Jacob Snyder

ACM104 Set 1

1. Consider:

$$AB \stackrel{?}{=} \sum_{k=1}^{p} a_k b^k = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1p} \\ a_{21}b_{11} & & & \\ \vdots & & \ddots & \\ a_{p1}b_{11} & & & a_{p1}b_{1p} \end{bmatrix} + \begin{bmatrix} a_{12}b_{21} & a_{12}b_{22} & \cdots & a_{12}b_{2p} \\ a_{22}b_{21} & & & \\ \vdots & & \ddots & \\ a_{p2}b_{21} & & & a_{p2}b_{2p} \end{bmatrix} + \cdots + \begin{bmatrix} a_{1p}b_{p1} & a_{1p}b_{p2} & \cdots & a_{1p}b_{pp} \\ a_{2p}b_{p1} & & & \\ \vdots & & \ddots & \\ a_{pp}b_{p1} & & & a_{pp}b_{pp} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1p}b_{p1}) & (a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1p}b_{p2}) & \cdots & (a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1p}b_{pp}) \\ (a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2p}b_{p1}) & & & \\ \vdots & & \ddots & \\ (a_{p1}b_{11} + a_{p2}b_{21} + \cdots + a_{pp}b_{p1}) & & & (a_{p1}b_{1p} + a_{p2}b_{2p} + \cdots + a_{pp}b_{pp}) \end{bmatrix}$$

Each entry of AB is clearly equal to $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$. Therefore it is true that this evaluation of AB is equivalent to multiplying the rows of A by the columns of B:

$$\boxed{AB = \sum_{k=1}^{p} a_k b^k}$$

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2.

$$A_{n\times n} = \begin{bmatrix} 0 & & a \\ & 0 & \\ \mathbf{0} & \ddots & \\ & & 0 \end{bmatrix}$$

where a is nonzero numbers

$$A^2 = \begin{bmatrix} 0 & & a \\ & 0 & \\ \mathbf{0} & \ddots & \\ & & 0 \end{bmatrix}\begin{bmatrix} 0 & & a \\ & 0 & \\ \mathbf{0} & \ddots & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & & b \\ & 0 & 0 & \\ \mathbf{0} & & \ddots & \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} = B$$

This is true because for any B_{ij} on the diagonal above the central diagonal, the first i elements of A_i will be zero and the last $n-i$ elements of A_j will be zero. Therefore $B_{ij} = 0$ for all values just above the diagonal.

$$A^4 = B^2 = \begin{bmatrix} 0 & 0 & & b \\ & 0 & 0 & \\ \mathbf{0} & & \ddots & \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & & b \\ & 0 & 0 & \\ \mathbf{0} & & \ddots & \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & c \\ & 0 & 0 & 0 & & \\ & & 0 & 0 & \ddots & \\ & & & 0 & \cdots & 0\;0\;0\;0 \\ \mathbf{0} & & & & & 0\;0\;0 \\ & & & & & 0\;0 \\ & & & & & 0 \end{bmatrix} = C$$

Using the same logic as we used to get B, we can get C. Therefore, if we apply A to itself enough times, eventually the diagonal of 0s will reach the top right and the matrix will go to zero. Following this pattern, it is guaranteed that $A^n = 0$.

Therefore, any strictly upper triangular matrix is nilpotent.

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3. Hypothesis based on MATLAB:

$$P_n = I_n \quad , \quad L_n = \begin{bmatrix} 1 & & & & & \\ -1/2 & 1 & & & 0 & \\ & -2/3 & 1 & & & \\ & & -3/4 & 1 & & \\ & 0 & & \ddots & \ddots & \\ & & & & -\frac{(n-1)}{n} & 1 \end{bmatrix} \qquad U_n = \begin{bmatrix} 2/1 & -1 & & & & \\ & 3/2 & -1 & & 0 & \\ & & 4/3 & -1 & & \\ & 0 & & 5/4 & -1 & \\ & & & & \ddots & -1 \\ & & & & & \frac{n+1}{n} \end{bmatrix}$$

Proof:

$$L_n U_n = \begin{bmatrix} 1 & & & & \\ -1/2 & 1 & & 0 & \\ & -2/3 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & 1 & \\ & & & -\frac{(n-1)}{n} & 1 \end{bmatrix} \begin{bmatrix} 2/1 & -1 & & & \\ & 3/2 & -1 & & 0 \\ & & 4/3 & -1 & \\ & & & \ddots & \ddots \\ & 0 & & & -1 \\ & & & & \frac{n+1}{n} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & & & \\ -1 & (1/2 + 3/2) & -1 & & \\ & -1 & (2/3 + 4/3) & & \\ & & \ddots & \ddots & \\ & & & & -1 \\ & & & \left(-\frac{(n-1)}{n}\right)\left(\frac{n}{n-1}\right) & \frac{(n-1)}{n} + \frac{(n+1)}{n} \end{bmatrix} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}$$

And
$$P_n A_n = A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Therefore $P_n A_n = L_n U_n$ so the above choices for P_n, L_n, and U_n are correct.

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4. a) We will prove that for a given permutation matrix P, $P^{-1} = P^T$:

Consider:

$P = P_1 P_2 P_3 \cdots P_k$ where $P_1, P_2, \ldots, P_k$ are elementary matrices of type 2

$$\rightarrow P^{-1} = (P_1 P_2 \cdots P_k)^{-1} = P_k^{-1} \cdots P_2^{-1} P_1^{-1} \qquad (1)$$

A given elementary matrix of type 2, P_a, is of the form

$$\begin{array}{c|c} \begin{array}{ccccccc} & & & i & & j & \end{array} & \\ \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & \cdots & 1 & \\ & & & 1 & \cdots & 0 & \\ & & & & & 1 & \\ & & & & & & \ddots \; 1 \end{bmatrix} & \begin{array}{c} \\ \\ \\ i \\ j \\ \\ \\ \end{array} \end{array}$$

and it interchanges the i^{th} and j^{th} rows of the matrix it's applied to.

Therefore it follows that the matrix required to reverse this operation, P_a^{-1}, would interchange rows i and j. That is, $\underline{P_a^{-1} = P_a}$

Also note that all elementary matrices of type 2 as described above are symmetric by definition: $\underline{P_a^T = P_a}$

Therefore, continuing from equation (1):

$$P^{-1} = P_k^{-1} \cdots P_2^{-1} P_1^{-1}$$
$$\rightarrow P^{-1} = P_k \cdots P_2 P_1$$
$$\rightarrow P^{-1} = P_k^T \cdots P_2^T P_1^T$$
$$\rightarrow P^{-1} = (P_1 P_2 \cdots P_k)^T$$
$$\rightarrow \boxed{P^{-1} = P^T}$$

Therefore all permutation matrizes are orthogonal.

(cont'd on back)

b) No, orthogonal matrices are not necessarily permutation matrices.

We will prove this by counterexample:

Suppose all orthogonal matrices are permutation matrices.

Consider:

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Check: $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$$A^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$\rightarrow$ $A^{-1} = A^T$ so A is orthogonal.

However, we can tell by observation that A is not a permutation matrix

Contradiction!

Therefore, orthogonal matrices are not necessarily permutation matrices.

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5. Let $C \in M_{n \times n}$ where c_{ij} is the element in the i^{th} row and j^{th} column of C.
Now suppose $A+B=C$. We will show that A can be represented as a symmetric matrix and B can be represented as a skew-symmetric matrix:

$$a_{ij} + b_{ij} = c_{ij}$$
$$a_{ji} + b_{ji} = c_{ji}$$

Let $a_{ij} = a_{ji} = \dfrac{c_{ij} + c_{ji}}{2}$ So $a_{ij} = a_{ji} =$ the midpoint of c_{ij} and c_{ji}

Note that because $a_{ij} = a_{ji}$, A is symmetric.

Let $b_{ij} = -b_{ji} = \dfrac{c_{ij} - c_{ji}}{2}$

Note that because $b_{ij} = -b_{ji}$, B is skew symmetric

Then: $a_{ij} + b_{ij} = \dfrac{c_{ij} + c_{ji} + c_{ij} - c_{ji}}{2} = c_{ij}$ ✓

and $a_{ji} + b_{ji} = \dfrac{c_{ij} + c_{ji} + c_{ji} - c_{ij}}{2} = c_{ji}$ ✓

$\boxed{\text{Therefore, given a matrix } C\text{, we can construct matrices } A \text{ and } B \text{ such that } A \text{ is symmetric and } B \text{ is skew-symmetric and } A+B=C.}$

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7. a)

$$A = \begin{bmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{bmatrix}$$

→ Subtract row 1 from all other rows:

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \\ \vdots & \vdots & & \vdots \\ n^2-n & n^2-n & \cdots & n^2-n \end{bmatrix}$$

→ Rows below row 2 are multiples of n, use row 2 to set all lower rows to 0:

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

→ Divide row 2 by $-n$:

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ -1 & -1 & \cdots & -1 \\ & 0 & & \end{bmatrix}$$

→ Add row 1 to row 2:

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ 0 & 1 & \cdots & n-1 \\ & 0 & & \end{bmatrix}$$

Therefore, the number of pivots in A's echelon form is 2. Therefore, the $\boxed{\text{rank of A is 2.}}$

b) The nonzero components of x are 0 and 0.01 (see attached code)