Mathematics I (BSM 101)

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Outlines

Integration by Parts

Let u and v be to functions of x, to integrate a product of u and v, one uses the product rule:

$$\int uvdx = u\int vdx - \int \frac{du}{dx}\left(\int vdx\right)dx$$

- **u** is the function $u(x)$
- **v** is the function $v(x)$
- $\mathbf{\frac{du}{dx}}$ is the derivative of the function $u(x)$

This gives us a rule for integration, called INTEGRATION BY PARTS, that allows us to integrate many products of functions of x.

Integration by Parts

Let u and v be to functions of x, to integrate a product of u and v, we can generate a formula from the product rule of differentation,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$$

where $u = u(x)$ and $v = v(x)$ are two functions of x. A slight rearrangement of the product rule gives

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx}v$$

Now, integrating both sides with respect to x results in

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}vdx \implies \int udv = uv - \int vdu$$

This gives us a rule for integration, called INTEGRATION BY PARTS, that allows us to integrate many products of functions of x.

Example 1

Integrate: $\int xe^x dx$

$$\int xe^x dx = x \cdot e^x - \int (1) \cdot e^x dx,$$

$$\text{i.e. take } u = x \implies \frac{du}{dx} = 1$$

$$\text{and take } v = e^x \implies \frac{dv}{dx} = e^x$$

$$\begin{aligned} \text{Now,} \quad \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C \\ &= (x-1)e^x + C \end{aligned}$$

Example

Integrate:

$$
\begin{aligned}
\int x^2 \ln x dx &= (\ln x) \cdot \left(\frac{1}{3}x^3\right) - \int \frac{1}{x} \cdot \left(\frac{1}{3}x^3\right) dx, \\
&\text{i.e. } u = \ln x \implies \frac{du}{dx} = \frac{1}{x} \\
&= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx \qquad\qquad \frac{dv}{dx} = x^2 \\
&= \frac{1}{3}x^3 \ln x - \frac{1}{3} \cdot \left(\frac{1}{3}x^3\right) + C \\
&= \frac{1}{9}x^3(3 \ln x - C
\end{aligned}
$$

Classwork

1. $\int \ln x dx$
2. $\int x \ln x dx$

Homework Problem

Use integration by parts to evaluate the integral.

1. $\int xe^{2x}dx$
2. $\int xe^{-x}dx$
3. $\int x^2 \ln x dx$
4. $\int x^3 \ln x dx$
5. $\int \sqrt{x} \ln x dx$
6. $\int x^3 e^x dx$

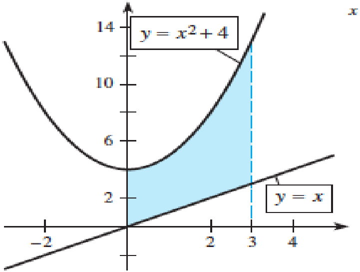
Use integration by parts to evaluate the integral. Note that evaluation may require integration by parts more than once.

1. $\int x^2 e^{-x} dx$
2. $\int 4x^2 e^x dx$
3. $\int 3x^3 e^{x^2} dx$
4. $\int x^3 e^x dx$

Area Between Two Curves

If f and g are continuous functions on $[a, b]$ and if $f(x) \geq g(x)$ on $[a, b]$, then the area of the region bounded by $y = f(x), y = g(x), x = a$, and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]dx$$



Example Find the area of the region bounded by $y = x^2 + 4, y = x, x = 0$, and $x = 3$.

Solution:

The graph of the region is shown in above figure. Because $y = x^2 + 4$ lies above $y = x$ in the interval from $x = 0$ to $x = 3$, the area is

$$\begin{aligned} A &= \int (\text{ top curve } - \text{ bottom curve})\, dx \\ A &= \int_0^3 \left[\left(x^2 + 4\right) - x\right] dx = \frac{x^3}{3} + 4x - \frac{x^2}{2}\Bigg|_0^3 \\ &= \left(9 + 12 - \frac{9}{2}\right) - (0 + 0 - 0) \\ &= 16\frac{1}{2} \text{ square units} \end{aligned}$$

We are sometimes asked to find the area enclosed by two curves. In this case, we find the points of intersection of the curves to determine a and b.

Example 2:

Find the area of the region R enclosed by the curves $y = x^3$ and $y = x^2$.

Solution:

To find the points where the curves intersect, solve the equations simultaneously as follows:

$$x^3 = x^2 \implies x^3 - x^2 = 0 \implies x^2(x-1) = 0 \implies x = 0, 1$$

The corresponding points $(0, 0)$ and $(1, 1)$ are the only points of intersection.

The region R enclosed by the two curves is bounded above by $y = x^2$ and below by $y = x^3$, over the interval $0 \leq x \leq 1$. The area of this region is given by the integral

$$A = \int_0^1 \left(x^2 - x^3\right) dx = \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1$$

$$= \left[\frac{1}{3}(1)^3 - \frac{1}{4}(1)^4\right] - \left[\frac{1}{3}(0)^3 - \frac{1}{4}(0)^4\right] = \frac{1}{12} \quad \text{square units}$$

Example 3

Find the area enclosed by $y = x^2$ and $y = 2x + 3$.

Solution We first find a and b by finding the x-coordinates of the points of intersection of the graphs. Setting the y-values equal gives

$$\begin{aligned} x^2 = 2x + 3 \implies x^2 - 2x - 3 = 0 \implies (x-3)(x+1) = 0 \\ \implies x = 3, \quad x = -1 \end{aligned}$$

Thus $a = -1$ and $b = 3$.

The area of the enclosed region is

$$A = \int_{-1}^{3} \left[(2x+3) - x^2\right] dx \implies x^2 + 3x - \frac{x^3}{3}\bigg|_{-1}^{3}$$

$$\implies (9 + 9 - 9) - \left(1 - 3 + \frac{1}{3}\right) \implies 10\frac{2}{3} \text{ square units}$$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Classwork Problems

Find the area enclosed by the curves:

1. $f(x) = x^3; \quad g(x) = x^2 + 2x$
2. $f(x) = x^3; \quad g(x) = 2x - x^2$
3. $f(x) = \frac{3}{x}; \quad g(x) = 4 - x$
4. $g(x) = 1 - x^2; \quad h(x) = x^2 + x$

Thank You