Mathematics I (BSM 101)

Manju Subedi

Gandaki University
Bachelor in Information Technology(BIT)
BSM 101

manjusubedi.061@gmail.com

December 29, 2022

Average rate of Change

Definition: The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h}$$

Average rate of Change

Definition: The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h}$$

Notice that the average rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ in geometry (look at the figure) a line joining two points of a curve is called a secant to the curve.

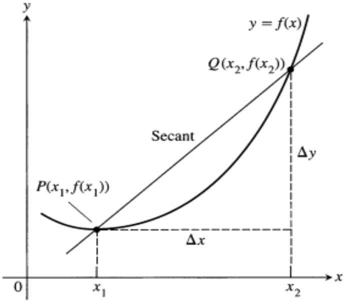
Average rate of Change

Definition: The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h}$$

Notice that the average rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ in geometry (look at the figure) a line joining two points of a curve is called a secant to the curve.



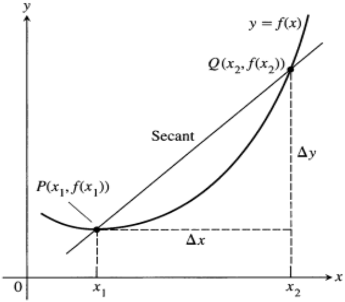
Average rate of Change

Definition: The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h}$$

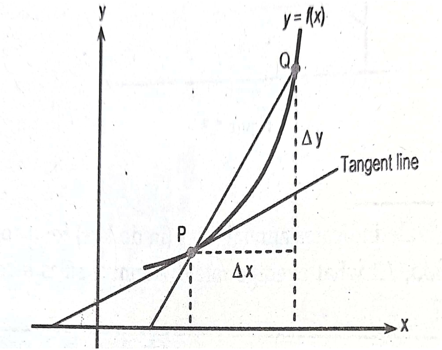
Notice that the average rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ in geometry (look at the figure) a line joining two points of a curve is called a secant to the curve.



A secant to the graph $y = f(x)$. Its slope is $\frac{\Delta y}{\Delta x}$, the average rate of change of f over the interval $[x_1, x_2]$ Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ.

Slope of a Curve at a Point (Tangent)

The slope of a curve at $x = a$ is the slope of the tangent line at $x = a$

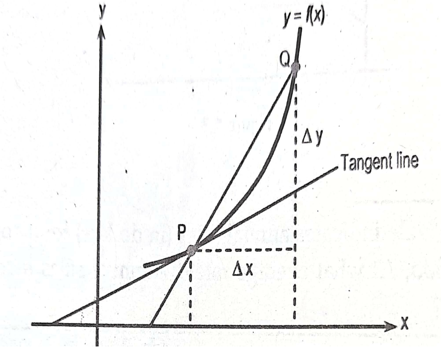


Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points in the curve. Then the slope of straight line PQ is given by

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of a Curve at a Point (Tangent)

The slope of a curve at $x = a$ is the slope of the tangent line at $x = a$



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points in the curve. Then the slope of straight line PQ is given by

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

As point Q moves towards P (i.e Δx approaches 0) the value of $\frac{\Delta y}{\Delta x}$ also change and will approach a limit, if it exist. This limit provides the slope of the curve at point P. This limit is also called the derivative of the function f(x) at the point P and written as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Derivative

Definition: The derivative of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.
If the limits on the right hand side exist (i.e $f'(x)$ exist) then we say f has a derivative (is differentiable) at x.
Notation: The derivative of $y = f(x)$ with respect to x is denoted as

$$y' \quad \text{or} \quad f'(x) \quad \text{or} \quad \frac{dy}{dx} \quad \text{or} \quad \frac{df}{dx} \quad \text{or} \quad \frac{d}{dx}f(x)$$

Example

Find the derivative of $y = \sqrt{x}$ for $x > 0$ using the definition of derivatives

Solution:

$$f(x) = \sqrt{x} \text{ and } f(x+h) = \sqrt{x+h}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \text{Multiply by } \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Some Important Rules Without Proof

- $$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \tag{1}$$
- $$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \tag{2}$$
- $$\sin(x+h) = \sin x \cos h + \cos x \sin h \tag{2}$$
- $$\cos(x+h) = \cos x \cos h - \sin x \sin h \tag{2}$$

The Derivative of the Sine:

To calculate the derivative of

$f(x) = \sin x \quad \text{and} \quad f(x+h) = \sin(x+h) = \sin x \cos h + \cos x \sin h$

We have

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(sinx) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

The Derivative of the Sine:

To calculate the derivative of

$f(x) = \sin x \quad \text{and} \quad f(x+h) = \sin(x+h) = \sin x \cos h + \cos x \sin h$

We have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(sinx) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

The Derivative of the Sine:

To calculate the derivative of

$f(x) = \sin x \quad \text{and} \quad f(x+h) = \sin(x+h) = \sin x \cos h + \cos x \sin h$

We have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(sinx) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

The Derivative of the Sine:

To calculate the derivative of

$f(x) = \sin x \quad \text{and} \quad f(x+h) = \sin(x+h) = \sin x \cos h + \cos x \sin h$

We have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(sinx) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h} \right)$$

The Derivative of the Sine:

To calculate the derivative of

$f(x) = \sin x \quad \text{and} \quad f(x+h) = \sin(x+h) = \sin x \cos h + \cos x \sin h$

We have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(sinx) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

The Derivative of the Sine:

To calculate the derivative of

$f(x) = \sin x \quad \text{and} \quad f(x+h) = \sin(x+h) = \sin x \cos h + \cos x \sin h$

We have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(sinx) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h} \right)$$

$$= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

The Derivative of the Sine:

To calculate the derivative of

$f(x) = \sin x \quad \text{and} \quad f(x+h) = \sin(x+h) = \sin x \cos h + \cos x \sin h$

We have

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(sinx) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

In short, the derivative of the sine is the cosine.

The Derivative of the Cosine:

Here, we have $f(x) = \cos x$

$f(x+h) = \cos(x+h) = \cos x \cos h - \sin x \sin h$,

then, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(cosx) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h \to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1 = -\sin x.$$

In short, the derivative of the cosine is the negative of the sine.

Formulas

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

1. Find the average rate of change of the function over the given interval or intervals.
   (i) $f(x) = x^3 + 1;$ (a) $[2, 3]$, (b) $[-1, 1]$
   (ii) $g(x) = x^2;$ (a) $[-1, 1]$, (b) $[-2, 0]$
   (iii) $h(t) = \cot t;$ (a) $[\pi/4, 3\pi/4]$, (b) $[\pi/6, \pi/2]$
2. Find the derivative of f. State the domain of f'
   (i) $f(x) = 5x + 3$
   (ii) $f(x) = \sqrt{x - 1}$
   (iii) $f(x) = \dfrac{1}{x^2}$

Thank You