Seoul National University

Lecture Note

Introduction to Stochastic Differential Equations

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Contents

Chapter 0

Introduction

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Office Hour: Tuesday 15:00 - 16:00
Grading

- Mid-terms 1 (15%, 10/10 or 17)
- Mid-terms 2 (15%, 11/7)
- Fianl-term (40%)
- Assignment (20%, 8-10 times)
- Attendance (10%, absent: -2%, late: -1%)

Let X be a standard normal random variable in $\mathbb{R}$. i.e., $\mathbb{P}[X \in [a,b]] = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.
(Central Limit Theorem) If $x_1, x_2, \cdots, x_n \in X, E(x_i) = m, Var(x_i) = \sigma^2$, then

$$\frac{\frac{x_1-m}{\sigma} + \frac{x_2-m}{\sigma} + \cdots + \frac{x_n-m}{\sigma}}{\sqrt{n}} \to X$$

In this class, we study dynamic version of this theorem. If $(W_t)_{t\geq 0}$ be a fluctuation, then $(W_t)_{t\geq 0}$ be a random variable in $C[0,T]$
Example. $\frac{dX_t}{dt} = rX_t; dX_1 = rX_t dt$. Then, $X_t = X_0 e^{rt}$ (unrisky assets, bank)
$dX_t = rX_t dt + \sigma X_t dW_t$, σ : volatility (risky assets, stock)
We will study:

1. Probability Space
2. Random Variable
3. Expectation

Textbooks:

1. Stochastic Calculus for Finance II (Shreve), covering chapter 1-3 or 4
2. Introduction to Stochastic Integration (Hui-Hsiung Kuo)

Chapter 1

General Probability Theory

1.1 Infinite Probability Spaces

There are three elements consisting probability space:

- S: Sample space
- $\mathcal{E}$: Family of events $\mathcal{E} \subseteq 2^S$ (σ-algebra in measure theory)
- $\mathbb{P}$: probability $\Rightarrow \mathbb{P}(E)$ is defined for all $E \in \mathcal{E}$ (μ with $\mu(S) = 1$)

Example.

1. Toss a coin twice (H for Head, T for Tail)
   Then, $S = \{HH, HT, TT, TT\}$
2. Uniform random variable in $[0,1]^3$
   Then, $S = [0,1]^3$. If $E = [0, \frac{1}{2}]^3$, then $\mathbb{P}(E) = Vol(E) = \frac{1}{8}$

How to define $\mathcal{E}$?
In example 2, let $\mathcal{E}$ = family of all subsets of $[0,1]^3$ naively. But Banach-Tarski Paradox says there are disjoint sets E, F with $\mathbb{P}(E \cup F) \neq \mathbb{P}(E) + \mathbb{P}(F)$ in this $\mathcal{E}$. Therefore we cannot naively set $\mathcal{E}$ (Use measure theory)

In example 1, suppose that we cannot see the second flip. If $\{HH\} \notin \mathcal{E}$ and $\{HT, HH\} \in \mathcal{E}$, then $\mathcal{E} = \{\phi, \{HH, HT\}, \{TH, TT\}, \{HH, HT, TH, TT\}\}$

Definition 1.1.1 (Measure)
Let Ω be a non-empty set and $\mathcal{F}$ be family of subsets of Ω with

1. $\phi \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$
3. $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

We say $\mathcal{F}$ as **σ-albegra** or **σ-field**, $A \in \mathcal{F}$ as **measurable**, and Ω as **measurable space**.

Exercises.

1) $\Omega \in \mathcal{F}$

2) $A_1, A_2, \cdots \in \mathcal{F}$, then $A_1 \cap A_2 \cdots \in \mathcal{F}$

3) $A_1, A_2, \cdots \in A_n \in \mathcal{F}$, then $A_1 \cup \cdots \cup A_n, A_1 \cap \cdots \cap A_n \in \mathcal{F}$.

4) $A, B \in \mathcal{F}$, then $A - B \in \mathcal{F}$

Definition 1.1.2 (Topological Space)
(See Rudin: *Real and Complex Analysis, Chapter 1.*) Let Θ be non-empty set and τ be family of subsets of Θ with

1. $\phi, \Theta \in \tau$

2. $V_1, \cdots V_n \in \tau \Rightarrow V_1 \cap \cdots \cap V_n \in \tau$

3. $V_\alpha \in \tau \ \ \forall \alpha \in I \Rightarrow \bigcup_{\alpha \in I} V_\alpha \in \tau$.

We say $V \in \tau$ be an **open set**, and (Θ, τ) be a **topological space**.

Definition 1.1.3 (Measurable Function)
$f : (\Omega, \mathcal{F}) \to (\Theta, \tau)$ is **measurable** if $f^{-1}(V) \in \mathcal{F} \ \ \forall V \in \tau$

Definition 1.1.4 (Positive Measure)
Let Ω be non-empty set and $\mathcal{F}$ be σ-algebra. Then $\mu : \mathcal{F} \to [0, \infty]$ is called **measurable** if

1. $A_1, A_2, \cdots$: disjoint members of $\mathcal{F} \Rightarrow \mu(A_1 \cup A_2 \cup \cdots) = \sum_{i=1}^{\infty} \mu(A_i)$

2. $\mu(A) < \infty$ for some $A \in \mathcal{F}$,

and $(\Omega, \mathcal{F}, \mu)$ is called a **measrue space**.

Definition 1.1.5 (probability space, random variable)

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is called as **probability space** if $\mathbb{P}(\Omega) = 1$.

2. X is called as **random varaible** if it is a function from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$

<u>Next Class</u>

- Borel sets on $\mathbb{R}$ or $\mathbb{R}^d$

- Lebesgue Measure

- Lebesgue Integral (Define Expectation of random variable)

Last class, we define a sample space Ω, a σ-algebra $\mathcal{F}$, and a (positive) measure $\mu : \mathcal{F} \to [0, \infty]$.

Exercises.

- $A_1 \subseteq A_2 \subseteq \cdots \Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n)$
- $A_1 \supseteq A_2 \supseteq \cdots, \mu(A_1) < \infty \Rightarrow \mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n)$

Theorem 1.1.6 (Rudin 1.10)
Let $\mathcal{F}_0$ be a collection of subset of Ω. Then, $\exists! \mathcal{F}^*$ minimal σ-algebra containing $\mathcal{F}_0$.

Proof. Let $\{\mathcal{F}_\alpha, \alpha \in I\}$ be a family of σ-algebra containing $\mathcal{F}_0$. Then, $\mathcal{F}^* = \bigcap_{\alpha \in I} F_\alpha$ satisfies the three condition: 1) contain $\mathcal{F}_0$ 2) σ-algebra 3) minimal (trivial, $\mathcal{F}^* \subseteq \mathcal{F}_\alpha$) □

Definition 1.1.7 (Borel measurable)
$\mathcal{B}$ is a **Borel σ-algebra** on topological space (Θ, τ) if $\mathcal{B}$ is minimal σ-algebra containing τ, and B is a **Borel measurable** if $B \in \mathcal{B}$.

Remark (Completion of measure space, Rudin 1.15).
Consider an extension $(\Omega, \mathcal{F}, \mu) \to (\Omega, \overline{\mathcal{F}}, \mu)$ where

1. $\overline{\mathcal{F}} = \{A \cup N : A \in \mathcal{F}, N \subseteq A_0 \in \mathcal{F}, \mu(A_0) = 0\}$
2. $\mu(A \cup N) = \mu(A)$

Then, (Check!)

1. (well-definedness) $A_1 \cup N_1 = A_2 \cup N_2 \Rightarrow \mu(A_1) = \mu(A_2)$
2. $\mu : \overline{\mathcal{F}}$ is σ-algebra.
3. $\mu : \overline{\mathcal{F}} \to [0, \infty]$ is a measure

Example.

1) $\mathbb{R}$

$$\mathcal{F}_0 = \tau \xrightarrow{1.10} \mathcal{B} \xrightarrow{\text{completion}} \overline{\mathcal{B}}$$

$$\mathcal{L} \xrightarrow{\text{Rudin CH 2}} \mathcal{L} \xrightarrow{\text{completion}} \mathcal{L}$$

2) $C[0, T] = \Omega = \{f; f : [0, T] \to \mathbb{R}, \text{continuous}\}$.
Define $\mathcal{F}_0 = \{\bigcup_{t_1, t_2, \cdots, t_k}(A_1, A_2, \cdots, A_k) : 0 \le t_1 < t_2 < \cdots < t_k \le T; A_1, \cdots A_k \in \overline{\mathcal{B}}\}$. We call $\{f \in C[0, T] : f(t_1) \in A_1, f(t_2) \in A_2, \cdots, f(t_k) \in A_k\}$ as **cylindrical set**. Consider

$$\begin{array}{ccccc} \mathcal{F}_0 & \stackrel{1.10}{\longrightarrow} & \mathcal{B} & \stackrel{\text{completion}}{\longrightarrow} & \overline{\mathcal{B}} \\ \mathbb{P}_{\text{BM}} & \stackrel{\text{KET}}{\longrightarrow} & \mathbb{P}_{\text{BM}} & \stackrel{\text{completion}}{\longrightarrow} & \mathbb{P}^*_{\text{BM}} \end{array}$$

(KET refers Kolmogorov's Extension Thm)

1.2 Random Variables and Distributions

Definition 1.2.1
$f : \Omega \to \mathbb{R}$ is measurable if $f^{-1}(V) \in \mathcal{F}$ for any open set $V \subseteq \mathbb{R}$.

Remark. $\mathcal{B}(\mathbb{R})$ = Borel σ-algebra in $\mathbb{R}$.

Remark. If f: measurable, then $f^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}(\mathbb{R})$.

Proof. Let $G = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \mathcal{F}\}$. Then, $\tau \subseteq G$, G : σ-algebra (check!), hence $\mathcal{B}(\mathbb{R}) \subseteq G$. □

Definition 1.2.2

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a **probability space** if $(\mathbb{P}(\Omega) = 1$.
- X is **random variable** if $X : \Omega \to \mathbb{R}$ is measruable.

Example.

1. Toss a coin Twice.
   $\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F} = 2^{\Omega} = \{\text{all subsets of } \Omega\}$, $\mathbb{P}(A) = \frac{1}{4}|A|,\ \ A \in \mathcal{F}$.
   Then, X = the number of H's is random variable with $X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 1$.
2. Uniform random variable in $[0, 1]$
   $\Omega = [0, 1]$, $\mathcal{F} = \{B \in \mathcal{B}(\mathbb{R}) : B \subseteq [0, 1]\}$, $\mathbb{P}(B) = \mathcal{L}(B)$ $(\mathbb{P}([0, 1]) = \mathcal{L}([0, 1]) = 1)$.
   Then, $X : [0, 1] \to \mathbb{R}$ with $X(x) = x$ be a (uniform) random variable in $[0, 1]$.

Remark. $\mathcal{L}$: Lebesgue measrue on $\mathbb{R}$. i.e.,$\mathcal{L}(a, b) = b - a$. Then, $\mathcal{L}(\{a\}) = 0$
$(\because \{a\} = \bigcap_{i=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n}) \Rightarrow \mathcal{L}(\{a\}) = \lim_{n\to\infty} \mathcal{L}((a - \frac{1}{n}, a + \frac{1}{n})) = 0)$
Similarly, $\mathcal{L}([a, b]) = \mathcal{L}([a, b)) = \mathcal{L}((a, b]) = b - a$, $\mathcal{L}(\mathbb{Q}) = \sum_{q\in\mathbb{Q}} \mathcal{L}(\{q\}) = 0$.

Return to uniform random variable,

$$\mathbb{P}[X \in (a, b)] = \mathbb{P}[\{x : X(x) \in (a, b)\}] = \mathbb{P}[(a, b)] = b - a.$$

Definition 1.2.3 (Distribution measure on X)
X is a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. μ_X is a **distribution measure** on X if μ_X is a probability measrue on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mu_X(B) = \mathbb{P}[X \in B] = \mathbb{P}[\{\omega : X(\omega) \in B\}] = \mathbb{P}[X^{-1}(B)]\ \forall B \in \mathcal{B}(\mathbb{R})$$

Definition 1.2.4 (Probability density function)
f is a **probability density function** of X if $\mu_X((a, b)) = \int_a^b f(x)dx$

Remark. There is a measure with no pdf: Dirac measrue

Remark. Lebesgue-Radon-Nikodym decomposition implies that any measure can be decomposed as density part and non-density part.

Example (Standard Normal random variable).
Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Define $F : (0,1) \to \mathbb{R}$ by $F(x) = N^{-1}(x)$ for $N(X) = \int_{-\infty}^{x} \phi(y)dy$.
Let $\Omega = (0,1), \mathcal{F} = \{B \in \mathcal{B}(\mathbb{R}) : B \subseteq (0,1)\}, \mathbb{P}(A) = \mathcal{L}(A) : A \in \mathcal{B}(\mathbb{R})$.
Then, $Y : \Omega \ni x \mapsto F(x) \in \mathbb{R}$ is a random variable with

$$\begin{aligned}
\mathbb{P}[Y \in (a,b)] &= \mathbb{P}[\{x : Y(x) \in (a,b)\}] \\
&= \mathbb{P}[\{x \in (N(a), N(b))\}] \\
&= N(b) - N(a) = \int_a^b \phi(x)dx,
\end{aligned}$$

and a density function is ϕ.

Previous Question: In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variable $X : \Omega \to \mathbb{R}$, the random element or random realization $\omega \in \Omega$ is a element of events in sample space. For example, $\omega = HHTTH$ is a random element in tossing a coin five times, and $X(\omega) = 3$. ($X(\omega)$ = # of Heads)
In the previous example(Standard Normal random variable), define $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1), \mathcal{B}(0,1), \mathbb{P})$, $\mathbb{P}((a,b)) = b - a$, $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$, $X : (0,1) \ni \omega \mapsto F^{-1}(\omega) \in \mathbb{R}$. Then, X is called a standard normal random variable.

1.3 Expectations

In the following, let $\Omega = (\Omega, \mathcal{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$. Then the expection $E(X)$ is a mean of $X(\omega)$ with respect to the randomness of ω (given by $\mathbb{P}$)

Definition 1.3.1 (Lebesgue Integration)
$(\Omega, \mathcal{F}, \mu)$ is a measure space, and $f : \Omega \to \mathbb{R}$ is a measurable function.

(1) $f : \Omega \to [0, \infty)$
Let $0 = y_0 < y_1 < y_2 < \cdots \to \mathbb{R}$ be a partition of $[0, \infty)$,
$\Pi = \{y_0, y_1, y_2, \cdots\} : \|\Pi\| = \sup_{i \geq 1} |y_i - y_{i-1}|$, and
$\mathrm{LS}_\pi = \sum_{i=0}^{\infty} y_i \mu[f^{-1}([y_i, y_{i+1}))]$.
In Rudin's book, $\lim_{\|\Pi\| \to 0} \mathrm{LS}_\pi$ converges to an element belonging to $[0, \infty]$.
Now, $\int f d\mu := \lim_{\|\Pi\| \to 0} \mathrm{LS}_\Pi$ is called a **Lebesgue Integral**.

(2) $f : \Omega \to \mathbb{R}$
Let $f^+ = \max\{f, 0\} \geq 0$, and $f^- = -\min\{f, 0\} \geq 0$. Then, $f = f^+ - f^-$, and $|f| = f^+ + f^-$. If $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$, then we say f is Lebesgue Integrable and $f \in L^1(\mu)$. The Lebesgue Integral of f $\int f d\mu$ is defined as $\int f^+ d\mu - \int f^- d\mu$

Remark.

1. $\int f^+ d\mu < \infty$ and $\int f^- d\mu = \infty$, then $\int f d\mu = -\infty$. The others are defined similarly.
2. $f \in L^1(\mu) \Leftrightarrow \int |f| d\mu < \infty$.

Example (Riemann vs Lebesgue Integral (p19-22)).

- $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{L})$ Lebesgue measure where $\mathcal{L}((a,b)) = b - a$.
- $f : \mathbb{R} \to \mathbb{R} \in L^1(\mathcal{L})$
- (Def) $A \subseteq \mathbb{R}$, $\int_A f d\mu := \int f \mathbb{1}_A d\mu$, where $\mathbb{1}_A(x) = 1$ if $x \in A$, and 0 otherwise.

If f is Riemann integrable, then $\int_{[a,b]} f d\mathcal{L} = \int_a^b f dx$.
Riemann integral is a limit of approximation by a partition of x-axis. On the other hand, Lebesgue integral is a limit of approximation by a partition of y-axis with preimage. Partition of x-axis is sensitive to fluctuation and restricted to Euclidean space, while partition of y-axis is not. For example, $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ is Lebesgue integrable, but it is not Riemann integrable since it is sensitive to fluctuation.

Definition 1.3.2 (Almost everywhere, 1.1.5 in Textbook)
$P(x)$ is a property at $x \in \mathbb{R}$. We say P holds **almost everywhere** (or a.e.) in $\mathbb{R}$ if and only if $\mathcal{L}(\{x : P(x) \text{ does not hold }\} = 0$.

Example. $f(x) = [x]$ is continuous almost everywhere.

Theorem 1.3.3
f is Riemann integrable if and only if f is continuous a.e.

Exercises. $f = g$ a.e. $\Rightarrow \int f d\mathcal{L} = \int g d\mathcal{L}$.

Definition 1.3.4 (Almost surely)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The event $A (\in \mathcal{F})$ occusrs **almost surely** (a.s.) if $\mathbb{P}(A) = 1$.

Example. Let X be a uniform random variable in $(0,1)$. Let $A = \{X(\omega) \neq \frac{1}{2}\}$; $\mathbb{P}(A) = 1$.

Definition 1.3.5 (Expectation, 1.3.3. in Textbook)
Expectation of $X : \Omega \to \mathbb{R}$ is defined by

$$E(X) := \int_\Omega X d\mathbb{P} \quad \text{if} \quad \int_\Omega |X| d\mathbb{P} < \infty$$

Theorem 1.3.6 (1.3.4 in Textbook)

1. *X takes finite number of values $\{x_1, x_2, \cdots, x_n\} \Rightarrow E(X) = \sum_{i=1}^n x_i \mathbb{P}(X = x_i)$*

2. *X, Y: random variables, $E(|X|), E(|Y|) < \infty$,*

   (i) *$X \leq Y$ a.s. (i.e. $\mathbb{P}[\{X(\omega) \leq Y(\omega)\}] = 1$), then $E(X) \leq E(Y)$*

   (ii) *$X = Y$ a.s. $\Rightarrow E(X) = E(Y)$*

3. *X, Y: random variables, $E(|X|), E(|Y|) < \infty \Rightarrow E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y)$.*

4. *Jensen's Inequality: $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function $\Rightarrow \phi(E(X)) \leq E(\phi(X))$ (c.f. $\phi(t) = t^2$)*

Proof of 4. Define $S_\phi = \{(a, b) \in \mathbb{R}^2 : a + bt \leq \phi(t) \ \ \forall t\}$. Then $\forall t \in \mathbb{R}, \phi(t) = \sup_{(a,b)\in S_\phi}\{a + bt\}$. In fact, it is a equivalent condition. Now,

$$\begin{aligned}
\phi(E(X)) &= \sup_{a,b \in S_\phi} \{a + bE(X)\} \\
&= \sup_{a,b \in S_\phi} E(a + bX) \\
&\leq E[\sup_{a,b \in S_\phi} (a + bx)] = E(\phi(X)) \quad \text{(Check!)}
\end{aligned}$$

□

Example (Dirac Measure in $\mathbb{R}$). $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \delta_y)$ $(y \in \mathbb{R})$ is a probability space with $\delta_y(A) = 1$ if $y \in A$, and 0 otherwise. Then, $\int_{\mathbb{R}} f d\delta_y = f(y)$ (Check!)

Consider modeling: X: random variable such that probability of $x_i = p_i$ with $\sum_{i=1}^n p_i = 1$. Then, $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ with $\mu = \sum_{i=1}^n p_i \delta_{x_i}$ is a probability space, and $P(X = x_i) = p_i$ for $X : \mathbb{R} \ni \omega \mapsto \omega \in \mathbb{R}$: Example of thm 1.3.4.