CASE 2: EQUAL & REAL EIGENVALUES

For the linear system,

$$\dot{\underline{x}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{x}$$

If $(a-d)^2 + 4bc = 0$, then the eigenvalues of the system will be equal, i.e.,

$$\boxed{\lambda_1 = \lambda_2 = \lambda}$$

Note that if $b \neq 0$ or $c \neq 0$, we can not transform the system into the following form $\dot{\underline{z}} \neq \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} z$

Insteadd, we will have

$$\dot{\underline{z}} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \underline{z}$$ where $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ is called JORDAN BLOCK

Then,

$$\left.\begin{aligned} \frac{dz_1}{dt} &= \dot{z}_1 = \lambda z_1 + z_2 \\ \frac{dz_2}{dt} &= \dot{z}_2 = \lambda z_2 \end{aligned}\right\} \Rightarrow \boxed{z_2 = C e^{\lambda z_1 / z_2}}$$

Equation of Phase Portraits in Canonic Coordinates

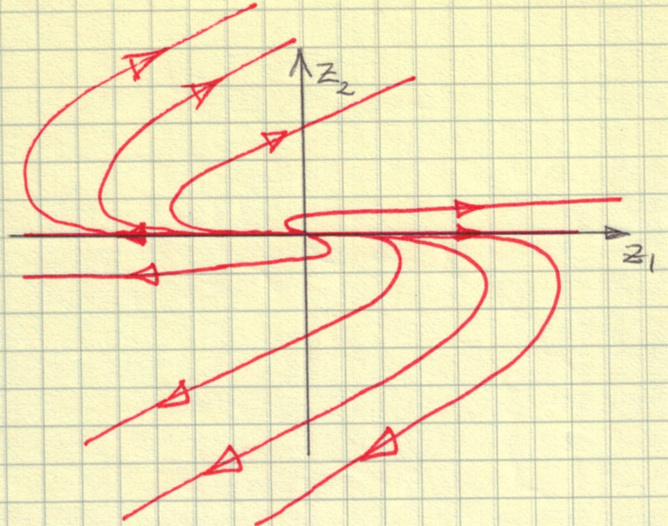
For EQUAL & REAL EIGENVALUES Case, we have 3 different alternatives;

- $\lambda_1 = \lambda_2 = \lambda \neq 0$
- $\lambda_1 = \lambda_2 = \lambda = 0$
- $b = 0$ and $c = 0$ $\Rightarrow$ $\lambda_1 = \lambda_2 = \lambda$ Decoupled Case

(i) $\lambda_1 = \lambda_2 = \lambda \neq 0$ DEGERATED NODE

Example: $\lambda_1 = \lambda_2 = 2$

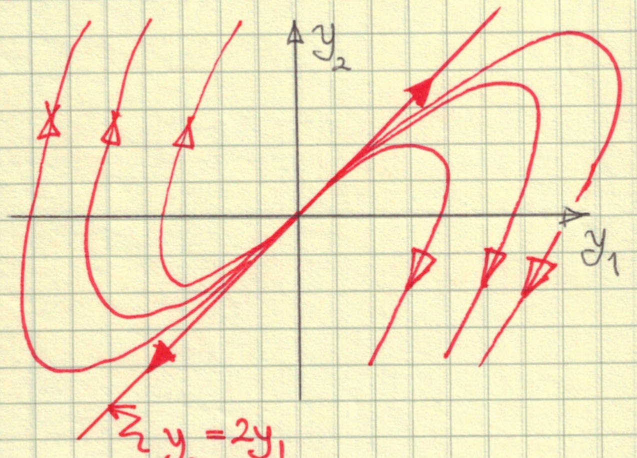
In Canonical Variables Form $\dot{\underline{z}} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \underline{z}$



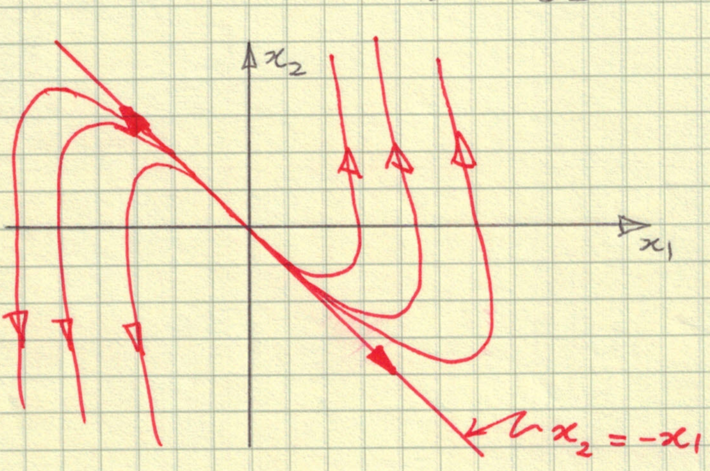
Note that the eigenvalues of the system are positive. (Unstable System)

In Phase Variable Form $\quad \dot{\underline{y}} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \underline{y}$ $\quad$ For $\lambda_1 = \lambda_2 = 2$

$y_2 = 2y_1$



In Arbitrary Form $\quad \dot{\underline{x}} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \underline{x}$ $\quad$ For $\lambda_1 = \lambda_2 = 2 \Rightarrow x_2 = -x_1$



(ii) The eigenvalues of the system are equal and zero, $\lambda_1 = \lambda_2 = 0$

In this case;

$$\dot{\underline{z}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z \quad \Rightarrow \quad \dot{z}_1 = z_2$$

$$\dot{z}_2 = 0 \quad \Rightarrow \quad \boxed{z_2 = c}$$

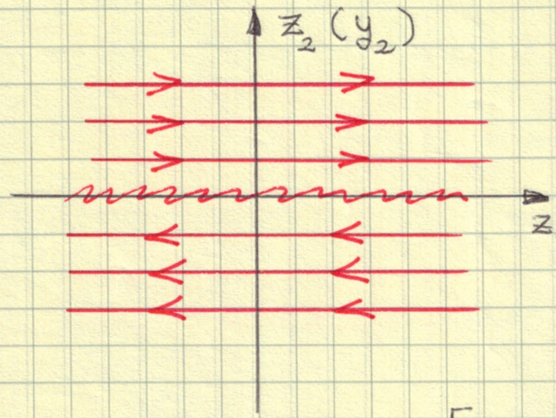
$$\dot{z}_1 = z_2 \quad \Rightarrow \quad z_1 \nearrow \text{ if } z_2 > 0$$

$$z_1 \searrow \text{ if } z_2 < 0$$

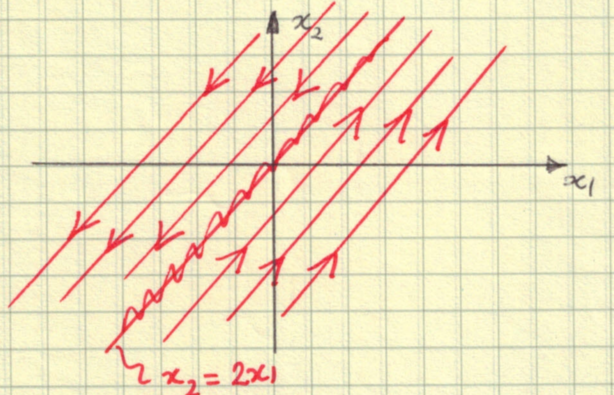
The system in the Phase Variable Form

is exactly the same as Canonical Variable Form, i.e.,

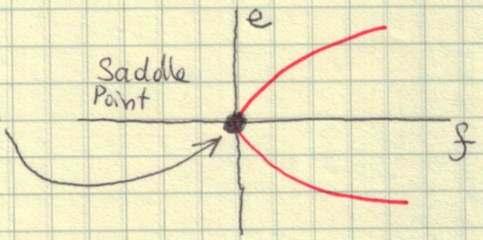
$$\dot{\underline{y}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{y}$$



In Arbitrary Form $\quad \dot{\underline{x}} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 0 \quad \Rightarrow x_2 = 2x$



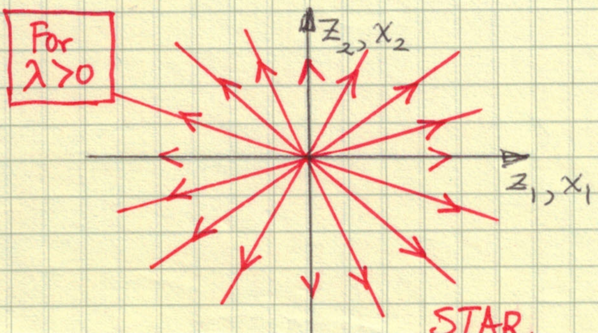
Notice that $\lambda_1 = \lambda_2 = 0$ means

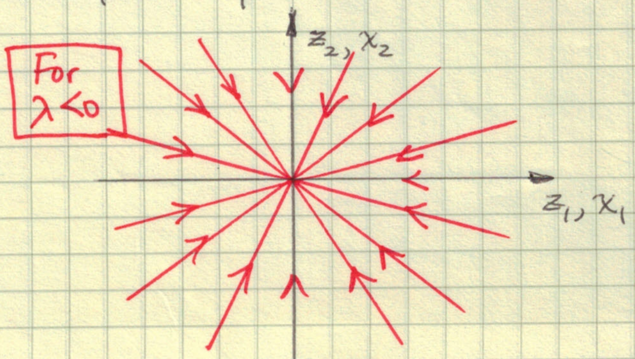


(iii) Diagonal Decoupled Case

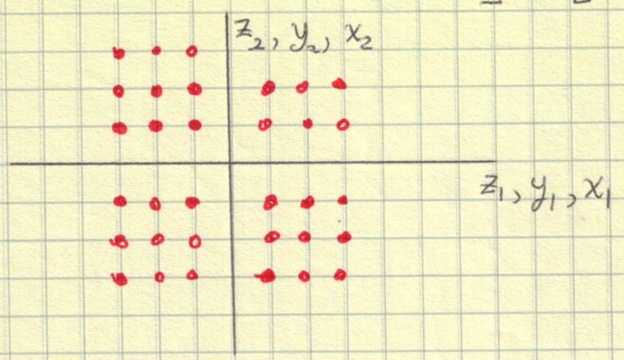
If $b = 0$ and $c = 0$, then the system in Arbitrary Form will be

$$\dot{\underline{x}} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \underline{x} \quad \text{or} \quad \frac{dx_2}{dx_1} = \frac{x_2}{x_1} \Rightarrow \boxed{x_2 = Cx_1}$$





For $\lambda_1 = \lambda_2 = 0$ $\quad \dot{\underline{x}} = \dot{\underline{z}} = \dot{\underline{y}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{x}$ (or $\underline{y}$ or $\underline{z}$)



CASE 3: COMPLEX EIGENVALUES

For the linear (or linearized) system

$$\dot{\underline{x}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{x}$$

If $(a-d)^2 + 4bc < 0$, then the eigenvalues of the system are complex conjugate, i.e.,

$$\boxed{\lambda_{1,2} = -\sigma \pm j\omega}$$

σ : Real Part

ω : Imaginary Part.

In this case, the system is represented in the canonical form as

$$\dot{\underline{z}} = \underline{\Lambda}\, \underline{z} \quad \text{where} \quad \underline{\Lambda} = \begin{bmatrix} -\sigma + j\omega & 0 \\ 0 & -\sigma - j\omega \end{bmatrix}$$

Instead of using complex numbers, we can use MODIFIED CANONICAL FORM. Then the system

$$\dot{\underline{z}} = \underline{\Lambda}_m \underline{z} \quad \text{where} \quad \underline{\Lambda}_m = \begin{bmatrix} -\sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Therefore we can write

$$\boxed{\begin{aligned} \dot{z}_1 &= -\sigma z_1 + \omega z_2 \\ \dot{z}_2 &= -\omega z_1 - \sigma z_2 \end{aligned}}$$

By using POLAR COORDINATES, $z_1 = r\cos\theta$ and

$z_2 = r\sin\theta$

The system is represented by the following equations:

$$\boxed{\begin{aligned} \dot{r} &= -\sigma r \\ \dot{\theta} &= -\omega \end{aligned}}$$

which forms SPIRALS (FOCUS).

The solution of the above differential equation is

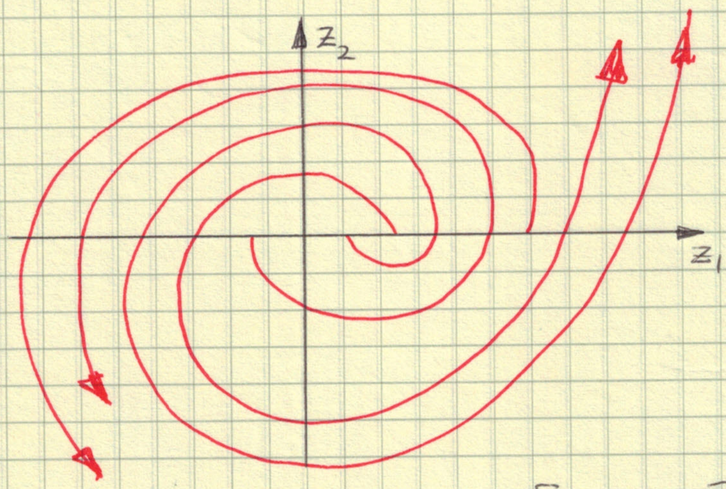
$$\boxed{r = r_0 \, e^{(\theta - \theta_0)\frac{\sigma}{\omega}}}$$

If the real part $\sigma = 0$, then $r = r_0 \Rightarrow$ CENTER (VERTEX - VORTEX)

Example: $\lambda_1, \lambda_2 = 1 \pm 3j$ Therefore $\sigma = 1$, $\omega = -3$

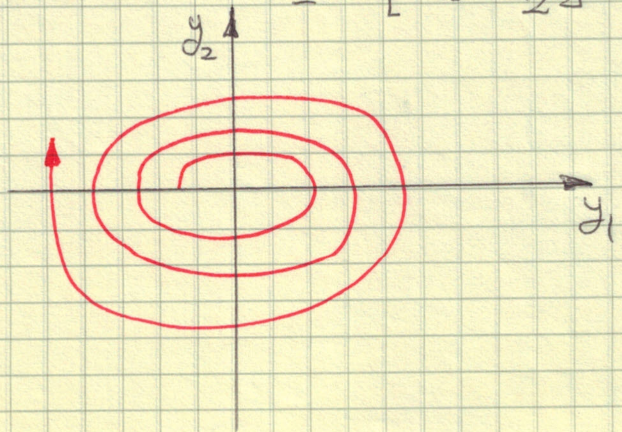
The MODIFIED CANONICAL FORM

$$\dot{\underline{z}} = \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} \underline{z}$$

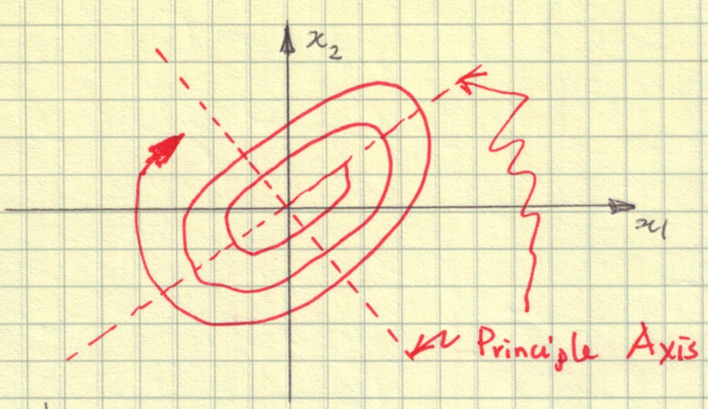


(Counter Clock Wise - CCW)

In Phase Variable Form $\quad \dot{\underline{y}} = \begin{bmatrix} 0 & 1 \\ -10 & 2 \end{bmatrix} \underline{y}$



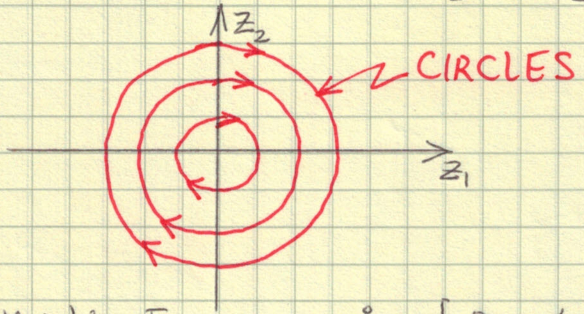
In Arbitrary Form $\quad \dot{\underline{x}} = \begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix} \underline{x}$



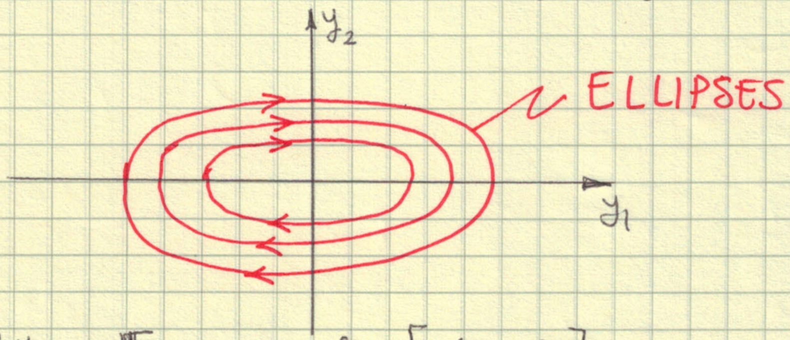
Example:

$$\lambda_{1,2} = \pm 3j$$

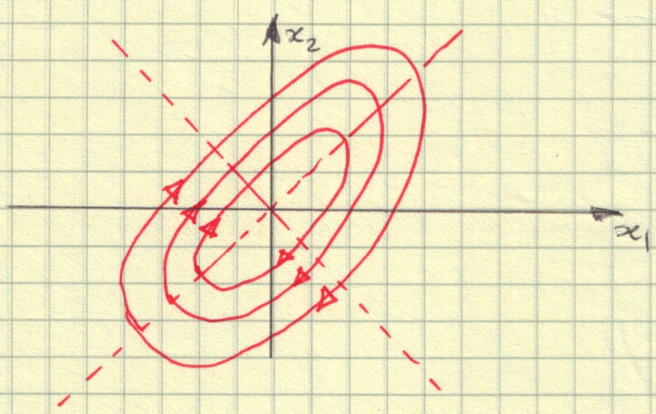
In Modified Canonical Form $\quad \dot{\underline{z}} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \underline{z}$



In Phase Variable Form $\quad \dot{\underline{y}} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \underline{y}$



In Arbitrary Form $\quad \dot{\underline{x}} = \begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix} \underline{x}$



DIRECTIONS FOR PRINCIPLE AXIS

$$\boxed{T = -\frac{b+c}{2d} \pm \sqrt{\left(\frac{b+c}{2d}\right)^2 + 1}}$$

PHASE PLANE ANALYSIS OF NONLINEAR SYSTEMS

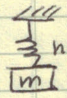
Phase Plane Analysis of linear Systems are valid only in the neighborhood of origin. Because many systems are, by nature, nonlinear. In order to construct the phase plane analysis of nonlinear systems, LOCAL ANALYSIS is necessary which leads us to the PHASE PLANE ANALYSIS of linear systems.

In fact, phase plane analysis of nonlinear systems is GLOBAL, or at least NON-LOCAL STATE PLANE ANALYSIS.

Example: Local Analysis is necessary in the neighborhood of singular points.

DUFFING'S EQUATION is given by

$$\ddot{x} + \omega_0^2 x - k^2 x^3 = 0$$

(

non-linear spring $\rightarrow m\ddot{x} + kx = 0$)
$(k_1 + k_2 x^2)$

In the state space form

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\omega_0^2 x_1 + k^2 x_1^3$$

Now let us find the perturbed system equations;

$$\left.\begin{aligned} \text{let } x_1 &= x_{1e} + \delta x_1 \\ x_2 &= x_{2e} + \delta x_2 \end{aligned}\right\} \text{ Sayfa 5}$$

Jacobian $\Rightarrow \begin{bmatrix} 0 & 1 \\ -\omega_0^2 + 3k^2 x_1^2 & 0 \end{bmatrix}$

Where

$$\underline{x}_{e_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \underline{x}_{e_2} = \begin{bmatrix} \omega_0/k \\ 0 \end{bmatrix}, \quad \underline{x}_{e_3} = \begin{bmatrix} -\omega_0/k \\ 0 \end{bmatrix}$$

Then the perturbed system equations can be written as

$$\delta \dot{x}_1 = \delta x_2$$

$$\delta \dot{x}_2 = -\omega_0^2 \delta x_1 + 3x_{1e}^2 k^2 \delta x_1 = (3x_{1e}^2 k^2 - \omega_0^2)\,\delta x_1$$

For $\underline{x}_{e_1}$:

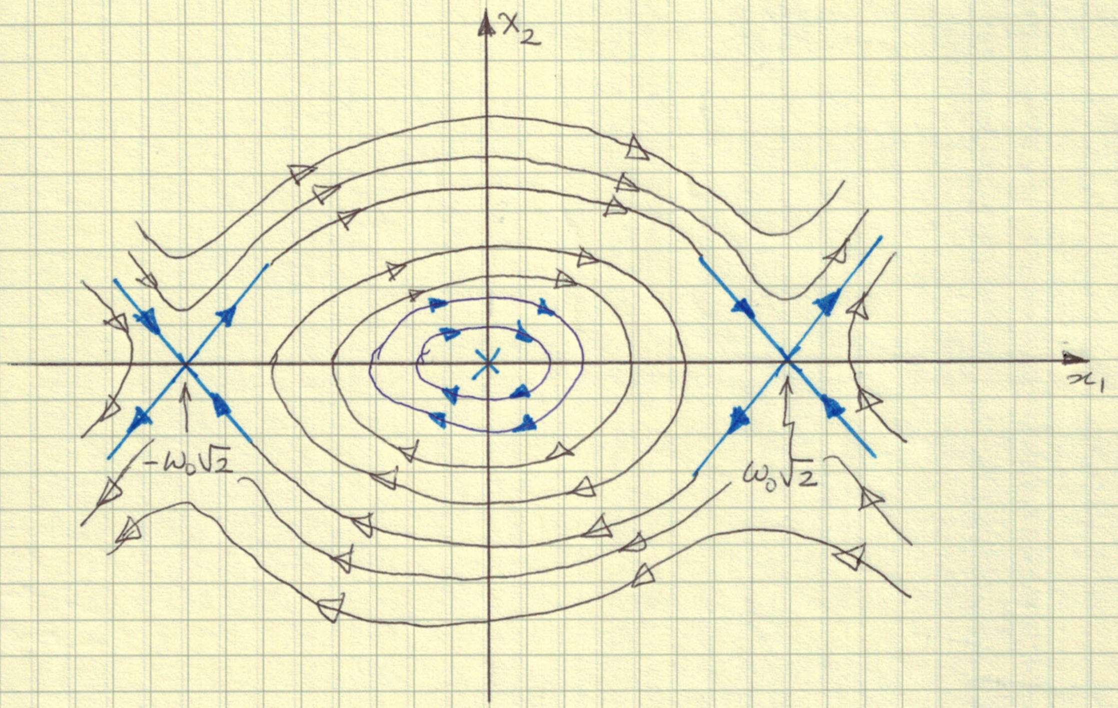
$$\left.\begin{aligned} \delta \dot{x}_1 &= \delta x_2 \\ \delta \dot{x}_2 &= -\omega_0^2 \delta x_1 \end{aligned}\right\} \delta \ddot{x} + \omega_0^2 \delta x = 0 \quad (\lambda_{1,2} = \pm j\omega_0$$

(CENTER)

For $\underline{x}_{e_2}$ and $\underline{x}_{e_3}$:

$$\left.\begin{aligned}\delta\dot{x}_1 &= \delta x_2\\ \delta\dot{x}_2 &= 2\omega_0^2\,\delta x_1\end{aligned}\right\} \Rightarrow \quad \delta\ddot{x} - 2\omega_0^2\,\delta x = 0 \qquad \lambda_{1,2} = \pm\,\omega_0\sqrt{2}$$

(SADDLE)



THEOREM: The structures of the trajectories in the neighborhood of the singular points at the origin for the linear differential equation

$$\dot{\underline{x}} = \underline{A}\,\underline{x}$$

and the nonlinear differential equation

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{g}(\underline{x})$$

are similar in cases of NODE, SPIRAL and SADDLE point provided that

$$\underline{g}(\underline{x}) = \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2)\end{bmatrix} = \begin{bmatrix} 0 & (x_1^2+x_2^2) \\ 0 & (x_1^2+x_2^2)\end{bmatrix}$$

Example:

$$\dot{x}_1 = \beta x_2 + \mu x_1 (x_1^2 + x_2^2)$$

$$\dot{x}_2 = -\beta x_1 + \mu x_2 (x_1^2 + x_2^2)$$

[The equation of motion of a nonlinear system]

$$\dot{x}_1 = 0 = \beta x_2 + \mu x_1 (x_1^2 + x_2^2)$$
$$\dot{x}_2 = 0 = -\beta x_1 + \mu x_2 (x_1^2 + x_2^2)$$

will give us singular point~~s~~, which is

$$\underline{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The perturbed system equation in the neighborhood of $\underline{x}_e$ is

$$\delta \dot{x}_1 = \beta \delta x_2$$
$$\delta \dot{x}_2 = -\beta \delta x_1$$

The Jacobian Matrix is $\underline{J}_{\underline{x}} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$

Since the eigenvalues of $\underline{J}_{\underline{x}}$ are $\pm \beta i$, we have a CENTER.
However the nonlinear system will have a different type of behavior.
If we transform the nonlinear system into polar coordinates,

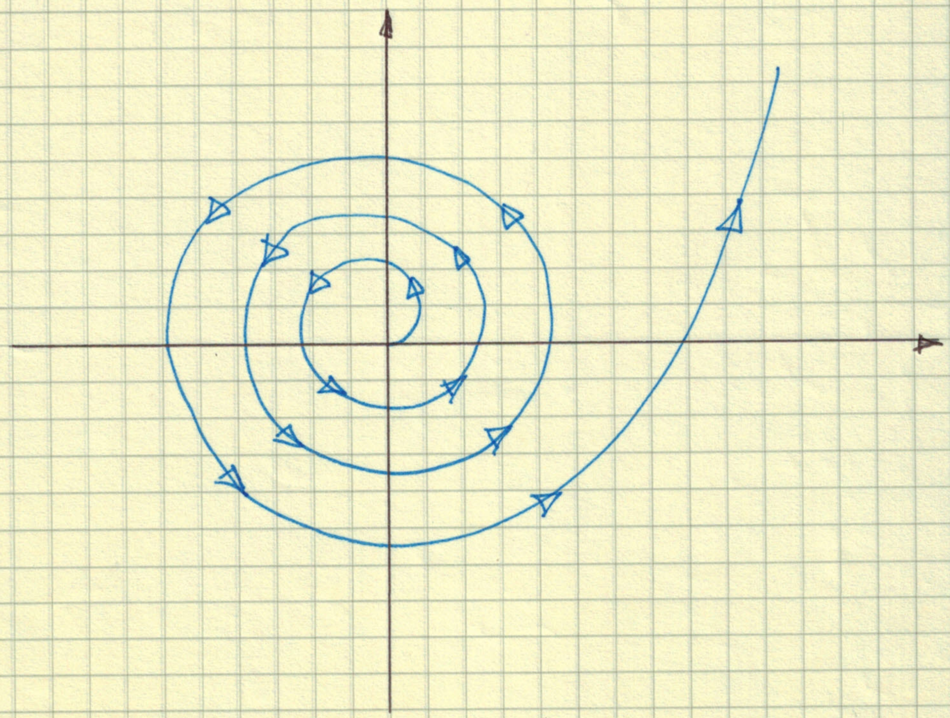
Let $x_1 = R\cos\theta \Rightarrow \dot{x}_1 = \dot{R}\cos\theta - R\dot{\theta}\sin\theta$

$x_2 = R\sin\theta \Rightarrow \dot{x}_2 = \dot{R}\sin\theta + R\dot{\theta}\cos\theta$

Then in the new coordinate system, we have

$$\boxed{\begin{aligned} \dot{R} &= \mu R^3 \\ \dot{\theta} &= \beta \end{aligned}}$$

which will give us UNSTABLE SPIRAL LIKE BEHAVIOUR.



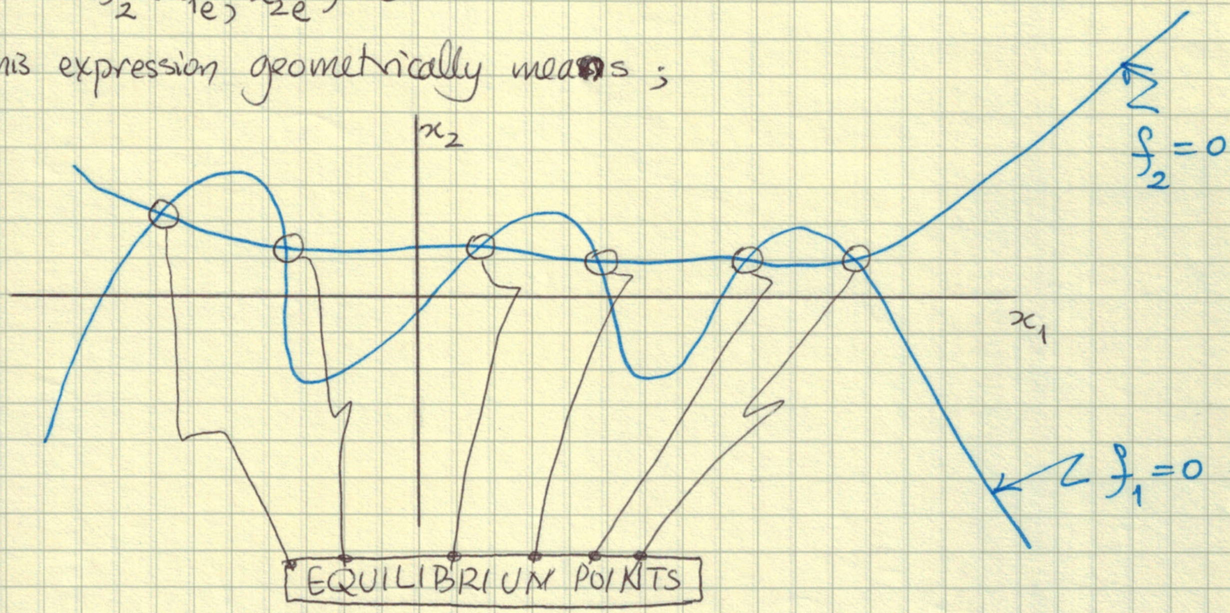
Since we are interested in equilibrium points of nonlinear systems, we have to solve two nonlinear functions simultaneously. For the system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

the equilibrium points $\underline{x}_e$ can be found, from

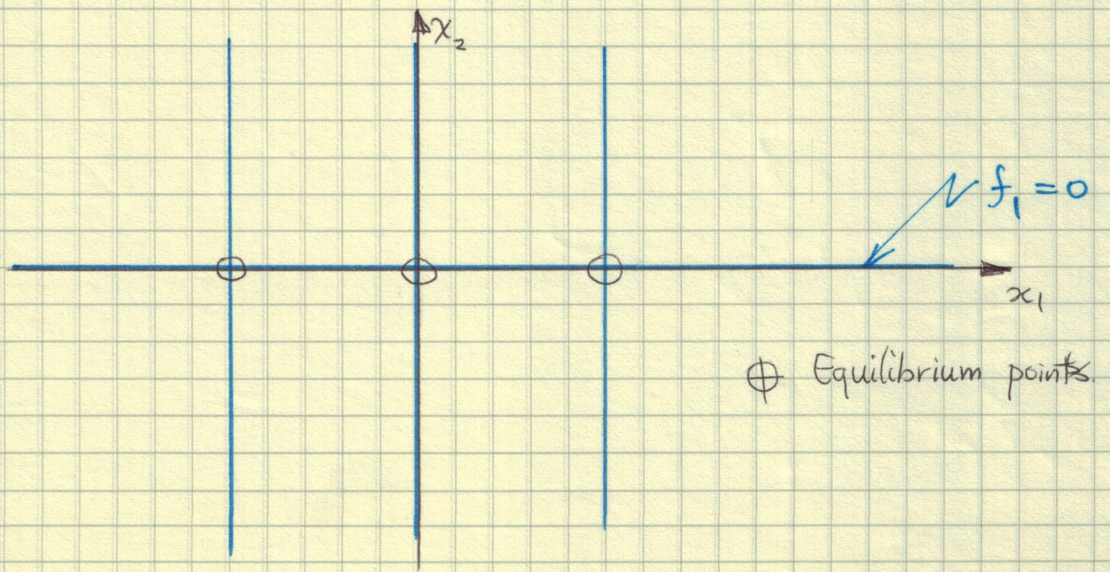
$$f_1(x_{1e}, x_{2e}) = 0$$
$$f_2(x_{1e}, x_{2e}) = 0$$

This expression geometrically means ;



For the previous example (DUFFING'S EQUATION) :

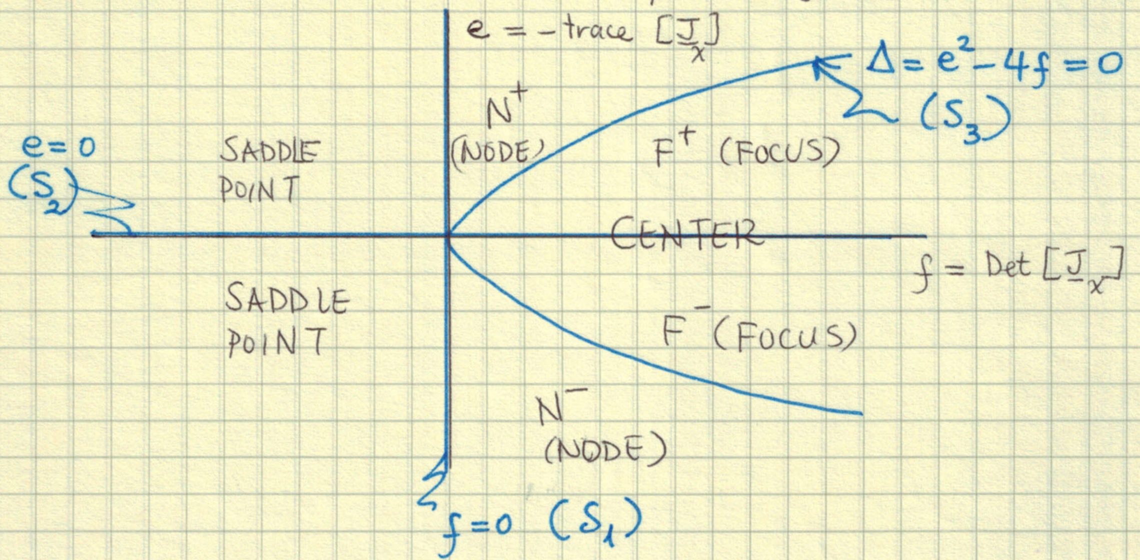
$$f_1(x_1, x_2) = x_2 = 0$$
$$f_2(x_1, x_2) = -\omega_0^2 x_1 + k^2 x_1^3 = 0$$



For the linear (or linearized) system

$$\dot{\underline{x}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{x} = \underline{J}_{\underline{x}} \underline{x}$$

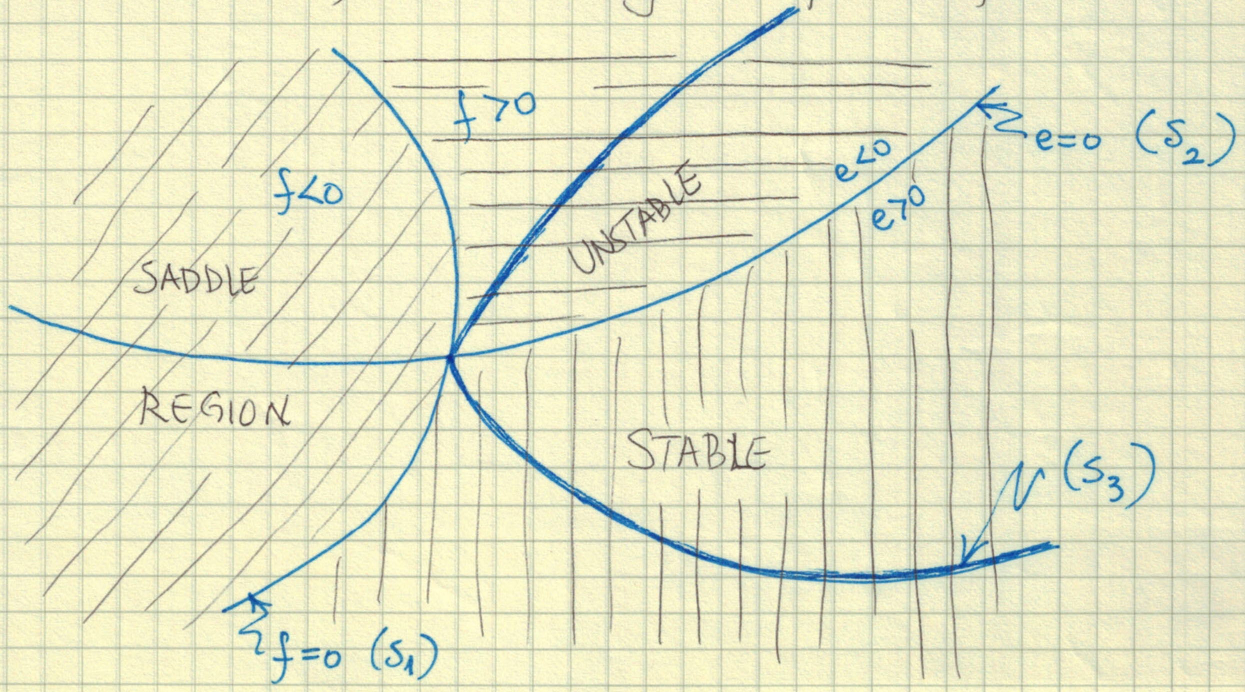
The characteristic equation is $\lambda^2 + e\lambda + f = 0$ where $e = -\text{trace}[\underline{J}]$; ↳ Köşegenlerin toplamı
and $f = \det[\underline{J}]$ are axis of stability regions, i.e.,



Note that we have 3 different cases which correspond to 3 different curves. These are

- $e = -\text{trace}[\underline{J}_x] = -(a+d) = 0 \implies S_2$ curve
- $f = \det[\underline{J}_x] = ad - bc = 0 \implies S_1$ curve
- $\Delta = e^2 - 4f = 0 \implies S_3$ curve

In the nonlinear case, these curves may be interpreted as follows:



Example: Consider again DUFFING's Equation

$$\ddot{x} + \omega_0^2 x - k^2 x^3 = 0$$

In the phase variable form;

$$x_1 = x \text{ and } x_2 = \dot{x} \Rightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega_0^2 x_1 + k^2 x_1^3 \end{aligned}$$

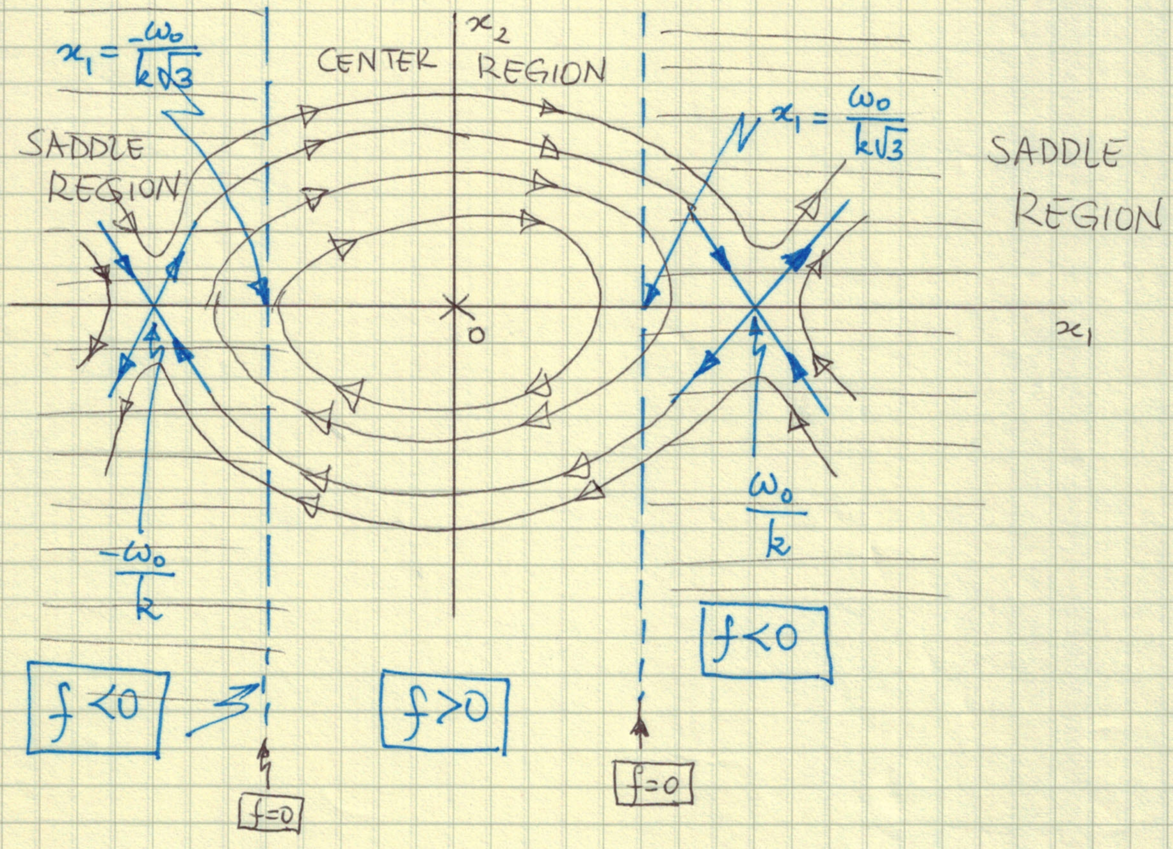
The Jacobian matrix is

$$\underline{J}_{\underline{x}} = \left[\begin{array}{c|c} 0 & 1 \\ -\omega_0^2 + 3k^2 x_1^2 & 0 \end{array}\right]$$

$\det[\underline{J}_{\underline{x}}] = \omega_0^2 - 3k^2 x_1^2 = 0 = f$ will give us $(S_1) \Rightarrow x_1 = \frac{\pm\omega_0}{k\sqrt{3}}$

$\text{trace}[\underline{J}_{\underline{x}}] = 0 = e \Rightarrow (S_2)$ WHOLE PHASE PLANE

$\Delta = e^2 - 4f = 0 - 4f = 0 \Rightarrow f = 0 \Rightarrow S_3$



Note that

$$\left.\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega_0^2 x_1 + k^2 x_1^3\end{aligned}\right\} \Rightarrow \frac{dx_2}{dx_1} = \frac{-\omega_0^2 x_1 + k^2 x_1^2}{x_2}$$

Then, $x_2\, dx_2 = (-\omega_0^2 x_1 + k^2 x_1^3)\, dx_1$

$$\boxed{\frac{1}{2} x_2^2 = \frac{1}{4} k^2 x_1^4 - \frac{1}{2}\omega_0^2 x_1^2 + C}$$

To find the value of c (which is the value at separation points),

For $x_1 = \pm \frac{\omega_0}{k} \Rightarrow x_2 = 0$

Then,

$$0 = \left[\frac{1}{4} \cdot k^2 \left(\frac{\omega_0^2}{k^2}\right)^{2} - \frac{1}{2}\omega_0^2 \cdot \left(\frac{\omega_0^2}{k^2}\right) + C\right]$$

$$\boxed{C = \frac{\omega_0^4}{4k^2}}$$

Then

$$\frac{1}{2} x_2^2 = \frac{1}{4} k^2 x_1^4 - \frac{1}{2}\omega_0^2 x_1^2 + \frac{\omega_0^4}{4k^2}$$

$$\boxed{x_2 = \pm \frac{k}{\sqrt{2}} \left[x_1^2 - \frac{\omega_0^2}{k^2}\right]}$$ Phase Plane Trajectory Equation.

Example: Consider the following nonlinear system

$$\left.\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= -2x_2 + x_1 x_2\end{aligned}\right\} J_x = \begin{bmatrix} x_2 - 1 & x_1 \\ x_2 & x_1 - 2 \end{bmatrix}$$

The equilibrium points can be found as

$$\underline{x}_{e_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{x}_{e_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The perturbed system equation is

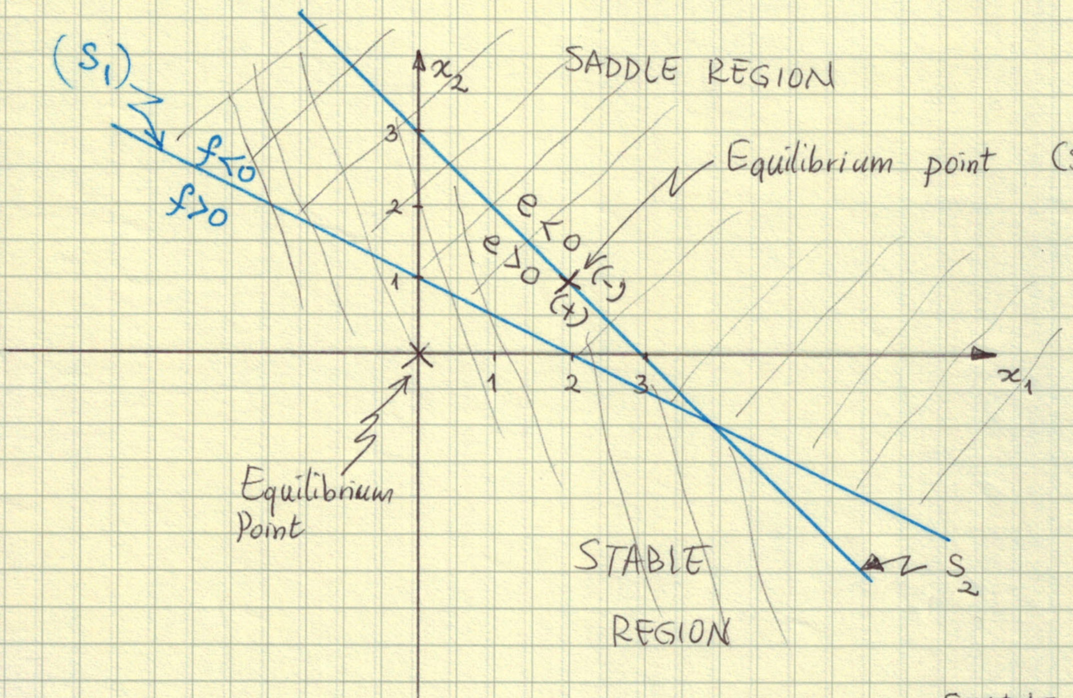
$$\delta\dot{x}_1 = (x_{2e} - 1)\,\delta x_1 + x_{1e}\,\delta x_2$$
$$\delta\dot{x}_2 = x_{2e}\,\delta x_1 + (x_{1e} - 2)\,\delta x_2$$

S_1 : $f = \det[\underline{J}_x] = (x_2-1)(x_1-2) - x_1 x_2 = 0$

S_2 : $e = -\text{trace}[\underline{J}_x] = -(x_2 - 1 + x_1 - 2) = 0$

S_3 : $\Delta = e^2 - 4b = (x_1 + x_2 - 3)^2 - 4(2 - x_1 - 2x_2) = 0$

Notice that $\underline{J}_x = \begin{bmatrix} x_2 - 1 & x_1 \\ x_2 & x_1 - 2 \end{bmatrix}$



For $\underline{x}_{e_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -2x_2 \end{matrix} \Bigg\} \Rightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -2 \end{matrix} \Bigg\}$ STABLE NODE (N^+)

x_1 and x_2 axes are the eigenvectors (CANONICAL Rep.).

For $\underline{x}_{e_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\begin{matrix} \dot{x}_1 = 2x_2 \\ \dot{x}_2 = x_1 \end{matrix} \Bigg\} \Rightarrow \lambda_{1,2} = \pm\sqrt{2}$ SADDLE

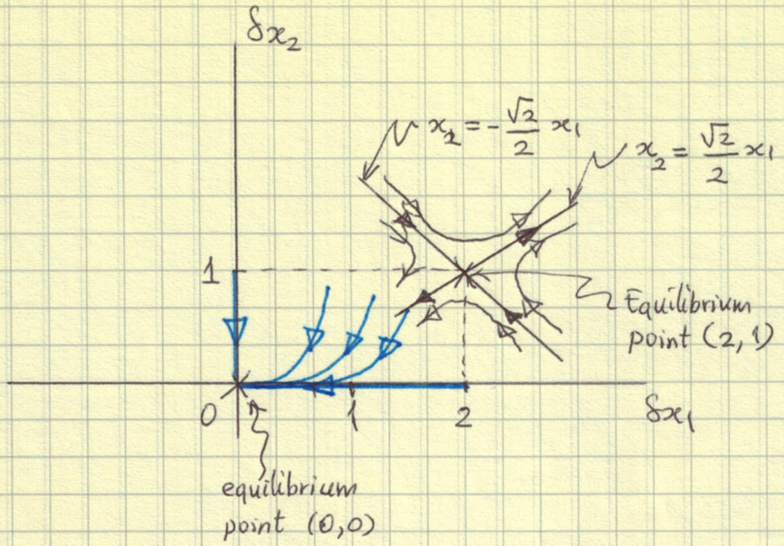
$x_2 = \pm \frac{\sqrt{2}}{2} x_1$ (Eigenvectors are also symmetrical)

The deviation equation (perturbed system)

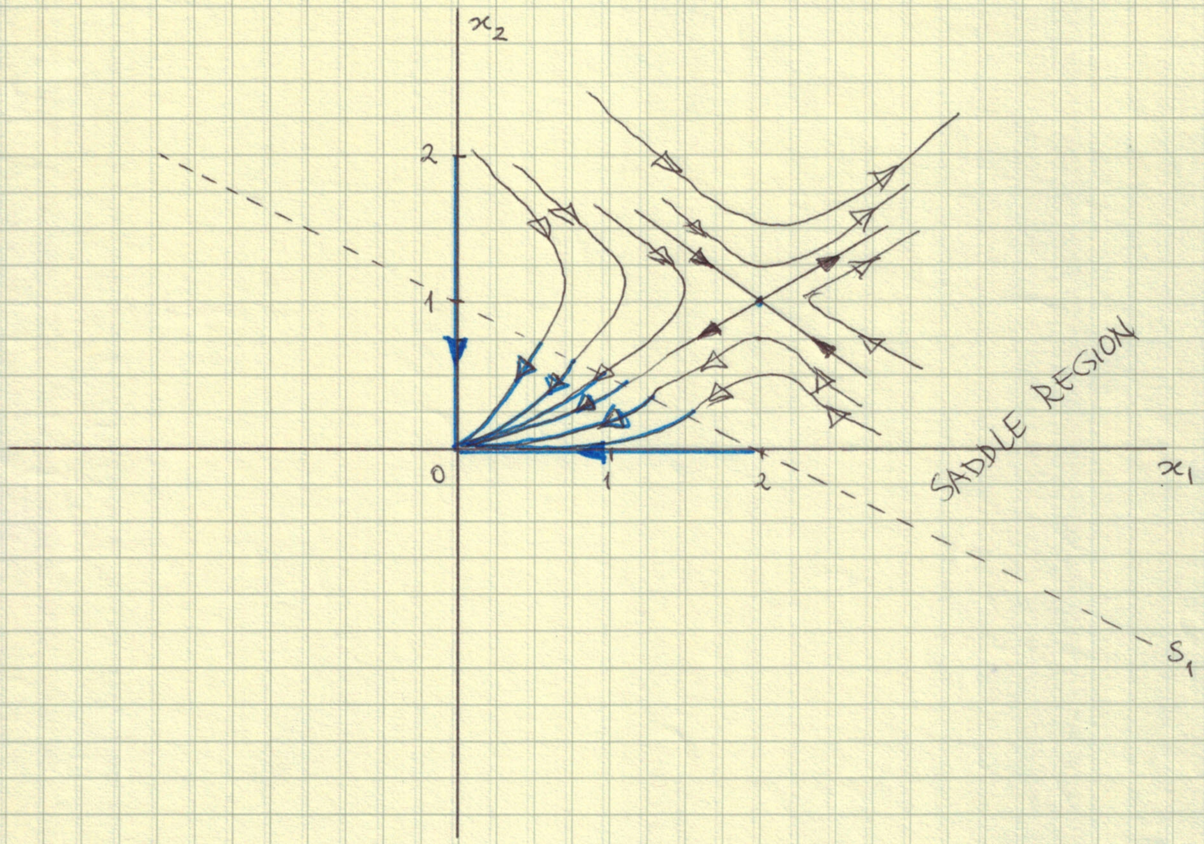
$$\delta\dot{x}_1 = (x_{2e} - 1)\,\delta x_1 + x_{1e}\,\delta x_2$$

$$\delta\dot{x}_2 = x_{2e}\,\delta x_1 + (x_{1e} - 2)\,\delta x_2$$

ii.



Note that by using the deviation equation (perturbed system equations), we can only find local phase plane. (Linear System)



To find the state plane (PHASE PLANE) TRAJECTORY Equations;

$$\frac{dx_2}{dx_1} = \frac{x_2(x_1-2)}{x_1(x_2-1)}$$

Then,

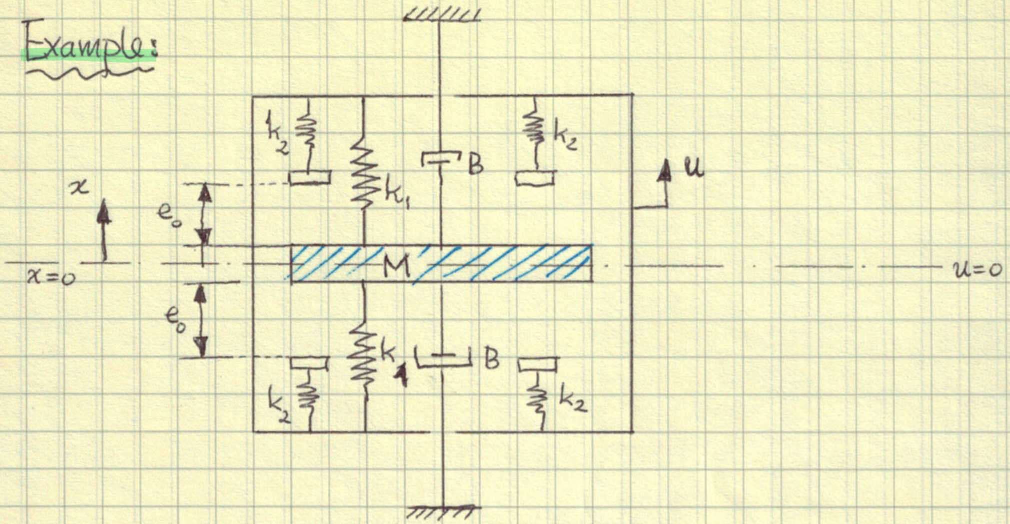
$$\frac{x_2-1}{x_2}dx_2 = \frac{x_1-2}{x_1}dx_1 \Rightarrow x_2 - \ln x_2 = x_1 - 2\ln x_1 + C$$

$$x_2 - x_1 = \ln\left(\frac{x_2}{x_1^2}\right) + C$$

Note that for $x_1 = 2$, $x_2 = 1$, then $C = -1 - \ln(1/4)$

$$\boxed{x_2 - x_1 = \ln\left(\frac{4x_2}{x_1^2}\right) - 1}$$ Phase Plane Trajectories.

Example:



Consider the system given on the left.

Depending on the difference between u and x, we get different equation of motion, which can be given in 3 different cases as follows:

$$M\ddot{x} + 2B\dot{x} + 2k_1(x-u) = 0 \qquad \text{for} \qquad |x-u| \leq e_0$$

$$M\ddot{x} + 2B\dot{x} + 2(k_1+k_2)(x-u-e_1) = 0 \quad \text{for} \quad x-u \geq e_0$$

$$M\ddot{x} + 2B\dot{x} + 2(k_1+k_2)(x-u+e_1) = 0 \quad \text{for} \quad x-u \leq e_0$$

The value of e_1 can be computed from the Potential Energy of the system.

$$E_p = \frac{1}{2}(2k_1)(x-u)^2 + \frac{1}{2}(2k_2)(x-u-e_0)^2$$

Then

$$\frac{dE_p}{dx} = 2k_1(x-u) + 2k_2(x-u-e_0)$$

$$= 2k_1(x-u) + 2k_2(x-u) - 2k_2 e_0$$

$$= 2(k_1+k_2)\left[x-u-\frac{k_2}{k_1+k_2}e_0\right]$$

Define now ;

$$e_1 \triangleq \frac{k_2}{k_1+k_2}e_0 < e_0$$

Now let $e = u - x$

Then the equation of motion is

$$M\ddot{x} + 2B\dot{x} = F(e)$$

where

$$F(e) = \begin{cases} 2k_1 e & |e| \leq e_0 \\ 2(k_1+k_2)(e+e_1) & e \leq e_0 \\ 2(k_1+k_2)(e-e_1) & e \geq e_0 \end{cases}$$

We can also express the equation of motion as;

$$T\ddot{x} + \dot{x} = f(e)$$

where

$$f(e) \triangleq F(e)/2B$$

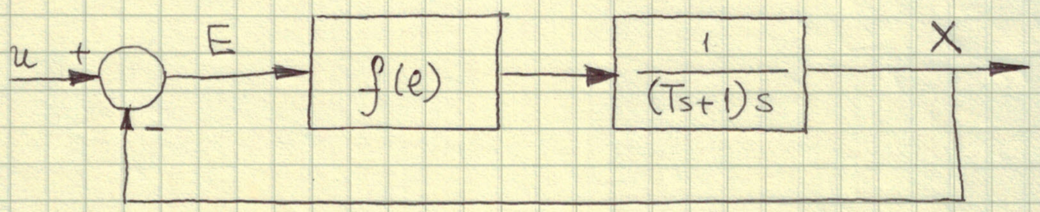
$$T \triangleq M/2B$$

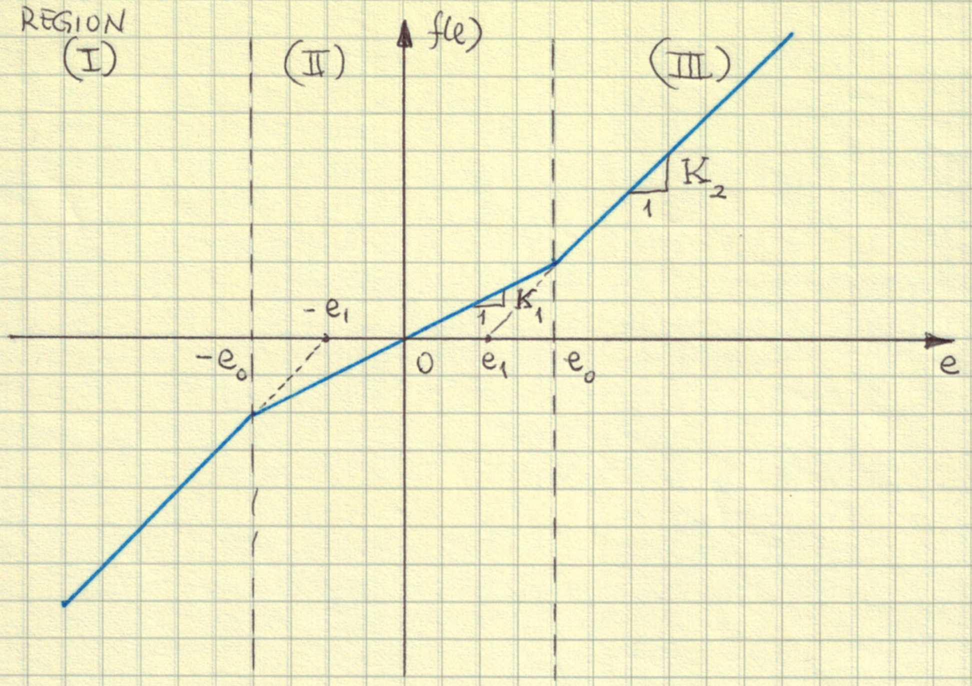
and

$$f(e) = \begin{cases} K_1 e & |e| \leq e_0 \\ K_2(e-e_1) & e \geq e_0 \\ K_2(e+e_1) & e \leq e_0 \end{cases}$$

where $K_1 = k_1/B$ and $K_2 = (k_1+k_2)/B$

We assume that $K_1 < K_2$. The system can be given in block diagram form as follows





Let us now analyze the system for different types of inputs.

STEP INPUT CASE

$u = u_0$ (Step Input , u_0 = step size ($\uparrow u$, u_0 $\rightarrow$)

$\dot{u} = 0$
$\ddot{u} = 0$ } (Since we have constant value , u_0)

Then

$$e = u - x \Rightarrow e = u_0 - x$$
$$\dot{e} = -\dot{x}$$
$$\ddot{e} = -\ddot{x}$$

The system equation becomes

$$T\ddot{x} + \dot{x} = f(e) \Rightarrow T.(-\ddot{e}) + (-\dot{e}) = f(e)$$

or

$$\boxed{T\ddot{e} + \dot{e} = -f(e)}$$

Let $x_1 = e$ and $x_2 = \dot{e}$. Then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-1}{T}\left[x_2 + f(x_1)\right]$$

Equilibrium Points:

$$\dot{x}_1 = 0 \Rightarrow \boxed{x_{2e} = 0}$$

$$\dot{x}_2 = 0 \Rightarrow \frac{-1}{T}\left[x_2 + f(x_1)\right] = 0 \Rightarrow \underline{f(x_{1e}) = 0}$$

From the graph of $f(e)$ versus e, we see that $\boxed{x_{1e} = 0}$

REGION (II):

$f(x_1) = K_1 x_1$, then

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-1}{T}\left[x_2 + K_1 x_1\right] \end{aligned}\right\} \Rightarrow$$

$$\lambda^2 + \frac{1}{T}\lambda + \frac{K_1}{T} = 0 \quad \text{(Characteristic Equation)}$$

$$\Delta = \frac{1}{T^2}\left(1 - 4K_1 T\right)$$

In this region we have $\underline{x}_{1e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Then

If $K_1 > \frac{1}{4T}$ $\Rightarrow$ we have F^+ (STABLE FOCUS) SPIRAL

If $K_1 < \frac{1}{4T}$ $\Rightarrow$ we have N^+ (STABLE NODE).

Notice that the eigenvalues are

$$\lambda_{1,2} = \underbrace{\frac{-1}{2T}}_{\text{Real and positive.}} \pm \frac{\sqrt{\Delta}}{2}$$

REGION (III)

$$f(x_1) = K_2(x_1 - e_1)$$

The equilibrium point is

$x_{1e} = e_1$ which is in REGION (II) - VIRTUAL

The state equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-1}{T}\left[K_2 x_1 + x_2 - K_2 e_1\right]$$

The characteristic equation is $\lambda^2 + \frac{1}{T}\lambda + \frac{K_2}{T} = 0$

The eigenvalues are

$$\lambda_{1,2} = \frac{-1}{2T} \pm \frac{\sqrt{\Delta}}{2} \qquad \text{where } \Delta = \frac{1}{T^2}\left(1 - 4K_2 T\right)$$

For $K_2 > \frac{1}{4T}$ $\Rightarrow$ we have F^+ (STABLE FOCUS) SPIRAL

$K_2 < \frac{1}{4T}$ $\Rightarrow$ we have N^+ (STABLE NODE)

We know that $K_1 < K_2$. Then there are 3 possibilities, which are

(F_I^+), F_{II}^+, (F_{III}^+)

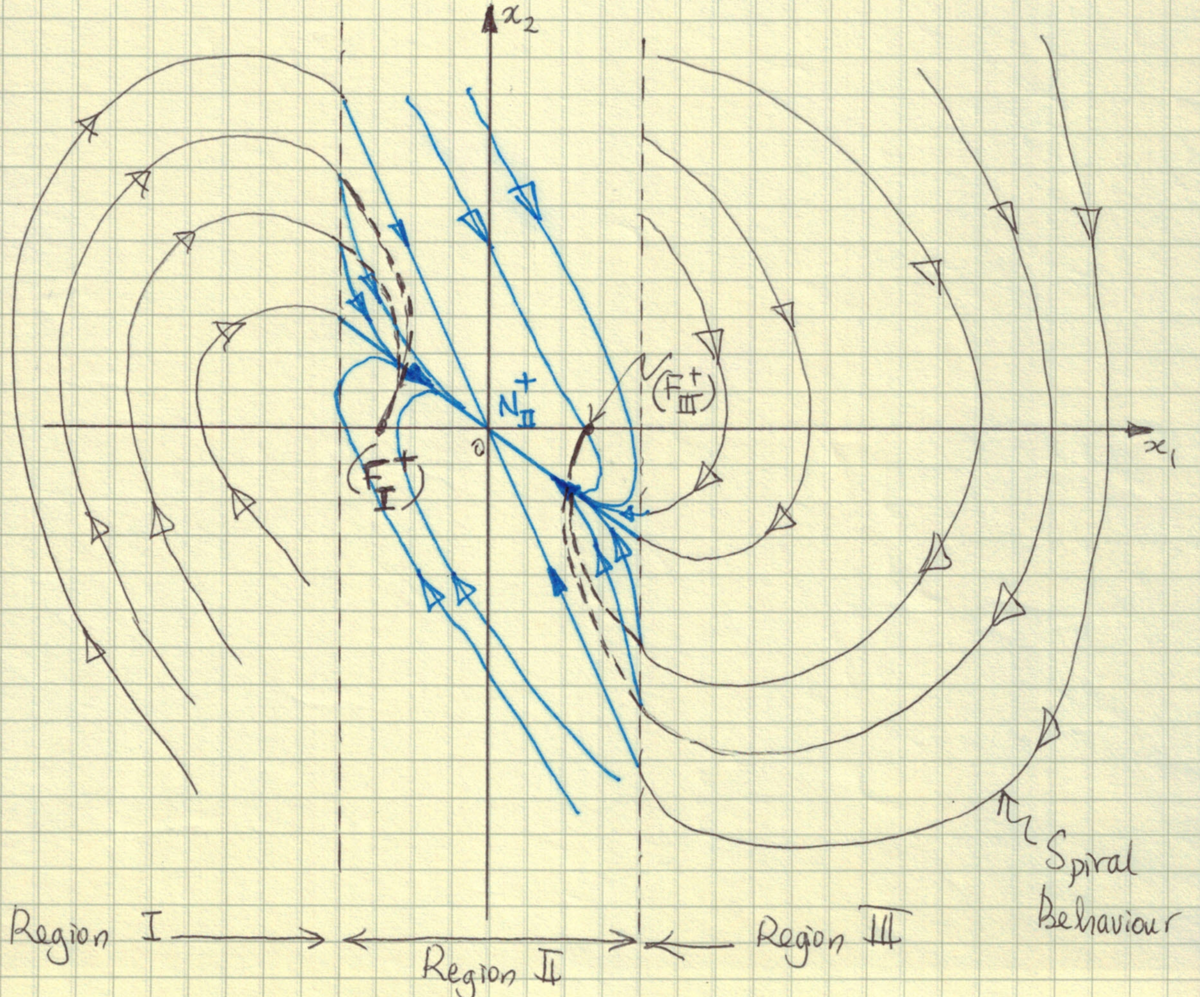
$\boxed{(F_I^+),\ N_{II}^+,\ (F_{III}^+)}$

(N_I^+), N_{II}^+, (N_{III}^+)

(F_I^+) : Virtual Focus (stable) in Region I

(N_{III}^+) : Virtual Stable Node in Region III.

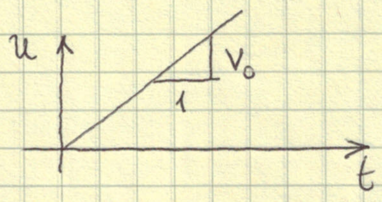
$\rightarrow$ Only this choice satisfies $K_1 < K_2$. Then,



Now let us consider another type of input;

RAMP INPUT:

In this case,



$$u = V_0 t$$

which yields

$$\dot{u} = V_0 \text{ and}$$

$$\ddot{u} = 0$$

Since $e = u - x$, then $\dot{e} = V_0 - \dot{x}$ and $\ddot{e} = -\ddot{x}$.

On the other hand,

$$T\ddot{x} + \dot{x} = f(e) \Rightarrow -T\ddot{e} + V_0 - \dot{e} = f(e) \qquad (*)$$

Re-arranging equation (*), we obtain

$$\ddot{e} = -\frac{1}{T}\left[\dot{e} - V_0 + f(e)\right]$$

Note that the steady-state value will be;

$$\dot{e} = 0 \text{ and } \ddot{e} = 0 \text{, then } \boxed{f(e) = V_0}$$

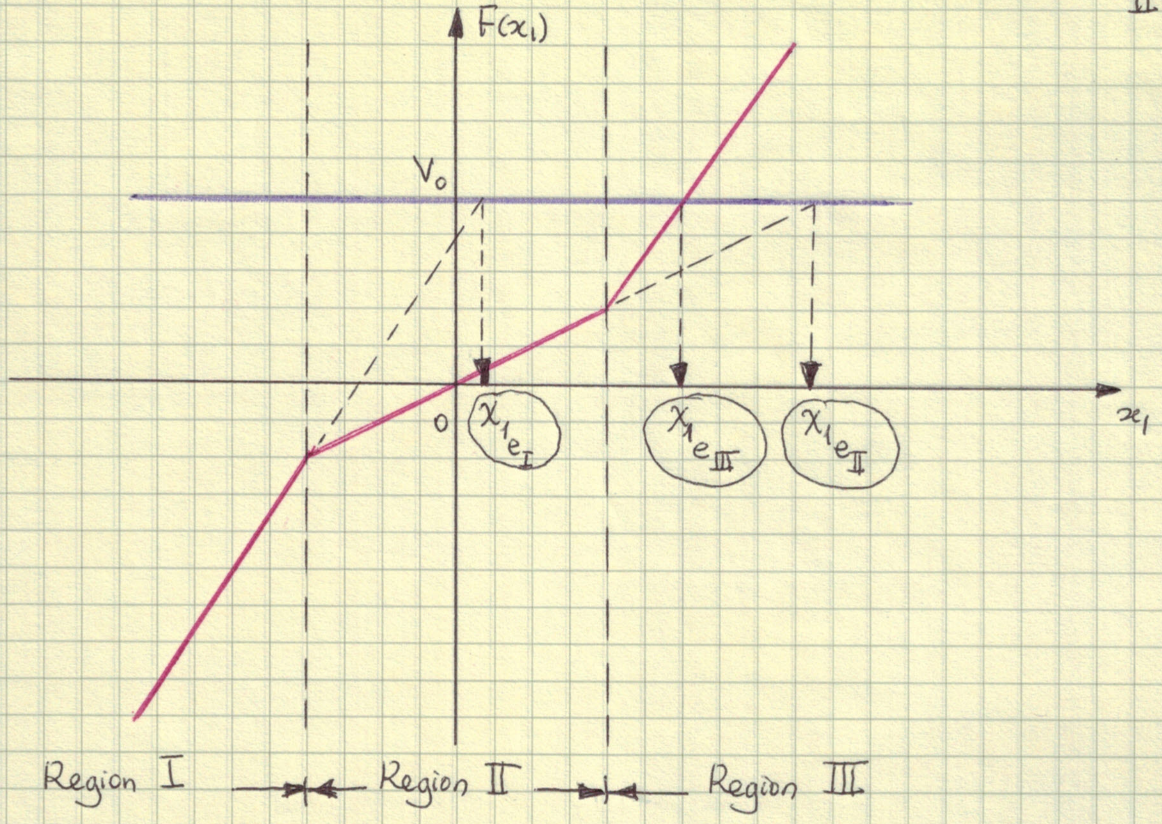
Now let

$$\left.\begin{array}{l} x_1 = e \\ x_2 = \dot{e} \end{array}\right\} \Rightarrow \frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\frac{1}{T}\left[x_2 - \left(V_0 - F(x_1)\right)\right]$$

The equilibrium point now will become

$$\boxed{\begin{array}{l} x_{2_e} = 0 \\ F(x_1) = V_0 \end{array}} \quad \begin{array}{l} (\text{From } \dot{x}_1 = 0) \\ (\text{From } \dot{x}_2 = 0 \text{ and } \dot{x}_1 = 0) \end{array}$$



Because of steady state value of $F(x_1) = V_0$ (in other words, equilibrium condition), the equilibrium points of each region is changed. That is

x_{1e_I} : Equilibrium point of Region I

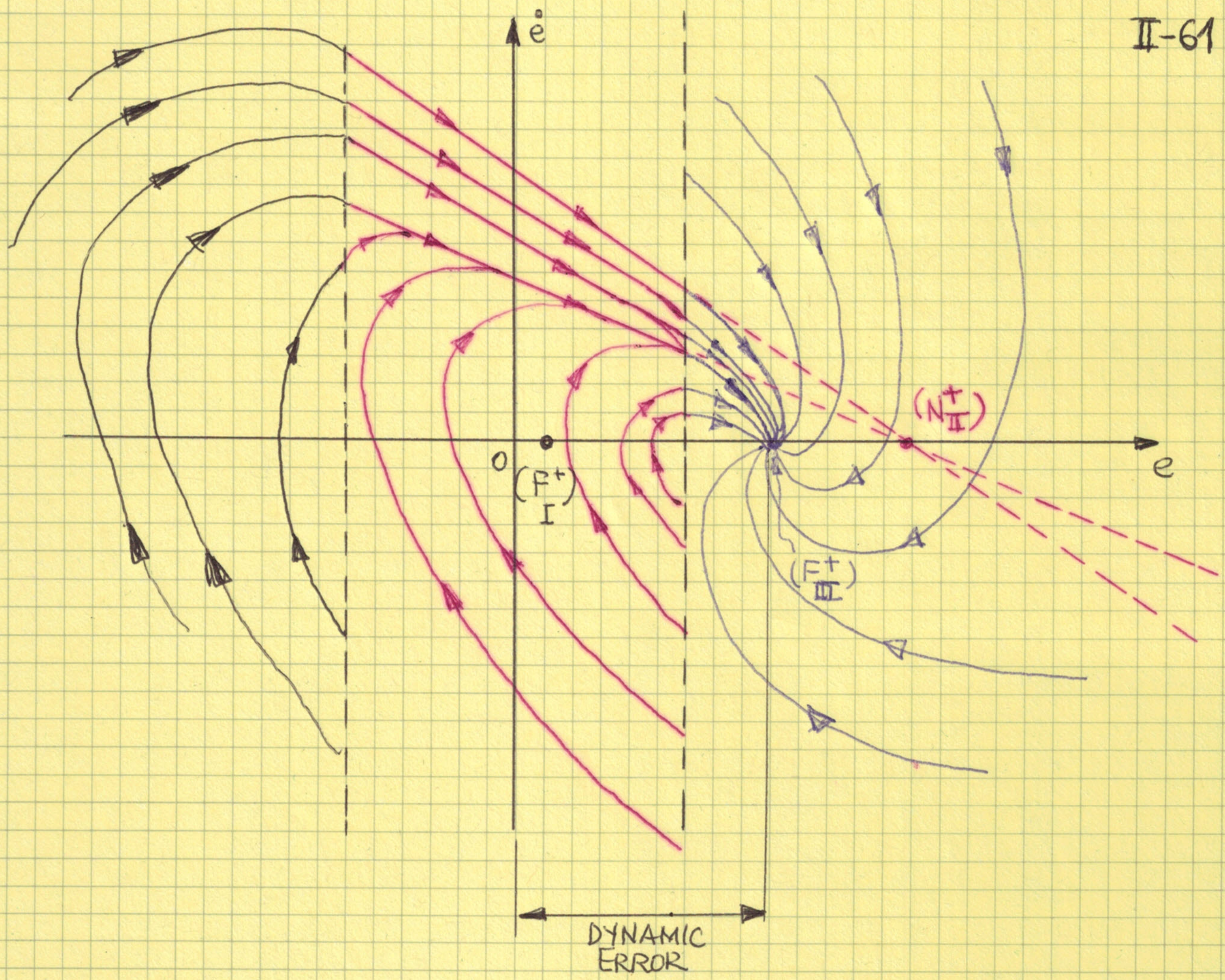
$x_{1e_{II}}$: Equilibrium point of Region II

$x_{1e_{III}}$: Equilibrium point of Region III

It can be shown that

x_{1e_I}, and $x_{1e_{III}}$ are STABLE FOCUS in virtual regions. In short, we have now (F_I^+), (F_{III}^+). On the other hand, $x_{1e_{II}}$ is a STABLE NODE, that is, (N_{II}^+) in virtual region.

The Phase Portraits of the system for RAMP INPUT are shown in the next graph.



ė
e
0
(F_I^+)
(N_{II}^+)
(F_{III}^+)
DYNAMIC ERROR

LIMIT CYCLES AND THEOREMS ON LIMIT CYCLES

In linear second-order autonomous systems, oscillations can take place for only particular combination of system parameters. Slight changes in system parameters will destroy the oscillation. When oscillations occur, the resulting state space trajectories will be closed curves surrounding the origin such as the trajectory for a center. The size of the closed curve, and hence the amplitude of oscillation, is not fixed and changes directly with the size of initial conditions.

In nonlinear systems, oscillations of the above type can also occur. But what is more interesting is that there can be oscillations that are independen of the size of the initial conditions. These are the oscillations of the LIMIT-CYCLE type which is the basis for any practical oscillator. A further property of limit-cycle type oscillations is that the oscillation is much less sensitive to system parameter variations. In particular, there usually exist finite ranges of parameter values over which the oscillation can be sustained.

In the phase plane a LIMIT CYCLE is defined as an ISOLATED CLOSED CURVE. The trajectory has to be both closed, indicating the periodic nature of the motion, and isolated, indicating the limiting nature of the cycle (with nearby trajectories converging or diverging from it.) Thus, while there are many closed curves in the phase portraits of the mass-spring-damper system, these are not considered Limit Cycles in this definition, because they are not isolated.

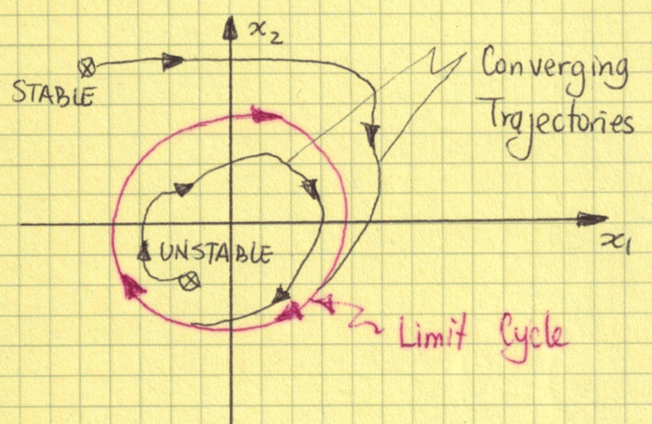
Depending on the motion patterns of the trajectories in the vicinity of the LIMIT CYCLE, one can distinguish three kinds of limit cycles.

(1) STABLE LIMIT CYCLES: all system trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$.

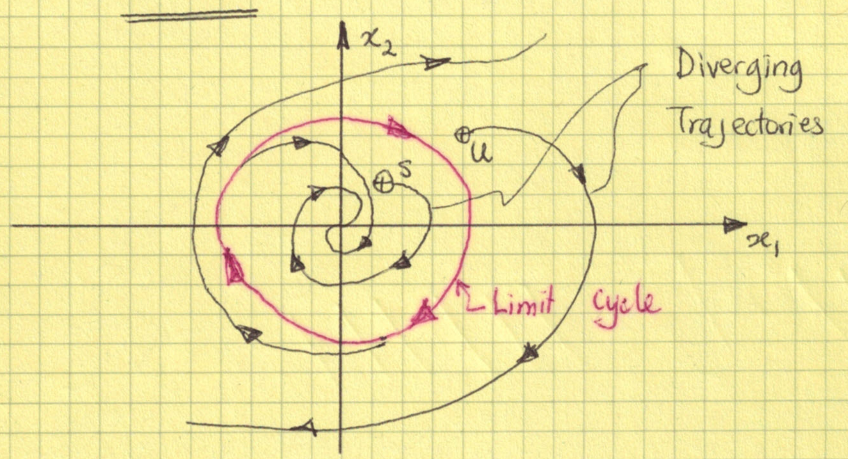
(2) UNSTABLE LIMIT CYCLES: All system trajectories in the vicinity of the limit cycle diverge from it as $t \to \infty$

(3) SEMI-STABLE LIMIT CYCLES: Some of the trajectories in the

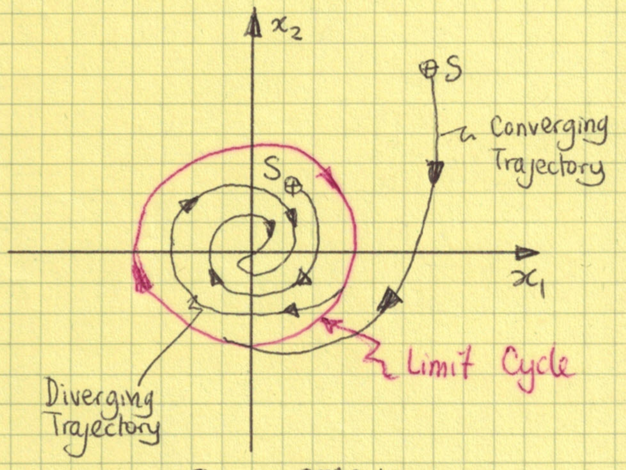
vicinity of the limit cycle converge to it, while the others diverge from it as $t \to \infty$.



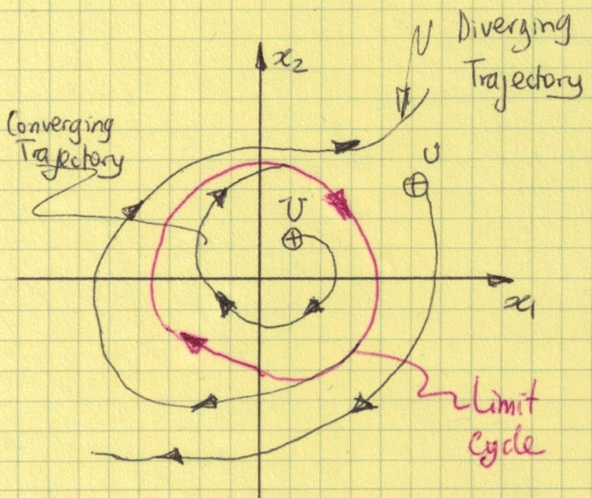
STABLE



UNSTABLE



SEMI-STABLE



SEMI-STABLE

Example: In this example we show analytically that the system

$$\dot{x}_1 = x_2 - x_1(x_1^2+x_2^2-1)$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2+x_2^2-1)$$

has a STABLE LIMIT CYCLE (LC) which is the unit circle in x_1-x_2 plane. To see this we transform the above equations to the polar coordinates by substituting

$$x_1 = r\cos\theta \qquad \left[\text{or } r = (x_1^2+x_2^2)^{1/2} \text{ and}\right.$$
$$x_2 = r\sin\theta \qquad \left.\theta = \tan^{-1}(x_2/x_1)\right]$$

with

$$\dot{x}_1 = \dot{r}\cos\theta - \dot{\theta} r \sin\theta$$
$$\dot{x}_2 = \dot{r}\sin\theta + \dot{\theta} r\cos\theta$$

to obtain

$$\dot{r}\cos\theta - \dot{\theta} r\sin\theta = r\sin\theta - r\cos\theta\,(r^2-1)$$
$$\dot{r}\sin\theta + \dot{\theta} r\cos\theta = -r\cos\theta - r\sin\theta\,(r^2-1)$$

After doing necessary simplifications to eliminate $\dot{r}$ and $\dot{\theta}$, we get

$$\boxed{\begin{aligned}\dot{r} &= -r\,(r^2-1)\\ \dot{\theta} &= -1\end{aligned}}$$

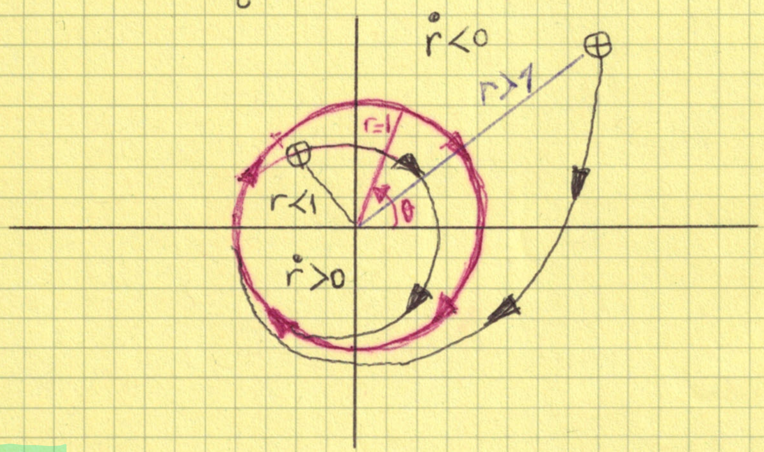
Note that

$$\dot{r} = -r\,(r^2-1) = \begin{cases} >0 & r<1 \\ =0 & r=1 \\ <0 & r>1 \end{cases}$$

which means that when the state starts on the unit circle, the above equation shows that $\dot{r}(t)=0$. Therefore, the state will circle around the origin with a period $1/2\pi$. When $r<1$, then $\dot{r}>0$. This implies that the state tends to the circle from inside. When $r>1$, then $\dot{r}<0$. This implies that the state tends toward the unit circle from outside. Therefore, the unit circle is a STABLE LIMIT CYCLE. This can also be concluded by examining the analytical solution of the equation.

$$r(t) = \frac{1}{(1+C_0 e^{-2t})^{1/2}} \quad ; \quad \theta(t) = \theta_0 - t$$

where $C_0 = \frac{1}{r_0^2} - 1$



Example: Similar to the above example, one can easily show that the following syste

$$\dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1)$$

has an UNSTABLE LIMIT CYCLE with $r = 1$

Example:

$$\dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)$$

has a SEMISTABLE LIMIT CYCLE with $r=1$

Amplitude & Frequency of Limit Cycles Through Harmonic Balancing

Note that Limit Cycle is the PERIODIC MOTION defined for nonlinear systems. Because of its oscillatory nature, we can define (or determine) the frequency and amplitude of limit cycles. The following example will demostrate this.

Example: Consider VAN DER POLL's equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

which can be represented in phase variables form as

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu(x^2-1)x_2 - x_1 \end{aligned}}$$

Define now the nonlinear term

$$z \overset{\Delta}{=} -x_1^2 x_2 = -(x_1^2)(sx_1) \quad \text{(in the s-domain)}$$

Then the whole system can be given in the s-domain as follows;

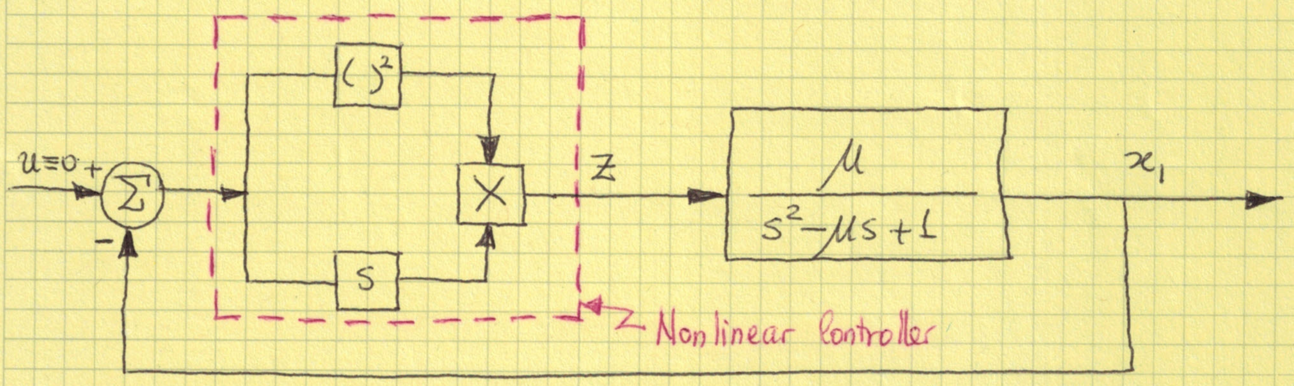
$$\boxed{\begin{aligned} sx_1 &= x_2 \\ sx_2 &= -x_1 + \mu x_2 + \mu z \end{aligned}} \qquad \text{with } z = -(x_1^2)(sx_1)$$

or,

$$s^2 x_1 = -x_1 + \mu s x_1 + \mu z$$

And the transfer function will be

$$(s^2 - \mu s + 1)\, x_1 = \mu z \quad \Rightarrow \quad \boxed{\frac{x_1}{z} = \frac{\mu}{s^2 - \mu s + 1}}$$



With this representation, the system can be considered as if there is a linear system given by

$$\frac{x_1}{z} = \frac{\mu}{s^2 - \mu s + 1}$$

and a nonlinear control element which is of the form

$$z = -(x_1^2)(sx_1)$$

If there is a Limit Cycle (an oscillatory motion, or periodic motion), then the response (output) should be

$$x_1(t) \cong X_1 \sin\omega t$$

where

$$\left.\begin{aligned} X_1 &= \text{Amplitude} \\ \omega &= \text{Frequency} \end{aligned}\right\} \text{of the motion}$$

Then

$$x_2(t) = \dot{x}_1(t) = X_1 \omega \cos\omega t$$

And nonlinear element,

$$z(t) = -(x_1)^2 x_2 = -\left(X_1 \sin\omega t\right)^2 \left(X_1 \omega \cos\omega t\right)$$
$$= -X_1^3 \omega \sin^2\omega t \cos\omega t$$
$$= -X_1^3 \omega \left(\cos\omega t - \cos^3\omega t\right)$$

On the other hand,

$$\cos^3\omega t \cong \frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t$$

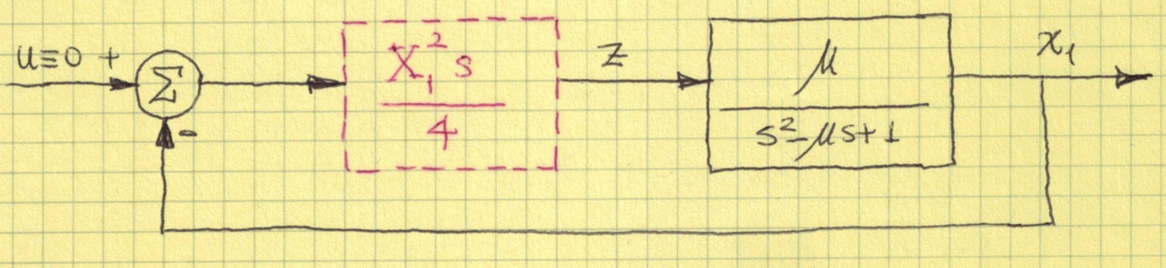
Then,

$$z(t) = -X_1^3 \omega \left[\cos\omega t - \cos^3\omega t\right] = -X_1^3 \omega \frac{1}{4}\left[\cos\omega t - \cos 3\omega t\right]$$

We assume that $\cos 3\omega t$ vanishes, because sufficiently higher order systems filter out these terms. Then

$$z(t) \cong -X_1^3 \frac{\omega}{4} \cos\omega t = -\frac{X_1^2}{4}(s X_1) = \left[-\frac{X_1^2 s}{4}\right] x_1$$

Therefore the nonlinear controller can be replaced as



Now, we have a linear system and the CHARACTERISTIC EQUATION of the system is

$$1 + \frac{X_1^2 s}{4} \cdot \frac{\mu}{s^2 - \mu s + 1} = 0$$

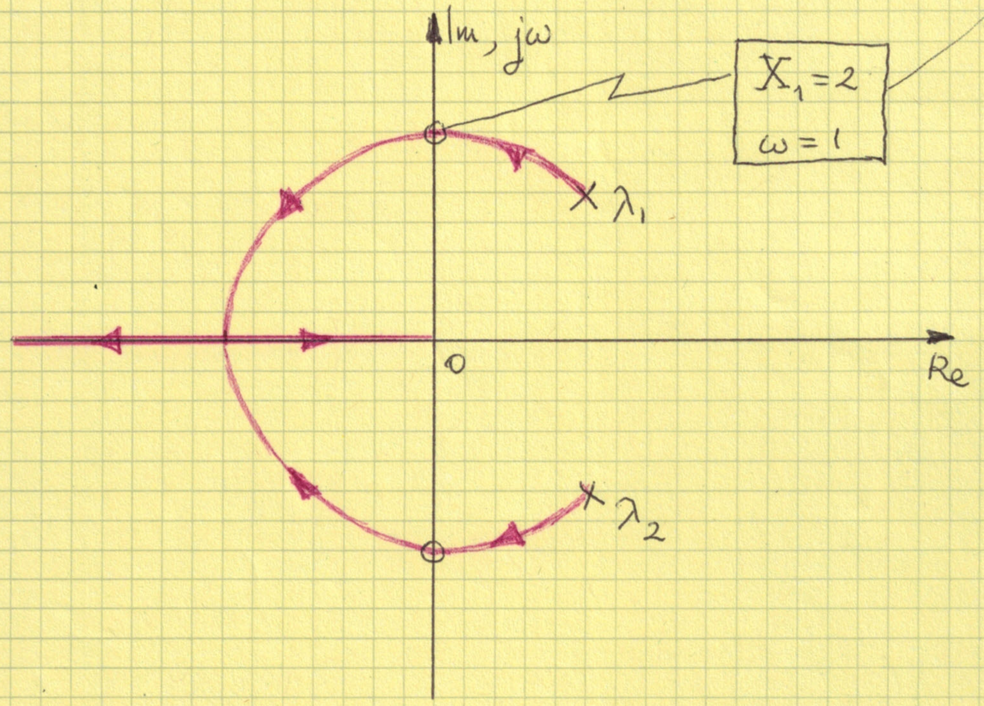
Or

$$4s^2 + \mu\left(X_1^2 - 4\right)s + 4 = 0$$

The roots of the characteristic equation

$$\lambda_{1,2} = -\frac{1}{2}\left[\frac{X_1^2}{4}-1\right]\mu \pm \frac{1}{2}\sqrt{\left[\frac{X_1^2}{4}-1\right]^2-4}$$

The ROOT LOCUS of the system



This is obtained from

$\mu(X_1^2-4)s=0$

$\boxed{X_1 = 2}$

Since

$4s^2+4=0$

$s^2+1=0$

$\omega = \sqrt{\frac{1}{1}} = 1$

EXISTENCE THEOREMS OF LIMIT CYCLES

There are several theorems of nonlinear mechanics which can sometimes be applied in specific cases to yield information on whether or not a limit cycle can or does exist in a given region of the state plane. Two theorems given by Poincaré, and two given by Bendixson, are selected as the most useful and informative. However, no attempt is made here to state or to prove these theorems rigorously.

The first theorem to be stated is due to Poincaré, who observed that there is a certain fixed relationship between a limit cycle and the type of singularities that it encloses. Before stating the theorem the <u>index</u> of a closed curve in the state plane should be defined.

POINCARE' INDEX:

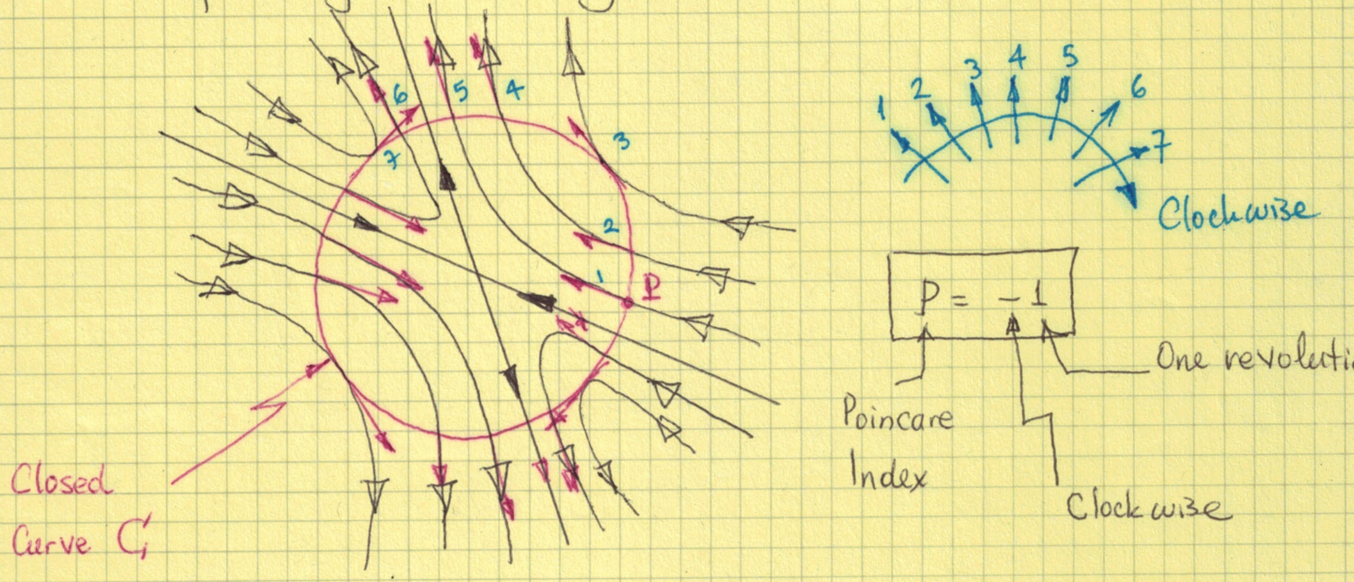
(1) Establish a closed curve C [Jordan curve; simple and closed curve]

(2) Let a point P move CCW (CounterClockWise) around C

(3) Let a vector rest on P and be carried with P as it moves

(4) Allow this vector to point always in the direction of the trajectory at P

(5) Count the numbers of revolutions of this vector in CCW for a complete turn around C. This number is the POINCARE INDEX of the region enclosed by C.



The POINCARE INDEX of a saddle point

Theorem 1 of Poincare'

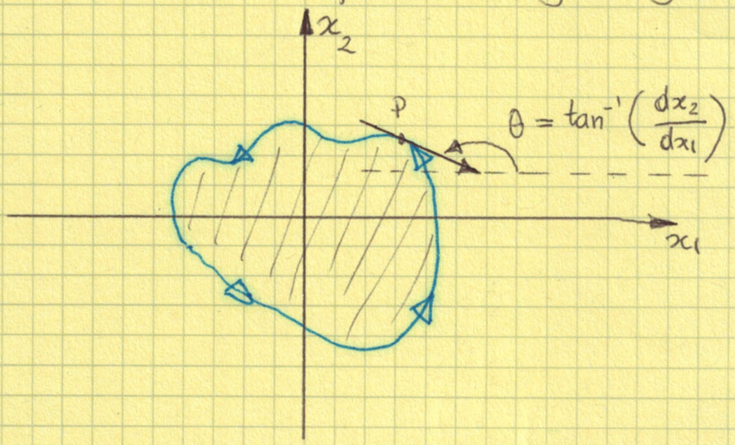
The number and nature of the singular (equilibrium) points enclosed by the closed curve C, in the state plane, is indicated by its index, in the following way:

no singular points enclosed ; $P = 0$
a center, node, or focus enclosed ; $P = +1$
a saddle point enclosed ; $P = -1$

and, in general $P = N - S$

where N = the total number of centers, foci, and nodes enclosed
S = the total number of saddle points enclosed

This theorem can be proved analytically;



Let a simple closed curve C and take a point P on it. Let also a vector rest on P with an angle θ, such that

$$\theta = \tan^{-1}\left(\frac{dx_2}{dx_1}\right)$$

Notice that

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}\right\} \Rightarrow \quad \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

The Poincaré index is now

$$P = \frac{1}{2\pi}\oint_C d\theta = \frac{1}{2\pi}\oint_C d\left[\tan^{-1}\left(\frac{dx_2}{dx_1}\right)\right] = \frac{1}{2\pi}\oint_C d\left[\tan^{-1}\left(\frac{f_2(x_1,x_2)}{f_1(x_1,x_2)}\right)\right]$$

Since

$$d\left[\tan^{-1}\left(\frac{f_2}{f_1}\right)\right] = \frac{d(f_2/f_1)}{1+(f_2/f_1)^2} = \frac{(f_1 df_2 - f_2 df_1)/f_1^2}{1+(f_2/f_1)^2}$$

Then

$$P = \frac{1}{2\pi}\oint_C \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}$$

or

$$P = \frac{1}{2\pi}\oint_C \frac{\left[f_1\frac{\partial f_2}{\partial x_1} - f_2\frac{\partial f_1}{\partial x_1}\right]dx_1 + \left[f_1\frac{\partial f_2}{\partial x_2} - f_2\frac{\partial f_1}{\partial x_2}\right]dx_2}{f_1^2 + f_2^2}$$

As a result;

(*) If the region enclosed by C' has no singular point, then $\boxed{P = 0}$

(*) If C' is a closed trajectory, then $\boxed{P = +1}$

(*) For the singular points Center, Node and Focus, $\boxed{P = +1}$

(*) For saddle points, $\boxed{P = -1}$

POINCARE THEOREM (Theorem 2 of Poincaré)

For a second order system, if a limit cycle exists, then $N - S = +1$. We can restate this

"For a second order system, if a limit cycle exists in a region then the Poincare Index $P = +1$ for this region".

In other words a limit cycle should enclose one and only one more center, node, or focus than saddle points, so that, quite apparently, a limit cycle must enclose at least one center, node, or focus.

The next theorem is due to BENDIXSON and is sometimes referred to as the first theorem of Bendixson. It gives a sufficient condition for the NONEXISTANCE of a limit cycle.

BENDIXSON'S 1st THEOREM (Theorem 1 of Bendixson)
(NEGATIVE CRITERION)

For a second order system

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

where $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ possess first partial derivatives with respect to x_1 and x_2, no limit cycle exist in any region of the state plane in which the divergence

$$I(x_1, x_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

does not change sign or does not vanish (is not identically zero) identica

Proof: Along any solution path, and in particular along a limit cycle, C,

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \quad \text{or} \quad f_1(x_1, x_2)\, dx_2 - f_2(x_1, x_2)\, dx_1 = 0$$

Therefore, around a limit cycle we can write

$$\oint_C \left(f_1\, dx_2 - f_2\, dx_1 \right) = 0$$

But from Gauss' Theorem (or Stokes' Theorem) around and inside C

$$\oint_C \left(f_1\, dx_2 - f_2\, dx_1 \right) = \int_{\text{inside}} \int_C \left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right] dx_1\, dx_2 = 0$$

and the only way this could be true

$\left[\text{if } \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} = I(x_1, x_2) \text{ were not identically zero} \right]$ $\blacksquare$

would be if the integrand of the double integral changed sign inside the limit cycle. $\blacksquare$

This theorem may be applied to define regions in the state-plane which can not possibly contain a limit cycle.

Example: Consider the following nonlinear system

$$\ddot{x} + 2a\dot{x} + \frac{b^2}{2} x + c^2 x^3 = 0$$

Define $x_1 \triangleq x$ and $x_2 \triangleq \dot{x}$ to get

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b^2}{2} x_1 - c^2 x_1^3 - 2a x_2 \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

$$f_1(x_1, x_2) = x_2 \Rightarrow \frac{\partial f_1}{\partial x_1} = 0$$

$$f_2(x_1,x_2) = -\frac{b^2}{2}x_1 - c^2x_1^3 - 2a x_2 \Rightarrow \frac{\partial f_2}{\partial x_2} = -2a$$

$$I(x_1,x_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - 2a = \text{CONSTANT}.$$

Therefore

For $a \neq 0$ $\Rightarrow$ No Limit Cycle in entire phase plane

For $a = 0$ $\Rightarrow$ Closed trajectories may exist.

Example: Consider now VAN DER POL's Equation

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$$

In the state space form

$$\dot{x}_1 = x_2 \qquad = f_1(x_1,x_2)$$
$$\dot{x}_2 = -x_1 - \mu(x_1^2-1)x_2 \qquad = f_2(x_1,x_2)$$

To check Bendixson's 1st Theorem;

$$f_1(x_1,x_2) = x_2 \quad \Rightarrow \quad \frac{\partial f_1}{\partial x_1} = 0$$

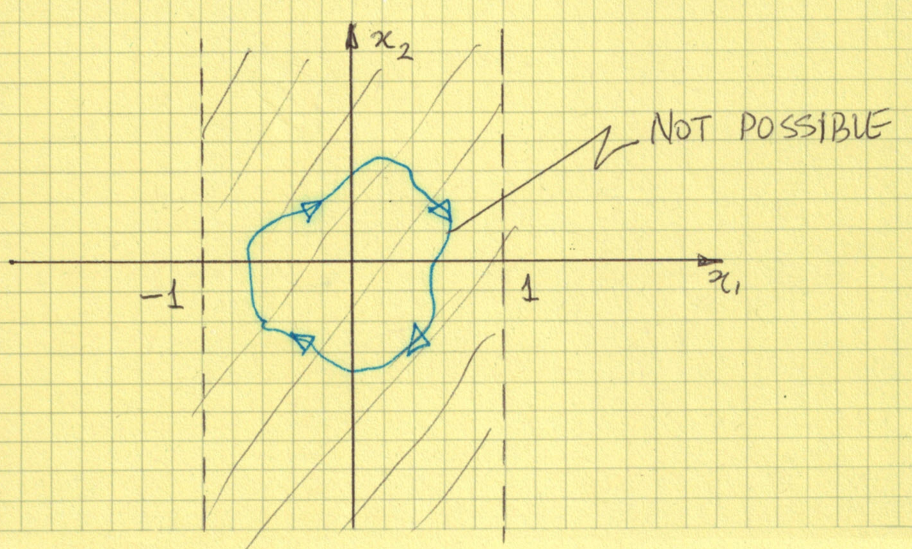
$$f_2(x_1,x_2) = -x_1 - \mu(x_1^2-1)x_2 \Rightarrow \frac{\partial f_2}{\partial x_2} = -\mu(x_1^2-1)$$

Then

$$I(x_1,x_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - \mu(x_1^2-1) = -\mu(x_1^2-1)$$

We can state that

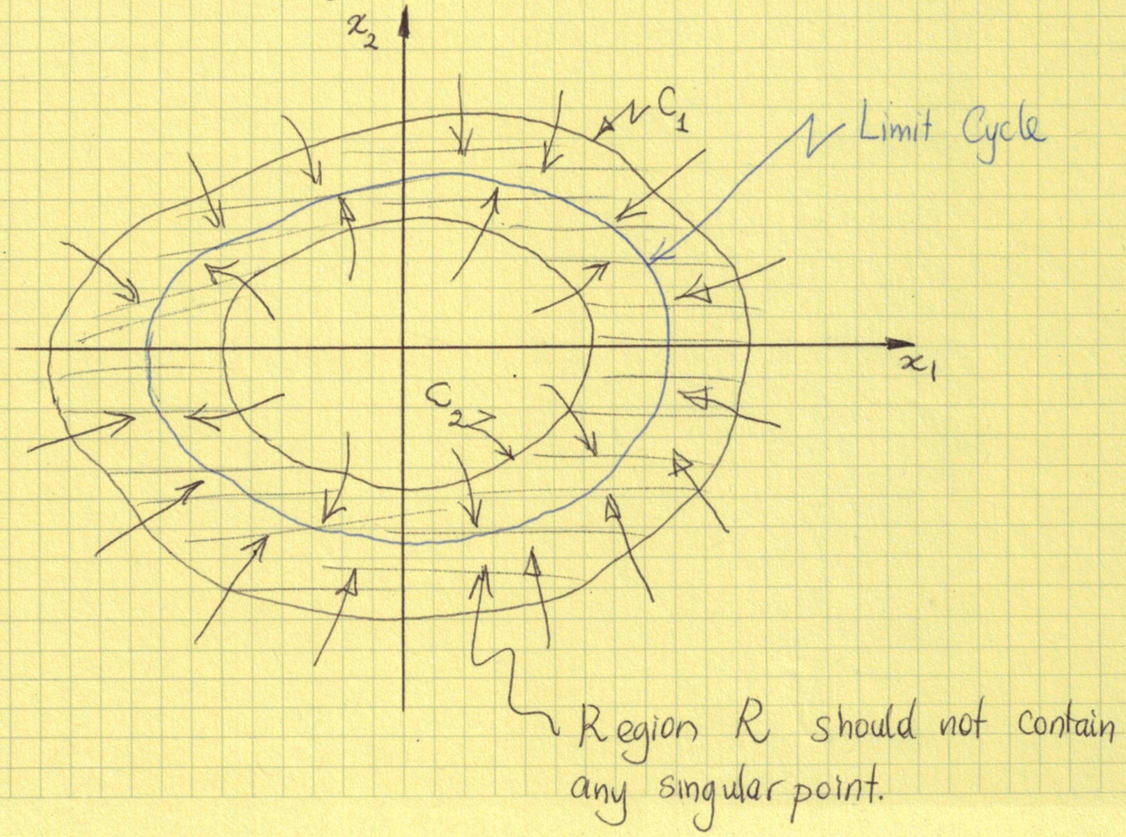
For $\mu \neq 0$ and $|x_1| < 1$ $\Rightarrow$ No Limit Cycle exists.



BENDIXSON'S 2ND THEOREM
(POINCARE' – BENDIXSON THEOREM)

If a trajectory of a second-order autonomous system remains inside a finite region without approaching any singular point, then the trajectory is either a closed curve or approach a closed trajectory (then it must either be a STABLE LIMIT CYCLE itself or it must tend to a stable limit cycle asymptotically.

Unfortunately, this last theorem is not often of very direct help in establishing the existence or nonexistence of limit cycles. It is, however, valuable in establishing concepts of the ways in which nonlinear control systems behave. One application is as follows: If in the state plane there are two closed curves C_1 and C_2, and C_1 encloses C_2 and it can be shown that all the trajectories enter the region enclosed by C_1 and leave the region enclosed by C_2 as time goes on, and there is no singular point in the region D between the two curves, then there is a limit cycle between C_1 and C_2. This theorem is illustrated in general in the figure given below. The theorem seems to be rather abvious, but its proof turns out to be quite involved and therefore is omitted.



The Poincare'- Bendixson Theorem suggests a way to determine the existence of a periodic orbit in the plane. Suppose we can find a closed bounded set M in R^2 such that M contains no equilibrium points of the system and is positively invariant; that is every solution of the system that starts in M remains in M for all $t \geq 0$. In such a case, we are assured from the Poincare'- Bendixson Theorem that M contains a periodic orbit.

Example: Consider the harmonic oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

and the function

$$V(x) = x_1^2 + x_2^2$$

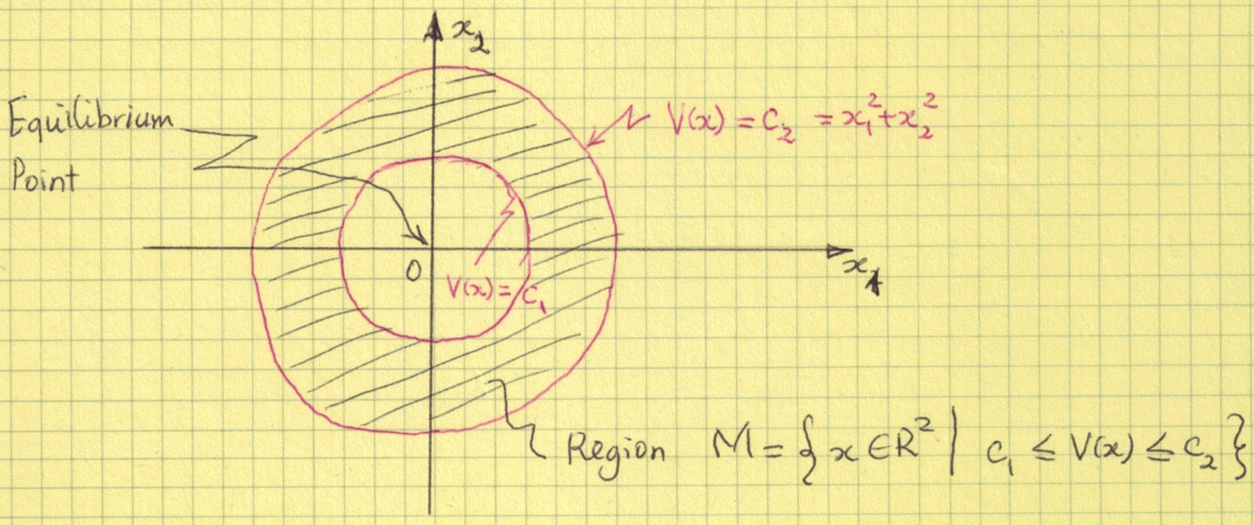
the derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(x_2) + 2x_2(-x_1) = 0$$

Hence, the trajectory of the system cannot cross the level surface $V(x) = c$ for any positive constant c. Define a set M as the annular region

$$M = \{x \in R^2 \mid c_1 \leq V(x) \leq c_2\}$$

where $c_2 > c_1 > 0$. Clearly, M is bounded and positively invariant. It is also free of equilibrium points since the only equilibrium point of the system is at the origin. Thus, we conclude that there is a periodic orbit in M.



This example emphasizes the fact that the Poincaré–Bendixson Theorem assures the existence of a periodic orbit but does not assure its uniqueness.

Example: Consider the system

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$$

$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

and the function

$$V(x) = x_1^2 + x_2^2$$

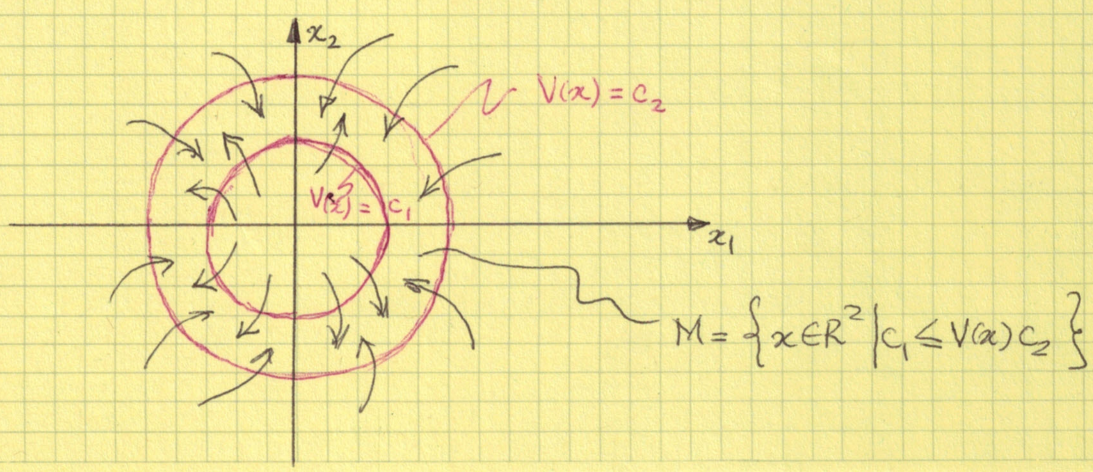
The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1x_2 + 2x_1^2(1 - x_1^2 - x_2^2) - 2x_1x_2 + 2x_2^2(1 - x_1^2 - x_2^2) \\ &= 2V(x)\left[1 - V(x)\right]\end{aligned}$$

The derivative $\dot{V}(x)$ is positive for $V(x) < 1$ and negative for $V(x) > 1$. Hence, on the level surface $V(x) = c_1$ with $0 < c_1 < 1$ all trajectories will be moving outword, while on the level surface $V(x) = c_2$ with $c_2 > 1$ all trajectories will be moving inward. This shows that the annulor region

$$M = \{x \in R^2 \mid c_1 \leq V(x) \leq c_2\}$$

is positively invariant. It is also closed, bounded, and free of equilibrium points since the origin $x = 0$ is the unique equilibrium point. Thus, from the Poincaré–Bendixson Theorem, we conclude that there is a periodic orbit in M.



Example: Consider the RAYLEIGH EQUATION given by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + 2\xi x_2 \left[1 - \frac{x_2^2}{3a}\right]$$

Slope of the trajectories in the phase plane can be defined as

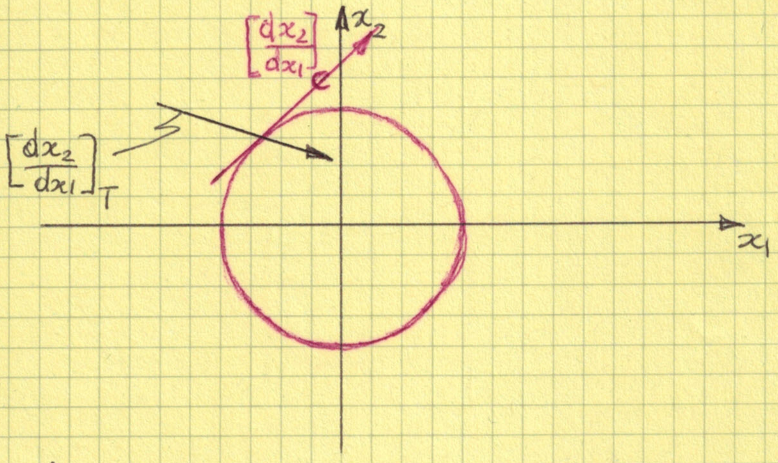
$$\left[\frac{dx_2}{dx_1}\right]_T = -\frac{x_1}{x_2} + 2\xi\left[1 - \frac{x_2^2}{3a}\right]$$

Now consider a simple closed curve, which is a circle with radius R, in $x_1 - x_2$ plane given by

$$x_1^2 + x_2^2 = R^2$$

The tangent vector at any point on the curve is

$$2x_1\,dx_1 + 2x_2\,dx_2 = 0 \Rightarrow \left[\frac{dx_2}{dx_1}\right]_C = -\frac{x_1}{x_2}$$



Note that if

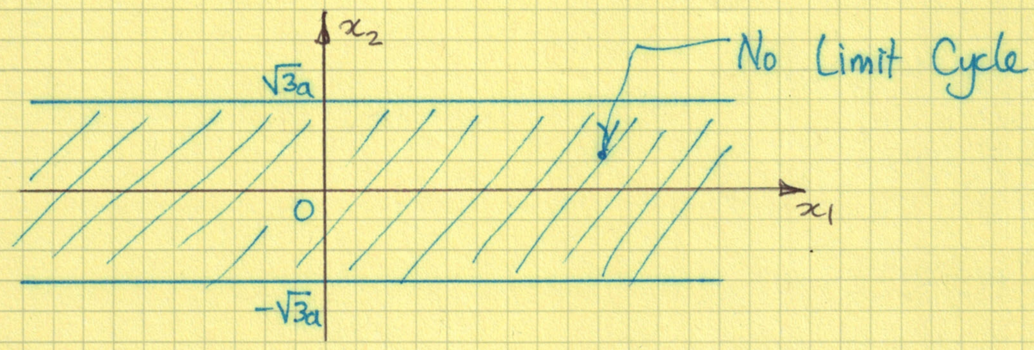
$$\delta = \left[\frac{dx_2}{dx_1}\right]_C - \left[\frac{dx_2}{dx_1}\right]_T$$

is positive, then there is an inside motion.

If $\delta < 0$, then the motion is in direction of outside. Therefore

$$\delta = \left[\frac{dx_2}{dx_1}\right]_C - \left[\frac{dx_2}{dx_1}\right]_T = -\frac{x_1}{x_2} - \left\{-\frac{x_1}{x_2} + 2\xi\left[1 - \frac{x_2^2}{3a}\right]\right\} = 2\xi\left[\frac{x_2^2}{3a} - \right.$$

For $x_2^2 < 3a \Rightarrow |x_2| < \sqrt{3a}$, $\delta < 0. \Rightarrow$ No Limit Cycle.



METHOD OF CONTACT CURVES (POINCARE')

Let us consider a general second-order system given by

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Let us also have a closed curve, which is a circle with R radius, be

$$x_1^2 + x_2^2 = R^2$$

Now we want to find the path where the slope of trajectory of the system is equal to the tangent of closed curve.

Locus of points at which the concentric circles are tangent to the trajectories are called CONTACT CURVE and can be determined from

$$\left[\frac{dx_2}{dx_1}\right]_T = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$: SLOPE of the TRAJECTORIES

$$\left[\frac{dx_2}{dx_1}\right]_C = -\frac{x_1}{x_2}$$: Tangents of the closed curve.

Therefore

$$\left[\frac{dx_2}{dx_1}\right]_T = \left[\frac{dx_2}{dx_1}\right]_C \Rightarrow \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = -\frac{x_1}{x_2}$$

or

$$\boxed{x_1 f_1(x_1, x_2) + x_2 f_2(x_1, x_2) = 0}$$ EQUATION OF THE CONTACT CURVE

Example: Consider the following nonlinear system

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_1 x_2 + x_2^2 - 1) \quad x_2 - x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_1$$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_1 x_2 + x_2^2 - 1) \quad -x_1 - x_2 x_1^2 - x_1 x_2^2 - x_2^3 + x_2$$

The system has only one equilibrium (singular) point at the origin. Thus,

$$\underline{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\delta f_1/\delta x_1 \Rightarrow -3x_1^2 + 2x_1x_2 + x_2^2 + 1$

$\delta f_1/\delta x_2 = 1 + x_1^2 + 2x_1 x_2$

$\delta f_2/\delta x_1 = -1 - 2x_1x_2 - x_2^2$

$\delta f_1/\delta x_2 = -x_1^2 - 2x_1x_2 - 3x_2^2 + 1$

The linearized system has the equation

$$\left.\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_2 \end{aligned}\right\} \Rightarrow \dot{\underline{x}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \underline{x} \Rightarrow \lambda_{1,2} = 1 \pm j$$

(which is an UNSTABLE FOCUS (F^-) $\Rightarrow P = +1$

Poincare' index

The system can be expressed in the polar coordinates as

$$x_1 = r\cos\theta$$
$$x_2 = r\sin\theta$$

Then

$$\frac{dr}{r\,d\theta} = (1 + 0.5\sin 2\theta)\,r^2 - 1$$

For large values of r $\quad (1+0.5\sin 2\theta)\,r^2 - 1 \cong (1 + 0.5\sin 2\theta)\,r^2$

Therefore

$$\frac{dr}{r\,d\theta} \cong (1+0.5\sin 2\theta)\,r^2 \Rightarrow \frac{dr}{r^3} \cong \underbrace{(1 + 0.5\sin 2\theta)\,d\theta}_{\triangleq\, c_1 > 0}$$

By integrating both sides;

$$-\frac{1}{2r^2} \cong c_1\theta + c_2 \Rightarrow \boxed{r^2 \cong \frac{1}{2c_1(\theta - \theta_0)}}$$

which shows us as $\theta \to$ increase, r^2 (or r) $\searrow$ decrease. Therefore at large the system is STABLE. There are two solutions:

(1) The local analysis shows that the origin is UNSTABLE (F^-)

(2) Nonlinear analysis (the system at large) shows that at large the system is STABLE

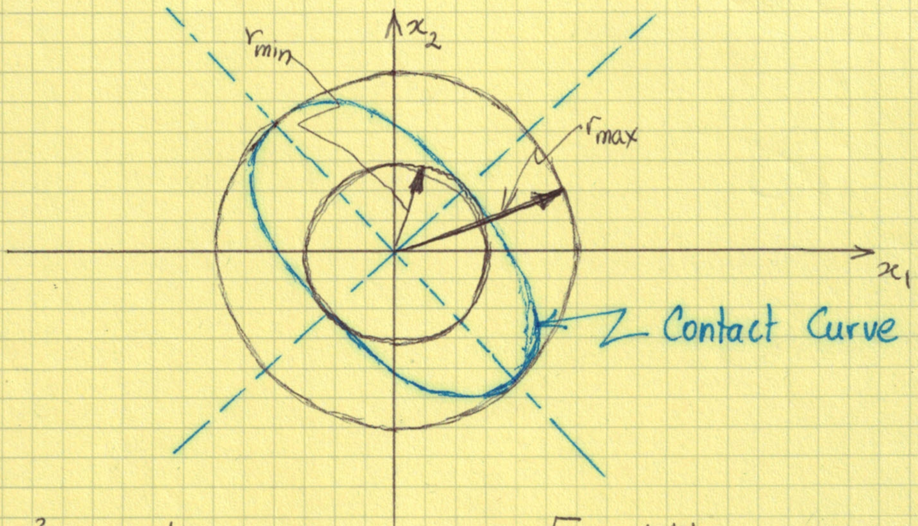
CONTACT CURVE:

$$x_1 f_1(x_1,x_2) + x_2 f_2(x_1,x_2) = x_1\left[x_2 - x_1\left(x_1^2 + x_1x_2 + x_2^2 - 1\right)\right]$$
$$+ x_2\left[-x_1 - x_2\left(x_1^2 + x_1x_2 + x_2^2 - 1\right)\right]$$
$$= -(x_1^2 + x_2^2)(x_1^2 + x_1x_2 + x_2^2 - 1) = 0$$

Since $x_1^2 + x_2^2 \geq 0$, then $\underline{x_1^2 + x_1x_2 + x_2^2 - 1 = 0}$ Contact Curve Equation

In polar coordinates

$$r^2\left[(1+0.5\sin 2\theta)\,r^2 - 1\right] = 0 \quad\Rightarrow\quad \boxed{r^2 = \frac{1}{1+0.5\sin 2\theta}}$$ (in polar coordinates)



: $\boxed{x_1^2 + x_1x_2 + x_2^2 = 1}$

$$r^2_{\substack{min\\max}} = \frac{1}{1 \pm 0.5} \quad\Rightarrow\quad \begin{aligned} r_{max} &= \sqrt{2} = 1.41 \\ r_{min} &= \sqrt{2/3} \cong 0.82 \end{aligned}$$

From now on, we will give some useful results for existence (and even stability conditions) of limit cycles.

THEOREM: A closed path C of a second order autonomous system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

is a STABLE LIMIT CYCLE if the following line integral

$$\oint_C \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dt$$

yields a negative number.

LIÉNARD SUFFICIENCY CONDITIONS FOR A LIMIT CYCLE

Liénard considered the following type nonlinear system

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

Here is the result:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

has a unique, stable limit cycle if the following sufficient ~~but~~ not necessary conditions are met:

(1) $f(x)$ & $g(x)$ are analytic.

(2) $g(x)$ is an odd-symmetric function $\rightarrow$ $\boxed{g(-x) = -g(x)}$
vanishing at origin $g(0) = 0$.

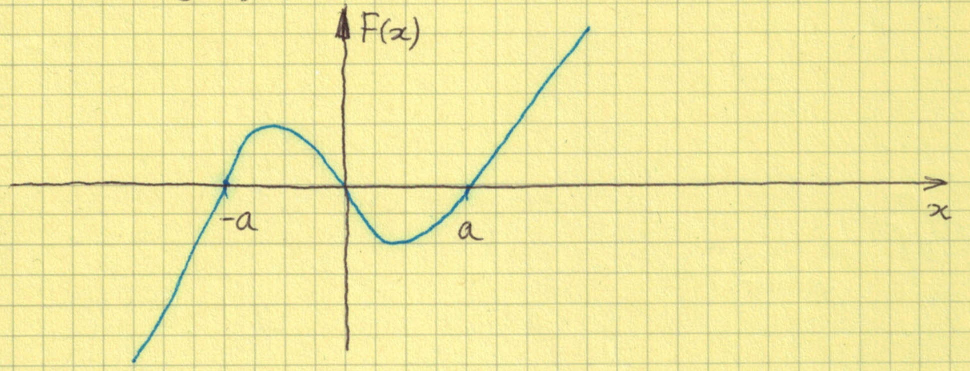
(3) $x\,g(x) \geq 0$

(4) $f(-x) = f(x)$

(5) $f(0) < 0$

(6) $F(x) \rightarrow \infty$ as $x \rightarrow \infty$ where $F(x) = \int_0^x f(x)\,dx$

(7) $F(x) = 0$ has a unique root at $x = a > 0$ & is monotonically increasing for $x > a$



Example: Consider VAN DER POL Equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \qquad \mu > 0$$

which can be re-written as

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

where

$$f(x) = \mu(x^2 - 1)$$

$$g(x) = x$$

Note that conditions (1), (2), (3), (4) & (5) are met.

For the condition number (6) ;

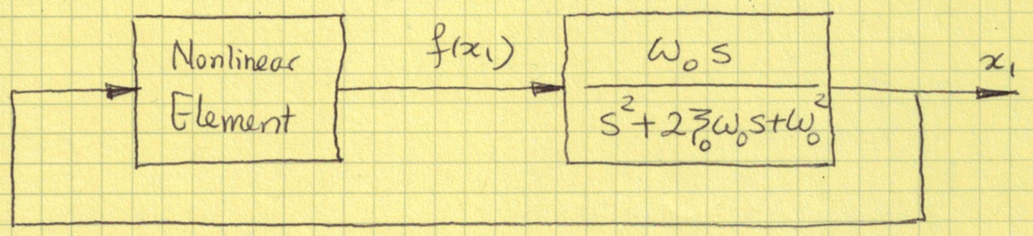
$$F(x) = \int_0^x f(x)\,dx = \int_0^x \mu(x^2-1)\,dx = \mu\left(\frac{x^3}{3} - x\right) \qquad F(x) \to \infty \text{ as } x \to \infty$$

For condition number (7) :

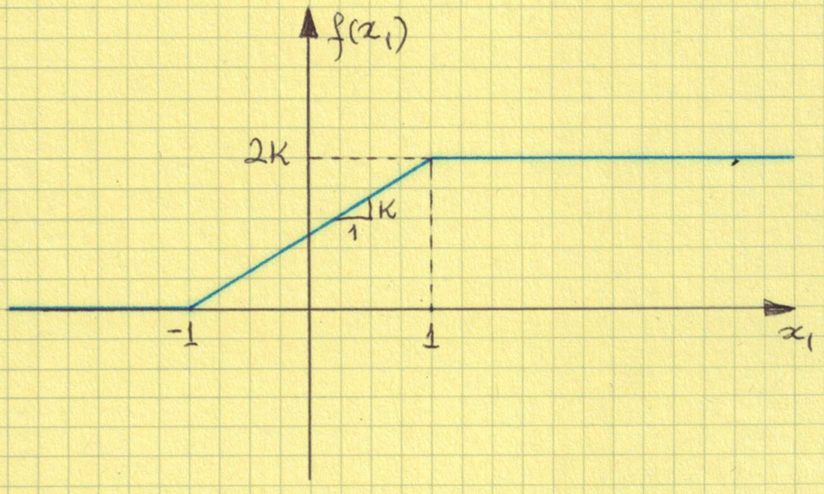
$$F(x) = 0 \Rightarrow \mu\left(\frac{x^3}{3} - x\right) = 0 \Rightarrow \underline{x = a = \sqrt{3} > 0}$$

and $F(x)$ is monotonically increasing for $x > a$

Example: Now let us consider a linear system attached to a nonlinear Control element.



The Nonlinear Element is given graphically below:



The transfer function between x_1 and $f(x_1)$ is

$$\frac{x_1}{f(x_1)} = \frac{\omega_0 s}{s^2 + 2\xi_0\omega_0 s + \omega_0^2}$$

Then, $[s^2 + 2\xi_0\omega_0 s + \omega_0^2]\, x_1(s) = \omega_0 s\, f(x_1)$

In time domain,

$$\frac{d^2 x_1}{dt^2} + 2\xi_0\omega_0 \frac{dx_1}{dt} + \omega_0^2 x_1 = \omega_0 \frac{d}{dt}\left[f(x_1)\right]$$

or

$$\ddot{x}_1 + 2\xi_0\omega_0\dot{x}_1 + \omega_0^2 x_1 = \omega_0 \underbrace{\frac{df_1(x_1)}{dx_1}}_{= f_1'(x_1)} \cdot \dot{x}_1$$

Then, we can write

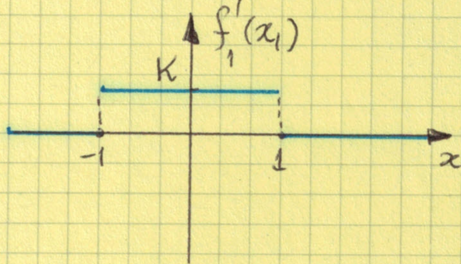
$$\ddot{x}_1 + \left(2\xi_0\omega_0 - \omega_0 f_1'(x_1)\right)\dot{x}_1 + \omega_0^2 x_1 = 0$$

Note that if

$\left(2\xi_0\omega_0 - \omega_0 f_1'(x_1)\right) < 0$, then the system will be UNSTABLE

On the other hand,

$$f_1'(x_1) = \begin{cases} K & |x_1| < 1 \\ 0 & |x_1| > 1 \end{cases}$$



If $|x_1| > 1$, then $f_1'(x_1) = 0 \Rightarrow \ddot{x}_1 + 2\xi_0\omega_0\dot{x}_1 + \omega_0^2 x_1 = 0$

The solution converges

If $|x_1| < 1$, then $f_1'(x_1) = K \Rightarrow \ddot{x}_1 + (2\xi_0\omega_0 - \omega_0 K)\dot{x}_1 + \omega_0^2 x_1 = 0$

For $K > 2\xi_0 \Rightarrow$ the system will be unstable, the solution diverges.

which means, there will be a <u>Limit Cycle</u>

This result can be obtained from Bendixson's 1st Theorem (Negative Criterion):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_0^2 x_1 - \left[2\xi_0 - f_1'(x_1)\right]\omega_0 x_2$$

$$I(x_1, x_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 + \left[-2\xi_0 + f'(x_1)\omega_0\right]$$

For $f'(x_1) = 0 \Rightarrow I(x_1, x_2) = -2\xi_0 \Rightarrow$ no limit cycle exists

$f'(x_1) = K = 2\xi_0 \Rightarrow I(x_1, x_2) = 0 \Rightarrow$ Closed trajectories may exist