Derivatives with Respect to the Parameters

define a function $T : \mathbb{R}^{r\times r} \times \dots \mathbb{R}^{r\times r} \mapsto \mathbb{R}^{m\times p}$

$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \dots$; Matrix

$$\underline{T} = \begin{bmatrix} D_1 & & & \\ C_2 B_1 & D_2 & & \\ \vdots & & \ddots & \\ C_n A \dots B_1 & \dots & & D_n \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} A & B \\ \hline C & D_1 \ 1 \end{array}\right]}_{T_N} \left[\begin{array}{c|c} A & B \\ \hline C & D_1 \ 1 \end{array}\right] \dots \underbrace{\left[\begin{array}{c|c} A & D \\ \hline C & 1 \ D \end{array}\right]}_{T_1}$$

$$\underset{\substack{\uparrow \\ \text{Derivative } k\text{-th stage}}}{D_{k,V}} T_N \dots T_k\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \dots T_1 = \frac{d}{dt}\Big|_{t=0} T_N \dots T_k\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} + t \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}\right) \dots T_1$$

$$\left[\begin{array}{c|c} A+tA' & B+tB' \\ \hline C+tC' & 1 \ \ D+tD' \end{array}\right]$$

no 1 on diagonal

$$= \Big|_{t=0} T_N \dots \tilde{T}_k\left(\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}\right) \dots T_1$$

$$= T_N \dots \hat{T}_k\left(\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}\right) \dots T_1$$

something like $\boxed{}$ $\operatorname{vec}\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$

$$\operatorname{vec}\Big(\underbrace{T_N \dots}_{\Delta_A} \hat{T}_k\left(\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}\right) \underbrace{\dots T_1}_{\Delta_B}\Big)$$

$$= \underbrace{\left(T_1^T \dots T_{k-1}^T \hat{P}_B^T \otimes T_N \dots T_{k+1} \hat{P}_A\right)}_{D_k} \operatorname{vec}\left(\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}\right)$$

$$\hat{T}_k\left(\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & 1 \\ & 0 \end{bmatrix}}_{\hat{P}_A} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \dots & & 0 \\ 0 & \dots & 1 & 0 \end{bmatrix}}_{\hat{P}_B}$$

$$D(\ ) = \underbrace{\begin{bmatrix} D_1 & \dots & D_n \end{bmatrix}}_{\downarrow} \begin{bmatrix} \operatorname{vec}\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}_1 \\ \vdots \\ \operatorname{vec}\begin{bmatrix} A & B \\ C & D \end{bmatrix}_n \end{bmatrix}$$

Kernel of this is T_n of a Manifold with $T(\dots) = T_{ref}$

$\Rightarrow$ all $\begin{bmatrix} A & B \\ C & D \end{bmatrix}_s$ that are equivalent

BRUNNEN

$$D_{\begin{bmatrix} A_k' & B_k' \\ C_k' & D_k' \end{bmatrix}} T = T_N \cdots T_{k+1} \begin{array}{cccc} x_{k-1} & & & x_k \\ \multicolumn{4}{c}{\begin{pmatrix} A_k' & 0 & B_k' & 0 \\ 0 & 1 & 0 & 0 \\ C_k' & 0 & D_k' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \end{array} T_{k-1} \cdots T_1 = \underbrace{T_N \cdots T_{k+1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{D_L} \begin{bmatrix} A_k' & B_k' \\ C_k' & D_k' \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} T_{k-1} \cdots T_1}_{D_R}$$

cut out collumns $(1, k+1)$

cut out rows $(1, k+1)$

reduces to 0-dim

$$D_L = \begin{bmatrix} A_N \cdots A_{k+1} & 0 \\ & 0 \\ 0 & 1 \\ \begin{bmatrix} 0_{k+1} \end{bmatrix} & 0 \end{bmatrix} \qquad D_R = \begin{bmatrix} A_{k-1} \cdots A_1 & [R_k] & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

also makes intuitive sense

$$\begin{array}{rcc} & \text{from state } x_k & \text{input } u_k \\ \text{state } x_{k+1} \leftarrow & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$}} \\ \text{output } y_{k+1} \leftarrow & & \end{array}$$