Seoul National University

Lecture Note

Introduction to Stochastic Differential Equations

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Contents

Chapter 0

Introduction

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Office Hour: Tuesday 15:00 - 16:00
Grading

- Mid-terms 1 (15%, 10/10 or 17)
- Mid-terms 2 (15%, 11/7)
- Fianl-term (40%)
- Assignment (20%, 8-10 times)
- Attendance (10%, absent: -2%, late: -1%)

Let X be a standard normal random variable in $\mathbb{R}$. i.e., $\mathbb{P}[X \in [a,b]] = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.
(Central Limit Theorem) If $x_1, x_2, \cdots, x_n \in X, E(x_i) = m, Var(x_i) = \sigma^2$, then

$$\frac{\frac{x_1-m}{\sigma} + \frac{x_2-m}{\sigma} + \cdots + \frac{x_n-m}{\sigma}}{\sqrt{n}} \to X$$

In this class, we study dynamic version of this theorem. If $(W_t)_{t\geq 0}$ be a fluctuation, then $(W_t)_{t\geq 0}$ be a random variable in $C[0,T]$

Example. $\frac{dX_t}{dt} = rX_t; dX_1 = rX_t dt$. Then, $X_t = X_0 e^{rt}$ (unrisky assets, bank)
$dX_t = rX_t dt + \sigma X_t dW_t$, σ : volatility (risky assets, stock)

We will study:

1. Probability Space
2. Random Variable
3. Expectation

Textbooks:

1. Stochastic Calculus for Finance II (Shreve), covering chapter 1-3 or 4
2. Introduction to Stochastic Integration (Hui-Hsiung Kuo)

Chapter 1

General Probability Theory

1.1 Infinite Probability Spaces

There are three elements consisting probability space:

- S: Sample space
- $\mathcal{E}$: Family of events $\mathcal{E} \subseteq 2^S$ (σ-algebra in measure theory)
- $\mathbb{P}$: probability $\Rightarrow \mathbb{P}(E)$ is defined for all $E \in \mathcal{E}$ (μ with $\mu(S) = 1$)

Example.

1. Toss a coin twice (H for Head, T for Tail)
   Then, $S = \{HH, HT, TT, TT\}$

2. Uniform random variable in $[0,1]^3$
   Then, $S = [0,1]^3$. If $E = [0, \frac{1}{2}]^3$, then $\mathbb{P}(E) = Vol(E) = \frac{1}{8}$

How to define $\mathcal{E}$?
In example 2, let $\mathcal{E}$ = family of all subsets of $[0,1]^3$ naively. But Banach-Tarski Paradox says there are disjoint sets E, F with $\mathbb{P}(E \cup F) \neq \mathbb{P}(E) + \mathbb{P}(F)$ in this $\mathcal{E}$. Therefore we cannot naively set $\mathcal{E}$ (Use measure theory)

In example 1, suppose that we cannot see the second flip. If $\{HH\} \notin \mathcal{E}$ and $\{HT, HH\} \in \mathcal{E}$, then $\mathcal{E} = \{\phi, \{HH, HT\}, \{TH, TT\}, \{HH, HT, TH, TT\}\}$

Definition 1.1.1 (Measure)
Let Ω be a non-empty set and $\mathcal{F}$ be family of subsets of Ω with

1. $\phi \in \mathcal{F}$

2. $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$

3. $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

We say $\mathcal{F}$ as **σ-algebra** or **σ-field**, $A \in \mathcal{F}$ as **measurable**, and Ω as **measurable space**.

Exercises.

1) $\Omega \in \mathcal{F}$

2) $A_1, A_2, \cdots \in \mathcal{F}$, then $A_1 \cap A_2 \cdots \in \mathcal{F}$

3) $A_1, A_2, \cdots \in A_n \in \mathcal{F}$, then $A_1 \cup \cdots \cup A_n, A_1 \cap \cdots \cap A_n \in \mathcal{F}$.

4) $A, B \in \mathcal{F}$, then $A - B \in \mathcal{F}$

Definition 1.1.2 (Topological Space)
(See Rudin: *Real and Complex Analysis, Chapter 1.*) Let Θ be non-empty set and τ be family of subsets of Θ with

1. $\phi, \Theta \in \tau$

2. $V_1, \cdots V_n \in \tau \Rightarrow V_1 \cap \cdots \cap V_n \in \tau$

3. $V_\alpha \in \tau \ \ \forall \alpha \in I \Rightarrow \bigcup_{\alpha \in I} V_\alpha \in \tau$.

We say $V \in \tau$ be an **open set**, and (Θ, τ) be a **topological space**.

Definition 1.1.3 (Measurable Function)
$f : (\Omega, \mathcal{F}) \to (\Theta, \tau)$ is **measurable** if $f^{-1}(V) \in \mathcal{F} \ \ \forall V \in \tau$

Definition 1.1.4 (Positive Measure)
Let Ω be non-empty set and $\mathcal{F}$ be σ-algebra. Then $\mu : \mathcal{F} \to [0, \infty]$ is called **measurable** if

1. $A_1, A_2, \cdots$: disjoint members of $\mathcal{F} \Rightarrow \mu(A_1 \cup A_2 \cup \cdots) = \sum_{i=1}^{\infty} \mu(A_i)$

2. $\mu(A) < \infty$ for some $A \in \mathcal{F}$,

and $(\Omega, \mathcal{F}, \mu)$ is called a **measure space**.

Definition 1.1.5 (probability space, random variable)

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space** if $\mathbb{P}(\Omega) = 1$.

2. X is called a **random variable** if it is a function from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$

<u>Next Class</u>

- Borel sets on $\mathbb{R}$ or $\mathbb{R}^d$
- Lebesgue Measure
- Lebesgue Integral (Define Expectation of random variable)

Last class, we define a sample space Ω, a σ-algebra $\mathcal{F}$, and a (positive) measure $\mu : \mathcal{F} \to [0, \infty]$.

Exercises.

- $A_1 \subseteq A_2 \subseteq \cdots \Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n)$
- $A_1 \supseteq A_2 \supseteq \cdots, \mu(A_1) < \infty \Rightarrow \mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n)$

Theorem 1.1.6 (Rudin 1.10)
Let $\mathcal{F}_0$ be a collection of subset of Ω. Then, $\exists! \mathcal{F}^*$ minimal σ-algebra containing $\mathcal{F}_0$.

Proof. Let $\{\mathcal{F}_\alpha, \alpha \in I\}$ be a family of σ-algebra containing $\mathcal{F}_0$. Then, $\mathcal{F}^* = \bigcap_{\alpha \in I} F_\alpha$ satisfies the three condition: 1) contain $\mathcal{F}_0$ 2) σ-algebra 3) minimal (trivial, $\mathcal{F}^* \subseteq \mathcal{F}_\alpha$) □

Definition 1.1.7 (Borel measurable)
$\mathcal{B}$ is a **Borel σ-algebra** on the topological space (Θ, τ) if $\mathcal{B}$ is a minimal σ-algebra containing τ, and B is a **Borel measurable** if $B \in \mathcal{B}$.

Remark (Completion of measure space, Rudin 1.15).
Consider an extension $(\Omega, \mathcal{F}, \mu) \to (\Omega, \overline{\mathcal{F}}, \mu)$ where

1. $\overline{\mathcal{F}} = \{A \cup N : A \in \mathcal{F}, N \subseteq A_0 \in \mathcal{F}, \mu(A_0) = 0\}$
2. $\mu(A \cup N) = \mu(A)$

Then, (Check!)

1. (well-definedness) $A_1 \cup N_1 = A_2 \cup N_2 \Rightarrow \mu(A_1) = \mu(A_2)$
2. $\mu : \overline{\mathcal{F}}$ is σ-algebra.
3. $\mu : \overline{\mathcal{F}} \to [0, \infty]$ is a measure

Example.

1) $\mathbb{R}$

$$\mathcal{F}_0 = \tau \xrightarrow{1.10} \mathcal{B} \xrightarrow{\text{completion}} \overline{\mathcal{B}}$$

$$\mathcal{L} \xrightarrow{\text{Rudin CH 2}} \mathcal{L} \xrightarrow{\text{completion}} \mathcal{L}$$

2) $C[0,T] = \Omega = \{f; f : [0,T] \to \mathbb{R}, \text{continuous}\}$.
Define $\mathcal{F}_0 = \{\bigcup_{t_1,t_2,\cdots,t_k}(A_1, A_2, \cdots, A_k) : 0 \le t_1 < t_2 < \cdots < t_k \le T; A_1, \cdots A_k \in \overline{\mathcal{B}}\}$. We call $\{f \in C[0,T] : f(t_1) \in A_1, f(t_2) \in A_2, \cdots, f(t_k) \in A_k\}$ as **cylindrical set**. Consider

$$\begin{array}{ccccc} \mathcal{F}_0 & \xrightarrow{1.10} & \mathcal{B} & \xrightarrow{\text{completion}} & \overline{\mathcal{B}} \\ \mathbb{P}_{\text{BM}} & \xrightarrow{\text{KET}} & \mathbb{P}_{\text{BM}} & \xrightarrow{\text{completion}} & \mathbb{P}^*_{\text{BM}} \end{array}$$

(KET refers Kolmogorov's Extension Thm)

1.2 Random Variables and Distributions

Definition 1.2.1
$f : \Omega \to \mathbb{R}$ is measurable if $f^{-1}(V) \in \mathcal{F}$ for any open set $V \subseteq \mathbb{R}$.

Remark. $\mathcal{B}(\mathbb{R})$ = Borel σ-algebra in $\mathbb{R}$.

Remark. If f: measurable, then $f^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}(\mathbb{R})$.

Proof. Let $G = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \mathcal{F}\}$. Then, $\tau \subseteq G$, G : σ-algebra (check!), hence $\mathcal{B}(\mathbb{R}) \subseteq G$. □

Definition 1.2.2

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a **probability space** if $(\mathbb{P}(\Omega) = 1$.
- X is a **random variable** if $X : \Omega \to \mathbb{R}$ is measurable.

Example.

1. Toss a coin Twice.
$\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F} = 2^\Omega = \{\text{all subsets of } \Omega\}$, $\mathbb{P}(A) = \frac{1}{4}|A|, \ \ A \in \mathcal{F}$.
Then, X = the number of H's is a random variable with $X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 1$.

2. Uniform random variable in $[0,1]$
$\Omega = [0,1]$, $\mathcal{F} = \{B \in \mathcal{B}(\mathbb{R}) : B \subseteq [0,1]\}$, $\mathbb{P}(B) = \mathcal{L}(B)$ ($\mathbb{P}([0,1]) = \mathcal{L}([0,1]) = 1$).
Then, $X : [0,1] \to \mathbb{R}$ with $X(x) = 1$ is a (uniform) random variable in $[0,1]$.

Remark. $\mathcal{L}$: Lebesgue measure on $\mathbb{R}$. i.e.,$\mathcal{L}(a,b) = b - a$. Then, $\mathcal{L}(\{a\}) = 0$
$(\because \{a\} = \bigcap_{i=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n}) \Rightarrow \mathcal{L}(\{a\}) = \lim_{n\to\infty} \mathcal{L}((a - \frac{1}{n}, a + \frac{1}{n})) = 0)$
Similarly, $\mathcal{L}([a,b]) = \mathcal{L}([a,b)) = \mathcal{L}((a,b]) = b - a$, $\mathcal{L}(\mathbb{Q}) = \sum_{q\in\mathbb{Q}} \mathcal{L}(\{q\}) = 0$.

Return to uniform random variable,

$$\mathbb{P}[X \in (a,b)] = \mathbb{P}[\{x : X(x) \in (a,b)\}] = \mathbb{P}[(a,b)] = b - a.$$

Definition 1.2.3 (Distribution measure on X)
X is a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. μ_X is a **distribution measure** on X if μ_X is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mu_X(B) = \mathbb{P}[X \in B] = \mathbb{P}[\{\omega : X(\omega) \in B\}] = \mathbb{P}[X^{-1}(B)] \; \forall B \in \mathcal{B}(\mathbb{R})$$

Definition 1.2.4 (Probability density function)
f is a **probability density function** of X if $\mu_X((a,b)) = \int_a^b f(x)dx$

Remark. There is a measure with no pdf: Dirac measure

Remark. Lebesgue-Radon-Nikodym decomposition implies that any measure can be decomposed as density part and singular part.

Example (Standard Normal random variable).
Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Define $F : (0,1) \to \mathbb{R}$ by $F(x) = N^{-1}(x)$ for $N(X) = \int_{-\infty}^{x} \phi(y)dy$.
Let $\Omega = (0,1), \mathcal{F} = \{B \in \mathcal{B}(\mathbb{R}) : B \subseteq (0,1)\}, \mathbb{P}(A) = \mathcal{L}(A) : A \in \mathcal{B}(\mathbb{R})$.
Then, $Y : \Omega \ni x \mapsto F(x) \in \mathbb{R}$ is a random variable with

$$\begin{aligned}\mathbb{P}[Y \in (a,b)] &= \mathbb{P}[\{x : Y(x) \in (a,b)\}] \\ &= \mathbb{P}[\{x \in (N(a), N(b))\}] \\ &= N(b) - N(a) = \int_a^b \phi(x)dx,\end{aligned}$$

and a density function is ϕ.

Previous Question: In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variable $X : \Omega \to \mathbb{R}$, the random element or random realization $\omega \in \Omega$ is a element of events in sample space. For example, $\omega = HHTTH$ is a random element in tossing a coin five times, and $X(\omega) = 3$. ($X(\omega)$ = # of Heads)
In the previous example(Standard Normal random variable), define $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1), \mathcal{B}(0,1), \mathbb{P})$, $\mathbb{P}((a,b)) = b - a$, $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$, $X : (0,1) \ni \omega \mapsto F^{-1}(\omega) \in \mathbb{R}$. Then, X is called a standard normal random variable.

1.3 Expectations

In the following, let $\Omega = (\Omega, \mathcal{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$. Then the expection $\mathbb{E}(X)$ is a mean of $X(\omega)$ with respect to the randomness of ω (given by $\mathbb{P}$)

Definition 1.3.1 (Lebesgue Integration)
$(\Omega, \mathcal{F}, \mu)$ is a measure space, and $f : \Omega \to \mathbb{R}$ is a measurable function.

(1) $f : \Omega \to [0, \infty)$
Let $0 = y_0 < y_1 < y_2 < \cdots \to \mathbb{R}$ be a partition of $[0, \infty)$,
$\Pi = \{y_0, y_1, y_2, \cdots\} : \|\Pi\| = \sup_{i \geq 1} |y_i - y_{i-1}|$, and
$\mathrm{LS}_\pi = \sum_{i=0}^{\infty} y_i \mu[f^{-1}([y_i, y_{i+1}))]$.
In Rudin's book, $\lim_{\|\Pi\| \to 0} \mathrm{LS}_\Pi$ converges to an element belonging to $[0, \infty]$.
Now, $\int f d\mu := \lim_{\|\Pi\| \to 0} \mathrm{LS}_\Pi$ is called a **Lebesgue Integral**.

(2) $f : \Omega \to \mathbb{R}$
Let $f^+ = \max\{f, 0\} \geq 0$, and $f^- = -\min\{f, 0\} \geq 0$. Then, $f = f^+ - f^-$, and $|f| = f^+ + f^-$. If $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$, then we say f is Lebesgue integrable and $f \in L^1(\mu)$. The Lebesgue integral of $f = \int f d\mu$ is defined as $\int f^+ d\mu - \int f^- d\mu$

Remark.

1. $\int f^+ d\mu < \infty$ and $\int f^- d\mu = \infty$, then $\int f d\mu = -\infty$. The others are defined similarly.

2. $f \in L^1(\mu) \Leftrightarrow \int |f| d\mu < \infty$.

Example (Riemann vs Lebesgue Integral (p19-22)).

- $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{L})$ Lebesgue measure where $\mathcal{L}((a, b)) = b - a$.

- $f : \mathbb{R} \to \mathbb{R} \in L^1(\mathcal{L})$

- (Def) $A \subseteq \mathbb{R}$, $\int_A f d\mu := \int f \mathbb{1}_A d\mu$, where $\mathbb{1}_A(x) = 1$ if $x \in A$, and 0 otherwise.

If f is Riemann integrable, then $\int_{[a,b]} f d\mathcal{L} = \int_a^b f dx$.
Riemann integral is a limit of approximation by a partition of x-axis. On the other hand, Lebesgue integral is a limit of approximation by a partition of y-axis with preimage. Partition of x-axis is sensitive to fluctuation and restricted to Euclidean space, while partition of y-axis is not. For example, $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ is Lebesgue integrable, but it is not Riemann integrable since it is sensitive to fluctuation.

Definition 1.3.2 (Almost everywhere, 1.1.5 in Textbook)
$P(x)$ is a property at $x \in \mathbb{R}$. We say P holds **almost everywhere** (or a.e.) in $\mathbb{R}$ if and only if $\mathcal{L}(\{x : P(x) \text{ does not hold }\} = 0$.

Example. $f(x) = [x]$ is continuous almost everywhere.

Theorem 1.3.3
f is Riemann integrable if and only if f is continuous a.e.

Exercises. $f = g$ a.e. $\Rightarrow \int f d\mathcal{L} = \int g d\mathcal{L}$.

Definition 1.3.4 (Almost surely)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The event $A(\in \mathcal{F})$ occurs **almost surely** (a.s.) if $\mathbb{P}(A) = 1$.

Example. Let X be a uniform random variable in $(0,1)$. Let $A = \{X(\omega) \neq \frac{1}{2}\}$; $\mathbb{P}(A) = 1$.

Definition 1.3.5 (Expectation, 1.3.3. in Textbook)
Expectation of $X : \Omega \to \mathbb{R}$ is defined by

$$\mathbb{E}(X) := \int_\Omega X d\mathbb{P} \quad \text{if} \quad \int_\Omega |X| d\mathbb{P} < \infty$$

Theorem 1.3.6 (1.3.4 in Textbook)

1. *X takes finite number of values $\{x_1, x_2, \cdots, x_n\} \Rightarrow \mathbb{E}(X) = \sum_{i=1}^n x_i \mathbb{P}(X = x_i)$*

2. *X, Y: random variables, $E(|X|), E(|Y|) < \infty$,*

   (i) $X \leq Y$ a.s. (i.e. $\mathbb{P}[\{X(\omega) \leq Y(\omega)\}] = 1$), then $\mathbb{E}(X) \leq \mathbb{E}(Y)$

   (ii) $X = Y$ a.s. $\Rightarrow \mathbb{E}(X) = \mathbb{E}(Y)$

3. *X, Y: random variables, $\mathbb{E}(|X|), \mathbb{E}(|Y|) < \infty \Rightarrow \mathbb{E}(\alpha X + \beta Y) = \alpha \mathbb{E}(X) + \beta \mathbb{E}(Y)$.*

4. *Jensen's Inequality: $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function $\Rightarrow \phi(\mathbb{E}(X)) \leq \mathbb{E}(\phi(X))$*
   (c.f. $\phi(t) = t^2$)

Proof of 4. Define $S_\phi = \{(a,b) \in \mathbb{R}^2 : a + bt \leq \phi(t) \ \ \forall t\}$. Then $\forall t \in \mathbb{R}, \phi(t) = \sup_{(a,b) \in S_\phi} \{a + bt\}$. In fact, it is a equivalent condition. Now,

$$\begin{aligned} \phi(\mathbb{E}[X]) &= \sup_{a,b \in S_\phi} \{a + b\mathbb{E}[X]\} \\ &= \sup_{a,b \in S_\phi} \mathbb{E}[a + bX] \\ &\leq \mathbb{E}[\sup_{a,b \in S_\phi} (a + bX)] = \mathbb{E}[\phi(X)] \quad \text{(Check!)} \end{aligned}$$

□

Example (Dirac Measure in $\mathbb{R}$). $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \delta_y)$ $(y \in \mathbb{R})$ is a probability space with $\delta_y(A) = 1$ if $y \in A$, and 0 otherwise. Then, $\int_{\mathbb{R}} f d\delta_y = f(y)$ (Check!)

Consider modeling: X: random variable such that probability of $x_i = p_i$ with $\sum_{i=1}^n p_i = 1$. Then, $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ with $\mu = \sum_{i=1}^n p_i \delta_{x_i}$ is a probability space, and $P(X = x_i) = p_i$ for $X : \mathbb{R} \ni \omega \mapsto \omega \in \mathbb{R}$: Example of thm 1.3.4.

Summary:

- Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
- Random variables: $X : \Omega \to \mathbb{R}$
- Expectation: $E(X) = \int X d\mathbb{P}$

1.4 Convention of Integrals

We will use this section when we define the Brownian motion.

Definition 1.4.1

(1) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $f, f_1, f_2, \cdots$ be measurable ($\Omega \to \mathbb{R}$). Then, $f_n \to f$ **almost everywhere** (a.e.) if

$$\mu[\{\omega : (f_n(\omega))_{n=1}^{\infty} \text{ does not converge to } f(\omega)\}] = 0$$

(2) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X, X_1, X_2, \cdots$ be random variables. Then, $X_n \to X$ **almost surely** (a.s.) if

$$\mathbb{P}[\{\omega : (X_n(\omega))_{n=1}^{\infty} \text{ does not converge to } X(\omega)\}] = 0$$

Question: $f_n \to$ a.e. Then, $\int f_n d\mu \to \int f d\mu$? $X_n \to X$ a.s. Then, $\mathbb{E}[X_n] \to \mathbb{E}[X]$?

Theorem 1.4.2 (Monotone Convergence Theorem. 1.4.5 in Textbook)
$0 \le f_1 \le f_2 \le \cdots$ *(or decreasing), and* $f_n \to f$ *a.e. Then,* $\int f_n d\mu \to \int f d\mu$.

Theorem 1.4.3 (Dominated Convergence Theorem. 1.4.9 in Textbook)
$\exists g \in L^1(\mu)$ *such that* $|f_n| \le g$ *for all* n*, and* $f_n \to f$ *a.e. Then,* $\int f_n d\mu \to \int f d\mu$.

Corollary 1.4.4
$\exists Y \in L^1(\mathbb{P})$ such that $|X_n| \le Y$ for all n, and $X_n \to X$ a.s. Then, $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

Example. Let $f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x \le \frac{1}{n}, \\ -n^2 x + n & \text{if } \frac{1}{n} < x \le \frac{2}{n}, \\ 0 & \text{otherwise.} \end{cases}$ Then, $f_n \to 0$ a.e. and $\int f_n dx = 1$.

1.5 Computation of Expectations

Notation: $(X : \Omega \ni \omega \mapsto X(\omega) \in \mathbb{R})$

- $\mathbb{E}[X] = \int X d\mathbb{P} = \int_\Omega X(\omega) d\mathbb{P}(\omega)$
- $\int_B X(\omega) d\mathbb{P}(\omega) := \int \mathbb{1}_B(\omega) X(\omega) d\mathbb{P}(\omega)$

Recall: X: random variable, μ_X: distribution measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\mu_X(B) = \mathbb{P}(X \in B)$.

Theorem 1.5.1
$g \in L^1(\mu_X)$. Then, $\mathbb{E}[g(X)] = \int_\mathbb{R} g(x) d\mu_X(x) (:= \int g d\mu_X)$.

Example. $g(x) = x$. $\int |x| d\mu_X(x) < \infty \Rightarrow \mathbb{E}[X] = \int x d\mu_X(x)$.

Proof. First, prove the thm holds for $g \geq 0$, then prove for general g by $g = g^+ - g^-$.

(1) $g = \mathbb{1}_B$
By thm 1.3.4. (1), $E[\mathbb{1}_B(X)] = 1 \cdot \mathbb{P}[\mathbb{1}_B(X) = 1] = \mathbb{P}(X \in B) = \mu_X(B) = \int \mathbb{1}_B(x) d\mu_X(x)$.

(2) $g = \sum_{k=1}^n \alpha_k \mathbb{1}_{B_k}$
Trivial by linearity.

(3) $g \geq 0$
By MCT. See *Rudin* chapter 1 for details.

□

Recall: X : random variable, X has density function f_X if

$$\mu_X((a,b)) = \int_a^b f_X(x) dx \quad \forall a, b.$$
$$\mu_X(B) = \int_B f_X d\mathcal{L} = \int_B f_X(x) d\mathcal{L}(x) = \int_B f_X(x) dx.$$

Theorem 1.5.2
$g \in L^1(\mu_X)$. Then, $E(g(X)) = \int_\mathbb{R} g(x) f_X(x) dx$.

Example. Let X be standard normal. i.e., $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ (regardless what X be). Then, $E(X^4) = \int_\mathbb{R} x^4 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = 3$.

Chapter 2

Information and Conditioning

2.1 Information and σ-algebras

Example. Toss a coin Three times. $\Omega = \{HHH, HHT, \cdots, TTT\}$.
$A_H = \{HHH, HHT, HTH, HTT\}$, $A_T = \{THT, THT, TTH, TTT\}$.
Let $\mathcal{F}(1) = \{\phi, \Omega, A_H, A_T\}$ so that it is a σ-algebra containing the randomness up to time 1.
Similarly, define $A_{HH}, A_{HT}, A_{TH}, A_{TT}$.
Let $\mathcal{F}(2) = \{\phi, \Omega, A_{HH}, A_{HT}, A_{TH}, A_{TT}, A_{HH} \cup A_{HT}, \cdots, A_{TT}^C\}$ so that it is a σ-algebra containing the randomness up to time 2, and define $\mathcal{F}(0)$ similarly, and let $\mathcal{F}(0) = \{\phi, \Omega\}$.
Then, $\mathcal{F}(0) \subseteq \mathcal{F}(1) \subseteq \mathcal{F}(2) \subseteq \mathcal{F}(3)$. Let X_t = # of heads until time t. Then, X_t is $\mathcal{F}(t)$-measurable for each t.
Now, $\{X_1 = 1\} = \{\omega : X_1(\omega) = 1\} = A_H$, and $\{X_1 = 0\} = \{\omega : X_1(\omega) = 0\} = A_T$.

Definition 2.1.1 (σ-algebra generated by X)
Ω is a set, $X : \Omega \to \mathbb{R}$. $\sigma(X) = \{A \subseteq \Omega : A = X^{-1}(B) \text{ for some } B \in \mathcal{B}(\mathbb{R})\}$. Then, $\sigma(X)$ is a $\sigma-$algebra(exercise) and it is called a **σ-algebra generated by** X.

Remark. X is a random variable in $(\Omega, \sigma(X))$.
X is a random variable in $(\Omega, \mathcal{F})$, then $\sigma(X) \subseteq \mathcal{F}$ (exercise)

Definition 2.1.2 ($\mathcal{F}$-measurable)
$(\Omega, \mathcal{F})$: measure space. $X : \Omega \to \mathbb{R}$. X is called **$\mathcal{F}$-measurable** if $\sigma(X) \subseteq \mathcal{F}$. i.e., X: measurable with respect to $(\Omega, \mathcal{F})$.

In example, $X(t)$ is $\mathcal{F}(t)$-measurable $\forall t$ (check!)
cf. $X(t) : \Omega \to \mathbb{R}$. $(X(t))^{-1}(B) \in \mathcal{F}(t) \ \ \forall B \in \mathcal{B}(\mathbb{R})$.

Enough to check $(X(t))^{-1}(\{0\}), (X(t))^{-1}(\{1\}), \cdots, (X(t))^{-1}(\{t\})$.
$\mathcal{F}(t)$ has enough information to determine $X(t)$ in the sense that $\{\omega : (X(t))(\omega) \in B\} \in \mathcal{F}(t) \ \ \forall B \in \mathcal{B}(\mathbb{R})$.

Definition 2.1.3 (Filtration, Stochastic Process)
Ω: non-empty set, $T > 0$.

1. If $\mathcal{F}(t)$ is a σ-algebra $\forall t \in [0, T] \in T$ and $s < t \Rightarrow \mathcal{F}(s) \subseteq \mathcal{F}(t)$, then $(\mathcal{F}(t) : t \in [0, T])$ is called a **filtration**
2. If $X(t) : \Omega \to \mathbb{R}$ is $\mathcal{F}(t)$-measurable $\forall t \in [0, T]$, then $(X(t) : t \in [0, T])$ is called **Stochastic Process adopted to the filtration** $\mathcal{F}(t)$.

2.2 Independence

$X : \Omega \to \mathbb{R}$, $\mathcal{F}$: σ-algebra on Ω.

1. $\mathcal{F}$ has full information to determine $X \Rightarrow X$ is $\mathcal{F}$-measurable. (2.1)
2. $\mathcal{F}$ has no information to determine $X \Rightarrow X$ is independent to $\mathcal{F}$. (2.2)
3. $\mathcal{F}$ has a partition information to determine $X \Rightarrow$ (2.3)

Definition 2.2.1 (independent)
$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. $A, B \in \mathcal{F}$ is **independent** if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Question: X, Y are random variables in $(\Omega, \mathcal{F}, \mathbb{P})$. If X, Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, but the converse does not hold.

Definition 2.2.2
$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ are sub σ-algebras of $\mathcal{F}$. $X, Y : \Omega \to \mathbb{R}$ are random variables in $(\Omega, \mathcal{F}, \mathbb{P})$.

1. $\mathcal{G}, \mathcal{H}$: independent iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \ \ \forall A \in \mathcal{G}, B \in \mathcal{H}$.
2. X, Y: independent iff $\sigma(X), \sigma(Y)$ are independent.
3. $X, \mathcal{G}$: independent iff $\sigma(X), \mathcal{G}$ are independent.

Definition 2.2.3
$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
$\mathcal{G}_1, \mathcal{G}_2, \cdots \mathcal{G}_n, \cdots$: sub σ-algebra of $\mathcal{F}$. $X_1, X_2, \cdots, X_n, \cdots$: random variable in $(\Omega, \mathcal{F}, \mathbb{P})$.

1. $\mathcal{G}_1, \cdots, \mathcal{G}_2$ are independent iff $\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdots \mathbb{P}(A_n)$ for $A_1 \in \mathcal{G}_1, \cdots, A_n \in \mathcal{G}_n$.

2. $X_1, \cdots, X_n$ are independent iff $\sigma(X_1) \sim \sigma(X_n)$ are independent.

3. $\mathcal{G}_1, \mathcal{G}_2, \cdots$ are independent iff $\mathcal{G}_1 \sim \mathcal{G}_n$ are independent $\forall n$.

4. $X_1, X_2, \cdots$ are independent iff $X_1 \sim X_n$ are independent $\forall n$.

Example. Toss a coin three times.

1. $X(2), X(3)$ are not independent.
   $\mathbb{P}(\{X(2) = 2\} \cap \{X(3) = 1\}) \neq \mathbb{P}(X(2) = 2)\mathbb{P}(X(3) = 1)$.

2. $X(2), X(3) - X(2)$ are independent.
   Why: $X(2)$ is an information at tossing first, second times, and $X(3)$ is an information at tossing third time.

Definition 2.2.4 (Joint distribution)
$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. X, Y are random variables in Ω. $(X, Y) : \Omega \ni \omega \mapsto (X(\omega), Y(\omega)) \in \mathbb{R}^2$

1. Joint Distribution Measure in $\mathbb{R}^2$
   $\mu_{X,Y}(C) = \mathbb{P}((X, Y) \in C)$ for $C \in \mathcal{B}(\mathbb{R}^2)$.
   (Note: We checked that $\{\omega : (X(\omega), Y(\omega)) \in C\} \in \mathcal{F}$ in real analysis.)

2. Joint Cumulative Distribution Function
   $F_{X,Y}(a, b) = \mathbb{P}(X \leq a, Y \leq b) = \mu_{X,Y}((-\infty, a] \times (-\infty, b])$ (check!)

3. Joint Probability Distribution Function
   If $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ is Borel-measurable and satisfies $\mu_{X,Y}(A \times B) = \int_B \int_A f_{X,Y}(x, y)dxdy$ for all $A, B \in \mathcal{B}(\mathbb{R})$, then $f_{X,Y}$ is called a joint probability density function (jpdf)

Theorem 2.2.5
$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, X, Y are random variables in Ω. Then, the followings are equivalent.

(i) X, Y *are independent*

(ii) $\mu_{X,Y}(A \times B) = \mu_X(A)\mu_Y(B) \;\; \forall A, B \in \mathcal{B}(\mathbb{R})$

(iii) $F_{X,Y}(a, b) = F_X(a)F_Y(b) \;\; \forall a, b \in \mathbb{R}$

(iv) $\mathbb{E}[e^{uX+vY}] = \mathbb{E}[e^{uX}]\mathbb{E}[e^{vY}]$

Remark. If JPDF $f_{X,Y}$ exists, then (i) to (iv) $\Leftrightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y)$ a.e.

Theorem 2.2.6
X, Y are independent if and only if $f, g : \mathbb{R} \to \mathbb{R}$ Borel-measurable, $\mathbb{E}[|f(X)g(Y)|] < \infty$ implies that $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$.

Remark. $f(x) = g(x) = x : \mathbb{E}[|XY|] < \infty \Rightarrow \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

Proof. Details are exercises.

(1) $f = \mathbb{1}_A, g = \mathbb{1}_B$

(2) f, g are simple functions

(3) $f, g \geq 0$

(4) f, g are general.

$\square$

<u>Review</u>
$\mathcal{G}, \mathcal{H}$ are independent if $\forall A \in \mathcal{G}, B \in \mathcal{H} \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
X, Y are independent if $\sigma(X), \sigma(Y)$ are independent.
* $\sigma(X) = \{A \in \Omega : A = X^{-1}(B)$ for some $B \in \mathcal{B}(\mathbb{R})\}$.
* $\mu_{X,Y}(C) = \mathbb{P}((X,Y) \in C) \ \ \forall C \in \mathcal{B}(\mathbb{R}^2)$.

Thm. T.F.A.E.C:

1. X, Y are independent
2. $\mu_{X,Y}(A \times B) = \mu_X(A)\mu_Y(B)$
3. $\mathcal{F}_{X,Y}(x,y) = \mathcal{F}_X(x)\mathcal{F}_Y(y)$
4. (If JPDF $f_{X,Y}$ exists) $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

Theorem 2.2.7
$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. X, Y are independent random variables, $f, g : \mathbb{R} \to \mathbb{R}$ are Borel measurable. Then, $f(X), g(Y)$ are independent.

Proof. $A \in \sigma(f(X))$; $A = (f \circ X)^{-1}(B)$ for some $B \in \mathbb{R} = X^{-1}(f^{-1}(B)) \in \sigma(X)$.
$\therefore \sigma(f(X)) \subseteq \sigma(X)$, $\sigma(g(Y)) \subseteq \sigma(Y) \Rightarrow \sigma(f(X)), \sigma(g(Y))$ are independent. $\square$

Corollary 2.2.8
$\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X))\mathbb{E}(g(Y)]$.

Definition 2.2.9
X, Y are random variables in $(\Omega, \mathcal{F}, \mathbb{P})$.

1. $Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
2. $std(X) = \sqrt{Var(X)}$
3. $Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
4. $corr(X, Y) = cov(X, Y)/(std(X)std(Y))$

Example.
- X: standard normal random variable ($N(0, 1^2)$)
- $\mathbb{P}(Z = 1) = \mathbb{P}(Z = -1) = \frac{1}{2}$ (X, Z are independent)
- $Y = XZ$. Then
    1) Y is standard normal,
    2) $corr(X, Y) = 0$.
    3) X, Y are not independent.

Definition 2.2.10 (Jointly normal)
X, Y are **jointly normal** with mean $m = (m_X, m_Y)$, $Var(C) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ if

$$f_{X,Y}(z) = \frac{1}{\sqrt{(2\pi)^2 \det C}} e^{-\frac{1}{2}(z-m)C^{-1}(z-m)^{\mathsf{T}}}$$

Theorem 2.2.11
X, Y are jointly normal and uncorrelated ($C_{12} = C_{21} = 0$). Then, they are independent.

2.3 Conditional Expectation

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. $\int_A X d\mathbb{P} := \int \mathbb{1}_A X d\mathbb{P} = \int \mathbb{1}_A(\omega) X(\omega) d\mathbb{P}(\omega)$.

Lemma. $\int_A X d\mathbb{P} = \int_A Y d\mathbb{P}$ for all $A \in \mathcal{F}$ if and only if $X = Y$ a.s.

Proof. $A_n = \{\omega : X(\omega) - Y(\omega) > \frac{1}{n}\}, B_n = \{\omega : X(\omega) - Y(\omega) < -\frac{1}{n}\}$. Then,

$$0 = \int_{A_n} (X - Y) d\mathbb{P} \geq \int_{A_n} \frac{1}{n} d\mathbb{P} = \frac{1}{n} \int \mathbb{1}_{A_n} d\mathbb{P} = \frac{1}{n} \mathbb{P}(A_n)$$

Thus, $\mathbb{P}(A_n) = 0$ $\forall n$. Similarly, $\mathbb{P}(B_n) = 0$. Now, $\{\omega : X(\omega) \neq Y(\omega)\} = (\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} B_n) \Rightarrow$ measure 0. □

Intuition. $(\Omega, \mathcal{F}, \mathbb{P})$ is given, $X : \mathcal{F}$-measurable random variable, $\mathcal{G} \subseteq \mathcal{F}$ is a sub σ-algebra. If we know nothing, then we expect X as $\mathbb{E}[X]$. If we know $\mathcal{F}$, then we expect X as X. Now, if we know $\mathcal{G}$, then we expect X as $\mathbb{E}[X|\mathcal{G}]$ (what is it?)

Definition 2.3.1 (Conditional Expectation)
$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. $X \in L^1(\mathbb{P})$ is a random variable. $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$. We define $\mathbb{E}[X|\mathcal{G}]$ as

1. $\mathcal{G}$-measurable random variable
2. $\int_A \mathbb{E}[X|\mathcal{G}](\omega) d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega)$.

Question. $\mathbb{E}[X|\mathcal{G}]$ exists? (Yes! proof skip). unique? (Yes! up to a.s.)

Remark. Lemma implies determine X (a.s.) is equivalent to know $\int_A X d\mathbb{P} \ \ \forall A \in \mathcal{F}$.
In this sense, conditional expectation $Y = \mathbb{E}[X|\mathcal{G}]$ is knowing $\int_A Y d\mathbb{P} = \int_A X d\mathbb{P} \ \ \forall A \in \mathcal{G}$.

Example. Toss a coin three times.
$\mathcal{F}(0) \subseteq \mathcal{F}(1) \subseteq \mathcal{F}(2) \subseteq \mathcal{F}(3)$. $X(t)$ is a number of heads until t times; $X(t)$ is $\mathcal{F}(t)$-measurable. If $\mathcal{F}(1) = \{\phi, \Omega, A_H, A_T\}$, then $\mathbb{E}[X(2)|\mathcal{F}(1)] = X(1) + \frac{1}{2}$, since we know the information of 1st flip.

Proof. Want: $\int_A (X(1) + \frac{1}{2}) d\mathbb{P} = \int_A X(2) d\mathbb{P}$ for all $A \in \mathcal{F}(1)$ (*c.f.* $\mathbb{P}(\omega) = \frac{1}{8} \ \forall \omega \in \Omega$).
For $A = A_H$, $\int \mathbb{1}_{A_H}(\omega)(X(1)(\omega) + \frac{1}{2}) d\mathbb{P}(\omega) = \frac{3}{2}\mathbb{P}(A_H) = \frac{3}{4}$.
$\int \mathbb{1}_{A_H}(\omega)(X(2))(\omega) d\mathbb{P}(\omega) = \sum_{\omega \in A_H} (X(2))(\omega)\mathbb{P}(\omega) = \frac{1}{8}(2+2+1+1) = \frac{3}{4}$.

□

Remark. $\mathcal{G} = \sigma(Y)$; $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X|\sigma(Y)] := \mathbb{E}[X|Y]$

Theorem 2.3.2
X, Y are independent random variable in $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$.

1. $\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$
2. *X is $\mathcal{G}$-measurable. Then,* $\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$.
3. *$\mathcal{H}$ is a sub σ-algebra of $\mathcal{G}$. Then,* $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$.
4. *$X, \mathcal{G}$ are independent, then* $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

Proof.
1. Exercise
2. We only need to show that $X \geq 0, Y \geq 0$ implies 2.

(a) $X = \mathbb{1}_B$
Want: $\mathbb{E}[\mathbb{1}_B Y|\mathcal{G}] = \mathbb{1}_B \mathbb{E}[Y|\mathcal{G}]$ for $B \in \mathcal{G}$.

(b) $X = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}$
Use linearity.

(c) $X \geq 0$
Use MCT

3. Want: $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$
Let $A \in \mathcal{H}$. Then, $\int_A \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}](\omega) d\mathbb{P}(\omega) = \int_A \mathbb{E}[X|\mathcal{G}] d\mathbb{P} = \int_A X d\mathbb{P} = \int_A \mathbb{E}[X|\mathcal{H}] d\mathbb{P}$

4. Can be shown similarly as in 2. Check $X = \mathbb{1}_B$ case. (Hint: $A \in \mathcal{G} \Rightarrow A, B$ are independent.)

□

Example (Revisit).

$$\begin{aligned}
\mathbb{E}[X(2)|\mathcal{F}(1)] &= \mathbb{E}[X(2) - X(1) + X(1)|\mathcal{F}(1)] \\
&= \mathbb{E}[X(2) - X(1)|\mathcal{F}(1)] + X(1) \\
&= \mathbb{E}[X(2) - X(1)] + X(1) \\
&= \frac{1}{2} + X(1)
\end{aligned}$$

Review

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space, $\mathcal{G} \subseteq \mathcal{F}$.
- X : $\mathcal{F}$-measurable random variable.
- $Y = \mathbb{E}[X|\mathcal{G}]$ if Y is $\mathcal{G}$-measurable.
- $\int_A Y(\omega) d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega) \ \ \forall A \in \mathcal{G}$.

Remark. $\mathbb{E}[X|\mathcal{G}]$ is an expectation of X when we know $\mathcal{G}$.

Remark. $Y = Z$ a.s. and Z is $\mathcal{G}$-measurable, then $Z = \mathbb{E}[X|\mathcal{G}]$.

Remark. $(X(t))_{t\in[0,T]}$ is stochastic process adapted to $(\mathcal{F}(t))_{t\in[0,T]}$. In this, $(X(t))_{t\in[0,T]}$ is a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$ and $X(t)$ is $\mathcal{F}(t)$-measurable. $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ for all $s < t$ and $\mathcal{F}(0) = \{\phi, \Omega\}$.

Remark. We can define $\mathcal{F}$ as $\mathcal{F}(t) = \bigcup_{s:s\leq t} \sigma(X(s))$

Definition 2.3.3 (Martingale, Markov Process)

1. Martingale $X(t)$
   $\mathbb{E}[X(t)|\mathcal{F}(s)] = X(s)$ for all $s < t$.

2. Markov Process $X(t)$
   For any borel measurable f, there exists some borel measurable g such that $\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s))$

Remark. In Martingale, if we know all the previous value, then the expectation of the future is as same as the expectation of the present.

Remark. Markov process is a generalization of Markov chain. We only have to know the present value.

Chapter 3

Brownian Motions

3.1 Introduction

To study Brownian Motions, we will study:

1. Random Walks
2. Definition of Brownian Motions and its basic property (We will change the text-book!)
3. Constuction of Brownian Motions

3.2 Scaled Random Walks

Definition 3.2.1 (Random Walk)

- Let $X_i = \begin{cases} 1 & \text{prob } \frac{1}{2} \\ -1 & \text{prob } \frac{1}{2} \end{cases}; X_1, X_2, \cdots$ independent.
- $M_n = X_1 + \cdots + X_n$ is called **random walk**
- $W_n(t) = \frac{1}{\sqrt{n}} M_{nt} (:= \frac{1}{\sqrt{n}} M_{[nt]})^v, \;\; t \in \{\frac{1}{n}k, k \in \mathbb{Z}_+\}$ is called **scaled random walk**

Proposition 3.2.2
Random walk holds the following properties:

1. Independent Increament
2. Martingale

3. Quadratic Variation

Proof.

1. See Def 3.2.3

2. Let $\mathcal{F}(n) = \sigma(X_1, X_2, \cdots, X_n)$ = smallest σ-algebra making$X_1 \sim X_n$ measurable. Then,

   - $M_n = X_1 + \cdots X_n$ is $\mathcal{F}(n)$-measurable
   - $(M_n)_{n\in\mathbb{N}}$ is stochastic process adapted to $(\mathcal{F}(n))_{n=0}^{\infty}$
   - $k < l \Rightarrow \mathbb{E}[M_l|\mathcal{F}(k)] = \mathbb{E}[M_l - M_k|\mathcal{F}(k)] + \mathbb{E}[M_k|\mathcal{F}(k)] = \mathbb{E}[M_l - M_k] + M_k = \mathbb{E}[X_{k+1} + \cdots + X_l] + M_k = \mathbb{E}[X_{k+1}] + \cdots + \mathbb{E}[X_l] + M_k = M_k$.

3. $\sum_{i=1}^{n}(M_i - M_{i-1})^2 = n$

□

Definition 3.2.3 (Independent increament)
M_n is **independent increament** if $M_{k_1}, M_{k_2} - M_{k_1}, \cdots, M_{k_m - k_{m-1}}$ are independent for any $k_1 < k_2 < \cdots < k_m$. Here, $M_{k_l} - M_{k_{l-1}}$ is called increament. If M_n is a random walk, then $M_{k_1} = \sum_{i=1}^{k_1} X_i, M_{k_2 - k_1} = \sum_{i=k_1}^{k_2} X_i, \cdots$ are independent.

Remark. Proposition 3.2.2 holds for scaled random variable $W_n(t) = \frac{1}{n} M_{nt}$ ($t \in \frac{1}{n}\mathbb{Z}_+$).

Proof.

1. Independent Increament
   For $t_1 < t_2 < \cdots < t_m$, $W_n(t_1) - W_n(0), W_n(t_2) - W_n(t_1), \cdots, W_n(t_m) - W_n(t_{m-1})$ are independent, since its increaments $W_n(t_{n+1}) - W_n(t_l) = \frac{1}{n}(M_{nt_{l+1}} - M_{nt_l})$ are independent by independent increament property of M_n.

2. Martingale
   Let $\mathcal{F}_n(t) = \sigma(X_1, X_2, \cdots, X_{nt})$. Then, $W_n(t) = \frac{1}{n}(X_1 + X_2 + \cdots + X_{nt})$ is $\mathcal{F}_n(t)$-measurable. Therefore, $(W_n(t))$ is stochastic process adapted to $(\mathcal{F}_n(t))$. With some computations as before, $\mathbb{E}[W_n(t)|\mathcal{F}_n(s)] = \cdots = W_n(s)$ for $s < t$.

3. Quadratic Variation
   $\sum_{i=1}^{nt}\left(W_n(\frac{i}{n}) - W_n(\frac{i-1}{n})\right)^2 = \sum_{i=1}^{nt}\left[\frac{1}{\sqrt{n}}(M_i - M_{i-1})\right]^2 = \sum_{i=1}^{nt}\frac{1}{n} \cdot 1 = t$

□

Example. Let $f \in C^1([0,t])$. Then,

$$\begin{aligned}
\sum_{i=1}^{nt} \left(f(\frac{i}{n}) - f(\frac{i-1}{n}) \right)^2 &= \sum_{i=1}^{nt} \left[\frac{1}{n} f'(\frac{x_i}{n}) \right]^2 \\
&= \frac{1}{n}\frac{1}{n} \sum_{i=1}^{nt} \left(f'(\frac{x_i}{n})^2 \right) \quad (\to \int_0^t [f'(x)]^2 dx) \\
&\leq \frac{c}{n} \quad (\to 0)
\end{aligned}$$

It is the most different property between random process and deterministic function: Q.V. of random variable is constant but Q.V. of C^1 function is zero.

Theorem 3.2.4 (Central Limit Theorem)
Let $Y_1, Y_2, \cdots$ are independent and identically distributed (called i.i.d.) with mean 0 and variation 1 $(\mathbb{E}(Y_i) = 0, Var(Y_i) = \mathbb{E}(Y_i^2) = 1)$. Then,

$$\frac{1}{\sqrt{n}} [Y_1 + \cdots Y_n] \to N(0, 1^2) \tag{$\bigstar$}$$

Remark. Meaning of $\bigstar$:

$$\mathbb{P}\left[\frac{1}{n}(Y_1 + \cdots + Y_n) \in [a,b] \right] \to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

$W_n(t) = \frac{1}{n} M_{nt} = \frac{1}{n}(X_1 + \cdots + X_{nt}) = \sqrt{t} \frac{1}{\sqrt{nt}}(X_1 + \cdots + X_{nt}) \sim N(0,t)$
cf. $N(\mu, \sigma^2)$ is a normal random variable with mean μ and variation σ^2. Using the above,

$$\lim_{n\to\infty} \mathbb{P}\left[W_n(t) \in [a,b] \right] = \int_a^b \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx.$$

$$W_n(t) = \frac{1}{n^{\frac{1}{2}+\alpha}} M_{nt} \begin{cases} \alpha < 0 & |W_n(t)| \to \infty \\ \alpha > 0 & |W_n(t)| \to 0 \end{cases}$$

Remark. $\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ is a heat kernel in PDE.

Summary

1. Independent Increament
2. Martingale
3. Markov Process
4. $W_n(t) \sim N(0,t)$
   $W_n(t) - W_n(s) \sim N(0, t-s)$

5. Q.V. in $[0,t] = t$.

Review

$X_1, X_2, \cdots$ are i.i.d. and $X_i = \begin{cases} \pm 1 & 1/2 \\ -1 & 1/2 \end{cases}$.

Random walk: $\mu_n = X_1 + \cdots + X_n$.

Scaled random walk: $W_n(t) = \frac{1}{\sqrt{n}} M_{nt}$. Then,

1. $W_n(0) = 0)$
2. Independent Increament
   $t_1 < t_2 < \cdots < t_n$, then $W_n(t_1), W_n(t_2) - W_n(t_1), \cdots, W_n(t_n) - W_n(t_{n-1})$ are independent.
3. Asymptotic Normal
   $W_n(t) - W_n(s) \sim N(0, t-s)$ as $n \to \infty$.

Chapter 2

Brownian Motion

2.1 Definition of Browinan Motion

Definition 2.1.1 (Stochastic Process)

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
- $[0, \infty)$ with Borel σ-algebra
- $X : [0, \infty) \times \Omega \to \mathbb{R}$, measurable.

Then, X is a **stochastic process** if

1. $X(t, \cdot) : \Omega \to \mathbb{R}$ is random variable
2. $X(\cdot, \omega) : [0, \infty) \to \mathbb{R}$ is measurable.

Remark. $X(t, \cdot) \Rightarrow X(t)$: random variable in Ω. $X(t) : \omega \mapsto [X(t)](\omega) = X(t, \omega)$.
For each $t \in [0, \infty)$ there exists random variable $X(t) : \Omega \to \mathbb{R}$. If we pick $\omega \in \Omega$, then each $X(t_i)$ is determined simultaneously by $X(t_i)(\omega)$.

Remark. We can work in $[0, T]$ instead of $[0, \infty)$. In fact, we can define in $[0, T]$ and extend to $[0, \infty)$, but it is extremly difficult.

Definition 2.1.2 (Brownian Motion in $[0, \infty)$)

- $t \in [0, \infty)$, $\omega \in \Omega((\Omega, \mathcal{F}, \mathbb{P})$: probability space)
- Stoch. Process $B(t, \omega)$

B is called **Brownian Motion** if

1. $B(0, \omega) = 0$ a.s. (i.e.,$\mathbb{P}\left[\{\omega : B(0, \omega) = 0\}\right] = 1$)

2. $B(\cdot, \omega) : [0, \infty) \to \mathbb{R}$ is a continuous function a.s.

3. $\forall 0 \leq s < t$, $B(t) - B(s) \sim N(0, t-s)$

4. Independent Increament

Remark. $B(t) \sim N(0, t)$ by 3 with $s = 0$.

Remark. $(B(t))_{t \geq 0} : \Omega \to \mathbb{R}$

- $B(t)$ itself is a normal distribution
- $B(t) - B(s) : \Omega \to \mathbb{R}$ is normal distribution with variance $t - s$.

Remark. Brwonian motion is a continuous version of random walk: random walk has property 1,4 and has property 3 with $n \to \infty$.

Theorem 2.1.3

1. *$s < t : \mathbb{E}[B(s)B(t)] = s$*

2. *$t_1 < t_2 < \cdots < t_n \Rightarrow (B(t_1), B(t_2), \cdots, B(t_n))$ is jointly normal with $\mu = (0, 0, \cdots, 0$ and $Var = C$. ($C_{ij} = t_{\min(i,j)} \ \forall 1, j$).*

Proof.

1. $\mathbb{E}[B(s)B(t)] = \mathbb{E}[B(s)(B(t) - B(s))] + \mathbb{E}[B(s)^2] = \mathbb{E}(B(s))\mathbb{E}(B(t) - B(s)) + s = s.$

2. Let $\vec{v} = (B(t_1), B(t_2) - B(t_1), \cdots, B(t_m) - B(t_{m-1}))$. Then,

$$\text{PDF of } \vec{v} = \frac{1}{\sqrt{2\pi t_1}} e^{-\frac{x_1^2}{2t_1}} \cdot \frac{1}{\sqrt{2\pi(t_2 - t_1)}} e^{-\frac{x_2^2}{2(t_2 - t_1)}} \cdots \frac{1}{\sqrt{2\pi(t_m - t_{m-1})}} e^{-\frac{x_m^2}{2(t_m - t_{m-1})}}.$$

Therefore, $\vec{v}$ is jointly normal with $\mu = 0$ and $Var = \operatorname{diag}(t_1, t_2 - t_1, \cdots, t_n - t_{n-1}) = D$, and,

$$\vec{W} = (B(t_1), \cdots, B(t_m)) = \vec{v} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \vec{v}E.$$

Thus, $\vec{W}$ is jointly normal with $\mu = (0, 0, \cdots, 0)$ and $Var = EDE^T = C$.

$\square$

Definition 2.1.4 (Filtration for Browinan Motion)

$$\begin{aligned}\mathcal{F}_t &= \sigma(B(s) : s \leq t)\\ &= \text{smallest } \sigma\text{-algebra containing } \{\omega : (B(s))(\omega) \in A\} \ \ \forall s \in [0,t], A : \text{Borel}\\ &= \text{smallest } \sigma\text{-algebra making } \forall B_s, s \in [0,t] \text{ measurable}\end{aligned}$$

Remark.

1. $(B(t))_{t\geq 0}$: Stochastic process adapted to the filtration $(\mathcal{F}_t)$.

2. $(B(t), \mathcal{F}(t))$: Martingale.

Proof. 1. By construction

2. $s < t \Rightarrow \mathbb{E}[B(t)|\mathcal{F}_s] = \mathbb{E}[B(t) - B(s)|\mathcal{F}_s] + \mathbb{E}[B(s)|\mathcal{F}_s]$

□

Appendix A

TA Session

Example (1.2.2). Let $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P})$ be the independent, infinite coin-toss space. Define stock price by

$$S_0(\omega) = 4 \quad \text{for all } \omega \in \Omega_\infty$$

$$S_1(\omega) = \begin{cases} 8 & \text{if } \omega_1 = H \\ 2 & \text{if } \omega_1 = T \end{cases}$$

$$S_2(\omega) = \begin{cases} 16 & \text{if } \omega_1 = \omega_2 = H \\ 4 & \text{if } \omega_1 \neq \omega_2 \\ 1 & \text{if } \omega_1 = \omega_2 = T \end{cases}$$

and in general

$$S_{n+1}(\omega) = \begin{cases} 2S_n(\omega) & \text{if } \omega_{n+1} = H \\ \frac{1}{2}S_n(\omega) & \text{if } \omega_{n+1} = T \end{cases}$$

Then, $S_0, S_1, \cdots$, are random variable.
For example, $\mathbb{P}(S_2 = 4) = \mathbb{P}(A_{HT} \cup A_{TH}) = 2pq$

Example (2.2.2). Let Ω be a three independent coin-toss space. Stock price random variables $S_0, S_1, \cdots$, are the same as the previous example. Let the probability measure $\mathbb{P}$ be given by

$$\mathbb{P}(HHH) = p^3, \mathbb{P}(HHT) = p^2 q, \cdots, \mathbb{P}(TTT) = q^3.$$

Assume $0 < p < 1$. Then, the random variables S_2 and S_3 are not independent.
$\because$ Consider the sets $\{S_3 = 32\} = \{HHH\}$ and $\{S_2 = 16\} = \{HHH, HHT\}$ whose probabilities are $\mathbb{P}(S_3 = 32) = p^3$ and $\mathbb{P}(S_2 = 16) = p^2$. In order to have Independence, $p^3 = \mathbb{P}(S_3 = 32) = \mathbb{P}(S_2 = 16 \text{ and } S_3 = 32) = \mathbb{P}(S_2 = 16)\mathbb{P}(S_3 = 32) = p^5 \Rightarrow\Leftarrow$.

The random variables S_2 and S_3/S_2 are independent. The σ-algebra generated by S_2 comprises ϕ, Ω, the atoms
$\{S_2 = 16\} = \{HHH, HHT\}, \{S_2 = 4\} = \{HTH, HTT, THH, THT\}, \{S_2 = 1\} = \{TTH, TTH\}$, and their unions.

The σ-algebra generated by S_3/S_2 comprises ϕ, Ω and
$\{S_3/S_2 = 2\} = \{HHH, HTH, THH, TTH\}, \{S_3/S_2 = \frac{1}{2}\} = \{HHT, HTT, THT, TTT\}$

For $A \in \sigma(S_2), B \in \sigma(S_3/S_2)$, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
ex) $p^3 = \mathbb{P}(S_2 = 16 \text{ and } S_3/S_2 = 2\} = \mathbb{P}(S_2 = 16)\mathbb{P}(S_3/S_2 = 2)) = p^2 p = p^3$

Example (2.2.10 Uncorrelated, dependent normal random variables).
Let X, Z be random variable satisfying

$$\begin{aligned} X &: \text{ standard normal random variable} \\ Z &: \text{ independent of } X,\ \mathbb{P}(Z = 1) = \frac{1}{2},\ \mathbb{P}(Z = -1) = \frac{1}{2} \end{aligned}$$

Define $Y = ZX$. Show

1. Y is standard normal random variable
2. X and Y are uncorrelated but they are dependent.

Proof.

1.
$$\begin{aligned} F_Y(b) &= \mathbb{P}(Y \leq b) \\ &= \mathbb{P}(Y \leq b \text{ and } Z = 1) + \mathbb{P}(Y \leq b \text{ and } Z = -1) \\ &= \mathbb{P}(X \leq b \text{ and } Z = 1) + \mathbb{P}(X \geq -b \text{ and } Z = -1) \\ &= \mathbb{P}(X \leq b)\mathbb{P}(Z = 1) + \mathbb{P}(X \geq -b)\mathbb{P}(Z = -1) \\ &= \frac{1}{2}N(b) + \frac{1}{2}N(b) \\ &= N(b) \end{aligned}$$

2. Since $\mathbb{E}X = \mathbb{E}Y = 0$,
$$Cov(X, Y) = \mathbb{E}[XY] = \mathbb{E}[ZX^2] = \mathbb{E}[Z]\mathbb{E}[X^2] = 0$$
$\therefore X$ and Y are uncorrelated.
If X and Y are independent, $|X|$ and $|Y|$ are independent. But $\mathbb{P}(|X| \leq 1, |Y| \leq 1) = \mathbb{P}(|X| \leq 1) = N(1) - N(-1)$, and $\mathbb{P}(|X| \leq 1, |Y| \leq 1) = \mathbb{P}(|X| \leq 1)\mathbb{P}(|Y| \leq 1) = (N(1) - N(-1))^2 \Rightarrow\Leftarrow$

$\square$

Let $\mu_{X,Y}$ be a joint distribution measure of (X,Y). Since $|X| = |Y|$, (X,Y) takes values only in the set $C = \{(x,y) : x = \pm y\}$.
It follows that for any measurable function f,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{1}_C(x,y) f_{X,Y}(x,y)dydx = 0$$

$\therefore$ There is no joint density $f_{X,Y}$ for (X,Y).

$$\begin{aligned}
F_{X,Y}(a,b) &= \mathbb{P}(X \le a, Y \le b)\\
&= \mathbb{P}(X \le a, X \le b, Z = 1) + \mathbb{P}(X \le a, -X \le b, Z = -1)\\
&= \frac{1}{2}\mathbb{P}(X \le a \wedge b) + \frac{1}{2}\mathbb{P}(-b \le X \le a)\\
&= \frac{1}{2}N(a \wedge b) + \frac{1}{2}((N(a) - N(-b)) \vee 0)
\end{aligned}$$

Example (2.2.12). Let (X,Y) be jointly normal with the density

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right)$$

Define $W = Y - \frac{\rho\sigma_2}{\sigma_1}X$. Then, X and W are independent.
Note that linear combination of jointly normal random variables are jointly normal (i.e., (X,W) is jointly normal).
Thus it suffices to show that $Cov(X,W) = 0$ (by Thm 2.2.9)

$$\begin{aligned}
Cov(X,W) &= \mathbb{E}[(X - \mathbb{E}X)(W - \mathbb{E}W)]\\
&= \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] - \mathbb{E}[\frac{\rho\sigma_2}{\sigma_1}(X - \mathbb{E}X)^2]\\
&= Cov(X,Y) - \frac{\rho\sigma_2}{\sigma_1}\sigma_1^2\\
&= 0
\end{aligned}$$

Let $f_{X,W}$ be joint density of X and W.

$$\begin{aligned}
\mathbb{E}[W] &= \mu_2 - \frac{\rho\sigma_2\mu_1}{\sigma_1} =: \mu_3\\
\mathbb{E}[(W - \mathbb{E}W)^2] &= \mathbb{E}[(Y - \mathbb{E}Y)^2] - \frac{2\rho\sigma_2}{\sigma_1}\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] + \frac{\rho^2\sigma_2^2}{\sigma_1^2}\mathbb{E}[(X - \mathbb{E}X)^2]\\
&= \sigma^2 - \frac{2\rho\sigma_2}{\sigma_1}\rho\sigma_1\sigma_2 + \frac{\rho^2\sigma_2^2}{\sigma_1^2}\sigma_1^2\\
&= (1-\rho^2)\sigma_2^2 =: \sigma_3^2
\end{aligned}$$

$\therefore f_{X,W}(x,w) = \frac{1}{2\pi\sigma_1\sigma_3}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(w-\mu_3)^2}{2\sigma_3^2}\right)$.
Note that we have decomposed Y into the linear combination $Y = \frac{\rho\sigma_2}{\sigma_1}X + W$ of a pair of independent normal random variables X and W.

Example (2.3.3). Let $\mathcal{G} = \sigma(X)$. Observe estimate Y based on X and error.

$$\begin{aligned}\mathbb{E}[Y|X] &= \frac{\rho\sigma_2}{\sigma_1} + \mathbb{E}[W] = \frac{\rho\sigma_2}{\sigma_1}(X-\mu_1) + \mu_2.\\ Y - \mathbb{E}[Y|X] &= W - \mathbb{E}[W]\end{aligned}$$

Note that the error is random variable with expected value zero and independent of the estimation $\mathbb{E}[Y|X]$.