}\text{ m}^2$

Find: ω_1, ω_2

Solution: Two linearly independent trial functions must be used. Both functions satisfy the boundary conditions. If f(x) is a trial function then $f(0) = 0, \frac{df}{dx}(1) = -\beta f(1)$ where $\beta = \frac{kL}{EA} = 0.01356$. The trial functions are generated from polynomials. In order for the boundary conditions to be satisfied the trial functions must be at least of order three.

$$f_1(x) = x^2 - \frac{4(1+\beta)}{1+2\beta}x$$

$$f_2(x) = x^3 - \frac{4(3+2\beta)}{1+2\beta}x$$

The mode shape is approximated by

$$w(x) = C_1 f_1(x) + C_2 f_2(x)$$

The nondimensional kinetic energy scalar product is calculated as

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$$(w,w)_T = \int_0^1 w^2(x)dx = \int_0^1 (C_1 f_1(x) + C_2 f_2(x))^2\, dx = 30.49C_1^2 + 9.50(2C_1C_2) + 2.96C_2^2$$

The potential energy scalar product is calculated as

$$(w,w)_V = \int_0^1 w^2(x)dx = \int_0^1 (C_1 f''_1(x) + C_2 f''_2(x))^2\, dx$$
$$= 377.14C_1^2 + 117.14(2C_1C_2) + 37.14C_2^2$$

Rayleigh's quotient is a minimum when

$$R\frac{\partial(w,w)_T}{\partial C_1} - \frac{\partial(w,w)_V}{\partial C_1} = 0$$

$$R\frac{\partial(w,w)_T}{\partial C_2} - \frac{\partial(w,w)_V}{\partial C_2} = 0$$

which leads to

$$(30.49R - 377.14)C_1 + (9.50R - 117.14)C_2 = 0$$

$$(9.50R - 117.14)C_1 + (2.96R - 37.14)C_2 = 0$$

The preceding equations have a non-trivial solution only if the determinant of the coefficient matrix is zero. This leads to

$$0.194R^2 - 26.0R + 285.0 = 0$$

whose solutions are

$$R = 12.04, 122.0$$

The natural frequency approximations are the square roots of the calculated values of R,

$$\omega_1 = 3.47, \omega_2 = 11.04$$

Problem 10.52 illustrates the use of the Rayleigh-Ritz method to approximate the lowest natural frequencies of a fixed-free beam. The lowest approximation is very close to the exact natural frequency of 3.52

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10.53 Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies for the system of Figure P10.53.

FIGURE P10.53

Given: $\rho = 6000\,\frac{\text{kg}}{\text{m}^3}, G = 60 \times 10^9\,\frac{\text{N}}{\text{m}^2}, L = 2\text{ m}, I = 7.1\text{kg}\cdot\text{m}^2, r = 35\text{ mm}$

Find: ω_1, ω_2

Solution: Two linearly independent trial functions must be used. Both functions satisfy the boundary conditions. If f(x) is a trial function then $\theta(0) = 0, \theta(L) = 0$. Two trial functions which satisfy the boundary conditions are

$$\theta_1(x) = \sin\frac{\pi x}{L}$$

$$\theta_2(x) = \sin\frac{2\pi x}{L}$$

Note that

$$J = \pi\frac{r^4}{2} = \pi\frac{(0.035\text{ m})^2}{2} = 2.357 \times 10^{-6}\text{ m}^4$$

The mode shape is approximated by

$$\theta(x) = C_1\theta_1(x) + C_2\theta_2(x)$$

The kinetic energy scalar product is

$$\begin{aligned}(\theta,\theta)_T &= \int_0^L \rho J\theta^2(x)dx + I\theta^2(L)\\ &= \int_0^L \rho J(C_1\sin\frac{\pi x}{L} + C_2\sin\frac{2\pi x^2}{L})dx + I(C_1\sin 0.6\pi + C_2\sin 1.2\,\pi)^2\\ &= [0.5\rho JL + I\,sin^2 0.6\pi]C_1^2 + 2I\sin\ \ 0.6\pi\sin\ \ 1.2\pi\,C_1C_2\\ &+ [0.5\rho JL + I\sin^2 1.2\pi]C_2^2\end{aligned}$$

The potential energy scalar product is calculated as

$$(\theta,\theta)_V = \int_0^L GJ\left(\frac{\partial\theta}{\partial x}\right)^2 dx = \int_0^L GJ(C_1\frac{\pi}{L}\cos\frac{\pi x}{L} + C_2\frac{2\pi}{L}\cos\frac{2\pi x}{L})^2\,dx = \frac{GJ\pi^2}{2L}C_1^2 + \frac{GJ\pi^2}{L}C_2^2$$

Rayleigh's quotient is a minimum when

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$$R\frac{\partial(\theta,\theta)_T}{\partial C_1}-\frac{\partial(\theta,\theta)_V}{\partial C_1}=0$$

$$R\frac{\partial(\theta,\theta)_T}{\partial C_2}-\frac{\partial(\theta,\theta)_V}{\partial C_2}=0$$

which leads to

$$\left(\rho JLR+1.809IR-\frac{GJ\pi^2}{L}\right)C_1-1.1180IRC_2=0$$

$$-1.180IRC_1+\left(\rho JLR+0.691IR-\frac{4GJ\pi^2}{L}\right)C_2=0$$

The preceding equations have a non-trivial solution only if the determinant of the coefficient matrix is zero. This leads to

$$0.717R^2-5.24\times10^7R+3.463\times10^{12}=0$$

whose solutions are

$$R=6.61\times10^4,1.53\times10^8$$

The natural frequency approximations are the square roots of the calculated values of R,

$$\omega_1=257.1\frac{\text{rad}}{\text{s}},\omega_2=12354\ \text{rad/s}$$

Problem 10.53 illustrates the use of the Rayleigh-Ritz method to approximate the natural frequencies of a torsional system with a discrete rotor.

10.54 Use the Rayleigh-Ritz method to approximate the three lowest natural frequencies of a fixed-pinned beam. Use polynomial of order six or less as trial functions.

Given: fixed-pinned beam

Find: Rayleigh-Ritz approximation to three lowest non-dimensional natural frequencies

Solution: Three polynomials which satisfy the boundary conditions are

$$w_1(x)=x^6-7x^3+6x^2$$

$$w_2(x)=x^5-4.5x^3+3.5x^2$$

$$w_1(x)=x^4-2.5x^3+1.5x^2$$

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The mass matrix is $m_{ij} = \int_0^1 w_i(x) w_j(x) dx$ and is calculated as

$$\mathbf{M} = \begin{bmatrix} 0.2203 & 0.111 & 0.0397 \\ 0.111 & 0.0588 & 0.0210 \\ 0.0397 & 0.0210 & 0.0075 \end{bmatrix}$$

The stiffness matrix is $k_{ij} = \int_0^1 \frac{d^2 w_i}{dx^2} \frac{d^2 w_j}{dx^2} dx$ and is calculated as

$$\mathbf{K} = \begin{bmatrix} 52.0 & 27.0 & 9.43 \\ 27.0 & 14.2 & 5.00 \\ 9.43 & 5.00 & 1.80 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. They are

$$\omega_1 = 15.42, \omega_2 = 50.61, \omega_3 = 109.12$$

Problem 10.54 illustrates the use of the Rayleigh-Ritz method to approximate the natural frequencies of a beam.

10.55 Use the Rayleigh-Ritz method to approximate the three lowest natural frequencies and their corresponding mode shapes of a fixed-free beam. Use polynomials of order six or less as trial functions.

Given: fixed-free beam

Find: Rayleigh-Ritz approximation to three lowest non-dimensional natural frequencies

Solution: Three polynomials which satisfy the boundary conditions are

$$w_1(x) = x^6 - 4x^3 + 3x^2$$

$$w_2(x) = x^5 - 3x^3 + 2x^2$$

$$w_1(x) = x^4 - 2x^3 + x^2$$

The mass matrix is $m_{ij} = \int_0^1 w_i(x) w_j(x) dx$ and is calculated as

$$\mathbf{M} = \begin{bmatrix} 0.0293 & 0.0171 & 0.0068 \\ 0.0171 & 0.0100 & 0.0040 \\ 0.0068 & 0.0040 & 0.0016 \end{bmatrix}$$

The stiffness matrix is $k_{ij} = \int_0^1 \frac{d^2 w_i}{dx^2} \frac{d^2 w_j}{dx^2} dx$ and is calculated as

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$$\mathbf{K} = \begin{bmatrix} 16.0 & 9.02 & 3.43 \\ 9.02 & 5.15 & 2.00 \\ 3.43 & 2.00 & 0.801 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. They are

$$\omega_1 = 22.38, \omega_2 = 63.02, \omega_3 = 128.04$$

Problem 10.55 illustrates the use of the Rayleigh-Ritz method to approximate the lowest natural frequencies of a fixed-free beam.

10.56 Use the Rayleigh-Ritz method to approximate the two lowest frequencies of transverse vibration of the system of Figure P10.56.

FIGURE P10.56

Given: pinned-pinned beam with attached mass

Find: approximations to two lowest natural frequencies

Solution: Two functions which satisfy the boundary conditions of a pinned-pinned beam are

$$w_1(x) = \sin \pi x$$

$$w_2(x) = \sin 2\pi x$$

The mass matrix is $m_{ij} = \int_0^1 \rho A w_i(x) w_j(x) dx + m w_i(0.7) w_j(0.7)$ and is calculated as

$$\mathbf{M} = \begin{bmatrix} 152.36 & -61.55 \\ -61.55 & 172.36 \end{bmatrix}$$

The stiffness matrix is $k_{ij} = \int_0^1 EI \frac{d^2 w_i}{dx^2} \frac{d^2 w_j}{dx^2} dx$ and is calculated as

$$\mathbf{K} = 10^8 \begin{bmatrix} 0.5455 & 0 \\ 0 & 8.7279 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. They are

$$\omega_1 = 2447, \omega_2 = 595.1$$

Problem 10.56 illustrates the use of the Rayleigh-Ritz method to approximate the lowest natural frequencies of a pinned-pinned beam with an attached mass.

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CHAPTER 11: FINITE-ELEMENT METHOD

Short Answer Questions

11.1 True: The function could satisfy only the geometric boundary conditions and still be an admissible function.

11.2 False: The boundary condition at a free end for a bar is a natural boundary condition.

11.3 True: The boundary conditions at a free end of a beam are the moment and shear is zero, both of which are a result of conservation laws. Thus they are natural boundary conditions.

11.4 True: The degrees of freedom of the beam element are the slopes and displacements of each end of the element.

11.5 False: For example a two element finite element model of a bar with a spring at each end predicts three natural frequencies, and a two-element model of a fixed-fixed bar predicts one natural frequency.

11.6 True: The model behaves as a discrete system model with the number of degrees of freedom governed by the global generalized coordinates.

11.7 True: The initial conditions can be expanded in a series of the basis functions (interpolation).

11.8 False: The local generalized coordinates coincide with the global generalized coordinates but for example w_3 from element 1 has the same global coordinate as w_1 from element 2.

11.9 False: The stiffness matrix for an interior element of length ℓ for a uniform bar is

$$k = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

11.10 False: The functions $w_1(x) = x + 1$ and $w_2(x) = x^2 + 1$ cannot be used as trial functions using the assumed modes method to predict the lowest natural frequencies of a fixed-free bar because they do not satisfy the geometric boundary condition that w(0) = 0.

11.11 An admissible function is a function which has appropriate continuity (displacements are continuous for bars, displacements and slopes are continuous for beams) and satisfies all geometric boundary conditions.

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11.12 Natural boundary conditions are those which are imposed due to a conservation law such as the boundary condition at the free end of a bar, the bar is not subject to any forces thus the stress on the end of the bar must be zero.

11.13 Using the assumed modes method one assumes a solution is an expansion of assumed modes which at least satisfy geometric boundary conditions. Lagrange's equations are applied to derive a discrete set of differential equations for the time dependent coefficients in the differential equations. Then the solution of the differential equations, including the free response, proceeds like that of a MDOF system.

11.14 A finite-element solution must satisfy the geometric boundary condition, w(0) = 0.

11.15 There are no geometric boundary conditions, thus a finite element model does not have to satisfy any boundary conditions.

11.16 The two degrees of freedom associated with the model of an element for a torsional bar are the angular displacements at each end of the element.

11.17 The local generalized coordinates associated with a beam element are the displacements at each end of the element and the slopes of the elastic curve at each end of the element.

11.18 There are three degrees of freedom in a three-element finite element model of a fixed-free bar (the displacements at the right end of each element).

11.19 There is only one degree of freedom in a two-element model of a fixed-fixed shaft with a rotor at its midspan (the angular displacement of the boundary of the two elements).

11.20 The are two degrees of freedom in a two-element model of a fixed-fixed beam (the displacement and slope at the boundary between the two elements).

11.21 There are three degrees of freedom in a two-element model of a fixed-pinned beam (the displacement and slope at the boundary between the two elements and the slope of the pinned end).

11.22 There are six degrees of freedom in a three-element model of a fixed-free beam (the displacements and slopes of the right ends of the elements).

11.23 There are six degrees of freedom in a three-element model of beam fixed at one end and attached to a linear spring at its other end (the displacements and slopes of the right ends of the elements including the end where the spring is attached).

11.24 A one-element finite-element model of a fixed-free bar attached to a spring of stiffness $\frac{EA}{2L}$ requires one-degree-of-freedom, the displacement of the end of the bar with the spring. The local coordinates are related to the global coordinates by $w_1 = 0$ and $w_2 = W_1$. The first row and first column of the element mass and stiffness matrices can be crossed out leaving

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$$\frac{\rho AL}{6}2\ddot{W}_1 + \left(\frac{EA}{L} + k\right)W_1 = 0$$

from which the natural frequency approximation is obtained as $\omega_1 = \sqrt{\frac{\frac{EA}{L}+k}{\frac{\rho AL}{3}}} = \sqrt{\frac{\frac{3EA}{L}\left(\frac{3}{2}\right)}{\rho AL}} = \sqrt{\frac{9E}{2\rho L^2}} = \frac{3}{L}\sqrt{\frac{E}{2\rho}}$.

11.25 A one-element finite-element model of the angular displacement of a fixed-free shaft attached to a spring of stiffness k_t requires one-degree-of-freedom, the displacement of the end of the bar with the spring. The local coordinates are related to the global coordinates by $\theta_1 = 0$ and $\theta_2 = W_1$. The first row and first column of the element mass and stiffness matrices can be crossed out leaving

$$\frac{\rho AL}{6}2\ddot{W}_1 + \left(\frac{GA}{L} + k_t\right)W_1 = 0$$

from which the natural frequency approximation is obtained as $\omega_1 = \sqrt{\frac{\frac{GA}{L}+k_t}{\frac{\rho AL}{6}}}$.

11.26 A one-element finite-element model of the angular displacement of a fixed-free shaft attached to a disk of mass moment of inertia I requires one-degree-of-freedom, the displacement of the end of the bar with the spring. The local coordinates are related to the global coordinates by $\theta_1 = 0$ and $\theta_2 = W_1$. The first row and first column of the element mass and stiffness matrices can be crossed out leaving

$$\left(\frac{\rho AL}{3} + I\right)\ddot{W}_1 + \frac{GA}{L}W_1 = 0$$

from which the natural frequency approximation is obtained as $\omega_1 = \sqrt{\frac{\frac{GA}{L}}{\frac{\rho AL}{3}+I}}$.

11.27 A one-element finite-element model of a fixed-free bar with a force $F_0 \sin \omega t$ at its free end requires one-degree-of-freedom, the displacement of the end of the bar. The local coordinates are related to the global coordinates by $w_1 = 0$ and $w_2 = W_1$. The force distribution is $F_0\delta(x - L)\sin \omega t$. The generalized force is $q_2 = \int_0^L \frac{F_0}{L}\delta(\xi - L)\sin \omega t\, d\xi = \frac{F_0}{L}\sin \omega t$. The first row and first column of the element mass and stiffness matrices can be crossed out leaving

$$\frac{\rho AL}{6}2\ddot{W}_1 + \frac{EA}{L}W_1 = \frac{F_0}{L}\sin \omega t$$

The steady-state solution of the differential equation is

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$$W_1 = \frac{3F_0}{\rho A L^2}\frac{1}{\frac{3E}{\rho L^2} - \omega^2}\sin\omega t$$

11.28 A bar that is circular in cross section has an area that varies over an element according to $A(\xi) = \pi\left(r_1 + \frac{r_2 - r_1}{\ell}\xi\right)^2$ which leads to a kinetic energy of $T = \frac{1}{2}\int_0^\ell \rho\pi\left(r_1 +$ *r2−r1ℓξ21ℓw2−w1ξ+w12dξ*

1.29 A bar that is circular in cross section has an area that varies over an element according to $A(\xi) = \pi\left(r_1 + \frac{r_2 - r_1}{\ell}\xi\right)^2$ which leads to a potential energy of $V = \frac{1}{2}\int_0^\ell E\pi\left(r_1 +$ *r2−r1ℓξ21ℓw2−w12dξ=12Eπ3ℓ3r23−r12r2−r1w22−2w2w1+w12* from which the element stiffness matrix is determined as

$$\mathbf{k} = \frac{E\pi}{3\ell^3}(r_2^3 - r_1^2)(r_2 - r_1)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

11.30 The kinetic energy of the bar is $T = \frac{1}{2}\int_0^\ell A\left(\rho_1 + \frac{\rho_2 - \rho_1}{\ell}\xi\right)^2\left[\frac{1}{\ell}(\dot{w}_2 - \dot{w}_1)\xi + \dot{w}_1\right]^2 d\xi$ from which the local mass matrix can be calculated.

11.31 A one element finite-element model of a beam that is fixed at one end and attached to a linear spring of stiffness k at the other end has two degrees of freedom. The generalized coordinates are the displacement and slope of the end where the spring is attached. The stiffness matrix is modified to take into account the potential energy of the spring. The differential equations governing the model are

$$\frac{\rho A L}{420}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix}\begin{bmatrix} \ddot{W}_1 \\ \ddot{W}_2 \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 12 + \frac{kL^2}{EI} & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequency approximations are roots of

$$|-\omega^2\mathbf{M} + \mathbf{K}| = \begin{vmatrix} -\omega^2 156\mu + 12\sigma + k & -22L\mu\omega^2 - 6L\sigma \\ -22L\mu\omega^2 - 6L\sigma & -\omega^2 4L^2\mu + 4L^2\sigma \end{vmatrix}$$
$$= (-\omega^2 156\mu + 12\sigma + k)(-\omega^2 4L^2\mu + 4L^2\sigma) - (-22L\mu\omega^2 - 6L\sigma)^2$$

where $\mu = \frac{\rho A L}{420}$ and $\sigma = \frac{EI}{L^3}$. Further algebra yields

$$140\mu^2\omega^4 - L^2\sigma(936\mu + 4k)\omega^2 + L^2\sigma(12\sigma + 4k) = 0$$

The lowest natural frequency is

$$\omega = \frac{L}{\mu}\sqrt{\frac{\sigma}{140}}\left[936\mu + 4k - \sqrt{(936 + 4k)^2 - 640\mu(12\sigma + 4k)}\right]^{1/2}$$

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11.32 A one-element finite-element model of a simply supported beam has two generalized coordinates, the displacement of each end. The work done by a force applied at the midspan gives $q_1 = q_3 = \frac{F_0 L}{2}\sin\omega t$. Thus the differential equations governing the approximation are

$$\frac{\rho AL}{420}\begin{bmatrix}156 & 54\\ 54 & 156\end{bmatrix}\begin{bmatrix}\ddot{W}_1\\ \ddot{W}_2\end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix}12 & -12\\ -12 & 12\end{bmatrix}\begin{bmatrix}W_1\\ W_2\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}\frac{F_0 L}{2}\sin\omega t$$

A solution of the form

$$\begin{bmatrix}W_1\\ W_2\end{bmatrix} = \begin{bmatrix}U_1\\ U_2\end{bmatrix}\sin\omega t$$

is assumed. Substituting into the differential equations leads to

$$\begin{bmatrix}-156\mu\omega^2 + 12\sigma & -54\mu\omega^2 - 12\sigma\\ -54\mu\omega^2 - 12\sigma & -156\mu\omega^2 + 12\sigma\end{bmatrix}\begin{bmatrix}U_1\\ U_2\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}\frac{F_0 L}{2}$$

where $\mu = \frac{\rho AL}{420}$ and $\sigma = \frac{EI}{L^3}$. The solution of the equations is

$$U_1 = U_2 = \frac{\frac{F_0 L}{2}(-102\mu\omega^2 + 12\sigma)}{21{,}420\mu^2\omega^4 - 5040\mu\sigma\omega^2} = \frac{\frac{F_0 L}{2}\left(-102\omega^2 + 5040\frac{EI}{\omega^2\rho AL^4}\right)}{21{,}420\omega^2 - 2{,}116{,}800\frac{EI}{\rho AL^4}}$$

The steady-state amplitude at the midspan of the beam is

$$u\left(\frac{L}{2}, t\right) = \frac{1}{2}U_1 + \frac{1}{2}U_2 = \frac{F_0 L\left(-102\omega^2 + 5040\frac{EI}{\omega^2\rho AL^4}\right)}{21{,}420\omega^2 - 2{,}116{,}800\frac{EI}{\rho AL^4}}$$

11.33 A one-element finite-element model of a beam that is fixed-free has two degrees of freedom. The generalized coordinates are the displacement and slope of the free end. The gerneralized forces are

$$q_1 = F_0 \sin\omega t \qquad q_2 = 0$$

where the spring is attached. The stiffness matrix is modified to take into account the potential energy of the spring. The differential equations governing the model are

$$\frac{\rho AL}{420}\begin{bmatrix}156 & -22L\\ -22L & 4L^2\end{bmatrix}\begin{bmatrix}\ddot{W}_1\\ \ddot{W}_2\end{bmatrix} + \frac{EI}{L^2}\begin{bmatrix}12 & -6L\\ -6L & 4L^2\end{bmatrix}\begin{bmatrix}W_1\\ W_2\end{bmatrix} = \begin{bmatrix}F_0\sin\omega t\\ 0\end{bmatrix}$$

A solution of the form

$$\begin{bmatrix}W_1\\ W_2\end{bmatrix} = \begin{bmatrix}U_1\\ U_2\end{bmatrix}\sin\omega t$$

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is assumed. Substituting into the differential equations leads to

$$\begin{bmatrix} -156\mu\omega^2 + 12\sigma & -22L\mu\omega^2 - 6L\sigma \\ -22L\mu\omega^2 - 6L\sigma & -4L^2\mu\omega^2 + 4L^2\sigma \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

The partial solution of the equations is

$$U_1 = \frac{F_0 L^3}{EI} \frac{-4\omega^2 + 1680\dfrac{EI}{\rho A L^4}}{140\omega^4 - 260{,}400\omega^2 \dfrac{EI}{\rho A L^4} + 2{,}116{,}800\left(\dfrac{EI}{\rho A L^4}\right)^2}$$

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Chapter Problems

11.1 The potential energy scalar product for a uniform bar is defined as

$$(f,g)_v = \int_0^L EAf(x)\frac{d^2g}{dx^2}dx$$

Consider the cases where (a) the bar is fixed at $x = 0$ and free at $x = L$ and (b) the bar is fixed at $x = 0$ and attached to a linear spring of stiffness k at $x = L$. Discuss, in each case, the implication of requiring $f(x)$ and $g(x)$ to satisfy only the geometric boundary conditions.

Solution: The integration by parts formula is

$$\int u dv = uv - \int v du$$

Applying integration by parts to the potential energy scalar product with

$$u = EAf(x)$$

$$dv = \frac{d^2g}{dx^2}$$

leads to

$$(f,g)_V = EAf(x)\frac{dg}{dx}\bigg|_0^L - \int_0^L \frac{d}{dx}(EAf(x))\frac{dg}{dx}dx$$

(a) Requiring f(x) and g(x) to satisfy only the geometric boundary condition of f(0)=0 leads to

$$(f,g)_V = EA(L)f(L)\frac{dg(L)}{dx} - \int_0^L \frac{d}{dx}(EAf)\frac{dg}{dx}dx$$

In this case the potential energy scalar product is not proportional to the potential energy due to the presence of the boundary term. An assumed modes or Rayleigh-Ritz solution will not satisfy the natural boundary condition at x = L and the total potential energy will differ from the potential energy predicted by the energy scalar product.

(b) Again requiring f(x) and g(x) only to satisfy the geometric boundary condition at x = 0 leads to the potential energy scalar product not being proportional to the potential energy of the system. However if the free end is attached to a linear spring, the potential energy of the spring can be added to the potential energy of the system.

Problem 11.1 illustrates the concept of geometric and natural boundary conditions.

11.2 Use the assumed modes method with trial functions

$$\omega_1(x) = \sin\left(\pi\frac{x}{L}\right) \qquad \omega_2(x) = \sin\left(2\pi\frac{x}{L}\right) \qquad \omega_3(x) = \sin\left(3\pi\frac{x}{L}\right)$$

to approximate the lowest natural frequency and its corresponding mode shape for a uniform fixed-fixed bar of length L.

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Solution: The assumed modes approximation to the mode shapes for the bar is

$$u(x) = C_1 w_1(x) + C_2 w_2(x) + C_3 w_3(x)$$

Application of the assumed-modes method leads to a system of equations of the form

$$\mathbf{M}\ddot{\mathbf{C}} + \mathbf{K}\mathbf{C} = \mathbf{0}$$

The approximations to the natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ where the elements of these matrices are

$$m_{ij} = \int_0^L \rho A w_i w_j \, dx$$

$$k_{ij} = \int_0^L EA \frac{dw_i}{dx}\frac{dw_j}{dx} \, dx$$

For example

$$m_{12} = \rho A \int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right) dx = 0$$

$$k_{23} = EA \int_0^l \left(\frac{\pi}{L}\right)\left(\frac{2\pi}{L}\right)\cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right) dx = 0$$

Using similar calculations the mass matrix is determined as

$$\mathbf{M} = \rho A \frac{L}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The elements of the stiffness matrix are determined as

$$\mathbf{K} = EA \frac{\pi^2}{2L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Then

$$\mathbf{M}^{-1}\mathbf{K} = \frac{\pi^2 E}{\rho L^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

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The natural frequency approximations are

$$\omega_1 = \pi\sqrt{\frac{E}{\rho L^2}} \quad \omega_2 = 2\pi\sqrt{\frac{E}{\rho L^2}} \quad \omega_3 = 3\pi\sqrt{\frac{E}{\rho L^2}}$$

The corresponding approximation for the first mode shape is

$$u_1(x) = \sin\left(\frac{\pi x}{L}\right)$$

The natural frequency and mode shape approximations are the same as the exact values for a fixed-fixed bar because the exact mode shapes are used as assumed modes.

Problem 11.2 illustrates application of the assumed modes method to determine the natural frequencies of a fixed-fixed bar.

11.3 Let $w_1(x)$, $w_2(x)$, $w_3(x)$, $w_4(x)$ be linearly independent polynomials of degree four or less that satisfy the geometric boundary conditions for a bar fixed at $x = 0$ and attached to a spring of stiffness k at $x = L$.

(a) Determine a set of $w_1(x)$, $w_2(x)$, $w_3(x)$, $w_4(x)$.
(b) Use the assumed modes method with the functions obtained in part (a) as trial functions and $kL/EA = 0.5$ to approximate the system's lowest natural frequencies and mode shapes.

Given: kL/EA = 0.5

Find: (a) $w_1(x)$, $w_2(x)$, $w_3(x)$, $w_4(x)$; (b) lowest ω and mode shapes

Solution: The most general form of a polynomial of degree four or less is

$$w(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

The geometric boundary condition is w(0) = 0. In order for all trail functions to satisfy this geometric condition $c_0 = 0$. Then a set of trial functions can be chosen as

$$w_1(x) = x/L \quad w_2(x) = (x/L)^2 \quad w_3(x) = (x/L)^3 \quad w_4(x) = (x/L)^4$$

A nondimensional variable is introduced as z = x/L. When the assumed modes method is used the elements of the mass and stiffness matrices are

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$$m_{ij} = \frac{1}{L}\int_0^1 \rho A w_i(z) w_j(z)\, dz$$

$$k_{ij} = \frac{1}{L}\int_0^1 EA \frac{dw_i}{dz}\frac{dw_j}{dz}\, dz + k w_i(1) w_j(1)$$

$$= \left(\frac{EA}{L}\right)\left(\int_0^1 \frac{dw_i}{dz}\frac{dw_j}{dz} + 0.5 w_i(1) w_j(1)\right)$$

For example

$$m_{13} = \frac{\rho A}{L}\int_0^1 (x)(x^3)\, dx = 0.2\frac{\rho A}{L}$$

$$k_{24} = \frac{EA}{L}\left(\int_0^1 (2x)(4x^3)\, dx + 0.5(1)(1)\right) = 2.3\frac{EA}{L}$$

The mass and stiffness matrices become

$$\mathbf{M} = \frac{\rho A}{L}\begin{bmatrix} 0.333 & 0.25 & 0.2 & 0.167 \\ 0.25 & 0.2 & 0.167 & 0.143 \\ 0.2 & 0.167 & 0.143 & 0.125 \\ 0.167 & 0.143 & 0.125 & 0.111 \end{bmatrix}$$

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} 1.5 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.833 & 2 & 2.3 \\ 1.5 & 2 & 2.3 & 2.5 \\ 1.5 & 2.3 & 2.5 & 2.786 \end{bmatrix}$$

The lowest natural frequency is approximated as

$$\omega_1 = 1.83\sqrt{\frac{E}{\rho}}$$

Problem 11.3 illustrates application of the assumed modes method.

11.4 Use the assumed modes method with trial functions

$$w_1(x) = x(x-L) \qquad w_2(x) = x(x-L)^2 \quad w_3(x) = x(x-L)^3$$

to approximate the two lowest natural frequencies and mode shapes for a simply supported beam.

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Solution: Using the given trial functions the assumed modes approximation for the mode shapes is

$$u(x) = C_1 w_1(x) + C_2 w_2(x) + C_3 w_3(x)$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ where the elements of these matrices are

$$m_{ij} = \int_0^L \rho A w_i(x) w_j(x)\,\mathrm{dx}$$

$$k_{ij} = \int_0^L EI \frac{d^2 w_i}{dx^2}\frac{d^2 w_j}{dx^2}\,\mathrm{dx}$$

For example

$$m_{12} = \rho A \int_0^L \left[x\left(1-\frac{x}{L}\right)\right]\left[x\left(1-\frac{x}{L}\right)^2\right]\mathrm{dx} = \frac{1}{60}\rho A L^3$$

$$k_{12} = EI\int_0^L \left(-\frac{2}{\mathrm{L}}\right)\left(-\frac{4}{\mathrm{L}}+\frac{6x}{L^2}\right)\mathrm{dx} = \frac{2EI}{L}$$

Similar calculations lead to the mass and stiffness matrices

$$\mathbf{M} = \rho A L^3 \begin{bmatrix} \frac{1}{30} & \frac{1}{60} & \frac{1}{105} \\ \frac{1}{60} & \frac{1}{105} & \frac{1}{168} \\ \frac{1}{105} & \frac{1}{186} & \frac{1}{152} \end{bmatrix} \quad \mathbf{K} = \frac{EI}{L}\begin{bmatrix} 4 & 2 & -14 \\ 2 & 4 & 8 \\ -14 & 8 & \frac{624}{5} \end{bmatrix}$$

The two lowest eigenvalues and the corresponding eigenvectors are

$$\lambda_1 = 1.589\frac{EI}{\rho A L^4} \quad \mathbf{X}_1 = \begin{bmatrix} -1.056 \\ 0.873 \\ -0.174 \end{bmatrix}$$

$$\lambda_2 = 1433\frac{EI}{\rho A L^4} \quad \mathbf{X}_2 = \begin{bmatrix} 7.131 \\ -16.26 \\ 1.58 \end{bmatrix}$$

thus the natural frequency and mode shape approximations are

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$$\omega_1 = \sqrt{\lambda_1} = 1.261\sqrt{\frac{EI}{\rho AL^4}}$$

$$u_1(x) = -1.056x\left(1-\frac{x}{L}\right) + 0.873x\left(1-\frac{x}{L}\right)^2 - 0.174x\left(1-\frac{x}{L}\right)^3$$

$$\omega_2 = \sqrt{\lambda_2} = 37.86\sqrt{\frac{EI}{\rho AL^4}}$$

$$u_2(x) = 7.131x\left(1-\frac{x}{L}\right) - 16.26x\left(1-\frac{x}{L}\right)^2 + 1.58x\left(1-\frac{x}{L}\right)^3$$

Problem 11.4 illustrates application of the assumed modes method to approximate the natural frequencies of a simply supported beam.

11.5 Repeat Chapter Problem 11.4 if the beam has a machine of mass $m = 2.0\rho AL$ where ρAL is the total mass of the beam. The machine is placed at the midspan of the beam.

Solution: Using the given trial functions the assumed modes approximation for the mode shapes is

$$u(x) = C_1 w_1(x) + C_2 w_2(x) + C_3 w_3(x)$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ where the elements of these matrices are

$$m_{ij} = \int_0^L \rho A w_i(x) w_j(x)\,\mathrm{dx} + 2.0\rho AL w_i(L/2) w_j(L/2)$$

$$k_{ij} = \int_0^L EI \frac{d^2 w_i}{dx^2}\frac{d^2 w_j}{dx^2}\,\mathrm{dx}$$

For example

$$m_{12} = \rho A\int_0^L \left[x\left(1-\frac{x}{L}\right)\right]\left[x\left(1-\frac{x}{L}\right)^2\right]\mathrm{dx} + 2.0\rho AL\left(\frac{L}{4}\right)\left(\frac{L}{8}\right) = 0.0792\rho AL^3$$

$$k_{12} = EI\int_0^L \left(-\frac{2}{\mathrm{L}}\right)\left(-\frac{4}{\mathrm{L}} + \frac{6x}{L^2}\right)\mathrm{dx} = \frac{2EI}{L}$$

Similar calculations lead to the mass and stiffness matrices

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$$\mathbf{M} = \rho A L^3 \begin{bmatrix} 0.1583 & 0.0792 & 0.0408 \\ 0.0792 & 0.0408 & 0.0216 \\ 0.0408 & 0.0216 & 0.0118 \end{bmatrix} \quad \mathbf{K} = \frac{EI}{L} \begin{bmatrix} 4 & 2 & -14 \\ 2 & 4 & 8 \\ -14 & 8 & \frac{624}{5} \end{bmatrix}$$

The two lowest eigenvalues and the corresponding eigenvectors are

$$\lambda_1 = 0.3499 \frac{EI}{\rho A L^4} \quad \mathbf{X}_1 = \begin{bmatrix} 0.7648 \\ -0.6318 \\ 0.1263 \end{bmatrix}$$

$$\lambda_2 = 1380 \frac{EI}{\rho A L^4} \quad \mathbf{X}_2 = \begin{bmatrix} 0.4213 \\ -0.9023 \\ 0.0916 \end{bmatrix}$$

thus the natural frequency and mode shape approximations are

$$\omega_1 = \sqrt{\lambda_1} = 0.5916 \sqrt{\frac{EI}{\rho A L^4}}$$

$$u_1(x) = 0.7648x\left(1 - \frac{x}{L}\right) - 0.6318x\left(1 - \frac{x}{L}\right)^2 + 0.1263x\left(1 - \frac{x}{L}\right)^3$$

$$\omega_2 = \sqrt{\lambda_2} = 37.16 \sqrt{\frac{EI}{\rho A L^4}}$$

$$u_2(x) = 0.4212x\left(1 - \frac{x}{L}\right) - 0.90236x\left(1 - \frac{x}{L}\right)^2 + 0.0916x\left(1 - \frac{x}{L}\right)^3$$

Problem 11.5 illustrates application of the assumed modes method to approximate the natural frequencies of a simply supported beam with a concentrated mass at its midspan.

11.6 The mode shapes of a uniform fixed-free bar of the form

$$\phi_n(x) = \sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad n = 1, 2, 3, \ldots$$

Use the assumed modes method with $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ as trial functions to approximate the lowest natural frequency and mode shapes for the tapered bar of Figure P11.6.

Given: r_0 = 5cm, $r(x) = r_0(1-\lambda)^2$, λ = 1cm/m, L = 3 m, E = 200 × 10^9 N/m^2, ρ = 7500 kg/m^3

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FIGURE P11.6

Find: lowest ω and mode shape

Solution: The trial functions are

$$\phi_1(x) = \sin\left(\frac{\pi x}{2L}\right)$$

$$\phi_2(x) = \sin\left(\frac{3\pi x}{2L}\right)$$

$$\phi_3(x) = \sin\left(\frac{5\pi x}{2L}\right)$$

The cross-section area is

$$A(x) = r_0^2(1-\lambda x)^4$$

When the assumed modes method is used the elements of the approximate mass and stiffness matrices are of the form

$$m_{ij} = \int_0^L \rho A(x)\phi_i(x)\phi_j(x)\,dx$$

$$k_{ij} = \int_0^L EA(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx$$

For example

$$m_{12} = \int_0^L \rho\pi r_0^2(1-\lambda x)^4 \sin\left(\frac{\pi x}{2L}\right)\sin\left(\frac{3\pi x}{2L}\right)dx$$

$$k_{23} = \int_0^L E\pi r_0^2(1-\lambda x)^4\left(\frac{\pi}{2L}\right)\cos\left(\frac{\pi x}{2L}\right)\left(\frac{3\pi}{2L}\right)\cos\left(\frac{3\pi x}{2L}\right)dx$$

Application of these equations leads to

$$\mathbf{M} = \begin{bmatrix} 81.89 & 2.03 & -0.205 \\ -0.205 & 82.984 & 2.048 \\ -0.205 & 2.048 & 83.13 \end{bmatrix}$$

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$$\mathbf{K} = 10^8 \begin{bmatrix} 6.234 & 0.4556 & 0.009202 \\ 0.4556 & 54.9 & 2.258 \\ 0.009202 & 2.258 & 152.2 \end{bmatrix}$$

The natural frequency approximations are the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The mode shape approximations are obtained using the corresponding eigenvector. The lowest natural frequency approximation and corresponding eigenvector are

$$\omega_1 = 2.771 \times 10^3 \text{ rad/s} \quad \mathbf{X_1} = \begin{bmatrix} 1 \\ -6.147 \times 10^{-3} \\ -6.502 \times 10^{-4} \end{bmatrix}$$

The assumed modes approximation for the mode shape for the lowest frequency mode is

$$u_1(x) = \sin\left(\frac{\pi x}{L}\right) - 6.147 \times 10^{-3} \sin\left(\frac{3\pi x}{2L}\right) - 6.502 \times 10^{-4} \sin\left(\frac{5\pi x}{2L}\right)$$

Problem 11.6 illustrates application of the assumed modes method to determine approximations to the natural frequencies of a non-uniform bar.

11.7 Use a one-element, finite-element model to approximate the lowest natural frequency of a uniform bar of mass density ρ, cross-sectional area A, elastic modulus E, and length L that is fixed at one end and has a block of mass m attached at one end.

Given: E, A, L ρ, m

Find: ω_1

Solution: The one-element finite-element model of a fixed-free bar with an end mass is illustrated below.

The one-element model has one-degree-of-freedom. The relationships between the element coordinates and the global coordinates are

$$w_1 = 0 \quad w_2 = W$$

The element stiffness matrix is

$$\mathbf{k} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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The element mass matrix is obtained by including the kinetic energy of the end mass. The total kinetic energy of the element with the end mass is

$$T = \frac{1}{2}\left[\frac{\rho AL}{3}\left(\dot{w}_1^2 + \dot{w}_1\dot{w}_2 + \dot{w}_2^2\right) + m\dot{w}_2^2\right]$$

leading to the element mass matrix of

$$\mathbf{m} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 + \dfrac{6m}{\rho AL} \end{bmatrix}$$

The differential equations become

$$\frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 + \dfrac{6m}{\rho AL} \end{bmatrix}\begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{bmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Substitution for the local coordinates in terms of the generalized coordinates leads to

$$\left(\frac{\rho AL}{3} + m\right)\ddot{W} + \frac{EA}{L}W = 0$$

$$\ddot{W} + \left(\frac{3EA}{\rho AL^2 + mL}\right)W = 0$$

from which the natural frequency approximation is obtained as

$$\omega_1 = \sqrt{\frac{3EA}{\rho AL^2 + mL}}$$

Problem 11.7 illustrates the use of a one-element finite element model to approximate the natural frequency of a bar with an end mass.

11.8 Derive the element stiffness and mass matrices for a tapered bar of rectangular cross-section, $A(x) = A_0(1 - \mu x)$.

Given: A(x)

Find: **m, k**

Solution: The element stiffness matrix is obtained from

$$V = \frac{1}{2}\int_0^{\ell} EA_0\,(1 - \mu\xi)\left[\frac{1}{\ell}(w_2 - w_1)\right]^2 dx = \frac{1}{2}\frac{EA_0}{\ell}\left(1 - \mu\frac{\ell}{2}\right)(w_2^2 - 2w_1w_2 + w_1^2)$$

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The stiffness matrix is determined as

$$\mathbf{k} = \frac{EA_0}{\ell}\left(1 - \mu\frac{\ell}{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The mass matrix is obtained from

$$T = \frac{1}{2}\int_0^{\ell} \rho A_0 (1 - \mu\xi)\left[\frac{1}{\ell}(\dot{w}_2 - \dot{w}_1)\xi + \dot{w}_1\right]^2 dx$$
$$= \frac{1}{2}\rho A_0 \left[\frac{\ell}{3}\left(1 - \frac{3\mu\ell}{4}\right)(\dot{w}_2^2 - 2\dot{w}_1\dot{w}_2 + \dot{w}_1^2) + \ell\left(1 - \frac{2\mu\ell}{3}\right)(\dot{w}_2 - \dot{w}_1)\dot{w}_1 + \ell\left(1 - \frac{\ell}{2}\right)\dot{w}_1\right]$$

The mass matrix is calculated as

$$\mathbf{m} = \rho A_0 \begin{bmatrix} \frac{\ell}{3}\left(1 - \frac{3\mu\ell}{4}\right) & \frac{5\ell}{2}\left(\frac{1}{6} - \frac{\mu\ell}{12}\right) \\ \frac{5\ell}{2}\left(\frac{1}{6} - \frac{\mu\ell}{12}\right) & \ell\left(2 - \frac{5\mu\ell}{6}\right) \end{bmatrix}$$

Problem 11.8 illustrates calculation of the mass and stiffness matrices for a non-uniform element.

11.9 Use a one-element, finite-element model to approximate the lowest nonzero torsional natural frequency of a uniform shaft of mass density ρ, polar moment of inertia J, shear modulus G, and length L that has a thin disk of mass moment of inertia I_1 attached at one end and a thin disk of mass moment of inertia I_2 attached at the other end.

Given: ρ, J, L, G, I_1, I_2

Find: ω_2

Solution: The one-element model of the shaft is shown below.

The one-element model has two-degrees-of-freedom. The local coordinates are the same as the global coordinates. The stiffness matrix for the element is

$$\mathbf{k} = \frac{JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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The kinetic energy of the element including the kinetic energy of the end disks is

$$T = \frac{1}{2}\left[\frac{\rho JL}{3}(\dot{w}_1^2 + \dot{w}_1\dot{w}_2 + \dot{w}_2^2) + I_1\dot{w}_1^2 + I_2\dot{w}_2^2\right]$$

which leads to the element mass matrix of

$$\mathbf{m} = \begin{bmatrix} \frac{\rho JL}{3} + I_1 & \frac{\rho JL}{6} \\ \frac{\rho JL}{6} & \frac{\rho JL}{3} + I_2 \end{bmatrix}$$

The differential equations for the motion of the shaft are

$$\begin{bmatrix} \frac{\rho JL}{3} + I_1 & \frac{\rho JL}{6} \\ \frac{\rho JL}{6} & \frac{\rho JL}{3} + I_2 \end{bmatrix}\begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{bmatrix} + \frac{JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The methods of chapter 6 are applied to determine the natural frequency approximations as

$$\omega_1 = 0 \quad \omega_2 = \sqrt{\frac{12G(\rho JL + I_1 + I_2)}{\rho L^2(\rho JL + 4I_1 + 4I_2)}}$$

Problem 11.9 illustrates a one-element finite-element model of a torsional shaft with end rotors.

11.10 Use a one-element, finite-element model to approximate the lowest natural frequencies of a uniform beam of mass density ρ, cross-sectional area A, cross-sectional moment of inertia I, elastic modulus E, and length L that is free at both ends.

Given: E, A, L, ρ

Find: ω_2

Solution: The one-element finite-element model of a free-free bar is shown below

The one-element finite-element model has two-degrees-of-freedom. The global coordinates are the same as the local coordinates. The local stiffness and mass matrices are respectively

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$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \mathbf{m} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The differential equations for the local coordinates are

$$\frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{bmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{m}^{-1}\mathbf{k}$ and are determined as

$$\omega_1 = 0 \qquad \omega_2 = 2\sqrt{\frac{3E}{\rho L^2}}$$

Problem 11.10 illustrates use of a one-element finite-element model to approximate the lowest non-zero natural frequency of a free-free bar.

11.11 Derive the element m_{34} of the element mass matrix for a beam element.

Solution: The quadratic form of the kinetic energy for the element matrix is

$$T = \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} m_{ij}\dot{w}_i\dot{w}_j$$

Hence $2m_{34}$ multiplies $1/2\dot{w}_3\dot{w}_4$ in the general expression for the kinetic energy. Using Eq.(11.30) in Eq.(11.31) leads to

$$m_{34} = \int_0^{\ell} \rho A\left(3\frac{\xi^2}{\ell^2} - 2\frac{\xi^3}{\ell^3}\right)\left(-\frac{\xi^2}{\ell^2} + \frac{\xi^3}{\ell^3}\right)\ell d\xi$$

Defining $\eta = \xi/\ell$ leads to

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$$m_{34} = \rho A \ell \int_0^1 \left(3\eta^2 - 2\eta^3\right)\left(-\eta^2 + \eta^3\right) \ell d\eta$$

$$m_{34} = \rho A \ell \int_0^1 (-2\eta^6 + 5\eta^5 - 3\eta^4)\, \ell d\eta$$

$$m_{34} = \rho A \ell^2 \left[-\frac{2}{7}\eta^7 + \frac{5}{6}\eta^6 - \frac{3}{5}\eta^5 \right]\Bigg|_{\eta=0}^{\eta=1}$$

$$m_{34} = \rho A \ell^2 \left(-\frac{2}{7} + \frac{5}{6} - \frac{3}{5} \right)$$

$$m_{34} = \rho A \ell \left(-\frac{22\ell}{420} \right)$$

Problem 11.11 illustrates the determination of an element of a local mass matrix for a beam element.

11.12 Derive the element k_{23} of the element stiffness matrix for a beam element.

Solution: The quadratic form of the potential energy for a beam element is

$$V = \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} k_{ij} w_i w_j$$

Thus the $2k_{23}$ multiplies $1/2 w_2 w_3$ in the quadratic form of the potential energy. Using Eq.(10.30) in Eq.(10.33) and noting that

$$\frac{\partial^2 u}{\partial \xi^2} = \left(-\frac{6}{\ell^2} + \frac{12\xi}{\ell^3} \right) w_1 + \left(-\frac{4}{\ell^2} + \frac{6\xi}{\ell^3} \right) \ell w_2$$
$$+ \left(\frac{6}{\ell^2} - \frac{12\xi}{\ell^3} \right) w_3 + \left(-\frac{2}{\ell^2} + \frac{6\xi}{\ell^3} \right) \ell w_4$$

leads to

$$k_{23} = \int_0^\ell EI \left(-\frac{4}{\ell^2} + \frac{6\xi}{\ell^3} \right) \left(\frac{6}{\ell^2} - \frac{12\xi}{\ell^3} \right) \ell \, d\xi$$

$$k_{23} = \frac{EI}{\ell^3} \int_0^\ell \left(-4 + 6\frac{\xi}{\ell} \right) \left(6 - 12\frac{\xi}{\ell} \right) d\xi$$

Defining $\eta = \xi / \ell$ leads to

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$$k_{23} = \frac{EI}{\ell^2}\int_0^1 (-4+6\eta)(6-12\eta)\,d\eta$$

$$k_{23} = \frac{12EI}{\ell^2}\int_0^1 (-2+7\eta-6\eta^2)\,d\eta$$

$$k_{23} = \frac{12EI}{\ell^2}\left(-2\eta+\frac{7}{2}\eta^2-2\eta^3\right)\Bigg|_{\eta=0}^{\eta=1}$$

$$k_{23} = \frac{12EI}{\ell^2}\left(-2+\frac{7}{2}-2\right)$$

$$k_{23} = -\frac{6EI}{\ell^2}$$

Problem 11.12 illustrates the determination of an element of the local stiffness matrix for a beam element.

11.13 Use a two-element, finite-element model to approximate the two lowest natural frequencies and their corresponding mode shapes for the system of Figure P11.13.

FIGURE P11.13

Given: E = 200 × 10^9 N/m^2, A = 1.6 × 10^{-4} m^2, L = 2.5 m, k = 1 × 10^7 N/m, ρ = 7600 kg/m^3

Find: ω_1, ω_2

Solution: The two-element finite-element model of the system, assuming each element is of equal length, is shown below.

The two-element model has two-degrees-of-freedom. The element stiffness and mass matrices are

Element 1: Local generalized coordinates $w_1 = 0$, $w_2 = W_1$. Local element matrices are

$$\mathbf{m}_1 = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{k}_1 = \frac{2EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global element matrices are

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$$\tilde{\mathbf{M}}_1 = \frac{\rho AL}{12}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{K}}_1 = \frac{2EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element 2: Local generalized coordinates $w_1 = W_1, w_2 = W_2$. Local element matrices are

$$\mathbf{m}_2 = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{k}_2 = \frac{2EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1+\dfrac{kL}{2EA} \end{bmatrix}$$

The global element matrices are

$$\tilde{\mathbf{M}}_2 = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \tilde{\mathbf{K}}_2 = \frac{2EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1+\dfrac{kL}{2EA} \end{bmatrix}$$

The system global matrices are

$$\mathbf{M} = \frac{\rho AL}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{K} = \frac{2EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1+\dfrac{kL}{2EA} \end{bmatrix}$$

Substitution of given values leads to

$$\mathbf{M} = \begin{bmatrix} 1.0133 & 0.2533 \\ 0.2533 & 0.5067 \end{bmatrix} \quad \mathbf{K} = 1x10^7\begin{bmatrix} 5.12 & -2.56 \\ -2.56 & 3.56 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The results are

$$\omega_1 = 4.17\times10^3 \text{ rad/s}, \quad \omega_2 = 1.22\times10^4 \text{ rad/s}$$

Problem 11.13 illustrates a two-element finite-element model of a bar.

11.14 Use a two-element, finite-element model to approximate the two lowest torsional natural frequencies for the system of Figure P11.14.

FIGURE P11.14

Given: $J = 3.2 \times 10^{-5}$ m^4, $G = 80 \times 10^9$ N/m^2, $\rho = 7000$ kg/m^3, $L = 4.2$ m, $I = 1.5$ kg · m^2

Find: ω_1, ω_2

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Solution: The two-element finite-element model of the system is shown below.

Element 1: Local generalized coordinates: $w_1 = 0, w_2 = W_1$. The local matrices are

$$\mathbf{m}_1 = \frac{\rho JL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{k}_1 = \frac{2JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global matrices for element 1 are

$$\tilde{\mathbf{M}}_1 = \frac{\rho JL}{12}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{K}}_1 = \frac{2JG}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Element 2: Local generalized coordinates: $w_1 = W_1, w_2 = W_2$. The local matrices are

$$\mathbf{m}_2 = \frac{\rho JL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 + \dfrac{12I}{\rho JL} \end{bmatrix} \quad \mathbf{k}_2 = \frac{2JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global matrices for element 2 are

$$\tilde{\mathbf{M}}_2 = \frac{\rho JL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 + \dfrac{12I}{\rho JL} \end{bmatrix} \quad \tilde{\mathbf{K}}_2 = \frac{2JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global matrices for the system are

$$\mathbf{M} = \frac{\rho JL}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 + \dfrac{12I}{\rho JL} \end{bmatrix} \quad \mathbf{K} = \frac{2JG}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Substitution of given values leads to

$$\mathbf{M} = \begin{bmatrix} 0.317 & 0.0784 \\ 0.0784 & 1.66 \end{bmatrix} \quad \mathbf{K} = 1\times 10^6 \begin{bmatrix} 2.44 & -1.22 \\ -1.22 & 1.22 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvlaues of $\mathbf{M}^{-1}\mathbf{K}$. The results are

$$\omega_1 = 5.78\times 10^2 \text{ rad/s}$$
$$\omega_2 = 2.94\times 10^3 \text{ rad/s}$$

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Problem 11.14 illustrates the use of a two-element finite element model for a fixed-free shaft with an end rotor.

11.15 Use a three-element, finite-element model to approximate the lowest natural frequency and its corresponding mode shape for the system of Figure P11.15.

$E = 200 \times 10^9$ N/m^2
$A = 3.5 \times 10^{-5}$ m^2
$L = 2.5$ m
$\rho = 7000$ kg/m^3

$m = 1.2$ kg
$k_1 = 2 \times 10^6$ N/m
$k_2 = 1.4 \times 10^6$ N/m

FIGURE P11.15

Given: E = 200 × 10^9 N/m^2, A = 3.5 × 10^{-5} m^2, L = 2.5 m, ρ = 7000 kg/m^3, m = 1.2 kg, k_1 = 2 × 10^6 N/m, k_2 = 1.4 ×10^6 N/m

Find: ω_1, $\mathbf{X_1}$

Solution: The three-element finite-element model is shown below.

The three-element model has four degrees of freedom where one-degree-of-freedom represents the displacement of the discrete mass. The length of each element is L/3

Element 1: Local generalized coordinates: $w_1 = 0, w_2 = W_1$. The global matrices for the element are

$$\tilde{\mathbf{M}}_1 = \frac{\rho J \ell}{6}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{K}}_1 = \frac{EA}{\ell}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: Local generalized coordinates: $w_1 = W_1,\ w_2 = W_2$. The global matrices for element 2 are

$$\tilde{\mathbf{M}}_2 = \frac{\rho J \ell}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{K}}_2 = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Element 3: Local generalized coordinates: $w_1 = W_2, w_2 = W_3$. The global matrices for element 3 are

$$\tilde{\mathbf{M}}_3 = \frac{\rho J \ell}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{K}}_3 = \frac{EA}{\ell}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The kinetic energy of the discrete mass is

$$T = \frac{1}{2} m \dot{W}_4^2$$

The potential energy of the discrete springs is

$$V = \frac{1}{2} k_1 (W_4 - W_3)^2 + \frac{1}{2} k_2 W_4^2$$

Thus the contributions to the global matrices from the discrete mass and springs are

$$\tilde{\mathbf{M}}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \quad \tilde{\mathbf{K}}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_1 & -k_1 \\ 0 & 0 & -k_1 & k_1 + k_2 \end{bmatrix}$$

The global mass and stiffness matrices are

$$\mathbf{M} = \frac{\rho J \ell}{6}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & \dfrac{6m}{\rho J \ell} \end{bmatrix}$$

$$\mathbf{K} = \frac{EA}{\ell}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 + \dfrac{k_1 \ell}{EA} & -\dfrac{k_1 \ell}{EA} \\ 0 & 0 & -\dfrac{k_1 \ell}{EA} & \dfrac{(k_1 + k_2)\ell}{EA} \end{bmatrix}$$

Substitution of given values leads to

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$$\mathbf{M} = \begin{bmatrix} 0.136 & 0.0341 & 0 & 0 \\ 0.0341 & 0.136 & 0.0341 & 0 \\ 0 & 0.0341 & 0.0684 & 0 \\ 0 & 0 & 0 & 1.2 \end{bmatrix} \quad \mathbf{K} = 1\times10^6 \begin{bmatrix} 16.8 & -8.4 & 0 & 0 \\ -8.4 & 16.8 & -8.4 & 0 \\ 0 & -8.4 & 10.4 & -2 \\ 0 & 0 & -2 & 3.4 \end{bmatrix}$$

The natural frequencies and mode shapes are the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$

The lowest natural frequency is determined as

$$\omega_1 = 1.58\times10^3 \text{ rad/s}$$

The corresponding mode shape vector is

$$\mathbf{X}_1 = \begin{bmatrix} 0.141 \\ 0.275 \\ 0.396 \\ 0.864 \end{bmatrix}$$

Problem 11.15 illustrates the use of a three-element finite-element model to predict the lowest natural frequency of a bar with an attached mass and spring system.

11.16 Use a three-element, finite-element model to approximate the steady-state response of the system of Figure P11.16.

FIGURE P11.15

Given: $L_1 = 2.1$ m, $G_1 = 40 \times 10^9$ N/m^2, $J_1 = 1.8 \times 10^{-5}$ m^4, $\rho_1 = 5000$ kg/m^3, $L_2 = 1.0$ m, $G_2 = 80 \times 10^9$ N/m^2, $J_2 = 4.3 \times 10^{-6}$ m^4, $\rho_2 = 7000$ kg/m^3, $I = 0.25$ kg · m^2, $T_0 = 100$ N · m, $\omega = 500$ rad/s

Find: steady-state response

Solution: A three-element finite-element model of the system is used. Two elements, each of length $\ell_1 = 1.05$ m are used over section AB and one element, of length $\ell_2 = 1.0$ m, used for section BC.

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The three-element model results in a three-degree-if-freedom system. The relations between the local generalized coordinates and the global generalized coordinates for each element are given below as well as the global matrices for each element. The local mass and stiffness matrices for each element are used, except that the mass matrix for the third element is modified to include the inertia of the attached rotor.

Element 1: Local generalized coordinates: $w_1 = 0,\ w_2 = W_1$

Global matrices:

$$\tilde{\mathbf{M}}_1 = \frac{\rho J \ell_1}{6}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.0315 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_1 = \frac{JG}{\ell_1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1\times 10^5 \begin{bmatrix} 6.86 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since no external forces are applied over element 1 the generalized force vector for the element is the zero vectors.

Element 2: Local generalized coordinates: $w_1 = W_1, w_2 = W_2$

Global matrices:

$$\tilde{\mathbf{M}}_2 = \frac{\rho J \ell_1}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.0315 & 0.0158 & 0 \\ 0.0158 & 0.0315 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{JG}{\ell_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1\times 10^5 \begin{bmatrix} 6.86 & -6.86 & 0 \\ -6.86 & 6.86 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since no external forces are applied over element 2, its generalized force vector is zero.

Element 3: Local generalized coordinates: $w_1 = W_2, \mathrm{w}_2 = W_3$

Global matrices:

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$$\tilde{\mathbf{M}}_3 = \frac{\rho J \ell_2}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2+\dfrac{6I}{\rho J \ell_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.0100 & 0.005 \\ 0 & 0.005 & 0.260 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_3 = \frac{JG}{\ell_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = 1\times 10^5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3.44 & -3.44 \\ 0 & -3.44 & 3.44 \end{bmatrix}$$

In local coordinates, the externally applied torque is expressed as

$$T(\xi, t) = T_0 \sin \omega t \, \delta(\xi - \ell_2)$$

which leads to a generalized force vector

$$\mathbf{Q} = \begin{bmatrix} 0 \\ 0 \\ T_0 \sin \omega t \end{bmatrix}$$

The assembled global matrices are

$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 + \tilde{\mathbf{M}}_3 = \begin{bmatrix} 0.0630 & 0.0158 & 0 \\ 0.0158 & 0.0415 & 0.0050 \\ 0 & 0.0050 & 0.2600 \end{bmatrix}$$

$$\mathbf{K} = \tilde{\mathbf{K}}_1 + \tilde{\mathbf{K}}_2 + \tilde{\mathbf{K}}_3 = 1\times 10^6 \begin{bmatrix} 1.371 & -0.686 & 0 \\ -0.686 & 1.030 & -0.344 \\ 0 & -0.344 & 0.344 \end{bmatrix}$$

The differential equations derived using the finite-element models are

$$\begin{bmatrix} 0.0630 & 0.0158 & 0 \\ 0.0158 & 0.0415 & 0.0050 \\ 0 & 0.0050 & 0.2600 \end{bmatrix}\begin{bmatrix} \ddot{W}_1 \\ \ddot{W}_2 \\ \ddot{W}_3 \end{bmatrix} + 10^6 \begin{bmatrix} 1.371 & -0.686 & 0 \\ -0.686 & 1.030 & -0.344 \\ 0 & -0.344 & 0.344 \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \sin 500t \end{bmatrix}$$

The steady-state solution of the differential equations can be written as

$$\mathbf{W} = \mathbf{U} \sin \omega t$$

Substitution into the differential equations leads to

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$$10^6\begin{bmatrix} 1.3557 & -0.6897 & 0 \\ -0.6897 & 1.0193 & -0.3453 \\ 0 & -0.3453 & 0.279 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$$

The solution to the above set of simultaneous equations is

$$\mathbf{U} = 10^{-3}\begin{bmatrix} 0.2610 \\ 0.5130 \\ 0.9932 \end{bmatrix}$$

Problem 11.16 illustrates use of the finite-element method to approximate the steady-state response of a non-uniform shaft with an end rotor subject to a harmonic torque.

11.17 Use a three-element finite element model to approximate the forced response of the system shown when the end of the bar is subject to the excitation shown in the figure.

FIGURE P11.15

FIGURE P11.17

Given: E = 200 × 10^9 N/m^2, A = 3.5 × 10^{-5} m^2, L = 2.5 m, ρ = 7000 kg/m^3, m = 1.2 kg, k_1 = 2 × 10^6 N/m, k_2 = 1.4 × 10^6 N/m

Find: forced response

Solution: The three element finite element model is developed in the solution of Chapter Problem 11.15 is shown below

The resulting mass and stiffness matrices are

$$\mathbf{M} = \begin{bmatrix} 0.136 & 0.0341 & 0 & 0 \\ 0.0341 & 0.136 & 0.0341 & 0 \\ 0 & 0.0341 & 0.0684 & 0 \\ 0 & 0 & 0 & 1.2 \end{bmatrix} \quad \mathbf{K} = 1\times10^6\begin{bmatrix} 16.8 & -8.4 & 0 & 0 \\ -8.4 & 16.8 & -8.4 & 0 \\ 0 & -8.4 & 10.4 & -2 \\ 0 & 0 & -2 & 3.4 \end{bmatrix}$$

The force is applied to the end of the bar, thus it is given the form of

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$$F(x,t) = 8\times 10^5[u(t) - 2u(t-0.01) + u(t-0.02)]\delta(x-L)$$

or applied to the end of the third element as $F(x,t) = \delta(x-\ell)$ which leads to $q_1 = 0$ and $q_2 = 8\times 10^8[u(t) - 2u(t-0.01) + u(t-0.02)]$ or in terms of global coordinates

$$\mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ 8\times 10^5[u(t) - 2u(t-0.01) + u(t-0.02)] \\ 0 \end{bmatrix}$$

The differential equations governing the finite-element approximation of the vibrations of the system are $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}$ which are

$$\begin{bmatrix} 0.136 & 0.0341 & 0 & 0 \\ 0.0341 & 0.136 & 0.0341 & 0 \\ 0 & 0.0341 & 0.0684 & 0 \\ 0 & 0 & 0 & 1.2 \end{bmatrix}\begin{bmatrix} \ddot{W}_1 \\ \ddot{W}_2 \\ \ddot{W}_3 \\ \ddot{W}_4 \end{bmatrix} + 1\times 10^6 \begin{bmatrix} 16.8 & -8.4 & 0 & 0 \\ -8.4 & 16.8 & -8.4 & 0 \\ 0 & -8.4 & 10.4 & -2 \\ 0 & 0 & -2 & 3.4 \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ 0 \\ 8\times 10^5[u(t) - 2u(t-0.01) + u(t-0.02)] \\ 0 \end{bmatrix}$$

The differential equations are solved using modal analysis. To this end the natural frequencies are calculated as the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ and the normalized mode shapes are the normalized eigenvectors.

Problem 11.17 illustrates the use of modal analysis to solve the discrete equations of a finite-element model.

11.18 Use a two-element, finite-element model to approximate the two lowest natural frequencies of transverse vibration of the beam of Figure P11.18.

FIGURE P11.18

Given: E, I, A, L, ρ

Find: ω_1, ω_2

Solution: Two elements of equal length are used to model the fixed-free beam. The two-element model is shown below.

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The local mass and stiffness matrices for each element are identical. The two-element finite-element model results in a four-degree-of-freedom model. The relations between the local generalized coordinates and the global generalized coordinates and the global matrices for each element are listed below.

Element 1: Local generalized coordinates: $w_1 = 0, w_2 = 0, w_3 = W_1, \mathrm{w}_4 = W_2$

Global matrices:

$$\tilde{\mathbf{M}}_1 = \frac{\rho AL}{840}\begin{bmatrix} 156 & -11L & 0 & 0 \\ -11L & L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{K}}_1 = \frac{8EA}{L^3}\begin{bmatrix} 12 & -3L & 0 & 0 \\ -3L & L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: Local generalized coordinates: $w_1 = W_1, w_2 = W_2, w_3 = W_3, w_4 = W_4$

Global matrices:

$$\tilde{\mathbf{M}}_2 = \frac{\rho AL}{840}\begin{bmatrix} 156 & 11L & 54 & -\frac{13L}{2} \\ 11L & L^2 & \frac{13L}{2} & -\frac{3L^2}{4} \\ 54 & \frac{13L}{2} & 156 & -11L \\ -\frac{13L}{2} & -\frac{3L^2}{4} & -11L & L^2 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{8EI}{L^3}\begin{bmatrix} 12 & 3L & -12 & 3L \\ 3L & L^2 & -3L & \frac{L^2}{2} \\ -12 & -3L & 12 & -3L \\ 3L & \frac{L^2}{2} & -3L & L^2 \end{bmatrix}$$

The global mass and stiffness matrices are

$$\mathbf{M} = \frac{\rho AL}{840}\begin{bmatrix} 312 & 0 & 54 & -\frac{3L}{2} \\ 0 & 2L^2 & \frac{13L}{2} & -\frac{3L^2}{4} \\ 54 & \frac{13L}{2} & 156 & -11L \\ -\frac{13L}{2} & -\frac{3L^2}{4} & -11L & L^2 \end{bmatrix}$$

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$$\mathbf{K} = \frac{8EA}{L^3}\begin{bmatrix} 24 & 0 & -12 & 3L \\ 0 & 2L^2 & -3L & \frac{L^2}{2} \\ -12 & -3L & 12 & -3L \\ 3L & \frac{L^2}{2} & -3L & L^2 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The two lowest natural frequencies are calculated as

$$\omega_1 = 3.517\sqrt{\frac{EI}{\rho A L^4}} \quad \omega_2 = 22.03\sqrt{\frac{EI}{\rho A L^4}}$$

which compare very favorably with the exact results reported in Table 10.4.

Problem 11.18 illustrates the use of a two-element finite-element model for a fixed-free beam.

11.19 Use a two-element, finite-element model to approximate the lowest natural frequencies of the beam of Figure P11.19.

FIGURE P11.19

Given: E, I, A, ρ, L

Find: ω_1, ω_2

Solution: The two-element model of the pinned-fixed beam is shown below. The elements are of equal length $\ell = L/2$.

The two-element finite-element model results in a three-degree-of-freedom system. The relations between the local generalized coordinates and the global generalized coordinates and the global matrices for each element are given below.

Element 1: Local generalized coordinates: $w_1 = 0, w_2 = W_1, w_3 = W_2, w_4 = W_3$.

Global matrices:

$$\tilde{\mathbf{M}}_1 = \frac{\rho A \ell}{420}\begin{bmatrix} 4\ell^2 & 13\ell & -3\ell^2 \\ 13\ell & 156 & -22\ell \\ -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix} \quad \tilde{\mathbf{K}}_1 = \frac{EI}{\ell^3}\begin{bmatrix} 4\ell^2 & -6\ell & 2\ell^2 \\ -6\ell & 12 & -6\ell \\ 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}$$

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Element 2: Local generalized coordinates: $w_1 = W_2, w_2 = W_3, w_3 = 0, w_4 = 0$.

Global matrices:

$$\tilde{\mathbf{M}}_2 = \frac{\rho A \ell}{420}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 156 & 22\ell \\ 0 & 22\ell & 4\ell^2 \end{bmatrix} \quad \tilde{\mathbf{K}}_2 = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 12 & 6\ell \\ 0 & 6\ell & 4\ell^2 \end{bmatrix}$$

The global mass and stiffness matrices are the sum of the element mass and stiffness matrices:

$$\mathbf{M} = \frac{\rho A \ell}{420}\begin{bmatrix} 4\ell^2 & 13\ell & -3\ell^2 \\ 13\ell & 312 & 0 \\ -3\ell^2 & 0 & 8\ell^2 \end{bmatrix} \quad \mathbf{K} = \frac{EI}{\ell^3}\begin{bmatrix} 4\ell^2 & -6\ell & 2\ell^2 \\ -6\ell & 24 & 0 \\ 2\ell^2 & 0 & 8\ell^2 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The first two natural frequencies are thus calculated as

$$\omega_1 = 3.89\sqrt{\frac{EI}{\rho A \ell^4}} = 15.56\sqrt{\frac{EI}{\rho A L^4}}$$

$$\omega_2 = 14.6\sqrt{\frac{EI}{\rho A \ell^4}} = 58.4\sqrt{\frac{EI}{\rho A L^4}}$$

Problem 11.19 illustrates the use of a two-element finite-element model to approximate the lowest natural frequencies of a fixed-pinned beam.

11.20 Use a two-element, finite-element model to approximate the two lowest natural frequencies of the system of Figure P11.20. Use elements of equal length.

FIGURE P11.20

Given: $k = 3EI/L^3$, $m = \rho AL/2$

Find: two lowest ω

Solution: The two-element finite element model of the system is shown below.

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A two-element finite-element model of the system has three degrees of freedom. The global mass and stiffness matrices for the two elements are given below with $\ell = L/2$.

Element 1: $w_1 = 0, w_2 = 0, w_3 = W_1, w_4 = W_2$

$$\tilde{\mathbf{M}}_1 = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & -22\ell & 0 \\ -22\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{K}}_1 = \frac{EI}{\ell^3}\begin{bmatrix} 12 & -6\ell & 0 \\ -6\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Element 2: $w_1 = W_1, w_2 = W_2, w_3 = 0, w_4 = 0$

$$\tilde{\mathbf{M}}_2 = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & 22\ell & 0 \\ 22\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{K}}_2 = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & 0 \\ 6\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The displacement of the end of the spring attached to the beam is the displacement of the midspan of element 2

$$u(\ell/2, t) = \frac{1}{2}w_1 + \frac{\ell}{8}w_2 + \frac{1}{2}w_3 - \frac{\ell}{8}w_4$$
$$u(\ell/2, t) = \frac{1}{2}W_1 + \frac{\ell}{8}W_2$$

The potential energy of the spring is

$$V = \frac{1}{2}k\left[W_3 - \left(\frac{1}{2}W_1 + \frac{\ell}{8}W_2\right)\right]^2$$

The stiffness matrix for the spring is

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$$\mathbf{K}_s = k\begin{bmatrix} \frac{1}{4} & \frac{\ell}{16} & -\frac{1}{2} \\ \frac{\ell}{16} & \frac{\ell^2}{64} & -\frac{\ell}{8} \\ -\frac{1}{2} & -\frac{\ell}{8} & 1 \end{bmatrix}$$

The mass matrix for the discrete mass is

$$\mathbf{M}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix}$$

The global mass and stiffness matrices are

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \frac{\rho A \ell}{420}\begin{bmatrix} 312 & 0 & 0 \\ 0 & 8\ell^2 & 0 \\ 0 & 0 & 420 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_s = \frac{EI}{\ell^3}\begin{bmatrix} 24.0313 & 0.0313\ell & -0.1875 \\ 0.0313\ell & 4.0059\ell^2 & -0.0469\ell \\ -0.1875 & -0.0469\ell & 0.3750 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The results are

$$\omega_1 = 0.617\sqrt{\frac{EI}{\rho A \ell^4}}$$

$$\omega_2 = 5.682\sqrt{\frac{EI}{\rho A \ell^4}}$$

Problem 11.20 illustrates the application of the finite-element method to determine approximations for the natural frequencies of a combined continuous and discrete system.

11.21 Use a three-element, finite-element model to approximate the three lowest natural frequencies of the system of Figure P11.21.

FIGURE P11.21

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Given: E = 200 × 10^9 N/m^2, I = 4.6 × 10^{-6} m^4, ρ = 7000 kg/m^3, A = 1.5 × 10^{-2} m^2, L = 3.3 m

Find: ω_1, ω_2, ω_3

Solution: A three-element finite-element model is shown below. Two elements are used between the supports and one element is used between the support and the end of the beam.

The three-element finite-element model leads to a six-degree-of-freedom system. The relations between the local generalized coordinates and the global generalized coordinates are given below. The global matrices for each element are given below.

Element 1: Local generalized coordinates $w_1 = 0, w_2 = W_1, w_3 = W_2, w_4 = W_3$

Global matrices:

$$\tilde{\mathbf{M}}_1 = \frac{\rho A \ell_1}{420}\begin{bmatrix} 4\ell_1^2 & 13\ell_1 & -3\ell_1^2 & 0 & 0 & 0 \\ 13\ell_1 & 156 & -22\ell_1 & 0 & 0 & 0 \\ -3\ell_1^2 & -22\ell_1 & 4\ell_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1.071 & 3.482 & -0.804 & 0 & 0 & 0 \\ 3.482 & 41.786 & -5.893 & 0 & 0 & 0 \\ -0.804 & -5.893 & 1.071 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_1 = \frac{EI}{\ell_1^3}\begin{bmatrix} 4\ell_1^2 & -6\ell_1 & 2\ell_1^2 & 0 & 0 & 0 \\ -6\ell_1 & 12 & -6\ell_1 & 0 & 0 & 0 \\ 2\ell_1^2 & -6\ell_1 & 4\ell_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 1\times 10^9 \begin{bmatrix} 0.368 & -0.552 & 0.184 & 0 & 0 & 0 \\ -0.552 & 1.104 & -0.552 & 0 & 0 & 0 \\ 0.184 & -0.552 & 0.368 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: Local generalized coordinates: $w_1 = W_2, w_2 = W_3, w_3 = 0, w_4 = W_4$

Global matrices:

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$$\tilde{\mathbf{M}}_2 = \frac{\rho A \ell_2}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22\ell_2 & -13\ell_2 & 0 & 0 \\ 0 & 22\ell_2 & 4\ell_2^2 & -3\ell_2^2 & 0 & 0 \\ 0 & -13\ell_2 & -3\ell_2^2 & 4\ell_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 41.786 & 5.893 & -3.482 & 0 & 0 \\ 0 & 5.893 & 1.071 & -0.804 & 0 & 0 \\ 0 & -3.841 & -0.804 & 1.71 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{EI}{\ell_2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6\ell_2 & 6\ell_2 & 0 & 0 \\ 0 & 6\ell_2 & 4\ell_2^2 & 2\ell_2^2 & 0 & 0 \\ 0 & 6\ell_2 & 2\ell_2^2 & 4\ell_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 1\times 10^9 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.104 & 0.552 & 0.552 & 0 & 0 \\ 0 & 0.552 & 0.368 & 0.184 & 0 & 0 \\ 0 & 0.552 & 0.184 & 0.368 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3: Local generalized coordinates: $w_1 = 0, w_2 = W_4, w_3 = W_5, \mathrm{w}_4 = W_6$

Global matrices:

$$\tilde{\mathbf{M}}_3 = \frac{\rho A \ell_3}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\ell_3^2 & 13\ell_3 & -3\ell_3^2 \\ 0 & 0 & 0 & 13\ell_3 & 156 & -22\ell_3 \\ 0 & 0 & 0 & -3\ell_3^2 & -22\ell_3 & 4\ell_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.354 & 5.885 & -1.765 \\ 0 & 0 & 0 & 5.885 & 54.321 & -9.959 \\ 0 & 0 & 0 & -1.764 & -9.959 & 2.354 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_3 = \frac{EI}{\ell_3^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\ell_3^2 & -6\ell_3 & 2\ell_3^2 \\ 0 & 0 & 0 & -6\ell_3 & 12 & -6\ell_3 \\ 0 & 0 & 0 & 2\ell_3^2 & -6\ell_3 & 4\ell_3^2 \end{bmatrix} = 1\times 10^8 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.831 & -3.266 & 1.415 \\ 0 & 0 & 0 & -3.266 & 5.025 & -3.266 \\ 0 & 0 & 0 & 1.415 & -3.266 & 2.831 \end{bmatrix}$$

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The global matrices are

$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 + \tilde{\mathbf{M}}_3 = \begin{bmatrix} 1.071 & 3.482 & -0.804 & 0 & 0 & 0 \\ 3.482 & 83.571 & 0 & -3.482 & 0 & 0 \\ -0.804 & 0 & 2.143 & -0.804 & 0 & 0 \\ 0 & -3.482 & -0.804 & 3.425 & 5.885 & -1.764 \\ 0 & 0 & 0 & 5.885 & 54.321 & -9.959 \\ 0 & 0 & 0 & -1.764 & -9.959 & 2.354 \end{bmatrix}$$

$$\mathbf{K} = \tilde{\mathbf{K}}_1 + \tilde{\mathbf{K}}_2 + \tilde{\mathbf{K}}_3 = 1\times10^9 \begin{bmatrix} -0.368 & -0.552 & 0.184 & 0 & 0 & 0 \\ -0.552 & 2.208 & 0 & 0.552 & 0 & 0 \\ 0.184 & 0 & 0.736 & 0.184 & 0 & 0 \\ 0 & 0.552 & 0.184 & 0.651 & -0.327 & 0.142 \\ 0 & 0 & 0 & -0.327 & 0.503 & -0.327 \\ 0 & 0 & 0 & 0.142 & -0.327 & 0.283 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The results are

$$\omega_1 = 1.017\times10^3 \text{ rad/s}$$
$$\omega_2 = 3.043\times10^3 \text{ rad/s}$$
$$\omega_3 = 9.712\times10^3 \text{ rad/s}$$

Problem 11.21 illustrates the use of a three-element finite-element model to approximate the natural frequencies of a pinned-pinned beam with an overhang.

11.22 Use a two-element, finite-element model to approximate the lowest natural frequency of the system of Figure P11.22.

$I_{AB} = 4.1 \times 10^{-6}$ m^4
$E_{AB} = 200 \times 10^9$ N/m^2
$A_{AB} = 6.3 \times 10^{-4}$ m^2
$\rho_{AB} = 7500$ kg/m^3
$E_{BC} = 140 \times 10^9$ N/m^2
$A_{BC} = 5.4 \times 10^{-5}$ m^2
$\rho_{BC} = 5600$ kg/m^3
$I_{BC} = 5.3 \times 10^{-7}$ m^4

FIGURE P11.22

Given: System shown

Find: ω_1

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Solution: A two-element finite-element model of the beam is illustrated below. One element is used to model section AB and one element is used to model section BC.

The two-element finite-element model leads to a three-degree-of-freedom system. The local matrices for each element are the standard mass and stiffness matrices for a beam element. The relations between the local generalized coordinates and the global generalized coordinates are given below as well as the global matrices for each element.

Element 1: Local generalized coordinates: $w_1 = 0, w_2 = 0, w_3 = W_1, w_4 = W_2$

Global matrices:

$$\tilde{\mathbf{M}}_1 = \frac{\rho_{AB} A_{AB} L_{AB}}{420} \begin{bmatrix} 156 & 22L_{AB} & 0 \\ 22L_{AB} & 4L_{AB}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_1 = \frac{E_{AB} I_{AB}}{L_{AB}^3} \begin{bmatrix} 12 & 6L_{AB} & 0 \\ 6L_{AB} & 4L_{AB}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Element 2: Local generalized coordinates: $w_1 = W_1\ w_2 = W_2, w_3 = 0, w_4 = W_3$

Global matrices:

$$\tilde{\mathbf{M}}_2 = \frac{\rho_{BC} A_{BC} L_{BC}}{420} \begin{bmatrix} 156 & 22L_{BC} & -13L_{BC} \\ 22L_{BC} & 4L_{BC}^2 & -3L_{BC}^2 \\ -13L_{BC} & -3L_{BC}^2 & 4L_{BC}^2 \end{bmatrix} = \begin{bmatrix} 0.2022 & 0.0513 & -0.0303 \\ 0.0513 & 0.0168 & -0.0126 \\ -0.0303 & -0.0126 & 0.0168 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{E_{BC} I_{BC}}{L_{BC}^3} \begin{bmatrix} 12 & 6L_{BC} & 6L_{BC} \\ 6L_{BC} & 4L_{BC}^2 & 2L_{BC}^2 \\ 6L_{BC} & 2L_{BC}^2 & 4L_{BC}^2 \end{bmatrix} = 10^5 \begin{bmatrix} 1.5267 & 1.3741 & 1.3741 \\ 1.3741 & 1.6489 & 0.8244 \\ 1.3741 & 0.8244 & 1.6489 \end{bmatrix}$$

The assembled global matrices are

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$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 = \begin{bmatrix} 2.3082 & -0.3051 & -0.0303 \\ -0.3051 & 0.0946 & -0.0126 \\ -0.0303 & -0.0126 & 0.0168 \end{bmatrix}$$

$$\mathbf{K} = \tilde{\mathbf{K}}_1 + \tilde{\mathbf{K}}_2 = 10^6 \begin{bmatrix} 5.8471 & -3.2793 & 0.1374 \\ -3.2793 & 2.8982 & 0.0824 \\ 0.1374 & 0.0824 & 0.1649 \end{bmatrix}$$

The approximations to the natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The lowest natural frequency is approximated as

$$\omega_1 = 970.5 \text{ rad/s}$$

Problem 11.22 illustrates the use of a two-element finite-element method to approximate the lowest natural frequency of a non-uniform fixed-pinned beam.

11.23 Use a two-element, finite-element model for the beam to approximate the two lowest natural frequencies of the system of Figure P11.23.

FIGURE P11.23

Given: ρ = 7600 kg/m^3, A = 4.5 × 10^{-3} m^2, m = 100 kg, E = 200 × 10^9 N/m^2, I = 1.8 × 10^{-6} m^4, L = 160 cm

Find: ω_1, ω_2

Solution: The two-element finite element model of the beam is shown below. Each element is of length $\ell = L/2 = 0.8\,\text{m}$.

The local mass and stiffness matrix for each element are those with the mass matrix for the first element modified to account for the discrete mass at the beam's midspan. The two-element finite-element model for this beam results in a four-degree-of-freedom system. The relations between the local generalized coordinates and the global generalized coordinates for each element are given below as well as the mass and stiffness matrices for each element written for the global coordinate system.

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Element 1: $w_1 = 0, w_2 = W_1, w_3 = W_2, w_4 = W_3$

Global matrices:

$$\tilde{\mathbf{M}}_1 = \frac{\rho A \ell}{420}\begin{bmatrix} 4\ell^2 & -6\ell & -3\ell^2 & 0 \\ -6\ell & 156 + \dfrac{420m}{\rho A \ell} & -22\ell & 0 \\ -3\ell^2 & -22\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.1668 & 0.6775 & -0.1251 & 0 \\ 0.6775 & 110.16 & -1.1465 & 0 \\ -0.1251 & -1.1465 & 0.1668 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_1 = \frac{EI}{\ell^3}\begin{bmatrix} 4\ell^2 & -6\ell & 2\ell^2 & 0 \\ -6\ell & 12 & -6\ell & 0 \\ 2\ell^2 & -6\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 10^6 \begin{bmatrix} 1.8000 & -3.3750 & 0.9000 & 0 \\ -3.3750 & 8.4375 & -3.3750 & 0 \\ 0.9000 & -3.3750 & 1.8000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: $w_1 = W_2, w_2 = W_3, w_3 = 0, w_4 = W_4$

Global matrices:

$$\tilde{\mathbf{M}}_2 = \frac{\rho A \ell}{420}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 156 & 22\ell & -13\ell \\ 0 & 22\ell & 4\ell^2 & -3\ell^2 \\ 0 & -13\ell & -3\ell^2 & 4\ell^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 10.1623 & 1.1465 & -0.6775 \\ 0 & 1.1465 & 0.1668 & -0.1251 \\ 0 & -0.6775 & -0.1251 & 0.1668 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 12 & 6\ell & 6\ell \\ 0 & 6\ell & 4\ell^2 & 2\ell^2 \\ 0 & 6\ell & 2\ell^2 & 4\ell^2 \end{bmatrix} = 10^6 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 8.4375 & 3.3750 & 3.3750 \\ 0 & 3.3750 & 1.8000 & 0.9000 \\ 0 & 3.3750 & 0.9000 & 1.8000 \end{bmatrix}$$

The assembled global matrices are

$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 = \begin{bmatrix} 0.1668 & 0.6775 & -0.1251 & 0 \\ 0.6775 & 120.32 & 0 & -0.6775 \\ -0.1251 & 0 & 0.3335 & -0.1251 \\ 0 & -0.6775 & -0.1251 & 0.1668 \end{bmatrix}$$

$$\mathbf{K} = \tilde{\mathbf{K}}_1 + \tilde{\mathbf{K}}_2 = 10^7 \begin{bmatrix} 0.1800 & -0.3375 & 0.0900 & 0 \\ -0.3375 & 1.6875 & 0 & 0.3375 \\ 0.0900 & 0 & 0.3600 & 0.0900 \\ 0 & 0.3375 & 0.0900 & 0.1800 \end{bmatrix}$$

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The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The approximations for the two lowest natural frequencies are

$$\omega_1 = 182.5 \text{ rad/s}$$
$$\omega_2 = 1.756 \times 10^3 \text{ rad/s}$$

Problem 11.23 illustrates the use of a two-element finite-element method to approximate the two lowest natural frequencies of a pinned-pinned beam with a machine at its midspan.

11.24 Use a two-element, finite-element model to approximate the two lowest natural frequencies of the system of Figure P11.24.

FIGURE P11.24

Given: ρ = 8000 kg/m^3, A = 1.1 × 10^{-4} m^2, E = 200 × 10^9 N/m^2, I_b = 1.9 × 10^{-6} m^4, L = 60 cm, m = 1.2 kg, I = 0.8 kg · m^2

Find: ω_1, ω_2

Solution: The two-element finite-element model of the beam is shown below. Each element is of length $\ell = L/2 = 0.3\,\text{m}$.

The two-element finite-element model leads to a four-degree-of-freedom system. The effect of the mass is added to the element in the third row and third column of the local mass matrix while the inertia of the mass is accounted for in the fourth row and fourth column of the local mass matrix. The relations between the local generalized coordinates and the global generalized coordinates for each element as well as the mass and stiffness matrices written for the global coordinates are given below.

Element 1: $w_1 = 0, w_2 = 0, w_3 = W_1, w_4 = W_2$

Global matrices:

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$$\tilde{\mathbf{M}}_1 = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & -22\ell & 0 & 0 \\ -22\ell & 4\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.0981 & -0.0041 & 0 & 0 \\ -0.0041 & 0.0002 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_1 = \frac{EI}{\ell^3}\begin{bmatrix} 12 & -6\ell & 0 & 0 \\ -6\ell & 4\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 10^8\begin{bmatrix} 1.6889 & -0.2533 & 0 & 0 \\ -0.2533 & 0.0507 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: $w_1 = W_1, w_2 = W_2, w_3 = W_3, w_4 = W_4$

Global matrices:

$$\tilde{\mathbf{M}}_2 = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 + \dfrac{420m}{\rho A \ell} & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 + \dfrac{420I}{\rho A \ell} \end{bmatrix}$$

$$= \begin{bmatrix} 0.0981 & 0.0041 & 0.0339 & -0.0025 \\ 0.0041 & 0.0002 & 0.0025 & -0.0002 \\ 0.0339 & 0.0025 & 1.2981 & -0.0041 \\ -0.0025 & -0.0002 & -0.0041 & 0.8002 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} = 10^8\begin{bmatrix} 1.6889 & 0.2533 & -1.6889 & 0.2533 \\ 0.2533 & 0.0507 & -0.2533 & 0.0253 \\ -1.6889 & -0.2533 & 1.6889 & -0.2533 \\ 0.2533 & 0.0253 & -0.2533 & 0.0507 \end{bmatrix}$$

The assembled global matrices are

$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 = \begin{bmatrix} 0.1961 & 0 & 0.0339 & -0.0025 \\ 0 & 0.0005 & 0.0025 & -0.0002 \\ 0.0339 & 0.0025 & 1.2981 & -0.0041 \\ -0.0025 & -0.0002 & -0.0041 & 0.8002 \end{bmatrix}$$

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$$\mathbf{K} = \tilde{\mathbf{K}}_1 + \tilde{\mathbf{K}}_2 = 10^8 \begin{bmatrix} 3788 & 0 & -1.6889 & 0.2533 \\ 0 & 0.1013 & -0.2533 & 0.0253 \\ -1.6889 & -0.2533 & 1.6889 & -0.2533 \\ 0.2533 & 0.0253 & -0.2533 & 0.0507 \end{bmatrix}$$

The approximations for the natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The results are

$$\omega_1 = 828.3 \text{ rad/s}$$
$$\omega_2 = 4.171 \times 10^3 \text{ rad/s}$$

Problem 11.24 illustrates the use of a two-element finite-element model to approximate the lowest natural frequencies of a fixed-freed beam with an end mass and inertia element.

11.25 Use a three-element, finite-element model to approximate the steady-state amplitude of the machine of the system of Figure P11.25.

FIGURE P11.25

Given: $I = 1.9 \times 10^{-5}$ m^4, $m = 210$ kg, $E = 200 \times 10^9$ N/m^2, $\rho = 7500$ kg/m^3, $A = 1.4 \times 10^{-2}$ m^2, $F_0 = 1500$ N, $\omega = 200$ rad/s, $L = 1.2$ m

Find: **X**

Solution: The three-element model of the beam is shown below.

The three-element model leads to a six-degree-of-freedom system. The relations between the local coordinates for each element and the global coordinates as well as the mass and stiffness matrices are given below where $\ell = L/3$.

Element 1: $w_1 = 0, w_2 = 0, w_3 = W_1, w_4 = W_2$

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$$\tilde{\mathbf{M}}_1 = \frac{\rho A \ell}{420}\begin{bmatrix} 12 & -22\ell & 0 & 0 & 0 & 0 \\ -22\ell & 4\ell^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{K}}_1 = \frac{EI}{\ell^3}\begin{bmatrix} 12 & -6\ell & 0 & 0 & 0 & 0 \\ -6\ell & 4\ell^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: $w_1 = W_1, \mathrm{w}_2 = W_2, \mathrm{w}_3 = W_3, \mathrm{w}_4 = W_4$

$$\tilde{\mathbf{M}}_2 = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell & 0 & 0 \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 & 0 & 0 \\ 54 & 13\ell & 156 & -22\ell & 0 & 0 \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell & 0 & 0 \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 & 0 & 0 \\ -12 & -6\ell & 12 & 6\ell & 0 & 0 \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3: $w_1 = W_3, \mathrm{w}_2 = W_4, \mathrm{w}_3 = W_5, \mathrm{w}_4 = W_6$. The mass matrix for element 3 includes the mass of the machine.

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$$\tilde{\mathbf{M}}_3 = \frac{\rho A \ell}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 156 & 22\ell & 53 & -13\ell \\ 0 & 0 & 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 0 & 0 & 54 & 13\ell & 156+\dfrac{420m}{\rho A\ell} & -22\ell \\ 0 & 0 & -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_3 = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 6\ell & -12 & 6\ell \\ 0 & 0 & 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ 0 & 0 & -12 & -6\ell & 12 & -6\ell \\ 0 & 0 & 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}$$

Substituting given values and adding the element matrices together leads to

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \begin{bmatrix} 31.2 & 0 & 5.4 & -0.52 & 0 & 0 \\ 0 & 0.128 & 0.52 & -0.048 & 0 & 0 \\ 5.4 & 0.52 & 31.2 & 0 & 5.4 & -0.52 \\ -0.52 & -0.048 & 0 & 0.128 & 0.52 & -0.048 \\ 0 & 0 & 5.4 & 0.52 & 225.6 & -0.88 \\ 0 & 0 & -0.52 & -0.048 & -0.88 & 0.064 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 = 10^9\begin{bmatrix} 1.25 & 0 & -0.7125 & 0.1425 & 0 & 0 \\ 0 & 0.0760 & -0.1425 & 0.0190 & 0 & 0 \\ -0.7125 & -0.1425 & 1.4250 & 0 & -0.7135 & 0.1425 \\ 0.1425 & 0.0190 & 0 & 0.0760 & -0.1425 & 0.0190 \\ 0 & 0 & -0.7125 & -0.1425 & 0.7125 & -0.1425 \\ 0 & 0 & 0.1425 & 0.0190 & -0.1425 & 0.0380 \end{bmatrix}$$

The force vector is

$$\mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F_0 \sin \omega t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1500 \sin 200t \\ 0 \end{bmatrix}$$

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The governing differential equations are of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}$$

A steady-state solution is assumed as

$$\mathbf{x} = \mathbf{X}\sin\omega t$$

Substitution into the differential equations leads to

$$\mathbf{X} = 10^{-3}\begin{bmatrix} -0.0757 \\ -0.3528 \\ -0.2620 \\ -0.5543 \\ -0.5005 \\ -0.6165 \end{bmatrix}$$

Thus the steady-state amplitude of the machine is 6.165 cm.

Problem 11.25 illustrates application of the finite-element method to determine the steady-state amplitude of a machine.

11.26 Use a three-element, finite-element model to approximate the steady-state amplitude of the machine of the system of Figure P11.26.

FIGURE P11.26

Given: Pinned-pinned beam with mass and spring

Find: Steady-state amplitude using three-element finite-element model of beam.

Solution: Three elements are used for the beam, each of equal length.

Element 1: $w_1 = 0, w_2 = W_1, w_3 = W_2, w_4 = W_3$

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$$\mathrm{M}_1 = \frac{\rho A \ell}{420}\begin{bmatrix} 4\ell^2 & 13\ell & -3\ell^2 & 0 & 0 & 0 \\ 13\ell & 156 & -22\ell & 0 & 0 & 0 \\ -3\ell^2 & -22\ell & 4\ell^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{K}_1 = \frac{EI}{\ell^3}\begin{bmatrix} 4\ell^2 & -6\ell & 2\ell^2 & 0 & 0 & 0 \\ -6\ell & 12 & -6\ell & 0 & 0 & 0 \\ 2\ell^2 & -6\ell & 4\ell^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The force vector is

$$F_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Element 2: $w_1 = W_2, w_2 = W_3, w_3 = W_4, w_4 = W_5$

The kinetic energy of the element is $T + \frac{1}{2}m\left[\dot{w}\left(\frac{L}{2}\right)\right]^2 = T + \frac{1}{2}m\left[\frac{1}{2}\dot{w}_1 + \frac{1}{8}\dot{w}_2 + \frac{1}{2}\dot{w}_3 - 18w4\right.$ from which the mass matrix is calculated as

$$\mathrm{M}_2 = \frac{\rho A \ell}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22\ell & 54 & -13\ell & 0 \\ 0 & 22\ell & 4\ell^2 & 13\ell & -3\ell^2 & 0 \\ 0 & 54 & 13 & 156 & -22\ell & 0 \\ 0 & -13\ell & -3\ell^2 & -22\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{m}{64}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 64 & 4 & 64 & -4 & 0 \\ 0 & 4 & 1 & 4 & -1 & 0 \\ 0 & 64 & 1 & 64 & -4 & 0 \\ 0 & -4 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The potential energy of the element is $V + \frac{1}{2}m\left[w\left(\frac{L}{2}\right)\right]^2 = T + \frac{1}{2}m\left[\frac{1}{2}w_1 + \frac{1}{8}w_2 + \frac{1}{2}w_3 - 18w4\right.$ from which the mass matrix is calculated as

$$\mathrm{K}_2 = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6\ell & -12 & 6\ell & 0 \\ 0 & 6\ell & 4\ell^2 & -6\ell & 2\ell^2 & 0 \\ 0 & -12 & -6\ell & 12 & -6\ell & 0 \\ 0 & 6\ell & 2\ell^2 & -6\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{k}{64}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 64 & 4 & 64 & -4 & 0 \\ 0 & 4 & 1 & 4 & -1 & 0 \\ 0 & 64 & 1 & 64 & -4 & 0 \\ 0 & -4 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The force written in terms of local coordinates is

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$$F(t) = F_0 \sin \omega t \, \delta\left(x - \frac{\ell}{2}\right)$$

which leads to

$$F_2 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{8} \\ \frac{1}{2} \\ -\frac{1}{8} \\ 0 \end{bmatrix} F_0 \sin \omega t$$

Element 3: $w_1 = W_4, w_2 = W_5, w_3 = 0, w_4 = W_6$

$$\mathrm{M}_3 = \frac{\rho A \ell}{420} \begin{bmatrix} 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 156 & & 22\ell & -13\ell \\ 0 & 0 & 0 & 22\ell & & 4\ell^2 & -3\ell^2 \\ 0 & 0 & 0 & -13\ell & & -3\ell^2 & 4\ell^2 \end{bmatrix}$$

$$\mathrm{K}_3 = \frac{EI}{\ell^3} \begin{bmatrix} 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 12 & & 6\ell & 6\ell \\ 0 & 0 & 0 & 6\ell & & 4\ell^2 & 2\ell^2 \\ 0 & 0 & 0 & 6\ell & & 2\ell^2 & 4\ell^2 \end{bmatrix}$$

The force vector is

$$F_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The global equations are

$$(\mathrm{M}_1 + \mathrm{M}_2 + \mathrm{M}_3)\ddot{\mathrm{W}} + (\mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_3)\mathrm{W} = \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3$$

They are solved by assuming

$$\mathrm{W} = \mathrm{U} \sin \omega t$$

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Problem 11.26 illustrates the use of a three-element finite element method to approximate the harmonic response of a beam.

11.27 The street light has a mass of 25 kg. The wind velocity is 60 m/s, but the force distribution is as shown in Figure P11.27. Use a three-element, finite-element model of the structure to approximate the steady-state amplitude of the light.

Given: street light

Find: Three-element finite element model

Solution: Three elements are used for the beam, each of equal length.

Element 1: $w_1 = 0, w_2 = 0, w_3 = W_1, w_4 = W_2$

Element 2: $w_1 = W_1, w_2 = W_2, w_3 = W_3, w_4 = W_4$

Element 2: $w_1 = W_3, w_2 = W_4, w_3 = W_5, w_4 = W_6$

FIGURE P11.27

In terms of global coordinates the force is equal to

$$F(x,t) = F_0 \sin \omega t \left(\frac{x}{L}\right)^2$$

Over element 1, in terms of local coordinates

$$F(x,t) = \frac{F_0}{9} \sin \omega t \left(\frac{\xi}{\ell}\right)^2$$

Over element 2 the local coordinate is related to the global coordinate by

$$x = \frac{L}{3\ell}(\xi + \ell)$$

and

$$F(\xi,t) = \frac{F_0}{9\ell^2} \sin \omega t \, (\xi + \ell)^2$$

Over element 3 the relation between the local coordinate and the global coordinate is

$$x = \frac{L}{3\ell}(\xi + 2\ell)$$

and

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$$F(\xi, t) = \frac{F_0}{9\ell^2} \sin \omega t \, (\xi + 2\ell)^2$$

The global mass and stiffness matrices are formed according to the elements. The global force vector is formed by calculating the force vector over each element and adding them together. A steady-state solution is assumed as

$$\mathbf{W} = \mathbf{U} \sin \omega t$$

Problem 11.27 illustrates the steady-state response of a system due to a distributed load.

11.28 Use a three-element, finite-element model to approximate the steady-state response of the system of Figure P11.28.

FIGURE P11.28

Given: F_0 = 800 N, ω = 120 rad/s, E = 200 × 10^9 N/m^2, A = 4.1 × 10^{-3} m^2, I = 8.6 × 10^{-5} m^4, ρ = 7500 kg/m^3, L = 3 m

Find: **X**

Solution: The three-element finite-element model of the system is shown below. Each element is of length $\ell = L/3 = 1\,\text{m}$

The three-element finite-element model leads to a six-degree-of-freedom system.

Element 1: $w_1 = 0, w_2 = W_1, w_3 = W_2, w_4 = W_3$

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$$\tilde{\mathbf{M}}_1 = \frac{\rho A \ell}{420}\begin{bmatrix} 4\ell^2 & 13\ell & -3\ell^2 & 0 & 0 & 0 \\ 13\ell & 156 & -22\ell & 0 & 0 & 0 \\ -3\ell^2 & -22\ell & 4\ell^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.2929 & 0.9518 & -0.2196 & 0 & 0 & 0 \\ 0.9518 & 11.421 & -1.6107 & 0 & 0 & 0 \\ -0.2916 & -1.6017 & 0.2929 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_1 = \frac{EI}{\ell^3}\begin{bmatrix} 4\ell^2 & -6\ell & 2\ell^2 & 0 & 0 & 0 \\ -6\ell & 12 & -6\ell & 0 & 0 & 0 \\ 2\ell^2 & -6\ell & 4\ell^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 10^8\begin{bmatrix} 0.688 & -1.032 & 0.344 & 0 & 0 & 0 \\ -1.032 & 2.064 & -1.032 & 0 & 0 & 0 \\ 0.344 & -1.032 & 0.688 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Elements of local generalized force vector

$$q_2 = \int_0^\ell \left(\frac{F_0\xi}{\ell}\sin\omega t\right)\left(\frac{\xi}{\ell} - 2\frac{\xi^2}{\ell^2} + \frac{\xi^3}{\ell^3}\right)\ell\, d\xi = \frac{F_0\ell^2}{30}\sin\omega t$$

$$q_3 = \int_0^\ell \left(\frac{F_0\xi}{\ell}\sin\omega t\right)\left(3\frac{\xi^2}{\ell^2} - 2\frac{\xi^3}{\ell^3}\right) d\xi = \frac{7F_0\ell}{20}\sin\omega t$$

$$q_4 = \int_0^\ell \left(\frac{F_0\xi}{\ell}\sin\omega t\right)\left(-\frac{\xi^2}{\ell^2} + \frac{\xi^3}{\ell^3}\right)\ell\, d\xi = -\frac{F_0\ell^2}{20}\sin\omega t$$

The global generalized force vector for element 1 is

$$\mathbf{Q}_1 = \begin{bmatrix} \frac{F_0\ell^2}{30} \\ \frac{7F_0\ell}{20} \\ -\frac{F_0\ell^2}{20} \\ 0 \\ 0 \\ 0 \end{bmatrix}\sin\omega t = \begin{bmatrix} 26.67 \\ 280.0 \\ -40 \\ 0 \\ 0 \\ 0 \end{bmatrix}\sin\omega t$$

Element 2: $w_1 = W_2, w_2 = W_3, w_3 = W_4, w_4 = W_5$

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$$\tilde{\mathbf{M}}_2 = \frac{\rho A \ell}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22\ell & 54 & -13\ell & 0 \\ 0 & 22\ell & 4\ell^2 & 13\ell & -3\ell^2 & 0 \\ 0 & 54 & 13\ell & 156 & -22\ell & 0 \\ 0 & -13\ell & -3\ell^2 & -22\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11.4214 & 1.6107 & 3.9536 & -0.9518 & 0 \\ 0 & 1.6107 & 0.2929 & 0.9518 & -0.2196 & 0 \\ 0 & 3.9536 & 0.9518 & 11.4214 & -1.6107 & 0 \\ 0 & -0.9518 & -0.2196 & -1.6107 & 0.2929 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6\ell & -12 & 6\ell & 0 \\ 0 & 6\ell & 4\ell^2 & -6\ell & 2\ell^2 & 0 \\ 0 & -12 & -6\ell & 12 & -6\ell & 0 \\ 0 & 6\ell & 2\ell^2 & -6\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 10^8 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.064 & 1.032 & -2.064 & 1.032 & 0 \\ 0 & 1.032 & 0.688 & -1.032 & 0.344 & 0 \\ 0 & -2.064 & -1.032 & 2.064 & -1.032 & 0 \\ 0 & 1.032 & 0.344 & -1.032 & 0.688 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The elements of the local generalized force vector are

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$$q_1 = \int_0^{\ell} (F_0 \sin \omega t)\left(1 - 3\frac{\xi^2}{\ell^2} + 2\frac{\xi^3}{\ell^3}\right) d\xi = \frac{F_0 \ell}{2} \sin \omega t$$

$$q_2 = \int_0^{\ell} (F_0 \sin \omega t)\left(\frac{\xi}{\ell} - 2\frac{\xi^2}{\ell^2} + \frac{\xi^3}{\ell^3}\right) \ell \, d\xi = \frac{F_0 \ell^2}{12} \sin \omega t$$

$$q_3 = \int_0^{\ell} (F_0 \sin \omega t)\left(3\frac{\xi^2}{\ell^2} - 2\frac{\xi^3}{\ell^3}\right) d\xi = \frac{F_0 \ell}{2} \sin \omega t$$

$$q_4 = \int_0^{\ell} (F_0 \sin \omega t)\left(-\frac{\xi^2}{\ell^2} + \frac{\xi^3}{\ell^3}\right) \ell \, d\xi = -\frac{F_0 \ell^2}{12} \sin \omega t$$

The global force vector for element 2 is

$$\mathbf{Q}_2 = F_0 \begin{bmatrix} 0 \\ \frac{\ell}{2} \\ \frac{\ell^2}{12} \\ \frac{\ell}{2} \\ -\frac{\ell^2}{12} \\ 0 \end{bmatrix} \sin \omega t = \begin{bmatrix} 0 \\ 400 \\ 66.67 \\ 400 \\ -66.67 \\ 0 \end{bmatrix}$$

Element 3: $w_1 = W_4, w_2 = W_5, w_3 = 0, w_4 = W_6$

$$\tilde{\mathbf{M}}_3 = \frac{\rho A \ell}{420} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 156 & 22\ell & -13\ell \\ 0 & 0 & 0 & 22\ell & 4\ell^2 & -3\ell^2 \\ 0 & 0 & 0 & -13\ell & -3\ell^2 & 4\ell^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 11.4214 & 1.6107 & -0.9518 \\ 0 & 0 & 0 & 1.6107 & 0.2929 & -0.2196 \\ 0 & 0 & 0 & -0.9518 & -0.2916 & 0.2929 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_3 = \frac{EI}{\ell^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 6\ell & 6\ell \\ 0 & 0 & 0 & 6\ell & 4\ell^2 & 2\ell^2 \\ 0 & 0 & 0 & 6\ell & 2\ell^2 & 4\ell^2 \end{bmatrix} = 10^8 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.064 & 1.032 & 1.032 \\ 0 & 0 & 0 & 1.032 & 0.688 & 0.344 \\ 0 & 0 & 0 & 1.032 & 0.344 & 0.688 \end{bmatrix}$$

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The local force vector for element 3 is the same as the local force vector for element 2. The global force vector for element 3 is

$$\mathbf{Q}_3 = F_0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\ell}{2} \\ \frac{\ell^2}{12} \\ -\frac{\ell^2}{12} \end{bmatrix} \sin \omega t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 400 \\ 66.67 \\ -66.67 \end{bmatrix} \sin \omega t$$

The elemental global matrices are summed leading to the assembled global matrices. The resulting differential equations are

$$\begin{bmatrix} 0.2929 & 0.9518 & -0.2196 & 0 & 0 & 0 \\ 0.9518 & 22.8429 & 0 & 3.9536 & -0.9518 & 0 \\ -0.2196 & 0 & 0.5857 & 0.9518 & -0.2196 & 0 \\ 0 & 3.9536 & 0.9518 & 22.8429 & 0 & -0.9518 \\ 0 & -0.9518 & -0.2196 & 0 & 0.5857 & -0.2196 \\ 0 & 0 & 0 & -0.9518 & -0.2196 & 0.2929 \end{bmatrix} \begin{bmatrix} \ddot{W}_1 \\ \ddot{W}_2 \\ \ddot{W}_3 \\ \ddot{W}_4 \\ \ddot{W}_5 \\ \ddot{W}_6 \end{bmatrix} +$$

$$10^8 \begin{bmatrix} 0.688 & -1.032 & 0.344 & 0 & 0 & 0 \\ -1.032 & 4.128 & 0 & -2.064 & 1.032 & 0 \\ 0.344 & 0 & 1.376 & -1.032 & 0.344 & 0 \\ 0 & -2.064 & -1.032 & 4.128 & 0 & 1.032 \\ 0 & 1.032 & 0.344 & 0 & 1.376 & 0.344 \\ 0 & 0 & 0 & 1.032 & 0.344 & 0.688 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{bmatrix} = \begin{bmatrix} 26.67 \\ 680.0 \\ 26.67 \\ 800.0 \\ 0 \\ -66.67 \end{bmatrix} \sin \omega t$$

The steady-state solution of the previous set of equations is assumed as

$$\mathbf{W} = \mathbf{U} \sin \omega t$$

which when substituted into the differential equations leads to the following set of simultaneous equations

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$$10^8\begin{bmatrix} 0.688 & -1.0321 & 0.344 & 0 & 0 & 0 \\ -1.0321 & 4.1427 & 0 & -2.0646 & 1.0321 & 0 \\ 0.344 & 0 & 1.3759 & -1.0321 & 0.344 & 0 \\ 0 & -2.0646 & -1.0321 & 4.1247 & 0 & 1.0321 \\ 0 & 1.0321 & 0.344 & 0 & 1.3759 & 0.344 \\ 0 & 0 & 0 & 1.0321 & 0.344 & 0.688 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix} = \begin{bmatrix} 26.67 \\ 680 \\ 26.67 \\ 800 \\ 0 \\ -26.67 \end{bmatrix}$$

The solution of these equations is

$$\mathbf{U} = 10^{-4}\begin{bmatrix} 0.4741 \\ 0.3944 \\ 0.2429 \\ 0.4017 \\ -0.2325 \\ -0.4961 \end{bmatrix}$$

Problem 11.28 illustrates application of the finite-element method to approximate the steady-state response of a beam with a non-uniform harmonic loading.

11.29 A plate and girder bridge is modeled as a simply supported beam, as illustrated in Figure P11.29. A vehicle is traveling across the bridge with the velocity v. Use a three-element, finite-element model of the bridge to determine the time-dependent response of the structure as the vehicle is crossing the bridge.

FIGURE P11.29

Solution: The force traveling across the bridge is modeled as a uniform load traveling at a speed of v. The mathematical form of the load is

$$F(x,t) = mg[u(vt - x) - u(vt - x + L_v]$$

Three elements are used for the beam, each of equal length.

Element 1: $w_1 = 0, w_2 = W_1, w_3 = W_2, w_4 = W_3$

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$$\mathrm{M}_1 = \frac{\rho A \ell}{420}\begin{bmatrix} 4\ell^2 & 13\ell & -3\ell^2 & 0 & 0 & 0 \\ 13\ell & 156 & -22\ell & 0 & 0 & 0 \\ -3\ell^2 & -22\ell & 4\ell^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{K}_1 = \frac{EI}{\ell^3}\begin{bmatrix} 4\ell^2 & -6\ell & 2\ell^2 & 0 & 0 & 0 \\ -6\ell & 12 & -6\ell & 0 & 0 & 0 \\ 2\ell^2 & -6\ell & 4\ell^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: $w_1 = W_2, w_2 = W_3, w_3 = W_4, w_4 = W_5$

$$\mathrm{M}_2 = \frac{\rho A \ell}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22\ell & 54 & -13\ell & 0 \\ 0 & 22\ell & 4\ell^2 & 13\ell & -3\ell^2 & 0 \\ 0 & 54 & 13 & 156 & -22\ell & 0 \\ 0 & -13\ell & -3\ell^2 & -22\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{K}_2 = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6\ell & -12 & 6\ell & 0 \\ 0 & 6\ell & 4\ell^2 & -6\ell & 2\ell^2 & 0 \\ 0 & -12 & -6\ell & 12 & -6\ell & 0 \\ 0 & 6\ell & 2\ell^2 & -6\ell & 4\ell^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3: $w_1 = W_4, w_2 = W_5, w_3 = 0, w_4 = W_6$

$$\mathrm{M}_3 = \frac{\rho A \ell}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 156 & 22\ell & -13\ell \\ 0 & 0 & 0 & 22\ell & 4\ell^2 & -3\ell^2 \\ 0 & 0 & 0 & -13\ell & -3\ell^2 & 4\ell^2 \end{bmatrix}$$

$$\mathrm{K}_3 = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 6\ell & 6\ell \\ 0 & 0 & 0 & 6\ell & 4\ell^2 & 2\ell^2 \\ 0 & 0 & 0 & 6\ell & 2\ell^2 & 4\ell^2 \end{bmatrix}$$

The global equations are

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$$(M_1 + M_2 + M_3)\ddot{W} + (K_1 + K_2 + K_3)W = F_1 + F_2 + F_3$$

They are solved by modal analysis.

Problem 11.29 illustrates the use of a three-element finite element method to approximate the time dependent response of a beam.

11.30 A simple model of a one-story frame structure is shown in Figure P11.30(a). Use one beam element to model each of the columns and two bar elements to model the girder. Determine the response of the structure if it is subject to the blast force of Figure P11.30(b).

FIGURE P11.30

Given: L = 5.1 m, h = 2.5 m, m_g = 2000 kg, E = 200 × 10^9 N/m^2, ρ = 7500 kg/m^3, A = 3.1 × 10^{-2} m^2, I = 1.7 × 10^{-5} m^4

Find: response of the structure

Solution: Beam elements are used for the columns and bar elements for the girder.

Element 1(beam element): $w_1 = 0, w_2 = 0, w_3 = W_1, w_4 = W_2$

$$\mathbf{M_1} = \frac{\left(7500\ \frac{kg}{m^3}\right)(3.1 \times 10^{-2}\ m^2)(2.5\ m)}{420}\begin{bmatrix} 156 & -22(2.5) & 0 & 0 & 0 \\ -22(2.5) & 4(2.5)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\mathbf{K_1} = \frac{\left(200 \times 10^6 \ \frac{N}{m^2}\right)(1.7 \times 10^{-5}\ m^4)}{(2.5\ )^3}\begin{bmatrix} 12 & -6(2.5) & 0 & 0 & 0 \\ -6(2.5) & 4(2.5)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2 (Bar element): $w_1 = W_1, w_2 = W_3$

$$\mathbf{M_2} = \frac{1000\ \text{kg}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K_2} = \frac{\left(200 \times 10^9 \ \frac{\text{N}}{\text{m}^2}\right)(3.1 \times 10^{-2}\ \text{m}^2)}{2.55\ \text{m}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3 (bar element): $w_1 = W_3, w_2 = W_4$

$$\mathbf{M_3} = \frac{1000\ \text{kg}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K_3} = \frac{\left(200 \times 10^9 \ \frac{\text{N}}{\text{m}^2}\right)(3.1 \times 10^{-2}\ \text{m}^2)}{2.55\ \text{m}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 4 (beam element): $w_1 = 0, w_2 = 0, w_3 = W_4, w_4 = W_5$

$$\mathbf{M_4} = \frac{\left(7500\ \frac{\text{kg}}{\text{m}^3}\right)(3.1 \times 10^{-2}\ \text{m}^2)(2.5\ \text{m})}{420}\begin{bmatrix} 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 156 & -22(2.5) & \\ 0 & 0 & 0 & -22.5(2.5) & 4(2.5)^2 & \end{bmatrix}$$

$$\mathbf{K_4} = \frac{\left(200 \times 10^6 \ \frac{\text{N}}{\text{m}^2}\right)(1.7 \times 10^{-5}\ \text{m}^4)}{(2.5\ )^3}\begin{bmatrix} 12 & -6(2.5) & & 0 & 0 & 0 \\ -6(2.5) & 4(2.5)^2 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & -6(2.5) & \\ 0 & 0 & 0 & -6(2.5) & 4(2.5)^2 & \end{bmatrix}$$

The matrices are assembled according to

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$$(M_1 + M_2 + M_3 + M_4)\ddot{W} + (K_1 + K_2 + K_3 + K_4)W = F_1 + F_2 + F_3 + F_4$$

where the force vectors are developed according to a uniform time dependent force of

$$F(t) = 7000(1 - 12.5t)$$

Problem 11.30 illustrates the use of a finite-element model to determine the transient response of a system.

11.31 Use the finite-element model of Chapter Problem 11.30 to determine the response of the structure if it is subject to the earthquake of Figure P11.31.

FIGURE P11.31

Given: L = 5.1 m, h = 2.5 m, m_g = 2000 kg, E = 200 × 10^9 N/m^2, ρ = 7500 kg/m^3, A = 3.1 × 10^{-2} m^2, I = 1.7 × 10^{-5} m^4

Find: response of the structure

Solution: The finite-element model of the structure is shown below. Beam elements are used for the columns and bar elements for the girder

Element 1(beam element):$w_1 = 0, w_2 = 0, w_3 = W_1, w_4 = W_2$

$$\mathbf{M_1} = \frac{\left(7500\,\frac{kg}{m^3}\right)(3.1\times10^{-2}\,m^2)(2.5\,m)}{420}\begin{bmatrix} 156 & -22(2.5) & 0 & 0 & 0 \\ -22(2.5) & 4(2.5)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\mathbf{K_1} = \frac{\left(200 \times 10^6 \ \frac{N}{m^2}\right)(1.7 \times 10^{-5}\ m^4)}{(2.5\ )^3}\begin{bmatrix} 12 & -6(2.5) & 0 & 0 & 0 \\ -6(2.5) & 4(2.5)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2 (Bar element): $w_1 = W_1, w_2 = W_3$

$$\mathbf{M_2} = \frac{1000\ \text{kg}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K_2} = \frac{\left(200 \times 10^9 \ \frac{\text{N}}{\text{m}^2}\right)(3.1 \times 10^{-2}\ \text{m}^2)}{2.55\ \text{m}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3 (bar element): $w_1 = W_3, w_2 = W_4$

$$\mathbf{M_3} = \frac{1000\ \text{kg}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K_3} = \frac{\left(200 \times 10^9 \ \frac{\text{N}}{\text{m}^2}\right)(3.1 \times 10^{-2}\ \text{m}^2)}{2.55\ \text{m}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 4 (beam element): $w_1 = 0, w_2 = 0, w_3 = W_4, w_4 = W_5$

$$\mathbf{M_4} = \frac{\left(7500\ \frac{\text{kg}}{\text{m}^3}\right)(3.1 \times 10^{-2}\ \text{m}^2)(2.5\ \text{m})}{420}\begin{bmatrix} 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 156 & -22(2.5) & \\ 0 & 0 & 0 & -22.5(2.5) & 4(2.5)^2 & \end{bmatrix}$$

$$\mathbf{K_4} = \frac{\left(200 \times 10^6 \ \frac{\text{N}}{\text{m}^2}\right)(1.7 \times 10^{-5}\ \text{m}^4)}{(2.5\ )^3}\begin{bmatrix} 12 & -6(2.5) & & 0 & 0 & 0 \\ -6(2.5) & 4(2.5)^2 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & -6(2.5) & \\ 0 & 0 & 0 & -6(2.5) & 4(2.5)^2 & \end{bmatrix}$$

The response is determined by solving

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$$(M_1 + M_2 + M_3 + M_4)\ddot{W} + (K_1 + K_2 + K_3 + K_4)W = F_1 + F_2 + F_3 + F_4$$

where the force vectors are developed according to a uniform time dependent force expressed in a Fourier series.

Problem 11.31 illustrates the use of a finite-element model to evaluate the response of a periodic exciation.

11.32 Use the finite-element model of Chapter Problem 11.30 to determine the response of the structure if HVAC equipment on the girder produces a lateral harmonic force of magnitude 3000 N at a frequency of 500 rpm.

Given: L = 5.1 m, h = 2.5 m, m_g = 2000 kg, E = 200 × 10^9 N/m^2, ρ = 7500 kg/m^3, A = 3.1 × 10^{-2} m^2, I = 1.7 × 10^{-5} m^4, 3000 N, ω = 500 rpm

Find: response of the structure

Solution: The finite-element model of the structure is shown below. Beam elements are used for the columns and bar elements for the girder

Element 1(beam element):$w_1 = 0, w_2 = 0, w_3 = W_1, w_4 = W_2$

$$\mathbf{M_1} = \frac{\left(7500\ \frac{\text{kg}}{\text{m}^3}\right)(3.1 \times 10^{-2}\ \text{m}^2)(2.5\ \text{m})}{420}\begin{bmatrix} 156 & -22(2.5) & 0 & 0 & 0 \\ -22(2.5) & 4(2.5)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K_1} = \frac{\left(200 \times 10^9\ \frac{\text{N}}{\text{m}^2}\right)(1.7 \times 10^{-5}\ \text{m}^4)}{(2.5\ )^3}\begin{bmatrix} 12 & -6(2.5) & 0 & 0 & 0 \\ -6(2.5) & 4(2.5)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2 (Bar element): $w_1 = W_1, w_2 = W_3$

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$$\mathbf{M_2} = \frac{1000\ \text{kg}}{6}\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K_2} = \frac{\left(200 \times 10^9\ \frac{\text{N}}{\text{m}^2}\right)(3.1 \times 10^{-2}\ \text{m}^2)}{2.55\ \text{m}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3 (bar element): $w_1 = W_3, w_2 = W_4$

$$\mathbf{M_3} = \frac{1000\ \text{kg}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K_3} = \frac{\left(200 \times 10^9\ \frac{\text{N}}{\text{m}^2}\right)(3.1 \times 10^{-2}\ \text{m}^2)}{2.55\ \text{m}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 4 (beam element): $w_1 = 0, w_2 = 0, w_3 = W_4, w_4 = W_5$

$$\mathbf{M_4} = \frac{\left(7500\ \frac{\text{kg}}{\text{m}^3}\right)(3.1 \times 10^{-2}\ \text{m}^2)(2.5\ \text{m})}{420}\begin{bmatrix} 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 156 & -22(2.5) & \\ 0 & 0 & 0 & -22.5(2.5) & 4(2.5)^2 & \end{bmatrix}$$

$$\mathbf{K_4} = \frac{\left(200 \times 10^9\ \frac{\text{N}}{\text{m}^2}\right)(1.7 \times 10^{-5}\ \text{m}^4)}{(2.5\ )^3}\begin{bmatrix} 12 & -6(2.5) & & 0 & 0 & 0 \\ -6(2.5) & 4(2.5)^2 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & -6(2.5) & \\ 0 & 0 & 0 & -6(2.5) & 4(2.5)^2 & \end{bmatrix}$$

Then

$$\mathbf{M} = \mathbf{M_1} + \mathbf{M_2} + \mathbf{M_3} + \mathbf{M_4} = \begin{bmatrix} 549.22 & -76.11 & & 0 & 0 & 0 \\ -76.11 & 34.59 & & 0 & 0 & 0 \\ 166.67 & 0 & 666.67 & 166.67 & 0 & \\ 0 & 0 & 166.67 & 549.22 & -76.11 & \\ 0 & 0 & 0 & -76.11 & 34.59 & \end{bmatrix}$$

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$$\mathbf{K} = \mathbf{K_1} + \mathbf{K_2} + \mathbf{K_3} + \mathbf{K_4} = 10^9 \begin{bmatrix} 2.434 & -0.0033 & -2.431 & 0 & 0 \\ -0.0033 & 0.0054 & 0 & 0 & 0 \\ -2.431 & 0 & 4.827 & -2.431 & 0 \\ 0 & 0 & -2.431 & 2.424 & -0.0033 \\ 0 & 0 & 0 & -0.0033 & 0.0054 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M^{-1}K}$. To this end

$$\omega_1 = 24.0\ \frac{\text{rad}}{\text{s}}, \omega_2 = 395.4\ \frac{\text{rad}}{\text{s}}, \omega_1 = 412.7\ \frac{\text{rad}}{\text{s}}, \omega_1 = 2531\ \frac{\text{rad}}{\text{s}}, \omega_1 = 5065\ \frac{\text{rad}}{\text{s}}$$

The modal matrix is the matrix whose columns are the normalized eigenvectors

$$\mathbf{P} = \begin{bmatrix} -0.0210 & -0.004 & -0.0060 & -0.0362 & 0.0346 \\ -0.125 & 0.1192 & -0.1286 & -0.0811 & 0.0765 \\ -0.210 & -0.0000 & -0.0062 & 0.0000 & -0.0380 \\ -0.210 & 0.0004 & -0.0060 & 0.0362 & 0.0346 \\ -0.125 & -0.1192 & -0.1286 & 0.0811 & 0.0765 \end{bmatrix}$$

The response of the system is obtained for a harmonic excitation.

Problem 11.32 illustrates the use of the finite-element method to determine the forced response of a beam.

11.34 Use one bar element to model each member of the truss of Figure P11.34 and approximate its two lowest natural frequencies.

FIGURE P11.34

Given: $E = 140 \times 10^9$ N/m^2, $\rho = 5000$ kg/m^3, $A = 1.6 \times 10^{-3}$ m^2

Find: ω_1, ω_2

Solution: The four-element finite-element model of the truss is shown.

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One element is used for each truss member. Geometry leads to

$$\ell_1 = 22.36\,\text{cm}$$
$$\ell_2 = 36.06\,\text{cm}$$
$$\theta_1 = 26.5°$$
$$\theta_2 = 56.3°$$

The relations between the local coordinates and the global generalized coordinates for each element are given below as well as the coordinate transformation matrices and the global mass and stiffness matrices. The local mass and stiffness matrices for each element are those for the standard bar element.

Element 1: $w_1 = 0,\ \mathrm{w}_2 = W_1 \cos\theta_1 + W_2 \sin\theta_1$

$$\mathbf{S_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \cos\theta_1 & \sin\theta_1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
\tilde{\mathbf{M}}_1 &= \mathbf{S_1^T m S_1} \\
&= \begin{bmatrix} 0 & \cos\theta_1 \\ 0 & \sin\theta_1 \\ 0 & 0 \\ 0 & \end{bmatrix} \frac{\rho A \ell_1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ \cos\theta_1 & \sin\theta_1 & 0 & 0 \end{bmatrix} \\
&= \frac{\rho A \ell_1}{3} \begin{bmatrix} \cos^2\theta_1 & \cos\theta_1 \sin\theta_1 & 0 & 0 \\ \cos\theta_1 \sin\theta_1 & \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\tilde{\mathbf{K}}_1 = \mathbf{S_1^T k S_1} = \frac{EA}{\ell_1} \begin{bmatrix} \cos^2\theta_1 & \cos\theta_1 \sin\theta_1 & 0 & 0 \\ \cos\theta_1 \sin\theta_1 & \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: $w_1 = 0,\ w_2 = W_1 \cos\theta_2 - W_2 \sin\theta_2$

$$\mathbf{S_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \cos\theta_2 & -\sin\theta_2 & 0 & 0 \end{bmatrix}$$

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$$\tilde{\mathbf{M}}_2 = \mathbf{S}_2^{\mathrm{T}}\mathbf{m}\mathbf{S}_2 = \frac{\rho A \ell_2}{3}\begin{bmatrix} \cos^2\theta_2 & -\cos\theta_2\sin\theta_2 & 0 & 0 \\ -\cos\theta_2\sin\theta_2 & \sin^2\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \mathbf{S}_2^{\mathrm{T}}\mathbf{k}\mathbf{S}_2 = \frac{EA}{\ell_2}\begin{bmatrix} \cos^2\theta_2 & -\cos\theta_2\sin\theta_2 & 0 & 0 \\ -\cos\theta_2\sin\theta_2 & \sin^2\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3: $w_1 = W_2, w_2 = W_4$

$$\tilde{\mathbf{M}}_3 = \frac{\rho A \ell_3}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_3 = \frac{EA}{\ell_3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Element 4: $w_1 = 0, w_2 = W_3$

$$\tilde{\mathbf{M}}_4 = \frac{\rho A \ell_4}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_4 = \frac{EA}{\ell_4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The assembled global mass and stiffness matrices are

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$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 + \tilde{\mathbf{M}}_3 + \tilde{\mathbf{M}}_4 = \begin{bmatrix} 0.7729 & -0.2052 & 0 & 0 \\ -0.2052 & 1.5849 & 0 & 0.4000 \\ 0 & 0 & 0.5333 & 0 \\ 0 & 0.4000 & 0 & 0.8000 \end{bmatrix}$$

$$\mathbf{K} = \tilde{\mathbf{K}}_1 + \tilde{\mathbf{K}}_2 + \tilde{\mathbf{K}}_3 + \tilde{\mathbf{K}}_4 = 10^9 \begin{bmatrix} 0.9926 & 0.1140 & 0 & 0 \\ 0.1140 & 0.6380 & 0 & -0.7467 \\ 0 & 0 & 1.1200 & 0 \\ 0 & -0.7467 & 0 & 0.7467 \end{bmatrix}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The two lowest natural frequencies are calculated as

$$\omega_1 = 1.3 \times 10^4 \text{ rad/s}$$
$$\omega_2 = 3.5 \times 10^4 \text{ rad/s}$$

Problem 11.34 illustrates the application of a finite-element model of a truss to determine its natural frequencies.

11.36 A beam is placed on an elastic foundation whose stiffness per unit length is k. Derive the element k_{23} of the local stiffness matrix for a beam element of length l including the stiffness of the elastic foundation.

Find: k_{23}

Solution: The total potential energy for a beam element resting on an elastic foundation of stiffness k is

$$V = \frac{1}{2}\int_0^{\ell}\left[EI\left(\frac{\partial^2 u}{\partial \xi^2}\right) + ku^2\right]d\xi$$

where u(ξ,t) is the displacement function for the element. Using the displacement function

$$V = \frac{1}{2}\int_0^{\ell}\left\{EI\left[\left(-\frac{6}{\ell}+12\frac{\xi}{\ell^2}\right)w_1 + \left(\frac{4}{\ell^2}+6\frac{\xi}{\ell^3}\right)\ell w_2 + \left(\frac{6}{\ell^2}-12\frac{\xi}{\ell^3}\right)w_3 + \left(-\frac{2}{\ell^2}+6\frac{\xi}{\ell^3}\right)\ell w_4\right]^2\right\}d\xi$$

$$+\frac{1}{2}k\int_0^{\ell}\left[\left(1-\frac{3\xi^2}{\ell^2}+2\frac{\xi^3}{\ell^3}\right)w_1 + \left(\frac{\xi}{\ell}-2\frac{\xi^2}{\ell^2}+\frac{\xi^3}{\ell^3}\right)\ell w_2 + \left(3\frac{\xi^2}{\ell^2}-2\frac{\xi^3}{\ell^3}\right)w_3 + \left(-\frac{\xi^2}{\ell^2}+\frac{\xi^3}{\ell^3}\right)w_4\right]^2 d\xi$$

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Using the quadratic form of the potential energy it can be seen that the stiffness element k_{23} is the coefficient multiplying w_2w_3 /4 in the quadratic form. To this end

$$k_{23} = \int_0^{\ell} EI\left(\frac{4}{\ell^2} + 6\frac{\xi}{\ell^3}\right)\left(\frac{6}{\ell^2} - 12\frac{\xi}{\ell^3}\right)\ell\, d\xi$$

$$+ k\int_9^{\ell}\left(\frac{\xi}{\ell} - 2\frac{\xi^2}{\ell^2} + \frac{\xi^3}{\ell^3}\right)\left(3\frac{\xi^2}{\ell^2} - 2\frac{\xi^3}{\ell^3}\right)\ell\, d\xi$$

$$= -\frac{6EI}{\ell^2} + \frac{13k\ell^2}{420}$$

Problem 11.36 illustrates the determination of stiffness matrix elements for a beam element on a continuous elastic foundation.

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CHAPTER 12: NONLINEAR VIBRATIONS

Short Answer Problems

12.1 False: The convolution integral cannot be applied to solve nonlinear problems (it is based upon the principle of linear superposition).

12.2 True: Coulomb damping can be modeled by a nonlinear function, $\frac{\mu mg|\dot{x}|}{\dot{x}}$.

12.3 False: The swinging spring is an example of a two-degree-of-freedom system with a quadratic nonlinearity.

12.4 True: The period and frequency of the free vibration response of a nonlinear system depends upon initial conditions.

12.5 False: The free response of a system with a cubic nonlinearity is at the natural frequency of the system which is stretched depending upon initial conditions.

12.6 False: A focus may be stable or unstable.

12.7 True: A saddle point is always unstable. The equilibrium point near the saddle point has one positive eigenvalue and one negative eigenvalue.

12.8 True: Secular terms must be removed from the response of a system. If not the system is non-uniform. As t gets large enough the second term is as large as the first term. Thus to the order approximated the system does not appear to be convergent.

12.9 True: The steady-state response consists of the forced response whose period is three times the linear natural period plus the free response whose frequency is adjusted to three times that of the excitation.

12.10 True: The effect of damping on nonlinear systems is similar to that of linear systems.

12.11 False: The forced response occurs at the frequency of excitation as well as superharmonic (r = 1/3) and subharmonic resonances (r = 3).

12.12 False: A MDOF system has an internal resonance when the parameters are such that one of the system's linear natural frequencies is in a certain combination with another of the system's natural frequencies.

12.13 True: A bifurcation occurs when a parameter splits the natural frequencies.

12.14 True: One route to chaos is through period doubling.

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12.15 The Laplace transform method cannot be applied to nonlinear systems because successful application of the Laplace transform method relies upon linearity of the transform. The Laplace transform cannot be applied successfully to nonlinear functions of x, for example $\mathcal{L}\{x^2\} \neq [\bar{X}(s)]^2$.

12.16 A spring has a cubic nonlinearity which is an example of a geometric nonlinearity.

12.17 A spring with a cubic nonlinearity equal to $-3x^3$ is an example of a softening nonlinearity.

12.18 (a) Stable focus (b) Center (c) Unstable node (d) Unstable saddle point

12.19 The equilibrium point is an unstable saddle point.

12.20 The equilibrium point is a stable focus.

12.21 The equilibrium point is a center.

12.22 The equilibrium point is a stable node.

12.23 The equilibrium point is an unstable focus.

12.24 The detuning parameter is used to express the nearness of the frequency of excitation to the frequency of interest. The relationship is substituted into the condition to eliminate secular terms. The result is a frequency response in terms of the detuning parameter.

12.25 (a) Softening spring (b) Points A and B are points of vertical tangency on the frequency response curve. As the frequency increases and reaches point B on the frequency response curve the frequency jumps to a point on the frequency response curve immediately above B. Similarly as the frequency decreases and reaches point A on the frequency response curve the frequency jumps. The points on the curve between A and B are unstable and can never be reached.

12.26 (a) Primary resonance when $\omega \approx 30$ rad/s. (b) Superharmonic resonance when $\omega \approx 10$ rad/s (c) Subharmonic resonance when $\omega \approx 90$ rad/s.

12.27 A combination resonance as $2\omega_2 - \omega_1 = 120$

12.28 A combination resonance as $2\omega_1 - \omega_2 = 120$

12.29 A combination resonance as $\omega_2 - \omega_1 - 120 = 120$

12.30 No resonances

12.31 A superharmonic resonance as $\omega_1 = \frac{1}{3}(120)$ and a combination resonance as $\omega_1 + \omega_2 - 120 = 120$

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12.32 A primary resonance as $\omega_1 = 120$ and a superharmonic resonance as $\omega_2 = \frac{1}{3}(120)$

12.33 A subharmonic resonance as $\omega_2 = 3(120)$ and a combination resonance as $2\omega_1 - \omega_1 = 120$

12.34 The steady state response consists of the forced response plus the free response. The period of the forced response is three times that of the linear natural period. The nonlinearity adjusts the frequency of the free response to three times that of the exciation.

12.35 An internal resonance occurs in a MDOF system when two or more of the natural frequencies combine to create a resonance condition.

12.36 The Poincaré section corresponding to a periodic function when the sampling interval is one-third of the period is three points.

12.37 The values of ω which have significant non-zero values of $f(\omega)$.

12.38 Fiegenbuam's constant is 4.669...

12.39 Given: $\Delta\ddot{x} + 2\Delta\dot{x} + 3\Delta x = 0$ The stability of the equilibrium point is determined by the solution of the differential equation. Assuming a solution of $e^{\beta t}$ leads to $\beta^2 + 2\beta + 3 = 0$. The solutions of the quadratic equation are $\beta = -1 \pm 2\sqrt{2}i$. The roots are complex conjugates with negative real parts. Thus the equilibrium point is a stable focus.

12.40 Given: $\Delta\ddot{x} - 2\Delta\dot{x} + 3\Delta x = 0$ The stability of the equilibrium point is determined by the solution of the differential equation. Assuming a solution of $e^{\beta t}$ leads to $\beta^2 - 2\beta + 3 = 0$. The solutions of the quadratic equation are $\beta = 1 \pm 2\sqrt{2}i$. The roots are complex conjugates with positive real parts. Thus the equilibrium point is an unstable focus.

12.41 Given: $\Delta\ddot{x} + 2\Delta\dot{x} - 3\Delta x = 0$ The stability of the equilibrium point is determined by the solution of the differential equation. Assuming a solution of $e^{\beta t}$ leads to $\beta^2 + 2\beta - 3 = 0$. The solutions of the quadratic equation are $\beta = -3, 1$. The roots are real and of opposite sign. Thus the equilibrium point is an unstable saddle point.

12.42 Given: $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$ (a) The equilibrium points are given by $\sin\theta = 0$ or $\theta = \pm n\pi$ where n is an integer. (b) The stability of the equilibrium points is examined by assuming $\theta = \pm n\pi + \Delta\theta$. Substitution into the differential equation leads to $\Delta\ddot{\theta} + \Delta\theta\cos(\pm n\pi) = 0$ or $\Delta\ddot{\theta} \pm \Delta\theta = 0$. The + sign is for n = 0, 2, 4,... and the − sign is for n = 1, 3, 5,.... Assuming a solution of $e^{\beta t}$ leads to $\beta^2 \pm 1 = 0$. The solutions of the quadratic equation are $\beta = \pm i$ for n = 0, 2, 4,... and $\beta = \pm 1$ for n = 1, 3, 5,.... Hence the equilibrium point is a center for n = 0, 2, 4.... and an unstable saddle point for n = 1, 3, 5...

12.43 Given: $\ddot{x} - 0.5\dot{x} + x - 0.1x^3 = 0$ (a) The equilibrium are the points for which $x - 0.1x^3 = 0$. This leads to $x(1 - 0.1x^2) = 0$ and the equilibrium points are $x = -0.316, 0, 0.316$. (b) First look at the equilibrium point x = 0. Assume $x = 0 + \Delta x$.

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Substitution into the differential equation yields $\Delta\ddot{x} - 0.5\Delta\dot{x} + \Delta x = 0$. Assuming a solution of $e^{\beta t}$ leads to $\beta^2 - 0.5\beta + 1 = 0$. The solutions of the quadratic equation are $\beta = 0.25 \pm 0.968i$. The roots are complex conjugates with positive real parts. Thus the equilibrium point is an unstable focus. Now consider the equilibrium point $x = -0.316$. Assume $x = -0.386 + \Delta x$. Substitution into the differential equation yields $\Delta\ddot{x} - 0.5\Delta\dot{x} + 0.955\Delta x = 0$. Assuming a solution of $e^{\beta t}$ leads to $\beta^2 - 0.5\beta + 0.955 = 0$. The solutions of the quadratic equation are $\beta = 0.25 \pm 0.944i$. The roots are complex conjugates with positive real parts. Thus the equilibrium point is an unstable focus. The equilibrium point x = 0.366 is also a focus.

12.44 Given: $\ddot{x} - x + 0.1x^3 = 0$ (a) The equilibrium are the points for which $x - 0.1x^3 = 0$. This leads to $x(1 - 0.1x^2) = 0$ and the equilibrium points are $x = -0.316, 0, 0.316$. (b) First look at the equilibrium point x = 0. Assume $x = 0 + \Delta x$. Substitution into the differential equation yields $\Delta\ddot{x} - \Delta x = 0$. Assuming a solution of $e^{\beta t}$ leads to $\beta^2 - 1 = 0$. The solutions of the quadratic equation are $\beta = \pm 1$ The roots are real and of different signs Thus the equilibrium point is an unstable saddle point. Now consider the equilibrium point $x = -0.316$. Assume $x = -0.386 + \Delta x$. Substitution into the differential equation yields $\Delta\ddot{x} - 0.955\Delta x = 0$. Assuming a solution of $e^{\beta t}$ leads to $\beta^2 - 0.955 = 0$. The solutions of the quadratic equation are $\beta = \pm 0.977$. The roots are real and of different signs Thus the equilibrium point is an unstable saddle point. The equilibrium point x = 0.366 is also an unstable saddle point.

12.45 (a) The equilibrium points are solutions to $\tan\theta = \frac{\omega^2}{g}$ (b) (i) The equilibrium point is $\theta = 26.6° = 0.463$ rad. Expanding about the equilibrium point, $\theta = 0.463 + \Delta\theta$, leads to $\Delta\ddot{\theta} + \frac{g}{R}(\cos(0.463)\,\Delta\theta + 0.5\sin(0.463)\Delta\theta) = \Delta\ddot{\theta} + 1.180\frac{g}{R}\Delta\theta = 0$. Hence the eigenvalues are both purely imaginary and the equilibrium point is a center. (ii) The equilibrium point satisfies $\tan\theta = 1$ or $\theta = \pi/4$. Expanding about the equilibrium point leads to $\Delta\ddot{\theta} + \frac{g}{R}\left(\cos\frac{\pi}{4}\Delta\theta + 0.5\sin\frac{\pi}{4}\,\Delta\theta\right) = 0$. The equilibrium point is a center. (iii) The equilibrium point satisfies $\tan\theta = 1.5$ or $\theta = 0.983$. Expanding about the equilibrium point leads to $\Delta\ddot{\theta} + 0.971\frac{g}{R}\Delta\theta = 0$. The equilibrium point is a center.

12.46 The initial conditions imply $\phi = \pi/2$ and $1 = A - \frac{0.01}{32}A^3$ whose solutions are $= 1.003,\ 56.0618,\ -57.0621$. Choosing A = 1.003, $w = t\left(1 + 0.1\frac{3}{8}1.003^2\right) = 1.0038t$. The solution is $x(t) = 1.003\sin\left(1.0038t + \frac{\pi}{2}\right) - \frac{0.01(1.003)^3}{32}\sin 3\left(1.0038t + \pi 2 = 1.003\sin 1.0038t + \pi 2 - 3.153\times 10 - 4\sin 3.0114t + 3\pi 2.\right.$

12.47 The solution to the differential equation is $x(t) = Ae^{-0.05t}\sin[(1 + 0.00375A^2)t + \phi]$. Substituting given initial conditions leads to $x(0) = 1 = A\sin\phi$ and $\dot{x}(0) = 0 = -0.05\,A\sin\phi + A(1 + 0.00375A^2)\cos\phi$. A trial and error solution leads to $\phi = 1.5211$ and $A = 1.0009$. The solution becomes $x(t) = 1.0009e^{-0.05t}\sin[1.00376t + \phi]$.

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12.48 The equation is Duffing's equation with a primary resonance. From the equation $= 0.01$, $\sigma = 1$, $\mu = 2.5$ and $\hat{F}_1 = 3$. Thus the equation for the steady-state amplitude becomes $4A^2\left[(2.5)^2 + \left(1 - \frac{3}{8}A^2\right)^2\right] = 9$. The only real solution is A = 0.100

12.49 The equation is Duffing's equation with a superharmonic resonance. From the equation $= 0.01$, $\sigma = -1$, $\mu = 2.5$ and $F_1 = 3$. Thus the equation for the steady-state amplitude becomes $\left[(2.5)^2 + \left(-1 - \frac{3}{8}A^2 - 27\right)^2\right]A^2 = 729$. The only real solution is A = 0.9491.

12.50 The equation is Duffing's equation with a subharmonic resonance. From the equation $= 0.01$, $\sigma = 6$, $\mu = 2.5$ and $F_1 = 3$. Thus the equation for the steady-state amplitude becomes $\left[9(2.5)^2 + \left(6 - \frac{9}{8}A^2 - 27\right)^2\right]A^2 = \frac{81}{16}9A^2$. No real solutions of the equation exist.

12.51 An example of an internal resonance for a fixed-free beam is $\omega_4 = 2\omega_3 - \omega_1 + 0.99$

12.52 An example of an internal resonance for a fixed-attached mass beam with $\beta = 0.25$ is $\omega_4 = 2\omega_3 - \omega_1 + 1.06$

12.53 The Fourier transform of f(t) is a value of a at $\omega = 3$ and 2a at $\omega = 4.5$.

12.54 (a) k_3 has units of N/m^3 (b) ε is dimensionless (c) σ is dimensionless

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Chapter Problems

12.1 The free-vibration response of a block hanging from a linear spring is the same as that of the block attached to the same spring, but sliding on a frictionless surface. Is the response the same if the spring has a force-displacement relation given by the following?

(a) $F = k_1x + k_3x^3$
(b) $F = k_1x + k_2x^2$
(c) $F = k_1x,\ x < x_0$
(d) $F = k_2x,\ x > x_0$

Given: force-displacement relations

Find: is hanging spring problem equivalent to sliding spring problem?

Solution: (a) Consider a block attached to a spring with a cubic nonlinearity sliding on a frictionless surface. The differential equation governing motion of a system is

$$m\ddot{x} + k_1x + k_3x^3 = 0$$

Now consider the same block hanging from the same spring. The spring has a static deflection described by

$$mg = k_1\Delta + k_3\Delta^3$$

Application of Newton's law to the free body diagram of the system at an arbitrary time leads to

$$m\ddot{x} = mg - k_1(x+\Delta) - k_3(x+\Delta)^3$$

which after using the static relation becomes

$$m\ddot{x} + k_1x + k_3x^3 = -3k_3(x\Delta^2 + x^2\Delta)$$

Hence since the governing differential equations are different, the free-vibrations responses of the two systems are not the same.

(b) In a similar fashion consider a block attached to a spring with a quadratic nonlinearity. The governing differential equation when the block is sliding on a frictionless surface is

$$m\ddot{x} + k_1x + k_2x^2 = 0$$

The governing differential equation when the block is hanging from a spring is

$$m\ddot{x} + k_1x + k_2x^2 = -2k_2x\Delta$$

Hence the free-vibrations responses of the systems are not the same.

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(c) Now consider a spring with different properties in tension and compression. In this case the governing differential equations for the block sliding on a frictionless surface are

$$m\ddot{x} + k_1 x = 0, \quad x > 0$$

$$m\ddot{x} + k_2 x = 0, \qquad x < 0$$

Now consider the block hanging from the same spring. Take x = 0 to be measures from the systems' static equilibrium position. The static deflection is given by

$$\Delta = \frac{mg}{k_2}$$

When x > 0, the spring is in tension and the static deflection cancels with gravity giving

$$m\ddot{x} + k_2 x = 0, \qquad x > 0$$

(d) However when x < 0, the spring could be either in tension or compression depending upon the relation between k_1, m, and x. If $-\Delta < x < 0$ the spring remains in tension and the preceding equation applies. However if $x < -\Delta$ then the spring is in compression and the differential equation becomes

$$m\ddot{x} + k_1(x - \Delta) = mg, \quad x < -\Delta$$

Hence the free-vibrations responses of the two systems are not the same.

Problem 12.1 illustrates that the static deflections term does not cancel with gravity when deriving the differential equation for a nonlinear system.

12.2 The system of Figure P12.2 is one of the few for which an exact solution is available. Its solution is obtained in a manner analogous to that of free vibrations with Coulomb damping. The block is displaced a distance $x_0 > \delta$ to the right from equilibrium and released. Determine the period of the resulting oscillations.

FIGURE P12.2

Given: m, k, x_0, δ

Find: T

Solution: The differential equations governing the motion of the system are

$$m\ddot{x} + kx = 0, \qquad x > -\delta \qquad (1)$$

$$m\ddot{x} + 2kx = 0, \qquad x < -\delta$$

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The system is nonlinear since different differential equations govern the motion depending upon the location of the block. The system is conservative, thus it is expected that the free vibrations are periodic. The initial conditions are

$$x(0) = x_0, \qquad \dot{x}(0) = 0$$

The first of equations (1) govern the motion initially. Its solution is

$$x(t) = C_1 cos\sqrt{\frac{k}{m}}t + C_2 sin\sqrt{\frac{k}{m}}\,\mathrm{t}$$

Application of the initial conditions leads to

$$x(t) = x_0 cos\sqrt{\frac{k}{m}}t \tag{2}$$

Equation (2) describes the motion until x = –δ. This occurs when

$$-\delta = x_0 cos\sqrt{\frac{k}{m}}t_1$$

$$t_1 = \sqrt{\frac{m}{k}}cos^{-1}\left(-\frac{\delta}{x_0}\right)$$

At this time the second of eqs. (1) becomes the governing differential equation. As the system moves during this period it is governed by a linear differential equation. It will return to this position after one half cycle of motion for this system, after an additional time

$$t_2 = \pi\sqrt{\frac{m}{2k}}$$

After the time $t_1 + t_2$, the system is again governed by the first of eqs. (1). The system eventually returns to its initial position in an additional time t_1. Hence the total time it takes for the system to return to equilibrium is

$$T = 2t_1 + t_2$$

$$= 2\sqrt{\frac{m}{k}}\,cos^{-1}\left(-\frac{\delta}{x_0}\right) + \pi\sqrt{\frac{m}{2k}}$$

Problem 12.2 illustrates that the period of oscillation of a nonlinear system depends upon the initial conditions.

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12.3 The block in Figure P12.3 is not attached to the springs. Determine the period of the resulting oscillations if the block is displaced a distance x_0 to the right from equilibrium and released.

FIGURE P12.3

Given: m, k_1, k_2, x_0

Find: T

Solution: The differential equations governing the motion of the system are

$$m\ddot{x} + k_2 x = 0, \quad x > 0 \tag{1}$$

$$m\ddot{x} + k_1 x = 0, \quad x < 0 \tag{2}$$

If the block is displaced a distance x_0 to the right and released, eq. (1) initially governs the motion. Its solution subject to the appropriate initial conditions is

$$x(t) = x_0 cos\sqrt{\frac{k_2}{m}}\, t \tag{3}$$

Equation (3) describes the motion until x = 0, which occurs at

$$t = \frac{\pi}{2}\sqrt{\frac{m}{k_2}}$$

Then equation (3) governs the motion of the system. It takes a time

$$t_2 = \pi\sqrt{\frac{m}{k_1}}$$

For the system to return back to x = 0. The eq.(1) again governs the motion. It takes the system an additional time t_1 to return to the initial position. Hence the period of the system is

$$T = \pi\sqrt{\frac{m}{k_2}} + \pi\sqrt{\frac{m}{k_1}}$$

Problem 12.3 illustrates the determination of the period of motion for a nonlinear system for which an exact solution exists.

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12.4 Without making linearizing assumptions, use Lagrange's equation to derive the nonlinear differential equation(s) governing the motion of the system shown. Use the generalized coordinates indicated in Figures P12.4.

FIGURE P12.4

Given: L, m, k, θ as generalized coordinate

Find: differential equation

Solution: The velocity of the mass center of the bar is

$$\vec{v} = \frac{L}{4}\dot{\theta}(-sin\theta\vec{\imath} - cos\theta\vec{\jmath})$$

Hence the total kinetic energy of the system at an arbitrary time is

$$T = \frac{1}{2}m\left(\frac{L^2}{16}cos^2\theta + \frac{L^2}{16}sin^2\theta\right) + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2$$

$$= \frac{1}{2}\frac{7}{48}mL^2\dot{\theta}^2$$

The potential energy of the system at an arbitrary time is

$$V = \frac{1}{2}k\left(\frac{3}{4}Lsin\theta\right)^2$$

$$= \frac{1}{2}\frac{9}{16}kL^2sin^2\theta$$

The Lagrangian is

$$L = \frac{1}{2}\left(\frac{7}{48}mL^2\dot{\theta}^2 - \frac{9}{16}kL^2sin^2\theta\right)$$

Application of Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

leads to

$$\frac{d}{dt}\left(\frac{7}{48}mL^2\dot{\theta}\right) + \frac{9}{16}kL^2sin\theta cos\theta = 0$$

$$\frac{7}{48}mL^2\ddot{\theta} + \frac{9}{16}kL^2sin\theta cos\theta = 0$$

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Problem 12.4 illustrates application of Lagrange's equation to derive the governing differential equation for a nonlinear one-degree-of-freedom system.

12.5 Without making linearizing assumptions, use Lagrange's equation to derive the nonlinear differential equation(s) governing the motion of the system shown. Use the generalized coordinates indicated in Figures P12.5.

FIGURE P12.5

Given: m, l, k, x as generalized coordinate

Find: differential equation

Solution: The kinetic energy of the system at an arbitrary time is

$$T = \frac{1}{2}m\dot{x}^2$$

The potential energy of the system at an arbitrary time is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k\left(\sqrt{\ell^2 + x^2} - \ell\right)^2$$

Hence the Lagrangian is

$$L = \frac{1}{2}\left[m\dot{x}^2 - kx^2 - \left(\sqrt{\ell^2 + x^2} - \ell\right)^2\right]$$

Application of Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

leads to

$$\frac{d}{dt}(m\dot{x}) + kx + k\left(\sqrt{\ell^2 + x^2} - \ell\right)\frac{x}{\sqrt{\ell^2 + x^2}} = 0$$

$$m\ddot{x} + 2kx - \frac{k\ell x}{\sqrt{\ell^2 + x^2}} = 0$$

Problem 12.5 illustrates application of Lagrange's equation to derive the nonlinear differential equation governing a one-degree-of-freedom system.

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12.6 Without making linearizing assumptions, use Lagrange's equation(s) to derive the nonlinear differential equations governing the motion of the system shown. Use the generalized coordinates indicated in Figures P12.6.

FIGURE P12.6

Given: m, ω, p, $y = px^2$, x as generalized coordinate

Find: differential equation

Solution: the velocity of the particle at an arbitrary instant is

$$\vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \omega x\vec{k}$$

$$= \dot{x}\vec{\imath} + 2px\dot{x}\vec{\jmath} + \omega x\vec{k}$$

Hence the kinetic energy of the particle at an arbitrary instant is

$$T = \frac{1}{2}m(\dot{x}^2 + 4p^2x^2\dot{x}^2 + \omega^2x^2)$$

Using y = 0 as the datum for potential energy calculations, the potential energy of the particle at an arbitrary instant is

$$V = mgy = mgpx^2$$

Hence the Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + 4p^2x^2\dot{x}^2 + \omega^2x^2) - mgpx^2$$

Application of Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

leads to

$$\frac{d}{dt}[m(1 + 4p^2x^2)\dot{x}] - 4mp^2x\dot{x}^2 - m\omega^2x + 2mgpx = 0$$

$$(1 + 4p^2x^2)\ddot{x} + (2gp - \omega^2)x + 4p^2x\dot{x}^2 = 0$$

Problem 12.6 illustrates application of Lagrange's equation to derive the nonlinear differential equation governing the motion of a one-degree-of-freedom system.

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12.7 Without making linearizing assumptions, use Lagrange's equations to derive the nonlinear differential equation(s) governing the motion of the system shown. Use the generalized coordinates indicated in Figures P12.7.

FIGURE P12.7

Given: m, k, l, x, and θ as generalized coordinates

Find: differential equations

Solution: The velocity of mass center of the block is

$$\vec{v}_b = \dot{x}\vec{\imath}$$

The velocity of the mass center of the bar is determined using the relative velocity equation

$$\vec{v}_G = \vec{v}_b + \dot{\theta}kx\left(\frac{\ell}{2}sin\theta\vec{\imath} - \frac{\ell}{2}cos\theta\vec{\jmath}\right)$$

$$= \left(\dot{x} + \frac{\ell}{2}\dot{\theta}cos\theta\right)\vec{\imath} + \frac{\ell}{2}\dot{\theta}sin\theta\vec{\jmath}$$

The total kinetic energy of the system at an arbitrary time is

$$T = \frac{1}{2}\dot{m}x^2 + \frac{1}{2}m\left[\left(\dot{x} + \frac{\ell}{2}\dot{\theta}cos\theta\right)^2 + \left(\frac{\ell}{2}\dot{\theta}sin\theta\right)^2\right] + \frac{1}{2}\frac{1}{12}m\ell^2\dot{\theta}^2$$

$$= \frac{1}{2}\left(2\dot{x}^2 + \ell\dot{x}\dot{\theta}cos\theta + \frac{\ell^2}{3}\dot{\theta}^2\right)$$

Using the center of the block as the datum for gravity potential energy calculations, the total potential energy of the system is

$$V = \frac{1}{2}kx^2 - mg\frac{\ell}{2}cos\theta$$

The Lagrangian is

$$L = \frac{1}{2}\left[m\left(2\dot{x}^2 + \ell\dot{x}\dot{\theta}cos\theta + \frac{\ell^2}{3}\dot{\theta}^2\right) - kx^2 + mg\frac{\ell}{2}cos\theta\right]$$

Application of Lagrange's equations for each of the generalized coordinates leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

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$$\frac{d}{dt}\left[m\left(2\dot{x}+\frac{1}{2}\ell\dot{\theta}cos\theta\right)\right]+kx=0$$

$$2m\ddot{x}+\frac{1}{2}m\ell\ddot{\theta}cos\theta-m\ell\dot{\theta}^2sin\theta+kx=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0$$

$$\frac{d}{dt}\left[m\left(\frac{1}{2}\ell\dot{x}cos\theta+\frac{1}{2}\ell^2\dot{\theta}\right)\right]-m\ell\dot{\theta}sin\theta+mg\frac{\ell}{2}sin\theta=0$$

$$m\frac{\ell^2}{3}\ddot{\theta}+\frac{1}{2}m\ell\ddot{x}cos\theta-m\ell\dot{x}\theta sin\theta+m\ell\dot{x}\theta sin\theta+mg\frac{\ell}{2}sin\theta=0$$

$$\frac{1}{3}m\ell^2\ddot{\theta}+\frac{1}{2}m\ell\ddot{x}cos\theta+mg\frac{\ell}{2}sin\theta=0$$

Problem 12.7 illustrates application of Lagrange's equations to derive the nonlinear differential equations governing the motion of a two-degree-of-freedom system.

12.8 A wedge of specific weight γ floats stably on the free surface of a fluid of specific weight γ_W (Figure P12.8). The wedge is given a vertical displacement δ, from the equilibrium position.

FIGURE P12.8

(a) Derive the differential equation governing the resulting free oscillations of the wedge. Neglect viscous effects and the added mass of the fluid.
(b) What is the equation of the trajectory in the phase plane which describes the resulting motion. Sketch the trajectory.
(c) Assume δ is small and use the method of renormalization to determine a two-term approximation for the frequency-amplitude relationship.

Given: L, b, h, γ, γ_W, δ

Find: (a) differential equations, (b) trajectory in phase plane, (c) two-term frequency-amplitude equation

Solution: (a) First consider the wedge as it floats in stable equilibrium on the surface. Assume the depth of the wedge into the water is a. The gravity force and the buoyancy force must balance when the system is in equilibrium, leading to

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$$\gamma_W L \frac{b}{2h} a^2 = \gamma L b \frac{h}{2}$$

$$a = h\sqrt{\frac{\gamma}{\gamma_W}}$$

Now assume the mass center is subject to a vertical displacement. Let x be the downward displacement of the mass center of the wedge, measured from its equilibrium position. Consider free body diagrams of the wedge at an arbitrary instant.

Summing forces acting on the wedge

$$W - F_b - m\ddot{x}$$

$$\gamma L b \frac{h}{2} - \gamma_W \frac{b}{2h}(a+x)^2 = m\ddot{x}$$

$$\gamma L b \frac{h}{2g}\ddot{x} + L b \gamma_W \sqrt{\frac{\gamma}{\gamma_W}} x + \gamma_W \frac{Lb}{2h} x^2 = 0$$

$$\ddot{x} + 2\frac{g}{h}\sqrt{\frac{\gamma_W}{\gamma}} + \frac{g}{h^2}\frac{\gamma_W}{\gamma} x^2 = 0$$

The differential equation is nondimensionalized by defining

$$x^* = \frac{x}{h}, \qquad t^* = \sqrt{2\frac{g}{h}\sqrt{\frac{\gamma_W}{\gamma}}}\, t$$

leading to

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$$\ddot{x} + x + \chi x^2 = 0, \quad \chi = \frac{1}{2}\sqrt{\frac{\gamma_W}{\gamma}}$$

(b) Let

$$v = \dot{x}$$

$$x = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Substituting into the governing equation leads to

$$v\frac{dv}{dx} + x + \chi x^2 = 0$$

$$\frac{v^2}{2} + \frac{x^2}{2} + \frac{\chi}{3}x^3 = C$$

Noting that at t = 0, v = 0, x = $\delta/h = \epsilon$ leads to

$$\frac{\epsilon^2}{2} + \frac{\chi}{3}\epsilon^3 = C$$

$$\frac{v^2}{2} + \frac{x^2}{2} + \frac{\chi}{3}x^3 = \frac{\epsilon^2}{2} + \frac{\chi}{3}\epsilon^3$$

(c) Assume ϵ is much less than 1. Then, a straightforward perturbation expansion is assumed,

$$x(t) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \ \ldots$$

Substituting the straightforward expansion into the governing differential equation leads to

$$\epsilon(\ddot{x}_1 + x_1) + \epsilon^2(x_2 + x_2 + \chi x_1^2) + \epsilon^3(\ddot{x}_3 + x_3 + 2\chi x_1 x_2) + \cdots = 0$$

Setting coefficients of like powers of ϵ to zero independently leads to

$$\ddot{x}_1 + x_1 = 0$$

$$\ddot{x}_2 + x_2 = -\chi x_1^2$$

$$\ddot{x}_3 + x_3 = -2\chi x_1 x_2$$

The solution of x_1 can be written as

$$x_1(t) = Asin(t + \phi)$$

Substitution into the equation for x_2 leads to

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$$\ddot{x}_2 + x_2 = -A^2\chi sin^2(t+\phi)$$

$$= -\chi\frac{A^2}{2}[1 - cos2(t+\phi)]$$

whose solution is

$$x_2(t) = -\chi\frac{A^2}{2} - \chi\frac{A^2}{6}cos2(t+\phi)$$

The equation for x_3 then becomes

$$\ddot{x}_3 + x_3 = \chi^2 A^3 sin(t+\phi)\left[1 + \frac{1}{3}cos2(t+\phi)\right]$$

$$= A^3\chi^2\left[\frac{1}{2}sin3(t+\phi) + \frac{5}{6}sin(t+\phi)\right]$$

The solution for x_3 is

$$x_3(t) = A^3\chi^2\left[-\frac{1}{16}sin3(t+\phi) - \frac{5}{12}tcos(t+\phi)\right]$$

Hence

$$x(t) = Asin(t+\phi) - \epsilon A^2\chi\left[\frac{1}{2} + \frac{1}{3}cos2(t+\phi)\right]$$
$$- \epsilon^2 A^3\chi^2\left[\frac{1}{16}sin3(t+\phi) + \frac{5}{12}tcos(t+\phi)\right] + \cdots$$

The preceding expansion is nonuniform, hence the method of renormalization is used,

$$t = 1 + \epsilon^2\lambda + \cdots$$

Substitution of the renormalized variable into the expansion for x(t) and using Taylor series expansions leads to

$$x = \epsilon Asin(w+\phi) - \epsilon^2 A^2\chi\left[\frac{1}{2} + \frac{1}{3}cos2(w+\phi)\right]$$
$$- \epsilon^3\left[\lambda Awcos(w+\phi) - A^3\frac{\chi^2}{16}sin3(w+\phi) - \frac{5}{12}A^3\chi^2 wcos(w+\phi)\right]$$
$$+ \cdots$$

Secular terms are eliminated if

$$\lambda = \frac{5A^2\chi^2}{12}$$

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Hence

$$t = w\left(1 + \epsilon^2 \frac{5A^2\chi^2}{12} + \cdots\right)$$

and

$$w = 1 - \frac{5A^2\chi^2}{12}\epsilon^2 + \cdots$$

Problem 12.8 illustrates (a) the nonlinear nature of the free oscillation of a wedge floating on the surface of a liquid, (b) the development of the trajectory in the state plane corresponding to the motion, and (c) the use of the method of renormalization to determine a two-term frequency amplitude relation.

12.9 Repeat Chapter Problem 12.8 for the inverted cone of Figure P12.9.

FIGURE P12.9

Given: γ, γ_W, r, h, δ

Find: (a) differential equation, (b) phase plane trajectory, and (c) two-term frequency amplitude relation

Solution: First consider the stable equilibrium position of the cone. When the cone is in equilibrium, the gravity force balances with buoyancy force, as shown on the following free-body diagram

Thus

$$\gamma\frac{\pi}{3}r^2h = \gamma_W\frac{\pi}{3}r^2h\left[1-\left(1-\frac{a}{h}\right)^3\right]$$

$$\frac{\gamma}{\gamma_W} = 1-\left(1-\frac{a}{h}\right)^3$$

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$$a = h\left[1 - \left(1 - \frac{\gamma}{\gamma_W}\right)^{\frac{1}{3}}\right]$$

Let x be the displacement of the mass center of the cone, measured positive downward from the cone, measured positive downward from the cone's equilibrium position. Consider free-body diagrams of the cone at an arbitrary instant.

Summing forces acting on the cone

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$

leads to

$$W - F_B = m\ddot{x}$$

$$\gamma\pi r^2\frac{h}{3} - \gamma_W\pi r^2\frac{h}{3}\left[1 - \left(1 - \frac{a+x}{h}\right)^3\right] = \frac{\gamma}{g}\pi r^2\frac{h}{3}\ddot{x}$$

$$\ddot{x} + \frac{\gamma_W}{\gamma}3g\left(1 - \frac{a}{h}\right)^2 x - 3g\frac{\gamma_W}{\gamma}\left(1 - \frac{a}{h}\right)\left(\frac{x}{h}\right)^2 + g\frac{\gamma_W}{\gamma}\left(\frac{x}{h}\right)^3 = 0$$

The preceding equation is nondimensionalized by defining

$$x^* = \frac{x}{h}, \qquad t^* = \sqrt{3\frac{g}{h}\frac{\gamma_W}{\gamma}\left(1 - \frac{a}{h}\right)^2}$$

Rewriting the differential equations using nondimensional variables and dropping the *s leads to

$$\ddot{x} + x - \alpha x^2 + \frac{1}{3}\alpha^2 x^3 = 0$$

where

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$$\alpha = \frac{1}{1 - \frac{a}{h}}$$

(b) Let

$$v = \frac{dx}{dt}, \quad \ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Thus the differential equation becomes

$$v\frac{dv}{dx} + x - \alpha x^2 + \frac{\alpha^2}{3}x^3 = 0$$

Integration with respect to x leads to

$$\frac{v^2}{2} + \frac{x^2}{2} - \frac{\alpha}{3}x^3 + \frac{\alpha^2}{12}x^4 = C$$

When t = 0, v = 0 and x = δ/h, thus

$$C = \frac{1}{2}\left(\frac{\delta}{h}\right)^2 - \frac{\alpha}{3}\left(\frac{\delta}{h}\right)^3 + \frac{\alpha^2}{12}\left(\frac{\delta}{h}\right)^4$$

(c) Now assume

$$\frac{\delta}{h} < 1$$

and define

$$\epsilon = \frac{\delta}{h}$$

A straightforward expansion for the solution of the differential equation is assumed as

$$x(t) = \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots$$

Substitution of the straightforward expansion into the governing differential equation leads to

$$\epsilon(\ddot{x}_1 + x_1) + \epsilon^2(\ddot{x}_2 + x_2 - \alpha x_1^2) + \epsilon^3\left(\ddot{x}_3 + x_3 - 2\alpha x_1 x_2 + \frac{1}{3}\alpha^2 x_1^3\right) \ldots$$

Setting coefficients of powers of ϵ to zero independently leads to

$$\ddot{x}_1 + x_1 = 0$$

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$$\ddot{x}_3 + x_3 = 2\alpha x_1 x_2 - \frac{\alpha^2}{3} x_1^3$$

The solution for x_1 can be written as

$$x_1 = Asint(t + \phi)$$

which when substituted into the second of the hierarchal equations leads to

$$\ddot{x}_2 + x_2 = \alpha A^2 sin^2(t + \phi) = \alpha \frac{A^2}{2}[1 - cos2(t + \phi)]$$

$$x_2 = \alpha \frac{A^2}{2} + \alpha \frac{A}{6} cos2(t + \phi)$$

Then the equation for x_3 becomes

$$\ddot{x}_3 + x_3 = 2\alpha^2 A^3 sin(t + \phi)\left[\frac{1}{2} + \frac{1}{6} cos2(t + \phi)\right] - \frac{\alpha^2}{3} A^3 sin^3(t + \phi)$$

$$\ddot{x}_3 + x_3 = \alpha^2 A^3 \left\{\frac{1}{2} sin3(t + \phi) + \frac{5}{6} sin(t + \phi) - \frac{1}{4} sin(t + \phi) + \frac{1}{12} sin3(t + \phi)\right\}$$

$$\ddot{x}_3 + x_3 = \frac{7}{12} \alpha^2 A^3 [sin3(t + \phi) + sin(t + \phi)]$$

$$x_3(t) = -\frac{7}{12} A^2 \alpha^2 \left[\frac{1}{8} sin3(t + \phi) + \frac{t}{2} sin(t + \phi)\right]$$

$$x(t) = \epsilon A sin(t + \phi) + \epsilon^2 A^2 \alpha \left[\frac{1}{2} + \frac{1}{6} cos2(t + \phi)\right] - \frac{7}{12} \epsilon^3 A^3 \alpha^2 \left[\frac{1}{8} sin3(t + \phi) + \frac{t}{2} cos(t + \phi)\right] + \cdots$$

Using the method of renormalization,

$$t = w(1 + \epsilon^2 \lambda + \cdots)$$

Which when substituted into the straightforward expansion, using Taylor series and recollecting terms on like powers of ϵ leads to

$$x = \epsilon A sin(w + \phi) + \epsilon^2 A^2 \alpha \left[\frac{1}{2} + \frac{1}{6} cos2(w + \phi)\right] + \epsilon^3 \left[wAcos(w + \phi) - \frac{7}{24} A^3 \alpha^2 wcos(w + \phi) - \frac{7}{96} A^3 \alpha^2 sin3(w + \phi)\right] + \cdots$$

Secular terms are eliminated if

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$$\lambda = \frac{7}{24} A^2 \alpha^2$$

Hence

$$t = w\left(1 + \frac{7}{24}\epsilon^2 A^2 \alpha^2 + \cdots\right)$$

Problem 12.9 illustrates (a) that static equilibrium position of a nonlinear system appears in the governing differential equation, (b) the derivation of the governing differential equation for the buoyancy induced oscillations of an inverted cone, (c) the development of a trajectory in the state plane, and (d) the use of the method of renormalization to derive a two-term frequency-amplitude relation for a system with both a quadratic nonlinearity and a cubic nonlinearity.

12.10 Determine the equation defining the state plane for the system of Figure P12.6. Sketch trajectories in the phase plane when the following are given. (a) p = 1.5 m^{-1}, ω = 5 rad/s, (b) p = 1.0 m^{-1}, ω = 5 rad/s, (c) p = 5.097 m^{-1}, ω = 10 rad/s.

FIGURE P12.6

Given: p, ω

Find: trajectories in state plane

Solution: The differential equation governing the motion of the system is derived in the solution of Problem 12.6 as

$$(1 + 4p^2x^2)\ddot{x} + (2gp - \omega^2)x + 4p^2x\dot{x}^2 = 0 \qquad (1)$$

Define

$$v = \dot{x}, \quad \ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \qquad (2)$$

Using v as the dependent variable in eq. (1) leads to

$$(1 + 4p^2x^2)v\frac{dv}{dx} + (2gp - \omega^2)x + 4p^2xv^2 = 0 \qquad (3)$$

Now define

$$u = v^2, \quad v\frac{dv}{dx} = \frac{1}{2}\frac{du}{dx} \qquad (4)$$

Using u as the dependent variable in the differential equation leads to

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$$\frac{1}{2}(1+4p^2x^2)\frac{du}{dx}+4p^2xu=-(2gp-\omega^2)x$$

$$\frac{du}{dx}+\frac{8p^2}{1+4p^2x^2}u=-\frac{2(2gp-\omega^2)x}{1+4p^2x^2} \tag{5}$$

Equation (5) is a first order linear differential equation with u as the dependent variable and x as the independent variable. Equation (5) can be solved using the integrating factor method. The appropriate integrating factor is

$$\mathrm{I}=\mathrm{e}^{\int\frac{8\mathrm{p}^2}{1+4\mathrm{p}^2\mathrm{x}^2}\mathrm{dx}}$$

$$=1+4p^2x^2$$

Hence

$$\frac{d}{dx}[u(1+4p^2x^2)]=-2(2gp-\omega^2)x$$

Integrating both sides of the preceding equation leads to

$$u(1+4p^2x^2)=C-(2gp-\omega^2)$$

where C is a constant. Replacing u by v^2 and solving for v leads to

$$v=\sqrt{\frac{C-(2gp-\omega^2)x^2}{1+4p^2x^2}}$$

for the parameters given, the appropriate trajectories are

(a) $v^2=\frac{C-4.93x^2}{1+9x^2}$

(b) $v^2=\frac{C+5.38x^2}{1+4x^2}$

(c) $v=0.981\frac{C}{x}$

The trajectories are sketched below.

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Problem 12.10 illustrates the state plane equation for a conservative one-degree-of-freedom system.

12.11 Plot the trajectory in the state plane corresponding to the motion of a mass attached to a linear spring free to slide on a surface with Coulomb damping when the mass is displaced from equilibrium and released from rest.

Given: m, k, μ, $x_0 = \delta$, $v_0 = 0$

Find: trajectory in state plane

Solution: The differential equation governing the motion of a system subject to Coulomb damping is

$$\ddot{x} + \omega x_n^2 = \pm \mu g$$

where

$$\omega_n = \sqrt{\frac{k}{m}}$$

and the right hand side is positive when the velocity is negative and vice versa. The velocity is defined as

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$$v = \frac{dx}{dx}$$

Rewriting the differential equation in terms of v and x leads to

$$v\frac{dv}{dx} + \omega_n^2 x = \pm\mu g x$$

$$v^2 + \omega_n^2 x = C \pm 2\mu g x$$

If the mass is given a positive displacement δ and released from rest, the velocity is initially negative. Note that motion ceases when the displacement becomes as small as μmg/k. This leads to the trajectory shown below.

Problem 12.11 illustrates the development of the state plane for free vibrations of a one-degree-of-freedom system with Coulomb damping.

12.12 Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} - x + \epsilon x^3 = 0 \qquad (1)$$

Given: Equation (1)

Find: Equilibrium points and their nature

Solution: Comparing eq. (1) to eq. (12.11),

$$g(\dot{x}) = 2\zeta\dot{x}, \quad f(x) = -x + \epsilon x^3$$

The equilibrium points in the state plane correspond to v = 0 and f(x) = 0. To this end

$$f(x) = 0 = x(\epsilon x^2 - 1)$$

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$$x = 0, \quad \pm\sqrt{\frac{1}{\epsilon}} \tag{2}$$

The point x = 0 is an equilibrium point regardless of the sign of ϵ. If $\epsilon < 0$, this is the only equilibrium point. The nature of the equilibrium point is obtained by noting

$$\frac{df}{dx} = -1 + 3\epsilon x^2, \quad \frac{dg}{d\dot{x}} = 2\zeta$$

$$\frac{df}{dx}(0) = -1$$

Then using eq. (12.12), the differential equation describing the motion when the system is perturbed from this equilibrium point is

$$\Delta\ddot{x} + 2\zeta\Delta\mathrm{x} - \Delta\mathrm{x} = 0 \tag{3}$$

The equation defining the exponents in the exponential solution of eq. (3) is

$$\beta^2 + 2\zeta\beta - 1 = 0$$

which leads to

$$\beta = -\zeta \pm \sqrt{\zeta^2 + 1}$$

Since both values of β are real of opposite signs, the equilibrium point x = 0 is a saddle point.

Now suppose $\epsilon > 0$. Then

$$\frac{df}{dx}\left(\pm\sqrt{\frac{1}{\epsilon}}\right) = 2$$

The differential equation describing the motion when the system is perturbed from either of these equilibrium points is

$$\Delta\ddot{x} + 2\zeta\Delta\dot{x} + 2\Delta x = 0$$

The equation defining the exponents in the exponential solutions of the above equation is

$$\beta^2 + 2\zeta\beta + 2 = 0$$

whose solutions are

$$\beta = -\zeta \pm \sqrt{\zeta^2 - 2}$$

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The nature of the equilibrium point depends upon the value of ζ. For $\zeta < \sqrt{2}$, the roots are complex conjugates of one another with a negative real part and hence the equilibrium points are stable foci. If $\zeta > \sqrt{2}$ the roots are both real and negative, hence the equilibrium points are stable nodes.

Problem 12.12 illustrates the determination of the equilibrium points and their nature for nonconservative system.

12.13 Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} - x - \epsilon x^3 = 0 \tag{1}$$

Given: Equation (1)

Find: Equilibrium points and their nature

Solution: Comparing eq. (1) to eq. (12.11),

$$g(\dot{x}) = 2\zeta\dot{x}, \quad f(x) = -x - \epsilon x^3$$

The equilibrium points in the state plane correspond to v = 0 and f(x) = 0. To this end

$$f(x) = 0 = -x(\epsilon x^2 + 1)$$

$$x = 0, \quad \pm\sqrt{\frac{1}{-\epsilon}} \tag{2}$$

The point x = 0 is an equilibrium point regardless of the sign of ϵ. If $\epsilon > 0$, this is the only equilibrium point. The nature of the equilibrium point is obtained by noting

$$\frac{df}{dx} = -1 - 3\epsilon x^2, \quad \frac{dg}{d\dot{x}} = 2\zeta$$

$$\frac{df}{dx}(0) = -1$$

Then using eq. (12.12), the differential equation describing the motion when the system is perturbed from this equilibrium point is

$$\Delta\ddot{x} + 2\zeta\Delta x - \Delta x = 0 \tag{3}$$

The equation defining the exponents in the exponential solution of eq. (3) is

$$\beta^2 + 2\zeta\beta - 1 = 0$$

which leads to

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$$\beta = -\zeta \pm \sqrt{\zeta^2 + 1}$$

Since both values of β are real of opposite signs, the equilibrium point x = 0 is a saddle point.

Now suppose $\epsilon < 0$. Then

$$\frac{df}{dx}\left(\pm\sqrt{\frac{1}{-\epsilon}}\right) = 2$$

The differential equation describing the motion when the system is perturbed from either of these equilibrium points is

$$\Delta\ddot{x} + 2\zeta\Delta\dot{x} + 2\Delta x = 0$$

The equation defining the exponents in the exponential solutions of the above equation is

$$\beta^2 + 2\zeta\beta + 2 = 0$$

whose solutions are

$$\beta = -\zeta \pm \sqrt{\zeta^2 - 2}$$

The nature of the equilibrium point depends upon the value of ζ. For $\zeta < \sqrt{2}$, the roots are complex conjugates of one another with a negative real part and hence the equilibrium points are stable foci. If $\zeta > \sqrt{2}$ the roots are both real and negative, hence the equilibrium points are stable nodes.

Problem 12.13 illustrates the determination of the equilibrium points and their nature for nonconservative system.

12.14 Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} - x + \epsilon x^2 = 0 \qquad (1)$$

Given: Equation (1)

Find: Equilibrium points and their nature

Solution: Comparing eq. (1) to eq. (12.11),

$$g(\dot{x}) = 2\zeta\dot{x}, \quad f(x) = -x + \epsilon x^2$$

The equilibrium points in the state plane correspond to v = 0 and f(x) = 0. To this end

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$$f(x) = 0 = -x(\epsilon x + 1)$$

$$x = 0, \quad -\frac{1}{\epsilon} \tag{2}$$

First consider the equilibrium point x = 0. The nature of this equilibrium point is obtained by noting

$$\frac{df}{dx} = 1 + 2\epsilon x, \quad \frac{dg}{d\dot{x}} = 2\zeta$$

$$\frac{df}{dx}(0) = 1$$

Then using eq. (12.12), the differential equation describing the motion when the system is perturbed from this equilibrium point is

$$\Delta\ddot{x} + 2\zeta\Delta\mathrm{x} + \Delta\mathrm{x} = 0 \tag{3}$$

The equation defining the exponents in the exponential solution of eq. (3) is

$$\beta^2 + 2\zeta\beta + 1 = 0$$

which leads to

$$\beta = -\zeta \pm \sqrt{\zeta^2 - 1}$$

The nature of the equilibrium point depends upon ζ. If $\zeta < 1$, then the values of β are complex conjugates with a negative real part, and hence x = 0 is a stable focus. If $\zeta > 1$, then both values of β are real and negative. Hence x = 0 is a stable focus for $\zeta > 1$.

Now consider the equilibrium point $x = \frac{1}{-\epsilon}$. Then

$$\frac{df}{dx}\left(\frac{1}{-\epsilon}\right) = -1$$

The differential equation describing the motion when the system is perturbed from this equilibrium point is

$$\Delta\ddot{x} + 2\zeta\Delta\dot{x} - \Delta x = 0$$

The equation defining the exponents in the exponential solutions of the above equation is

$$\beta^2 + 2\zeta\beta - 1 = 0$$

whose solutions are

$$\beta = -\zeta \pm \sqrt{\zeta^2 + 1}$$

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For this equilibrium point, both roots are real with opposite signs. Hence $x = \frac{-1}{\epsilon}$ is a saddle point.

Problem 12.14 illustrates the determination of the equilibrium points and their nature for nonconservative system.

12.15 Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} + x - \epsilon x^2 = 0 \tag{1}$$

Given: Equation (1)

Find: Equilibrium points and their nature

Solution: Comparing eq. (1) to eq. (12.11),

$$g(\dot{x}) = 2\zeta\dot{x}, \quad f(x) = x - \epsilon x^2$$

The equilibrium points in the state plane correspond to v = 0 and f(x) = 0. To this end

$$f(x) = 0 = x(1 - \epsilon x)$$

$$x = 0, \quad \frac{1}{\epsilon} \tag{2}$$

First consider the equilibrium point x = 0. The nature of this equilibrium point is obtained by noting

$$\frac{df}{dx} = 1 - 2\epsilon x, \quad \frac{dg}{d\dot{x}} = 2\zeta$$

$$\frac{df}{dx}(0) = 1$$

Then using eq. (12.12), the differential equation describing the motion when the system is perturbed from this equilibrium point is

$$\Delta\ddot{x} + 2\zeta\Delta\mathrm{x} + \Delta\mathrm{x} = 0 \tag{3}$$

The equation defining the exponents in the exponential solution of eq. (3) is

$$\beta^2 + 2\zeta\beta + 1 = 0$$

which leads to

$$\beta = -\zeta \pm \sqrt{\zeta^2 - 1}$$

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The nature of the equilibrium point depends upon ζ. If $\zeta < 1$, then the values of β are complex conjugates with a negative real part, and hence x = 0 is a stable focus. If $\zeta > 1$, then both values of β are real and negative. Hence x = 0 is a stable focus for $\zeta > 1$.

Now consider the equilibrium point $x = \frac{1}{\epsilon}$. Then

$$\frac{df}{dx}\left(\frac{1}{\epsilon}\right) = -1$$

The differential equation describing the motion when the system is perturbed from this equilibrium point is

$$\Delta\ddot{x} + 2\zeta\Delta\dot{x} - \Delta x = 0$$

The equation defining the exponents in the exponential solutions of the above equation is

$$\beta^2 + 2\zeta\beta - 1 = 0$$

whose solutions are

$$\beta = -\zeta \pm \sqrt{\zeta^2 + 1}$$

For this equilibrium point, both roots are real with opposite signs. Hence $x = \frac{1}{\epsilon}$ is a saddle point.

Problem 12.15 illustrates the determination of the equilibrium points and their nature for nonconservative system.

12.16 The equation of motion for the free oscillations of a pendulum subject to quadratic damping is

$$\ddot{\theta} + 2\zeta\dot{\theta}^2 + sin\theta = 0 \qquad (1)$$

(a) Determine an exact equation defining the state plane.
(b) Determine the equilibrium points and their type.

Given: Eq. (1)

Find: (a) equation of state plane, (b) equilibrium points and their type

Solution: The angular velocity of the pendulum is

$$\omega = \dot{\theta}$$

Then note that

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$$\ddot{\theta} = \dot{\omega} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} \tag{2}$$

Substituting eq.(2) into eq. (1) leads to

$$\omega\frac{d\omega}{d\theta} + 2\zeta\omega^2 + sin\theta = 0 \tag{3}$$

Equation (3) is a first order ordinary differential equation that can be solved for ω as a function of θ. To this end, let

$$v = \omega^2, \quad \frac{dv}{d\theta} = 2\omega\frac{d\omega}{d\theta}$$

Then using v as the dependent variable, eq. (3) becomes

$$\frac{1}{2}\frac{dv}{d\theta} + 2\zeta v = -sin\theta$$

$$\frac{dv}{d\theta} + 4\zeta v = -2sin\theta \tag{4}$$

Equation (4) is a first order linear ordinary differential equation. Its solution is

$$v(\theta) = Ce^{-4\zeta\theta} + \frac{2}{1+16\zeta^2}(cos\theta - 4\zeta sin\theta)$$

where C is an arbitrary constant. Hence the equation defining the state plane is

$$\omega^2 = Ce^{-4\zeta\theta} + \frac{2}{1+16\zeta^2}(cos\theta - 4\zeta sin\theta)$$

(b) Comparing eq.(1) with eq. (12.10),

$$g(\dot{\theta}) = 2\zeta\dot{\theta}^2, \quad f(\theta) = sin\theta$$

The equilibrium points in the state plane are those points such that $\omega = 0$ and $f(\theta) = 0$. Then

$$f(\theta) = 0 = sin\theta \rightarrow \theta = 0, \pm\pi, \pm 2\pi, \ldots$$

The nature of the equilibrium points are determined by examining the equation

$$\Delta\ddot{x} + \frac{dg}{dx}(0)\Delta\dot{x} + \frac{df}{dx}(\theta_0)\Delta x = 0$$

$$\Delta\ddot{x} + \cos(\pm n\pi)\,\Delta x = 0$$

$$\Delta\ddot{x} + (-1)^n\Delta x = 0$$

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If n is odd, the previous equation becomes,

$$\Delta\ddot{x} - \Delta x = 0$$

whose solutions are

$$e^{t}, \quad e^{-t}$$

Thus the equilibrium points corresponding to odd multiples of π are saddle points.

If n is even,

$$\Delta\ddot{x} + \Delta x = 0$$

whose solutions are

$$e^{it}, \quad e^{-it}$$

and hence the equilibrium points corresponding to even multiples of π are centers.

Problem 12.16 illustrates (a) the development of the equation defining the state plane for an autonomous system, and (b) the determination of the equilibrium points and their type.

12.17 Determine the period of oscillation of a mass attached to a hardening spring with a cubic nonlinearity.

Given: $F = x + \epsilon x^3$

Find: T

Solution: From eq. (12.19)

$$T = \frac{4}{\sqrt{2}} \int_{x_0}^{0} \frac{d\lambda}{\int_{\lambda}^{x_0} (\eta + \epsilon\eta^3) d\eta}$$

$$= \frac{4}{\sqrt{2}} \int_{x_0}^{0} \frac{d\lambda}{\sqrt{\frac{x_0^2}{2} + \epsilon\frac{x_0^4}{4} - \frac{\lambda^2}{2} - \epsilon\frac{\lambda^4}{4}}}$$

Let

$$\phi = \frac{\lambda}{x_0}$$

Then

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$$T = \frac{4x_0}{\sqrt{2}} \int_0^1 \frac{d\phi}{\sqrt{\frac{x_0^2}{2} + \epsilon\frac{x_0^4}{4} - x_0^2\frac{\phi^2}{2} - \epsilon x_0^4\frac{\phi^4}{4}}}$$

$$= \frac{4\sqrt{2}}{x_0\sqrt{\epsilon}} \int_0^1 \frac{d\phi}{\sqrt{\frac{2}{\epsilon x_0^2} + 1 - \frac{2}{\epsilon x_0^2} - \phi^4}}$$

$$T = \frac{4\sqrt{2}}{x_0\epsilon} \int_0^1 \frac{d\phi}{\sqrt{(\phi^2 - 1)\left[\phi^2 - \left(\frac{2 + \epsilon x_0^2}{\epsilon x_0^2}\right)\right]}}$$

The above expression is related to elliptic integrals.

Problem 12.17 illustrates the use of eq. (12.19) to determine an integral expression for the exact period of a nonlinear system.

12.18 Determine an integral expression for the period of oscillation of the system Figure P12.6.

FIGURE P12.6

Given: m, p, ω

Find: T

Solution: The differential equation governing the motion of the system is derived in the solution of Problem 12.6 as

$$(1 + 4p^2x^2)\ddot{x} + (2gp - \omega^2)x + 4p^2x\dot{x}^2 = 0 \qquad (1)$$

Define

$$v = \dot{x}, \quad \ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \qquad (2)$$

Using v as the dependent variable in eq. (1) leads to

$$(1 + 4p^2x^2)v\frac{dv}{dx} + (2gp - \omega^2)x + 4p^2xv^2 = 0 \qquad (3)$$

Now define

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$$u = v^2, \quad v\frac{dv}{dx} = \frac{1}{2}\frac{du}{dx} \tag{4}$$

Using u as the dependent variable in the differential equation leads to

$$\frac{1}{2}(1 + 4p^2x^2)\frac{du}{dx} + 4p^2xu = -(2gp - \omega^2)x$$

$$\frac{du}{dx} + \frac{8p^2}{1+4p^2x^2}u = -\frac{2(2gp-\omega^2)x}{1+4p^2x^2} \tag{5}$$

Equation (5) is a first order linear differential equation with u as the dependent variable and x as the independent variable. Equation (5) can be solved using the integrating factor method. The appropriate integrating factor is

$$\mathrm{I} = \mathrm{e}^{\int \frac{8\mathrm{p}^2}{1+4\mathrm{p}^2\mathrm{x}^2}\mathrm{dx}}$$

$$= 1 + 4p^2x^2 \tag{6}$$

Hence

$$\frac{d}{dx}[u(1 + 4p^2x^2)] = -2(2gp - \omega^2)x \tag{7}$$

Integrating both sides of eq. (7) leads to

$$u(1 + 4p^2x^2) = C - (2gp - \omega^2)x^2$$

where C is a constant. Replacing u by v^2 and solving for v leads to

$$v = \sqrt{\frac{C - (2gp - \omega^2)x^2}{1 + 4p^2x^2}}$$

Then using eq.(2) and integrating

$$t = \int_0^x \sqrt{\frac{1 + 4p^2x^2}{C - (2gp - \omega^2)x^2}}\, dx$$

The above integral can be used to develop the period for a given set of initial conditions.

Problem 12.18 illustrates the development of an integral expression for the period of oscillation of a nonlinear conservative system.

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12.19 Use the method of renormalization to determine a two-term approximation for the frequency-amplitude relation for the system of Figure P12.4. If the bar is rotated 4° from equilibrium and released, what is the period for L = 4 m, k = 1000 N/m, and m = 10 kg?

FIGURE P12.4

Given: m = 10 kg, L = 4 m, k = 1000 N/m, $\theta_0 = 4°$

Find: T

Solution: The differential equation for this system is derived in the solution of problem 12.4 as

$$\frac{7}{48}mL^2\ddot{\theta} + \frac{9}{16}kL^2 sin\theta cos\theta = 0$$

$$\ddot{\theta} + \frac{27}{7}\frac{k}{m}\frac{1}{2}sin2\theta = 0$$

Expanding the trigonometric term using a Taylor series expansion, and keeping only through the cubic terms leads to

$$\ddot{\theta} + \omega^2\theta - \frac{2}{3}\omega^2\theta^3 = 0$$

where

$$\omega = \sqrt{\frac{27}{7}\frac{k}{m}}$$

is the natural frequency of the linearized system.

Assume the initial displacement is small, say ϵ. The response is expanded in a series of powers of ϵ as

$$\theta(t) = \epsilon\theta_1(t) + \epsilon^3\theta_3(t) + \cdots$$

Which when substituted into the governing differential equation leads to

$$\epsilon(\ddot{\theta}_1 + \omega^2\theta_1) + \epsilon^3\left(\ddot{\theta}_3 + \omega^2\theta_3 - \frac{2}{3}\omega^2\theta_1^3\right) + \cdots$$

Setting coefficients of powers of ϵ to zero independently leads to

$$\ddot{\theta}_1 + \omega^2\theta_1 = 0$$

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$$\ddot{\theta}_3 + \omega^2\theta_3 = \frac{2}{3}\omega^2\theta_1^3$$

The solution of the first equation is

$$\theta_1(t) = Asin(\omega t + \phi)$$

which when substituted into the second equation gives

$$\ddot{\theta}_3 + \omega^2\theta_3 = \frac{2}{3}A^3\omega^2 sin^3(\omega t + \phi) = \frac{1}{6}\omega^2 A^3[3sin(\omega t + \phi) - sin3(\omega t + \phi)]$$

whose particular solution is

$$\theta_3 = \frac{1}{6}A^3\omega^2\left[-\frac{3}{2\omega}tcos(\omega t + \phi) - \frac{1}{8\omega^2}sin3(\omega t + \phi)\right]$$

Hence

$$\theta(t) = \epsilon Asin(\omega t + \phi) + \epsilon^3\left[-\frac{1}{4}\omega A^3 tcos(\omega t + \phi) - \frac{1}{48}A^3 sin3(\omega t + \phi)\right] + \cdots$$

Unfortunately, the preceding expansion is nonuniform, thus the method of renormalization is used,

$$\omega t = w(1 + \epsilon^2\lambda + \cdots)$$

which when substituted into the preceding expansion, using the Taylor series expansion, and recollecting coefficients of like powers of ϵ leads to

$$\theta(t) = \epsilon Asin(w + \phi) + \epsilon^3\left[-\frac{1}{4}A^3 wcos(w + \phi) - \frac{1}{48}A^3 sin3(w + \phi) + \lambda Awcos(w + \phi)\right] + \cdots$$

Secular terms are removed if

$$\lambda = \frac{A^3}{4}$$

Hence

$$\omega t = \left(1 + \epsilon^2\frac{A^2}{4}\right)w$$

$$w = \omega\left(1 - \epsilon^2\frac{A^2}{4}\right)t$$

and

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$$\theta(t) = \epsilon A sin(w + \phi) + -\frac{1}{48}\epsilon^3 A^3 sin3(w + \phi) + \cdots$$

(b) The linearized natural frequency is

$$\omega = \sqrt{\frac{k}{m}} = 10\ \frac{rad}{sec}$$

The initial conditions are

$$\theta(0) = 4° = 0.0698\, rad, \quad \dot{\theta}(0) = 0$$

which when applied to the solution gives

$$\phi = \frac{\pi}{2}$$

$$0.0698 = \epsilon A + \frac{1}{48}\epsilon^3 A^3$$

and the initial amplitude is approximately

$$\epsilon A = 0.0698$$

Then the nonlinear frequency is

$$\omega\left(1 - \frac{1}{4}\epsilon^2 A^2\right) = 10\left[1 - \frac{1}{4}(0.0698)^2\right] = 9.988\ \frac{rad}{sec}$$

and the period is

$$T = \frac{2\pi}{9.988\ \frac{rad}{sec}} = 0.629\ sec$$

Problem 12.19 illustrates (a) the use of Taylor series expansions to develop differential equations whose solution can be approximated using perturbation methods, (b) the use of the method of renormalization, and (c0 the calculation of the period of free vibration for a nonlinear system.

12.20 A 25-kg mass is attached to a hardening spring with k_1 = 1000 N/m and k_3 = 4000 N/m^3. The mass is displaced 15 mm from equilibrium and released from rest. What is the period of the resulting oscillations?

Given: m = 25 kg, k_1 = 1000 N/m, k_3 = 4000 N/m, δ = 15 mm

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Find: T

Solution: Using the notations of eqs.(10.8) and (10.9)

$$\Delta = \frac{mg}{k_1} = \frac{(25\ kg)\left(9.81\ \frac{m}{sec^2}\right)}{1000\ \frac{N}{m}} = 0.245\ m$$

$$\epsilon = \frac{k_3}{k_1}\Delta^2 = \frac{4000\frac{N}{m^3}}{1000\frac{N}{m}}(0.245\ m)^2 = 0.241$$

The nondimensional initial displacement is

$$\delta = \frac{0.015\ m}{0.245\ m} = 0.0612$$

The nondimensional amplitude is obtained by solving

$$\delta = A - \frac{\epsilon}{32}A^3$$

$$A \approx 0.0612$$

The nondimensional frequency is

$$\omega^* = 1 + \frac{3}{8}\epsilon A^3$$

The dimensional frequency is

$$\omega = \omega^*\sqrt{\frac{k_1}{m}}$$

The period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{1000\ \frac{\mathrm{N}}{\mathrm{m}}}{25\ \mathrm{kg}}\left[1 + \frac{3}{8}(0.241)(0.0612)^3\right]}} = 0.993\ \mathrm{sec}$$

Problem 12.20 illustrates determination of the period of nonlinear oscillation for a block attached to a hardening cubic spring.

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12.21 Suppose the mass of Chapter Problem 12.20 is subject to an impulse which imparts a velocity of 3.1 m/s to the mass when the mass is in equilibrium. What is the period of the ensuing oscillations?

Given: m = 25 kg, k_1 = 1000 N/m, k_3 = 4000 N/m, v = 3.2 m/sec

Find: T

Solution: Using the notations of eqs.(12.8) and (12.9)

$$\Delta = \frac{mg}{k_1} = \frac{(25\text{ kg})\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right)}{1000\ \frac{\text{N}}{\text{m}}} = 0.245\text{ m}$$

$$\epsilon = \frac{k_3}{k_1}\Delta^2 = \frac{4000\frac{\text{N}}{\text{m}^3}}{1000\frac{\text{N}}{\text{m}}}(0.245\text{ m})^2 = 0.241$$

$$\omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{1000\ \frac{\text{N}}{\text{m}}}{25\text{ kg}}} = 6.32\ \frac{\text{rad}}{\text{sec}}$$

The nondimensional velocity is

$$\frac{dx}{dt} = \frac{d(x^*\Delta)}{dt^*}\frac{dt^*}{dt} = \Delta\omega_n\frac{dx^*}{dt^*}$$

$$\frac{dx^*}{dt^*} = \frac{1}{\Delta\omega_n}\frac{dx}{dt}$$

Hence the initial nondimensional velocity is

$$\frac{dx^*}{dt^*}(0) = \frac{3.1\ \frac{\text{m}}{\text{sec}}}{0.245\left(6.32\ \frac{\text{rad}}{\text{sec}}\right)} = 2.00$$

$$A \approx 0.0612$$

The nondimensional displacement is given by eq. (12.30) as

$$x = Asin(w+\phi) - \frac{\epsilon}{32}A^3 sin3(w+\phi)$$

where

$$w = t\left(1 + \frac{3}{8}\epsilon A^3\right)$$

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Since x(0) = 0, $\phi = 0$. Applying the initial velocity condition

$$\frac{dx}{dt}(0) = 2.00 = \left(1 + \frac{3}{8}\epsilon A^3\right)\left(A - \frac{3}{32}\epsilon A^3\right)$$

$$= A + \frac{9}{32}\epsilon A^3$$

where terms higher than cubic have been neglected. Solving the above equation for A leads to

$$A = 1.672$$

Hence the nondimensional period is

$$T = \frac{2\pi}{\omega_n\left(1 + \frac{3}{8}\epsilon(A)^3\right)} = \frac{2\pi}{\left(6.32\ \frac{\text{rad}}{\text{sec}}\right)\left[1 + \frac{3}{8}(0.241)(1.672)^3\right]} = 0.699 \text{ sec}$$

Problem 12.21 illustrates (a) application of initial conditions to determine the free-vibration amplitude for a system governed by Duffing's equation and (b) determination of the period of a nonlinear system governed by Duffing's equation.

12.22 Suppose the mass of Chapter Problem 12.20 is attached to the same spring when a 50-N force is statically applied and suddenly removed. What is the period of the ensuing oscillations?

Given: m = 25 kg, k_1 = 1000 N/m, k_3 = 4000 N/m, F = 50 N

Find: T

Solution: Consider a free body diagram of the initial position. Application of the equation of static equilibrium leads to

$$k_1x_0 + k_3x_0^3 = F$$

$$1000x_0 + 4000x_0^3 = 50$$

$$x_0 = 0.04951 \text{ m}$$

Using the nondimensional variables and parameters of eqs. (12.8) and (12.9),

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$$\Delta = \frac{mg}{k_1} = \frac{(25\text{ kg})\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right)}{1000\ \frac{\text{N}}{\text{m}}} = 0.245\text{ m}$$

$$\epsilon = \frac{k_3}{k_1}\Delta^2 = \frac{4000\frac{\text{N}}{\text{m}^3}}{1000\frac{\text{N}}{\text{m}}}(0.245\text{ m})^2 = 0.241$$

$$\omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{1000\ \frac{\text{N}}{\text{m}}}{25\text{ kg}}} = 6.32\ \frac{\text{rad}}{\text{sec}}$$

The nondimensional initial displacement becomes

$$\delta = \frac{x_0}{\Delta} = \frac{0.04951\text{ m}}{0.245\text{ m}} = 0.202$$

The equation for the nondimensional amplitude for a mass attached to a cubic spring subject to an initial displacement but no initial velocity is

$$\delta = A - \frac{\epsilon}{32}A^3$$

$$0.202 = A - 0.00753A^3$$

$$A \approx 0.202$$

The period is given by

$$T = \frac{2\pi}{\omega_n\left(1 + \frac{3}{8}\epsilon A^3\right)} = \frac{2\pi}{\left(6.32\ \frac{\text{rad}}{\text{sec}}\right)\left[1 + \frac{3}{8}(0.241)(0.202)^3\right]} = 0.992\text{ sec}$$

Problem 12.22 illustrates (a) the determination of the equilibrium position when a force is applied to a nonlinear spring, and (b) the determination of the period of oscillation of a block attached to a spring with cubic nonlinearity.

12.23 Use the method of renormalization to determine a two-term frequency-amplitude relationship for the particle on the rotating parabola of Figure P12.6, assuming the amplitude is small.

Given: p, ω

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Find: ω_n

Solution: The differential equation governing the motion of the system is derived in the solution of Problem 12.6 as

$$(1+4p^2x^2)\ddot{x}+(2gp-\omega^2)x+4p^2x\dot{x}^2=0 \tag{1}$$

Assume the amplitude of free oscillations is small, say of the order ϵ where ϵ is a small dimensionless parameter. A straightforward expansion for the particle displacement is assumed as

$$x(t)=\epsilon x_1(t)+\epsilon^2x_2(t)+\epsilon^3x_3(t)+\cdots \tag{2}$$

Substitution of eq. (2) into eq. (1) leads to

$$\begin{aligned}&(1+4p^2(\epsilon x_1+\epsilon^2x_2+\epsilon^3x_3+\cdots)^2)(\epsilon\ddot{x}_1+\epsilon^2\ddot{x}_2+\epsilon^3\ddot{x}_3+\cdots)\\&+(2gp-\omega^2)(\epsilon x_1+\epsilon^2x_2+\epsilon^3x_3+\cdots)\\&+4p^2(\epsilon x_1+\epsilon^2x_2+\epsilon^3x_3+\cdots)(\epsilon\dot{x}_1+\epsilon^2\dot{x}_2+\epsilon^3\dot{x}_3+\cdots)^2=0\end{aligned} \tag{3}$$

Collecting coefficients of like powers of ϵ leads to

$$\begin{aligned}&\epsilon[\ddot{x}_1+(2gp-\omega^2)x_1]+\epsilon^2[\ddot{x}_2+(2gp-\omega^2)x_2]\\&\quad+\epsilon^3[\ddot{x}_3+(2gp-\omega^2)x_3+4p^2x_1^2\ddot{x}_1-4p^2x_1\dot{x}_1^2]+\cdots=0\end{aligned}$$

Setting coefficients of powers of ϵ to zero independently leads to

$$\ddot{x}_1+(2gp-\omega^2)x_1=0 \tag{3}$$

$$\ddot{x}_2+(2gp-\omega^2)x_2=0 \tag{4}$$

$$\ddot{x}_3+(2gp-\omega^2)x_3=-4p^2x_1^2\ddot{x}_1-4p^2x_1\dot{x}_1^2 \tag{5}$$

The solution of eq. (3) is

$$x_1(t)=Asin(\omega_0t+\phi) \tag{6}$$

where

$$\omega_0=\sqrt{2gp-\omega^2}$$

Equation (4) has the trivial solution, and $x_2=0$. Substitution of eq. (6) into eq. (5) leads to

$$\begin{aligned}\ddot{x}_3+(2gp-\omega^2)x_3&=4p^2A^3\omega_0^2[-sin^3(\omega_0t+\phi)+sin(\omega_0t+\phi)cos^2(\omega_0t+\phi)]\\&=4p^2A^3\omega_0^2sin(\omega_0t+\phi)[cos^2(\omega_0t+\phi)-sin^2(\omega_0t+\phi)]\\&=4p^2A^3\omega_0^2sin(\omega_0t+\phi)cos2(\omega_0t+\phi)\end{aligned}$$

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$$= 2p^2A^3\omega_0^2[sin^3(\omega_0 t + \phi) - sin(\omega_0 t + \phi)]$$

The solution of the above equation is

$$x_3 = p^2A^3\left[-\frac{1}{4}sin3(\omega_0 t + \phi) + \omega_0 t cos(\omega_0 t + \phi)\right]$$

Hence

$$x(t) = \epsilon A sin(\omega_0 t + \phi) + \epsilon^3 p^2 A^3\left[-\tfrac{1}{4}sin3(\omega_0 t + \phi) + \omega_0 t cos(\omega_0 t + \phi)\right] + \cdots \qquad (7)$$

The preceding expansion is nonuniform, thus the method of renormalization is used. To this end, let

$$t = \omega_0(t + \epsilon^2\lambda_1 + \cdots)$$

Then eq. (7) becomes

$$\begin{aligned} x = \epsilon A sin(w + \epsilon^2 w\lambda_1 + \cdots + \phi) \\ + \epsilon^3 p^2 A^3\left[-\frac{1}{4}sin3(w + \epsilon^2 w\lambda_1 + \cdots + \phi)\right. \\ \left. + (w + \epsilon^2 w\lambda_1 + \cdots)cos(w + \epsilon^2 w\lambda_1 + \cdots + \phi)\right] \end{aligned}$$

Using a Taylor series expansion, the preceding equation becomes through terms of ϵ^3

$$x = \epsilon A sin(w + \phi) + \epsilon^3\left[w\lambda_1 A cos(w + \phi) - \frac{p^2A^3}{4}sin3(w + \phi) + p^2A^3 w cos(w + \phi)\right] \ldots$$

Hence secular terms are eliminated if

$$\lambda_1 = -p^2A^2$$

Hence

$$\omega_0 t = w(1 - \epsilon^2 p^2 A^2 + \cdots)$$

$$\omega = \omega_0(1 - \epsilon^2 p^2 A^2 + \cdots)$$

Problem 12.23 illustrates the use of the method of renormalization to determine a two term frequency amplitude relation for a system with cubic nonlinearity.

12.24 Use the method of renormalization to determine a two-term frequency-amplitude relationship for a block of mass *m* attached to a spring with a quadratic nonlinearity. When nondimensionalized the differential equation governing free vibrations of the system is

$$\ddot{x} + \omega^2 x + \epsilon x^2 = 0, \quad \epsilon \ll 1 \qquad (1)$$

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Given: equation (1)

Find: frequency-amplitude relationship

Solution: Assume a straightforward expansion of the form

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \tag{2}$$

Substitution of eq. (2) into eq. (1) leads to

$$(\ddot{x}_0 + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2 + \cdots) + \omega^2(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) + \epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 = 0$$

Collecting coefficients of like powers of ϵ leads to

$$(\ddot{x}_0 + \omega^2 x_0) + \epsilon(\ddot{x}_1 + \omega^2 x_1 + x_0^2) + \epsilon^2(\ddot{x}_2 + \omega^2 x_2 + 2x_0 x_1) + \cdots = 0$$

Setting coefficients of powers of ϵ to zero independently leads to

$$\ddot{x}_0 + \omega^2 x_0 = 0 \tag{3}$$

$$\ddot{x}_1 + \omega^2 x_1 = -x_0^2 \tag{4}$$

$$\ddot{x}_2 + \omega^2 x_2 = -2x_0 x_1 \tag{5}$$

The solution of eq. (3) is

$$x_0(t) = A sin(\omega t + \phi) \tag{6}$$

Substitution of eq. (6) into eq. (4) leads to

$$\ddot{x}_1 + \omega^2 x_1 = -A^2 sin^2(\omega t + \phi) = -\frac{A^2}{2}[1 - cos2(\omega t + \phi)]$$

The particular solution for x_1 is

$$x_1(t) = -\frac{A^2}{2\omega^2} - \frac{A^2}{6\omega^2} cos2(\omega t + \phi) \tag{7}$$

Substitution of Esq.(6) and (7) in eq.(5) leads to

$$\begin{aligned}\ddot{x}_2 + \omega^2 x_2 &= 2A sin(\omega t + \phi)\left[\frac{A^2}{2\omega^2} + \frac{A^2}{6\omega^2} cos2(\omega t + \phi)\right] \\ &= \frac{A^3}{\omega^2}\left[sin(\omega t + \phi) + \frac{1}{3} sin(\omega t + \phi) cos2(\omega t + \phi)\right] \\ &= \frac{A^3}{\omega^2}\left[\frac{1}{2} sin3(\omega t + \phi) + \frac{5}{6} sin(\omega t + \phi)\right]\end{aligned}$$

The solution of the above equation is

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$$x_2(t) = \frac{A^3}{\omega^2}\left[-\frac{1}{16\omega^2}sin3(\omega t+\phi) - \frac{5}{12\omega}tcos(\omega t+\phi)\right]$$

Hence

$$x(t) = Asin(\omega t+\phi) - \epsilon\frac{A^2}{\omega^2}\left[\frac{1}{2}+\frac{1}{3}cos2(\omega t+\phi)\right]$$

$$-\epsilon^2\frac{A^2}{\omega^2}\left[\frac{1}{16\omega^2}sin3(\omega t+\phi)+\frac{5}{12\omega}tcos(\omega t+\phi)\right]\ldots \qquad (8)$$

The preceding expansion is nonuniform, hence the method of renormalization is used,

$$\omega t = w(1+\epsilon^2\lambda_1+\cdots) \qquad (9)$$

Substitution of eq.(9) into eq.(8) and using Taylor series expansions leads to

$$x = Asin(w+\phi) - \epsilon\frac{A^2}{\omega^2}\left[\frac{1}{2}+\frac{1}{3}cos2(w+\phi)\right]$$

$$-\epsilon^2\left[\lambda_1 Awcos(w+\phi) - \frac{A^3}{16\omega^4}sin3(w+\phi)+\frac{5A^3}{12\omega^4}wcos(w+\phi)\right]\ldots$$

Secular terms are eliminated if

$$\lambda_1 = \frac{5A^2}{12\omega^4}$$

Hence

$$\omega t = w\left(1+\epsilon^2\frac{5A^2}{12\omega^4}+\cdots\right)$$

and

$$w = \omega t\left(1-\frac{5}{12}\epsilon^2\frac{A^3}{\omega^2}+\cdots\right)$$

Problem 12.24 illustrates the use of the method of renormalization to determine a two-term approximation to the frequency of a system with a quadratic nonlinearity.

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12.25 If $F(t) = F_0 \sin \omega t$, what values of ω will lead to the presence of the following?

(a) A primary resonance
(b) A superharmonic resonance
(c) A subharmonic resonance

FIGURE P12.25

Given: m = 17.8 kg, k_1 = 1000 N/m, k_3 = 950 N/m^3, c = 20 N · sec/m

Find: values of ω leading to resonance conditions

Solution: The natural frequency of the block attached to a linear spring of the stiffness of k_1 is

$$\omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{1000\ \frac{\text{N}}{\text{m}}}{17.8\ \text{kg}}} = 7.495\ \frac{\text{rad}}{\text{sec}}$$

(a) A primary resonance occurs when the excitation frequency is close to ω_n,

$$\omega \approx 7.495\ \frac{\text{rad}}{\text{sec}}$$

(b) A superharmonic resonance occurs when the excitation frequency is close to $\omega_n/3$

$$\omega \approx 2.498\ \frac{\text{rad}}{\text{sec}}$$

(c) A subharmonic resonance occurs when the excitation frequency is close to $3\omega_n$

$$\omega \approx 22.486\ \frac{\text{rad}}{\text{sec}}$$

Problem 12.25 illustrates the frequencies for primary, superharmonic, and subharmonic resonances of a system with a cubic nonlinearity.

12.26 When $F(t) = 5 \sin 8t$ N, a primary resonance condition occurs. Determine the amplitude of the forced response.

FIGURE P12.25

Given: m = 17.8 kg, k_1 = 1000 N/m, k_3 = 950 N/m^3, c = 20 N · sec/m, F_0 = 5 N, ω = 8 rad/sec

Find: A

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Solution: The using the definitions of eqs.(12.8) and (12.9),

$$\omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{1000\ \frac{N}{m}}{17.8\ kg}} = 7.495\ \frac{\text{rad}}{\text{sec}}$$

$$\Delta = \frac{mg}{k_1} = \frac{(17.8\ \text{kg})\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right)}{1000\ \frac{\text{N}}{\text{m}}} = 0.1746\ \text{m}$$

$$\epsilon = \frac{k_3}{k_1}\Delta^2 = \frac{1000\ \frac{\text{N}}{\text{m}}}{950\ \frac{\text{N}}{\text{m}^3}}(0.1746\ \text{m})^2 = 0.0321$$

$$\Lambda = \frac{F_0}{m\omega_n^2\Delta} = \frac{5\ \text{N}}{(17.8\ \text{kg})\left(7.4953\ \frac{\text{rad}}{\text{sec}}\right)^2(0.1746\ \text{m})} = 0.02863$$

$$\dot{F} = \frac{\Lambda}{\epsilon} = \frac{0.02863}{0.0321} = 0.892$$

$$\mu = \frac{c}{2\epsilon m\omega_n} = \frac{20\ \frac{\text{N}\cdot\text{sec}}{\text{m}}}{2(0.0321)(17.8\ \text{kg})\left(7.4953\ \frac{\text{rad}}{\text{sec}}\right)} = 2.335$$

The detuning parameter is calculated from

$$\frac{\omega}{\omega_n} = 1 + \varepsilon\sigma$$

$$\frac{8\ \frac{\text{rad}}{\text{sec}}}{7.4953\ \frac{\text{rad}}{\text{sec}}} = 1 + 0.0321\sigma$$

$$\sigma = 2.098$$

The equation for steady-state amplitude in the presence of a primary resonance is

$$4A^2\left[\mu^2 + \left(\sigma - \frac{3}{8}A^2\right)^2\right] = \dot{F}^2$$

Substituting given and calculated values

$$4A^2\left[(2.335)^2 + \left(2.098 - \frac{3}{8}A^2\right)^2\right] = (0.892)^2$$

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$$0.1407A^6 - 1.574A^4 + 9.853A^2 - 0.199 = 0$$

Only one real root of the preceding equation exists

$$A = 0.143$$

This leads to a dimensional amplitude of

$$x = \Delta A = (0.1746\ \text{m})(0.14) = 0.0250\ \text{m}$$

Problem 12.26 illustrates the frequency-amplitude response for a system with a cubic nonlinearity subject to a primary resonance.

12.27 When $F(t)$ = 150 sin 2.5t N, a superharmonic resonance condition occurs. Determine the amplitude of the forced response.

FIGURE P12.25

Given: m = 17.8 kg, k_1 = 1000 N/m, k_3 = 950 N/m^3, c = 20 N · sec/m, F_0 = 150 N, ω = 2.5 rad/sec

Find: A

Solution: The Using the definitions of eqs.(12.8) and (12.9),

$$\omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{1000\ \frac{\text{N}}{\text{m}}}{17.8\ \text{kg}}} = 7.495\ \frac{\text{rad}}{\text{sec}}$$

$$\Delta = \frac{mg}{k_1} = \frac{(17.8\ \text{kg})\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right)}{1000\ \frac{\text{N}}{\text{m}}} = 0.1746\ \text{m}$$

$$\epsilon = \frac{k_3}{k_1}\Delta^2 = \frac{1000\ \frac{\text{N}}{\text{m}}}{950\ \frac{\text{N}}{\text{m}^3}}(0.1746\ \text{m})^2 = 0.0321$$

$$\Lambda = \frac{F_0}{m\omega_n^2\Delta} = \frac{150\ \text{N}}{(17.8\ \text{kg})\left(7.4953\ \frac{\text{rad}}{\text{sec}}\right)^2(0.1746\ \text{m})} = 0.859$$

$$\dot{F} = \frac{\Lambda}{\epsilon} = \frac{0.02863}{0.0321} = 0.892$$

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$$\mu = \frac{c}{2\epsilon m\omega_n} = \frac{20\ \frac{\text{N}-\text{sec}}{\text{m}}}{2(0.0321)(17.8\ \text{kg})\left(7.4953\ \frac{\text{rad}}{\text{sec}}\right)} = 2.335$$

The detuning parameter is calculated from

$$\frac{\omega}{\omega_n} = \frac{1}{3} + \varepsilon\sigma$$

$$\frac{2.5\ \frac{\text{rad}}{\text{sec}}}{7.4953\ \frac{\text{rad}}{\text{sec}}} = \frac{1}{3} + 0.0321\sigma$$

$$\sigma = 0.00651$$

The equation for steady-state amplitude in the presence of a superharmonic resonance is

$$\left[\mu^2 + \left(\sigma - 3\Lambda^2 - \frac{3}{8}A^2\right)^2\right]A^2 = \Lambda^6$$

Substituting given and calculated values

$$\left[(2.335)^2 + \left(0.00651 - 3(0.859)^2 - \frac{3}{8}A^2\right)^2\right] = (0.859)^6$$

$$0.1407A^6 - 1.65A^4 + 10.32A^2 - 0.4017 = 0$$

Only one real root of the preceding equation exists

$$A = 0.197$$

This leads to a dimensional amplitude of

$$x = \Delta A = (0.1746\ m)(0.197) = 0.0344\ m$$

Problem 12.27 illustrates the frequency-amplitude response for a system with a cubic nonlinearity subject to a superharmonic resonance.

12.28 If $F(t) = F_0 \sin \omega t$ N, for what value of ω will a jump in amplitude occur when ω is increased slightly beyond this value when

(a) $F_0 = 5$ N and a primary resonance occurs.
(b) $F_0 = 150$ N and a superharmonic resonance occurs.

FIGURE P12.25

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Given: m = 17.8 kg, k_1 = 1000 N/m, k_3 = 950 N/m^3, c = 20 N · sec/m, (a) F_0 = 5 N, (b) F_0 = 150 N

Find: ω for jump phenomenon

Solution: The Using the definitions of eqs.(12.8) and (12.9),

$$\omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{1000\ \frac{\mathrm{N}}{\mathrm{m}}}{17.8\ \mathrm{kg}}} = 7.495\ \frac{\mathrm{rad}}{\mathrm{sec}}$$

$$\Delta = \frac{mg}{k_1} = \frac{(17.8\ \mathrm{kg})\left(9.81\ \frac{\mathrm{m}}{\mathrm{sec}^2}\right)}{1000\ \frac{\mathrm{N}}{\mathrm{m}}} = 0.1746\ \mathrm{m}$$

$$\epsilon = \frac{k_3}{k_1}\Delta^2 = \frac{1000\ \frac{\mathrm{N}}{\mathrm{m}}}{950\ \frac{\mathrm{N}}{\mathrm{m}^3}}(0.1746\ \mathrm{m})^2 = 0.0321$$

$$\Lambda = \frac{F_0}{m\omega_n^2\Delta} = \frac{\mathrm{F}_0}{(17.8\ \mathrm{kg})\left(7.4953\ \frac{\mathrm{rad}}{\mathrm{sec}}\right)^2(0.1746\ \mathrm{m})} = 0.00573\mathrm{F}_0$$

$$\mu = \frac{c}{2\epsilon m\omega_n} = \frac{20\ \frac{\mathrm{N}\cdot\mathrm{sec}}{\mathrm{m}}}{2(0.0321)(17.8\ \mathrm{kg})\left(7.4953\ \frac{\mathrm{rad}}{\mathrm{sec}}\right)} = 2.335$$

(a) For the primary resonance, define

$$\dot{F} = \frac{\Lambda}{\epsilon} = \frac{(0.00573)(5\ N)}{0.0321} = 0.892$$

The frequency amplitude curve is given by

$$4A^2\left[\mu^2 + \left(\sigma - \frac{3}{8}A^2\right)^2\right] = \dot{F}^2$$

where

$$r = 1 + \epsilon\sigma$$

The jump phenomenon occurs when the frequency response curve has a point of vertical tangency. Thus

$$\frac{dA}{d\sigma} = \infty$$

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Which leads to

$$\dot{F}^4 = \frac{9}{4}A^6\dot{F}^2 - 9A^8\mu^2$$

The above equation has no real solution for A. Thus the jump phenomenon is not present for this particular combination of parameters. Perhaps the damping ratio is too large.

(b) For the superharmonic resonance,

$$\Lambda = 0.00573(150\ N) = 0.859$$

The frequency response curve for the superharmonic resonance is

$$A^2\left[\mu^2 + \left(\sigma - 3\Lambda^2 - \frac{3}{8}A^2\right)^2\right] = \Lambda^6$$

$$\sigma = 3\Lambda^2 + \frac{3}{8}A^2 + \sqrt{\frac{\Lambda^6}{A^2} - \mu^2}$$

The point of vertical tangency is achieved when

$$\frac{d\sigma}{dA} = 0 \rightarrow A^6 - 13.57A^8 = 1.777$$

which also has no real solution. Note that the coefficient of the A8 term is large because of the large damping ratio. Hence the jump phenomenon does not occur for this damping ratio.

Problem 12.28 illustrates (a) the frequency response curves for the primary resonance and the superharmonic resonance of Duffing's equation, and (b) that the jump phenomenon does not occur for all values of parameters.

12.29 If $F(t) = 25 \sin 22t$ N, will a nontrivial subharmonic response exist?

FIGURE P12.25

Given: m = 17.8 kg, k_1 = 1000 N/m, k_3 = 950 N/m^3, c = 20 N · sec/m, F_0 = 25 N, ω = 22 rad/sec

Find: will a non-trivial subharmonic response exist

Solution: The using the definitions of eqs.(12.8) and (12.9),

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$$\omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{1000\ \frac{\text{N}}{\text{m}}}{17.8\ \text{kg}}} = 7.4953\ \frac{\text{rad}}{\text{sec}}$$

$$\Delta = \frac{mg}{k_1} = \frac{(17.8\ \text{kg})\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right)}{1000\ \frac{\text{N}}{\text{m}}} = 0.1746\ \text{m}$$

$$\epsilon = \frac{k_3}{k_1}\Delta^2 = \frac{1000\ \frac{\text{N}}{\text{m}}}{950\ \frac{\text{N}}{\text{m}^3}}(0.1746\ \text{m})^2 = 0.0321$$

$$\mu = \frac{c}{2\epsilon m\omega_n} = \frac{20\ \frac{\text{N}\cdot\text{sec}}{\text{m}}}{2(0.0321)(17.8\ \text{kg})\left(7.4953\ \frac{\text{rad}}{\text{sec}}\right)} = 2.335$$

For the subharmonic resonance the detuning parameter is calculated from

$$\frac{\omega}{\omega_n} = 3 + \varepsilon\sigma$$

$$\frac{22\ \frac{\text{rad}}{\text{sec}}}{7.4953\ \frac{\text{rad}}{\text{sec}}} = 3 + 0.0321\sigma$$

$$\sigma = -2.019$$

Note that

$$\left(\frac{\sigma}{\mu}\right)^2 = \left(\frac{-2.109}{2.335}\right)^2 = 0.748$$

Since this value is less than 63, the boundaries of the inequality in eq.(12.45) are imaginary and thus a subharmonic resonance does not exist.

Problem 12.29 illustrates the conditions for which a non-trivial subharmonic resonance will exist.

12.30 If $F(t) = 30 \sin 15t + 25 \sin \omega t$ N, what values of ω lead to a combination resonance?

Given: m = 17.8 kg, k_1 = 1000 N/m, k_3 = 950 N/m^3, c = 20 N · sec/m, F_1 = 30 N, ω_1 = 15 rad/sec, F_2 = 25N

FIGURE P12.25

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Find: ω such that combination resonances exist

Solution: The natural frequency of the linear system composed of the block attached to a linear spring of stiffness k_1 is

$$\omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{1000\ \frac{\text{N}}{\text{m}}}{17.8\ \text{kg}}} = 7.4953\ \frac{\text{rad}}{\text{sec}}$$

The frequency ratios are defined as

$$r_1 = \frac{\omega_1}{\omega_n} = \frac{15\ \frac{\text{rad}}{\text{sec}}}{7.4953\ \frac{\text{rad}}{\text{sec}}} = 2.001$$

$$r_2 = \frac{\omega_2}{7.4953}$$

Each of the conditions leading to combination resonances indicated in section 12.6 are explored

(1) $2r_2 + r_1 = 1$ This condition cannot be met as $r_1>1$

(2) $2r_1 - r_2 = \pm 1$, $r_2 = 3.003, 5.002$, $\omega = 22.50$ rad/sec, 37.49 rad/sec

(3) $2r_2 - r_1 = \pm 1$, $r_2 = 1.5005, 0.5005$, $\omega = 11.25$ rad/sec, 3.751 rad/sec

(4) $r_1 - r_2 + 1 = -1$, $r_2 = 4.001$, $\omega = 29.99$ rad/sec

(5) $r_1 - r_2 - 1 = \pm 1$, $r_2 - r_1 - 1 = \pm 1$, all cases already considered

(6) $r_1 + r_2 - 1 = 1$, cannot be met since $r_1 > 2$

Problem 12.30 illustrates combination resonance conditions for systems with cubic nonlinearities.

12.31 If $F(t) = 30 \sin 2.5t + 25 \sin \omega t$ N, what values of ω lead to simultaneous resonances?

FIGURE P12.25

Given: m = 17.8 kg, k_1 = 1000 N/m, k_3 = 950 N/m^3, c = 20 N · sec/m, F_1 =30 N, ω_1 = 2.5 rad/sec, F_2 = 25N

Find: ω such that simultaneous resonances exist

Solution: The natural frequency of the block attached to a linear spring of stiffness k_1 is

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$$\omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{1000\ \frac{\text{N}}{\text{m}}}{17.8\ \text{kg}}} = 7.4953\ \frac{\text{rad}}{\text{sec}}$$

The nondimensional frequencies are defined as

$$r_1 = \frac{\omega_1}{\omega_n} = \frac{2.5\ \frac{\text{rad}}{\text{sec}}}{7.4953\ \frac{\text{rad}}{\text{sec}}} = 0.333$$

$$r_2 = \frac{\omega}{\omega_n} = \frac{\omega}{7.4953\frac{\text{rad}}{\text{sec}}}$$

Note that since r_1 is approximately 1/3, a superharmonic resonance exists. Simultaneous resonances exist if r_2 is of a form to cause a primary resonance, a subharmonic resonance, or a combination resonance.

Primary resonance: $r_2 = 1$, $\omega = 7.4953$ rad/sec

Subharmonic resonance: $r_2 = 3$, $\omega = 22.49$ rad/sec

Simultaneous resonance:

(a) $2r_2 + r_1 = 1$ This gives $r_2 = 1/3$, but a superharmonic resonance already exists

(b) $2r_1 - r_2 = \pm 1$, $r_2 = 5/3$, $\omega = 12.49$ rad/sec

(c) $2r_2 - r_1 = \pm 1$, $r_2 = 4/3$, $\omega = 9.99$ rad/sec

(4) $r_1 - r_2 + 1 = -1$, $r_2 - r_1 + 1 = \pm 1$, $r_2 = 7/3$, $\omega = 17.49$ rad/sec

(5) $r_1 - r_2 - 1 = \pm 1$, $r_2 - r_1 - 1 = \pm 1$, $r_2 = 7/3$, 1/3

(6) $r_1 + r_2 - 1 = 1$, $r_2 = 5/3$

Problem 12.31 illustrates simultaneous resonance developed in a system with a cubic nonlinearity.

12.32 If $m_2 = 10$ kg, for what values of k_2 will internal resonance exist?

FIGURE P12.32

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Given: $m_1 = 10$ kg, $m_2 = 10$ kg, $k_1 = 1000$ N/m, $k_3 = 950$ N/m^3

Find: k_2 for internal resonance

Solution: Consider free-body diagrams of the system at an arbitrary instant.

Application of Newton's law

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$

To the free-body diagrams leads to the differential equations

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 + k_3x_1^3 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = F(t)$$

The matrix form of the linearized system is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. Substituting for m_1 and k_1 and developing the characteristic equation leads to

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 0.1 & 0 \\ 0 & \dfrac{1}{m_2} \end{bmatrix}\begin{bmatrix} 1000 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 100 + 0.1k_2 & -0.1k_2 \\ -\dfrac{k_2}{m_2} & \dfrac{k_2}{m_2} \end{bmatrix}$$

The quadratic formula is used to determine

$$det\begin{bmatrix} 100 + 0.1k_2 - \lambda & -0.1k_2 \\ -\dfrac{k_2}{m_2} & \dfrac{k_2}{m_2} - \lambda \end{bmatrix} = 0$$

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$$\lambda^2 - \left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)\lambda + 100\frac{k_2}{m_2} = 0$$

$$\lambda = \frac{1}{2}\left(\frac{k_2}{m_2} + 100 + 0.1k_2 \pm \sqrt{\left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)^2 - 400\frac{k_2}{m_2}}\right)$$

Note that the preceding can be written as

$$\lambda = \frac{1}{2}\left(a \pm \sqrt{b}\right)$$

Then the natural frequencies become

$$\omega_1 = \left[\frac{1}{2}\left(a - \sqrt{b}\right)\right]^{\frac{1}{2}}$$

$$\omega_2 = \left[\frac{1}{2}\left(a + \sqrt{b}\right)\right]^{\frac{1}{2}}$$

An internal resonance occurs if

$$\omega_2 = 3\omega_1$$

$$\omega_2^2 = 9\omega_1^2$$

$$\frac{1}{2}\left(a + \sqrt{b}\right) = \frac{9}{2}\left(a - \sqrt{b}\right)$$

$$\frac{4}{5}a = \sqrt{b}$$

$$\frac{16}{25}a^2 = b$$

or

$$\frac{16}{25}\left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)^2 = \left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)^2 - 400\frac{k_2}{m_2}$$

$$\frac{9}{25}\left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)^2 = 400\frac{k_2}{m_2}$$

For $m_2 = 10$ kg, the preceding equation becomes

$$(0.2k_2 + 100)^2 = 111.11k_2$$

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The above equation is rearranged and the quadratic formula is used to solve yielding

$$k_2 = 154\ \frac{N}{m}, \quad 1624\ \frac{N}{m}$$

Problem 12.32 illustrates internal resonance conditions for a two-degrees-of-freedom system.

13.33 For what values of m_2 are internal resonances possible? If an internal resonance is possible in terms of m_2, for what values of k_2 will they exist?

FIGURE P12.32

Given: $m_1 = 10$ kg, $k_1 = 1000$ N/m, $k_3 = 950$ N/m^3

Find: m_2 for internal resonance

Solution: Consider free-body diagrams of the system at an arbitrary instant.

Application of Newton's law

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$

To the free-body diagrams leads to the differential equations

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 + k_3x_1^3 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = F(t)$$

The matrix form of the linearized system is

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$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. Substituting for m_1 and k_1 and developing the characteristic equation leads to

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 0.1 & 0 \\ 0 & \dfrac{1}{m_2} \end{bmatrix}\begin{bmatrix} 1000 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 100 + 0.1k_2 & -0.1k_2 \\ -\dfrac{k_2}{m_2} & \dfrac{k_2}{m_2} \end{bmatrix}$$

The quadratic formula is used to determine

$$det\begin{bmatrix} 100 + 0.1k_2 - \lambda & -0.1k_2 \\ -\dfrac{k_2}{m_2} & \dfrac{k_2}{m_2} - \lambda \end{bmatrix} = 0$$

$$\lambda^2 - \left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)\lambda + 100\frac{k_2}{m_2} = 0$$

$$\lambda = \frac{1}{2}\left(\frac{k_2}{m_2} + 100 + 0.1k_2 \pm \sqrt{\left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)^2 - 400\frac{k_2}{m_2}}\right)$$

Note that the preceding can be written as

$$\lambda = \frac{1}{2}\left(a \pm \sqrt{b}\right)$$

Then the natural frequencies become

$$\omega_1 = \left[\frac{1}{2}\left(a - \sqrt{b}\right)\right]^{\frac{1}{2}}$$

$$\omega_2 = \left[\frac{1}{2}\left(a + \sqrt{b}\right)\right]^{\frac{1}{2}}$$

An internal resonance occurs if

$$\omega_2 = 3\omega_1$$

$$\omega_2^2 = 9\omega_1^2$$

$$\frac{1}{2}\left(a + \sqrt{b}\right) = \frac{9}{2}\left(a - \sqrt{b}\right)$$

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$$\frac{4}{5}a = \sqrt{b}$$

$$\frac{16}{25}a^2 = b$$

or

$$\frac{16}{25}\left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)^2 = \left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)^2 - 400\frac{k_2}{m_2}$$

$$\frac{9}{25}\left(\frac{k_2}{m_2} + 100 + 0.1k_2\right)^2 = 400\frac{k_2}{m_2}$$

Rearranging leads to

$$k_2^2(1 + 0.1m_2^2) + (-9.11.11m_2 + 20m_2^2)k_2 = 10{,}000m_2^2 = 0$$

Which for given m_2 can be solved for k_2 using the quadratic formula as

$$k_2 = \frac{9.11.11 - 20m_2^2}{1 + 0.1m_2^2} \pm \frac{\sqrt{(-911.11m_2 + 20m_2^2)^2 - 4(1 + 0.1m_2^2)(10000m_2^2)}}{1 + 0.1m_2^2}$$

Hence an internal resonance will exist for values of m_2 and k_2 that satisfy the above equation. However, for certain values of m_2, the discriminant is negative, yielding no values of k_2 for which an internal resonance exists. The values of m_2 for which an internal resonance exists are determined by setting the discriminant greater than zero. This leads to

$$790{,}000 - 36440m_2 - 3600m_2^2 > 0$$

The values of m2 which satisfy the preceding equation are

$$m_2 < 20.71\ kg$$

Problem 12.33 illustrates the development of an internal resonance for a two-degrees-of-freedom system.

13.34 Consider the system with m_2 = 10 kg and k_2 = 2000 N/m. The right mass is displaced 10 mm from equilibrium while the left mass is held in place. The system is released from rest from this configuration.

(a) Determine the natural frequencies, mode shapes, and principal coordinates for the linearized system.
(b) Write the nonlinear differential equations governing the system using the principal coordinates of the linearized system as dependent variables.

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FIGURE P12.32

Given: $m_1 = 10$ kg, $m_2 = 10$ kg, $k_1 = 1000$ N/m, $k_3 = 950$ N/m^3, $k_2 = 2000$ N/m

Find: (a) ω_1, ω_2, $\mathbf{X_1}$, $\mathbf{X_2}$, p_1, p_2 (b) differential equations using p_1 and p_2 as dependent variables

Solution: Consider free-body diagrams of the system at an arbitrary instant.

Application of Newton's law

$$\left(\sum F\right)_{ext} = \left(\sum F\right)_{eff}$$

To the free-body diagrams leads to the differential equations

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 + k_3x_1^3 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = F(t)$$

The matrix form of the linearized system is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. Substituting for m_1 and k_1 and developing the characteristic equation leads to

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}\begin{bmatrix} 3000 & -2000 \\ -2000 & 2000 \end{bmatrix} = \begin{bmatrix} 300 & -200 \\ -200 & 200 \end{bmatrix}$$

The characteristic equation is

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$$\det\begin{bmatrix}300-\lambda & -200\\ -200 & 200-\lambda\end{bmatrix}=0$$

$$\lambda^2-500\lambda+20000=0$$

whose solutions are

$$\lambda=43.85,\qquad 456.1$$

which lead to natural frequencies of

$$\omega_1=6.62\,\frac{\text{rad}}{\text{sec}},\quad \omega_2=21.36\,\frac{\text{rad}}{\text{sec}}$$

For a calculated eigenvalue, the corresponding mode shape is obtained by solving

$$\begin{bmatrix}300-\lambda & -200\\ -200 & 200-\lambda\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Since the preceding equations represented by the matrix system are dependent, only the first is used, yielding

$$X_2=\left(\frac{300-\lambda}{200}\right)X_1$$

Arbitrarily choosing X_1 = 1, the mode shapes are determined to be

$$\boldsymbol{X_1}=C_1\begin{bmatrix}1\\ 1.281\end{bmatrix},\quad \boldsymbol{X_2}=C_2\begin{bmatrix}1\\ -0.781\end{bmatrix}$$

The modes shapes are normalized with respect to the kinetic energy scalar product, leading to

$$\boldsymbol{X_1}=\begin{bmatrix}0.195\\ 0.249\end{bmatrix},\quad \boldsymbol{X_2}=\begin{bmatrix}0.249\\ -0.195\end{bmatrix}$$

Hence the modal matrix is

$$\boldsymbol{P}=\begin{bmatrix}0.195 & 0.249\\ 0.249 & -0.195\end{bmatrix}$$

The principal coordinates are related to the chosen generalized coordinates by

$$\boldsymbol{x}=\boldsymbol{P}\boldsymbol{p}$$

or

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0.195 & 0.249\\ 0.249 & -0.195\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}$$

The nonlinear differential equations can be written in matrix form as

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$$\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\begin{bmatrix}3000 & -2000\\ -2000 & 3000\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}950x_1^3\\ 0\end{bmatrix}=\begin{bmatrix}0\\ F(t)\end{bmatrix}$$

It is noted that

$$x_1^3=(0.195p_1+0.249p_2)^3=0.00741p_1^3+0.284p_1^2p_2+0.0363p_1p_2^2+0.154p_2^3$$

Using the principal coordinates as dependent variables and multiplying by $\mathbf{P}^{\mathbf{T}}$ leads to

$$\begin{bmatrix}p_1+\omega_1^2p_1\\ p_2+\omega_2^2p_2\end{bmatrix}+\begin{bmatrix}0.195 & 0.249\\ 0.249 & -0.195\end{bmatrix}\begin{bmatrix}7.035p_1^3+23.56p_1^2p_2+34.48p_1p_2^2+14.63p_2^3\\ 0\end{bmatrix}=\begin{bmatrix}0.249F(t)\\ =0.195F(t)\end{bmatrix}$$

and the governing differential equations become

$$\ddot{p}_1+43.85p_1+1.372p_1^3+4.59p_1^2p_2+6.72p_1p_2^2+2.85p_2^3=0.249F(t)$$

$$\ddot{p}_2+456.1p_2+1.75p_1^3+5.87p_1^2p_2+8.59p_1p_2^2+3.64p_2^3=-0.195F(t)$$

Problem 12.34 illustrates the coupling present in a two-degrees-of-freedom system with a cubic nonlinearity.

12.35 If m_2 = 10 kg, k_2 = 1000 N/m, and $F(t)$ = 150 sin ωt N, for what values of ω will the following resonances exist?

(a) Primary resonance
(b) Superharmonic resonance
(c) Subharmonic resonance
(d) Combination resonance

Given: m_1 = 10 kg, m_2 = 10 kg, k_1 = 1000 N/m, k_3 = 950 N/m^3, k_2 = 1000 N/m

Find: ω for resonances

Solution: The differential equations governing the forced motion of the system are

$$m_1\ddot{x}_1+(k_1+k_2)x_1-k_2x_2+k_3x_1^3=0$$

$$m_2\ddot{x}_2-k_2x_1+k_2x_2=F_0 sin\omega t$$

The matrix formulation of the corresponding linear system for free vibrations is

$$\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\begin{bmatrix}k_1+k_2 & -k_2\\ -k_2 & k_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ F(t)\end{bmatrix}$$

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The natural frequencies of the linear system are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. Substituting given values

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} 2000 & -1000 \\ -1000 & 1000 \end{bmatrix} = \begin{bmatrix} 200 & -100 \\ -100 & 100 \end{bmatrix}$$

The characteristic equation becomes

$$det \begin{bmatrix} 200-\lambda & -100 \\ -100 & 100-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 300\lambda + 10000 = 0$$

whose solutions are

$$\lambda = 38.2, \qquad 261.8$$

which lead to natural frequencies of

$$\omega_1 = 6.18\,\frac{\text{rad}}{\text{sec}}, \quad \omega_2 = 16.18\,\frac{\text{rad}}{\text{sec}}$$

(a) Primary resonances occur when the excitation frequency is close to either of the linear natural frequencies

$$\omega_1 = 6.18\,\frac{\text{rad}}{\text{sec}}, \quad \omega_2 = 16.18\,\frac{\text{rad}}{\text{sec}}$$

(b) Superharmonic resonances occur when

$$\omega \approx \frac{\omega_i}{3}$$

which for this problem leads to

$$\omega \approx 2.06\,\frac{\text{rad}}{\text{sec}}, \quad 5.39\,\frac{\text{rad}}{\text{sec}}$$

(c) Subharmonic resonances occur when

$$\omega \approx 3\omega_i$$

which for this problem leads to

$$\omega \approx 18.4\,\frac{\text{rad}}{\text{sec}}, \quad 48.54\,\frac{\text{rad}}{\text{sec}}$$

(d) Combination resonances occur for a system with a cubic nonlinearity when

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$$\omega \approx 2\omega_1 + \omega_2, \quad \omega_1 + 2\omega_2, \quad 2\omega_2 - \omega_1, \quad \frac{1}{2}(\omega_1 + \omega_2), \quad \frac{1}{2}(\omega_1 - \omega_2)$$

which for this problem leads to

$$\omega \approx 28.54\frac{\text{rad}}{\text{sec}}, \quad 18.54\ \frac{\text{rad}}{\text{sec}}, \quad 26.18\ \frac{\text{rad}}{\text{sec}}, \quad 11.18\ \frac{\text{rad}}{\text{sec}}, \quad 5.0\ \frac{\text{rad}}{\text{sec}}$$

Problem 12.35 illustrates primary, subharmonic, superharmonic, and combination resonance conditions for a two degree-of-freedom system with a cubic nonlinearity.

12.36 Consider the system of Figure P12.36.

(a) Derive the nonlinear differential equations governing the motion of the system using the generalized coordinates shown.
(b) Expand trigonometric functions of the generalized coordinates using Taylor series expansions. Rewrite the differential equations keeping only quadratic and cubic nonlinearities.
(c) For what values of l in terms of the other parameters will an internal resonance exist?
(d) In the absence of an internal resonance, for what values of ω will resonance conditions exists?

FIGURE P12.36

Given: m, l, k, x, and θ as generalized coordinates

Find: (a) differential equations, (c) values of l for internal resonance, and values of ω for other resonances

Solution: (a) the differential equations governing the motion of this system are derived, using Lagrange's equations, in the solution of Problem 12.7. They are

$$2m\ddot{x} + \frac{1}{2}m\ell\ddot{\theta}cos\theta - m\ell\dot{\theta}^2 sin\theta + kx = F(t)$$

$$\frac{1}{3}m\ell^2\ddot{\theta} + \frac{1}{2}m\ell\ddot{x}cos\theta + mg\frac{\ell}{2}sin\theta = 0$$

(b) The Taylor series expansion of the trigonometric terms are

$$sin\theta = \theta - \frac{1}{6}\theta^3 + \cdots$$

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$$cos\theta = 1 - \frac{1}{2}\theta^2 + \cdots$$

Substituting into the differential equations and keeping only through cubic terms leads to

$$2m\ddot{x} + \frac{1}{2}m\ell\ddot{\theta} + kx - \frac{1}{4}m\ell\ddot{\theta}\dot{\theta}^2 - m\ell\dot{\theta}^2\theta = F(t)$$

$$\frac{1}{3}m\ell^2\ddot{\theta} + \frac{1}{2}m\ell\ddot{x} + \frac{1}{2}mg\ell\theta - \frac{1}{4}m\ell\ddot{x}\theta^2 - \frac{1}{12}mg\ell\theta^3 = 0$$

(c) The matrix form of the linearized differential equations is

$$\begin{bmatrix} 2m & \frac{1}{2}m\ell \\ \frac{1}{2}m\ell & \frac{1}{3}m\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & \frac{1}{2}mg\ell \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}$$

The natural frequencies are the reciprocals of the square roots of the eigenvalues of $\mathbf{K}^{-1}\mathbf{M}$. To this end

$$\boldsymbol{K^{-1}M} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \frac{2}{mg\ell} \end{bmatrix}\begin{bmatrix} 2m & \frac{1}{2}m\ell \\ \frac{1}{2}m\ell & \frac{1}{3}m\ell^2 \end{bmatrix} = \begin{bmatrix} \frac{2m}{k} & \frac{m\ell}{2k} \\ \frac{1}{g} & \frac{2}{3}\frac{\ell}{g} \end{bmatrix}$$

The characteristic equation becomes

$$\det\begin{bmatrix} \frac{2m}{k} - \lambda & \frac{m\ell}{2k} \\ \frac{1}{g} & \frac{2}{3}\frac{\ell}{g} - \lambda \end{bmatrix} = 0$$

$$\lambda^2 - \left(\frac{2m}{k} + \frac{2}{3}\frac{\ell}{g}\right)\lambda + \frac{5}{6}\frac{m\ell}{gk} = 0$$

The eigenvalues are

$$\lambda = \frac{1}{2}\left(\frac{2m}{k} + \frac{2}{3}\frac{\ell}{g} \pm \sqrt{\left(\frac{2m}{k} + \frac{2}{3}\frac{\ell}{g}\right)^2 - \frac{10}{3}\frac{m\ell}{gk}}\right)$$

which can be written in the form of

$$\lambda = \frac{1}{2}\left(a \pm \sqrt{b}\right)$$

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The system has a cubic nonlinearity. Hence an internal resonance occurs when

$$\omega_2 = 3\omega_1$$

$$\omega_2^2 = 9\omega_1^2$$

$$\frac{2}{\left(a-\sqrt{b}\right)} = \frac{18}{a+\sqrt{b}}$$

$$\sqrt{b} = \frac{4}{5}a$$

$$b = \frac{16}{25}a^2$$

which leads to

$$\left(\frac{2m}{k}+\frac{2}{3}\frac{\ell}{g}\right)^2 - \frac{10}{3}\frac{m\ell}{gk} = \frac{16}{25}\left(\frac{2m}{k}+\frac{2}{3}\frac{\ell}{g}\right)^2$$

or

$$\frac{4}{9}\frac{\ell^2}{g^2} - \frac{178}{27}\frac{m\ell}{gk} + \frac{4m^2}{k^2} = 0$$

whose solutions are

$$\ell = 2.13\frac{mg}{k}, \quad 12.7\frac{mg}{k}$$

(d) In the absence of an internal resonance, the other resonance conditions are:

(1) Primary resonance

$$\omega \approx \omega_1, \quad \omega_2$$

(2) Superharmonic resonance

$$\omega \approx \frac{1}{3}\omega_1, \quad \frac{1}{3}\omega_2$$

(c) Subharmonic resonances occur when

$$\omega \approx 3\omega_1, \quad 3\omega_2$$

(d) Combination resonances

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$$\omega \approx 2\omega_1 + \omega_2, \quad 2\omega_1 - \omega_2, \quad \omega_2 - 2\omega_1, \quad \omega_1 + 2\omega_2, \quad \frac{1}{2}(\omega_1 + \omega_2), \quad \frac{1}{2}(\omega_1 - \omega_2)$$

Problem 12.36 illustrates (a) the use of Taylor series expansions to write nonlinear differential equations in terms of cubic nonlinearities when nonlinearities are geometric, (b) the presence of internal resonances, and (c) the presence of primary, subharmonic, superharmonic, and combination resonances for a two-degree-of-freedom system.

12.37 Show that the coefficient multiplying $p_2^2 p_1$ for a pinned-pinned beam is zero in Equation (11.50).

Given: pinned-pinned beam, eq.(11.50)

Show: coefficient on $p_2^2 p_1$ term is zero

Solution: The triple summation has three terms which lead to $p_2^2 p_1$. First consider k = 2, l = 2, m = 1. The coefficient is

$$\left(\phi_j, \frac{\partial^2 \phi_2}{\partial x^2}\right) \int_0^1 \frac{\partial \phi_2}{\partial x} \frac{\partial \phi_1}{\partial x} dx$$

Note that for a pinned-pinned beam

$$\phi_1 = \sqrt{2} sin\pi x, \quad \phi_2 = \sqrt{2} sin 2\pi x$$

Hence

$$\int_0^1 \frac{\partial \phi_2}{\partial x} \frac{\partial \phi_1}{\partial x} dx = \int_0^1 \sqrt{2}\pi cos\pi x \left(\sqrt{2} 2\pi cos 2\pi x\right) dx$$

$$= 4\pi^2 \int_0^1 cos\pi x cos 2\pi\pi\pi x \, dx$$

$$= 0$$

Note that when k = 2, l = 1, m = 2, the same coefficients is obtained. Note a similar argument occurs when k = 1, l = 2, and m = 2 when

$$\left(\phi_j, \frac{\partial^2 \phi_2}{\partial x^2}\right) = 0$$

for any j.

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Problem 12.37 illustrates that even though the frequency terms are correct for the presence of an internal resonance for a pinned-pinned beam, an internal resonance does not exist because the nonlinear term involving the frequencies in question is identically zero.

12.38 A fixed-free rectangular steel beam (ρ = 7850 kg/m^3, E = 210 × 10^9 N/m^2) with a length of 1 m, base of 2 cm, and height of 5 cm is subject to a single-frequency harmonic excitation. List all excitation frequencies that should be avoided to avoid all primary, secondary, and combination resonances involving the three lowest modes.

Given: ρ = 7850 kg/m^3, E = 210 × 10^9 N/m^2, L = 1 m, b = 2 cm, h = 5 cm

Find: Ω to avoid

Solution: The geometric properties of the beam are

$$A = bh = (0.02\text{ m})(0.05\text{ m}) = 0.001\text{ m}$$

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(0.02\text{ m})(0.05\text{ m})^3 = 2.08 \times 10^{-7}\text{ m}^4$$

From Table 9.4 the three lowest nondimensional frequencies of a fixed-free beam are

$$\omega_1^* = 3.51 \quad \omega_2^* = 22.03, \quad \omega_3^* = 61.70$$

The dimensional frequencies are related to the nondimensional frequencies by

$$\omega_i = \omega_i^* \sqrt{\frac{EI}{\rho A L^4}}$$

where

$$\sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{\left(210 \times 10^9 \frac{\text{N}}{\text{m}^2}\right)(2.08 \times 10^{-7}\text{ m}^4)}{\left(7850 \frac{\text{kg}}{\text{m}^3}\right)(0.001\text{ m}^2)(1\text{ m})^4}} = 74.59 \frac{\text{rad}}{\text{sec}}$$

Hence the beam's dimensional natural frequencies are

$$\omega_1 = 261.8 \frac{\text{rad}}{\text{sec}}, \qquad \omega_2 = 1643 \frac{\text{rad}}{\text{sec}}, \qquad \omega_3 = 4602 \frac{\text{rad}}{\text{sec}}$$

In order to avoid a primary resonance the excitation frequency should not be near a natural frequency. Hence

$$\Omega \neq 261.8 \frac{\text{rad}}{\text{sec}}, \qquad 1643 \frac{\text{rad}}{\text{sec}}, \qquad 4602 \frac{\text{rad}}{\text{sec}}$$

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In order to avoid a superharmonic resonance, the excitation frequency should not be near one third of any of the natural frequencies,

$$\Omega \neq 87.3\ \frac{\text{rad}}{\text{sec}}, \qquad 547.7\ \frac{\text{rad}}{\text{sec}}, \qquad 1534\ \frac{\text{rad}}{\text{sec}}$$

In order to avoid a subharmonic resonance, the excitation frequency should not be near three times any of the natural frequencies,

$$\Omega \neq 785.4\ \frac{\text{rad}}{\text{sec}}, \qquad 4929\ \frac{\text{rad}}{\text{sec}}, \qquad 1381\ \frac{\text{rad}}{\text{sec}}$$

There are several forms of combination resonance to avoid:

(1) $\Omega = 2\omega_i \pm \omega_j$:

$$2\omega_1 + \omega_2 = 2167\ \frac{\text{rad}}{\text{sec}}, \quad 2\omega_1 + \omega_3 = 5126\ \frac{\text{rad}}{\text{sec}}$$

$$2\omega_2 - \omega_1 = 3014\ \frac{\text{rad}}{\text{sec}}, \quad 2\omega_2 + \omega_3 = 7888\ \frac{\text{rad}}{\text{sec}}$$

$$2\omega_3 - \omega_1 = 8943\ \frac{\text{rad}}{\text{sec}}, \quad 2\omega_3 - \omega_2 = 7562\ \frac{\text{rad}}{\text{sec}}$$

(2) $\Omega = \omega_i \pm \omega_j \pm \omega_k$:

$$\omega_1 + \omega_2 + \omega_3 = 6507\ \frac{\text{rad}}{\text{sec}}, \quad \omega_3 + \omega_2 - \omega_3 = 5983\ \frac{\text{rad}}{\text{sec}}$$

$$\omega_{13} - \omega_2 + \omega_{31} = 3221\ \frac{\text{rad}}{\text{sec}}, \quad \omega_3 - \omega_2 - \omega_1 = 2697\ \frac{\text{rad}}{\text{sec}}$$

(3) $\Omega = (\omega_i \pm \omega_j)/2$:

$$\frac{\omega_1 + \omega_2}{2} = 952\ \frac{\text{rad}}{\text{sec}}, \quad \frac{\omega_1 + \omega_3}{2} = 2432\ \frac{\text{rad}}{\text{sec}}$$

$$\frac{\omega_2 - \omega_1}{2} = 691\ \frac{\text{rad}}{\text{sec}}, \quad \frac{\omega_2 + \omega_3}{2} = 3123\ \frac{\text{rad}}{\text{sec}}$$

$$\frac{\omega_3 - \omega_2}{2} = 1479\ \frac{\text{rad}}{\text{sec}}, \quad \frac{\omega_3 - \omega_1}{2} = 2170\ \frac{\text{rad}}{\text{sec}}$$

Problem 12.38 illustrates primary, secondary, and combination resonance conditions for a continuous system.

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12.39 If the beam of Chapter Problem 12.38 is fixed-fixed, which of the following excitation frequencies should be avoided and why?

(a) 180 rad/s
(b) 1530 rad/s
(c) 2200 rad/s
(d) 7940 rad/s

Given: $\rho = 7850$ kg/m^3, $E = 210 \times 10^9$ N/m^2, L = 1 m, b = 2 cm, h = 5 cm

Find: Ω to avoid

Solution: The geometric properties of the beam are

$$A = bh = (0.02\text{ m})(0.05\text{ m}) = 0.001\text{ m}$$

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(0.02\text{ m})(0.05\text{ m})^3 = 2.08 \times 10^{-7}\text{ m}^4$$

From Table 9.4 the five lowest nondimensional frequencies of a fixed-fixed beam are

$$\omega_1^* = 22.37, \quad \omega_2^* = 61.66, \quad \omega_3^* = 120.9, \quad \omega_4^* = 199.9, \quad \omega_5^* = 298.6$$

The dimensional frequencies are related to the nondimensional frequencies by

$$\omega_i = \omega_i^* \sqrt{\frac{EI}{\rho A L^4}}$$

where

$$\sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{\left(210 \times 10^9 \frac{\text{N}}{\text{m}^2}\right)(2.08 \times 10^{-7}\text{ m}^4)}{\left(7850 \frac{\text{kg}}{\text{m}^3}\right)(0.001\text{ m}^2)(1\text{ m})^4}} = 74.59 \frac{\text{rad}}{\text{sec}}$$

Hence the beam's dimensional natural frequencies are

$$\omega_1 = 1668 \frac{\text{rad}}{\text{sec}}, \quad \omega_2 = 4600 \frac{\text{rad}}{\text{sec}}, \quad \omega_3 = 9017 \frac{\text{rad}}{\text{sec}},$$

$$\omega_4 = 14910 \frac{\text{rad}}{\text{sec}}, \quad \omega_5 = 21601 \frac{\text{rad}}{\text{sec}}$$

From the given frequencies (b) 1530 rad/sec should be avoided because it leads to a superharmonic resonance involving the second mode; (c) 2200 rad/sec should be avoided because it leads to a combination resonance involving the first and second mode

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$(\omega_2 - \omega_1)/2$ and; (d) 7940 rad/sec should be avoided because it leads to a combination resonance involving the first and second modes $(2\omega_1 + \omega_2)$.

Problem 12.39 illustrates secondary and combination resonance conditions for continuous systems.

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Mechanical Engineering/ College of Engineering

CHAPTER 13: RANDOM VIBRATIONS

Short answer questions

13.1 True: The Rayleigh distribution is only applicable for random variables the can have only positive values.

13.2 False: A stationary process is one in which the mean and standard deviation are the same at any time.

13.3 True: The Weiner-Khintchine equations imply that the power spectral density is the Fourier transform of the autocorrelation function.

13.4 True: The normalized Gaussian distribution has a mean of zero and a standard deviation of one.

13.5 False: The probability density function is the derivative of the probability distribution function.

13.6 True: For a stationary process $R(-\tau) = R(\tau)$.

13.7 True: The sinusoidal transfer function is the transfer function with s replaced by $i\omega$ which is the same as the transfer function defined by the Fourier transform.

13.8 False: If $x(t) = A \sin 5t$ then for $|x| < A$.

13.9 True: The mean of a random variable is E(x) which can be calculated by $\mu = \int_{-\infty}^{\infty} xp(x)dx$for a stationary ergodic process.

13.10 False: The standard deviation is the positive square root of the variance.

13.11 True: A narrow band process has a limited number of frequencies in the process and hence a narrow band over which the power spectral density is defined.

13.12 False: For a stationary process the autocorrelation function $R(\tau)$ is an even function of τ.

13.13 An ensemble is a set of measurements of the same random function.

13.14 A stationary process is one where the important statistics such as mean and variance are independent of time.

13.15 An ergodic process is one where the mean, variance and other statistics are independent of which member of the ensemble is used.

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13.16 The wind induced vibrations of a bridge is more likely to be a random process.

13.17 The total area under the curve of a probability density function is one.

13.18 The Central Limit Theorem implies that the mean of a random process, no matter what the distribution of the process is a random variable that is normally distributed.

13.19 The power spectral density for ideal white noise is a constant, S_0.

13.20 The autocorrelation function for ideal white noise is $S_0\delta(\tau)$.

13.21 The Fourier transform of $\delta(t)$ is one.

13.22 $P(x) = \int_{-\infty}^{x} p(x)dx$

13.23 P(0) = 0.5 for the normalized Gaussian distribution.

13.24 The probability of the maximum value of the response of a system follows the Gaussian probability distribution.

13.25 The probability of the absolute value of the response of a system follows the Rayleigh probability distribution.

13.26 $P(\infty) = 1$.

13.27 The probability that $x > b$ is 1 – P(b).

13.28 The probability that $-1 \leq x < 3$ is P(3) – P(–1).

13.29 $S_x(\omega) = |H(\omega)|^2 S_F(\omega)$

13.30 The mean square value of the process is $\int_{-\infty}^{\infty} S(\omega)|H(\omega)|^2 d\omega$.

13.31 (a) $Pr(z < 1) = 0.8413$ (b) $\Pr(-2 < z < 1) = p(1) - p(-2) = 0.8413 - 0.0027 = 0.8386$ (c) $\Pr(z > 0.5) = 1 - \Pr(z < 0.5) = 1 - 0.6915 = 0.3085$

13.32 Given: $\mu = -1.3$ and $\sigma = 2.8$. The normalized random variable is $z = \frac{x+1.3}{2.8}$.

(a) $\Pr(x < -3.3) = \Pr\left(z < \frac{-3.3+1.3}{2.8} = -0.714\right) = 0.2389$

(b) $\Pr(x > 3.3) = \Pr\left(z > \frac{3.3+1.3}{2.8} = 1.64\right) = 0.0505$

(c) $\Pr(0 < x < 6.3) = \Pr\left(\frac{1.3}{6.8} = 0.191 < z < \frac{6.3+1.3}{2.8} = 2.71\right) = \Pr(z < 2.71) - \Pr(z < 0.191) = 0.9966 - 0.5753 = 0.4213$

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13.33 Given: Rayleigh distribution $\mu = 3.1$

(a) $\mu = \sqrt{\frac{\pi\alpha}{2}} \Longrightarrow \alpha = \frac{2\mu^2}{\pi} = \frac{2(3.1)^2}{\pi} = 6.12.$ $\Pr(x > 3.1) = \Pr\left(\frac{y^2}{2\alpha} > \frac{3.1^2}{2(6.12)} = 0.785\right) = 0.45$

(b) $\Pr(x < 2.3) = \Pr\left(\frac{y^2}{2\alpha} < \frac{2.3^2}{2(6.12)} = 0.432\right) = 0.36$

(c) $\Pr(2.9 < x < 3.3) = \Pr\left(0.687 < \frac{y^2}{2\alpha} < 0.890\right) = 0.58 - 0.48 - 0.1$

13.34 Given: $p(x) = \frac{1}{\pi(1+x^2)}$

(a) $p(x) = \frac{dP}{dx} \Longrightarrow P(x) = \int_{-\infty}^{x} p(x)dx = \int_{-\infty}^{x} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi}\left(\tan^{-1} x + \frac{\pi}{2}\right)$

(b) $E(x) = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx = 0$

(c) $E(x^2) = \int_{-\infty}^{\infty} \frac{x^2}{\pi(1+x^2)} dx = \infty$

13.35 Given: $P(x) = 1 - e^{-x^\gamma}$; $p(x) = \frac{dP}{dx} = x^{\gamma-1}e^{x^\gamma}$

13.36 (a) $(\omega) = \frac{1}{-m\omega^2 + ic\omega}$; (b) $|H(\omega)| = \frac{1}{\sqrt{(m\omega^2)^2 + (c\omega)^2}}$; (c)S_0;

(d) $S_x(\omega) = \frac{S_0}{(m\omega^2)^2 + (c\omega)^2}$; (e)$\bar{x}^2 = \int_{-\infty}^{\infty} \frac{S_0}{(m\omega^2)^2 + (c\omega)^2} d\omega = \infty$

13.37 (a) $\mathrm{N}^2 \cdot \mathrm{m/s}$ (b) $\mathrm{m}^2/(\mathrm{s} \cdot \mathrm{rad})$ (c) $\mathrm{N} \cdot \mathrm{s}$ (d) $\mathrm{m} \cdot \mathrm{s}$ (e) $\mathrm{m/N}$ (f) m^2 (g) N^2 (h) m (i) N (j) Dimensionless (k) $1/\mathrm{m}$

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Chapter Problems

13.1 Determine the autocorrelation function for $x(t) = A\cos 2t$.

Given: $x(t) = A\cos 2t$

Find: $R(\tau)$

Solution: The autocorrelation function is given by

$$R(\tau) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A\cos 2t\,[A\cos 2(t+\tau)]\,d\tau = A^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2t\cos 2(t+\tau)\,d\tau$$

$$= A^2\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\tau\cos^2 2t\,d\tau + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\ 2\tau\sin 2t\cos 2t\,d\tau\right]$$

$$= \frac{A^2}{2}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\ 2\tau\,(1-\cos 2t)d\tau + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\ 2\tau\sin 4t\,d\tau\right]$$

$$= \frac{A^2}{4}\cos 2\tau$$

Problem 13.1 illustrates the autocorrelation function.

13.2 Determine the autocorrelation function for the rectangular wave shown in Figure P13.2.

FIGURE P13.2

Given: T, rectangular wave

Find: $R(\tau)$

Solution:

$$R(\tau) = \frac{1}{T}\int_0^T y(t)y(t+\tau)dt = \int_0^{\frac{T}{8}} Ay(t+\tau)dt \qquad (a)$$

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where

$$y(t+\tau) = \begin{cases} A & 0 < \frac{T}{8} - \tau \\ 0 & \frac{T}{8} < \tau < \frac{7T}{8} \\ A & \tau - \frac{7T}{8} < T \end{cases} \qquad (b)$$

For $0 < \tau < \frac{T}{8}$

$$R(\tau) = \frac{1}{T}\int_0^{\frac{T}{8}-\tau} A^2 dt = A^2\left(\frac{1}{8} - \frac{\tau}{T}\right) \qquad (c)$$

For $\frac{T}{8} < \tau < \frac{7T}{8}$,

$$R(\tau) = 0 \qquad (d)$$

For $\frac{7T}{8} < \tau < T$,

$$R(\tau) = \frac{1}{T}\int_{T-\tau}^{\frac{T}{8}} A^2 dt = A^2\left(\frac{\tau}{T} - \frac{7}{8}\right) \qquad (e)$$

The autocorrelation is periodic of period T and one period is illustrated below.

Problem 13.2 illustrates the autocorrelation function.

13.3 Determine the autocorrelation function for the rectangular wave shown in Figure P13.3.

FIGURE P13.3

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Given: F(t)

Find: $R(\tau)$

Solution: The autocorrelation function is defined by

$$R(\tau) = \frac{1}{T}\int_0^T F(t)F(t+\tau)dt$$

where

$$F(t) = \begin{cases} A & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} < t < T \end{cases}$$

Hence

$$R(\tau) = \frac{1}{T}\int_0^{\frac{T}{2}} AF(t+\tau)dt$$

Thus for $0 < \tau < \frac{T}{2}$

$$R(\tau) = \frac{1}{T}\left[\int_0^{\frac{T}{2}-\tau} A^2 dt\right] = \frac{A^2}{T}\left(\frac{T}{2}-\tau\right) = A^2\left(\frac{1}{2}-\frac{\tau}{T}\right)$$

For $\frac{T}{2} < \tau < T$

$$R(\tau) = \frac{1}{T}\left[\int_{T-\tau}^{\frac{T}{2}} A^2 dt\right] = \frac{A^2}{T}\left(\tau-\frac{T}{2}\right) = A^2\left(\frac{\tau}{T}-\frac{1}{2}\right)$$

The autocorrelation is periodic of period T and one period is illustrated below.

Problem 13.3 illustrates computation of the autocorrelation function.

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13.4 Determine the autocorrelation function for the triangular wave shown in Figure P13.4.

FIGURE P13.4

Given: F(t)

Find: $R(\tau)$

Solution: The autocorrelation function is defined by

$$R(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(t)F(t+\tau)dt$$

where

$$F(t) = \begin{cases} A\left(1+\frac{2t}{T}\right) & -\frac{T}{2} \le t \le T \\ A\left(1-\frac{2t}{T}\right) 0 & 0 < t < \frac{T}{2} \end{cases}$$

Hence

$$R(\tau) = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{0} A\left(1+\frac{2t}{T}\right)F(t+\tau)dt + \int_{0}^{\frac{T}{2}} A\left(1-\frac{2t}{T}\right)F(t+\tau)dt\right]$$

For $0 < \tau < \frac{T}{2}$

$$R(\tau) = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{-\tau} A\left(1+\frac{2t}{T}\right)A\left[1+\frac{2(t+\tau)}{T}\right]dt + \int_{-\tau}^{0} A\left(1+\frac{2t}{T}\right)A\left[1-\frac{2(t+\tau)}{T}\right]dt\right.$$
$$+ \int_{0}^{\frac{T}{2}-\tau} A\left(1-\frac{2t}{T}\right)A\left[1-\frac{2(t+\tau)}{T}\right]dt$$
$$\left.+ \int_{\frac{T}{2}-\tau}^{\frac{T}{2}} A\left(1-\frac{2t}{T}\right)A\left[1+\frac{2(t+\tau)}{T}\right]dt\right]$$

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For $\frac{T}{2} < \tau < T$

$$R(\tau) = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{0} A\left(1+\frac{2t}{T}\right)A\left[1-\frac{2(t+\tau)}{T}\right]dt + \int_{0}^{\frac{T}{2}} A\left(1-\frac{2t}{T}\right)A\left[1+\frac{2(t+\tau)}{T}\right]dt\right]$$

The autocorrelation is periodic of period T.

Problem 13.4 illustrates computation of the autocorrelation function.

13.5 Determine the autocorrelation function for the triangular wave shown in Figure P13.5.

FIGURE P13.5

Given: F(t)

Find: $R(\tau)$

Solution: The autocorrelation function is defined by

$$R(\tau) = \frac{1}{T}\int_{0}^{T} F(t)F(t+\tau)dt$$

where

$$F(t) = \begin{cases} A\frac{2t}{T} & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} < t < T \end{cases}$$

Hence

$$R(\tau) = \frac{1}{T}\int_{0}^{\frac{T}{2}} A\frac{2t}{T}F(t+\tau)dt$$

For $0 < \tau < \frac{T}{2}$

$$R(\tau) = \frac{1}{T}\int_{0}^{\frac{T}{2}-\tau} A\frac{2t}{T}A\frac{2(t+\tau)}{T}dt = A^2\left(\frac{1}{6} - \frac{\tau}{2T} - \frac{2}{2T^2} + \frac{2\tau^3}{3T^3}\right)$$

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For $\frac{T}{2} < \tau < T$

$$R(\tau) = \frac{1}{T}\int_{\frac{T}{2}-\tau}^{\frac{T}{2}} A\frac{2t}{T}A\frac{2(t+\frac{T}{2}-\tau)}{T}dt = A^2\left[\frac{5}{12} - \frac{\tau}{2T} - \frac{1}{2}\left(\frac{T}{2}-\tau\right)^3\right]$$

The autocorrelation is periodic of period T.

Problem 13.5 illustrates computation of the autocorrelation function.

13.6 Determine the autocorrelation function for the triangular wave shown in Figure P13.6.

FIGURE P13.6

Given: F(t)

Find: $R(\tau)$

Solution: The autocorrelation function is defined by

$$R(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(t)F(t+\tau)dt$$

where

$$F(t) = \begin{cases} A\left(2+\frac{4t}{T}\right) & -\frac{T}{2} \le t \le -\frac{T}{4} \\ -A\frac{4t}{T} & -\frac{T}{4} \le t \le \frac{T}{4} \\ A\left(-2+\frac{4t}{T}\right) & \frac{T}{4} \le t \le \frac{T}{2} \end{cases}$$

Hence

$$R(\tau) = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{-\frac{T}{4}} A\left(2+\frac{4t}{T}\right)F(t+\tau)dt + \int_{-\frac{T}{4}}^{\frac{T}{4}} -A\frac{4t}{T}F(t+\tau)dt + \int_{\frac{T}{4}}^{\frac{T}{2}} A\left(-2+\frac{4t}{T}\right)F(t+\tau)dt\right]$$

For $0 < \tau < \frac{T}{2}$

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$$R(\tau)=\frac{1}{T}\left[\int_{-\frac{T}{2}}^{-\tau}A\left(1+\frac{2t}{T}\right)A\left[1+\frac{2(t+\tau)}{T}\right]dt+\int_{-\tau}^{0}A\left(1+\frac{2t}{T}\right)A\left[1-\frac{2(t+\tau)}{T}\right]dt\right.$$
$$+\int_{0}^{\frac{T}{2}-\tau}A\left(1-\frac{2t}{T}\right)A\left[1-\frac{2(t+\tau)}{T}\right]dt$$
$$\left.+\int_{\frac{T}{2}-\tau}^{\frac{T}{2}}A\left(1-\frac{2t}{T}\right)A\left[1+\frac{2(t+\tau)}{T}\right]dt\right]$$

For $\frac{T}{2}<\tau<T$

$$R(\tau)=\frac{1}{T}\left[\int_{-\frac{T}{2}}^{0}A\left(1+\frac{2t}{T}\right)A\left[1-\frac{2(t+\tau)}{T}\right]dt+\int_{0}^{\frac{T}{2}}A\left(1-\frac{2t}{T}\right)A\left[1+\frac{2(t+\tau)}{T}\right]dt\right]$$

The autocorrelation is periodic of period T.

Problem 13.6 illustrates computation of the autocorrelation function.

13.7 A sine wave has the form $x(t)=3-2\sin 4t$. Determine the expected value of x and x^2.

Given: $x(t)=3-2\sin 4t$

Find: $E(x), E(x^2)$

Solution: The period of the function is $T=\frac{2\pi}{4}=\frac{\pi}{2}$. Thus

$$E(x)=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(3-\sin 2t)\,dt=\left(3t-\frac{1}{2}\cos 2t\right)_{t=-\frac{\pi}{4}}^{t=\frac{\pi}{4}}=\frac{3\pi}{2}-\frac{1}{2}\left(\cos\frac{\pi}{2}-\cos-\frac{\pi}{2}\right)=\frac{3\pi}{2}$$

and

$$E(x^2)=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(3-\sin 2t)^2\,dt=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(9-6\sin 2t+\sin^2 2t)dt$$
$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{19}{2}-6\sin 2t-\frac{1}{2}\cos 4t\right)dt=19\pi$$

Problem 13.7 illustrates the expected value of a function of time.

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13.8 Assume that *t* is uniformly distributed,

(a) Determine the probability density function $p(x)$ for the function in Chapter Problem 13.7.
(b) Determine the probability distribution function $P(x)$ for the function in Chapter Problem 13.7.

Solution: The formula for p(x) is

$$p(x) = \frac{p(t)}{\sum_{i=1}^{n} \left|\frac{dx}{dt}(t_i)\right|}$$

where the t_i are all times when $x(t_i) = x$. Assume that it is equally likely that a time t is chosen over one period. Thus

$$p(t) = \frac{1}{T} = \frac{2}{\pi}$$

The solution of

$$3 - 2\sin(4t_i) = x$$

is

$$t_i = \frac{1}{4}\sin^{-1}\left(\frac{3-x}{2}\right)$$

for $1 \le x \le 5$. Then

$$\frac{dx}{dt}(t_i) = -2\cos 4t_i = -8(1-\sin^2 4t_i)^{1/2} = -8\left[1-\left(\frac{3-x}{2}\right)^2\right]^{1/2}$$

There are two values of t over each cycle where 3 – 2sin 4t = x. Thus according to the formula

$$p(x) = \frac{1}{4\pi\left[1-\left(\frac{3-x}{2}\right)^2\right]^{\frac{1}{2}}} = \begin{cases} \frac{1}{4\pi}\left[1-\left(\frac{3-x}{2}\right)^2\right]^{-1/2} & 1 \le x \le 5 \\ 0 & x < -1 \quad \text{or} \quad x > 5 \end{cases}$$

(b) The probability distribution is

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$$P(x) = \int_{-\infty}^{x} p(x)dx = \begin{cases} 0 & x < 1 \\ \int_{1}^{x} \frac{1}{4\pi}\left[1 - \left(\frac{3-x}{2}\right)^2\right]^{-1/2} dx & 1 < x < 5 \\ 1 & x > 5 \end{cases}$$

$$P(x) = \begin{cases} 0 & x < 1 \\ \left[1 - \frac{2}{\pi}\sin^{-1}\left(\frac{3-x}{2}\right)\right] & 1 < x < 5 \\ 1 & x > 5 \end{cases}$$

Problem 13.8 illustrates the probability density function and the probability distribution.

13.9 Determine the probability density function for the periodic function, one period of which is shown in Figure P13.9.

FIGURE P13.9

Given: P(t)

Find: Probability density function

Solution: Assume that it is equally likely that a time t is chosen over one period. Thus

$$p(t) = \frac{1}{T}$$

Then for x≠ A

$$p(x) = \frac{p(t)}{\sum_{i=1}^{n}\left|\frac{dx}{dt}(t_i)\right|}$$

and

$$p(x) = \begin{cases} 0 & x < 0 \\ \frac{2}{T} & 0 < x < A \\ \frac{1}{2} & x = A \\ 0 & x > A \end{cases}$$

Problem 13.9 illustrates the probability distribution function.

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13.10 Determine the probability density function for the half-period cosine wave of Figure P13.10.

FIGURE P13.10

Given: F(t)

Find: p(x)

Solution: The formula for p(x) is

$$p(x) = \frac{p(t)}{\sum_{i=1}^{n} \left|\frac{dx}{dt}(t_i)\right|}$$

where the t_i are all times when $x(t_i) = x$. Assume that it is equally likely that a time t is chosen over one period. Thus

$$p(t) = \frac{1}{T}$$

The solution of

$$\text{Acos}\left(\frac{\pi t}{T}\right) = x$$

is

$$t_i = \frac{T}{\pi}\cos^{-1}\frac{x}{A}$$

for $0 \le x \le A$. Then

$$\frac{dx}{dt}(t_i) = -A\frac{\pi}{T}\sin t_i\frac{\pi}{T} = -A\frac{\pi}{T}\left(1-\cos^2\frac{\pi}{T}t_i\right)^{1/2} = -A\left[1-\left(\frac{x}{A}\right)^2\right]^{1/2}$$

There are two values of t over each cycle where $\text{Acos}\left(\frac{\pi t}{T}\right) = x$. Thus according to the formula

$$p(x) = \begin{cases} \frac{1}{2TA}\left[1-\left(\frac{x}{A}\right)^2\right]^{-1/2} & 0 \le x \le A \\ 0 \qquad x < 0 \quad \text{or} \quad x > A \end{cases}$$

Problem 13.10 illustrates the probability density function.

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13.11 Determine the Fourier transform of the rectangular pulse of Figure P13.11.

FIGURE P13.11

Given: T, rectangular pulse

Find: $F(\omega)$

Solution: The Fourier transform of a function F(t) is

$$F(\omega) = \int_{-\infty}^{\infty} F(t)e^{i\omega t}dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} F_0 e^{i\omega t}dt = \frac{F_0}{i\omega}\left(e^{i\omega t}\right)_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{F_0}{i\omega}\left(e^{i\omega T/2} - e^{-i\omega T/2}\right) = \frac{2F_0}{\omega}\sin\frac{\omega T}{2}$$

Problem 13.11 illustrates calculation of the Fourier transform.

13.12 Determine the Fourier transform for the triangular pulse of Figure P13.12.

FIGURE P13.12

Given: T, triangular pulse

Find: $F(\omega)$

Solution: The Fourier transform of a function F(t) is

$$F(\omega) = \int_{-\infty}^{\infty} F(t)e^{i\omega t}dt = \int_{-\frac{T}{2}}^{0} F_0\left(\frac{2}{T}t - 1\right)e^{i\omega t}dt + \int_{0}^{\frac{T}{2}} F_0\left(1 - \frac{2}{T}t\right)e^{i\omega t}dt$$

$$F_0\left[\frac{2}{T}\left(\frac{t}{i\omega} + \frac{1}{\omega^2}\right)e^{i\omega t} - \frac{1}{i\omega}e^{i\omega t}\right]_{-\frac{T}{2}}^{0} + F_0\left[\frac{1}{i\omega}e^{i\omega t} - \frac{2}{T}\left(\frac{t}{i\omega} + \frac{1}{\omega^2}\right)e^{i\omega t}\right]_{0}^{\frac{T}{2}}$$

$$= \frac{4}{T\omega^2}\left(1 - \cos\frac{\omega T}{2}\right) + \frac{4}{\omega}\sin\frac{\omega T}{2} + i\frac{2}{\omega}\left(1 - \cos\frac{\omega T}{2}\right)$$

Problem 13.12 illustrates the Fourier transform.

13.13 Determine the Fourier transform of the half-period cosine wave of Figure P13.10.

FIGURE P13.10

Given: T, half-period cosine wave

Find: $F(\omega)$

Solution: The Fourier transform of a function F(t) is

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$$F(\omega) = \int_{-\infty}^{\infty} F(t)e^{i\omega t}dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} F_0 \cos\frac{\pi t}{T} e^{i\omega t}dt$$

$$= \frac{F_0}{\omega^2\left(\frac{\pi^2}{T^2}-1\right)}\left[e^{i\omega t}\left(i\omega\cos\frac{\pi t}{T}+\frac{\pi}{T}\sin\frac{\pi t}{T}\right)\right]_{t=-\frac{T}{2}}^{t=\frac{T}{2}}$$

$$= \frac{2F_0}{\omega^2\pi\left(\frac{\pi^2}{T^2}-1\right)}\cos\frac{\omega T}{2}$$

Problem 13.13 illustrates the Fourier transform.

13.14 Determine the power spectral density of the wave shown in Figure P13.2.

FIGURE P13.2

Given: F(t)

Find: $S(\omega)$

Solution: The power spectral density is the Fourier transform of the autocorrelation function divided by 2π which is determined during the solution of Chapter Problem 13.2.

$$R(\tau) = \begin{cases} A^2\left(\frac{1}{8}-\frac{\tau}{T}\right) & 0 \le \tau \le \frac{T}{8} \\ 0 & \frac{T}{8} \le \tau \le \frac{7T}{8} \\ A^2\left(\frac{\tau}{T}-\frac{7}{8}\right) & \frac{7T}{8} \le \tau \le T \end{cases}$$

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{i\omega\tau}d\tau = \frac{1}{2\pi}\left[\int_0^{\frac{T}{8}} A^2\left(\frac{1}{8}-\frac{\tau}{T}\right)e^{i\omega\tau}d\tau + \int_{\frac{7T}{8}}^{T} A^2\left(\frac{\tau}{T}-\frac{7}{8}\right)e^{i\omega\tau}d\tau\right]$$

$$= \frac{A^2}{\pi\omega}\left[\frac{1}{\omega T}\left(-1+e^{-i\omega\frac{T}{8}}-e^{-i\omega t}+e^{-i\omega\frac{7T}{8}}\right)+\frac{i}{8}\left(-1+7e^{-i\omega T}+2e^{-i\omega\frac{T}{8}}-14e^{-i\omega\frac{7T}{8}}\right)\right]$$

Problem 13.14 illustrates the power spectral density.

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13.15 Determine the power spectral density of the wave shown in Figure P13.3.

FIGURE P13.3

Given: F(t)

Find: $S(\omega)$

Solution: The power spectral density is the Fourier transform of the autocorrelation function, which is determined during the solution of Chapter Problem 13.3, divided by 2π

$$R(\tau) = \begin{cases} A^2\left(\frac{1}{2} - \frac{\tau}{T}\right) & 0 \le \tau \le \frac{T}{2} \\ A^2\left(\frac{\tau}{T} - \frac{1}{2}\right) & \frac{T}{2} \le \tau \le T \end{cases}$$

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{i\omega\tau}d\tau = \frac{1}{2\pi}\left[\int_0^{\frac{T}{2}} A^2\left(\frac{1}{2} - \frac{\tau}{T}\right)e^{-i\omega\tau}d\tau + \int_{\frac{T}{2}}^{T} A^2\left(\frac{\tau}{T} - \frac{1}{2}\right)e^{-i\omega\tau}d\tau\right]$$

$$= \frac{A^2}{2\pi\omega}\left[\frac{i}{2}\left(-1 + e^{-i\omega T}\right) + \frac{1}{\omega T}\left(1 - 2e^{-i\omega\frac{T}{2}} + e^{-i\omega T}\right)\right]$$

Problem 13.15 illustrates the power spectral density function.

13.16 Determine the power spectral density of the wave shown in Figure P13.4.

FIGURE P13.4

Given: F(t)

Find: $S(\omega)$

Solution: The power spectral density is the Fourier transform of the autocorrelation function, which is determined during the solution of Chapter Problem 13.4, divided by 2π

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{i\omega\tau}d\tau$$

Problem 13.16 illustrates the power spectral density.

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13.17 Determine the power spectral density of the wave shown in Figure P13.6.

FIGURE P13.6

Given: F(t)

Find: $S(\omega)$

Solution: The power spectral density is the Fourier transform of the autocorrelation function, which is determined during the solution of Chapter Problem 13.6, divided by 2π

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{i\omega\tau}d\tau$$

Problem 13.17 illustrates the power spectral density.

13.18 A force has band limited white noise with frequency bounds of $\omega_1 = 100\frac{\text{rad}}{\text{s}}$ and $\omega_2 = 500\frac{\text{rad}}{\text{s}}$ and magnitude $S_0 = 2\times 10^2\ \frac{\text{N}^2\cdot\text{s}}{\text{rad}}$. Determine the following.

(a) The autocorrelation function for the force
(b) The expected mean square value of the force
(c) Assuming the mean is zero, what is the probability that the magnitude of the force is greater than 1000 N?

Given: $\omega_1 = 100\frac{\text{rad}}{\text{s}}$, $\omega_2 = 500\frac{\text{rad}}{\text{s}}$, $S_0 = 2\times 10^2\ \frac{\text{N}^2\cdot\text{s}}{\text{rad}}$

Find: $R(\tau)$, $\bar{F}^2$, Pr $(F > 1000)$

Solution: (a) The autocorrelation for band limited white noise is

$$R(\tau) = 2\int_{\omega_1}^{\omega_2} S_0\ \cos\omega\tau\, d\omega = \frac{2S_0}{\tau}(\sin\omega_2\tau - \sin\omega_1\tau) = \frac{2(2\times 10^2)}{\tau}(\sin 500\tau - \sin 100\tau)$$

(b) The expected mean square value of the force is

$$\bar{F}^2 = \int_{\omega_1}^{\omega_2} S_0 d\omega = S_0(\omega_2 - \omega_1) = \left(2\times 10^2\ \frac{\text{N}^2\cdot\text{s}}{\text{rad}}\right)(500\text{ rad} - 100\text{ rad}) = 8\times 10^4\ \text{N}^2\cdot\text{s}$$

(c) If the mean is zero the expected value is the variance. Assuming a Rayleigh distribution for the magnitude of the force,

$$z = \frac{1000^2}{2(8\times 10^4)} = 6.25$$

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$$\Pr(F > 1000) = \Pr(z > 6.25) = 1 - P(6.25) = 1 - 0.9999 = 0.0001$$

Problem 13.18 illustrates (a) The autocorrelation function for band limited white noise (b) the mean square average of a random variable and (c) the Gaussian distribution.

13.19 A SDOF system with mass 20 kg, $\zeta = 0.1$ and $\omega_n = 100\ \frac{\text{rad}}{\text{s}}$ is subject to white noise with $S_0 = 1 \times 10^{-2}\ \frac{\text{N}^2\cdot\text{s}}{\text{rad}}$. What is the power spectral density of the response $S_x(\omega)$?

Given: m = 20 kg, $\zeta = 0.1$, $\omega_n = 100\ \frac{\text{rad}}{\text{s}}$, $S_0 = 1 \times 10^{-2}\ \frac{\text{N}^2\cdot\text{s}}{\text{rad}}$

Find: $S_x(\omega)$

Solution: The transfer function of the system is

$$H(\omega) = \frac{1}{20(-\omega^2 + 20i\omega + 10000)}$$

The power spectral density of the response is

$$S_x(\omega) = S_0|H(\omega)|^2 = \frac{1 \times 10^{-2}}{(20)^2[(10000 - \omega^2)^2 + (20\omega)^2]} = \frac{2.5 \times 10^{-5}}{(10000 - \omega^2)^2 + (20\omega)^2}$$

Problem 13.19 illustrates power spectral density of the response of a system.

13.20 A SDOF system with mass 30 kg, $\zeta = 0.05$ and $\omega_n = 200\ \frac{\text{rad}}{\text{s}}$ is subject to white noise with $S_0 = 1 \times 10^{-2}\ \frac{\text{N}^2}{\text{Hz}}$. What is the power spectral density of the response $W_x(f)$?

Given: m = 30 kg, $\zeta = 0.05$, $\omega_n = 200\ \frac{\text{rad}}{\text{s}}$, $S_0 = 1 \times 10^{-2}\ \frac{\text{N}^2}{\text{Hz}}$

Find: $S_x(\omega)$

Solution: The transfer function of the system is

$$H(\omega) = \frac{1}{30(-\omega^2 + 20i\omega + 40000)}$$

The power spectral density of the response is

$$S_x(\omega) = S_0|H(\omega)|^2 = \frac{1 \times 10^{-2}}{(30)^2[(40000 - \omega^2)^2 + (20\omega)^2]} = \frac{3.08 \times 10^{-6}}{(10000 - \omega^2)^2 + (20\omega)^2}\ \frac{\text{m}^2 \cdot \text{s}}{\text{rad}}$$

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The power spectral density written in terms of f is

$$W_x(f) = 2\pi S_x(f) = \frac{1.94 \times 10^{-5}}{(10000 - \omega^2)^2 + (20\omega)^2} \frac{\mathrm{m}^2 \cdot \mathrm{s}}{\mathrm{Hz}}$$

Problem 13.20 illustrates power spectral density of the response of a system.

13.21 A SDOF system with a mass 20 kg, $\zeta = 0.1$ and $\omega_n = 100 \frac{\mathrm{rad}}{\mathrm{s}}$ is subject to white noise with $S_0 = 1 \times 10^{-2} \frac{\mathrm{N}^2 \cdot \mathrm{s}}{\mathrm{rad}}$.

(a) What is the mean square value of the response of the system?
(b) What is the probability of the response exceeding 5 mm?
(c) What is the probability of the maximum of the response exceeding 5 mm?

Given: m = 20 kg, $\zeta = 0.1$, $\omega_n = 100 \frac{\mathrm{rad}}{\mathrm{s}}$, $S_0 = 1 \times 10^{-2} \frac{\mathrm{N}^2 \cdot \mathrm{s}}{\mathrm{rad}}$

Find: $\overline{x^2}$, Pr(x > 5 mm), Pr(x_{max} > 5 mm)

Solution: (a) The transfer function of the system is

$$H(\omega) = \frac{1}{20(-\omega^2 + 20i\omega + 10000)}$$

The power spectral density of the response is

$$S_x(\omega) = S_0|H(\omega)|^2 = \frac{1 \times 10^{-2}}{(20)^2[(10000 - \omega^2)^2 + (20\omega)^2]} = \frac{2.5 \times 10^{-5}}{(10000 - \omega^2)^2 + (20\omega)^2}$$

The mean square of the response is

$$\overline{x^2} = \int_{-\infty}^{\infty} \frac{2.5 \times 10^{-5}}{(10000 - \omega^2)^2 + (20\omega)^2} d\omega = \frac{\pi(2.5 \times 10^{-5})}{2(10000)(20)} = 1.96 \times 10^{-10}\ \mathrm{m}^2$$

(b) Assuming a mean of zero, the variance is equal to the mean square value. Assuming a Gaussian distribution the normalized variable is

$$z = \frac{5 \times 10^{-6}}{\sqrt{1.96 \times 10^{-10}}} = 0.999$$

$$\Pr(x > 0.005\ \mathrm{m}) = \Pr(z > 0.999) = 1 - P(0.999) = 1 - 0.4999 = 0.5001$$

(c) Assuming a Rayleigh distribution

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$$\frac{(5\times10^{-6})^2}{2(1.96\times10^{-10})}=0.06377$$

Then

$$\Pr(x_{max}>0.005\text{ m})=\Pr(z>0.0637)=1-P(0.06377)=1-0.06=0.94$$

Problem 13.21 illustrates the use of the power spectral density in calculating the mean square value of the response.

13.22 The SDOF system of Figure 13.22 is subject to a white noise with $S_0=1\times10^{-2}\ \frac{\text{m}^2}{\text{rad}\cdot\text{s}}$ (the power spectral density of the acceleration of the base). Calculate the mean square value of the acceleration of the 20 kg block.

FIGURE P13.22

Solution: The transfer function for the block is

$$H(\omega)=\frac{1000\omega i+1\times10^6}{-\omega^2+1\times10^6+1000\omega i}$$

The mean square response of the system is

$$\bar{x}^2=\int_{-\infty}^{\infty}1\times10^{-2}|H(\omega)|^2d\omega=\int_{-\infty}^{\infty}1\times10^{-2}\frac{(1\times10^{12})^2+(1000\omega^3)^2}{(\omega^4-1\times10^6\omega^2+1\times10^{12})^2}d\omega$$

$$=62.832$$

Problem 13.22 illustrates use of the transfer function to find the mean square response.

13.23 The SDOF system of Figure P13.23 is subject to a white noise with $S_0=1\times10^{-3}\ \frac{\text{N}^2\cdot\text{s}}{\text{rad}}$. What is the mean square value of the response of the 300 kg mass?

FIGURE P13.23

Given: System shown

Find: $\bar{x}^2$

Solution: The transfer function for the system is

$$H(\omega)=\frac{2000\omega i+1\times10^6}{-\omega^2+2\times10^6+2000\omega i}$$

The mean square value of the response is

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$$\bar{x}^2 = \int_{-\infty}^{\infty} 1 \times 10^{-3} |H(\omega)|^2 d\omega$$

$$= \int_{-\infty}^{\infty} 1x10^{-3} \frac{(4 \times 10^6 \omega^2 + 2 \times 10^{12})^2 + (-2000\omega^3 + 2 \times 10^9 \omega)^2}{(\omega^4 + 4 \times 10^{12})^2} d\omega$$

$$= 7.609$$

Problem 13.23 illustrates the use of the transfer function in finding the mean square value of the response.

13.24 Solve Chapter Problem 13.21, assuming the power spectral density is band limited with $\omega_1 = 50 \frac{\text{rad}}{\text{s}}$ and $\omega_2 = 200 \frac{\text{rad}}{\text{s}}$.

Given: m = 20 kg, $\zeta = 0.1$, $\omega_n = 100 \frac{\text{rad}}{\text{s}}$, $S_0 = 1 \times 10^{-2} \frac{\text{N}^2 \cdot \text{s}}{\text{rad}}$

Find: $\overline{x^2}$, Pr(x > 5 mm), Pr(x_{max} > 5 mm)

Solution: (a) The transfer function of the system is

$$H(\omega) = \frac{1}{20(-\omega^2 + 20i\omega + 10000)}$$

The power spectral density of the response is

$$S_x(\omega) = S_0 |H(\omega)|^2 = \frac{1 \times 10^{-2}}{(20)^2[(10000 - \omega^2)^2 + (20\omega)^2]} = \frac{2.5 \times 10^{-5}}{(10000 - \omega^2)^2 + (20\omega)^2}$$

The mean square of the response is

$$\overline{x^2} = \int_{50}^{200} \frac{2.5 \times 10^{-5}}{(10000 - \omega^2)^2 + (20\omega)^2} d\omega = \frac{\pi(2.5 \times 10^{-5})}{2(10000)(20)} = 1.96 \times 10^{-10} \text{ m}^2$$

(b) Assuming a mean of zero, the variance is equal to the mean square value. Assuming a Gaussian distribution the normalized variable is

$$z = \frac{5 \times 10^{-6}}{\sqrt{1.96 \times 10^{-10}}} = 0.357$$

Then

$$\Pr(z > 1.129) = 1 - \Pr(z < 0.357) = 1 - 0.640 = 0.360$$

(c) Assuming a Rayleigh distribution for the maximum, the value of the Rayleigh variable is

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$$u = \frac{(5 \times 10^{-6})^2}{2(1.96 \times 10^{-10})} = 0.0637$$

Then

$$\Pr(x_{max} > 5 \times 10^{-6}) = \Pr(u > 0.0637) = 1 - \Pr(u < 0.0637) = 1 - 0.06 = 0.94$$

Problem 13.24 illustrates the use of the mean square response to calculate the probability of exceeding a threshold.

13.25 Solve Chapter Problem 13.21, assuming the force is narrowband with a power spectral density given by $S_F(\omega) = \frac{3\times10^{-3}}{2+5\omega^2}$.

Given: m = 20 kg, $\zeta = 0.1$, $\omega_n = 100\ \frac{\text{rad}}{\text{s}}$, $S_F(\omega) = \frac{3\times10^{-3}}{2+5\omega^2}\ \frac{\text{N}^2\cdot\text{s}}{\text{rad}}$

Find: $\overline{x^2}$, Pr(x > 5 mm), Pr(x_{max} > 5 mm)

Solution: (a) The transfer function of the system is

$$H(\omega) = \frac{1}{20(-\omega^2 + 20i\omega + 10000)}$$

The power spectral density of the response is

$$S_x(\omega) = S_F(\omega)|H(\omega)|^2 = \frac{1}{(20)^2[(10000 - \omega^2)^2 + (20\omega)^2]}\left[\frac{3 \times 10^{-3}}{2 + 5\omega^2}\right]$$

The mean square of the response is

$$\overline{x^2} = \int_{-\infty}^{\infty} \frac{7.5 \times 10^{-6}}{[(10000 - \omega^2)^2 + (20\omega)^2](2 + 5\omega^2)} d\omega = 2.519 \times 10^{-12}$$

(b) Assuming a mean of zero, the variance is equal to the mean square value. Assuming a Gaussian distribution the normalized variable is

$$z = \frac{5 \times 10^{-6}}{\sqrt{2.519 \times 10^{-12}}} = 3.15$$

Then

$$\Pr(x > 3.15) = 1 - \Pr(x < 3.15) = 1 - 0.9992 = 0.0008$$

(c) Assuming a Rayleigh distribution

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$$\frac{(5\times10^{-6})^2}{2(2.519\times10^{-12})}=4.96$$

Then

$$\Pr(x_{max}>0.005\text{ m})=\Pr(z>0.637)=1-P(4.96)=1-0.9928=0.00701$$

Problem 13.25 illustrates the mean square value, the Gaussian and the Rayleigh distribution.

13.26 Solve Chapter Problem 13.22, assuming the acceleration is band limited with $\omega_1=10\frac{\text{rad}}{\text{s}}$ and $\omega_2=30\frac{\text{rad}}{\text{s}}$.

FIGURE P13.22

Solution: The transfer function for the acceleration of the block is

$$H(\omega)=\frac{1000\omega i+1\times10^6}{-\omega^2+1\times10^6+1000\omega i}$$

The mean square value due to band limited white noise is

$$\bar{x}^2=\int_{\omega_1}^{\omega_2}1\times10^{-2}|H(\omega)|^2d\omega=\int_{10}^{30}1\times10^{-2}\frac{(1\times10^{12})^2+(1000\omega^3)^2}{(\omega^4-1\times10^6\omega^2+1\times10^{12})^2}d\omega=0.2$$

Problem 13.26 illustrates the response of a system due to band limited white noise.

13.27 A two-degree-of-freedom system has governing differential equations

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\\ddot{x}_2\end{bmatrix}+\begin{bmatrix}5&-2\\-2&2\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}+\begin{bmatrix}200&-100\\-100&300\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}F(t)\\0\end{bmatrix}$$

where $F(t)$ is random with a power spectral density of $S_0=5\times10^{-2}\frac{\text{N}^2\cdot\text{s}}{\text{rad}}$.

(a) Determine the mean square value of x_1.
(b) Determine the mean square value of x_2.

Given: differential equations, S_0

Find: $\bar{x}_1^2$ and $\bar{x}_2^2$

Solution: The Fourier transforms of the differential equations are

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$$\begin{bmatrix} -\omega^2 + 5i\omega + 200 & -2i\omega - 100 \\ -2i\omega - 100 & -\omega^2 + 2i\omega + 300 \end{bmatrix} \begin{bmatrix} X_1(\omega) \\ X_1(\omega) \end{bmatrix} = \begin{bmatrix} F(\omega) \\ 0 \end{bmatrix}$$

and leads to

$$H_1(\omega) = \frac{\begin{vmatrix} 1 & -2i\omega - 100 \\ 0 & -\omega^2 + 2i\omega + 300 \end{vmatrix}}{\begin{vmatrix} -\omega^2 + 5i\omega + 200 & -2i\omega - 100 \\ -2i\omega - 100 & -\omega^2 + 2i\omega + 300 \end{vmatrix}} = \frac{(300 - \omega^2) + 2i\omega}{\omega^4 - 514\omega^2 + 50000 + i(-12\omega^3 + 1900\omega)}$$

and

$$H_2(\omega) = \frac{\begin{vmatrix} -\omega^2 + 5i\omega + 200 & 1 \\ -2i\omega - 100 & 0 \end{vmatrix}}{\begin{vmatrix} -\omega^2 + 5i\omega + 200 & -2i\omega - 100 \\ -2i\omega - 100 & -\omega^2 + 2i\omega + 300 \end{vmatrix}} = \frac{2i\omega + 100}{\omega^4 - 514\omega^2 + 50000 + i(-12\omega^3 + 1900\omega)}$$

The mean square value for x_1 is

$$\bar{x}^2{}_1 = \int_{-\infty}^{\infty} 5 \times 10^{-2} |H_1(\omega)|^2 d\omega$$
$$= \int_{-\infty}^{\infty} 5 \times 10^{-2} \frac{(-\omega^6 + 790\omega^4 - 160400\omega^2 + 3 \times 10^6)^2 + (-10\omega^5 + 2978\omega^3 - 470000)^2}{[(\omega^4 - 514\omega^2 + 50000)^2 + (-12\omega^3 + 1900\omega)^2]^2} d\omega$$
$$= 5.839 \times 10^{-5}$$

The mean square value for x_2 is

$$\bar{x}^2{}_2 = \int_{-\infty}^{\infty} 5 \times 10^{-2} |H_2(\omega)|^2 d\omega$$
$$= \int_{-\infty}^{\infty} 5 \times 10^{-2} \frac{(76\omega^4 - 47600\omega^2 + 5 \times 10^6)^2 + (2\omega^5 + 172\omega^3 + 10000\omega)^2}{[(\omega^4 - 514\omega^2 + 50000)^2 + (-12\omega^3 + 1900\omega)^2]^2} d\omega$$
$$= 1.442 \times 10^{-6}$$

Problem 13.27 illustrates the use of transfer functions to determine the mean square value of the response for a random excitation.

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