Make Reachable

$R_1 = [\,]$ $\quad R_2 = [B_1] = [U S V^T]$

sub ↗ ↙ reduced

$[\hat{B}_1] = [\hat{S}^{-\frac{1}{2}} \hat{V}^T]$ $\quad \Rightarrow$ reduced

$\Rightarrow \hat{C}_2 = C_2 \hat{U} \hat{S}^{-\frac{1}{2}}$

$\hat{A}_2 = A_2 \hat{U} \hat{S}^{-\frac{1}{2}}$

$R_3 = [B_2 \,|\, \hat{A}_2 \hat{B}_1] = [B_2 \,|\, \underbrace{A_2 U S^{-\frac{1}{2}}}_{\hat{A}_2} \underbrace{S^{-\frac{1}{2}} V^T}_{\hat{B}_1}] = [B_2 \,|\, A_2 B_1]$

$= [A_2 B_2] \cdot \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & 1 \end{bmatrix}$ ← rows independent, full Rank

$= U S V^T \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & 1 \end{bmatrix} = U S^{\frac{1}{2}} S^{\frac{1}{2}} V^T \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & 1 \end{bmatrix}$

$\hat{S}^{-\frac{1}{2}} \hat{V}^T$

$[\hat{A}_2 \,\hat{B}_2]$

General

loop up from $i = 1$ to $n-1$

$U S V^T = [A_i \,|\, B_i]$

crop smaller $\sigma_s \Rightarrow \hat{U} \hat{S} \hat{V}^T$

$[\hat{A}_i \,|\, \hat{B}_i] = \hat{S}^{\frac{1}{2}} \hat{V}^T$

$A_{i+1} = A_{i+1} \hat{U} \hat{S}^{\frac{1}{2}}, \quad C_{i+1} = C_{i+1} \hat{U} \hat{S}^{\frac{1}{2}}$

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does it interfere with observ.

$\hat{O}_k = \begin{bmatrix} C_k \\ C_{k+1} A_k \\ \vdots \end{bmatrix} \hat{U} \hat{S}^{\frac{1}{2}}$

↑ ↑

Cm independent

$\det((OU)^T (OU)) =$

$\det(U^T O^T O U) > 0$

$\det > 0$ since O is

$\Leftrightarrow OU$ lin. independent

$\Rightarrow$ take subset

$\Rightarrow$ scale later

Make Observable

$$O_n = [C_n] = [U S V^T]$$

$$C_n = \hat{U} \hat{S}^{-\frac{1}{2}}$$

$$\hat{B}_{n-1} = \hat{S}^{-\frac{1}{2}} \hat{V}^T B_{n-1}$$
$$\hat{A}_{n-1} = \hat{S}^{-\frac{1}{2}} \hat{V}^T A_{n-1}$$

$$O_{n-1} = \begin{bmatrix} C_{n-1} \\ C_n A_{n-1} \end{bmatrix}$$

full rank $\Leftrightarrow$ independent columns

$$= \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & C_n \end{bmatrix} \begin{bmatrix} C_{n-1} \\ A_{n-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & C_n \end{bmatrix} U S V^T$$

$$= \begin{bmatrix} \mathbb{1} & 0 \\ 0 & C_n \end{bmatrix} \hat{U} \hat{S}^{\frac{1}{2}} \hat{S}^{\frac{1}{2}} \hat{V}^T$$

$$\begin{bmatrix} \hat{C}_{n-1} \\ \hat{A}_{n-1} \end{bmatrix}$$

General:

Cholesky factor to R

$$U S V^T = \begin{bmatrix} C_i \\ A_i \end{bmatrix}$$

cut small $\sigma_s \Rightarrow \hat{U} \hat{S} \hat{V}^T$

$$\begin{bmatrix} \hat{C}_i \\ \hat{A}_i \end{bmatrix} = \hat{U} \hat{S}^{\frac{1}{2}}$$

$$\hat{B}_{i-1} = \hat{S}^{\frac{1}{2}} \hat{V}^T \hat{B}_{i-1} \qquad \hat{A}_{i-1} = \hat{S}^{\frac{1}{2}} \hat{V}^T \hat{A}_{i-1}$$