Jacob Snyder

ACM106B Set 7 Problem 1

$u_t = a u_x$

1. $$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = 0$$

$$\rightarrow \quad v_m^{n+1} - v_m^n = \frac{ak}{2h}\left[v_{m-1}^{n+1} - v_{m+1}^{n+1}\right]$$

$$\rightarrow \quad v_m^n = \frac{ak}{2h}\left[v_{m+1}^{n+1} - v_{m-1}^{n+1}\right] + v_m^{n+1}$$

$$\rightarrow \quad \hat{v}^n e^{i\omega x_m} = \frac{ak}{2h}\left[\hat{v}^{n+1} e^{i\omega x_{m+1}} - \hat{v}^{n+1} e^{i\omega x_{m-1}}\right] + \hat{v}^{n+1} e^{i\omega x_m}$$

$$\rightarrow \quad \hat{v}^n = \hat{v}^{n+1}\left[\frac{ak}{2h}\left[e^{i\omega h} - e^{-i\omega h}\right] + 1\right]$$

$$\rightarrow \quad \hat{v}^n = \hat{v}^{n+1}\left[\frac{ak}{2h}\left[i2\sin(\omega h)\right] + 1\right]$$

$$\rightarrow \quad \hat{v}^{n+1} = \hat{v}^{n}\left[\frac{iak\sin(\omega h)}{h} + 1\right]^{-1}$$

$$\rightarrow \quad \hat{Q} = \left[\frac{iak}{h}\sin(\omega h) + 1\right]^{-1}$$

$$\rightarrow \quad |\hat{Q}| = \left[\left|\frac{iak}{h}\sin(\omega h) + 1\right|\right]^{-1} = \Big[\underbrace{\frac{a^2k^2}{h^2}\sin^2(\omega h) + 1}_{\text{we note that this } \geq 1\text{, so } |\hat{Q}| \leq 1}\Big]^{-1}$$

so the scheme is unconditionally stable

$$v(x_m, t_n) = u(x_m, t_{n+1}) - k u_t(x_m, t_{n+1}) + O(k^2) = u(x_m, t_{n+1}) + k a u_x(x_m, t_{n+1}) + O(k^2)$$

$$v(x_{m-1}, t_{n+1}) = u(x_m, t_{n+1}) - h u_x(x_m, t_{n+1}) + O(h^2)$$

$$v(x_{m+1}, t_{n+1}) = u(x_m, t_{n+1}) + h u_x(x_m, t_{n+1}) + O(h^2)$$

$$\Rightarrow \quad \frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = \frac{-kau_x(x_m, t_{n+1}) + O(k^2)}{k} + a\frac{2hu_x(x_m, t_{n+1}) + O(h^2)}{2h}$$

$$= -au_x(x_m, t_{n+1}) + au_x(x_m, t_{n+1}) + O(h+k) = O(h+k) \rightarrow 0 \text{ as } h, k \rightarrow 0$$

So the scheme is consistent.

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1.2. $u_t = -a u_x$

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

$\rightarrow v_m^{n+1} = -\frac{ak}{2h}\left[v_{m+1}^n - v_{m-1}^n\right] + v_m^n$

$\rightarrow \hat{v}^{n+1} e^{i\omega x_m} = -\frac{ak}{2h}\left[\hat{v}^n e^{i\omega x_{m+1}} - \hat{v}^n e^{i\omega x_{m-1}}\right] + \hat{v}^n e^{i\omega x_m}$

$\rightarrow \hat{v}^{n+1} = \hat{v}^n\left[1 - \frac{ak}{2h}\left[e^{i\omega h} - e^{-i\omega h}\right]\right]$

$\rightarrow \hat{v}^{n+1} = \hat{v}^n\left[1 - \frac{ak}{2h}\left[2i\sin(\omega h)\right]\right]$

So $\hat{Q} = 1 - \frac{iak}{h}\sin(\omega h)$

$\rightarrow |\hat{Q}| = \left[1 + \frac{a^2k^2}{h^2}\sin^2(\omega h)\right]^{1/2}$

$\rightarrow$ So long as $\frac{k}{h^2}$ is bounded by some j, we can select K s.t. $|\hat{Q}| \le 1 + Kk$ by setting $K = a^2 j$. So the scheme is stable for bounded $\frac{k}{h^2}$, as desired.

$v(x_m, t_{n+1}) = v(x_m, t_n) + k v_t(x_m, t_n) + O(k^2)$

$v(x_{m+1}, t_n) = v(x_m, t_n) + h v_x(x_m, t_n) + O(h^2)$

$v(x_{m-1}, t_n) = v(x_m, t_n) - h v_x(x_m, t_n) + O(h^2)$

$\rightarrow$ Scheme $= \frac{k v_t(x_m, t_n)}{k} + a\frac{2h v_x(x_m, t_n)}{2h} + O(k+h)$

$= O(h+k) \rightarrow 0$ as $h, k \rightarrow 0$

So the scheme is consistent.

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NOTE part (b) is first. Sorry!

1.3. b) $v_m^{n+1} = \frac{1}{2}(v_{m+1}^n + v_{m-1}^n) - \frac{1}{2}akh^{-3}(v_{m+2}^n - 2v_{m+1}^n + 2v_{m-1}^n - v_{m-2}^n)$

$\rightarrow \hat{v}^{n+1}e^{i\omega x_m} = \frac{1}{2}(\hat{v}^n e^{i\omega x_{m+1}} + \hat{v}^n e^{i\omega x_{m-1}}) - \frac{1}{2}akh^{-3}(\hat{v}^n e^{i\omega x_{m+2}} - 2\hat{v}^n e^{i\omega x_{m+1}} + 2\hat{v}^n e^{i\omega x_{m-1}} - \hat{v}^n e^{i\omega x_{m-2}})$

$\rightarrow \hat{v}^{n+1} = \frac{1}{2}(\hat{v}^n[e^{i\omega h} + e^{-i\omega h}]) - \frac{1}{2}akh^{-3}\hat{v}^n[e^{2i\omega h} - 2e^{i\omega h} + 2e^{-i\omega h} - e^{-2i\omega h}]$

$\rightarrow \hat{v}^{n+1} = \hat{v}^n[\cos(\omega h) - \frac{i}{2}akh^{-3}[2\sin(2\omega h) - 4\sin(\omega h)]]$

$\rightarrow \hat{Q} = \cos(\omega h) - iakh^{-3}[\sin(2\omega h) - 2\sin(\omega h)]$

$\rightarrow |\hat{Q}| = [\cos^2(\omega h) + a^2k^2h^{-6}[64\sin^6(\frac{\omega h}{2})\cos^2(\frac{\omega h}{2})]]^{1/2}$

We require $\frac{k}{h^6}$ to be bounded by some j $(\frac{k}{h^6} \leq j)$ so that we set $K = a^2 j$ so that

$|\hat{Q}| \leq 1 + Kk$

$\rightarrow |\hat{Q}| \leq 1 + a^2(k)\left[\frac{k}{h^6}\right] \leq 1 + Kk$ as desired.

a) $v_{m+1}^n = v(x_m,t_n) + hv_x(x_m,t_n) + \frac{1}{2}h^2 v_{xx}(x_m,t_n) + \frac{1}{6}h^3 v_{xxx}(x_m,t_n) + O(h^4)$

$v_{m-1}^n = v(x_m,t_n) - hv_x(x_m,t_n) + \frac{1}{2}h^2 v_{xx}(x_m,t_n) - \frac{1}{6}h^3 v_{xxx}(x_m,t_n) + O(h^4)$

$v_{m+2}^n = v(x_m,t_n) + 2hv_x(x_m,t_n) + 2h^2 v_{xx}(x_m,t_n) + \frac{8}{6}h^3 v_{xxx}(x_m,t_n) + O(h^4)$

$v_{m-2}^n = v(x_m,t_n) + 2hv_x(x_m,t_n) + 2h^2 v_{xx}(x_m,t_n) - \frac{8}{6}h^3 v_{xxx}(x_m,t_n) + O(h^4)$

$\rightarrow v_m^{n+1} = v(x_m,t_n) + \frac{1}{2}h^2 v_{xx}(x_m,t_n) + O(h^4)$

$\quad - \frac{1}{2}akh^{-3}[4hv_x(x_m,t_n) + \frac{16}{6}h^3 v_{xxx}(x_m,t_n) - 4hv_x(x_m,t_n) - \frac{4}{6}h^3 v_{xxx}(x_m,t_n) + O(h^5)]$

$\Rightarrow v_m^{n+1} = v(x_m,t_n) + kv_t(x_m,t_n) + O(k^2) = v(x_m,t_n) - akv_{xxx}(x_m,t_n) + O(k^2)$

$\rightarrow -akv_{xxx}(x_m,t_n) + O(k^2) = -akh^{-3}(h^3)v_{xxx}(x_m,t_n) + O(h^2k) +$

$\rightarrow 0 = O(\frac{h^2}{k})$ as desired, the scheme is consistent so long as $\frac{h^2}{k} \to 0$ when $h,k \to 0$

c) This scheme cannot be both consistent and stable since if $\frac{h^2}{k} \to 0$ when $h,k \to 0$, then $\frac{k}{h^2} \to +\infty \Rightarrow \frac{k}{h^6} \to +\infty$ so $\frac{k}{h^6}$ cannot be bounded. So the conditions for consistency and stability cannot be simultaneously satisfied.

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1.4. $u_t + a u_x = 0$

$$\frac{1}{2k}\left[(v_m^{n+1} + v_{m+1}^{n+1}) - (v_m^n + v_{m+1}^n)\right] + \frac{a}{2h}\left[(v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)\right] = 0$$

Stability

$$\frac{1}{2k}\left[(\hat{v}^{n+1} e^{i\omega x_m} + \hat{v}^{n+1} e^{i\omega x_{m+1}}) - (\hat{v}^n e^{i\omega x_m} + \hat{v}^n e^{i\omega x_{m+1}})\right]$$
$$+ \frac{a}{2h}\left[(\hat{v}^{n+1} e^{i\omega x_{m+1}} - \hat{v}^{n+1} e^{i\omega x_m}) + (\hat{v}^n e^{i\omega x_{m+1}} - \hat{v}^n e^{i\omega x_m})\right] = 0$$

$$\rightarrow \frac{1}{2k}\hat{v}^{n+1}\left[1 + e^{i\omega h}\right] - \frac{1}{2k}\hat{v}^n\left[1 + e^{i\omega h}\right] + \frac{a}{2h}\left[\hat{v}^{n+1}(e^{i\omega h} - 1) + \hat{v}^n(e^{i\omega h} - 1)\right] = 0$$

$$\rightarrow \hat{v}^{n+1}\left[\frac{1}{2k}(1 + e^{i\omega h}) + \frac{a}{2h}(e^{i\omega h} - 1)\right] + \hat{v}^n\left[\frac{a}{2h}(e^{i\omega h} - 1) - \frac{1}{2k}(1 + e^{i\omega h})\right] = 0$$

$\rightarrow \hat{v}^{n+1} = \hat{v}^n \rightarrow \hat{Q} = 1 \rightarrow |\hat{Q}| = 1$ so the scheme is stable unconditionally.

Consistency

$$v(x_{m+1/2}, t_n) = v(x_{m+1/2}, t_{n+1/2}) - \frac{k}{2} v_t(x_{m+1/2}, t_{n+1/2}) + O(k^2)$$
$$v(x_{m+1/2}, t_{n+1}) = v(x_{m+1/2}, t_{n+1/2}) + \frac{k}{2} v_t(x_{m+1/2}, t_{n+1/2}) + O(k^2)$$
$$v(x_{m+1}, t_n) = v(x_{m+1/2}, t_{n+1/2}) + \frac{ka}{2} v_x(x_{m+1/2}, t_{n+1/2}) + \frac{h}{2}\left[v_x(x_{m+1/2}, t_{n+1/2}) + \frac{ka}{2} v_{xx}(x_{m+1/2}, t_{n+1/2})\right] + O(k^2 + h^2k^2)$$
$$v(x_m, t_n) = v(x_{m+1/2}, t_{n+1/2}) + \frac{ka}{2} v_x(x_{m+1/2}, t_{n+1/2}) - \frac{h}{2}\left[v_x(x_{m+1/2}, t_{n+1/2}) + \frac{ka}{2} v_{xx}(x_{m+1/2}, t_{n+1/2})\right] + O(k^2 + h^2k^2)$$
$$v(x_{m+1}, t_{n+1}) = v(x_{m+1/2}, t_{n+1/2}) - \frac{ka}{2} v_x(x_{m+1/2}, t_{n+1/2}) + \frac{h}{2}\left[v_x(x_{m+1/2}, t_{n+1/2}) - \frac{ka}{2} v_{xx}(x_{m+1/2}, t_{n+1/2})\right] + O(k^2 + h^2k^2)$$
$$v(x_m, t_{n+1}) = v(x_{m+1/2}, t_{n+1/2}) - \frac{ka}{2} v_x(x_{m+1/2}, t_{n+1/2}) - \frac{h}{2}\left[v_x(x_{m+1/2}, t_{n+1/2}) - \frac{ka}{2} v_{xx}(x_{m+1/2}, t_{n+1/2})\right] + O(k^2 + h^2k^2)$$

$$\rightarrow \frac{1}{2k}\left[2v(x_{m+1/2}, t_{n+1/2}) - ka v_x(x_{m+1/2}, t_{n+1/2}) - (2v(x_{m+1/2}, t_{n+1/2}) + ka v_x(x_{m+1/2}, t_{n+1/2})) + O(k^2 + h^2k^2)\right]$$
$$+ \frac{a}{2h}\left[(h[v_x(x_{m+1/2}, t_{n+1/2}) - \frac{ka}{2} v_{xx}(x_{m+1/2}, t_{n+1/2})]) + (h(v_x(x_{m+1/2}, t_{n+1/2}) + \frac{ka}{2} v_{xx}(x_{m+1/2}, t_{n+1/2}))\right] + O(k^2 + h^2k^2) = 0$$

$$\rightarrow a v_x(x_{m+1/2}, t_{n+1/2}) - a v_x(x_{m+1/2}, t_{n+1/2}) = O(k^2 + h^2k^2)$$

$\rightarrow 0 = O(k^2 + h^2k^2)$ as $k, h \to 0$, $k^2 + h^2k^2 \to 0$

So the scheme is unconditionally consistent.

$\Rightarrow$ By the Lax-Richtmyer Equivalence Theorem, the scheme is convergent for all a.

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ACM106B Set 7 Problem 2

2.1. This scheme is 2nd order in space and time. The scheme is stable, unconditionally.

2.2. This scheme is 2nd order in space and time. The scheme is unstable, unconditionally. We see when implementing the scheme that it quickly grows out of control. This is expected since the scheme is not stable.

Problem 3

3.1) We can see how when the wave bounces back from the right boundary it does so in highly oscillatory parasitic nodes.

2)a) We can see the wave bounce back (in the wrong direction), but the magnitude of the parasitic nodes (oscillations) is much decreased. In addition, now the wave is able to move forwards in the [-1, 0] range.

b) The wave no longer bounces back in the incorrect direction. However it quickly loses amplitude (maybe this is intended?)