Jacob Snyder

ACM106b Set 1 Problem 1

1. 1) $u_{tt} = u_{xx}$

We propose $u(x,t) = F(x+t) + G(x-t)$

Let $\xi = x+t,\ \eta = x-t \rightarrow u(\xi,\eta) = F(\xi) + G(\eta)$

$\rightarrow \frac{d\xi}{dx} = 1 \quad \frac{d\eta}{dx} = 1$

$\frac{d\xi}{dt} = 1 \quad \frac{d\eta}{dt} = -1$

$\rightarrow u_{tt} = u_{xx}$

$\rightarrow \frac{d^2F}{dx^2} + \frac{d^2G}{dx^2} = \frac{d^2F}{dt^2} + \frac{d^2G}{dt^2}$

$\rightarrow \frac{d^2F}{d\xi^2} + \frac{d^2G}{d\eta^2} = \frac{d^2F}{d\xi^2} + \frac{d^2G}{d\eta^2}(-1)^2$

$\rightarrow \frac{d^2F}{d\xi^2} + \frac{d^2G}{d\eta^2} = \frac{d^2F}{d\xi^2} + \frac{d^2G}{d\eta^2}$ ✓

2) We consider the boundary condition at $x+t=1$ vs at $x+t=-1$:

$\underline{x+t=1}$ $\quad$ $\underline{x+t=-1}$

$f_2 = F(1) + G(x-t)$ $\quad$ $f_1 = F(-1) + G(x-t)$

$\rightarrow f_2 - F(1) = f_1 - F(-1) \rightarrow \boxed{f_2 - f_1 = F(1) - F(-1)}$

Now at $x-t=1$ and at $x-t=-1$:

$\underline{x-t=1}$ $\quad$ $\underline{x-t=-1}$

$f_4 = F(x+t) + G(1)$ $\quad$ $f_3 = F(x+t) + G(-1)$

$\rightarrow f_4 - G(1) = f_3 - G(-1) \rightarrow \boxed{f_4 - f_3 = G(1) - G(-1)}$

We note that $F(1), F(-1), G(1), G(-1)$ are all constants so boundary conditions across from each other must differ by only a constant.

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ACM106b Set 1 Problem 2,3

2\. 1) a) $\partial_x u(x_i) = au_i + bu_{i+1} + cu_{i+2}$

$= au(ih) + bu(ih+h) + cu(ih+2h)$

Taylor expanding yields: (about ih)

$\partial_x u(x_i) = (a+b+c)u(ih) + u'(ih)(b+2c)h + \frac{1}{2}u''(ih)h^2(b+4c) + O(h^3)$

So we have $a+b+c=0 \rightarrow a = \frac{1}{2h} - \frac{2}{h} = \frac{-3}{2h}$

$b+2c = \frac{1}{h} \rightarrow -2c = \frac{1}{h} \rightarrow c = \frac{-1}{2h}$

$b+4c = 0 \rightarrow b = -4c \rightarrow b = \frac{2}{h}$

$\Rightarrow \quad \partial_x u(x_i) = \frac{-3}{2h}u_i + \frac{2}{h}u_{i+1} - \frac{1}{2h}u_{i+2} + O(h^3)$

$\rightarrow \partial_x u(x_i) = \dfrac{-3u_i + 4u_{i+1} - u_{i+2}}{2h} + O(h^3)$ as desired. $p=3$.

b) $\partial_x = \frac{1}{h}\log(I + D_+)$

$= \frac{1}{h}[D_+ - \frac{1}{2}D_+^2 + O(D_+^3)]$

$= \frac{1}{h}[u_{i+1} - u_i - \frac{1}{2}[u_{i+2} - u_{i+1} - u_{i+1} + u_i]] + O(h^2)$

$= \frac{1}{h}[-\frac{u_{i+2}}{2} + 2u_{i+1} - \frac{3}{2}u_i] + O(h^2)$

$= \dfrac{-3u_i + 4u_{i+1} - u_{i+2}}{2h} + O(h^2)$ as desired.

2) a) $\partial_x^2 = \frac{1}{h^2}(\Delta_0^2 - \frac{1}{12}\Delta_0^4) + O(h^4)$ (Lecture notes 1 page 58)

$\partial_x^4 = \frac{1}{h^4}(\Delta_0^4) + O(h^2)$

We note that $\Delta_0^2 = D_+D_- \rightarrow \Delta_0^4 = (D_+D_-)(D_+D_-)$

$\rightarrow \partial_x^2 + \frac{h^2}{12}\partial_x^4 = \frac{1}{h^2}(D_+D_- - \frac{1}{12}h^4(D_+D_-)^2) + \frac{1}{12h^2}((D_+D_-)^2) + O(h^4)$

$\rightarrow \boxed{\partial_x^2 + \frac{h^2}{12}\partial_x^4 = \frac{1}{h^2}D_+D_- + O(h^4)}$ as desired.

b) $\partial_x^2 = \frac{1}{h^2}D_+D_- - \frac{h^2}{12}\partial_x^4 + O(h^4) = \frac{1}{h^2}[D_+D_- - \frac{h^4}{12}[\frac{1}{h^4}((D_+D_-)^2) + O(h^2)]] + O(h^4)$

$= \frac{1}{h^2}[D_+D_-(1 - \frac{D_+D_-}{12})] + O(h^4)$ as desired.

c) $(1 + \frac{1}{12}D_+D_-)\partial_x^2 u = [1 + \frac{1}{12}D_+D_-][\frac{1}{h^2}[D_+D_- - \frac{1}{12}(D_+D_-)^2] + O(h^4)]$

$= \frac{1}{h^2}[D_+D_-][1 + \frac{1}{12}D_+D_-][1 - \frac{1}{12}D_+D_-] = \frac{1}{h^2}[D_+D_-][1 - \frac{1}{144}(D_+D_-)^2] + O(h^4)$

$\approx \boxed{\frac{1}{h^2}[D_+D_-] + O(h^4)}$ as desired, so we conclude that (3) also holds

3\. 2) We can see that as we decrease h by a factor of 2, the error improves by a factor of roughly 1.5 every time. We also see that $\gamma \ll 1$ for every n, k pair so the error bound $\|e\|_{2,h} \lesssim \frac{h^2}{12\pi^2}\|f''\|_{2,h}$ is definitely satisfied. We note that γ decreases with increases in k.