Seoul National Uiniversity

M1522.000900 Data Structure

Homework 1: Mathematical Preliminaries (Chapter 2)

Computer Science & Engineering

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Question 1

If an relation is an equivalence relation, it is reflexive, symmetric, and transitive.

(1) For integers a and b, a≡b if and if only a + b is even.

k, p, q, r are integers.

Reflexive: k+k=2k

Symmetric: p+q = q+p

Transtive: if p+q and q+r are even then p+r is even(if p, q, r is even number or not)

This relation is an equivalance relation.

(2) For integers a and b, a≡b if and if only a + b is odd.

Not Reflexive: k+k=2k(k is an integer). 2k is even.

This relation is not an equivalance relation.

(3) For nonzero rational numbers a and b, a≡b if and only if a * b >0 .

p, q, r are nonzero rational numbers

Reflexive: p*p>0

Symmetric: p*q = q*p

Transtive: if p*q>0 and q*r>0, then p*r>0(if p, q, r is positive or not)

This relation is an equivalance relation.

(4) For nonzero rational numbers a and b, a≡b if and only if a / b is an integer.

Reflexive: a/a=1

Not Symmetric: for example a=1 and b=2, a/b=1/2 and b/a=2

This relation is not an equivalance relation.

(5) For rational numbers a and b, a≡b if and only if a – b is an integer.

p, q, r are rational numbers.

Reflexive: p-p=0. 0 is an integer.

Symmetric: if p-q=k(k is an integer), then q-p=-k(integer)

Transtive: if p-q=k(k is an integer) and q-r=l(l is an integer), then p-r=k+l(integer)

This relation is an equivalance relation.

(6) For rational numbers a and b, a≡b if and only if |a – b| ≤ 2.

p, q, r are rational numbers.

Reflexive: |p-p|=0≤2

Symmetric: |p-q|=|q-p|

Not Transtive: for example p=4, q=2, r=0, |p-q|=2, |q-r|=2, but |p-r|=4(larger than 2)

This relation is not an equivalance relation.

Question 2

If an relation is a partial order, it is antisymmetric and transitive.

Let a, b, c are people.

(1) "is father of" on the set of people

Antisymmetric: "a is father of b" and "b is father of a" can't be true at the same time. Because assumption is false, it is antisymmetric.

Not Transitive: if a is father of b and b is father of c, then a is not father of c.

This relation is not a partial order.

(2) "is ancestor of" on the set of people

Antisymmetric: "a is ancestor of b" and "b is ancestor of a" can't be true at the same time. Because assumption is false, it is antisymmetric.

Transitive: if a is ancestor of b and b is ancestor of c, then a is ancestor of c.

This relation is a partial order.

(3) "is older than" on the set of people

Antisymmetric: "a is older than b" and "b is older than a" can't be true at the same time. Because assumption is false, it is antisymmetric.

Transitive: if a is older than b and b is older than c, then a is older than c.

This relation is a partial order.

(4) "is sister of" on the set of people

Not Antisymmetric: if a is sister of b and b is sister of a, then a and b is not same person.

Transitive: if a is sister of b and b is sister of c, then a is sister of c.

This relation is not a partial order.

(5) {〈a, b〉, 〈a, a〉, 〈b, a〉} on the set {a, b}

Not Antisymmetric: if aRb and bRa, then a and b are not equal.

This relation is not a partial order.

(6) {〈2, 1〉, 〈1, 3〉, 〈2, 3〉} on the set {1, 2, 3}

Antisymmetric: Because assumption is false, it is antisymmetric. (2R1 exists, but 1R2 does not. 1R3 exists, but 3R1 does not. 2R3 exists, but 3R2 does not.)

Transitive: if 2R1 and 1R3, then 2R3.

This relation is a partial order.

Question 3

(1) T(10) = 1023

(2) T(n) is the number of disk movements to finish the game with n disks. So T(n+1) is the number of disk movements to finish the game with n+1 disks. To move n+1 disks, first move n disks. The number of disks movements with n disks from leftmost pole to middle pole is T(n). The number of disk movement with largest disk from leftmost pole to rightmost pole is 1. The number of disks movements with n disks from middle pole to rightmost pole is T(n). As a result, T(n+1) = 2T(n)+1.

(3) T(n+1) = 2T(n)+1

T(n+1) – a = 2(T(n) – a)

T(n+1) = 2T(n) – a

a = -1

T(n+1) + 1 = 2(T(n) + 1)

$\frac{\mathrm{T(n+1)+1}}{\mathrm{T(n)+1}}$=2

T(n)+1 is a geometric sequence with common ratio 2.

T(n) + 1 = (T(1) + 1) * 2^{n-1}

T(n) + 1 = (1 + 1) * 2^{n-1}

T(n) = 2^{n}-1 : closed form solution.

Question 4

S(n) = $2S\left(\frac{n}{2}\right)+n$ for $n=2^{i}$, where i>0; S(1)=1

(1)

S(n) = 2S(n/2) + n that is equal to S(2^{i}) = 2S(2^{i-1}) + 2^{i} $(n=2^{i})$

S(2^{i}) = 2 S(2^{i-1}) + 2^{i}

= 2(2 S(2^{i-2}) + 2^{i-1}) + 2^{i}

$= 2^2S(2^{i-2}) + 2 * 2^i$

$= 2^2(2S(2^{i-3}) + 2^{i-2}) + 2 * 2^i$

$= 2^3S(2^{i-3}) + 3 * 2^i$

It can be

$S(2^i) = 2^iS(2^{i-i}) + i * 2^i$

$= 2^i + i * 2^i$

$= (i+1)2^i$

Because $n=2^i$, i is $\log_2 n$.

$S(n) = (\log_2 n+1)n$

(2)

$S(2) = 2S(1) + 2 = 4 = (\log_2 2+1)2$

(3)

The induction hypothesis is

$S(2^k) = 2S(2^{k-1}) + 2^k = (k+1)2^k$ $(k>0)$ $(n = 2^k)$

The induction hypothesis states that $S(2^k) = (k+1)2^k$, and because $S(2^k) = 2S(2^{k-1}) + 2^k$, we can substitute for $S(2^k)$ to get

$S(2^{k+1}) = 2S(2^k) + 2^{k+1}$

$= 2(k+1) 2^k + 2^{k+1}$

$= (k+1) 2^{k+1} + 2^{k+1}$

$= (k+2) 2^{k+1}$

Thus, by mathematical induction,

$S(n) = (\log_2 n+1)n$

Question 5

T(n) = $\begin{cases} 2 & if\ n = 2 \\ 2T\left(\frac{n}{2}\right) + n & if\ n = 2^k, for\ k > 1 \end{cases}$

To prove by mathematical induction, go through the following steps.

1 Check the base case. The base case of n=2 is true. T(2) = 2.

2 State the induction hypothesis. The induction hypothesis is

T(n/2) = 2T(n/2^2) + n/2 = n lg n that is equal to T(2^{i-1}) = 2T(2^{i-2}) + 2^{i-1} = (i-1)*2^{i-1} (i>2) ($n = 2^i$)

3 Use the assumption from the induction hypothesis n = 2^{i-1} to show that the result is true for n = 2^i. Combining the definition of the recurrence with the induction hypothesis we see immediately that

T(n) = 2T(n/2) + n that is equal to T(2^i) = 2T(2^{i-1}) + 2^i ($n = 2^i$)

T(2^i) = 2T(2^{i-1}) + 2^i

= 2 (i-1)*2^{i-1} + 2^i

= (i-1)*2^i + 2^i

= i*2^i

Because n=2^i, i is lg n.

T(n) = n lg n.