Jacob Snyder

ACM106a Set 1 Problem 1

1. a)

Step 1: Suppose $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$ and $\|\cdot\|_b$ is equivalent to $\|\cdot\|_c$ on a linear space S.
Since $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$, we have that there exists some $c_1 > 0$ such that for any $x \in S$:

$$\frac{1}{c_1}\|x\|_a \leq \|x\|_b \leq c_1\|x\|_a \qquad (1)$$

Similarly, since $\|\cdot\|_b$ is equivalent to $\|\cdot\|_b$, there exists $c_2 > 0$ such that for any $x \in S$:

$$\frac{1}{c_2}\|x\|_b \leq \|x\|_c \leq c_2\|x\|_b \qquad (2)$$

From (1), we have that $\frac{1}{c_1}\|x\|_a \leq \|x\|_b \rightarrow \frac{1}{c_1 c_2}\|x\|_a \leq \frac{1}{c_2}\|x\|_b$

From (2), we also have that $\frac{1}{c_2}\|x\|_b \leq \|x\|_c$, so it follows that:

$$\frac{1}{c_1 c_2}\|x\|_a \leq \|x\|_c \qquad (3)$$

From (1), we have that $\|x\|_b \leq c_1\|x\|_a \rightarrow c_2\|x\|_b \leq c_1 c_2\|x_a\|$

From (2), we have that $\|x\|_c \leq c_2\|x\|_b$, so it follows that

$$\|x\|_c \leq c_1 c_2\|x_a\| \qquad (4)$$

From (3) and (4), we have that

$$\frac{1}{c_1 c_2}\|x\|_a \leq \|x\|_c \leq c_1 c_2\|x\|_a \qquad (5)$$

Therefore, $\|\cdot\|_a$ is equivalent to $\|\cdot\|_c$ on S, as desired.

Step 2

In order to prove continuity of $f(x)$, we must prove the following:
for every $\varepsilon > 0$, there exists a $\delta > 0$ s.t.

$$\|x-y\|_2 < \delta \Rightarrow \big|\|x\|_c - \|y\|_c\big| < \varepsilon$$

By the triangle inequality, we have that

$$\|x-y+y\|_c \leq \|x-y\|_c + \|y\|_c$$
$$\rightarrow \|x\|_c - \|y\|_c \leq \|x-y\|_c$$
$$\rightarrow \big|\|x\|_c - \|y\|_c\big| \leq \|x-y\|_c$$

We decompose x and y as follows, where e_i is the unit vector of the i^{th} coordinate:

$$x = \Sigma_i x_i e_i, \quad y = \Sigma_i y_i e_i$$

by Triangle Inequality

$$\rightarrow \|x-y\|_c = \|\Sigma_i (x_i - y_i)e_i\|_c \leq \Sigma_i \|(x_i - y_i)e_i\|_c = \Sigma_i |x_i - y_i|\,\|e_i\|_c$$

By the Cauchy-Schwarz inequality we have $\Sigma_i |x_i - y_i|\,\|e_i\|_c \leq \sqrt{\Sigma_i (x_i-y_i)^2}\sqrt{\Sigma_i \|e_i\|_c^2} = \|x-y\|_2\sqrt{\Sigma_i \|e_i\|_c^2}$

So we have $\|x-y\|_c \leq \|x-y\|_2\sqrt{\Sigma_i \|e_i\|_c^2} \leq \|x-y\|_2 \cdot \sqrt{n}\max_i \|e_i\|_c \qquad (6)$

So setting $\delta = \frac{\varepsilon}{\sqrt{n}\max_i \|e_i\|_c}$, it follows that $\|x-y\|_2 \leq \frac{\varepsilon}{\sqrt{n}\max_i \|e_i\|_c}$

$\Rightarrow \|x-y\|_c \leq \|x-y\|_2 \cdot \sqrt{n}\max_i \|e_i\|_c$ (from (6)) $\Rightarrow \|x-y\|_c \leq \varepsilon$ as desired.

Therefore $f(x) = \|x\|_c$ is continuous.

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1.a) Part 3

$D = \{x : \|x\|_2 = 1\}$ is a compact set since it is bounded (trivially, no coordinates can exceed 1) and closed (since the coordinates satisfy an algebraic equation for a unit sphere, all limit points are trivially contained within the set). Thus $f(x)$ achieves its maxima f_{max} and minima f_{min} on D.

Trivially, $\|x\|_c \geq 0$ by defintion, so $f_{min} \geq 0$.

Suppose $f_{min} = 0 \Rightarrow \|x\|_c = 0 \Rightarrow x = 0 \Rightarrow \|x\|_2 = 0 \Rightarrow x \notin D$.

Therefore, $f_{min} > 0$.

$f_{min} \leq f(x) \leq f_{max} \qquad \forall x \in D$

Define $C = \max\{f_{max}, \frac{1}{f_{min}}\} > 0$

So $C \geq f_{max}$, $\frac{1}{C} \leq f_{min}$

$\rightarrow \quad \frac{1}{C} \leq f(x) \leq C \qquad \forall x \in D \qquad (7)$

We need to prove that $\frac{1}{C}\|x\|_2 \leq \|x\|_c \leq C\|x\|_2 \qquad \forall x \in S$

So we divide by $\|x\|_2$:

$\rightarrow \quad \frac{1}{C} \leq \|\frac{x}{\|x\|_2}\|_c \leq C$

We note, however, that $\|\frac{x}{\|x\|_2}\|_2 = \frac{\|x\|_2}{\|x\|_2} = 1$, so $\frac{x}{\|x\|_2} \in D$, so we have already proven this (see (7)).

Therefore, we conclude that $\|\cdot\|_2$ is equivalent to $\|\cdot\|_c$ on S where $\|\cdot\|_c$ is any arbitrary norm on S. It then follows from Step 1's conclusion that all norms on S are equivalent, as desired.

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1. b) $1 \le q \le p \le +\infty$

i) $\|x\|_p \le \|x\|_q$? (8)

$\rightarrow \quad 1 \le \left\| \frac{x}{\|x\|_p} \right\|_q$

$\rightarrow \quad 1^q \le \left\| \frac{x}{\|x\|_p} \right\|_q^q$

$\rightarrow \quad 1^p \le \sum_i \left| \frac{x_i}{\|x\|_p} \right|^q$

$\rightarrow \quad \sum_i \left| \frac{x_i}{\|x\|_p} \right|^p \le \sum_i \left| \frac{x_i}{\|x\|_p} \right|^q$ (9)

We note that $\left| \frac{x_i}{\|x\|_p} \right| \le 1 \quad \forall i$, so we know it is true that

$\left| \frac{x_i}{\|x\|_p} \right|^p \le \left| \frac{x_i}{\|x\|_p} \right|^q \quad \forall i$, so it follows that (9) is true, so it follows that (8) is true. $\therefore$

Equality example: $x = [1, 0, 0, \dots, 0] \in \mathbb{R}^n$

↳ Trivially, $\|x\|_p = \|x\|_q = 1$

ii) $\|x\|_q^q = \sum_i |x_i|^q$

By Holder's Inequality we have

$\sum_i |x_i|^q (1) \le \left(\sum [|x_i|^q]^{p/q} \right)^{q/p} \left(\sum_i 1^{\frac{p}{p-q}} \right)^{1-q/p}$

$\rightarrow \quad \sum_i |x_i|^q \le \left[\sum_i |x_i|^p \right]^{q/p} n^{(1-q/p)}$

$\Rightarrow \quad \left[\sum_i |x_i|^q \right]^{1/q} \le \left[\sum_i |x_i|^p \right]^{1/p} n^{(1/q - 1/p)}$

$\Rightarrow \quad \|x\|_q \le n^{(1/q - 1/p)} \|x\|_p$ as desired.

Equality: consider $q = p = 2$, $x = [1, 0]$; $\|x\|_q \le n^{1/q - 1/p} \|x\|_p \rightarrow 1 = 2^0 (1) \rightarrow 1 = 1$ ✓

iii) $\|x\|_p \le n^{1/p} \|x\|_\infty$?

$\rightarrow \quad \sum_i |x_i|^p \le n \left[\max_i |x_i| \right]^p$

This is necessarily true since $\max_i |x_i| \ge x_i \quad \forall i$.

Equality: $p = 1$, $x = [1, 1]$

$\rightarrow \quad \sum_i |x_i|^p \le n \left[\max_i |x_i| \right]^p \rightarrow 2 \le 2 \rightarrow 2 = 2$ ✓

iv) $\|x\|_\infty \le \|x\|_p \rightarrow \max_i |x_i| \le \left[\sum_i |x_i|^p \right]^{1/p} \rightarrow \left[\max_i |x_i| \right]^p \le \sum_i |x_i|^p$

$\rightarrow$ This is trivially true since $[\max_i |x_i|]^p$ is contained within the sum on the LHS.

Equality: $p = 1$, $x = [1, 0] \rightarrow 1 \le 1 + 0 \rightarrow 1 = 1$ ✓

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1.b) v) $\|A\|_1 \le n\|A\|_\infty$?

$$\rightarrow \sup_{x\in\mathbb{R}^n, x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} \le n \sup_{x\in\mathbb{R}^n, x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

$$\rightarrow \sup_{x\in\mathbb{R}^n} \frac{\sum_i |Ax_i|}{\sum_i |x_i|} \le \sup_{x\in\mathbb{R}^n} \frac{[\max_i |Ax_i|]\,n}{\max_i |x_i|}$$

It is clear that $\sum_i |x_i| \ge \max_i |x_i|$, and that $\sum_i |Ax_i| \le n\,[\max_i |Ax_i|]$, therefore it follows that

$$\sup_{x\in\mathbb{R}^n\, x\neq 0} \frac{\sum_i |Ax_i|}{\sum_i |x_i|} \le \sup_{x\in\mathbb{R}^n\, x\neq 0} \frac{[\max_i |Ax_i|]\,n}{\max_i |x_i|}$$ as desired.