Mathematics I (BSM 101)

Manju Subedi

Gandaki University
Bachelor in Information Technology(BIT)
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manjusubedi.061@gmail.com

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Outlines

- Concept of Limits
- $\epsilon - \delta$ Definition of Limits
- Limits: A Numerical Approach
- Limits: A Graphical Approach
- Limits: An Algebraic Approach
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- Limits at ∞

Concept of Limits

Definition:
Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that $f(x)$ approaches the limit L as x approaches c, and we write

$$\lim_{x \to c} f(x) = L.$$

Also, The **left hand limit** of $f(x)$ as x approaches c is equal to L. we write

$$\lim_{x \to c^-} f(x) = L$$

And, The **right hand limit** of $f(x)$ as x approaches c is equal to L. We write

$$\lim_{x \to c^+} f(x) = L$$

Theorem

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

$\epsilon - \delta$ Definition of Limits

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \epsilon$$

Limits: A Numerical Approach

Example: Guess the value of $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

Solution: Notice that function $f(x) = \frac{x^2 - 1}{x - 1}$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \to c} f(x)$ says that we consider values of x that are colse to c but not equal to c.

Values of x and $f(x)$ below 1

x	0.9	0.99	0.999	0.99999	0.999999
f(x)	1.9	1.99	1.999	1.99999	1.999999

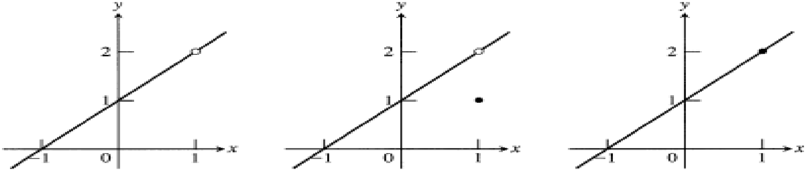
Values of x and $f(x)$ above 1

x	1.1	1.01	1.001	1.0001	1.000001
f(x)	2.1	2.01	2.001	2.0001	2.000001

Exercise: Determine the value of $\lim_{x \to 1}(3x - 5)$ by numerical approach.

Limits: A Graphical Approach

Limits: A Graphical Approach



(a) $f(x) = \frac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

(a) The existence of a limit as $x \to c$ does not depend on how the function may be defined at c. The function f in above fig. has limit 2 as $x \to 1$ even though f is not defined at $x = 1$.
(b) The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$.
(c) The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$. For h we have $\lim_{x \to 1} h(x) = h(1)$.

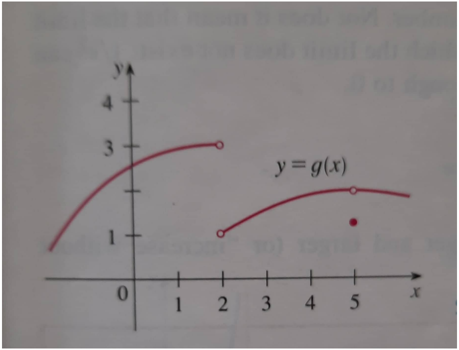
Limits: An Graphical Approach

Exercise: The graph of a function $y = g(x)$ is shown in figure use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$

(c) $\lim_{x \to 2} g(x)$ (d) $\lim_{x \to 5^-} g(x)$

(e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$



Limits: An Algebraic Approach

Example: Find $f(x) = \dfrac{x^2 - 1}{x - 1}$

Solution: Here, we can not find the limit by substituting $x = 1$ because $f(1)$ is not defined. Instead we need to do some preliminary algebra.

$f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x-1)(x+1)}{x-1} = x + 1$

Now, $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \lim\limits_{x \to 1} (x + 1) = 2$

Exercise: $f(x) = \dfrac{x - 2}{x^2 + 4x - 3}$ find $\lim\limits_{x \to 1} f(x)$

The Sandwich Theorem: Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then, $\lim\limits_{x \to c} f(x) = L.$

Properties of Limits

The following rules hold if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M (L$ and M real numbers).

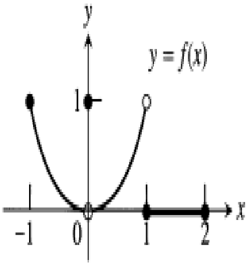
- Sum Rule: $\lim_{x\to c}[f(x) + g(x)] = L + M$
- Difference Rule: $\lim_{x\to c}[f(x) - g(x)] = L - M$
- Product Rule: $\lim_{x\to c} f(x) \cdot g(x) = L \cdot M$
- Constant Multiple Rule: $\lim_{x\to c} kf(x) = kL$ (any number k)
- Quotient Rule: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
- Power Rule: If m and n are integers, then $\lim_{x\to c}[f(x)]^{m/n} = L^{m/n}$, provided $L^{m/n}$ is a real number.
  - $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$ provided that $L > 0$ when n is even.
  - If r is a positive constant, then $\lim_{x\to c}[f(x)]^r = [\lim_{x\to c} f(x)]^r = L^r$

Other Important Limits:

- $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$, $a > 0$
- $$\lim_{x\to\infty} a^x = \begin{cases} \infty, & a > 0 \\ 0, & 0 < a < 1 \end{cases}$$
- $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$, $\lim_{x\to\infty} e^{-x} = 0$, $\lim_{x\to-\infty} e^{-x} = \infty$
- The limit of a function as x approaches c is independent of the value of the function at c. When $\lim_{x\to c} f(x)$ exists, the value of the function at c may be
  (i) same as the limit, (ii) undefined, or (iii) defined but different from the limit.
- The limit is said to exist only if the following conditions are satisfied:
  (i) The limit L is a finite value (real number).
  (ii) The limit as x approaches c from the left equals the limit as x approaches c from the right. That is, we must have
  $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$

- If $\lim_{x\to\infty} f(x)$ is not defined and takes the form: $\frac{\infty}{\infty}$,
  Divide the numerator and denminator of $f(x)$ by the highest power(degree) of x that has appeared either in the numerator or in the denominator of $f(x)$, then apply limit.
- If $\lim_{x\to\infty} f(x)$ is not defined and takes the form: $\infty - \infty$,
  simplify or rationalize the numerator of $f(x)$, then apply limit.
  End Behavior: The behavior of a function as $x \to \pm\infty$ is called the function's end behavior. At each of the function's ends, the function could exhibit one of the following types of behavior:
- The function $f(x)$ approaches a horizontal asymptote $y = L$.
- The function $f(x) \to \infty$ or $f(x) \to -\infty$.
- The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior. Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim_{x\to -1^+} f(x) = 1$

b) $\lim_{x\to 0^-} f(x) = 0$

c) $\lim_{x\to 0^-} f(x) = 1$

d) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

e) $\lim_{x\to 0} f(x)$ exists

f) $\lim_{x\to 0} f(x) = 0$

g) $\lim_{x\to 0} f(x) = 1$

h) $\lim_{x\to 1} f(x) = 1$

i) $\lim_{x\to 1} f(x) = 0$

j) $\lim_{x\to 2^-} f(x) = 2$

k) $\lim_{x\to -1^-} f(x)$ does not exist.

l) $\lim_{x\to 2^+} f(x) = 0$

Classwork

Evaluate the following limits:

1. $\lim\limits_{x\to 1} \frac{x^2+x-2}{x^2-x}$
2. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$
3. $\lim\limits_{x\to 3} \dfrac{x^3-27}{x^4-81}$
4. $\lim\limits_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$.
5. $\lim\limits_{x\to \pm\infty} \frac{3x^2+2x-1}{5x^2-4x+7}$
6. $\lim\limits_{x\to \infty} \sqrt{3x}-\sqrt{x-a}$

Thank You