수학 문제 연구회

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13, December 2024

Let $M_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ be a set of real matrices. Solve the following statements:

1. Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, show that $J^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -I$.

2. Show that for $x, y, u, v \in \mathbb{R}$,

$$(xI + yJ)(uI + vJ) = (xu - vy)I + (uy + vx)J.$$

   Let $z = x + iy$, $w = u + iv$, then the correspondence $z \leftrightarrow xI + yJ$, $w \leftrightarrow uI + vJ$ gives isomorphism between fields

$$\mathbb{C} \leftrightarrow M_C := \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x, y \in \mathbb{R} \right\} \subset M_2(\mathbb{R}).$$

3. For a complex-valued function $f(z) = u(z) + v(z)i$, where each $u(z)$, $v(z)$ is a real-valued function, the necessary and sufficient condition for differentiability of $f(z)$ in domain D is to satisfy the following PDE in the specified domain:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   translate the above condition of differentiability for complex functions to the condition for functions defined on M_C.

Here is an additional question: Are there any subspaces of $M_2(\mathbb{R})$ that form a field? If so, what can be said about the properties (e.g. conditions of differentiability) of functions on these fields?