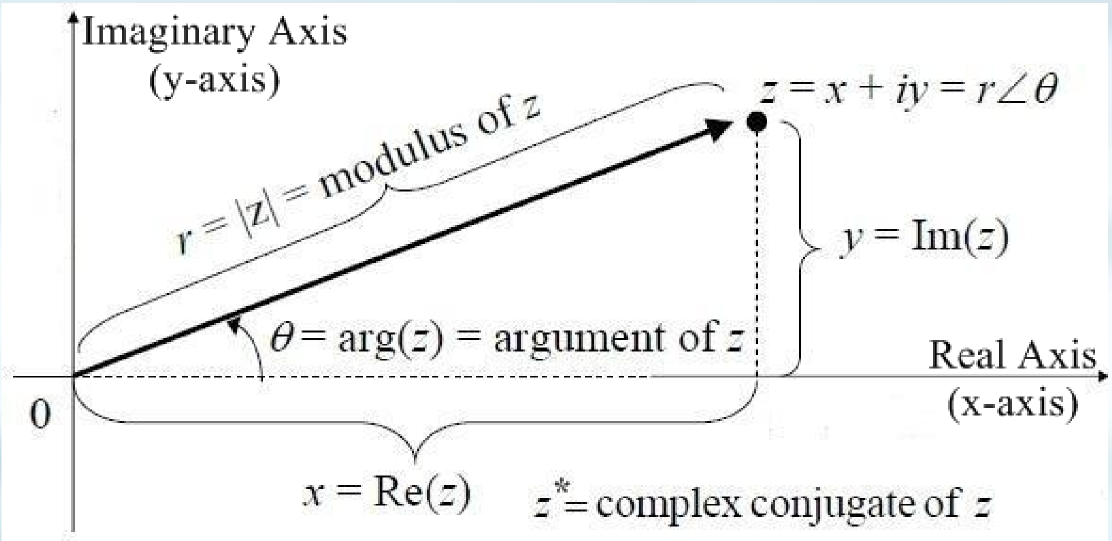
COMPLEX NUMBERS

KEY TOPICS:

- ❖ DEFINITION OF A COMPLEX NUMBER
- ❖ REPRESENTATION OF A COMPLEX NUMBER IN POLAR FORM
- ❖ COMPLEX CONJUGATE
- ❖ EULER'S FORMULA AND DE MOIVRE'S THEOREM
- ❖ ALGEBRAIC RULES FOR COMPLEX NUMBERS

DEFINITION OF A COMPLEX NUMBER



A Complex Number z is denoted as:

$z = x + jy$ **in which:**

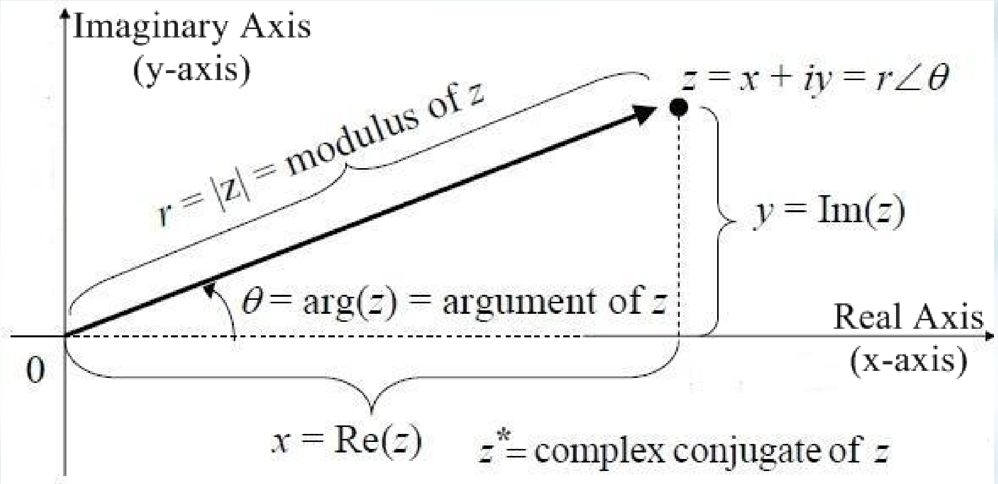
x **is called the REAL PART of z, often written as** $x = \mathrm{Re}(z)$

y **is called the IMAGINARY PART of z, often written as** $y = \mathrm{Im}(z)$

$j = \sqrt{-1}$

x**-axis is called the REAL AXIS and y-axis is called the IMAGINARY AXIS.**

REPRESENTATION OF A COMPLEX NUMBER IN POLAR FORM



A complex number $z = x + jy$ **can be expressed in Polar form as:**

$z = r(\cos\theta + j\sin\theta)$ **in which:**

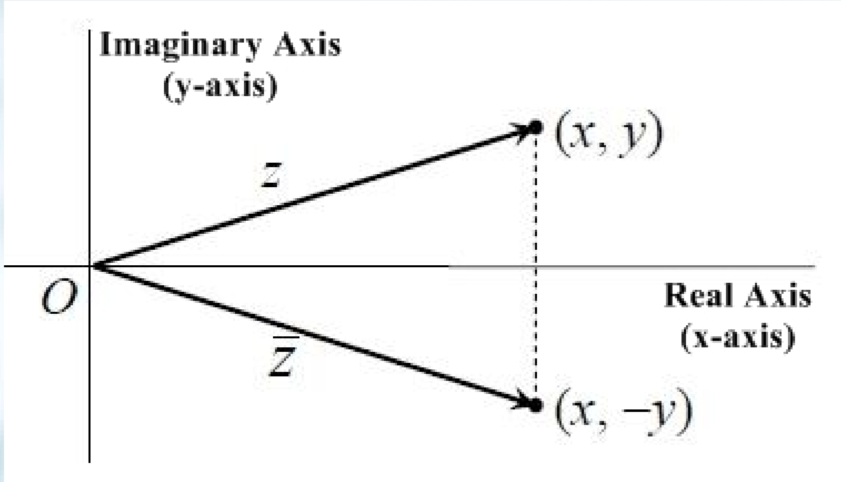
θ **is the ARGUMENT of z, often written as:** $\theta = \arg(z)$

r **is the MODULUS of** z**, often written as:** $r = |z|$

It can be seen from the figure that:

$$x = r\cos\theta,\ y = r\sin\theta \qquad r = |z| = \sqrt{x^2 + y^2} = |z^*| \qquad \theta = \arg(z) = \tan^{-1}\frac{y}{x}\,rad$$

COMPLEX CONJUGATE



The COMPLEX CONJUGATE of $z = x + jy$ is denoted as $\bar{z}$ or z^* and is defined as

$$\bar{z} = z^* = x - jy$$

The COMPLEX CONJUGATE of $z = r(\cos\theta + j\sin\theta)$ is defined as

$$\bar{z} = z^* = r(\cos\theta - j\sin\theta)$$

In Engineering Electromagnetics, notation z^* is used instead of $\bar{z}$.

NOTES on COMPLEX CONJUGATE:

1. $\mathrm{Re}(z) = \dfrac{1}{2}(z + z^*)$

2. $\mathrm{Im}(z) = \dfrac{1}{2i}(z - z^*)$

3. $zz^* = x^2 + y^2 = |z|^2$

4. $\dfrac{z_1}{z_2} = \dfrac{z_1(z_2{}^*)}{|z_2|^2}$

Examples:

Q.1. Prove that $z = x + jy = r(\cos\theta + j\sin\theta)$

Solution:

$$z = x + jy$$

$$z = r\cos\theta + jr\sin\theta$$

$$z = r(\cos\theta + j\sin\theta)$$

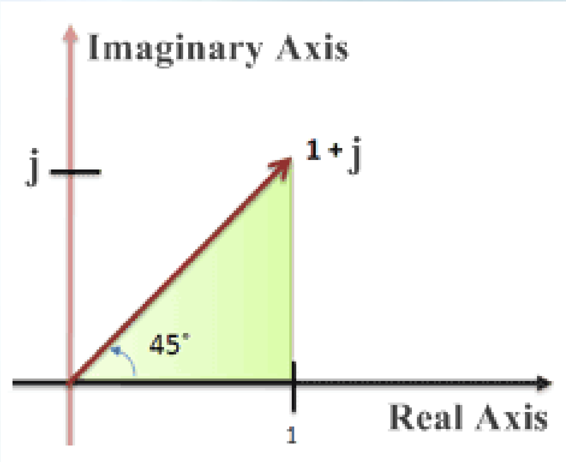
$$\therefore z = x + jy = r(\cos\theta + j\sin\theta)$$

Q.2. Let $z = 1 + j$

a. Express z **in Polar form**

b. Express the complex conjugate of z **in** $x + jy$ **form and Polar form**

Solution:



a. $r = |z| = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{1}{1} = \dfrac{\pi}{4} rad$

Polar form of z is:

$$z = r(\cos\theta + j\sin\theta) = \sqrt{2}(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4})$$

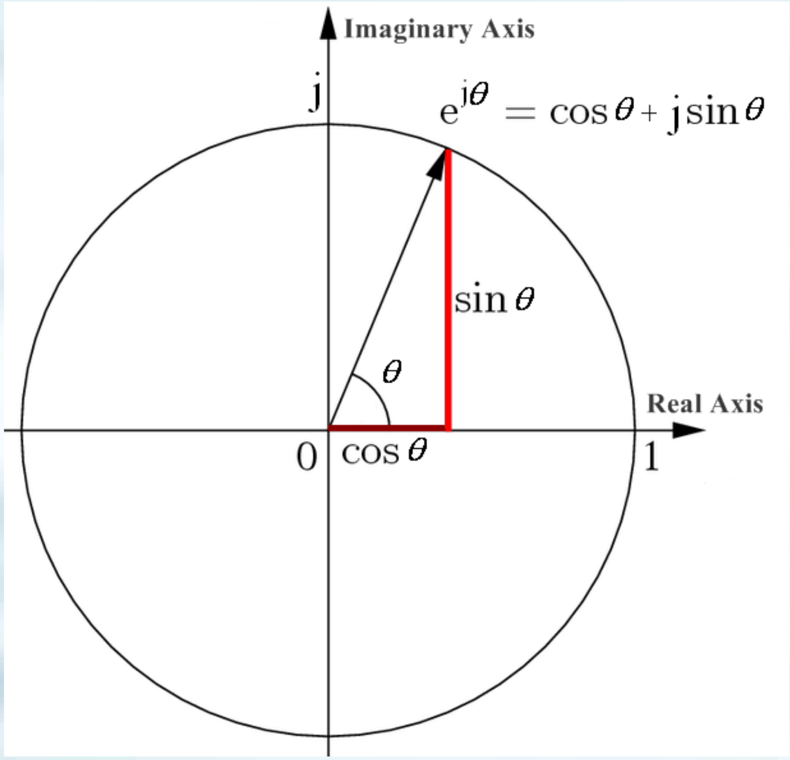
b. **Complex conjugate of z in $x + jy$ form is:**

$$z^* = x - jy = 1 - j$$

Complex conjugate of z in Polar form is:

$$z^* = r(\cos\theta - j\sin\theta) = \sqrt{2}(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4})$$

EULER'S FORMULA



The identity $e^{j\theta} = \cos\theta + j\sin\theta$ **is called Euler's Formula.**

Using Euler's Formula, we can write $z = r(\cos\theta + j\sin\theta)$ **as** $z = re^{j\theta}$ **or** $z = r\angle\theta$

NOTES on EULER'S FORMULA:

1. $$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

2. $$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

3. $(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$**, also known as DE MOIVRE'S THEOREM**

ALGEBRAIC RULES FOR COMPLEX NUMBERS

Let $z_1 = a + jb = r_1\angle\theta_1 = r_1 e^{j\theta_1}$ **and** $z_2 = c + jd = r_2\angle\theta_2 = r_2 e^{j\theta_2}$ **and** $r_2 \neq 0$
then:

ADDITION

$$z_1 + z_2 = (a + c) + j(b + d)$$

SUBTRACTION

$$z_1 - z_2 = (a - c) + j(b - d)$$

MULTIPLICATION

$$z_1 z_2 = (ac - bd) + j(ad + bc)$$

or

$$z_1 z_2 = r_1 e^{j\theta_1} . r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)} = r_1 r_2 \angle(\theta_1 + \theta_2)$$

DIVISION

$$\frac{z_1}{z_2} = \frac{ac+bd}{c^2+d^2} + j\frac{bc-ad}{c^2+d^2}$$

or

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} = \frac{r_1}{r_2}\angle(\theta_1-\theta_2)$$

POWERS

Let us say, $z = re^{j\theta}$**, thus:**

$$z^n = (re^{j\theta})^n = r^n e^{j(\theta+\theta+\ldots+\theta)}$$

$$\therefore z^n = r^n e^{jn\theta}$$

Examples:

Q.1. Let us say $a = (2 + 3j)$ **and** $b = (6 - 2j)$**, compute the following:**

a. $a + b$ **b.** $a - b$ **c.** ab **d.** $\dfrac{a}{b}$

Solution:

a. $a + b = (2 + 3j) + (6 - 2j) = (2 + 6) + j(3 - 2)$

$\therefore a + b = 8 + j$

b. $a - b = (2 + 3j) - (6 - 2j) = (2 - 6) + j(3 - (-2))$

$\therefore a - b = -4 + 5j$

c. $ab = (2 + 3j)(6 - 2j) = (2)(6) + (2)(-2j) + (3j)(6) + (3j)(-2j)$

$ab = 12 - 4j + 18j + 6$

$\therefore ab = 18 + 14j$

d. $\dfrac{a}{b} = \dfrac{2 + 3j}{6 - 2j}\dfrac{6 + 2j}{6 + 2j}$

$= \dfrac{12 + 4j + 18j + 6j^2}{36 + 4} = \dfrac{6 + 22j}{40}$

$\therefore \dfrac{a}{b} = \dfrac{3}{20} + \dfrac{11j}{20}$

Q.2. Let $z_1 = 1 + j$ **and** $z_2 = \sqrt{3} - j$**. Compute**

a. $z_1 z_2$ **b.** $\dfrac{z_1}{z_2}$

Solution:

$$z_1 = 1 + j = \sqrt{2} e^{j\pi/4},\ z_2 = \sqrt{3} - j = 2e^{-j\pi/6}$$

a. $$\therefore z_1 z_2 = 2\sqrt{2}\exp\left(\frac{j\pi}{4} - \frac{j\pi}{6}\right) = 2\sqrt{2}\exp\left(\frac{j\pi}{12}\right)$$

or

$$z_1 z_2 = 2\sqrt{2}\exp\left(\frac{j\pi}{12}\right) = 2\sqrt{2}\left(\cos\frac{\pi}{12} + j\sin\frac{\pi}{12}\right)$$

$$\therefore z_1 z_2 \approx 2.73 + 0.73j$$

b. $$\therefore \frac{z_1}{z_2} = \frac{\sqrt{2}e^{j\pi/4}}{2e^{-j\pi/6}} = \frac{\sqrt{2}}{2}\mathrm{e}^{(\frac{5\pi j}{12})}$$

or

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\mathrm{e}^{(\frac{5\pi j}{12})} \approx 0.707(\cos\frac{5\pi}{12} + j\sin\frac{5\pi}{12})$$

$$\therefore \frac{z_1}{z_2} \approx 0.183 + 0.683j$$

Note on Complex Numbers:

1. $$\sqrt{j} = \sqrt{e^{j\pi/2}} = e^{j\pi/4}$$

$$\therefore \sqrt{j} = \frac{1+j}{\sqrt{2}}$$