

Ch2. Gaussian Processes (Part 1)

Bayesian Optimization Seminar

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1. Review of Chapter 1: Introduction
2. Definition and Basic Properties
3. Inference with Exact and Noisy Observations
4. Joint Gaussian Processes
5. Summary

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The Optimization Problem

Goal

Find the global optimum of an objective function $f : \mathcal{X} \rightarrow \mathbb{R}$:

$$x^* = \arg \max_{x \in \mathcal{X}} f(x); \quad f^* = \max_{x \in \mathcal{X}} f(x)$$

Setting:

- Objective function f is **expensive** to evaluate (time, cost, resources)
- We can only access f through **sequential observations**
- Observations may be **noisy**: $y = f(x) + \varepsilon$

Challenge: How do we decide where to observe next, given limited budget?

Observation Model

Observations are realized by a stochastic mechanism:

$$p(y \mid x, \phi), \quad \text{where } \phi = f(x)$$

Common model — Additive Gaussian noise:

$$y = \phi + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, \sigma_n^2)$$

$$\Rightarrow p(y \mid x, \phi, \sigma_n) = \mathcal{N}(y; \phi, \sigma_n^2)$$

Assumption: Multiple observations are **conditionally independent** given the objective function values:

$$p(\mathbf{y} \mid \mathbf{x}, \phi) = \prod_i p(y_i \mid x_i, \phi_i)$$

The Bayesian Approach: Key Idea

Core Principle

Treat the unknown objective function f as a **random variable** and use **Bayesian inference** to reason about it.

Bayesian Inference Refresher:

1. Start with a **prior** $p(\phi \mid x)$ encoding initial beliefs
2. Observe data $\mathcal{D} = (x, y)$ via the **likelihood** $p(y \mid x, \phi)$
3. Update to the **posterior** via Bayes' rule:

$$p(\phi \mid \mathcal{D}) = \frac{p(y \mid x, \phi) p(\phi \mid x)}{p(y \mid x)}$$

The posterior captures what we now believe about ϕ after seeing the data.

Inference of the Objective Function

To reason about the *entire* objective function $f : \mathcal{X} \rightarrow \mathbb{R}$, we need a **stochastic process** — a probability distribution over functions.

Specifying a Stochastic Process

We specify the distribution of function values $\phi = f(\mathbf{x})$ for any finite set of locations $\mathbf{x} \subset \mathcal{X}$:

$$p(\phi \mid \mathbf{x})$$

Gaussian Processes: The family where all such finite-dimensional distributions are **multivariate Gaussian** — mathematically convenient and widely used in Bayesian optimization.

Posterior Predictive Distribution

After observing data \mathcal{D} , we can predict the outcome of a new observation at x :

Posterior Predictive Distribution

$$p(y' \mid x, \mathcal{D}) = \int p(y' \mid x, \phi) p(\phi \mid x, \mathcal{D}) d\phi$$

Interpretation:

- Integrates over all possible values of $\phi = f(x)$
- Weights by their plausibility under the posterior
- Naturally accounts for **uncertainty** in the objective function

This distribution is *instrumental* for making informed decisions about where to observe next.

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What is a Gaussian Process?

A **Gaussian process (GP)** extends the multivariate normal distribution to model functions on infinite domains.

Key Idea

We model an objective function $f : \mathcal{X} \rightarrow \mathbb{R}$ as an infinite collection of random variables, one for each point in the domain. The **Kolmogorov extension theorem** allows us to specify this distribution through finite-dimensional marginals.

GPs inherit convenient mathematical properties of the multivariate normal distribution while remaining computationally tractable.

Recall: Kolmogorov's Extension Theorem

Question

What is the *consistency* property in Kolmogorov's Extension Theorem?

Kolmogorov's Extension Theorem [Durrett, 2019]

Suppose we are given probability measures μ_n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ that are *consistent*, that is,

$$\mu_{n+1}((a_1, b_1] \times \cdots \times (a_{n+1}, b_{n+1}] \times \mathbf{R}) = \mu_n((a_1, b_1] \times \cdots \times (a_n, b_n])$$

Then there is a unique probability measure P on $(\mathbb{R}^{\mathbf{N}}, \mathcal{B}(\mathbb{R}^{\mathbf{N}}))$ with

$$P(\omega : \omega_i \in (a_i, b_i], 1 \leq i \leq n) = \mu_n((a_1, b_1] \times \cdots \times (a_n, b_n])$$

where $\mathbf{N} = \{1, 2, \dots\}$ and $\mathcal{B}(\mathbb{R}^{\mathbf{N}}) = \{(\omega_1, \omega_2, \dots) : \omega_i \in \mathcal{B}(\mathbb{R})\}$

In [Durrett, 2019], theorem is not complete to use in our senario because P defines on countable index set \mathbf{N} .

Q&A: Countable vs. Uncountable Domains

Question

What are the characteristics of modeling on countable vs uncountable domains?

Definition: Sample path [Le Gall, 2016]

Let $(X_t)_{t \in \mathcal{X}}$ be a random process with values in E . The *sample paths* of X are the mappings $\mathcal{X} \ni t \mapsto X_t(\omega)$ obtained when fixing $\omega \in \Omega$. The sample paths of X is thus form a collection of mappings from \mathcal{X} into E indexed by $\omega \in \Omega$.

Countable infinite domain (e.g., $\mathcal{X} = \mathbb{Z}$):

- GP reduces to specifying consistent MVN distributions on all finite subsets
- Sample paths are sequences $\{f(x_i)\}_{i=1}^{\infty}$
- No notion of “continuity” in the classical sense

Q&A: Countable vs. Uncountable Domains

Question

What are the characteristics of modeling on countable vs uncountable domains?

Definition: Sample path [Le Gall, 2016]

Let $(X_t)_{t \in \mathcal{X}}$ be a random process with values in E . The *sample paths* of X are the mappings $\mathcal{X} \ni t \mapsto X_t(\omega)$ obtained when fixing $\omega \in \Omega$. The sample paths of X is thus form a collection of mappings from \mathcal{X} into E indexed by $\omega \in \Omega$.

Uncountable infinite domain (e.g., $\mathcal{X} = \mathbb{R}^d$):

- Sample paths are actual functions
- Continuity, differentiability become meaningful properties
- Requires careful treatment (measurability issues, sample path properties)

Kolmogorov Extension Theorem: The Foundation

Question

What is the *consistency* property in Kolmogorov's Extension Theorem?

Kolmogorov's Extension Theorem : Stochastic process ver. [Oksendal, 2003]

For all $t_1, \dots, t_k \in T$, $k \in \mathbb{N}$ let ν_{t_1, \dots, t_k} be probability measures on \mathbb{R}^{nk} s.t.

$$\nu_{t_{\sigma(1)}, \dots, t_{\sigma(k)}}(F_1 \times \dots \times F_k) = \nu_{t_1, \dots, t_k}(F_{\sigma^{-1}(1)} \times \dots \times F_{\sigma^{-1}(k)}) \quad (\text{K1})$$

for all permutations σ on $\{1, 2, \dots, k\}$ and

$$\nu_{t_1, \dots, t_k}(F_1 \times \dots \times F_k) = \nu_{t_1, \dots, t_k, t_{k+1}, \dots, t_{k+m}}(F_1 \times \dots \times F_k \times \mathbb{R}^n \times \dots \times \mathbb{R}^n) \quad (\text{K2})$$

for all $m \in \mathbb{N}$.

Kolmogorov Extension Theorem: The Foundation

Kolmogorov's Extension Theorem : Stochastic process ver. (cont.) [Oksendal, 2003]

Then there exists a probability space (Ω, \mathcal{F}, P) and a stochastic process $\{X_t\}$ on Ω , $X_t : \Omega \rightarrow \mathbb{R}^n$, s.t.

$$\nu_{t_1, \dots, t_k}(F_1 \times \dots \times F_k) = P[X_{t_1} \in F_1, \dots, X_{t_k} \in F_k],$$

for all $t_i \in T$, $k \in \mathbb{N}$ and all Borel sets F_i .

Consistency Conditions:

1. **Permutation invariance** (K1): The joint distribution is unchanged by reordering indices
2. **Marginalization consistency** (K2): If $\mathbf{x} \subset \mathbf{x}'$, then marginalizing $p(\phi' \mid \mathbf{x}')$ over $\phi' \setminus \phi$ yields $p(\phi \mid \mathbf{x})$

For GPs, these are automatically satisfied because the MVN satisfies them (the marginal of a MVN is MVN with the corresponding submatrix of the covariance).

GP Specification: Mean and Covariance Functions

A GP on f is specified by:

$$p(f) = \mathcal{GP}(f; \mu, K)$$

- **Mean function** $\mu : \mathcal{X} \rightarrow \mathbb{R}$: determines the expected function value

$$\mu(x) = \mathbb{E}[\phi \mid x], \quad \text{where } \phi = f(x)$$

- **Covariance function (kernel)** $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$: encodes the correlation structure

$$K(x, x') = \text{cov}[\phi, \phi' \mid x, x'], \quad \text{where } \phi' = f(x')$$

The covariance function must be **symmetric** and **positive semidefinite**.

Q&A: Extension from Countable Dense Subsets

Question

If f is a.s. continuous and $D \subset \mathcal{X}$ is countable dense, can $p(f \mid D)$ determine $p(f)$?

Proof.

Let f and g be two continuous functions on \mathcal{X} . Suppose $f(x) = g(x)$ for $\forall x \in D$. For any $x \in \mathcal{X}$, there exists a sequence $\{q_n\} \subset D$ such that $q_n \rightarrow x$. ($\because D$ is dense in \mathcal{X}) By continuity,

$$f(x) = f\left(\lim_{n \rightarrow \infty} q_n\right) = \lim_{n \rightarrow \infty} f(q_n) = \lim_{n \rightarrow \infty} g(q_n) = g\left(\lim_{n \rightarrow \infty} q_n\right) = g(x)$$

Thus, $f(x) = g(x), \forall x \in \mathcal{X}$. Since sample paths are a.s. continuous, the process is uniquely determined by its values on D . □

Finite-Dimensional Marginals

For any finite set of points $\mathbf{x} \subset \mathcal{X}$, the corresponding function values $\phi = f(\mathbf{x})$ follow a multivariate normal distribution:

$$p(\phi \mid \mathbf{x}) = \mathcal{N}(\phi; \boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where

$$\boldsymbol{\mu} = \mathbb{E}[\phi \mid \mathbf{x}] = \mu(\mathbf{x}); \quad \boldsymbol{\Sigma} = \text{cov}[\phi \mid \mathbf{x}] = K(\mathbf{x}, \mathbf{x})$$

Gram Matrix

$K(\mathbf{x}, \mathbf{x})$ is the matrix formed by evaluating K for each pair of points:

$$\Sigma_{ij} = K(x_i, x_j)$$

Example: Squared Exponential Covariance

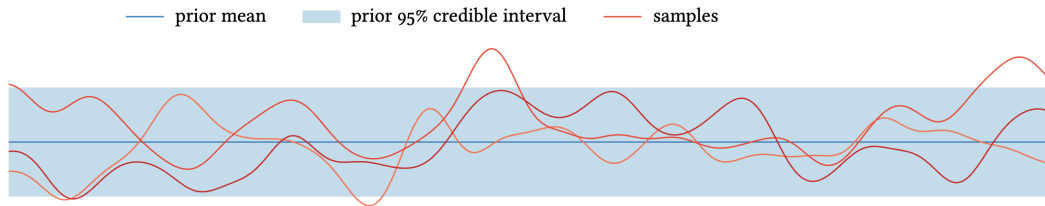


Figure 1: Example of Gaussian Process [Garnett, 2023]

Consider $\mathcal{X} = [0, 30]$ with:

- Mean function: $\mu \equiv 0$ (constant central tendency)
- Covariance function (squared exponential):

$$K(x, x') = \exp\left(-\frac{1}{2}|x - x'|^2\right)$$

Example: Squared Exponential Covariance

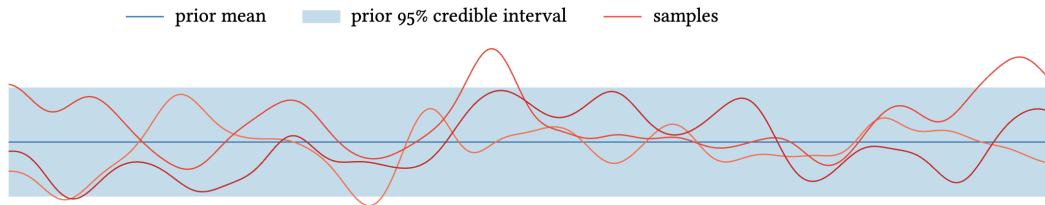


Figure 1: Example of Gaussian Process [Garnett, 2023]

Properties:

- $\text{var}[\phi \mid x] = K(x, x) = 1$ at every point
- Correlation decreases with distance: nearby values are highly correlated, distant values are nearly independent
- This encodes a statistical notion of **continuity**

Sampling from a Gaussian Process (Appendix A.2)

To sample from a GP with mean μ and covariance K :

1. Choose a finite grid of points $\mathbf{x} = (x_1, \dots, x_n)$
2. Compute $\boldsymbol{\mu} = \mu(\mathbf{x})$ and $\boldsymbol{\Sigma} = K(\mathbf{x}, \mathbf{x})$
3. Factor: $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^\top$ (Cholesky decomposition)
4. Sample $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
5. Compute $\boldsymbol{\phi} = \boldsymbol{\mu} + \mathbf{L}\mathbf{z}$

The resulting sample $\boldsymbol{\phi}$ represents function values at the chosen grid points, respecting the correlation structure encoded by K .

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General Framework: Jointly Gaussian Observations

We can condition a GP $p(f) = \mathcal{GP}(f; \mu, K)$ on any vector \mathbf{y} sharing a joint Gaussian distribution with f :

$$p(f, \mathbf{y}) = \mathcal{GP} \left(\begin{bmatrix} f \\ \mathbf{y} \end{bmatrix}; \begin{bmatrix} \mu \\ \mathbf{m} \end{bmatrix}, \begin{bmatrix} K & \boldsymbol{\kappa}^\top \\ \boldsymbol{\kappa} & \mathbf{C} \end{bmatrix} \right)$$

where:

- $p(\mathbf{y}) = \mathcal{N}(\mathbf{y}; \mathbf{m}, \mathbf{C})$: marginal distribution of observations
- $\boldsymbol{\kappa}(x) = \text{cov}[\mathbf{y}, \phi \mid x]$: cross-covariance function

Q&A: What Observations \mathbf{y} Are Jointly Gaussian with f ?

Question

Besides function values, what else can \mathbf{y} be?

Any of the following are jointly Gaussian with a GP:

- Function values: $\mathbf{y} = f(\mathbf{x})$ (the basic case)
- Affine transformations: $\mathbf{y} = \mathbf{A}f(\mathbf{x}) + \mathbf{b}$
- Limits of affine transformations:
 - Partial derivatives: $\frac{\partial f}{\partial x_i}(x)$
 - Integrals/expectations: $\int f(x)p(x)dx$
- Any of the above + independent Gaussian noise

Q&A: What Observations \mathbf{y} Are Jointly Gaussian with f ?

Question

Besides function values, what else can \mathbf{y} be?

Other examples: Based on linear (affine) operator, these are joint Gaussian distribution.

- Line/Path Integral Observations [Särkkä, 2011]

$$\mathbf{y} = \int_{\gamma} f(x) ds + \varepsilon, \quad \kappa(x') = \int_{\gamma} K(x, x') ds$$

Used at Tomographic reconstruction (CT/MRI), Measure expected path of moving sensor

- Convolved/Smoothed Observations by smoothing kernel G [Alvarez et al., 2011]

$$\mathbf{y} = (f * G)(x_0) = \int f(x') G(x_0 - x') dx', \quad \kappa(x) = \int G(x_0 - x') K(x', x) dx'$$

Used at Sensor spatial averaging, image blur (PSF), multi-output GP via convolution

Posterior Gaussian Process

Conditioning on observations $\mathcal{D} = \mathbf{y}$ yields a GP posterior:

$$p(f \mid \mathcal{D}) = \mathcal{GP}(f; \mu_{\mathcal{D}}, K_{\mathcal{D}})$$

Posterior Mean and Covariance

$$\mu_{\mathcal{D}}(x) = \mu(x) + \kappa(x)^{\top} \mathbf{C}^{-1}(\mathbf{y} - \mathbf{m})$$

$$K_{\mathcal{D}}(x, x') = K(x, x') - \kappa(x)^{\top} \mathbf{C}^{-1} \kappa(x')$$

Inference procedure:

1. Compute marginal distribution of \mathbf{y}
2. Derive cross-covariance function κ
3. Apply the posterior formulas

Handling Additive Gaussian Noise

Suppose we observe $\mathbf{z} = \mathbf{y} + \varepsilon$ where $\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{N})$ is independent noise.

Then:

$$p(\mathbf{z} \mid \mathbf{N}) = \mathcal{N}(\mathbf{z}; \mathbf{m}, \mathbf{C} + \mathbf{N}); \quad \text{cov}[\mathbf{z}, \phi \mid x] = \kappa(x)$$

Key Result

Simply replace \mathbf{C} with $\mathbf{C} + \mathbf{N}$ in the posterior formulas!

As $\mathbf{N} \rightarrow \mathbf{0}$, the posterior converges to that from direct observation of \mathbf{y} .

Inference with Exact Function Evaluations

Suppose we observe f at locations \mathbf{x} , revealing $\phi = f(\mathbf{x})$. We know that $p(\phi|\mathbf{x}) = \mathcal{N}(\phi; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ so $\kappa(x) = \text{cov}[\phi, \phi|\mathbf{x}, x]$.

The posterior is $p(f | \mathcal{D}) = \mathcal{GP}(f; \mu_{\mathcal{D}}, K_{\mathcal{D}})$ with:

$$\begin{aligned}\mu_{\mathcal{D}}(x) &= \mu(x) + K(x, \mathbf{x})\boldsymbol{\Sigma}^{-1}(\phi - \boldsymbol{\mu}) \\ K_{\mathcal{D}}(x, x') &= K(x, x') - K(x, \mathbf{x})\boldsymbol{\Sigma}^{-1}K(\mathbf{x}, x')\end{aligned}$$

where $\boldsymbol{\Sigma} = K(\mathbf{x}, \mathbf{x})$ and $\boldsymbol{\mu} = \mu(\mathbf{x})$.

Key properties:

- Posterior mean **interpolates** through observed points
- Posterior variance **vanishes** at observed locations
- Uncertainty remains unchanged far from observations

Inference with Noisy Function Evaluations

Suppose observations are corrupted: $\mathbf{y} = \phi + \varepsilon$ with $\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{N})$.

Common noise models:

- **Homoskedastic:** $\mathbf{N} = \sigma_n^2 \mathbf{I}$ (constant noise)
- **Heteroskedastic:** $\mathbf{N} = \text{diag}(\sigma_n^2(\mathbf{x}))$ (location-dependent)

The posterior formulas become:

$$\begin{aligned}\mu_{\mathcal{D}}(x) &= \mu(x) + K(x, \mathbf{x})(\Sigma + \mathbf{N})^{-1}(\mathbf{y} - \mu) \\ K_{\mathcal{D}}(x, x') &= K(x, x') - K(x, \mathbf{x})(\Sigma + \mathbf{N})^{-1}K(\mathbf{x}, x')\end{aligned}$$

The posterior mean no longer interpolates exactly; extreme values may be “explained away” as noise.

Inference with Noisy Function Evaluations

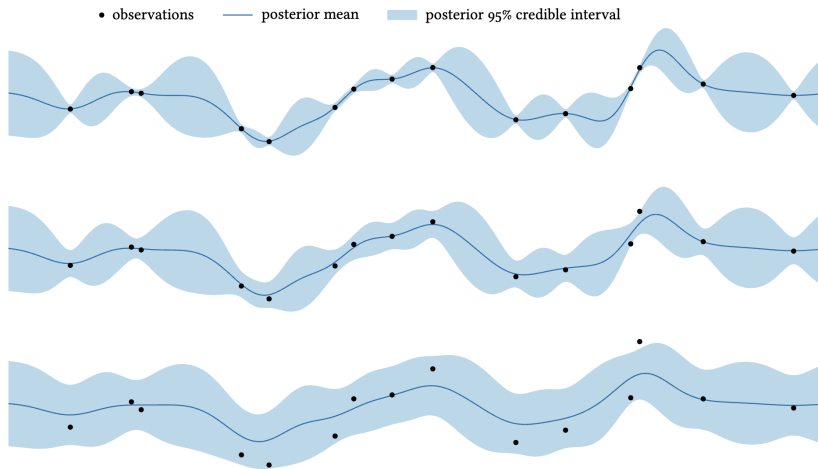


Figure 2: Conditioning GP on data corrupted by increasing levels of homoskedastic noise [Garnett, 2023]

Interpretation of Posterior Moments

Consider a single observation y with distribution $\mathcal{N}(y; m, s^2)$ and define:

- z-score: $z = \frac{y-m}{s}$
- Correlation: $\rho = \text{corr}[y, \phi \mid x] = \frac{\kappa(x)}{\sigma s}$

Posterior Moments (Scalar Case)

Posterior mean of ϕ : $\mu + \sigma \rho z$

Posterior std of ϕ : $\sigma \sqrt{1 - \rho^2}$

Intuition:

- Mean shifts proportionally to z-score and correlation strength
- Variance reduction depends only on correlation $|\rho|$

Interpretation of Posterior Moments: Vector Case

For vector-valued observations \mathbf{y} , factor the covariance as $C = SPS$ where:

- S : diagonal with $S_{ii} = \sqrt{C_{ii}} = \text{std}[y_i]$
- $P = \text{corr}[\mathbf{y}]$: observation correlation matrix

Define vectors of z-scores and cross-correlations:

$$z_i = \frac{y_i - m_i}{s_i}, \quad \rho_i = \frac{[\kappa(x)]_i}{\sigma s_i}$$

Posterior Moments (Vector Case)

Posterior mean of ϕ : $\mu + \sigma \boldsymbol{\rho}^\top P^{-1} \mathbf{z}$

Posterior std of ϕ : $\sigma \sqrt{1 - \boldsymbol{\rho}^\top P^{-1} \boldsymbol{\rho}}$

Q&A: Differential Entropy as Global Uncertainty

Question

Why is differential entropy a measure of global uncertainty?

Definition: Differential Entropy

For a random variable ω with density $p(\omega)$, *differential entropy* is defined as:

$$H[\omega] = - \int p(\omega) \log p(\omega) d\omega$$

For Gaussian distributions:

- Univariate: $H[\mathcal{N}(\mu, \sigma^2)] = \frac{1}{2} \log(2\pi e \sigma^2)$
- Multivariate: $H[\mathcal{N}(\mu, \Sigma)] = \frac{1}{2} \log |2\pi e \Sigma|$

For GPs: The joint entropy of function values at locations \mathbf{x} is:

$$H[\phi \mid \mathbf{x}] = \frac{1}{2} \log |2\pi e K(\mathbf{x}, \mathbf{x})|$$

Q&A: Why Differential Entropy Measures Global Uncertainty

Key insight: Entropy depends on the **determinant of the covariance matrix**.

Why “Global”?

- Marginal variance $\sigma_i^2 = \Sigma_{ii}$: uncertainty of **individual** variables
- Determinant $|\Sigma|$: **joint** uncertainty of **all** variables together
- Also reflects correlations: high correlation \Rightarrow smaller $|\Sigma|$

Q&A: Entropy Reduction in the Posterior

Prior vs Posterior Covariance:

$$\text{Prior : } K(\mathbf{x}, \mathbf{x})$$

$$\text{Posterior : } K_{\mathcal{D}}(\mathbf{x}, \mathbf{x}) = K(\mathbf{x}, \mathbf{x}) - \kappa(\mathbf{x})^{\top} C^{-1} \kappa(\mathbf{x})$$

Global Uncertainty Reduction

For positive semidefinite matrices A, B : $|A| \leq |A + B|$, hence:

$$|K_{\mathcal{D}}(\mathbf{x}, \mathbf{x})| \leq |K(\mathbf{x}, \mathbf{x})|$$

Therefore, posterior entropy is **always** \leq **prior entropy**.

Interpretation:

- Independent observations ($\rho = 0$): no entropy change
- More precise observations (larger C^{-1}): greater entropy reduction \Rightarrow more information gained

Posterior Predictive Distribution

For the latent function value $\phi = f(x)$:

$$p(\phi \mid x, \mathcal{D}) = \mathcal{N}(\phi; \mu_{\mathcal{D}}(x), K_{\mathcal{D}}(x, x))$$

For a **noisy observation** y at location x (with noise variance σ_n^2):

$$p(y \mid x, \mathcal{D}, \sigma_n) = \mathcal{N}(y; \mu_{\mathcal{D}}(x), K_{\mathcal{D}}(x, x) + \sigma_n^2)$$

The predictive credible intervals for noisy measurements are inflated compared to the latent function, reflecting observation uncertainty.

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Topics Covered in Remaining Sections

The remainder of Chapter 2 covers more specialized topics:

- **§2.4 Joint Gaussian Processes:** Modeling multiple correlated functions
- **§2.5 Continuity:** Conditions for continuous sample paths
- **§2.6 Differentiability:** Conditions for differentiable sample paths; derivative observations
- **§2.7 Existence/Uniqueness of Global Maxima:** Theoretical guarantees
- **§2.8 Non-Gaussian Observations:** Approximate inference methods

Motivation: Modeling Multiple Functions

In some settings, we need to jointly reason about **multiple related functions**:

- An objective function and its gradient
- An expensive objective and cheaper surrogates (multifidelity)
- Multiple objectives (multiobjective optimization)

Key Idea

“Paste together” multiple functions into a single function on a larger domain, then construct a standard GP on this combined function.

Definition of Joint Gaussian Process

Consider functions $\{f_i : \mathcal{X}_i \rightarrow \mathbb{R}\}$. Define the **disjoint union**:

$$\bigsqcup f : \mathcal{X} \rightarrow \mathbb{R}, \quad \mathcal{X} = \bigsqcup \mathcal{X}_i$$

such that $\bigsqcup f|_{\mathcal{X}_i} \equiv f_i$.

A **joint Gaussian process** is a GP on $\bigsqcup f$:

$$p(\bigsqcup f) = \mathcal{GP}(\bigsqcup f; \mu, K)$$

The mean and covariance functions on \mathcal{X} encode both:

- Marginal behavior of each function
- Cross-correlations between functions

Decomposed Notation

For two functions $f : \mathcal{F} \rightarrow \mathbb{R}$ and $g : \mathcal{G} \rightarrow \mathbb{R}$:

$$p(f, g) = \mathcal{GP} \left(\begin{bmatrix} f \\ g \end{bmatrix} ; \begin{bmatrix} \mu_f \\ \mu_g \end{bmatrix}, \begin{bmatrix} K_f & K_{fg} \\ K_{gf} & K_g \end{bmatrix} \right)$$

Components:

- μ_f, K_f and μ_g, K_g : marginal GP parameters
- $K_{fg}(x, x') = \text{cov}[\phi, \gamma \mid x, x']$: cross-covariance
- $K_{gf} = K_{fg}^\top$

Marginal property: Each function has a marginal GP distribution:

$$p(f) = \mathcal{GP}(f; \mu_f, K_f); \quad p(g) = \mathcal{GP}(g; \mu_g, K_g)$$

Q&A: Designing Cross-Covariance Functions

Question

How do we choose the cross-covariance function K_{fg} ?

Common approaches:

1. **Scaled base kernel:** $K_{fg}(x, x') = \rho \cdot K(x, x')$ where $|\rho| < 1$
 - Simple, interpretable: ρ is the correlation at any point
2. **Linear Model of Coregionalization (LMC)** [Alvarez et al., 2011]:

$$K_{fg}(x, x') = \sum_{q=1}^Q a_q^{(f)} a_q^{(g)} K_q(x, x')$$

3. **Convolution processes:** Define via convolution with a smoothing kernel

Constraint: The full covariance matrix must remain positive semidefinite!

Example: Correlated Functions

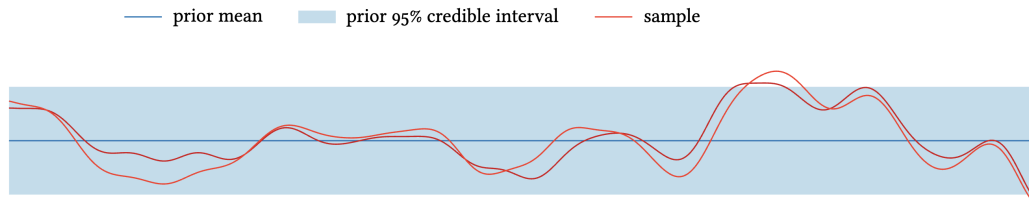


Figure 3: Example of Joint Gaussian Process [Garnett, 2023]

Consider $f, g : [0, 30] \rightarrow \mathbb{R}$ with:

- Same marginal: $\mu \equiv 0$, squared exponential covariance K
- Cross-covariance: $K_{fg}(x, x') = 0.9 \cdot K(x, x')$

Example: Correlated Functions

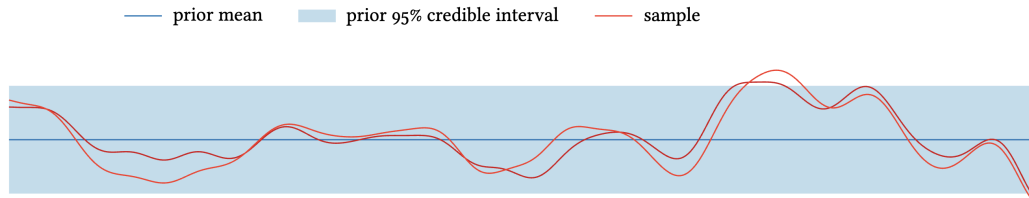


Figure 3: Example of Joint Gaussian Process [Garnett, 2023]

For any point x , the correlation between $\phi = f(x)$ and $\gamma = g(x)$ is:

$$\text{corr}[\phi, \gamma \mid x] = 0.9$$

Consequence: Samples from the joint distribution show strong coupling, the functions “move together.”

Q&A: Why Disjoint Union for Joint GP Domains?

Question

Is disjoint union the standard way to define joint GP domains?

Yes, mathematically this is the cleanest approach:

$$\bigsqcup f : \mathcal{X} = \bigsqcup_i \mathcal{X}_i \rightarrow \mathbb{R}, \quad \text{where } (\bigsqcup f)|_{\mathcal{X}_i} \equiv f_i$$

Why disjoint union?

- Ensures each point (x, i) uniquely identifies *which* function
- In previous example, if $\mathcal{X}_1 = \mathcal{X}_2 = [0, 30]$, we distinguish $f(10)$ from $g(10)$
 - 엄밀하게는 multi-output GP
- The kernel can then define both within-function and cross-function covariance

Q&A: Domain Size and Function Influence in Joint GPs

Question

Can domain size differences affect function influence in joint GPs?

Yes, this is a real concern in practice!

- Function with more observations may dominate inference
- Different scales of domains may require different length scales
- Numerical conditioning can suffer from imbalanced data

Mitigation strategies:

1. **Output scaling:** Normalize each function to similar variance
2. **Weighted likelihoods:** Down-weight abundant data sources
3. **Hierarchical priors:** Place priors on correlation parameters

Inference for Joint GPs

The joint GP construction allows us to condition on observations of **any** of the functions using the standard inference procedure.

Examples

Given observations of f on the left side of the domain and observations of g on the right side:

- Observations of f inform our belief about g (and vice versa)
- Information propagates through the cross-covariance structure
- Strong correlation \Rightarrow strong information transfer

This is particularly useful for **multifidelity optimization**: cheap surrogate evaluations inform our belief about the expensive objective.

Contents

1. Review of Chapter 1: Introduction
2. Definition and Basic Properties
3. Inference with Exact and Noisy Observations
4. Joint Gaussian Processes
5. Summary

Summary of Key Ideas

1. **GP Definition:** Specified by mean μ and covariance K functions; finite marginals are multivariate Gaussian
2. **Exact Inference:** Conditioning on jointly Gaussian observations yields a GP posterior with closed-form mean and covariance
3. **Noisy Inference:** Replace \mathbf{C} with $\mathbf{C} + \mathbf{N}$ to handle additive Gaussian noise
4. **Posterior Interpretation:** Mean update \propto (correlation \times z-score); variance reduction depends on correlation strength
5. **Joint GPs:** Model multiple correlated functions; enable information sharing across related tasks

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Thank You