수학 문제 연구회

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Let $\mathcal{A} := \{f(z) : f : \mathbb{C} \to \mathbb{C}, f \text{ is differentiable on entire domain}\}$ be a set of analytic functions, and $\mathbb{A} := \{(a_n)_{n=0,\ldots} | a_n \in \mathbb{C}, n \in \mathbb{N}\}$ be a set of complex-valued sequences. Prove following statements:

1. Each $\mathcal{A}$, $\mathbb{A}$ is $\mathbb{C}$-vector space.
2. Set of sequences satisfying $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$ is subspace of $\mathbb{A}$.
3. Let denote the set in statement 2 as $\mathbb{A}'$, then $\mathcal{A} \simeq \mathbb{A}'$ as vector space.

Here's an additional question: By fundamental theorem of algebra, if $M := \max\{n | a_n \neq 0\} < \infty$ then, function $\sum a_n z^n$ has exactly M zeros (counting multiplicities). But, if $M = \infty$, there exists function without zeros (e.g. e^z). Is there a corresponding condition in $\mathbb{A}'$ for non-constant function to have no zeros?