Cognitive Imprecision and Strategic Behavior[*]

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October 4, 2021

Abstract

We propose and experimentally test a theory of strategic behavior in which players are *cognitively imprecise* and perceive a fundamental parameter with noise. We focus on 2×2 coordination games, which generate multiple equilibria when perception is precise. When adding a small amount of cognitive imprecision to the model, we obtain a unique equilibrium where players use a simple cutoff strategy. The model further predicts that behavior is *context-dependent*: players implement the unique equilibrium strategy with noise, and the noise decreases in fundamental volatility. Our experimental data strongly support this novel prediction and reject several alternative game-theoretic models that do not predict context-dependence. We also find that subjects are aware of other players' imprecision, which is key to generating strategic uncertainty. Our framework has important implications for the literature on global games and, more broadly, illuminates the role of perception in generating both random and context-dependent behavior in games.

Keywords: Perception, Efficient Coding, Coordination, Global Games
JEL Codes: C72, C92, D91, E71

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1   Introduction

Over the past decade, economists have begun investigating the effects of *cognitive imprecision* on individual decision-making (see Woodford 2020 for a review). This agenda proposes that the decision-maker's perception of the economic environment is noisy, and does not coincide with the objective environment. To date, the emphasis of cognitive imprecision in economics has largely been in the domains of choice under risk and intertemporal choice. For example, theory has shown that even when an agent has linear utility and perfectly patient time preferences, noisy perception of payoffs can lead agents to behave as if they are risk averse (Khaw, Li and Woodford, 2021) and as if they discount the future (Gabaix and Laibson, 2017). Initial experimental tests of these theories have produced encouraging results (Enke and Graeber, 2019; Gershman and Bhui, 2020; Khaw, Li and Woodford, 2021; Frydman and Jin, Forthcoming).

Motivated by the evidence from individual decision-making studies, it is natural to ask whether cognitive imprecision also affects strategic behavior. This question is important not only to test whether cognitive imprecision extends into other environments, but also because noisy perception can fundamentally affect equilibrium predictions. Indeed, it is well known that in games often used to model bank runs, currency attacks, and revolutions, there can be multiple equilibria.[1] This multiplicity largely comes from the assumption that players can perfectly perceive a fundamental parameter, which then serves as a coordination device. However, as pointed out by Woodford (2020), if one adds a small amount of cognitive imprecision to the game so that each player perceives a slightly different fundamental, then coordination becomes more difficult and multiplicity breaks down. In this manner, noisy perception can help *refine* equilibrium predictions. This line of reasoning follows directly from the vast literature on global games (Carlsson and Van Damme, 1993; Morris and Shin, 2003; Angeletos and Lian, 2016) with the important distinction that, here, we interpret the noise as arising from perceptual errors rather than from traditional sources of asymmetric information (e.g., different opportunities that agents have to acquire information about an uncertain state of the world).

In this paper, we theoretically develop and experimentally test the hypothesis that perceptual noise systematically affects strategic behavior. The hypothesis enables us to apply standard results from the global games literature without explicitly introducing private information. Specifically, we analyze a 2×2 simultaneous move game where players can choose to invest or not in an asset. Each player's payoff depends on the value of a fundamental

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[1]For examples of game theoretic models in these domains, see Diamond and Dybvig (1983), Obstfeld (1996), Morris and Shin (1998), Atkeson (2000), Goldstein and Pauzner (2005), and Edmond (2013). See also He et al. (2019) for a related model of debt rollover.

and on the action of the other player. While theory predicts multiple equilibria for a range of fundamental values in the complete information version of the game, a small amount of perceptual noise generates a unique equilibrium: each player invests if and only if their noisy perception of the fundamental crosses a threshold. This is the classic conclusion from the global games literature, under the view that the noise in observing fundamentals captures asymmetric information.[2]

Because we adopt the view that noise arises from perceptual error, we can leverage principles from psychology to generate even sharper equilibrium predictions. Specifically, we draw on the principle of *efficient coding*, which states that cognitive resources devoted to accurate perception of stimuli, will optimally shift with the distribution of stimuli. In our setting, efficient coding implies that perception of a given fundamental value will be noisier when fundamental volatility is larger. The intuition is that each player has a limited set of cognitive resources, and she optimally allocates these resources towards perceiving those fundamental values that she expects to occur more frequently.[3] Thus, when the distribution of fundamentals becomes more volatile, cognitive resources are dispersed more broadly, which leads to larger perceptual errors (for values near the center of the distribution). Efficient coding therefore predicts *context-dependent* behavior: each player invests if and only if their signal crosses a threshold, where signal noise is increasing in fundamental volatility.

To test this prediction, we conduct a pre-registered experiment in which subjects are randomly matched into pairs and play a simultaneous move game in each of three hundred rounds. The game is characterized by the value of a fundamental parameter, which is clearly displayed to both subjects on each round as a two-digit Arabic numeral, such as "45". We rely on subjects' inherent cognitive imprecision to transform this "public" signal into a private signal, owing to idiosyncratic perceptual errors. The perceptual error induces uncertainty over the fundamental. More importantly, perceptual error also induces *strategic uncertainty* over the other player's action, which is key to breaking the multiplicity of equilibria. Our key experimental treatment is to manipulate the volatility of the fundamental across a high volatility and a low volatility condition, in order to test for context-dependent behavior.

Our data strongly support the novel prediction that equilibrium outcomes are context-

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[2]Several previous experimental studies have found empirical support for the global games prediction that the probability of investing is monotonic in the fundamental (Heinemann, Nagel and Ockenfels, 2004, 2009; Cabrales, Nagel and Armenter, 2007; Avoyan, 2019; Szkup and Trevino, 2020; Goryunov and Rigos, 2020). As we describe further below, one feature that differentiates our study from the existing experimental literature on global games, is that we invoke the global games arguments without explicitly endowing subjects with private information

[3]Such an assumption has been validated in many papers on sensory perception (Girshick et al., 2011; Wei and Stocker, 2015; Payzan-LeNestour and Woodford, Forthcoming) and in economic decision making (Polania, Woodford and Ruff, 2019; Frydman and Jin, Forthcoming).

dependent. Specifically, we observe that the probability of investing is monotone in the fundamental, and we find that this monotonic relationship is significantly stronger in the low volatility condition than in the high volatility condition. In light of our model, we interpret the observed treatment effect as a consequence of more accurate perception of fundamentals in the low volatility condition. Further evidence comes from the distribution of response times, which indicates that subjects choose their action significantly faster when they are adapted to the low volatility distribution of fundamentals.

We emphasize that, in both experimental conditions, the strong monotonic relationship that we observe between fundamentals and investing is not predicted under the complete information version of the game. As such, our data suggest that even when subjects receive no explicit private signals from the experimenter, private information is inherent in the game because subjects encode the fundamental with idiosyncratic perceptual noise. Our framework therefore provides a new explanation for earlier experimental papers that find a high correlation between behavior and fundamentals, regardless of whether subjects received explicit private signals (Heinemann, Nagel and Ockenfels, 2004; Van Huyck, Viriyavipart and Brown, 2018).[4] Such a result may appear puzzling because, in the absence of private information, there are multiple equilibria and behavior should not vary smoothly with fundamentals. However, one can explain the result by taking a broader view of the potential sources of private information to also include perceptual errors.[5]

The data from our experiment can also separate between cognitive imprecision and several alternative models from game theory. Perhaps the closest model to cognitive imprecision is Quantal Response Equilibrium (QRE) (McKelvey and Palfrey, 1995, 1998). In QRE, each player stochastically best responds to their opponent. Cognitive imprecision and QRE share the prediction that in equilibrium, behavior is random (even when players are not indifferent between actions). However, a unique prediction generated by cognitive imprecision, that is not shared by QRE, is that behavior depends on the distribution from which the fundamental is drawn. In fact, this context dependence further separates cognitive imprecision from a broader class of theories including level-k thinking (Stahl and Wilson, 1994, 1995; Nagel,

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[4]Moreover, when discussing an experiment where there is no explicit private information about payoffs, Heinemann, Nagel and Ockenfels (2009) argue that "Of course, players know the true payoff. Their uncertainty about others' behavior makes them behave *as if* they are uncertain about payoffs" (p. 203). Our results indicate that it may well be the case that subjects do not know the true payoff. Heinemann, Nagel and Ockenfels (2009) also estimate the standard deviation of *as if* private signals in their experimental data and find that the model implies a fairly large standard deviation; one interpretation of their result is that a portion of the large estimate of noise in private signals is driven by errors in perceiving the fundamental.

[5]Van Huyck and Stahl (2018) conduct an experiment by simultaneously varying both the range and the mean of payoffs in a stag hunt game. However, because the fundamental in their experiment never takes on values in the "dominance regions", one cannot interpret the experimental results through the theory of global games.

1995). We discuss these differences across theories in more detail in Section 5.

In our model, perceptual error generates a unique equilibrium because it makes a player uncertain about their opponent's perception of the fundamental. Our theory therefore relies on the assumption that each player is aware of their opponent's imprecision. To investigate the validity of this assumption, we conduct a second experiment, where subjects are asked to classify whether a two-digit number is greater than a reference level of 55 (which is chosen to be the same as the threshold in the unique equilibrium of the game in our first experiment). We then incentivize subjects to report their beliefs about (i) the average accuracy of all other subjects in the experiment and (ii) their own accuracy

We find that subjects are aware of their own errors and, more importantly, they are aware of others' errors in the classification task. Subjects also report beliefs that discriminating between a number close to the threshold, say "54", is harder than discriminating between a number far from the threshold, say "47." This property has been shown theoretically to have important implications for equilibrium selection (Morris and Yang, 2021), and has recently been formalized in the rational inattention literature using a so-called "neighborhood cost function" (Hébert and Woodford, Forthcoming). The data from our second experiment therefore provide novel evidence supporting the assumption that subjects are aware of others' errors and that nearby states are harder to distinguish compared to far-away states.

In both of our experiments, the fundamental is represented by a two-digit Arabic numeral that is clearly displayed to each subject. Thus, any cognitive imprecision that we observe in our experiments is likely to be a lower bound on the imprecision in perceiving more complex, but publicly available, stimuli in the field (e.g., market prices). This is important because previous theoretical work has shown that the amount of noise in public signals, relative to that in private signals, determines whether there exists a unique equilibrium (Morris and Shin, 2003; Angeletos and Werning, 2006; Hellwig, Mukherji and Tsyvinski, 2006).

While a major goal of recent work has been to test whether principles of cognitive imprecision are active in individual economic decision-making (Enke and Graeber, 2019; Gershman and Bhui, 2020; Khaw, Li and Woodford, 2021; Frydman and Jin, Forthcoming; Polania, Woodford and Ruff, 2019), there has yet to be a similar set of tests in strategic settings. Our results provide clear evidence that cognitive imprecision does indeed extend into strategic settings, and that subjects are aware of this feature of their own and others' perception. By testing whether similar cognitive mechanisms apply in individual and strategic decision-making, our work is related to recent brain imaging evidence from Nagel, Brovelli, Heinemann and Coricelli (2018) who show that common neural circuits are activated during lottery choice and games with strategic uncertainty.

The remainder of the paper proceeds as follows: Section 2 presents the model and derives

the theoretical predictions for our experimental manipulation. Sections 3 and 4 describe the experimental design and report the experimental results for Experiment 1 (the simultaneous move game) and Experiment 2 (the number classification task), respectively. Section 5 discusses some assumptions of our theoretical framework and its connection with existing behavioral game theory models. Section 6 concludes.

2   Model

In this section, we present a model in which players imprecisely perceive their strategic environment. We encode this imprecision in the model by assuming that each player forms a noisy perception of a "fundamental" payoff in the game. As a consequence of noisy perception, each player's perception of the game does not necessarily coincide with the objective game. In particular, a player's perception of the fundamental payoff will, in general, differ from the true fundamental. We illustrate the strategic implications of noisy perception in the setting of a 2×2 simultaneous move game. We focus our analysis of the game on parameters that capture the essential features of a coordination game. Consider the following game, where $b > a$:

	Not Invest	Invest
Not Invest	θ, θ	θ, a
Invest	a, θ	b, b

Table 1: The Game

In what follows, we always assume that a and b are perceived precisely (i.e., without any noise) by both players, and we are interested in the effect of imprecise perception of θ.[6] As a benchmark, we first consider the predictions of a model in which θ is perceived precisely, and then we relax this assumption to investigate the implications of cognitive imprecision.

2.1   Benchmark: Precise Perception

When both players perceive θ precisely, the game is one of complete information and its Nash equilibria depend on the true value of θ, as outlined below:

- If $\theta > b$, then Invest is a strictly dominated action for each player, and (Not Invest, Not Invest) is the unique Nash (and dominant strategy) equilibrium.

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[6]Our assumption that a and b are perceived without noise can be justified, for example, through a learning mechanism. In our experiment, we keep a and b constant across all rounds, so the amount of noise in perceiving a and b is arguably minimal.

- If $\theta < a$, then Not Invest is a strictly dominated action for each player, and (Invest, Invest) is the unique Nash (and dominant strategy) equilibrium.

- If $a \leq \theta \leq b$, then there are two Nash equilibria in pure strategies: (Not Invest, Not Invest) and (Invest, Invest). There also exists one Nash equilibrium in mixed strategies.

Thus, when θ takes on values in the intermediate range $[a, b]$, there are multiple pure strategy Nash equilibria. This prediction relies on the assumption that each player precisely perceives θ. Precise perception generates common knowledge about θ, which enables coordination and gives rise to multiple self-fulfilling equilibria. The predictions change dramatically, however, when we relax the assumption that players can perceive θ precisely.

2.2 Cognitive Imprecision

Suppose now that players perceive θ with noise. This assumption is backed up by a large literature in numerical cognition, which finds that people encode numerical quantities with noise, even when the quantities are presented symbolically (see Dehaene 2011 for a review). To model the imprecision, we assume that all players have a common prior that θ is distributed normally: $\theta \sim \mathcal{N}\left(\mu_\theta, \sigma_\theta^2\right)$. Our key assumption is that, instead of precisely observing the realized value of θ, each player only has access to a noisy internal representation of θ.

Assumption 1 (Cognitive Imprecision) *Each player i, $i = \{1, 2\}$, observes a noisy internal representation of θ, $S_i = m(\theta) + \epsilon_i$, where each ϵ_i is independently and normally distributed: $\epsilon_i \sim \mathcal{N}\left(0, \sigma_S^2\right)$, with $\sigma_S^2 > 0$.*

Assumption 1 says that each player's internal representation is conditionally independent, and depends on θ through an encoding function, $m(\theta)$. We follow Khaw, Li and Woodford (2021), and assume a linear encoding function with a power constraint:[7]

Assumption 2 (Encoding Function) *The encoding function is linear: $m(\theta) = \xi + \psi\theta$. In addition, there is a power constraint, $E[m^2] \leq \Omega^2 < \infty$.*

The parameters of the encoding function, (ξ, ψ), are allowed to vary with the player's environment – which we characterize by the prior distribution of θ. Thus, the conditional distribution of noisy signals can vary across environments. This assumption is built on substantial empirical evidence, mainly from the literature on sensory perception, which demonstrates that the distribution of noisy internal representations is *optimally* adapted to the statistical

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[7]Khaw, Li and Woodford (2021) assume a slightly different specification of the encoding function, which is linear in the logarithm of the stimulus value. See their Appendix C for details.

regularities of the environment. This principle is called *efficient coding*, and recent work has empirically documented effects of efficient coding in economic choices (Polania, Woodford and Ruff, 2019; Frydman and Jin, Forthcoming). To close the efficient coding model, we need to specify the performance objective which drives the player's optimal choice of the encoding function parameters.

Assumption 3 (Performance Objective) *The player chooses the encoding function which minimizes the mean squared error between θ and its conditional mean, $E[\theta|s_i]$.*

The previous three assumptions describe the perceptual constraints and the objective function of each player. Given these constraints and objectives, we can derive the efficient encoding function that each player optimally chooses.[8]

Proposition 1 (Efficient Coding) *Given Assumptions 1-3, the optimal encoding function is $m^\star(\theta) = \xi^\star + \psi^\star\theta = -\frac{\Omega\mu_\theta}{\sigma_\theta} + \frac{\Omega}{\sigma_\theta}\theta$. Consider the transformed internal representation, $Z_i \equiv (S_i - \xi^\star)/\psi^\star$. The conditional distribution of this transformed internal representation is $N(\theta, \omega\sigma_\theta^2)$, where $\omega = \sigma_S^2/\Omega^2$. The variance of the transformed internal representation is proportional to the variance of θ.*

At an intuitive level, efficient coding implies that perceptual resources are allocated so as to better discriminate between different values of θ that are expected to occur more frequently under the player's prior beliefs. Specifically, as the volatility of the prior decreases, perceptual resources are reallocated towards the center of the distribution.

Given the optimal encoding function in Proposition 1, we can now solve for equilibria of the game. We restrict our analyses to monotone equilibria of the incomplete information game, that is, equilibria in which actions are monotonic in the transformed internal representation, Z_i. In a *monotone equilibrium*, players' mutual best response is to choose Invest if and only if their transformed internal representation is below a threshold $k^\star$. Adapting results from the global games literature (Carlsson and Van Damme, 1993; Morris and Shin, 2003; Morris, 2010) to the game in Table 1, with the further assumption that $\mu_\theta = (a+b)/2$ (as in our experiment), we can establish there exists a monotone equilibrium such that player i invests if and only if $Z_i \leq \mu_\theta$, for any value of σ_θ, σ_S and Ω. Furthermore, if the noise in the transformed internal representation is sufficiently small, then this is the unique monotone equilibrium.

Proposition 2 (Equilibrium Existence and Uniqueness) *Suppose Assumptions 1-3 and $\mu_\theta = (a+b)/2$. There exists an equilibrium of the game where each player invests if and*

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[8]In Section 5, we discuss the robustness of our main theoretical results to different assumptions about the player's performance objective.
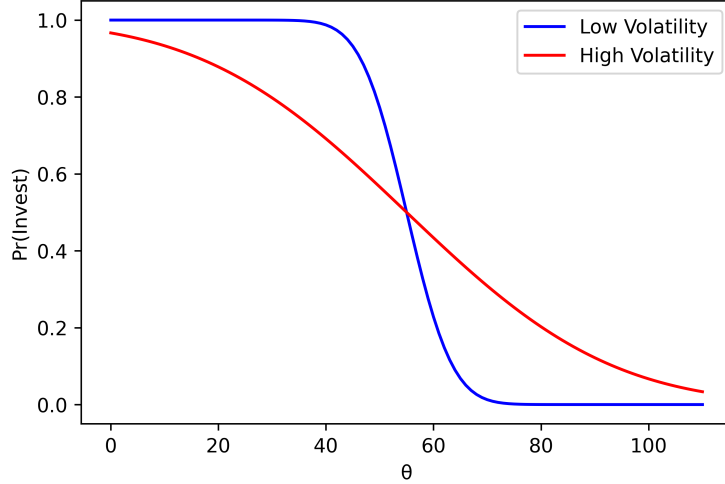
Figure 1: Predicted Probability of Investing as a Function of θ. Note: The blue line denotes the prediction for a low volatility distribution with $\theta \sim N(55, 20)$; the red line denotes the prediction for a high volatility distribution with $\theta \sim N(55, 400)$; we set the following parameter values: $a = 47$, $b = 63$, and $\omega = 1.5$.

only if $Z_i \leq \mu_\theta$ (or, equivalently, $E[\theta|Z_i] \leq \mu_\theta$). Moreover, if $\sqrt{\omega(1+\omega)} < \frac{\sqrt{6\pi}}{(b-a)}\sigma_\theta^2$, this is the unique monotone equilibrium of the game.

In deriving Proposition 2, we assume common knowledge of the distribution of internal representations. However, precise knowledge of the underlying information structure is not necessary for this equilibrium to arise. As we show in Appendix B, in a simpler model where $m(\theta) = \theta$, it is enough to assume that (i) $\mu_\theta = (a+b)/2$, (ii) $E[\epsilon_i] = 0$, (iii) the distribution of ϵ_i is symmetric, quasiconcave and independent of the realized value of θ, and (iv) the distribution of θ is symmetric and continuous on $\mathbb{R}$.

Proposition 2 implies a particular set of comparative statics with respect to θ. The probability of investing is pinned down by the distribution of the transformed internal representation: $Pr[\text{Invest}|\theta] = Pr[Z_i \leq \mu_\theta|\theta] = \Phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)$, where $\Phi(\cdot)$ is the cumulative density function of the standard normal. This result indicates that, in the unique monotone equilibrium, the probability of investing is monotonically decreasing in θ. Moreover, the probability of investing depends not only on θ but also on the prior distribution from which θ is drawn: σ_θ modulates the optimal encoding rule and, thus, the noise in implementing the unique monotone equilibrium strategy of the game. Thus, if we experimentally manipulate the volatility of the prior, we should see that, in the unique equilibrium of the game obtained when ω is sufficiently small, the probability of investing is more sensitive to θ when the prior volatility is smaller. This prediction is summarized in the following proposition.

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Proposition 3 (Comparative Statics) *Suppose Assumptions 1-3, $\mu_\theta = (a + b)/2$, and $\sqrt{\omega(1 + \omega)} < \frac{\sqrt{6\pi}}{(b-a)}\sigma_\theta^2$. In the unique monotone equilibrium of the game, the probability that each player invests for a given value of θ is $Pr[Invest|\theta] = \Phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)$. This probability is decreasing in θ. Moreover, increasing the precision of the distribution of θ (that is, $1/\sigma_\theta$) increases the sensitivity of choices to θ (that is, the rate at which $Pr[Invest|\theta]$ decreases with θ) for values of θ close to μ_θ.*

We illustrate the implications of Proposition 3 in Figure 1. The figure shows that the prior variance of θ strongly affects choice sensitiivty to θ. This stark difference in equilibrium behavior for different distributions of θ motivates our experimental design.

3 Experiment 1: Simultaneous-Move Game

3.1 Experimental Design

In this experiment, we test the model by incentivizing subjects to play a simultaneous-move game, and we manipulate the distribution that generates the fundamental payoff, θ. We pre-register the experiment and recruit 300 subjects from the online data collection platform, Prolific.[9] We restrict our sample to subjects who, at the time of data collection, (i) were UK nationals and residents, (ii) did not have any previous "rejected" submissions on Prolific, and (iii) answered all comprehension quiz question correctly. Subjects are paid 2 GBP ($\sim$ 2.8 USD) for completing the experiment, and they have the opportunity to receive additional earnings based on their choices and the choices of other participants.

The experiment consists of 300 rounds, and each subject participates in all rounds. In each round, a subject is randomly matched with another subject and, together, they play the simultaneous-move game in Table 1. In all rounds, we set the payoff parameters $a = 47$ and $b = 63$. The only feature of the game that varies across rounds is the value θ, which is drawn from the condition-specific distribution $f(\theta)$. In each round, both subjects observe the same realization of θ. In order to shut down learning about other participants' behavior, we choose not to provide subjects with feedback about their payoff or their opponent's choice in a given round. At the end of the experiment, one round is selected at random, and subjects are paid according to the number of points they earned in that round, which in turn, depends on their action, their opponent's action, and the (round-specific) value of θ. Points are converted to GBPs using the rate 20:1. The average duration of the experiment was $\sim$ 25 minutes and average earnings, including the participation fee, were $\sim$ 5.5 GBP ($\sim$ 7.7 USD).

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[9]The pre-registration document is available in Appendix D.

Subjects are randomized into one of two experimental conditions: a *high volatility* condition or a *low volatility* condition, which differ only based on the distribution of θ. In the high volatility condition, $f(\theta)$ is normally distributed with mean 55 and variance 400. In the low volatility condition, $f(\theta)$ is normally distributed with mean 55 and variance 20. In both conditions, after drawing θ from its respective distribution, we round θ to the nearest integer, and we re-draw θ if the rounded value is less than 11 or greater than 99. We implement these modifications to the normal distribution to control complexity and ensure that θ is a two-digit number in each round. We do not give subjects any explicit information about $f(\theta)$ in the instructions, as our intention is to test whether a subject's perceptual system can adapt to the statistical properties of the environment without explicit top-down information. Moreover, we believe that such a design is more natural than explicitly telling subjects the distribution of parameters they will experience, as this could artificially direct their attention to the distribution (a similar design feature is used in Frydman and Jin (Forthcoming)). Each condition contains an identical set of instructions and comprehension quiz.[10]

Recall that, in the complete information version of the game that subjects face, there are multiple equilibria when $\theta \in$ is in the range, [47, 63]. We therefore focus our analyses on games for which θ lies in this range, which occurs on 93% of rounds in the low volatility condition and on 31% of rounds in the high volatility condition. We pre-register that our main analyses are restricted to those rounds for which $\theta \in [47, 63]$, which we call "common rounds." This is a crucial feature of our design, because it allows us to compare behavior across conditions, using the exact same set of games, and varying only the context — which is characterized by the distribution of games. In choosing the variance of $f(\theta)$ for each condition, we thus strike a balance between generating a substantial number of common rounds to analyze, and creating a large difference in prior variance across conditions; the latter is necessary to generate different predictions across conditions. As outlined in our pre-registration, we also exclude the first 30 rounds from our analyses, in order to allow subjects time to adapt to the distribution of θ.

Figure 2 provides a screenshot of a single round shown to subjects. In order to avoid framing effects, we label the two options "Option A" and "Option B", and the left-right location of each option is randomized across rounds. The number "45" is the realized value of θ on the specific round shown in Figure 2. We emphasize that while the number is clearly displayed to all subjects, and thus would traditionally be interpreted as public information, here we rely on cognitive imprecision to transform the fundamental value into private information. In other words, we assume that the constraints on a subject's perceptual system make it impossible to perfectly perceive the fundamental value (Assumption 1). Further-

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[10]The experimental instructions are available in Appendix E.

Option A	Option B
45	**47** if other participant chooses A **63** if other participant chooses B

Figure 2: Sample Screenshot Shown to Participants in Experiment 1. Note: In this round, the realized value of θ is 45, which is clearly and explicitly displayed to both subjects. Subjects choose either "Option A" or "Option B" by pressing one of two different keys on the keyboard.

more, efficient coding implies that the amount of imprecision varies endogenously with the distribution of θ across conditions.

3.2 Experimental Results

Choice Behavior

Following our pre-registration, we restrict our analysis to common rounds in which subjects execute a decision with a response time greater than 0.5 seconds. After applying this restriction, we are left with 50,129 decisions, of which 73.0% are in the low volatility condition and 27.0% are in the high volatility condition. Across both conditions, subjects choose to invest on 58.9% of rounds, and exhibit an average response time of 1.64 seconds.

In Figure 3, we plot the probability of investing as a function of the fundamental, separately for the two experimental conditions. One can see that, in both conditions, the aggregate data are consistent with the hypothesis that subjects implement strategies that are monotone in θ. This finding is in line with previous experimental results on coordination games (Heinemann, Nagel and Ockenfels, 2004, 2009; Szkup and Trevino, 2020). Importantly, we observe a smooth decreasing relationship between θ and the probability of investing, even without explicitly endowing subjects with any private signals about θ. We interpret this result as subjects implementing a noisy threshold strategy in both experimental conditions, which is consistent with the comparative static between θ and the probability of investing in Proposition 3.

In order to provide a more targeted test of cognitive imprecision, we exploit the variation

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Dependent Variable: Pr(Invest)	(1)	(2)	(3)	(4)
$(\theta - 55)$	-0.458***	-0.467***	-0.577***	-0.481***
	(0.033)	(0.039)	(0.051)	(0.037)
$(\theta - 55)$ x Low	-0.499***	-0.351***	-0.487***	-0.374***
	(0.063)	(0.059)	(0.076)	(0.061)
Low	-0.182	-0.335	-0.170	-0.275
	(0.386)	(0.343)	(0.423)	(0.360)
Late				-0.022
				(0.121)
$(\theta - 55)$ x Late				0.013
				(0.021)
Low x Late				0.092
				(0.154)
Low x $(\theta - 55)$ x Late				-0.065*
				(0.034)
Constant	1.351***	1.316***	1.465***	1.292***
	(0.221)	(0.224)	(0.222)	(0.229)
Observations	50,129	13,196	12,861	25,864
Rounds	31-300	31-100	231-300	(31-100)
				& (231-300)

Table 2: Mixed Effects Logistic Regression Results. Note: Observations are at the subject-round level. The dependent variable takes value 1 if the subject chooses to Invest, and 0 otherwise. The variable *Low* takes the value 1 if the round belongs to the low volatility condition and 0 otherwise. The variable *Late* takes the value 1 if the round number is 231 or greater, and 0 otherwise. Only data from rounds where $46 < \theta < 64$ are included in the regressions. There are random effects on $(\theta - 55)$ and the intercept. Standard errors of the fixed effect estimates are clustered at the subject level. ***, **, * denote statistical significance at the 1%, 5%, and 10% levels, respectively.
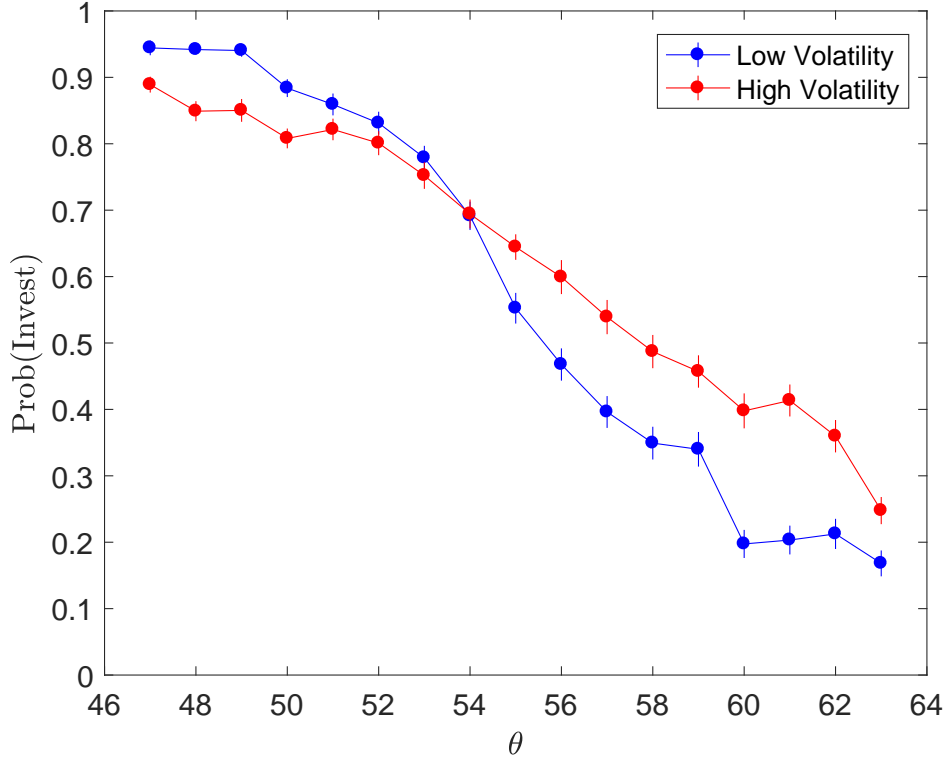
Figure 3: Observed Probability of Investing as a Function θ. Note: For each value of θ between 47 and 63, we plot the proportion of rounds in which a subject chooses to invest, separately for each of the two experimental conditions. Data are pooled across subjects and are shown for rounds 31-300, after an initial 30-round adaptation period. Vertical bars inside each data point denote two standard errors of the mean. Standard errors are clustered by subject.

in the distribution of θ across our two experimental conditions. Efficient coding predicts context-dependent behavior, where the threshold strategy from the unique monotone equilibrium is implemented with more precision in the low volatility condition. Figure 3 provides evidence consistent with this prediction: the frequency of investing is more sensitive to the fundamental in the low volatility condition, compared to the high volatility condition.

To formally test the difference in slope, we estimate a series of mixed effects logistic regressions. Column (1) of Table 2 confirms our main result: the coefficient on the interaction term $(\theta - 55)$ x *Low* is significantly negative, indicating that the probability of investing decreases in the fundamental more rapidly when a subject is adapted to the low volatility condition. Columns (2) and (3) show that this result holds in both early (first 70 trials after adaptation) and late (last 70 rounds of the session) subsamples. Column (4) indicates that the treatment effect becomes moderately stronger over the course of the experiment.

The strengthening of the treatment effect over the course of the experiment suggests that subjects have not fully adapted to the distribution by round 100, and additional rounds of play provide the opportunity for further adaptation.

Dependent Variable: Pr(Invest)	(1)	(2)	(3)	(4)
$(\theta - 55)$	-0.384***	-0.389***	-0.364***	-0.447***
	(0.039)	(0.041)	(0.036)	(0.047)
$(\theta - 55)$ x Low	-0.266***	-0.275***	-0.285***	-0.299***
	(0.051)	(0.054)	(0.052)	(0.063)
Low	-0.317	-0.356	-0.129	-0.207
	(0.276)	(0.285)	(0.283)	(0.317)
Constant	1.067***	1.202***	0.993***	1.174***
	(0.181)	(0.199)	(0.192)	(0.217)
Observations	4,263	4,053	3,677	3,255
Rounds of Experience with Game (θ)	3	4	5	6

Table 3: Mixed Effects Logistic Regressions Estimates. Note: The dependent variable takes value 1 if the subject chooses to Invest, and 0 otherwise. The variable *Low* takes value 1 if the round belongs to the low volatility condition and 0 otherwise. Only data from rounds where $46 < \theta < 64$ are included. There are random effects on $(\theta - 55)$ and the intercept. Standard errors of the fixed effect estimates are clustered at the subject level. ***, **, * denote statistical significance at the 1%, 5%, and 10% levels, respectively.

While subjects do not receive feedback after each round, it is still possible that they learn about the strategic environment through repeated exposure to the game, as in Weber (2003) and Rick and Weber (2010). Note that our experimental manipulation implies that subjects in different conditions will experience the same game, characterized by θ, a different number of times (because we vary the size of the set we draw θ from). This raises the potential concern that our observed treatment effect is due to the differential ability to learn (rather than to cognitive imprecision). To investigate this alternative explanation, Table 3 presents results when we restrict to subsamples where subjects have identical experience with a given game in both conditions. In particular, the first column restricts to those rounds on which subjects in both conditions have previously observed 3 games with the same value of θ as in the current round. Columns (2) - (4) further restrict the data based on more and more experience with a given game. The regression results indicate that our main treatment effect obtains among each of the different subsamples. These results indicate that learning cannot be driving the entire treatment effect we observe.

It is important to point out that the results in Figure 3 are aggregated across subjects. Thus, while the data are consistent the prediction that, at the individual subject level, sig-

nals are drawn from a noisier distribution in the high volatility condition, there is another potential explanation based on aggregation. Specifically, suppose that subjects perceive θ perfectly, that they use a (potentially non-equilibrium) cutoff strategy, and that there is heterogeneity with respect to the cutoff they adopt. If some subjects use low cutoffs, while others use high cutoffs, then this heterogeneity would give rise to the decreasing relationship observed in both aggregate curves in Figure 3. Furthermore, if the variance in cutoff strategies across subjects is larger in the high volatility condition, then this alternative hypothesis could explain the weaker relationship between θ and the probability of investing in the high volatility condition. To test this alternative hypothesis, based on heterogeneity of cutoff strategies, we can structurally estimate the model to obtain subject-specific cutoffs and measures of perceptual noise.

Structural Estimation

According to the model described in Section 2, subject i chooses the parameters of the encoding rule, $m_i(\theta) = \xi_i + \psi_i \theta$. She then observes a noisy internal representation, $S_i = m_i(\theta) + \epsilon_i$. If we define a transformed version of the noisy internal representation, as $Z_i = (S_i - \xi_i)/\psi_i$, then for a cutoff $Z_i^\star$, our model predicts that she invests if and only if $Z_i \leq Z_i^\star$. The model in Section 2 shows that, in the unique monotone equilibrium of the game with cognitive imprecision, all subjects in the same treatment choose the same $(\xi_i, \psi_i, Z_i^\star)$. Here, we allow subjects to make heterogeneous (non-equilibrium) choices and we structurally estimate these parameters using behavior observed in the experiment.

Consider subject i who adopts a cutoff value of $Z_i^\star$ and, in round t, receives a noisy internal representation $S_{it} = \xi_i + \psi_i \theta_t + \epsilon_{it}$. The probability that subject i invests in round t is the probability that her transformed noisy internal representation is below her cutoff:

$$\mathbb{P}(\text{Invest}|\theta_t, \sigma_S, \psi_i, Z_i^\star) = \Phi\left(\frac{Z_i^\star - \theta_t}{\sigma_S/\psi_i}\right) \tag{1}$$

(Note that θ_t is not observable to the subject, but the transformed noisy internal representation is normally distributed with mean θ_t and standard deviation σ_S/ψ_i).

We structurally estimate the model using maximum likelihood estimation. In particular, for each subject, we estimate the standard deviation of the transformed noisy internal representation, $\sigma_i = \sigma_S/\psi_i$, and the cutoff $Z_i^\star$.[11] We maximize the following log-likelihood

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[11]Note that, with our data, we cannot separately identify σ_S and ψ_i (since these two parameters are perfect substitutes in the conditional density of Z_i). At the same time, while ψ_i is an endogenous variable, we interpret σ_S as an exogenous parameter, capturing the degree of a subject's perceptual acuity. In Section 2, we assumed σ_S is homogeneous across subjects. Even if we allowed for heterogeneity in this exogenous parameter across subjects, the randomization into experimental conditions would guarantee a

function over $(\sigma_i, Z_i^\star)$, using behavior in rounds 31 - 300:

$$LL\left(\sigma_i, Z_i^\star, \mathbf{y}_i\right) = \sum_{t=31}^{300} y_{it} \cdot \log\left(\mathbb{P}(\text{Invest}|\theta_t, \sigma_i, Z_i^\star)\right) + (1 - y_{it}) \cdot \log(1 - \mathbb{P}(\text{Invest}|\theta_t, \sigma_i, Z_i^\star)),$$
(2)

where $\mathbf{y}_i \equiv \{y_{it}\}_{t=31}^{300}$ and y_{it} denotes the subject's choice in round t, with $y_{it} = 1$ if the subject chooses to invest and $y_{it} = 0$ if the subject chooses not to invest. We maximize the log-likelihood function in (2) by searching over grid values of $[\sigma_i, Z_i^\star] \in [0.1, 50.1] \times [11, 99]$, in increments of 0.5 along each dimension.

Figure 4 plots the distribution of estimated parameters for the 300 subjects (150 in each condition). Beginning with the upper panel, we see that, for most subjects, the estimated cutoff lies between 50 and 60. The mean cutoff in the high volatility condition is 58.5 and the mean cutoff in the low volatility condition is 57.2. These means are not significantly different from one another ($p = 0.15$). The average cutoff in each condition is, however, significantly greater than 55. As can be seen from the figure, this difference relative to 55 is driven mainly by the right tail of the distribution, which captures a small fraction of subjects who almost always choose to invest.

More importantly, we find that the standard deviation of estimated cutoffs is not significantly different across conditions (8.4 in high volatility vs. 7.5 in low volatility, $p = 0.43$ Levene's test). This suggests that heterogeneity in cutoffs is *not* driving our main result. If it were, we would have observed a more concentrated distribution of cutoffs in the low volatility condition and, thus, a significantly lower standard deviation of estimated cutoffs in the low volatility condition.

Instead, the lower panel of Figure 4 reveals that the difference in behavior across conditions stems from the standard deviation of the noisy internal representations. The mean estimated value of σ_i is significantly higher in the high volatility condition (14.4 vs. 5.9, $p < 0.001$). One can easily see from the figure that this effect holds not only on average, but across the whole distribution. In summary, while the aggregate data in Figure 3 are consistent with subjects in the high volatility condition exhibiting (i) a wider dispersion of cutoffs or (ii) a higher amount of noise in the internal representation of the fundamental, our structural estimation indicates that the effect is coming only through the second channel, as predicted by the theory developed in Section 2.

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similar distribution of σ_S in the two sub-populations. For this reason, in what follows, we attribute any difference in the distribution of the estimated σ_i's across conditions to differences in its endogenous element, ψ_i.
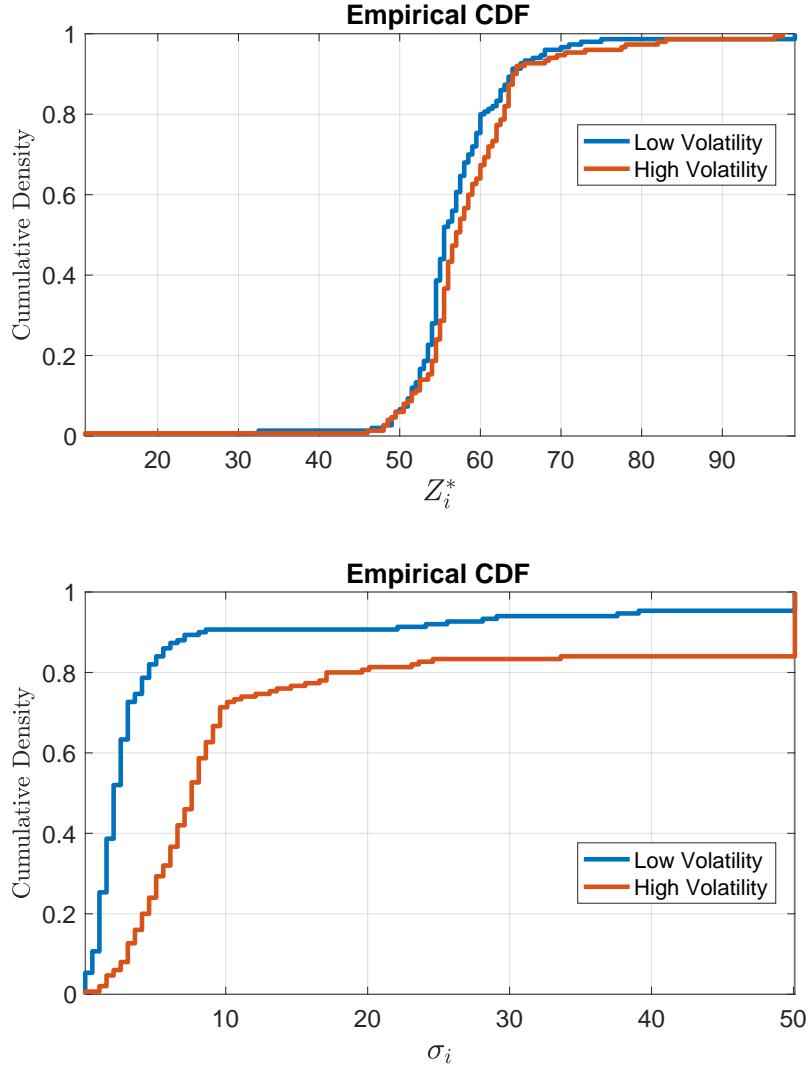
Figure 4: Empirical CDFs of Subject-Level Structural Estimates. Note: Upper panel is empirical CDF of estimated cutoffs. Lower panel is empirical CDF of estimated standard deviations of noisy internal representations.

Response Times

Here we analyze the distribution of response times in both conditions. As outlined in our pre-registration, we hypothesize that if subjects are implementing cutoff strategies, then response times should peak at the cutoff value. Figure 5 shows that response times are longer in the high volatility condition. Moreover, in the high volatility condition, the peak response time is at 55, whereas in the low volatility condition, the peak is at 54. If subjects
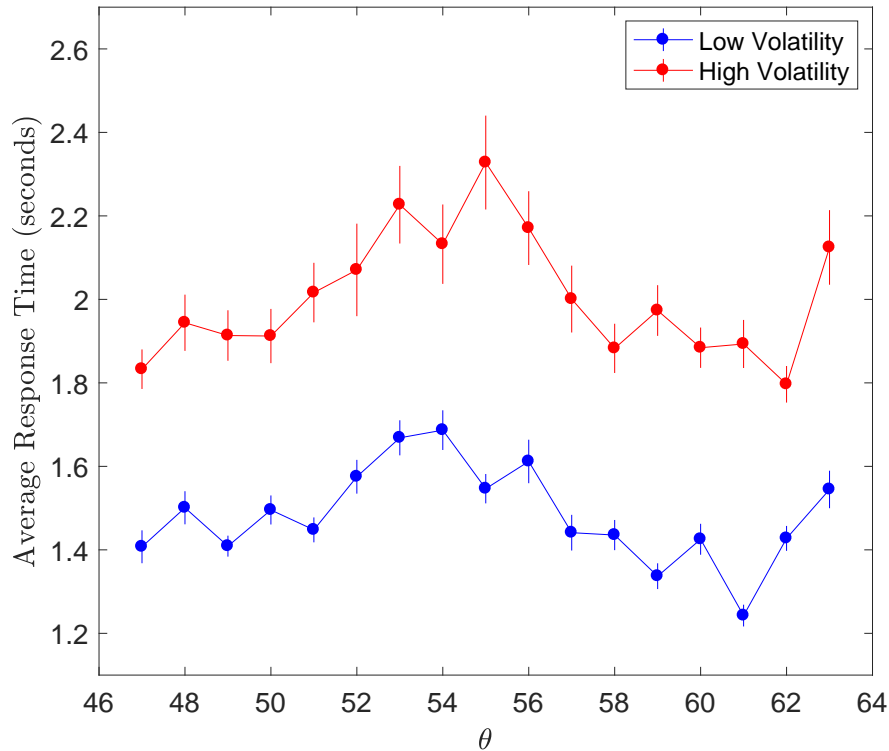
Figure 5: Average Response Time as a Function of θ in Experiment 1. Note: Response times are averaged across subjects and across rounds. Vertical bars denote two standard errors of the mean. Standard errors are clustered by subject.

are implementing the unique equilibrium threshold strategy, which involves discriminating whether the fundamental is above or below 55, then models of sequential sampling from the mathematical psychology literature (Ratcliff, 1978; Krajbich, Armel and Rangel, 2010) would predict that response times should peak at the predicted threshold of 55, since these are the most "difficult" discrimination problems. The response time data provide some support for this prediction.

4   Experiment 2: Number Classification Task

4.1   Experimental Design

Because we rely on cognitive imprecision to endogenously transform the complete information game into an incomplete information game, our theoretical analysis also assumes common knowledge of cognitive imprecision. While we cannot directly test the common knowledge assumption, here we report results from a second experiment that is designed to investigate

whether subjects are aware of their own imprecision and the imprecision of others. If subjects are not aware of the cognitive imprecision of others, then this would shut down the channel that generates strategic uncertainty in our model, which is key to generating the unique equilibrium.

Our method for studying awareness of imprecision is to create a simplified version of the previous experiment, but one that retains the core individual decision-making prediction that subjects play a threshold strategy. Specifically, we employ a task from the numerical cognition literature where subjects are incentivized to quickly and accurately classify whether a two-digit number is larger or smaller than the number 55. Note that this strategy is identical to the theoretical prediction in the unique equilibrium from the previous experiment; here we exogenously impose the strategy on subjects without any strategic considerations or equilibrium requirements. We then incentivize subjects to report beliefs about errors in their own classification and in the classification of others.

For this second experiment, we recruit 300 subjects from Prolific who did not participate in Experiment 1. We pay subjects 1 GBP for completing the study, in addition to earnings from three phases of the experiment. In Phase 1, on each of 150 rounds, subjects are incentivized to quickly and accurately classify whether a two-digit Arabic numeral on the experimental display screen is larger or smaller than 55. In particular, subjects earn $(1.5 \times$ accuracy $- 1 \times$ speed) GBPs, where 'accuracy' is the percentage of trials where the subject classifies the number correctly, and 'speed' is the average response time in seconds. As in Experiment 1, there are two conditions, and the only difference across conditions is the distribution from which the two-digit Arabic numeral (which we again denote by θ) is drawn. Furthermore, we use the same two distributions as in Experiment 1: in the high volatility condition θ is normally distributed with mean 55 and variance 400; in the low volatility condition, θ is normally distributed with mean 55 and variance 20. We also round each value of θ to the nearest integer and re-draw if the rounded integer is less than 11 or greater than 99 (again, to ensure that each number contains exactly two digits).

It is worth noting that one difference in incentives between Experiment 1 and Experiment 2 involves speed. Here, in Experiment 2, we additionally penalize subjects based on the time it takes them to respond.[12] The reason we impose the speed incentive in Experiment 2 comes from the well known "speed-accuracy tradeoff" in perceptual decision-making: one can obtain higher accuracy in classification as decision speed slows down. Thus, in order to increase statistical power to detect how accuracy differs for values of θ close and far from the threshold, we jointly reward speed and accuracy.

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[12]See Frydman and Jin (Forthcoming) who use a similar inventive structure in a number classification task.

In Phase 2 of the experiment, subjects are incentivized to report beliefs about others' performance in the task. Not only are we interested in eliciting awareness of imprecision, we also collect data to test whether subjects believe that others are more imprecise when the number on screen is closer to the reference level of 55. We implement this latter test because the specific noise structure in perception plays an important role in generating the unique equilibrium we explored in the previous experiment; in particular, recent theoretical work has shown that an important feature of the noise structure in generating a unique equilibrium is that nearby states are harder to discriminate compared to far away states (Morris and Yang, 2021; Hébert and Woodford, Forthcoming). To investigate this property of the noise structure, we ask subjects to consider the 149 other participants in their experimental condition of the study, who also just completed Phase 1. We then ask subjects the following two questions:

1. Consider only trials where the number on screen was equal to 47. In what percentage of these trials do you think the other participants gave a correct answer, that is, they correctly classified whether the number was smaller or larger than 55?

2. Consider only trials where the number on screen was equal to 54. In what percentage of these trials do you think the other participants gave a correct answer, that is, they correctly classified whether the number was smaller or larger than 55?

For each of the two questions, we pay the subject 0.5 GBP if their forecast is within 1% of the true percentage.[13] Question 1 elicits beliefs about others' imprecision when the distance between the number is far from the threshold (47 vs. 55), whereas Question 2 elicits beliefs about others' imprecision when the distance is close (54 vs. 55). While we could have asked subjects about their beliefs about others' imprecision for a range of numbers — rather than the single numbers 47 and 55 — this would have introduced a confound, since the distribution of numbers is different across conditions.

In Phase 3, we ask subjects about their own performance on the number classification task (that they completed in Phase 1). This question is not trivial because we do not provide subjects with feedback after any round in Phase 1 (nor after the end of Phase 1). Here, we are also interested in subjects' awareness of their own imprecision for numbers that are close and far from the threshold. Specifically, we ask subjects the following two questions:

1. Consider only trials where the number on screen was between 52 and 58. In what

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[13]Following Hartzmark, Hirshman and Imas (Forthcoming), we chose to use this elicitation procedure as opposed to more complex mechanisms such as the Binarized Schoring Rule (BSR) due to recent evidence showing that the BSR can systematically bias truthful reporting (Danz, Vesterlund and Wilson, 2020).

percentage of these trials do you think you correctly classified whether the number was smaller or larger than 55?

2. Consider only trials where the number on screen was less than 52 or greater than 58. In what percentage of these trials do you think you correctly classified whether the number was smaller or larger than 55?

For each of these two questions, we again reward subjects with 0.50 GBP if they provide an answer that is within 1% of their true accuracy. All subjects first go through Phase 1, and the order of Phase 2 and Phase 3 is randomized across subjects. We note that one potential concern with our design, is that when asking subjects about their performance in Phase 1, we are testing memory, not *ex-ante* beliefs. This is a reasonable concern, and an alternative is to have subjects forecast their performance before undertaking the classification task. However, under this alternative design, subjects' classification performance would be endogenous to their beliefs, and would invalidate the incentive compatibility of our belief elicitation procedure. For this reason, we opt to implement Phase 1 first for all subjects.

4.2   Experimental Results

We first report results from subjects' behavior in Phase 1, which are shown in Figure 6. In the left panel, we replicate the classic results from previous experiments on number discrimination, whereby subjects exhibit errors, and these errors increase as the number on screen approaches the threshold (Dehaene, Dupoux and Mehler, 1990). Moreover, we see that, for numbers between 47 and 63, errors are systematically higher in the high volatility condition (Frydman and Jin, Forthcoming). Similar patterns are reflected in the response times shown in the right panel of Figure 6: response times increase as the number approaches the threshold of 55, and response times are systematically longer in the high volatility condition.

The purpose of Phase 1 is to create a dataset about performance, over which we can ask subjects about their beliefs in Phases 2 and 3. In the left panel of Figure 7, we see that subjects believe their performance in the classification task exhibits imprecision (that is, beliefs about accuracy are less than 100%). Moreover, we see that subjects are aware that mistakes are more likely for numbers closer to the threshold (greater than 52 and less than 58) than for numbers farther from the threshold (less than 52 or greater than 58; $p < 0.001$). In the middle panel, we see that subjects report similar beliefs about other subjects' imprecision; beliefs about others' errors when discriminating 54 vs. 55 are greater than when discriminating 47 vs. 55 ($p < 0.001$). Thi

Our data also enable us to test one other feature of beliefs about others' imprecision. As outlined in our pre-registration, we test whether beliefs about others' accuracy on rounds
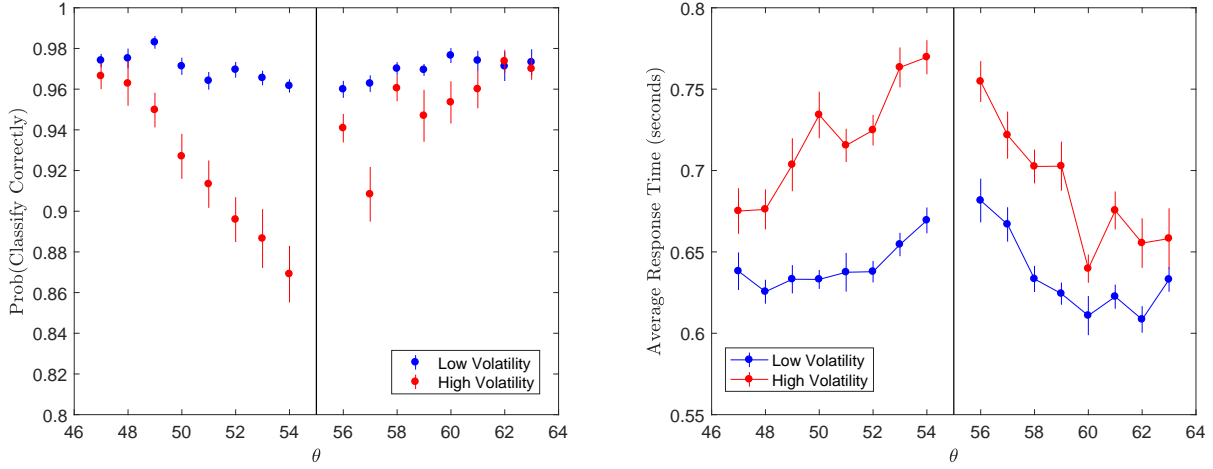
Figure 6: Accuracy and Response Times in the Classification Task in Experiment 2. Note: Left panel shows the proportion of rounds on which subjects correctly classified the stimulus as greater than or less than the reference level of 55. Right panel shows the average response time on rounds where subjects correctly classified the stimulus. In both panels, the vertical bars denote two standard errors of the mean. Standard errors are clustered by subject.

when $\theta = 54$ is higher for those subjects who experience the low volatility distribution in Phase 1.[14] Such a test investigates the hypothesis that subjects are aware that others' perception of a given number varies as a function of the experienced distribution. Indeed, the right panel of Figure 7 shows that, for $\theta = 54$, subjects who experience the high volatility distribution in Phase 1 report that others make more errors, compared to those subjects who experience the low volatility distribution in Phase 1 ($p = 0.048$).

In summary, we find that subjects are aware of their own imprecision and the imprecision of others. Furthermore, subjects are aware that the structure of their own and others' imprecision is consistent with the noise structure that we assume in our model. That is, our additive noise assumption gives rise to the prediction that nearby states should be harder to distinguish from far-away states. Finally, the fact that subjects are aware of others' imprecision is consistent with the mechanism in our model that generates strategic uncertainty, and pins down the unique equilibrium.

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[14]Pre-registration documents are available in Appendix D.

Figure 7: Beliefs about Own and Others' Accuracy in the Classification Task. Note: Panel (a) shows the average belief about own accuracy for values of θ that are far ($\theta < 52$ or $\theta > 58$) and close ($51 < \theta < 59$) to the threshold 55. Panel (b) shows the average belief about others' accuracy for values of θ that are far ($\theta = 47$) and close ($\theta = 54$) to the threshold 55. Panel (c) shows the average belief about others' accuracy when $\theta = 54$, split by experimental condition. In all panels, vertical bars denote two standard errors of the mean.

5 Discussion

5.1 Efficient Coding Assumption

Here we revisit the assumption about efficient coding in our model. The specific performance objective that we assume in Section 2 is only one of several plausible specifications (Ma and Woodford, 2020). In particular, there are other possible objective functions that players may have, besides minimizing the mean squared error of the estimate of θ. For example, a prominent alternative efficient coding objective from the literature on sensory perception is to maximize the mutual information between the state and its noisy internal representation. In the proof of Proposition 1, we confirm that the coding rule we use in our model is robust to this alternative objective.

Yet another alternative objective that has been examined in the economics literature is maximization of expected reward. In Appendix **??**, we show that the result in Proposition 1 is robust to using this alternative objective function. Specifically, we maintain the constraints in Assumption 2 and we analyze a two-stage game. In the first stage, each player optimally chooses parameters of the encoding function. In the second stage, players choose strategies in the simultaneous move game, conditional on their chosen encoding function from the first stage. We show that the optimal encoding function still takes the form characterized in Proposition 1. Thus, our theoretical predictions are robust to three performance objectives: (i) minimizing mean squared error of the estimate of θ, (ii) maximizing mutual information

between the noisy internal representation and θ and (iii) maximizing expected reward.

5.2   Comparison with Quantal Response Equilibrium

In our model, stochastic perception of θ immediately generates stochastic strategic behavior. As such, our model is related to Quantal Response Equilibrium (QRE; McKelvey and Palfrey 1995, 1998, Goeree, Holt and Palfrey 2016), which is a workhorse model of stochastic behavior in experimental game theory.[15] In QRE, the utility associated with a given action is subject to a random shock, and each player best responds given the distribution of these shocks. For some parameter values, the models of QRE and cognitive imprecision deliver similar predictions, in that both theories predict that the probability of investing decreases monotonically in θ.

However, there is one fundamental difference in the assumptions of the two theories, which generates distinguishing predictions. When applying QRE to our experimental games, player i's utility from not investing is $\theta + \epsilon_i$, where ϵ_i is mean zero noise. Crucially, player i is assumed to know both θ and his private shock ϵ_i. This implies that player i believes that his opponent's utility from not investing is centered at θ (even if player i does not know player j's shock, ϵ_j). In this sense, θ is common knowledge. In contrast, in our model of cognitive imprecision, the stochastic element of the theory is embedded in the perception of θ, rather than in the utility of not investing. Therefore, in cognitive imprecision, player i does not observe θ, but a noisy signal $Z_i = \theta + \epsilon_i$. This implies that player i believes that his opponent's perceived utility of not investing is centered at $\theta + \epsilon_i$. Thus, in contrast to QRE, there is no common knowledge of θ in cognitive imprecision.

This difference in assumptions leads to two important distinguishing predictions. The first difference is that, in QRE, each player has precise perception of θ, and thus there is no role for a prior distribution over θ. The prior does, however, play a key role in cognitive imprecision, since it impacts the perception of θ and, more importantly, player i's beliefs about player j's perception of θ. Our main result, displayed in Figure 3, clearly shows that the prior distribution has a systematic effect on behavior, which supports the prediction of cognitive imprecision.

While the noise structure in QRE is almost alway taken to be exogenous, efficient coding may offer one such source for noisy responses in QRE. Interestingly, in the original QRE paper, McKelvey and Palfrey (1995) propose that *"...to the extent that we can find observable independent variables that a priori one would expect to be correlated with the precision of*

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[15]For other models of strategic interaction with stochastic choice, see Chen, Friedman and Thisse (1997), Goeree and Holt (2004), Friedman and Mezzetti (2005), Gonçalves (2020), and Goeree and Louis (Forthcoming).

these [expected payoff] estimates, one can make predictions about the effects of different experimental treatments that systematically vary these independent variables" (pg. 7-8). Efficient coding provides one such independent variable, which is the volatility of the payoff distribution.[16]

The second difference between QRE and cognitive imprecision involves the theoretical conditions that are sufficient to generate a unique equilibrium. As shown in Proposition 3, cognitive imprecision generates a unique equilibrium when the noise in perception is sufficiently small. One interpretation of this condition, is that when players pay sufficient attention to the coordination game, so that the variance of the internal representation Z_i is below a threshold, then uniqueness obtains under our theory of cognitive imprecision. In contrast, QRE delivers a unique equilibrium when the variance of the shock to utility is sufficiently *large* (Ui, 2006). While our data do not enable us to test between this difference in conditions for uniqueness, one implication is that when players devote a substantial amount of attention to the coordination game, the multiplicity of equilibria is more likely to be eliminated under cognitive imprecision, compared with QRE.

5.3 Comparison with Level-k Thinking

Our results also relate to another behavioral theory of games called Level-k Thinking (Stahl and Wilson, 1994, 1995; Nagel, 1995). In one prominent version of this theory, there are different types of players, and each type best responds to another type who exhibits one less degree of strategic sophistication. For example, a Level-0 type would be characterized by no strategic sophistication and, thus, would exhibit purely random behavior. A Level-1 type would then best respond to a Level-0 player, and a Level-2 player would best respond to a Level-1 player, and so on. What are the predictions of Level-k Thinking the game in our first experiment? Given that Level-0 players randomize, the expected utility of a Level-1 player from Invest is

$$EU_{L1}(\text{Invest}) = \frac{1}{2}a + \frac{1}{2}b$$

Thus, $EU_{L1}(\text{Invest}) > EU(\text{Not Invest})$ if and only if $\theta < (a + b)/2$. Next, under the assumption that Level-2 players believe they are facing a Level-1 opponent, the expected

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[16]In related work, Friedman (2020) proposes a model that endogenizes the precision parameter in QRE, though it is the set of payoffs in the *current game* that determine the precision parameter — rather than the distribution of payoffs within a class of games, as in our model.

utility from Invest for a Level-2 player is

$$EU_{L2}(\text{Invest}) = \begin{cases} b \text{ if } \theta < (a+b)/2 \\ a \text{ if } \theta > (a+b)/2 \end{cases}$$

When $\theta < (a+b)/2$, then $EU_{L2}(\text{Invest}) = b > \theta$. Conversely, when $\theta > (a+b)/2$, then $EU_{L2}(\text{Invest}) = a < \theta$. Thus, Level-2 players choose Invest if and only if $\theta < (a+b)/2$. Using the same logic, we obtain the same prediction for all upper levels.

In sum, the fraction of subjects who choose Invest is:

$$\Pr[\text{Invest}] = \begin{cases} \Pr[L_0]\frac{1}{2} + (1 - \Pr[L_0]) \text{ if } \theta < (a+b)/2 \\ \Pr[L_0]\frac{1}{2} \qquad\qquad\qquad \text{ if } \theta > (a+b)/2 \end{cases}$$

where $\Pr[L_0]$ is the fraction of Level-0 players in the population. The theory therefore predicts that, in the aggregate, the probability of investing is monotone in θ and exhibits a sharp decrease at $\theta = (a+b)/2$. However, Level-k Thinking does not predict any difference across our experimental treatments; thus the theory would need to be augmented with some extra feature in order to explain the clear context-dependence we observe in our data.

5.4 Interpretation of Signals in Global Games

One theme that emerges from both our theoretical and experimental analyses is that the noise in global games can be interpreted literally as irreducible error stemming from perceptual constraints. This theme is related to the idea from Heinemann, Nagel and Ockenfels (2009) that behavior in a complete information coordination game can be interpreted *as if* players are observing a fundamental parameter with noise. Like us, Heinemann, Nagel and Ockenfels (2009) structurally estimate a global games model, and find a sizable standard deviation of private signals. However, Heinemann, Nagel and Ockenfels (2009) argue that the (only) source of the estimated standard deviation of private signals is strategic uncertainty. In contrast, we argue that the standard deviation of private signals is driven by cognitive imprecision. By adopting an "as is" interpretation of noise in private signals, we are able to generate and test novel hypotheses about how the standard deviation of private signals varies across environments.

Another important implication for the literature on global games has to do with the role of public vs. private signals. A series of papers has argued that when an institution like the government or financial markets can generate public signals, then it is possible a unique equilibrium may no longer obtain in a global games model (Angeletos and Werning 2006,

Atkeson 2000, Hellwig, Mukherji and Tsyvinski 2006). The argument is that a sufficiently precise public signal can act as a coordination device, and thus restore multiple equilibria. However, our theory and experimental results suggest that there is an important difference between *access* to a public signal and actually using the public signal as a coordination device. Specifically, even if all players have access to the public signal, each player's perceptual system will process the same public signal slightly differently. This perceptual friction therefore transforms the public signal into private information. In sum, mere access to a public signal is not enough to overturn the classic global games result. The ability to precisely perceive public information is also necessary and, as we have shown, this cannot be taken for granted.

6   Conclusion

We have provided and experimentally validated a framework for analyzing strategic behavior when players have imprecise perception of a fundamental parameter. Our results are in line with previous experiments on global games, which find evidence consistent with an equilibrium where all players invest once the fundamental crosses a threshold (Heinemann, Nagel and Ockenfels, 2004, 2009; Cabrales, Nagel and Armenter, 2007; Szkup and Trevino, 2020; Goryunov and Rigos, 2020). At the same time, our experimental data suggest that the predictions from the global games literature may be more applicable than previously thought: even when there is no explicit private information given to players, imprecise perception can serve as a source of private information. Interestingly, the particular manner in which we model imprecise perception is closely connected to the noise structure used in the global games literature to select a unique equilibrium.

We also find empirical evidence of *context-dependent* strategic behavior, which is consistent with efficient coding. This context-dependence rules out several alternative game-theoretic models. We argue that the unstable strategic behavior that we observe across experimental conditions is a consequence of the efficient use of cognitive resources. In our setting of a 2×2 coordination games, efficient coding provides a mechanism that modulates the probability that two players coordinate and play the same action. One important direction for future research is to understand the implications of cognitive imprecision in more general games. The idea that "public" payoffs may be perceived as private information, is likely to have important implications for other strategic behavior outside the coordination games we study here.

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A   Proofs

Proof of Proposition 1

Here we adapt the theoretical derivation of efficient coding from Khaw, Li and Woodford (2021) to our framework where the distribution of θ is normal rather than lognormal. Suppose that the internal representation S of θ is drawn from

$$S|\theta \sim N(m(\theta), \sigma_S^2) \tag{3}$$

where the *encoding rule*, $m(\theta)$, is a linear transformation of θ, $m(\theta) = \xi + \psi\theta$. Parameters ξ and ψ are endogenous while the precision parameter σ_S is exogenous. We assume that the cognitive process producing the internal representation is subject to a "power constraint"

$$E[m^2] \leq \Omega^2 < \infty \tag{4}$$

The efficient coding hypothesis requires that the encoding rule $m(\theta)$ is chosen (among all linear functions satisfying the constraint) so as to maximize the system's objective function, for a given prior distribution of θ. As in Khaw, Li and Woodford (2021), we assume that the system produces an estimate of θ on the basis of S, $\tilde{\theta}(S)$, and that the goal of the design problem is to have a system that achieves as low as possible a mean squared error of this estimate. Given a noisy internal representation, the estimate which minimizes the mean squared error is the mean of the posterior distribution of θ, that is, $\tilde{\theta}(S) = E[\theta|S]$ for all S. The goal of the design problem is, thus, to minimize the variance of the posterior distribution of θ.

Consider the transformed internal representation, $Z \equiv (S - \xi)/\psi$. The distribution of the transformed internal representation conditional on θ is $Z|\theta \sim N(\theta, \sigma_S^2/\psi^2)$. Thus, the distribution of θ given the (transformed) internal representation is

$$\theta|Z \sim N\left(\mu_\theta + \frac{\sigma_\theta^2}{\sigma_\theta^2 + (\sigma_S^2/\psi^2)}(Z - \mu_\theta), \frac{\sigma_\theta^2(\sigma_S^2/\psi^2)}{\sigma_\theta^2 + (\sigma_S^2/\psi^2)}\right) \tag{5}$$

The variance of the posterior distribution of θ is strictly increasing in the variance of Z, σ_S^2/ψ^2. Thus, it is desirable to make ψ as large as possible (in order to make the mean squared error of the estimate as small as possible) consistent with the power constraint. When the distribution of θ is normal, we have

$$E[m^2] = \xi^2 + \psi^2 E[\theta^2] + 2\xi\psi E[\theta] = (\xi + \psi\mu_\theta)^2 + \psi^2\sigma_\theta^2 \leq \Omega \tag{6}$$

The largest value of ψ consistent with this constraint is achieved when

$$\xi = -\psi\mu_\theta \ , \ \psi = \frac{\Omega}{\sigma_\theta} \tag{7}$$

Thus, $m^\star(\theta) = -\frac{\Omega}{\sigma_\theta}\mu_\theta + \frac{\Omega}{\sigma_\theta}\theta$ and

$$Z|\theta \ \sim \ N\left(\theta, \frac{\sigma_S^2}{\Omega^2}\sigma_\theta^2\right) \tag{8}$$

The same optimal coding rule obtains under an alternative goal of the system. Consider the more conventional hypothesis from sensory perception literature, whereby the encoding rule is assumed to maximize the Shannon mutual information between the objective state θ and its subjective representation S. Denote with ρ_θ the precision of θ and with ρ_S the precision of S. We have $\theta \sim N\left(\mu_x, \frac{1}{\rho_\theta}\right)$, $S|\theta \sim N\left(\xi + \psi\theta, \frac{1}{\rho_S}\right)$, $Z|\theta \sim \left(\theta, \frac{1}{\rho_Z}\right)$, and $\theta|Z \sim N\left(\frac{\rho_\theta\mu_\theta + \rho_Z Z}{\rho_\theta + \rho_Z}, \frac{1}{\rho_\theta + \rho_Z}\right)$, where $Z = \frac{S-\xi}{\psi}$ and $\rho_Z = \psi^2/\sigma_S^2$. The Shannon mutual information between θ and Z is

$$I(\theta, Z) = \frac{1}{2}\log_2\left(\frac{\sigma_\theta^2}{\sigma_{\theta|Z}^2}\right) = \frac{1}{2}\log_2\left(1 + \frac{\rho_Z}{\rho_\theta}\right) \tag{9}$$

which is strictly increasing in ρ_Z and, thus, strictly decreasing in σ_Z^2. This means that, as for the previous goal, it is desirable to make ψ as large as possible (consistent with the power constraint).

Proof of Proposition 2

First, we show that, when the conditions in the statement of the Proposition are satisfied, there exists a unique monotone equilibrium of the game.

Remember that the distribution of Z_i conditional on θ is normal with mean equal to θ and variance equal to σ_Z^2, where $\sigma_Z^2 = \omega\sigma_\theta^2 = (\sigma_S^2/\Omega^2)\sigma_\theta^2$. Since the likelihood function of Z_i is conjugate to the prior distribution of θ, we have a closed form solution for the distribution of player i's posterior beliefs over θ.

In particular, player 1's posterior distribution of θ given Z_1 is

$$\theta|Z_1 \sim \mathcal{N}\left(\frac{\sigma_Z^2}{\sigma_\theta^2 + \sigma_Z^2}\mu_\theta + \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_Z^2}Z_1, \frac{\sigma_\theta^2\sigma_Z^2}{\sigma_\theta^2 + \sigma_Z^2}\right) \tag{10}$$

Thus, we have:

$$EU[\text{Not Invest}|Z_1] = E[\theta|Z_1] = \frac{\sigma_Z^2\mu_\theta + \sigma_\theta^2 Z_1}{\sigma_\theta^2 + \sigma_Z^2} \tag{11}$$

On the other hand, player 1's expected utility from investing is

$$EU[\text{Invest}|Z_1] = a + (b - a)\Pr[\text{Opponent Invests}|Z_1] \tag{12}$$

Assume player 1 believes his opponent uses a monotone strategy with threshold k. In this case, player 1's expectation that the opponent invests is $\Pr[Z_2 \leq k|Z_1]$. Player 1's distribution of Z_2 given Z_1 is:

$$Z_2|Z_1 \sim \mathcal{N}\left(E[\theta|Z_1] = \frac{\sigma_Z^2}{\sigma_\theta^2 + \sigma_Z^2}\mu_\theta + \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_Z^2}Z_1, \frac{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}{\sigma_\theta^2 + \sigma_Z^2}\right)$$

Thus, we have:

$$\Pr[Z_2 \leq k|Z_1] = \Phi\left(\frac{k\sqrt{\sigma_\theta^2 + \sigma_Z^2}}{\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}} - \frac{\sigma_Z^2\mu_\theta + \sigma_\theta^2 Z_1}{\sqrt{\sigma_\theta^2 + \sigma_Z^2}\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}}\right) = \Phi\left(\frac{(\sigma_\theta^2 + \sigma_Z^2)k - \sigma_Z^2\mu_\theta - \sigma_\theta^2 Z_1}{\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\right)$$

where $\Phi(\cdot)$ is the cumulative distribution of the standard normal.

Player 1's best response is to invest if and only if

$$\begin{aligned}
EU[\text{Not Invest}|Z_1] &\leq EU[\text{Invest}|Z_1, k] \\
\frac{\sigma_Z^2\mu_\theta + \sigma_\theta^2 Z_1}{\sigma_\theta^2 + \sigma_Z^2} &\leq a + (b - a)\Phi\left(\frac{(\sigma_\theta^2 + \sigma_Z^2)k - \sigma_Z^2\mu_\theta - \sigma_\theta^2 Z_1}{\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\right)
\end{aligned} \tag{13}$$

If we write $\overline{Z}(k)$ for the unique value of Z_1 such that player 1 is indifferent between investing and not investing (this is well defined since player 1's expected payoff from not investing is strictly increasing in Z_1 and player 1's expected payoff from investing is strictly decreasing in Z_1), the best response of player 1 is to follow a monotone strategy with threshold equal to $\overline{Z}(k)$, that is, to invest if and only if $Z_1 \leq \overline{Z}(k)$.

Observe that as $k \to -\infty$ (player 2 never invests), $EU[\text{Invest}|Z_1, k]$ tends to a, so $\overline{Z}(k)$ tends to $\frac{(\sigma_\theta^2 + \sigma_Z^2)a - \sigma_Z^2\mu_\theta}{\sigma_\theta^2}$. As $k \to \infty$ (player 2 always invests), $EU[\text{Invest}|Z_1]$ tends to b, so $\overline{Z}(k)$ tends to $\frac{(\sigma_\theta^2 + \sigma_Z^2)b - \sigma_Z^2\mu_\theta}{\sigma_\theta^2}$. A fixed point of $\overline{Z}(k)$ — that is a value $k^\star$ such that $\overline{Z}(k^\star) = k^\star$ — is a monotone equilibrium of the game where each player invests if and only if his signal is below $k^\star$. Since $\overline{Z}(k)$ is a mapping from $\mathbb{R}$ to itself and is continuous in k, there exists $k \in \left[\frac{(\sigma_\theta^2 + \sigma_Z^2)a - \sigma_Z^2\mu_\theta}{\sigma_\theta^2}, \frac{(\sigma_\theta^2 + \sigma_Z^2)b - \sigma_Z^2\mu_\theta}{\sigma_\theta^2}\right]$, such that $\overline{Z}(k) = k$ and a threshold equilibrium of this game exists.

When is there a unique equilibrium? Define $W(\overline{Z}(k), k)$ as

$$W(\overline{Z}(k), k) = \frac{\sigma_Z^2 \mu_\theta + \sigma_\theta^2 \overline{Z}(k)}{\sigma_\theta^2 + \sigma_Z^2} - a - (b-a)\Phi\left(\frac{(\sigma_\theta^2 + \sigma_Z^2)\,k - \sigma_Z^2 \mu_\theta - \sigma_\theta^2 \overline{Z}(k)}{\sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\right)$$

At a fixed point, $\overline{Z}(k^\star) = k^\star$. Thus, we have:

$$W(k^\star) = \frac{\sigma_Z^2 \mu_\theta + \sigma_\theta^2 k^\star}{\sigma_\theta^2 + \sigma_Z^2} - a - (b-a)\Phi\left(\frac{\sigma_Z^2}{\sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\,(k^\star - \mu_\theta)\right)$$

Then,

$$\frac{\partial W(k^\star)}{\partial k^\star} = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_Z^2} - \phi\left(\frac{\sigma_Z^2}{\sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\,(k^\star - \mu_\theta)\right)\frac{\sigma_Z^2(b-a)}{\sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}$$

And there is a unique fixed point if and only if $\frac{\partial W(k^\star)}{\partial k^\star} > 0$ at the fixed point. When $\frac{\partial W(k^\star)}{\partial k^\star} < 0$, there are at least three fixed points.

Since $\phi(y) \leq \frac{1}{\sqrt{2\pi}}$, this is a sufficient condition for $\frac{\partial W(k^\star)}{\partial k^\star} > 0$:

$$\frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_Z^2} > \frac{1}{\sqrt{2\pi}}\frac{\sigma_Z^2(b-a)}{\sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}} \tag{14}$$

$$\frac{\sigma_\theta^2 \sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}}{(b-a)\sigma_Z^2 \sqrt{\sigma_\theta^2 + \sigma_Z^2}} > \frac{1}{\sqrt{2\pi}} \tag{15}$$

$$\sqrt{2\pi} > \frac{(b-a)\sigma_Z^2 \sqrt{\sigma_\theta^2 + \sigma_Z^2}}{\sigma_\theta^2 \sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}} \tag{16}$$

We can find the same sufficient condition if we use implicit differentiation to derive a condition which guarantees that $\frac{\partial \overline{Z}(k)}{\partial k} \in (0, 1)$ and, thus, $\overline{Z}(k)$ is a contraction.

$$\frac{\partial \overline{Z}(k)}{\partial k} = -\frac{\partial W/\partial k}{\partial W/\partial \overline{Z}(k)} = \frac{\phi\left(\frac{(\sigma_\theta^2 + \sigma_Z^2)k - \sigma_Z^2 \mu_\theta - \sigma_\theta^2 \overline{Z}(k)}{\sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\right)\frac{(b-a)\sigma_\theta^2 + (b-a)\sigma_Z^2}{\sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}}{\frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_Z^2} + \phi\left(\frac{(\sigma_\theta^2 + \sigma_Z^2)k - \sigma_Z^2 \mu_\theta - \sigma_\theta^2 \overline{Z}(k)}{\sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\right)\frac{(b-a)\sigma_\theta^2}{\sqrt{2\sigma_\theta^2 \sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}}$$

Note that, since $\phi(\cdot) \in (0, 1)$, $\frac{\partial \overline{Z}(k)}{\partial k}$ is always positive. It is also less than 1 if the following

condition is satisfied:

$$\frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_Z^2} > \phi\left(\frac{(\sigma_\theta^2 + \sigma_Z^2)k - \sigma_Z^2\mu_\theta - \sigma_\theta^2\overline{Z}(k)}{\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\right)\frac{(b-a)\sigma_Z^2}{\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}$$

$$\frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_Z^2}\frac{\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}{(b-a)\sigma_Z^2} > \phi\left(\frac{(\sigma_\theta^2 + \sigma_Z^2)k - \sigma_Z^2\mu_\theta - \sigma_\theta^2\overline{Z}(k)}{\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\right)$$

$$\frac{\sigma_\theta^2\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}}{(b-a)\sigma_Z^2\sqrt{\sigma_\theta^2 + \sigma_Z^2}} > \frac{1}{\sqrt{2\pi}} \qquad (17)$$

When the condition in equation (17) is satisfied, there exists a constant $G < 1$ such that $|\partial\overline{Z}(k)/\partial k| \leq G < 1$ for all $k \in \mathbb{R}$. This means that, when the condition in equation (17) is satisfied, $\overline{Z}(k)$ is a contraction and, thus, by the Banach fixed point (or contraction mapping) theorem, $\overline{Z}(k)$ has exactly one solution.

The condition $\sqrt{\omega(1 + \omega)} < \frac{\sqrt{6\pi}}{(b-a)}\sigma_\theta^2$ is obtained by replacing $\sigma_Z = \omega\sigma_\theta$ in the condition for a unique monotone equilibrium from Proposition 2 and re-arranging terms. Thus, this shows that, when the conditions in the statement of the Proposition are satisfied, there exists a unique monotone equilibrium of the game.

Second, we show that, when $\mu_\theta = \frac{(a+b)}{2}$, there exists a monotone equilibrium of the game where $k^\star = \mu_\theta$ for any value of σ_θ, σ_S and ω (or, equivalently, for any value of σ_θ and σ_Z). When the third condition from Proposition 2 is satisfied, this is the unique monotone equilibrium of the game.

Assume player 2 uses a threshold strategy where he invests if and only if $Z_2 \leq k = \mu_\theta$. Is this an equilibrium, that is, is $\overline{Z}(\mu_\theta) = \mu_\theta$? $\overline{Z}(\mu_\theta)$ is the value of Z_1 such that the following equation is satisfied with equality:

$$\frac{\sigma_Z^2\mu_\theta + \sigma_\theta^2 Z_1}{\sigma_\theta^2 + \sigma_Z^2} = a + (b-a)\Phi\left(\frac{(\sigma_\theta^2 + \sigma_Z^2)k - \sigma_Z^2\mu_\theta - \sigma_\theta^2 Z_1}{\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\right)$$

$$\frac{\sigma_Z^2\mu_\theta + \sigma_\theta^2 Z_1}{\sigma_\theta^2 + \sigma_Z^2} = a + (b-a)\Phi\left(\frac{\sigma_\theta^2\mu_\theta - \sigma_\theta^2 Z_1}{\sqrt{2\sigma_\theta^2\sigma_Z^2 + \sigma_Z^4}\sqrt{\sigma_\theta^2 + \sigma_Z^2}}\right)$$

If we set $\overline{Z}(\mu_\theta) = \mu_\theta$, we get:

$$\mu_\theta = a + (b-a)\Phi(0)$$
$$\mu_\theta = \frac{(a+b)}{2}$$

Proof of Proposition 3

From Proposition 2 and the condition in the statement of Proposition 3, we know that there exists a unique monotone equilibrium of the game where each player invests if and only if his transformed internal representation is smaller than μ_θ. In this equilibrium, $Pr[Invest|\theta] = Pr\left[Z_i \le \mu_\theta|\theta\right] = \Phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)$ and $\frac{\partial Pr[Invest|\theta]}{\partial \theta} = -\phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)\left(\frac{1}{\omega\sigma_\theta}\right)$. Thus, $Pr[Invest|\theta]$ grows with σ_θ if $\theta < \mu_\theta$ and it decreases with σ_θ is $\theta > \mu_\theta$. Moreover, the sensitivity of choices to θ decreses with σ_θ for values of θ around the cutoff.

Indeed, we have

$$
\begin{aligned}
\frac{\partial Pr\left[Invest|\theta\right]}{\partial\theta\partial\sigma_\theta} &= \phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)\left(\frac{1}{\omega\sigma_\theta^2}\right) + \phi'\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta^2}\right)\left(\frac{1}{\omega\sigma_\theta}\right) \\
&= \phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)\left(\frac{1}{\omega\sigma_\theta^2}\right) - \left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)\phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta^2}\right)\left(\frac{1}{\omega\sigma_\theta}\right) \\
&= \phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)\left(\frac{1}{\omega\sigma_\theta^2}\right) - \phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)\left(\frac{(\mu_\theta - \theta)^2}{\omega^3\sigma_\theta^4}\right) \\
&= \phi\left(\frac{\mu_\theta - \theta}{\omega\sigma_\theta}\right)\left(\frac{\omega^2\sigma_\theta^2 - (\mu_\theta - \theta)^2}{\omega^3\sigma_\theta^4}\right)
\end{aligned}
$$

which is positive if and only if $(\mu_\theta - \theta)^2 < \omega^2\sigma_\theta^2$.

(In the second line, we used the fact that $\phi'(x) = -x\phi(x)$.)

B   Robustness of Monotone Equilibrium with $k^\star = \mu_\theta$

Let us introduce the following definitions from Chambers and Healy (2012):

Definition 1 *A random variable with cumulative density function F and mean μ is **symmetric** if, for every $a \geq 0$, $F(\mu + a) = 1 - \lim_{x \to a^-} F(\mu - a)$.*

Definition 2 *A random variable is **quasiconcave** (or unimodal) if it has a density function f such that for all $x, x' \in \mathbb{R}$ and $\lambda \in (0, 1)$, $f(\lambda x + (1 - \lambda)x') \geq \min\{f(x), f(x')\}$.*

Definition 3 *The error term ϵ_i satisfies **symmetric dependence** with respect to the random variable θ if, for each realization of θ, $\epsilon_i | \theta$ has a continuous density function $f_{\epsilon_i | \theta}$ satisfying $f_{\epsilon_i | \theta}(\epsilon_i | \mu_\theta + a) = f_{\epsilon_i | \theta}(\epsilon_i | \mu_\theta - a)$ for almost every ϵ_i and a in $\mathbb{R}$. (Note that error terms that are independent of θ satisfy this definition).*

Consider the following assumptions:

(A1) $S_i = \theta + \epsilon_i$

(A2) $E[\theta] = \mu_\theta < \infty$

(A3) θ is a symmetric random variable and its density is continuous on $\mathbb{R}$

(A4) $E[\epsilon_i | \theta] = 0$ for each θ

(A5) ϵ_i is a symmetric and quasiconcave random variable

(A6) ϵ_i satisfies symmetric dependence with respect to θ

Lemma 1 (Chambers and Healy 2012, Proposition 2) *Assume A1-A6. A Bayesian agent updates his beliefs over θ in the direction of the signal, that is, for almost every $S_i \in \mathbb{R}$, there exists some $\alpha \geq 0$ such that $E[\theta | S_i] = \alpha S_i + (1 - \alpha)\mu_\theta$.*

Proposition 4 *Assume A1-A6 and $\mu_\theta = (a + b)/2$. There exists a monotone equilibrium of the game where $k^\star = \mu_\theta$.*

Proof of Proposition 4   The proof can be carried out with general values for a and b (such that $b > a$). For ease of exposition, we focus on the experimental parameters: $a = 47$, $b = 63$, $\mu_\theta = 55$. Assume that player j uses threshold $k_j = 55$, that is, he invests if and only if $S_j < 55$. We want to show that player i's best response is to use the same threshold, $k_i = 55$. Player i prefers to invest if and only if $EU[\text{Not Invest}|S_i] < EU[\text{Invest}|S_i, k_j]$.

Thus, we want to show that (1) when $S_i = 55$, $EU[\text{Not Invest}|S_i] = EU[\text{Invest}|S_i, k_j = 55]$; (2) when $S_i < 55$, $EU[\text{Not Invest}|S_i] < EU[\text{Invest}|S_i, k_j = 55]$; and (3) when $S_i > 55$, $EU[\text{Not Invest}|S_i] > EU[\text{Invest}|S_i, k_j = 55]$.

By Lemma 1, $EU[\text{Not Invest}|S_i] = E[\theta|S_i] = \alpha S_i + (1-\alpha)\mu_\theta$ where $\alpha \geq 0$. Note also that $EU[\text{Invest}|S_i, k_j = 55] = 47 + (63 - 47)Pr[S_j < k_j = 55|S_i]$. First, we prove (1). Assume $S_i = 55$. We want to show that $EU[\text{Not Invest}|S_i] = EU[\text{Invest}|S_i, k_j = 55]$. By Lemma 1, $EU[\text{Not Invest}|S_i = 55] = \alpha S_i + (1-\alpha)\mu_\theta = \alpha(55) + (1-\alpha)(55) = 55$. Thus, the equality we want to show becomes $55 = 47 + (63 - 47)Pr[S_j < k_j = 55|S_i = 55]$. This equality is satisfied if and only if $Pr[S_j < k_j = 55|S_i = 55] = 1/2$. By A1 and A4 (and linearity of expectation), $E[S_j|S_i] = E[\theta|S_i] = 55$. By A5, the density of of $S_j|S_i$ is symmetric. Thus, the probability S_j takes a value below its posterior mean (55) is $1/2$. This proves (1).

Second, we prove (2). Assume $S_i < 55$. By Lemma 1, $EU[\text{Not Invest}|S_i] = \alpha S_i + (1-\alpha)55$. This is smaller than 55 for any positive α. This also means that, by A1 and A4, $E[S_j|S_i] = E[\theta|S_i] < 55$. The probability that the opponent invests is the posterior probability that his signal is below 55 (given S_i). Since the conditional distribution of the opponent's signal is symmetric around its mean (by A5), the median is equal to the mean. This means that the conditional CDF of the opponent signal equals $1/2$ at the posterior mean, is greater than $1/2$ for values of S_j above the mean and is lower than $1/2$ for values of S_j below the mean. Since the posterior mean of the opponent's signal is lower than 55, the probability that player j's signal is lower than 55 (conditional on $S_i < 55$) is greater than $1/2$. Thus, $EU[\text{Invest}|S_i, k_j = 55] = 47 + (63 - 47)Pr[S_j < k_j = 55|S_i] > 55$. This proves that $EU[\text{Invest}|S_i, k_j = 55] > 55 > EU[\text{Not Invest}|S_i]$. (3) can be proven analogously.

C   Alternative Model of Efficient Coding

Assumption 4 (Alternative Performance Objective) *Players choose the encoding function which maximizes their expected reward in the simultaneous-move game.*

Consider the following two-stage game: in stage 1, each player $i = \{1, 2\}$ chooses simultaneously and independently the parameters of his encoding function, (xi_i, ψ_i), to maximize the performance objective in Assumption 4 under the constraints in Assumption 2; in stage 2, players participate to the simultaneous-move game described in Table 1 endowed with the encoding functions chosen in the previous stage. We solve this game by backward induction.

Stage 2: Simultaneous-Move Game (with Exogenous Encoding Functions)

For each player $i = \{1, 2\}$, we have:

$$S_i | \theta \sim N\left(m_i(\theta), \sigma_S^2\right)$$

where $m_i(\theta) = \xi_i + \psi_i \theta$.

Consider the transformed internal representation $Z_i = (S_i - \xi_i)/\psi_i$. We have:

$$Z_i | \theta \sim N\left(\theta, \beta_i^2\right)$$

where $\beta_i = (\sigma_S/\psi_i)$.

Proposition 5 *Suppose Assumptions 1, 2, 4 and $\mu_\theta = (a + b)/2$. Regardless of σ_θ, σ_S, (ξ_1, ψ_1), and (ξ_2, ψ_2), there exists an equilibrium of the game where each player invests if and only if $Z_i \leq \mu_\theta$. Moreover, if $\frac{\sigma_\theta^2 \sqrt{\beta_i^2(2\sigma_\theta^2 + \beta_i^2)}}{(b-a)\beta_i^2 \sqrt{\sigma_\theta^2 + \beta_i^2}} > \frac{1}{\sqrt{2\pi}}$ for all $i = \{1, 2\}$, this is the unique monotone equilibrium of the game.*

Proof. Since the likelihood function of Z_i is conjugate to the prior distribution of θ, we have a closed form solution for the distribution of player i's posterior beliefs over θ. In particular, player 1's posterior distribution of θ given Z_1 is

$$\theta | Z_1 \sim \mathcal{N}\left(\frac{\beta_1^2 \mu_\theta + \sigma_\theta^2 Z_1}{\sigma_\theta^2 + \beta_1^2}, \frac{\sigma_\theta^2 \beta_1^2}{\sigma_\theta^2 + \beta_1^2}\right)$$

Thus, we have:

$$EU[\text{Not Invest}|Z_1] = E[\theta | Z_1] = \frac{\beta_1^2 \mu_\theta + \sigma_\theta^2 Z_1}{\sigma_\theta^2 + \beta_1^2}$$

On the other hand, player 1's expected utility from investing is

$$EU[\text{Invest}|Z_1] = a + (b-a)\Pr[\text{Opponent Invests}|Z_1]$$

Assume player 1 believes his opponent uses a monotone strategy with threshold k_2. In this case, player 1's expectation that the opponent invests is $\Pr[Z_2 \leq k_2|Z_1]$. Player 1's belief over the distribution of Z_2 conditional on Z_1 is:

$$Z_2|Z_1 \sim \mathcal{N}\left(\frac{\beta_1^2\mu_\theta + \sigma_\theta^2 Z_1}{\sigma_\theta^2 + \beta_1^2}, \frac{\sigma_\theta^2(\beta_1^2 + \beta_2^2) + \beta_1^2\beta_2^2}{\sigma_\theta^2 + \beta_1^2}\right)$$

Thus, we have:

$$\Pr[Z_2 \leq k_2|Z_1] = \Phi\left(\frac{k_2(\sigma_\theta^2 + \beta_1^2) - \beta_1^2\mu_\theta - \sigma_\theta^2 Z_1}{\sqrt{\sigma_\theta^2 + \beta_1^2}\sqrt{\sigma_\theta^2(\beta_1^2 + \beta_2^2) + \beta_1^2\beta_2^2}}\right)$$

where $\Phi(\cdot)$ is the cumulative distribution of the standard normal.

Player 1's best response is to invest if and only if

$$EU[\text{Not Invest}|Z_1] \leq EU[\text{Invest}|Z_1, k]$$
$$\frac{\beta_1^2\mu_\theta + \sigma_\theta^2 Z_1}{\sigma_\theta^2 + \beta_1^2} \leq a + (b-a)\Phi\left(\frac{k_2(\sigma_\theta^2 + \beta_1^2) - \beta_1^2\mu_\theta - \sigma_\theta^2 Z_1}{\sqrt{\sigma_\theta^2 + \beta_1^2}\sqrt{\sigma_\theta^2(\beta_1^2 + \beta_2^2) + \beta_1^2\beta_2^2}}\right)$$

Assume $k_2 = \mu_\theta$. We want to show that player's best response is to use the same cutoff. In this case, player 1's best response is to invest if and only if

$$EU[\text{Not Invest}|Z_1] \leq EU[\text{Invest}|Z_1, k]$$
$$\frac{\beta_1^2\mu_\theta + \sigma_\theta^2 Z_1}{\sigma_\theta^2 + \beta_1^2} \leq a + (b-a)\Phi\left(\frac{\sigma_\theta^2(\mu_\theta - Z_1)}{\sqrt{\sigma_\theta^2 + \beta_1^2}\sqrt{\sigma_\theta^2(\beta_1^2 + \beta_2^2) + \beta_1^2\beta_2^2}}\right)$$

First, note that the LHS is a convex combination of μ_θ and Z_1 and that, thus, it is a) equal to μ_θ when $Z_1 = \mu_\theta$, b) smaller than μ_θ when $Z_1 < \mu_\theta$, and c) larger than μ_θ when $Z_1 > \mu_\theta$. Second, remember that $\mu_\theta = (a+b)/2$ and note that the RHS is a) equal to μ_θ when the argument of $\Phi(\cdot)$ is 0 (that is, when $Z_1 = \mu_\theta$, since the denominator is strictly positive); b) larger than μ_θ when the argument of $\Phi(\cdot)$ is strictly positive (that is, when $Z_1 < \mu_\theta$), and c) smaller than μ_θ when the argument of $\Phi(\cdot)$ is strictly negative (that is, when $Z_1 > \mu_\theta$). This means that, when player 2 invests if and only if $Z_2 \leq k_2 = \mu_\theta$, then player 1's best response is to invest if and only if $Z_1 \leq \mu_\theta$. This proves that there exists an equilibrium where both players use a monotone strategy with cutoff equal to μ_θ for any

value of (ξ_1, ψ_1), (ξ_2, ψ_2), σ_S and σ_θ.

Finally, to show that, when the condition in the statement of the proposition is satisfied, this is the unique equilibrium of the game, we can use the same steps in the proof of Proposition 2 to show that the best response mapping is a contraction (and that, thus, we can apply the contraction mapping theorem). In particular, it is sufficient to show that the derivative of the best response function of player 1 with respect to k_2 and the derivative of the best response function of player 2 with respect to k_1 have both an absolute value strictly smaller than 1 (see Theorem 5 in Cachon and Netessine 2006). ∎

Stage 1: Encoding Function Choice

When deriving the optimal choice of the encoding function in stage 1, we assume that, in stage 2, players use the cutoff strategy in the (unique) equilibrium from Proposition 5.

Proposition 6 *Suppose Assumptions 1, 2, 4, and $\mu_\theta = (a + b)/2$. The optimal encoding function is the same for both players and is given by $m^\star(\theta) = \xi^\star + \psi^\star \theta = -\frac{\Omega \mu_\theta}{\sigma_\theta} + \frac{\Omega}{\sigma_\theta}\theta$.*

Proof. In stage 2, each player $i = \{1, 2\}$ invests if and only if $Z_i \leq \mu_\theta$. Given the conditional distribution of Z_i, the probability player i invests for a given θ and encoding function is

$$\mathbb{P}_i(\text{Invest}|\theta, \psi_i) = \Phi\left(\frac{\mu_\theta - \theta}{\sigma_S/\psi_i}\right)$$

Thus, the expected utility player i gets from the game with a given value of θ is

$$
\begin{aligned}
EU_i(\theta, \psi_i) &= \mathbb{P}_i(\text{Invest}|\theta, \psi_i)\left(a + \mathbb{P}_{-i}(\text{Invest}|\theta, \psi_{-i})(b - a)\right) + (1 - \mathbb{P}_i(\text{Invest}|\theta, \psi_i))\theta \\
&= \theta + \mathbb{P}_i(\text{Invest}|\theta, \psi_i)\left(a + \mathbb{P}_{-i}(\text{Invest}|\theta, \psi_{-i})(b - a) - \theta\right) \\
&= \theta + \Phi\left(\frac{\mu_\theta - \theta}{\sigma_S/\psi_i}\right)\left(a + \Phi\left(\frac{\mu_\theta - \theta}{\sigma_S/\psi_{-i}}\right)(b - a) - \theta\right)
\end{aligned}
$$

where we use $-i$ to denote i's opponent. How does this expected utility change with ψ_i (taking ψ_{-i} as given)?

$$\frac{\partial EU_i(\theta, \psi_i)}{\partial \psi_i} = \phi\left(\frac{\mu_\theta - \theta}{\sigma_S/\psi_i}\right)\left(\frac{\mu_\theta - \theta}{\sigma_S}\right)\left(a + \Phi\left(\frac{\mu_\theta - \theta}{\sigma_S/\psi_{-i}}\right)(b - a) - \theta\right) \qquad (18)$$

Since $\phi(\cdot)$ is strictly positive for any argument, the sign of equation (18) is determined by the product of its second and third term. First, note that the second term is a) equal to 0 when $\theta = \mu_\theta$, b) strictly positive when $\theta < \mu_\theta$ and c) strictly negative when $\theta > \mu_\theta$. Second, note that — since $\mathbb{P}_{-i}(\text{Invest}|\theta, \psi_{-i})$ is greater than 1/2 if and only if $\theta < \mu_\theta$ and

$\mu_\theta = (a+b)/2$ — the third term is a) strictly positive when $\theta < \mu_\theta$ and b) strictly negative when $\theta > \mu_\theta$. This means that the product of the second and third term of equation (18) is always positive, with the exception of the case when $\theta = \mu_\theta$, in which case it is 0.

We have shown that the expected payoff in a game with a given θ is strictly increasing in ψ_i for any value of $\theta \neq \mu_\theta$ and it is constant in ψ_i for $\theta = \mu_\theta$. This means that, from an ex-ante perspective (that is, when a player knows the distribution of θ but does not know its actual realization), each player's expected reward from the simultaneous-move game — that is, $EU_i(\psi_i) = \int EU_i(\theta, \psi_i) f(\theta) d\theta$ — is strictly increasing in ψ_i. Therefore, it is desirable to make ψ_i as large as possible consistent with the power constraint. When the distribution of θ is normal, we have

$$E[m^2] = \xi^2 + \psi^2 E[\theta^2] + 2\xi\psi E[\theta] = (\xi + \psi\mu_\theta)^2 + \psi^2 \sigma_\theta^2 \leq \Omega$$

The largest value of ψ consistent with this constraint in Assumption 2 is achieved when

$$\xi = -\psi\mu_\theta \ , \ \psi = \frac{\Omega}{\sigma_\theta}$$

Thus, $m^\star(\theta) = -\frac{\Omega}{\sigma_\theta}\mu_\theta + \frac{\Omega}{\sigma_\theta}\theta.$ ∎

D   Experimental Pre-Registrations

Experiment 1

CONFIDENTIAL - FOR PEER-REVIEW ONLY
Cognitive Imprecision and Strategic Behavior - PROLIFIC (#58473)

Created: 02/16/2021 09:11 AM (PT)
Shared: 06/30/2021 11:57 AM (PT)

1) Have any data been collected for this study already?
No, no data have been collected for this study yet.

2) What's the main question being asked or hypothesis being tested in this study?
We will test a theory of strategic behavior in which a subject's perception of a game payoff is based on a noisy internal representation of the payoff (due to information processing constraints in the brain). Specifically, we focus on 2x2 simultaneous-move games and we model perception using the principle of efficient coding, which implies that perception becomes more precise as the dispersion of encountered stimuli (i.e., payoffs) is reduced. Assuming that a game payoff is perceived noisily and that players are aware of the noisy perception means that the game is one of incomplete information. Equilibrium analysis of this incomplete information game delivers our main hypothesis: manipulating the prior distribution from which a game payoff is drawn during an experimental session will affect the distribution of choices in a given game (that is, keeping the payoffs of the game fixed). In particular, we hypothesize that increasing the volatility of the prior distribution of a game payoff will make observed choices less responsive to that payoff.

3) Describe the key dependent variable(s) specifying how they will be measured.
In each round of a simultaneous-move game, a subject can choose one of two actions: "Option A" or "Option B". This binary action is the key dependent variable of the study. We will also collect response times for each round.

4) How many and which conditions will participants be assigned to?
The game played on each round is symmetric, and the only difference across rounds is the payoff X. Specifically, if a subject chooses "Option A," they receive X points, regardless of their opponent's action. If a subject chooses "Option B," they receive 47 points if their opponent chooses Option A, and 63 points if their opponent chooses Option B. There are two conditions: a high volatility condition and a low volatility condition; subjects are randomly assigned to one of the two conditions. In each condition there are 300 rounds. In the high volatility condition, we draw X from a normal distribution with mean 55 and variance 400. In the low volatility condition, we draw X from a normal distribution with mean 55 and variance 20. We then implement the following two steps:

1) X is rounded to the nearest integer
2) We re-draw X if the nearest integer is less than 11 or greater than 99, which ensures that X contains exactly 2-digits

5) Specify exactly which analyses you will conduct to examine the main question/hypothesis.
We will test whether the degree of noise in subjects' action selection is greater in the high volatility condition compared to the low volatility condition, for values of X in the range [47, 63] in all rounds 31-300. Specifically, we will conduct regressions where the dependent variable is a dummy variable equal to 1 if the choice is "Option B" (and 0 otherwise), and the main independent variable is X. Our main test of interest is whether the probability that subjects choose "Option B" decreases more rapidly as a function of X in the low volatility condition compared to the high volatility condition.

6) Describe exactly how outliers will be defined and handled, and your precise rule(s) for excluding observations.
After reading the experimental instructions and before the first round of the game, subjects will take a comprehension quiz, which consists of 3 questions. Any subject who does not answer all 3 questions correctly will be routed out of the experiment and excluded from the dataset. Of those subjects who pass the comprehension quiz, we will exclude observations for which response times are excessively fast, as defined by a response time that is less than 500 milliseconds.

7) How many observations will be collected or what will determine sample size? No need to justify decision, but be precise about exactly how the number will be determined.
We will collect data until we have N=300 subjects (150 subjects in each of the two conditions) who have correctly passed the comprehension quiz (defined as answering all 3 questions correctly).

8) Anything else you would like to pre-register? (e.g., secondary analyses, variables collected for exploratory purposes, unusual analyses planned?)
As a secondary analysis, we will test whether response times are faster in the low volatility condition compared to the high volatility condition (conditional on X). We will also explore whether the peak of the response time distribution is associated with the equilibrium threshold predicted by a theory in which a game payoff is based on a subject's noisy internal representation. These secondary analyses will be restricted to rounds that satisfy both of the following criteria: 1) X is in the range [47, 63]; 2) the round number is greater than 30

Available at https://aspredicted.org/blind.php?x=7se6hz

Experiment 2

CONFIDENTIAL - FOR PEER-REVIEW ONLY
Awareness of Own and Others' Cognitive Imprecision - PROLIFIC (#60101)

Created: 03/05/2021 07:16 AM (PT)
Shared: 06/30/2021 11:58 AM (PT)

1) Have any data been collected for this study already?
No, no data have been collected for this study yet.

2) What's the main question being asked or hypothesis being tested in this study?
We will test whether subjects are aware of their own and others' cognitive imprecision. We will study this question in the setting of a classic number discrimination task where a subject is incentivized to classify whether a two-digit Arabic numeral is greater than or smaller than a reference number (e.g., Dehaene, Dupoux and Mehler 1990). The innovation of this experiment is to also incentivize subjects to provide their beliefs about their own cognitive imprecision and the cognitive imprecision of other participants. In particular, we hypothesize that subjects are aware that (i) they make mistakes in such a number discrimination task, (ii) other subjects make mistakes in such a number discrimination task, and (iii) both own and others' mistakes are more frequent as the stimulus number becomes closer to the reference number.

3) Describe the key dependent variable(s) specifying how they will be measured.
There are three parts to the experiment:

Part 1: Subjects will go through a number discrimination task in which they are incentivized to accurately and quickly discriminate whether a 2-digit number that we denote as X is bigger or smaller than the reference number of "55". We will collect accuracy and response time data for each of the one hundred fifty trials in this part of the task.

Part 2: We will ask the subject to give beliefs about their own accuracy in Part 1, over the domains (X<52 or X>58) and (51< X< 59). Specifically, for each of the two domains, we will incentivize subjects to report the percentage of trials on which they believe they made accurate choices.

Part 3: We will ask the subject to give beliefs about other subjects' accuracy in Part 1, for the values of X=54 and X=47 (in the same experimental condition, see below). Specifically, for each of the two values of X, we will incentivize the subject to report the percentage of trials on which he believes all subjects (who went through the same condition as him) made accurate choices.

4) How many and which conditions will participants be assigned to?
There are two conditions, and the only difference across conditions is in Part 1 (though the order of Parts 2 and 3 will be randomized at the subject level). Subjects are randomized into the two conditions. The two conditions are "high volatility" and "low volatility" and they refer to the different distributions from which numbers are drawn in Part 1. In each condition there are 150 rounds. In the high volatility condition, we draw X from a normal distribution with mean 55 and variance 400. In the low volatility condition, we draw X from a normal distribution with mean 55 and variance 20. We then implement the following two steps:

1) X is rounded to the nearest integer
2) We re-draw X if the nearest integer is less than 11 or greater than 99, which ensures that X contains exactly 2-digits

5) Specify exactly which analyses you will conduct to examine the main question/hypothesis.
We will test whether (i) the stated beliefs about own and others' accuracy from Parts 2 and 3 are statistically different from 100%, (ii) the stated belief about own accuracy in "far comparisons" (that is, for X in the range [11, 51] or in the range [59, 99]) is greater than the stated belief about own accuracy in "close comparisons" (that is, for X in the range [52, 58]), and (iii)) the stated belief about others' accuracy for X=47 is greater than the stated belief about others' accuracy for X=54.

6) Describe exactly how outliers will be defined and handled, and your precise rule(s) for excluding observations.
Subjects will take a comprehension quiz after reading the experimental instructions for Part 1, which consists of 1 question. Any subject who does not answer the question correctly will be routed out of the experiment and excluded from the dataset.

7) How many observations will be collected or what will determine sample size? No need to justify decision, but be precise about exactly how the number will be determined.
We will collect data until we have N=300 subjects (150 subjects in each of the two conditions) who have correctly passed the comprehension quiz (defined as answering the single quiz question correctly).

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8) Anything else you would like to pre-register? (e.g., secondary analyses, variables collected for exploratory purposes, unusual analyses planned?)

As a secondary analysis, we will assess whether (i) actual accuracy in the number classification task in the range [47, 63], on trials 31-150, is higher in the low volatility condition compared to the high volatility condition, conditional on X, (ii) response times in the number classification task are faster in the low volatility condition compared to the high volatility condition in the range [47, 63], on trials 31-150, and conditional on X, and (iii) perceived others' accuracy is higher for X=54 in the low volatility condition compared to the high volatility condition.

E   Experimental Instructions

Experiment 1

Welcome!

You will earn £2 for completing this study and will have the opportunity to **earn more money** depending on your decisions during the study.

Specifically, at the end of the study, the computer will randomly select one question. You will receive points from the randomly selected question and the number of points depends on your decision and the decision of another participant. Points will be converted to pounds using the rate 20 points = £1. For example, if you earned 60 points for the selected question, you would then earn 60/20 = £3 (in addition to the completion fee).

All questions are equally likely to be selected so **make all choices carefully**.

The next pages give detailed instructions. Following the instructions, you will take a quiz on them. You will be allowed to continue and will be entitled to payment **only if you answer all questions on the quiz correctly**.

Instructions (1/2)

The study is separated into 6 parts of 50 rounds each.

In each round, you are randomly matched with another participant, who we call your **opponent**.

In each round, both you and your opponent will be asked to choose between **two options**:

"Option A" or "Option B"

Here is how to earn points:

- If you choose Option A, the number of points you receive does not depend on whether your opponent chooses Option A or B. The amount of points you receive for choosing Option A can be different in different rounds and will be displayed on your screen.
- If you choose Option B, the number of points you receive depends on your opponent's decision: if your opponent chooses Option A, you will receive 47 points; if your opponent also chooses Option B you will receive 63 points.

Importantly, your opponent is reading these same exaxt instructions. This means that:

- If your opponent chooses Option A, his/her payoff does not depend on your decision and the number of points he/she earns are those given by Option A.
- If your opponent chooses Option B, the number of points he/she receives depends on your decision: if you choose Option A, your opponent will receive 47 points; if you also choose Option B, your opponent will receive 63 points.

Instructions (2/2)

Below is an example screen from the study:

Option A	**Option B**
53	**47** if other participant chooses A **63** if other participant chooses B

In this example, Option A is on the LEFT side of the screen and Option B is on the RIGHT.

In each round, you will choose one of the two options by pressing either the "A" key on your keyboard for the LEFT option or the "L" key on your keyboard for the RIGHT option. On some rounds, Option A will be on the LEFT, and in other rounds it will be on the RIGHT.

In the example above:

- Option A pays you 53 points regardless of your opponent's decision, while Option B pays you 47 points if your opponent chooses Option A and 63 points if your opponent chooses Option B.
- Note also that, if your opponent chooses Option A, he/she earns 53 points regardless of your decision. If your opponent, instead, chooses Option B, he/she earns 47 points if you choose Option A and 63 points if you choose Option B.

Experiment 2

Thank you for participating in this study!

Before we begin, please close all other applications on your computer and put away your cell phone. This study will last approximately **10 minutes**. During this time, we ask your complete and undistracted attention. You will earn £1 for completing the study and you will have the opportunity to **earn more money** depending on your answers during the study.

This study consists of **two phases**. The instructions for Phase 1 are given in the next page. After you go through Phase 1, you will be given a new set of instructions for Phase 2.

When you are ready to continue, press ENTER.

In **Phase 1**, you will see a series of numbers and will be asked to classify whether each number is **larger or smaller than 55**. If the number displayed is smaller than 55, press the "A" key on your keyboard. If the number displayed is larger than 55, press the "L" key.

Your bonus payment will depend on the speed and accuracy of your classification. Specifically:

Bonus Payment = £ (1.5 x accuracy – 1 x speed)

where "accuracy" is the percentage of trials where you correctly classified the number as larger or smaller than 55, and "speed" is the average amount of time it takes you to classify the number on all trials throughout the study, in seconds.

Thus, you make the most money by answering as **quickly** and as **accurately** as possible.

For example, if you correctly classified the number on all trials and it took you 0.3 seconds to respond to each question, you would earn £(1.5 x 100% - 10 x 0.3) = £1.20. If instead you only classified 70% of the numbers correctly and took 0.8 seconds to respond to each question, you would earn £(1.5 x 70% - 10 x 0.8) = £0.25.

Phase 1 will be separated into 3 parts of 50 trials each. In between, you can take a short break.

Before starting with the classification task, you will be asked a question to check your understanding of the instructions. You will be allowed to continue **only if you answer this question correctly**.

When you are ready to continue with the comprehension question, press ENTER.

This is **Phase 2** of the study.

Phase 2 consists of four questions, two on this page and two on the next one.

There are 99 other participants in this study.

Consider the task completed by **the other participants** in Phase 1.

Question 1

Consider only trials where the number on the screen was **equal to 47**. In what percentage of these trials do you think **the other participants** gave a correct answer, that is, they correctly classified whether the number was smaller or larger than 55? Give us your forecast on a scale between 0% and 100%, where 0% means you believe no answer in these trials was correct and 100% means you believe all answers in these trials were correct. If your forecast is within plus or minus 1% of the true percentage, you will earn £0.5.

Question 2

Consider only trials where the number on the screen was **equal to 54**. In what percentage of these trials do you think **the other participants** gave a correct answer, that is, they correctly classified whether the number was smaller or larger than 55? Give us your forecast on a scale between 0% and 100%, where 0% means you believe no answer in these trials was correct and 100% means you believe all answers in these trials were correct. If your forecast is within plus or minus 1% of the true percentage, you will earn £0.5.

Press ENTER to confirm your answers.

Consider the task **you** completed in Phase 1.

Question 3

Consider only trials where the number on the screen was **between 52 and 58**. In what percentage of these trials do you think **you** gave a correct answer, that is, you correctly classified whether the number was smaller or larger than 55? Give us your forecast on a scale between 0% and 100% where 0% means you believe no answer in these trials was correct and 100% means you believe all answers in these trials were correct. If your forecast is within plus or minus 1% of your true accuracy, you will earn £0.5.

Question 4

Consider only trials where the number on the screen was **smaller than 52 or larger than 58**. In what percentage of these trials do you think **you** gave a correct answer, that is, you correctly classified whether the number was smaller or larger than 55? Give us your forecast on a scale between 0% and 100% where 0% means you believe no answer in these trials was correct and 100% means you believe all answers in these trials were correct. If your forecast is within plus or minus 1% of your true accuracy, you will earn £0.5.

Press ENTER to confirm your answers.