Mathematics I (BSM 101)

Manju Subedi

Gandaki University
Bachelor in Information Technology(BIT)
BSM 101

manjusubedi.061@gmail.com

January 12, 2023

Application Of Derivatives

- Maximum and minimum of a function of one variable
- Indeterminant forms
- Rolle's theorem
- Mean value theorem
- Taylor's and Maclaurin's series

A Limit Process: While computing the limits

$$\lim_{x \to a} f(x)$$

of certain functions $f(x)$, we may come across the following situations like,

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \times \infty, \quad \infty - \infty, \quad 1^{\infty}, \quad 0^0, \quad \infty^0$$

is known as indeterminant forms. Finding the limits of the function which has indeterminant form by using the derivatives is discovered by John Bernoulli. The rule is known as l'Hopital's Rule (pronounced as Lho pi tal Rule) named after Guillaume de l'Hopital's , a french mathematician.

Suppose $f(x)$ and $g(x)$ are differentiable functions and $g\left(x'\right) \neq 0$ with

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x). \text{ Then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to a} f(x) = \pm\infty = \lim_{x \to a} g(x). \text{ Then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

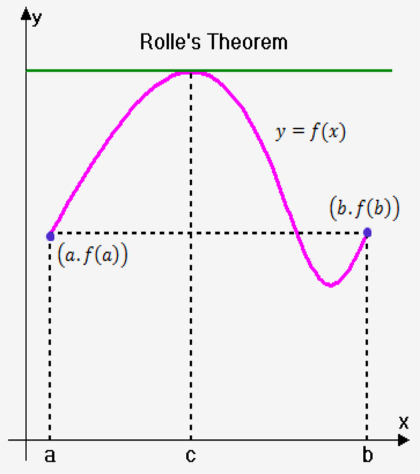
Evaluate the limits:

1. $\lim\limits_{x \to 1} \left(\dfrac{x^2 - 3x + 2}{x^2 - 4x + 3} \right)$ 2. $\lim\limits_{x \to a} \left(\dfrac{x^n - a^n}{x - a} \right)$

3. $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right)$ 4. $\lim\limits_{x \to 0} (x \log x)$ 5. $\lim\limits_{x \to 0} \frac{3x - \sin x}{x}$

6. $\lim\limits_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$ 7. $\lim\limits_{x \to 0} \frac{x - \sin x}{x^3}$

Suppose $f(x)$ is a function that satisfies all of the following.
1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).
3. $f(a) = f(b)$
Then there is a number c such that $a < c < b$ and $f'(c) = 0$.
Or, in other words $f(x)$ has a critical point in (a, b).



...

Example:

1. Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for $f(x) = x^2 - 2x - 8$ on $[-1, 3]$.

Solution:

To veirfy the Rolle's theorem, first check all the conditions. The function is a polynomial which is continuous and differentiable everywhere and so will be continuous on $[-1, 3]$ and differentiable on $(-1, 3)$

Next, $f(a) = f(-1) = -5 = f(3) = f(b)$.

Now, all we need to do is take the derivative,

$f'(x) = 2x - 2$

and then solve $f'(c) = 0$

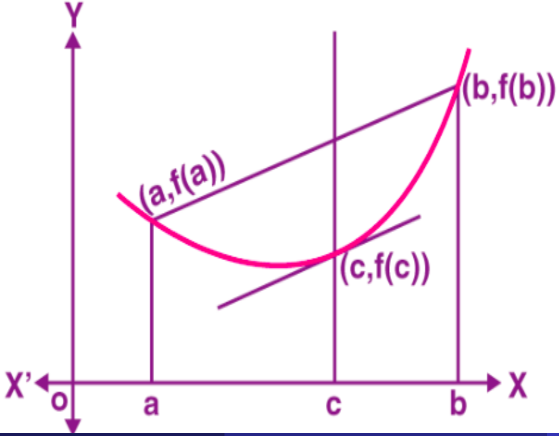
$2c - 2 = 0 \quad \Rightarrow \quad c = 1$

So, we found a single value and it is in the interval so the required value is, $c = 1$

Mean Value Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Examples:

1. Verify Mean Value Theorem for the following function in the given interval.

a. $f(x) = x^2 - 4x - 3 \quad [1, 4]$

Solution: Since, $f(x)$ is a polynomial, it is continuous on $[1, 4]$ and differentiable on $(1, 4)$.

Now, $f'(x) = 2x - 4$

$f'(c) = 2c - 4$

$f(b) = f(4) = 4^2 - 4 \cdot 4 - 3 = -3$

$f(a) = f(1) = 1^2 - 4 \cdot 1 - 3 = -6$

By Mean Value Theorem:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\implies 2c - 4 = \frac{-3 - (-6)}{4 - 1}$$

$$\implies 2c - 4 = 1$$

$\implies c = \frac{5}{2}$ Thus, required value of $c = \frac{5}{2} \in [1, 4]$

Exercise

Verify Mean Value Theorem for the following function in the given interval.

b. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
c. $f(x) = x^{2/3}, \quad [0, 1]$
c. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$
d. $f(x) = \sqrt{x - 1}, \quad [1, 3]$

Taylor's Series and Maclaurin's Series

Taylor Series: The Taylor series of $f(x)$ about $x = 0$ is the power series $\sum_{n=0}^{\infty} a_n x^n$, where

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Taylor Series about x = a: The Taylor series of $f(x)$ about $x = a$ is the power series

$$\sum_{n=0}^{\infty} a_n (x - a)^n$$

$$a_n = \frac{f^{(n)}(a)}{n!}$$

Evaluate the limits:

a. $\lim\limits_{x\to 1}\left(\dfrac{x^2-3x+2}{x^2-4x+3}\right)$ b. $\lim\limits_{x\to a}\left(\dfrac{x^n-a^n}{x-a}\right)$ c. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$

d. $\lim\limits_{x\to 0}(x\log x)$ e. $\lim\limits_{x\to 0}\frac{3x-\sin x}{x}$ f. $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

g. $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}$ h. $\lim\limits_{x\to\infty} x^2e^{-x}$ i. $\lim\limits_{x\to\infty} x^-2e^x$

Verify Mean Value Theorem for the following function in the given interval.

b. $f(x)=x^2+2x-1, \quad [0,1]$

c. $f(x)=x^{2/3}, \quad [0,1]$

c. $f(x)=x+\frac{1}{x}, \quad \left[\frac{1}{2},2\right]$

d. $f(x)=\sqrt{x-1}, \quad [1,3]$

Thank You