

SEOUL NATIONAL UNIVERSITY

LECTURE NOTE

Introduction to Stochastic Differential Equations

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Chapter 0

Introduction

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Grading

- Mid-terms 1 (15%, 10/10 or 17)
- Mid-terms 2 (15%, 11/7)
- Final-term (40%)
- Assignment (20%, 8-10 times)
- Attendance (10%, absent: -2%, late: -1%)

Let X be a standard normal random variable in \mathbb{R} . i.e., $\mathbb{P}[X \in [a, b]] = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

(Central Limit Theorem) If $x_1, x_2, \dots, x_n \in X, E(x_i) = m, Var(x_i) = \sigma^2$, then

$$\frac{\frac{x_1-m}{\sigma} + \frac{x_2-m}{\sigma} + \dots + \frac{x_n-m}{\sigma}}{\sqrt{n}} \rightarrow X$$

In this class, we study dynamic version of this theorem. If $(W_t)_{t \geq 0}$ be a fluctuation, then $(W_t)_{t \geq 0}$ be a random variable in $C[0, T]$

Example. $\frac{dX_t}{dt} = rX_t; dX_t = rX_t dt$. Then, $X_t = X_0 e^{rt}$ (unrisky assets, bank)

$dX_t = rX_t dt + \sigma X_t dW_t, \sigma$: volatility (risky assets, stock)

We will study:

1. Probability Space
2. Random Variable
3. Expectation

Textbooks:

1. Stochastic Calculus for Finance II (Shreve), covering chapter 1-3 or 4
2. Introduction to Stochastic Integration (Hui-Hsiung Kuo)

Chapter 1

Probability Space

1.1 Measure

There are three elements consisting probability space:

- S : Sample space
- \mathcal{E} : Family of events $E \subseteq S$ (σ -algebra in measure theory)
- \mathbb{P} : probability $\Rightarrow \mathbb{P}(E)$ is defined for all $E \subseteq \mathcal{E}$ (μ with $\mu(S) = 1$)

Example. 1. Toss a coin twice (H for Head, T for Tail)

Then, $S = \{HH, HT, TT, TH\}$

2. Uniform random variable in $[0, 1]^3$

Then, $S = [0, 1]^3$. If $E = [0, \frac{1}{2}]^3$, then $\mathbb{P}(E) = Vol(E) = \frac{1}{8}$

How to define \mathcal{E} ?

In example 2, let $\mathcal{E} =$ family of all subsets of $[0, 1]^3$ naively. But Banach-Tarski Paradox says there are disjoint sets E, F with $\mathbb{P}(E \cup F) \neq \mathbb{P}(E) + \mathbb{P}(F)$ in this \mathcal{E} . Therefore we cannot naively set \mathcal{E} (Use measure theory)

In example 1, suppose that we cannot see the second flip. If $\{HH\} \notin \mathcal{E}$ and $\{HT, HH\} \in \mathcal{E}$, then $\mathcal{E} = \{\emptyset, \{HH, HT\}, \{TH, TT\}, \{HH, HT, TH, TT\}\}$

Definition 1.1 (Measure)

Let Ω be non-empty set and \mathcal{F} be family of subsets of Ω with

1. $\emptyset \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$
3. $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

We say \mathcal{F} as σ -algebra or σ -field, $A \subseteq \mathcal{F}$ as measurable, and Ω as measurable space.

Exercises.

- 1) $\Omega \in \mathcal{F}$
- 2) $A_1, A_2, \dots \in \mathcal{F}$, then $A_1 \cap A_2 \cap \dots \in \mathcal{F}$
- 3) $A_1, A_2, \dots \in \mathcal{F}$, then $A_1 \cup \dots \cup A_n, A_1 \cap \dots \cap A_n \in \mathcal{F}$.
- 4) $A, B \in \mathcal{F}$, then $A - B \in \mathcal{F}$

Definition 1.2 (Topological Space)

(See Rudin: *Real and Complex Analysis*, Chapter 1.) Let Θ be non-empty set and τ be family of subsets of Θ with

1. $\emptyset, \Theta \in \tau$
2. $V_1, \dots, V_n \in \tau \Rightarrow V_1 \cap \dots \cap V_n \in \tau$
3. $V_\alpha \in \tau \ \forall \alpha \in I \Rightarrow \bigcup_{\alpha \in I} V_\alpha \in \tau$.

We say $V \in \tau$ be **open set**, and (Θ, τ) be **topological space**.

Definition 1.3 (Measurable Function)

$f : (\Omega, \mathcal{F}) \rightarrow (\Theta, \tau)$ is **measurable** if $f^{-1}(V) \in \mathcal{F} \ \forall V \in \tau$

Definition 1.4 (Positive Measure)

Let Ω be non-empty set and \mathcal{F} be σ -algebra. Then $\mu : \mathcal{F} \rightarrow [0, \infty]$ is called **measurable** if

1. A_1, A_2, \dots : disjoint members of $\mathcal{F} \Rightarrow \mu(A_1 \cup A_2 \cup \dots) = \sum_{i=1}^{\infty} \mu(A_i)$
2. $\mu(A) < \infty$ for some $A \in \mathcal{F}$,

and $(\Omega, \mathcal{F}, \mu)$ is called **measure space**.

Definition 1.5 (probability space, random variable)

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is called as **probability space** if $\mathbb{P}(\Omega) = 1$.
2. X is called as **random variable** if it is a function from $(\Omega, \mathcal{F}, \mathbb{P})$ to \mathbb{R}

Next Class

- Borel sets on \mathbb{R} or \mathbb{R}^d
- Lebesgue Measure
- Lebesgue Integral (Define Expectation of random variable)

Last class, we define a sample space Ω , a σ -algebra \mathcal{F} , and a (positive) measure $\mu : \mathcal{F} \rightarrow [0, \infty]$.

Exercises.

- $A_1 \subseteq A_2 \subseteq \cdots \Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n \rightarrow \infty} \mu(A_n)$
- $A_1 \subseteq A_2 \subseteq \cdots, \mu(A_1) < \infty \Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n \rightarrow \infty} \mu(A_n)$

Theorem 1.6 (Rudin 1.10)

Let \mathcal{F}_0 be a collection of subset of Ω . Then, $\exists! \mathcal{F}^*$ minimal σ -algebra containing \mathcal{F}_0 .

Proof. Let $\{\mathcal{F}_\alpha, \alpha \in I\}$ be family of σ -algebra containing \mathcal{F}_0 . Then, $\mathcal{F}^* = \bigcap_{\alpha \in I} \mathcal{F}_\alpha$ satisfies the three condition: 1) contain \mathcal{F}_0 2) σ -algebra 3) minimal (trivial, $\mathcal{F}^* \subseteq \mathcal{F}_\alpha$) \square

Definition 1.7 (Borel measurable)

\mathcal{B} is called a **Borel σ -algebra** on topological space (Θ, τ) if \mathcal{B} is minimal σ -algebra containing τ , and B is called **Borel measurable** if $B \in \mathcal{B}$.

Remark (Completion of measure space, Rudin 1.15).

Consider an extension $(\Omega, \mathcal{F}, \mu) \rightarrow (\Omega, \overline{\mathcal{F}}, \mu)$ where

1. $\overline{\mathcal{F}} = \{A \cup N : A \in \mathcal{F}, N \subseteq A_0 \subseteq \mathcal{F}, \mu(A_0) = 0\}$
2. $\mu(A \cup N) = \mu(A)$

Then, (Check!)

1. (well-definedness) $A_1 \cup N_1 = A_2 \cup N_2 \Rightarrow \mu(A_1) = \mu(A_2)$
2. $\mu : \overline{\mathcal{F}}$ is σ -algebra.
3. $\mu : \overline{\mathcal{F}} \rightarrow [0, \infty]$ is a measure

Example.

- 1) \mathbb{R}

$$\begin{array}{ccc} \mathcal{F}_0 = \tau & \xrightarrow{1.10} \mathcal{B} & \xrightarrow{\text{completion}} \overline{\mathcal{B}} \\ \mathcal{L} & \xrightarrow{\text{Rudin CH2}} \mathcal{L} & \xrightarrow{\text{completion}} \mathcal{L} \end{array}$$

- 2) $C[0, T] = \Omega = \{f; f : [0, T] \rightarrow \mathbb{R}, \text{continuous}\}$.

Define $\mathcal{F}_0 = \{\bigcup_{t_1, t_2, \dots, t_k} (A_1, A_2, \dots, A_k) : 0 \leq t_1 < t_2 < \dots < t_k \leq T; A_1, \dots, A_k \in \overline{\mathcal{B}}\}$. We call $\{f \in C[0, T] : f(t_1) \in A_1, f(t_2) \in A_2, \dots, f(t_k) \in A_k\}$ as **cylindrical set**. Consider

$$\begin{array}{ccc} \mathcal{F}_0 & \xrightarrow{1.10} \mathcal{B} & \xrightarrow{\text{completion}} \overline{\mathcal{B}} \\ \mathbb{P}_{\text{BM}} & \xrightarrow{\text{KET}} \mathbb{P}_{\text{BM}} & \xrightarrow{\text{completion}} \mathbb{P}_{\text{BM}}^* \end{array}$$

(KET refers Kolmogorov's Extension Thm)

1.2 Random variable

Definition 1.8

$f : \Omega \rightarrow \mathbb{R}$ is measurable if $f^{-1}(V) \in \mathcal{F}$ for any open set $V \subseteq \mathbb{R}$.

Remark. $\mathcal{B}(\mathbb{R})$ = Borel σ -algebra in \mathbb{R} .

Remark. If f : measurable, then $f^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}(\mathbb{R})$.

Proof. Let $G = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \mathcal{F}\}$. Then, $\tau \subseteq G$, G : σ -algebra (check!), hence $\mathcal{B}(\mathbb{R}) \subseteq G$. \square

Definition 1.9

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a **probability space** if $(\mathbb{P}(\Omega) = 1)$.
- X is **random variable** if $X : \Omega \rightarrow \mathbb{R}$ is measruable.

Example.

1. Toss a coin Twice.

$\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F} = 2^\Omega = \{\text{all subsets of } \Omega\}$, $\mathbb{P}(A) = \frac{1}{4}|A|$, $A \in \mathcal{F}$.

Then, $X = \#$ of H's is random variable with $X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 1$.

2. Uniform random variable in $[0, 1]$

$\Omega = [0, 1]$, $\mathcal{F} = \{B \in \mathcal{B}(\mathbb{R}) : B \subseteq [0, 1]\}$, $\mathbb{P}(B) = \mathcal{L}(B)$ ($\mathbb{P}([0, 1]) = \mathcal{L}([0, 1]) = 1$).

Then, $X : [0, 1] \rightarrow \mathbb{R}$ with $X(x) = x$ be a (uniform) random variable in $[0, 1]$.

Remark. \mathcal{L} : Lebesgue measrue on \mathbb{R} . i.e., $\mathcal{L}(a, b) = b - a$. Then, $(a) = 0$

($\because \{a\} = \bigcap_{i=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}) \Rightarrow \mathcal{L}(\{a\}) = \lim_{n \rightarrow \infty} \mathcal{L}((a - \frac{1}{n}, a + \frac{1}{n})) = 0$)

Similarly, $\mathcal{L}([a, b]) = \mathcal{L}([a, b)) = \mathcal{L}((a, b]) = b - a$, $\mathcal{L}(\mathbb{Q}) = \sum_{q \in \mathbb{Q}} \mathcal{L}(\{q\}) = 0$.

Return to uniform random variable,

$$\mathbb{P}[X \in (a, b)] = \mathbb{P}[\{x : X(x) \in (a, b)\}] = \mathbb{P}[(a, b)] = b - a.$$

Definition 1.10 (Distribution measure on X)

X is a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. μ_X is a **distribution measure on X** if μ_X is a probability measrue on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mu_X(B) = \mathbb{P}[X \in B] \quad \forall B \in \mathcal{B}(\mathbb{R}) = \mathbb{P}[\{\omega : X(\omega) \in B\}] = \mathbb{P}[X^{-1}(B)]$$

Definition 1.11 (Probability density function)

f is a **probability density function** of X if $\mu_X((a, b)) = \int_a^b f(x)dx$

Remark. Radon-Nikodym-Lebesgue decomposition implies that any measure can be decomposed as density part and non-density part.

Example (Standard Normal random variable).

Let $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Define $F : (0,1) \rightarrow \mathbb{R}$ by $F(x) = N^{-1}(x)$ for $N(X) = \int_{-\infty}^x \phi(y)dy$.

Let $\Omega = (0,1)$, $\mathcal{F} = \{B \in \mathcal{B}(\mathbb{R}) : B \subseteq (0,1)\}$, $\mathbb{P}(A) = \mathcal{L}(A) : A \in \mathcal{B}(\mathbb{R})$.

Then, $Y : \Omega \ni x \mapsto F(x) \in \mathbb{R}$ is a random variable with

$$\begin{aligned}\mathbb{P}[Y \in (a,b)] &= \mathcal{P}[\{x : Y(x) \in (a,b)\}] \\ &= \mathbb{P}[\{x \in (N(a), N(b))\}] \\ &= N(b) - N(a) = \int_a^b \phi(x)dx,\end{aligned}$$

and a density function is ϕ .