

SEOUL NATIONAL UNIVERSITY

LECTURE NOTE

Introduction to Stochastic Differential Equations

Lecture by Seo Insuk

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Contents

Chapter 0

Introduction

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Office Hour: Tuesday 15:00 - 16:00

Grading

- Mid-terms 1 (15%, 10/10 or 17)
- Mid-terms 2 (15%, 11/7)
- Final-term (40%)
- Assignment (20%, 8-10 times)
- Attendance (10%, absent: -2%, late: -1%)

Let X be a standard normal random variable in \mathbb{R} . i.e., $\mathbb{P}[X \in [a, b]] = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.
(Central Limit Theorem) If $x_1, x_2, \dots, x_n \in X, E(x_i) = m, Var(x_i) = \sigma^2$, then

$$\frac{\frac{x_1-m}{\sigma} + \frac{x_2-m}{\sigma} + \dots + \frac{x_n-m}{\sigma}}{\sqrt{n}} \rightarrow X$$

In this class, we study dynamic version of this theorem. If $(W_t)_{t \geq 0}$ be a fluctuation, then $(W_t)_{t \geq 0}$ be a random variable in $C[0, T]$

Example. $\frac{dX_t}{dt} = rX_t; dX_t = rX_t dt$. Then, $X_t = X_0 e^{rt}$ (unrisky assets, bank)

$dX_t = rX_t dt + \sigma X_t dW_t$, σ : volatility (risky assets, stock)

We will study:

1. Probability Space
2. Random Variable
3. Expectation

Textbooks:

1. Stochastic Calculus for Finance II (Shreve), covering chapter 1-3 or 4
2. Introduction to Stochastic Integration (Hui-Hsiung Kuo)

Chapter 1

General Probability Theory

1.1 Infinite Probability Spaces

There are three elements consisting probability space:

- S : Sample space
- \mathcal{E} : Family of events $\mathcal{E} \subseteq 2^S$ (σ -algebra in measure theory)
- \mathbb{P} : probability $\Rightarrow \mathbb{P}(E)$ is defined for all $E \in \mathcal{E}$ (μ with $\mu(S) = 1$)

Example.

1. Toss a coin twice (H for Head, T for Tail)
Then, $S = \{HH, HT, TT, TH\}$
2. Uniform random variable in $[0, 1]^3$
Then, $S = [0, 1]^3$. If $E = [0, \frac{1}{2}]^3$, then $\mathbb{P}(E) = \text{Vol}(E) = \frac{1}{8}$

How to define \mathcal{E} ?

In example 2, let $\mathcal{E} =$ family of all subsets of $[0, 1]^3$ naively. But Banach-Tarski Paradox says there are disjoint sets E, F with $\mathbb{P}(E \cup F) \neq \mathbb{P}(E) + \mathbb{P}(F)$ in this \mathcal{E} . Therefore we cannot naively set \mathcal{E} (Use measure theory)

In example 1, suppose that we cannot see the second flip. If $\{HH\} \notin \mathcal{E}$ and $\{HT, HH\} \in \mathcal{E}$, then $\mathcal{E} = \{\emptyset, \{HH, HT\}, \{TH, TT\}, \{HH, HT, TH, TT\}\}$

Definition 1.1.1 (Measure)

Let Ω be a non-empty set and \mathcal{F} be family of subsets of Ω with

1. $\emptyset \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$
3. $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

We say \mathcal{F} as σ -algebra or σ -field, $A \in \mathcal{F}$ as **measurable**, and Ω as **measurable space**.

Exercises.

- 1) $\Omega \in \mathcal{F}$
- 2) $A_1, A_2, \dots \in \mathcal{F}$, then $A_1 \cap A_2 \cap \dots \in \mathcal{F}$
- 3) $A_1, A_2, \dots \in \mathcal{F}$, then $A_1 \cup \dots \cup A_n, A_1 \cap \dots \cap A_n \in \mathcal{F}$.
- 4) $A, B \in \mathcal{F}$, then $A - B \in \mathcal{F}$

Definition 1.1.2 (Topological Space)

(See Rudin: *Real and Complex Analysis*, Chapter 1.) Let Θ be non-empty set and τ be family of subsets of Θ with

1. $\emptyset, \Theta \in \tau$
2. $V_1, \dots, V_n \in \tau \Rightarrow V_1 \cap \dots \cap V_n \in \tau$
3. $V_\alpha \in \tau \ \forall \alpha \in I \Rightarrow \bigcup_{\alpha \in I} V_\alpha \in \tau$.

We say $V \in \tau$ be an **open set**, and (Θ, τ) be a **topological space**.

Definition 1.1.3 (Measurable Function)

$f : (\Omega, \mathcal{F}) \rightarrow (\Theta, \tau)$ is **measurable** if $f^{-1}(V) \in \mathcal{F} \ \forall V \in \tau$

Definition 1.1.4 (Positive Measure)

Let Ω be non-empty set and \mathcal{F} be σ -algebra. Then $\mu : \mathcal{F} \rightarrow [0, \infty]$ is called **measurable** if

1. A_1, A_2, \dots : disjoint members of $\mathcal{F} \Rightarrow \mu(A_1 \cup A_2 \cup \dots) = \sum_{i=1}^{\infty} \mu(A_i)$
2. $\mu(A) < \infty$ for some $A \in \mathcal{F}$,

and $(\Omega, \mathcal{F}, \mu)$ is called a **measure space**.

Definition 1.1.5 (probability space, random variable)

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is called as **probability space** if $\mathbb{P}(\Omega) = 1$.
2. X is called as **random variable** if it is a function from $(\Omega, \mathcal{F}, \mathbb{P})$ to \mathbb{R}

Next Class

- Borel sets on \mathbb{R} or \mathbb{R}^d
- Lebesgue Measure
- Lebesgue Integral (Define Expectation of random variable)

Last class, we define a sample space Ω , a σ -algebra \mathcal{F} , and a (positive) measure $\mu : \mathcal{F} \rightarrow [0, \infty]$.

Exercises.

- $A_1 \subseteq A_2 \subseteq \dots \Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n \rightarrow \infty} \mu(A_n)$
- $A_1 \supseteq A_2 \supseteq \dots, \mu(A_1) < \infty \Rightarrow \mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{n \rightarrow \infty} \mu(A_n)$

Theorem 1.1.6 (Rudin 1.10)

Let \mathcal{F}_0 be a collection of subset of Ω . Then, $\exists! \mathcal{F}^*$ minimal σ -algebra containing \mathcal{F}_0 .

Proof. Let $\{\mathcal{F}_\alpha, \alpha \in I\}$ be a family of σ -algebra containing \mathcal{F}_0 . Then, $\mathcal{F}^* = \bigcap_{\alpha \in I} \mathcal{F}_\alpha$ satisfies the three condition: 1) contain \mathcal{F}_0 2) σ -algebra 3) minimal (trivial, $\mathcal{F}^* \subseteq \mathcal{F}_\alpha$) \square

Definition 1.1.7 (Borel measurable)

\mathcal{B} is a **Borel σ -algebra** on topological space (Θ, τ) if \mathcal{B} is minimal σ -algebra containing τ , and B is a **Borel measurable** if $B \in \mathcal{B}$.

Remark (Completion of measure space, Rudin 1.15).

Consider an extension $(\Omega, \mathcal{F}, \mu) \rightarrow (\Omega, \overline{\mathcal{F}}, \mu)$ where

1. $\overline{\mathcal{F}} = \{A \cup N : A \in \mathcal{F}, N \subseteq A_0 \in \mathcal{F}, \mu(A_0) = 0\}$
2. $\mu(A \cup N) = \mu(A)$

Then, (Check!)

1. (well-definedness) $A_1 \cup N_1 = A_2 \cup N_2 \Rightarrow \mu(A_1) = \mu(A_2)$
2. $\mu : \overline{\mathcal{F}}$ is σ -algebra.
3. $\mu : \overline{\mathcal{F}} \rightarrow [0, \infty]$ is a measure

Example.

1) \mathbb{R}

$$\mathcal{F}_0 = \tau \xrightarrow{1.10} \mathcal{B} \xrightarrow{\text{completion}} \overline{\mathcal{B}}$$

$$\mathcal{L} \xrightarrow{\text{Rudin CH 2}} \mathcal{L} \xrightarrow{\text{completion}} \mathcal{L}$$

2) $C[0, T] = \Omega = \{f; f : [0, T] \rightarrow \mathbb{R}, \text{continuous}\}$.

Define $\mathcal{F}_0 = \{\bigcup_{t_1, t_2, \dots, t_k} (A_1, A_2, \dots, A_k) : 0 \leq t_1 < t_2 < \dots < t_k \leq T; A_1, \dots, A_k \in \overline{\mathcal{B}}\}$. We call $\{f \in C[0, T] : f(t_1) \in A_1, f(t_2) \in A_2, \dots, f(t_k) \in A_k\}$ as **cylindrical set**. Consider

$$\begin{array}{ccccc} \mathcal{F}_0 & \xrightarrow{1.10} & \mathcal{B} & \xrightarrow{\text{completion}} & \overline{\mathcal{B}} \\ \mathbb{P}_{\text{BM}} & \xrightarrow{\text{KET}} & \mathbb{P}_{\text{BM}} & \xrightarrow{\text{completion}} & \mathbb{P}_{\text{BM}}^* \end{array}$$

(KET refers Kolmogorov's Extension Thm)

1.2 Random Variables and Distributions

Definition 1.2.1

$f : \Omega \rightarrow \mathbb{R}$ is measurable if $f^{-1}(V) \in \mathcal{F}$ for any open set $V \subseteq \mathbb{R}$.

Remark. $\mathcal{B}(\mathbb{R})$ = Borel σ -algebra in \mathbb{R} .

Remark. If f : measurable, then $f^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}(\mathbb{R})$.

Proof. Let $G = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \mathcal{F}\}$. Then, $\tau \subseteq G$, G : σ -algebra (check!), hence $\mathcal{B}(\mathbb{R}) \subseteq G$. \square

Definition 1.2.2

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a **probability space** if $\mathbb{P}(\Omega) = 1$.
- X is **random variable** if $X : \Omega \rightarrow \mathbb{R}$ is measruable.

Example.

1. Toss a coin Twice.

$\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F} = 2^\Omega = \{\text{all subsets of } \Omega\}$, $\mathbb{P}(A) = \frac{1}{4}|A|$, $A \in \mathcal{F}$.

Then, X = the number of H's is random variable with $X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 1$.

2. Uniform random variable in $[0, 1]$

$\Omega = [0, 1]$, $\mathcal{F} = \{B \in \mathcal{B}(\mathbb{R}) : B \subseteq [0, 1]\}$, $\mathbb{P}(B) = \mathcal{L}(B)$ ($\mathbb{P}([0, 1]) = \mathcal{L}([0, 1]) = 1$).

Then, $X : [0, 1] \rightarrow \mathbb{R}$ with $X(x) = x$ be a (uniform) random variable in $[0, 1]$.

Remark. \mathcal{L} : Lebesgue measrue on \mathbb{R} . i.e., $\mathcal{L}(a, b) = b - a$. Then, $\mathcal{L}(\{a\}) = 0$

($\because \{a\} = \bigcap_{i=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}) \Rightarrow \mathcal{L}(\{a\}) = \lim_{n \rightarrow \infty} \mathcal{L}((a - \frac{1}{n}, a + \frac{1}{n})) = 0$)

Similarly, $\mathcal{L}([a, b]) = \mathcal{L}([a, b)) = \mathcal{L}((a, b]) = b - a$, $\mathcal{L}(\mathbb{Q}) = \sum_{q \in \mathbb{Q}} \mathcal{L}(\{q\}) = 0$.

Return to uniform random variable,

$$\mathbb{P}[X \in (a, b)] = \mathbb{P}[\{x : X(x) \in (a, b)\}] = \mathbb{P}[(a, b)] = b - a.$$

Definition 1.2.3 (Distribution measure on X)

X is a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. μ_X is a **distribution measure** on X if μ_X is a probability measrue on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mu_X(B) = \mathbb{P}[X \in B] = \mathbb{P}[\{\omega : X(\omega) \in B\}] = \mathbb{P}[X^{-1}(B)] \quad \forall B \in \mathcal{B}(\mathbb{R})$$

Definition 1.2.4 (Probability density function)

f is a **probability density function** of X if $\mu_X((a, b)) = \int_a^b f(x)dx$

Remark. There is a measure with no pdf: Dirac measrue

Remark. Lebesgue-Radon-Nikodym decomposition implies that any measure can be decomposed as density part and non-density part.

Example (Standard Normal random variable).

Let $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Define $F : (0, 1) \rightarrow \mathbb{R}$ by $F(x) = N^{-1}(x)$ for $N(X) = \int_{-\infty}^x \phi(y)dy$.

Let $\Omega = (0, 1)$, $\mathcal{F} = \{B \in \mathcal{B}(\mathbb{R}) : B \subseteq (0, 1)\}$, $\mathbb{P}(A) = \mathcal{L}(A) : A \in \mathcal{B}(\mathbb{R})$.

Then, $Y : \Omega \ni x \mapsto F(x) \in \mathbb{R}$ is a random variable with

$$\begin{aligned}\mathbb{P}[Y \in (a, b)] &= \mathbb{P}[\{x : Y(x) \in (a, b)\}] \\ &= \mathbb{P}[\{x \in (N(a), N(b))\}] \\ &= N(b) - N(a) = \int_a^b \phi(x)dx,\end{aligned}$$

and a density function is ϕ .

Previous Question: In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variable $X : \Omega \rightarrow \mathbb{R}$, the random element or random realization $\omega \in \Omega$ is a element of events in sample space. For example, $\omega = HHTTH$ is a random element in tossing a coin five times, and $X(\omega) = 3$. ($X(\omega) = \#$ of Heads)

In the previous example(Standard Normal random variable), define $(\Omega, \mathcal{F}, \mathbb{P}) = ((0, 1), \mathcal{B}(0, 1), \mathbb{P})$, $\mathbb{P}((a, b)) = b - a$, $F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy$, $X : (0, 1) \ni \omega \mapsto F^{-1}(\omega) \in \mathbb{R}$. Then, X is called a standard normal random variable.

1.3 Expectations

In the following, let $\Omega = (\Omega, \mathcal{F}, \mathbb{P})$. Let $X : \Omega \rightarrow \mathbb{R}$. Then the expectation $E(X)$ is a mean of $X(\omega)$ with respect to the randomness of ω (given by \mathbb{P})

Definition 1.3.1 (Lebesgue Integration)

$(\Omega, \mathcal{F}, \mu)$ is a measure space, and $f : \Omega \rightarrow \mathbb{R}$ is a measurable function.

(1) $f : \Omega \rightarrow [0, \infty)$

Let $0 = y_0 < y_1 < y_2 < \dots \rightarrow \mathbb{R}$ be a partition of $[0, \infty)$,

$\Pi = \{y_0, y_1, y_2, \dots\} : \|\Pi\| = \sup_{i \geq 1} |y_i - y_{i-1}|$, and

$LS_\Pi = \sum_{i=0}^{\infty} y_i \mu[f^{-1}([y_i, y_{i+1}))]$.

In Rudin's book, $\lim_{\|\Pi\| \rightarrow 0} LS_\Pi$ converges to an element belonging to $[0, \infty]$.

Now, $\int f d\mu := \lim_{\|\Pi\| \rightarrow 0} LS_\Pi$ is called a **Lebesgue Integral**.

(2) $f : \Omega \rightarrow \mathbb{R}$

Let $f^+ = \max\{f, 0\} \geq 0$, and $f^- = -\min\{f, 0\} \geq 0$. Then, $f = f^+ - f^-$, and $|f| = f^+ + f^-$. If $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$, then we say f is Lebesgue Integrable and $f \in L^1(\mu)$. The Lebesgue Integral of f $\int f d\mu$ is defined as $\int f^+ d\mu - \int f^- d\mu$

Remark.

1. $\int f^+ d\mu < \infty$ and $\int f^- d\mu = \infty$, then $\int f d\mu = -\infty$. The others are defined similarly.
2. $f \in L^1(\mu) \Leftrightarrow \int |f| d\mu < \infty$.

Example (Riemann vs Lebesgue Integral (p19-22)).

- $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{L})$ Lebesgue measure where $\mathcal{L}((a, b)) = b - a$.
- $f : \mathbb{R} \rightarrow \mathbb{R} \in L^1(\mathcal{L})$
- (Def) $A \subseteq \mathbb{R}$, $\int_A f d\mu := \int f \mathbb{1}_A d\mu$, where $\mathbb{1}_A(x) = 1$ if $x \in A$, and 0 otherwise.

If f is Riemann integrable, then $\int_{[a,b]} f d\mathcal{L} = \int_a^b f dx$.

Riemann integral is a limit of approximation by a partition of x -axis. On the other hand, Lebesgue integral is a limit of approximation by a partition of y -axis with preimage. Partition of x -axis is sensitive to fluctuation and restricted to Euclidean space, while partition of y -axis is not. For example, $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ is Lebesgue integrable, but it is not Riemann integrable since it is sensitive to fluctuation.

Definition 1.3.2 (Almost everywhere, 1.1.5 in Textbook)

$P(x)$ is a property at $x \in \mathbb{R}$. We say P holds **almost everywhere** (or a.e.) in \mathbb{R} if and only if $\mathcal{L}(\{x : P(x) \text{ does not hold}\}) = 0$.

Example. $f(x) = [x]$ is continuous almost everywhere.

Theorem 1.3.3

f is Riemann integrable if and only if f is continuous a.e.

Exercises. $f = g$ a.e. $\Rightarrow \int f d\mathcal{L} = \int g d\mathcal{L}$.

Definition 1.3.4 (Almost surely)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The event $A(\in \mathcal{F})$ occurs **almost surely** (a.s.) if $\mathbb{P}(A) = 1$.

Example. Let X be a uniform random variable in $(0, 1)$. Let $A = \{X(\omega) \neq \frac{1}{2}\}$; $\mathbb{P}(A) = 1$.

Definition 1.3.5 (Expectation, 1.3.3. in Textbook)

Expectation of $X : \Omega \rightarrow \mathbb{R}$ is defined by

$$E(X) := \int_{\Omega} X d\mathbb{P} \quad \text{if} \quad \int_{\Omega} |X| d\mathbb{P} < \infty$$

Theorem 1.3.6 (1.3.4 in Textbook)

1. X takes finite number of values $\{x_1, x_2, \dots, x_n\} \Rightarrow E(X) = \sum_{i=1}^n x_i \mathbb{P}(X = x_i)$

2. X, Y : random variables, $E(|X|), E(|Y|) < \infty$,

(i) $X \leq Y$ a.s. (i.e. $\mathbb{P}[\{X(\omega) \leq Y(\omega)\}] = 1$), then $E(X) \leq E(Y)$

(ii) $X = Y$ a.s. $\Rightarrow E(X) = E(Y)$

3. X, Y : random variables, $E(|X|), E(|Y|) < \infty \Rightarrow E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y)$.

4. Jensen's Inequality: $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function $\Rightarrow \phi(E(X)) \leq E(\phi(X))$ (c.f. $\phi(t) = t^2$)

Proof of 4. Define $S_\phi = \{(a, b) \in \mathbb{R}^2 : a + bt \leq \phi(t) \ \forall t\}$. Then $\forall t \in \mathbb{R}, \phi(t) = \sup_{(a,b) \in S_\phi} \{a + bt\}$. In fact, it is a equivalent condition. Now,

$$\begin{aligned} \phi(E(X)) &= \sup_{a,b \in S_\phi} \{a + bE(X)\} \\ &= \sup_{a,b \in S_\phi} E(a + bX) \\ &\leq E\left[\sup_{a,b \in S_\phi} (a + bx)\right] = E(\phi(X)) \quad (\text{Check!}) \end{aligned}$$

□

Example (Dirac Measure in \mathbb{R}). $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \delta_y)$ ($y \in \mathbb{R}$) is a probability space with $\delta_y(A) = 1$ if $y \in A$, and 0 otherwise. Then, $\int_{\mathbb{R}} f d\delta_y = f(y)$ (Check!)

Consider modeling: X : random variable such that probability of $x_i = p_i$ with $\sum_{i=1}^n p_i = 1$. Then, $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ with $\mu = \sum_{i=1}^n p_i \delta_{x_i}$ is a probability space, and $P(X = x_i) = p_i$ for $X : \mathbb{R} \ni \omega \mapsto \omega \in \mathbb{R}$: Example of thm 1.3.4.