Optimum Frobenius Norm

$$f(T) = \|M - T\|_2^2 = tr\left((M-T)(M-T)^T\right)$$

$M \Rightarrow$ original matrix

$T \Rightarrow$ operator

$$D_V \|M - T\|_2^2 = 2\,tr\left(V\,(T-M)^T\right)$$

V: change in operator

Parameter(s): $\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}_k$ for Stage k

existing system

$$D_P \|M - T(P)\|_2^2 = 2\,tr\left(D_P\{T(P)\}\,(T(P) - M)^T\right)$$

Embedding

$$\begin{bmatrix} \hat A & \hat B \\ \hat C & \hat D \\ & 1 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \\ & 1 \end{bmatrix} = \begin{bmatrix} \hat A - A & 0 & \hat B - B \\ \hat C - C & 0 & \hat D - D \\ & & 0 \end{bmatrix}$$

optimization

$$= 2\,tr\left(T_{N-k+1}\,\tilde P_A \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \tilde P_B\, T_{k-1 \dots 1} \left(T_{N-k+1}\,\tilde P_A \left(\begin{bmatrix} \hat A & \hat B \\ \hat C & \hat D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \tilde P_B\, T_{k-1 \dots 1}\right)^T\right)$$

Insert O_{k+1} & R_k

$R_k R_k^T = 1$

$$= 2\,tr\left(\begin{bmatrix} 0 & 0 \\ & 1 \\ T & 0 \\ O_{k+1} & 0 \\ \vdots & \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} \leftarrow R_k \rightarrow & & 0 \\ 0 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ & 1 \\ T & 0 \\ O_{k+1} & \\ \vdots & 0 \end{bmatrix} \left(\begin{bmatrix} \hat A & \hat B \\ \hat C & \hat D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \begin{bmatrix} \leftarrow R_k \rightarrow & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)^T\right)$$

Cyclic property

Transpose

$$= 2\,tr\left(\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} \leftarrow R_k \rightarrow & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \uparrow & \\ R_k^T & 0 \\ \downarrow & \\ & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} \hat A & \hat B \\ \hat C & \hat D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)^T \begin{bmatrix} 0 & \leftarrow O_{k+1}^T \rightarrow \\ 0 \; 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ T & \\ & 0 \\ O_{k+1} & \\ \vdots & \end{bmatrix}\right)$$

$R_k R_k^T = 1$

$O_{k+1}^T O_{k+1} = 1$

$$= 2\,tr\left(\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \left(\begin{bmatrix} \hat A & \hat B \\ \hat C & \hat D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2\,tr\left(\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \left(\begin{bmatrix} \hat A & \hat B \\ \hat C & \hat D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)^T\right)$$

$$\Rightarrow D \|M - T(P)\|_2^2 = 2\left(\begin{bmatrix} \hat A & \hat B \\ \hat C & \hat D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$$

$\Rightarrow$ this should be 0 for $\hat A = A$ etc. this is obviously true

But since we want a reduced state dim for $x_k \iff \begin{bmatrix} \hat A \\ \hat C \end{bmatrix}$ is a low rank matrix $\begin{bmatrix} \hat A \\ \hat C \end{bmatrix} = \hat U \hat \Sigma \hat V^T$

$$\iff x_{k\,new} = \underbrace{S}_{\text{invertible matrix}} \hat V\, x_k \iff x_k = \hat V^T S^{-1} x_{k\,new}$$

As there is no restriction on $\hat B$ and $\hat D$ we can set $\hat B = B$ and $\hat D = D$

$$\Rightarrow \begin{bmatrix} \hat A & B \\ \hat C & D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \hat A - A & 0 \\ \hat C - C & 0 \end{bmatrix}$$

this gives the reduced derivative

$$D_{red} = \begin{bmatrix} \hat A \\ \hat C \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix}$$

Now we calculate the Riemannian gradient on the manifold M_k

for this we project the Gradient on the Tangent space $T_x M_x$

if $\begin{bmatrix}\hat{A}\\ \hat{C}\end{bmatrix} = \hat{U}\hat{\Sigma}\hat{V}^T$ we get with [Vandereycken] (p. 1221) 1218

$$P_{T_{\left[\hat{A};\hat{C}\right]} M_k} = P_{\hat{U}} Z P_{\hat{V}} + (1-P_U) Z P_V + P_U \left(Z (1-P_V)\right)^T$$

select some σ_s

we obtain $U \Sigma V^T = \text{svd}\left(\begin{bmatrix}A\\ C\end{bmatrix}\right)$ and set $\hat{U} = U[:, abc]$ etc...

Thus general gradient becomes $\begin{bmatrix}\hat{A}\\ \hat{C}\end{bmatrix} - \begin{bmatrix}A\\ C\end{bmatrix} = U\Sigma V^T - \hat{U}\Sigma\hat{V}^T = U(\Sigma - \hat{\Sigma}_{org}) V^T$

$$= U \begin{bmatrix}0 & 0\\ 0 & \Sigma_r\end{bmatrix} V^T$$

$$D = \hat{U}\underbrace{\hat{U}^T U}_{\begin{bmatrix}1 & 0\end{bmatrix}} \begin{bmatrix}0 & 0\\ 0 & \Sigma_r\end{bmatrix} \underbrace{V^T \hat{V}}_{\begin{bmatrix}1\\ 0\end{bmatrix}} \hat{V}^T + (1-\hat{U}\hat{U}^T) U \begin{bmatrix}0 & 0\\ 0 & \Sigma_r\end{bmatrix} \underbrace{V^T\hat{V}}_{\begin{bmatrix}1\\ 0\end{bmatrix}}\hat{V}^T + \hat{U}\underbrace{\hat{U}^T U}_{\begin{bmatrix}1 & 0\end{bmatrix}} \begin{bmatrix}0 & 0\\ 0 & \Sigma_r\end{bmatrix} (1-\hat{V}\hat{V}^T)$$

$$= 0 \qquad = 0 \qquad = 0$$

$\Rightarrow$ is critical point $\rightarrow$ which one has lowest value?

$$\| M - T \|_2^2 = tr\left((M-T)(M-T)^T\right) =$$

$$tr\left(\begin{bmatrix}0 & 0\\ 0 & 1\\ I & 0\\ O_{k+n} & \\ I & 0\end{bmatrix}\left(\begin{bmatrix}\hat{A} & \hat{B}\\ \hat{C} & \hat{D}\end{bmatrix} - \begin{bmatrix}A & B\\ C & D\end{bmatrix}\right)\underbrace{\begin{bmatrix}I & R_k & -1 & 0 & 0\\ 0 & & & 1 & 0\end{bmatrix}\begin{bmatrix}0 & I\\ 0 & R_k^T\\ & I\\ 1 & 0\\ 0 & 0\end{bmatrix}}_{=1}\left(\begin{bmatrix}\hat{A} & \hat{B}\\ \hat{C} & \hat{D}\end{bmatrix} - \begin{bmatrix}A & B\\ C & D\end{bmatrix}\right)\begin{bmatrix}I & O_{k+n}^T & -1 & 0 & 0\\ 0 & - & 0 & 1 & 0\end{bmatrix}\right)$$

$$= tr\left(\left(\begin{bmatrix}\hat{A} & \hat{B}\\ \hat{C} & \hat{D}\end{bmatrix} - \begin{bmatrix}A & B\\ C & D\end{bmatrix}\right)\left(\begin{bmatrix}\hat{A} & \hat{B}\\ \hat{C} & \hat{D}\end{bmatrix} - \begin{bmatrix}A & B\\ C & D\end{bmatrix}\right)^T\right) = tr\left(U \begin{bmatrix}0 & 0\\ 0 & \Sigma_r\end{bmatrix} V^T V \begin{bmatrix}0 & 0\\ 0 & \Sigma_r\end{bmatrix} U^T\right)$$

$= \sum \sigma_r$ ← sum over cut σ_s → choose smallest σ_s

Are these the σ_s of the Hankel operator?

$$H_k = O_k R_k = \begin{bmatrix}1\\ I\\ O_{k+n}\\ I\end{bmatrix}\begin{bmatrix}A_k\\ C_k\end{bmatrix} R_k = \underbrace{\begin{bmatrix}1\\ I\\ O_{k+n}\\ I\end{bmatrix} U}_{U'} \Sigma \underbrace{V^T R_k}_{V'} \Rightarrow \text{is svd of } H \quad \square$$