Understanding the Volume of a Cone

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For our purposes, a cone is a solid obtained by connecting the edge of a two-dimensional surface to a single point not on the same plane. See examples in fig. 1

fig. 1

a. commonly called a "right circular cone"

b. any of this cone's four faces can be considered a "base"

c. this also qualifies as a cone by our definition

Curiously, the volumes of all such cones can be calculated by the formula:

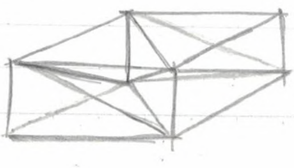
$$V = \frac{1}{3} A h \quad (1)$$

Where the volume, V, is equal to one-third the product of the area of its base, A, and the height of its tip measured perpendicular from the plane of its base, h.

To better understand this, let's choose a type of cone that is simple to calculate the volume of without knowing such a formula. A cone comprised of four triangular faces suits this purpose very well for several reasons, some of which will soon become apparent.

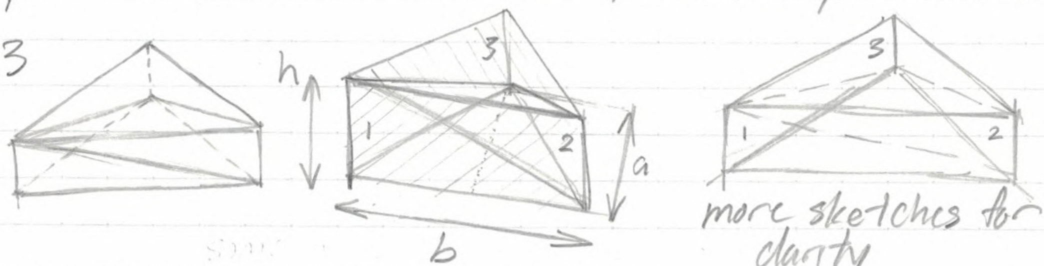
To derive the formula for this class of four-faced cone, better known as a tetrahedron, we try to find a relationship between its perpendicular dimensions and its actual volume. In other words, we try to fit tetrahedra into a rectangular prism whose volume can be found trivially. With some spatial juggling, one might eventually realize that rectangular prisms can be perfectly split up into six tetrahedra as shown in fig. 2.

fig. 2



From the mess that is fig. 2, you may have noticed that half the prism is "redundant." By deleting one half of a diagonally split rectangular prism, we are left with three tetrahedra comprising a triangular prism whose volume is still quite easily calculated.

fig. 3



more sketches for clarity

The volume of the entire triangular prism is given by the formula $\frac{1}{2}abh$, the product of the area of its triangular base, $\frac{1}{2}ab$, and its height, h. Convince yourself that all three tetrahedra are congruent. Finally, we see that a single tetrahedron has volume:

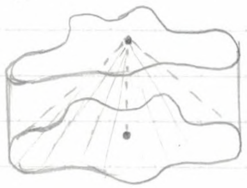
$$V = \frac{1}{3}\left(\frac{1}{2}abh\right) = \frac{1}{6}abh \qquad (2)$$

For the sake of understanding (1) intuitively, knowing (2) is not particularly important, rather, the reasoning we used to find it will be applied more generally. Equation (1) has quite sensible variables, it ought to make sense that a cone's volume be directly proportional to the area of its base and its height. Notice that this product, Ah, also gives us the general volume of a solid created by extruding the base some height, h. The mystery, really, is in the constant factor of $\frac{1}{3}$. With our tetrahedron/prism example, we were able to easily rationalize this division by 3 by showing geometrically how a triangular prism fits exactly 3 congruent tetrahedra. Now, how might we extend this to such cones as that shown in fig. 1a and 1b? Clearly these shapes cannot be so easily fit together.

The answer, it turns out, requires a vague conceptual understanding of limits. One might progress slowly through increasingly complex cones, but we shall skip straight to a cone similar to fig. 1c.

To avoid unnecessary complications that we will later address, we can slide the vertex of the cone in fig. 1c so that lies in the region of space directly above the base. Then, we drop a line from the vertex down to the base and we also extrude the base up to the vertex so that our diagram now looks something like this:

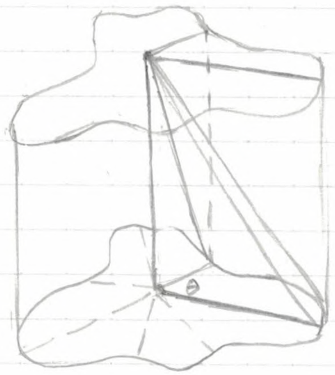
fig. 3a.



diagonal lines are used here to indicate the continuous sloped surface now embedded within our vertically extruded shape.

Now, we cut this solid up like a cake, with all our cuts running radially from the vertical line from the vertex.

fig. 3b



only one of several slices is depicted here

From here, Calculus students likely see where this is going. We increase the number of slices and say that as the number of slices approaches infinity, each slice looks more and more like our triangular prism from earlier. Also, the section of our cone stuck inside the slice looks more and more like our tetrahedra, with which we are quite familiar by now. So concludes a loose explanation of where the 1/3 in equation (1) comes from.

ADDENDUM: Addressing Complications

A. Bizarre base shapes

fig. 4

In general, our previous heuristic cannot work for a shape where it is not true that an imaginary person standing at any point on the base can walk in a straight line to any other point without falling off.

fig. 5

cannot fall off

falls off sometimes

The shape in fig. 3 actually does not satisfy this condition; we had to choose a special point near the middle to avoid issues. When examining cones with non-negotiable vertex positions and shapes such as in fig. 4 where no such special point exists, we can simply think of having to subtract a smaller cone from a larger one where each has a base that satisfies our conditions.

B. Distant vertices / Vertices not over base

fig. 6

just apply Cavalieri's principle