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AcM157 Set 3

1. By definition,

$$\mathrm{Cov}(\hat{F}_n(x), \hat{F}_n(y)) = \mathbb{E}[\hat{F}_n(x)\hat{F}_n(y)] - \mathbb{E}[\hat{F}_n(x)]\,\mathbb{E}[\hat{F}_n(y)]$$

$$= \mathbb{E}[\hat{F}_n(x)\hat{F}_n(y)] - F_n(x)F_n(y)$$

$$= \frac{1}{n^2}\sum_{i,j}^{n} H(x-X_i)H(y-X_j) - F_n(x)F_n(y)$$

$$= \frac{1}{n^2}\Big[\underbrace{nF_n(\min(x,y))}_{\text{for } i=j} + \underbrace{n(n-1)F_n(x)F_n(y)}_{\text{for } i\neq j}\Big] - F_n(x)F_n(y)$$

$$= \frac{F_n(\min(x,y))}{n} + \left(\frac{n-1}{n} - 1\right)F_n(x)F_n(y) = \frac{F_n(\min(x,y))}{n} + \left(\frac{-1}{n}\right)(F_n(x)F_n(y))$$

$$= \boxed{\frac{1}{n}\left(F_n(\min(x,y)) - F_n(x)F_n(y)\right)}$$

2. $$\kappa_F = \frac{\int (x-\mu_F)^3\, dF(x)}{\left(\int (x-\mu_F)^2\, dF(x)\right)^{3/2}}$$

$$\hat{\kappa}_F = \frac{\int (x-\mu_{\hat{F}})^3\, d\hat{F}(x)}{\left(\int (x-\mu_{\hat{F}})^2\, d\hat{F}(x)\right)^{3/2}} = \frac{\int \left(x-\frac{1}{n}\sum_{i=1}^{n} x_i\right)^3 d\hat{F}(x)}{\left(\int \left(x-\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 d\hat{F}(x)\right)^{3/2}}$$

$$= \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)^{3/2}} = \boxed{\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^3}{\hat{\sigma}^3}}$$

4. a) $\hat{\theta}_n = \min\{x : \hat{F}(x)=1\} = X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$

b) $\mathbb{B}[\hat{\theta}_n] = \mathbb{E}[\hat{\theta}_n] - \theta = \frac{n\theta}{n+1} - \theta = \boxed{\theta\left[\frac{n}{n+1} - 1\right]}$

c) $\hat{\theta}_n^J = n\hat{\theta}_n - (n-1)\bar{\theta}_n^J = nX_{(n)} - \frac{n-1}{n}\left[(n-1)X_{(n)} + X_{(n-1)}\right]$ ← 2nd largest value

$$= X_{(n)}\left[n - \frac{(n-1)^2}{n}\right] - X_{(n-1)}\left[\frac{n-1}{n}\right]$$

d) $\mathbb{B}[\hat{\theta}_n^J] = \mathbb{E}[\hat{\theta}_n^J] - \theta = \mathbb{E}[X_{(n)}]\left[n - \frac{(n-1)^2}{n}\right] - \mathbb{E}[X_{(n-1)}]\left[\frac{n-1}{n}\right] - \theta$

$$= \frac{n\theta}{n+1}\left[n - \frac{(n-1)^2}{n}\right] - \left[\frac{(n-1)\theta}{n+1}\right]\left[\frac{n-1}{n}\right] - \theta$$

$$= \theta\left[\frac{n}{n+1}\left[\frac{2n-1}{n}\right] - \frac{(n-1)^2}{n(n+1)} - 1\right] = \theta\left[\frac{2n^2 - n - n^2 + 2n - 1 - n^2 - n}{n(n+1)}\right]$$

$$= \theta\left[\frac{-1}{n(n+1)}\right]$$

(cont'd on back)

5. a) $\mathbb{B}[\hat{\theta}_n] = \mathbb{E}[\hat{\theta}_n] - \theta = \mathbb{E}[e^{\bar{X}_n}] - e^{\mu} = e^{\mu + \frac{1}{2n}} - e^{\mu}$

$= e^{\mu}\sqrt[2n]{e} - e^{\mu} = e^{\mu}(\sqrt[2n]{e} - 1)$

$\rightarrow$ Taylor expand $e^{1/2n}$:

$e^{1/2n} = 1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{48n^3} + \dots$

$\rightarrow e^{\mu}(e^{1/2n} - 1) = \frac{e^{\mu}}{2n} + \frac{e^{\mu}}{8n^2} + \frac{e^{\mu}}{48n^3} + \dots$

so $a = \frac{e^{\mu}}{2}$, $b = \frac{e^{\mu}}{8}$, and the rest of the terms are $O\left(\frac{1}{n^3}\right)$.