Jacob Snyder

ACM106B Set 6 Problem 1

1.1. $$\frac{u_m^{n+1}-u_m^{n-1}}{2k} + a\left(1+\frac{h^2}{6}D_+D_-\right)^{-1} D_0 u_m^n = 0$$

$$\rightarrow D_0 u = \frac{u_m^{n+1}-u_m^{n-1}}{2ka}\left(1+\frac{h^2}{6}D_+D_-\right)$$

Note:

$$u_m^{n+1} = u_m^n + u_m^{n'}(k) + \tfrac{1}{2}u_m^{n''}(k)^2 + O(h^3)$$
$$u_m^{n-1} = u_m^n - u_m^{n'}(k) + \tfrac{1}{2}u_m^{n''}(k)^2 + O(h^3)$$

$$\rightarrow D_0 u = \frac{2u_m^{n'}k + O(h^3)}{2ka}\left[1+\frac{h^2}{6}D_+D_-\right]$$

$$\rightarrow D_0 u = \frac{1}{a}[u_x]\left[1+\frac{h^2}{6}D_+D_-\right] + O(h^4)$$ as desired.

1.2. $$\left(1+\frac{h^2}{6}D_+D_-\right)\left[\frac{u_m^{n+1}-u_m^{n-1}}{2k}\right] + aD_0 u_m^n = 0$$

$$\rightarrow u_m^{n+1} - u_m^{n-1} + \frac{h^2}{6}D_+D_- u_m^{n+1} - \frac{h^2}{6}D_+D_- u_m^{n-1} + 2ka D_0 u_m^n = 0$$

$$\rightarrow \hat{u}^{n+1}e^{i\omega x} - \hat{u}^{n-1}e^{i\omega x} - \frac{2}{3}\sin^2\left(\frac{\omega x}{2}\right)\hat{u}^{n+1}e^{i\omega x} + \frac{2}{3}\sin^2\left(\frac{\omega x}{2}\right)\hat{u}^{n-1}e^{i\omega x} + \frac{2kai}{h}\sin(\omega x)\hat{u}^n e^{i\omega x} = 0$$

$$\rightarrow \hat{u}^{n+1}\left[1-\frac{2}{3}\sin^2\left(\frac{\omega x}{2}\right)\right] = \hat{u}^{n-1}\left[-1+\frac{2}{3}\sin^2\left(\frac{\omega x}{2}\right)\right] + \hat{u}^n\left[\frac{2kai}{h}\sin(\omega x)\right]$$

set $\hat{u}^n(\omega) = z^n$

$$\rightarrow z^2 = -1 + z\left(2a\lambda i\sin(\omega x)\right)\left(1-\frac{2}{3}\sin^2\left(\frac{\omega x}{2}\right)\right)^{-1}$$

$$\rightarrow z^2 = z\left(2ia\lambda\sin(\omega x)\right)\left(\frac{2}{3}+\cos(\omega x)\right)^{-1} - 1$$

So we have

$$g_\pm = \left[\frac{ia\lambda\sin(\omega x)}{\frac{2}{3}+\cos(\omega x)}\right] \pm \left[\left[\frac{2ia\lambda\sin(\omega x)}{2/3+\cos(\omega x)}\right]^2 - 4\right]^{1/2}$$

We require that $|g_+|, |g_-| \le 1$

It follows that $|a\lambda|^2 < \frac{\left(\frac{2}{3}\right)}{2} \rightarrow |a\lambda|^2 < \frac{1}{3} \rightarrow |a\lambda| < \frac{1}{\sqrt{3}}$ as desired.

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ACM106B Set 6 Problem 2

2.i) $u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) + \frac{ak}{12h}(u_{j+2}^n - 8u_{j+1}^n + 8u_{j-1}^n - u_{j-2}^n)$

$\rightarrow \hat{u}^{n+1} e^{iwx} = \frac{1}{2}\left[\hat{u}^n e^{iw(x-\Delta x)} + \hat{u}^n e^{iw(x+\Delta x)}\right] + \frac{ak}{12h}\left(\hat{u}^n e^{iw(x+2\Delta x)} - 8\hat{u}^n e^{iw(x+\Delta x)} + 8\hat{u}^n e^{iw(x-\Delta x)} - \hat{u}^n e^{iw(x-2\Delta x)}\right)$

$\rightarrow \hat{u}^{n+1} = \frac{1}{2}\left[\hat{u}^n\left[e^{-iw\Delta x} + e^{iw\Delta x}\right]\right] + \frac{ak}{12h}\left[\hat{u}^n\left[e^{2iw\Delta x} - 8e^{iw\Delta x} + 8e^{-iw\Delta x} - e^{-2iw\Delta x}\right]\right]$

$\rightarrow \hat{u}^{n+1} = \hat{u}^n\left[\frac{1}{2}(2\cos(w\Delta x) + \frac{ak}{12h}\left[2i\sin(2w\Delta x) - 16i\sin(w\Delta x)\right]\right]$

so $g(w) = \cos(wh) + \frac{a\lambda i}{6}\left[\sin(2wh) - 8\sin(wh)\right]$

$= \cos(wh) + \frac{a\lambda i}{3}\left[\sin(wh)[\cos(wh) - 4]\right]$

$\rightarrow |g(w)|^2 = \left|\cos^2(wh) - \frac{a^2\lambda^2}{9}\left[\sin^2(wh)\left[\cos(wh) - 4\right]^2\right] + \frac{2a\lambda i}{3}\cos(wh)\sin(wh)\left[\cos(wh) - 4\right]\right|$

We note that the terms following $\cos^2(wh)$ can be expressed as a function of a, λ, w, so indeed this can be represented as $\cos^2(w) + h(w)\sin^2(w)$, as desired.

2.ii) From the Finite Difference Methods and Stability Analysis lecture notes, page 43, we have $|at_n| \leq nh$

$\rightarrow |a\lambda| \leq 1$

2.iii) Suppose $\sin(w) = 0$. Then $\cos(w) = \pm 1$ so $g(w) = \pm 1$ so the condition $|g(w)| \leq 1$ is met.

If $\sin(w) \neq 0$, then we require that $h(w)\sin^2(w) \leq \sin^2(w)$ (since $\cos^2(w) + \sin^2(w) = 1$, if $h(w)\sin^2(w) \leq \sin^2(w)$, then $|g(w)|^2 \leq 1$ so $|g(w)| \leq 1$)

$h(w) = \frac{a^2\lambda^2}{9}[\cos(wh) - 4]^2 + \frac{2a\lambda}{3}\frac{\cos(w)}{\sin(w)}[\cos(w) - 4]$

Since $\sin(w) \neq 0$, $|\cos(w)| < 1 \rightarrow \frac{[\cos(w) - 4]^2}{25} < 1$

so $\frac{a^2\lambda^2}{9} \leq \frac{1}{25} \rightarrow \boxed{|a\lambda| \leq \frac{3}{5}}$

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ACM106B Set 6 Problem 3

3.1. We can use the diagonal form:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_t + \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_x = 0$$

Where $w_1 = u_1 + u_2$, $w_2 = u_1 - u_2$

3.2. $w_1(x,0) = u_1(x,0) + u_2(x,0) = f(x) + g(x)$

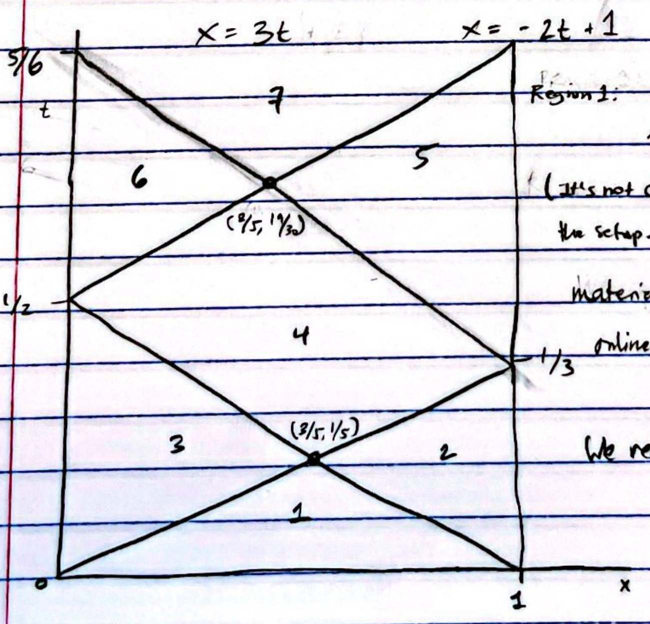
$w_2(x,0) = u_1(x,0) - u_2(x,0) = f(x) - g(x)$

3.3. Since the two families of characteristics propagate in opposite directions, we need one boundary condition at each boundary $x=0$, $x=1$:

$$w_1(0,t) = +w_2(0,t) + 2\alpha(t)$$

$$w_2(1,t) = w_1(0,t) - 2\beta(t)$$

3.4. The families of characteristics are given by:



Region 1: $u_1 = f(x) + 2[\alpha(1/2) - f(1)]t$

$u_2 = g(x) + 3[\beta(1/3) - g(0)]t$

(It's not clear to me how the solution by region follows from the setup... this seems right for region 1 but I can't find any helpful material in the slides or in lecture recordings or online so I'm a bit stuck here)

We require that $f(0) = \alpha(0)$, $g(1) = \beta(0)$.