Seoul National University

Lecture Note

Introduction to Stochastic Differential Equations

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September 3, 2018

Chapter 0

Introduction

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Grading

- Mid-terms 1 (15%, 10/10 or 17)
- Mid-terms 2 (15%, 11/7)
- Fianl-term (40%)
- Assignment (20%, 8-10 times)
- Attendance (10%, absent: -2%, late: -1%)

Let X be a standard normal random variable in $\mathbb{R}$. i.e., $\mathbb{P}[X \in [a,b]] = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.
(Central Limit Theorem) If $x_1, x_2, \cdots, x_n \in X, E(x_i) = m, Var(x_i) = \sigma^2$, then

$$\frac{\frac{x_1-m}{\sigma} + \frac{x_2-m}{\sigma} + \cdots + \frac{x_n-m}{\sigma}}{\sqrt{n}} \to X$$

In this class, we study dynamic version of this theorem. If $(W_t)_{t\geq 0}$ be a fluctuation, then $(W_t)_{t\geq 0}$ be a random variable in $C[0,T]$
Example. $\frac{dX_t}{dt} = rX_t; dX_1 = rX_t dt$. Then, $X_t = X_0 e^{rt}$ (unrisky assets, bank)
$dX_t = rX_t dt + \sigma X_t dW_t$, σ : volatility (risky assets, stock)
We will study:

1. Probability Space
2. Random Variable
3. Expectation

Textbooks:

1. Stochastic Calculus for Finance II (Shreve), covering chapter 1-3 or 4
2. Introduction to Stochastic Integration (Hui-Hsiung Kuo)

Chapter 1

Probability Space

There are three elements consisting probability space:

- S: Sample space
- $\mathcal{E}$: Family of events $E \subseteq S$ (σ-algebra in measure theory)
- $\mathbb{P}$: probability $\Rightarrow \mathbb{P}(E)$ is defined for all $E \subseteq \mathcal{E}$ (μ with $\mu(S) = 1$)

Example. 1. Toss a coin twice (H for Head, T for Tail)
Then, $S = \{$HH, HT, TT, TT$\}$

2. Uniform random variable in $[0,1]^3$
Then, $S = [0,1]^3$. If $E = [0, \frac{1}{2}]^3$, then $\mathbb{P}(E) = Vol(E) = \frac{1}{8}$

How to define $\mathcal{E}$?
In example 2, let $\mathcal{E}$ = family of all subsets of $[0,1]^3$ naively. But Banach-Tarski Paradox says there are disjoint sets E, F with $\mathbb{P}(E \cup F) \neq \mathbb{P}(E) + \mathbb{P}(F)$ in this $\mathcal{E}$. Therefore we cannot naively set $\mathcal{E}$ (Use measure theory)

In example 1, suppose that we cannot see the second flip. If $\{HH\} \notin \mathcal{E}$ and $\{HT, HH\} \in \mathcal{E}$, then $\mathcal{E} = \{\phi, \{HH, HT\}, \{TH, TT\}, \{HH, HT, TH, TT\}\}$

Definition 1.1 (Measure)
Let Ω be non-empty set and $\mathcal{F}$ be family of subsets of Ω with

1. $\phi \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$
3. $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

We say $\mathcal{F}$ as σ-albegra or σ-field, $A \subseteq \mathcal{F}$ as measurable, and Ω as measurable space.

Exercises.

1) $\Omega \in F$

2) $A_1, A_2, \cdots \in \mathcal{F}$, then $A_1 \cap A_2 \cdots \in \mathcal{F}$

3) $A_1, A_2, \cdots \in A_n \in \mathcal{F}$, then $A_1 \cup \cdots \cup A_n, A_1 \cap \cdots \cap A_n \in \mathcal{F}$.

4) $A, B \in \mathcal{F}$, then $A - B \in \mathcal{F}$

Definition 1.2 (Topological Space)
(See Rudin: *Real and Complex Analysis, Chapter 1.*) Let Θ be non-empty set and τ be family of subsets of Θ with

1. $\phi, \Theta \in \tau$

2. $V_1, \cdots V_n \in \tau \Rightarrow V_1 \cap \cdots \cap V_n \in \tau$

3. $V_\alpha \in \tau \ \ \forall \alpha \in I \Rightarrow \bigcup_{\alpha \in I} V_\alpha \in \tau$.

We say $V \in \tau$ be **open set**, and (Θ, τ) be **topological space**.

Definition 1.3 (Measurable Function)
$f : (\Omega, \mathcal{F}) \to (\Theta, \tau)$ is **measurable** if $f^{-1}(V) \in \mathcal{F} \ \ \forall V \in \tau$

Definition 1.4 (Positive Measure)
Let Ω be non-empty set and $\mathcal{F}$ be σ-algebra. Then $\mu : \mathcal{F} \to [0, \infty]$ is called **measurable** if

1. $A_1, A_2, \cdots$: disjoint members of $\mathcal{F} \Rightarrow \mu(A_1 \cup A_2 \cup \cdots) = \sum_{i=1}^{\infty} \mu(A_i)$

2. $\mu(A) < \infty$ for some $A \in \mathcal{F}$,

and $(\Omega, \mathcal{F}, \mu)$ is called **measrue space**.

Definition 1.5

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is called as **probability space** if $\mathbb{P}(\Omega) = 1$.

2. X is called as **random varaible** if it is a function from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$

<u>Next Class</u>

- Borel sets on $\mathbb{R}$ or $\mathbb{R}^d$

- Lebesgue Measure

- Lebesgue Integral (Define Expectation of random variable)