Jacob Snyder

ACM106a Set 8 Problem 1

1.1 $\int_0^{2\pi} q(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} q(x)\,dx$ by the composite trapezoidal rule

$\approx \frac{h}{2}\left[q(0) + 2\sum_{i=1}^{n} q(x_i) + q(2\pi)\right]$ (where $q(x)$ is any 2π-periodic function)

Since q is 2π-periodic, $q(0) = q(2\pi)$:

$= \frac{2\pi}{2(n+1)}\left[2q(0) + 2\sum_{i=1}^{n} q(x_i)\right]$

$= \frac{2\pi}{n+1}\left[\sum_{i=0}^{n} q(x_i)\right]$ (1)

Consider $p(x)$ defined in the problem, $p(x) = \sum_{k=-n}^{n} c_k e^{ikt}$ term by term:

$\int_0^{2\pi} c_k e^{ikt} = \left[\frac{c_k}{ik} e^{ikt}\right]_0^{2\pi} = 0$ for $k \neq 0$, $k \in \mathbb{Z}$

$\searrow = 2\pi c_0$ for $k=0$

Returning to (1), we can show that for $q(x) = c_k e^{ikt}$ the composite trapezoidal rule yields $2\pi c_0$ for $k=0$ term:

$\frac{2\pi}{n+1}\sum_{i=0}^{n} c_0 = \frac{2\pi}{n+1}(n+1)(c_0) = 2\pi c_0$

And we can show that the composite trapezoidal rule yields 0 for $k \neq 0$ terms of the sum (denoted $q_k(x_i)$):

$\frac{2\pi}{n+1}\sum_{i=0}^{n} q_k(x_i) = \frac{2\pi}{n+1}\left[\sum_{j=0}^{n} c_k[\cos(kx_j) + i\sin(kx_j)]\right]$

Since cos and sin are 2π-periodic, it can be shown that the sin terms cancel with each other at the index pairs $(j=1, j=n)$, $(j=2, j=n-1)$, etc, using angle sum/difference trig identities.*1

Same with the cos terms, but at the index pairs $(i=0, i=n/2)$, $(i=1, i=n/2+1)$,*2 etc

So $\rightarrow \sum_{i=0}^{n} q(x_i) = 0$

so $\frac{h}{2}\left[p(0) + 2\sum_{i=1}^{n} p(x_i) + p(2\pi)\right] = \int_0^{2\pi} p(x)\,dx = 2\pi c_0$ as desired.

(cont'd on next page).

*1: If n is odd. Since $\sin(kx_0)=0$ and $\sin(kx_a) = -\sin(kx_{n-a+1})$ for $k \in \mathbb{Z}\setminus 0$
If n is even, then $\sin(kx_{n/2}) = 0$, and the terms pair off similarly without the $n/2$ term.

*2: If n is even. Since $\cos(kx)$ over $[0,\pi]$ is equivalent to $-\cos(kx)$ over $[\pi, 2\pi]$.
If n is odd, then to be honest I don't know how to show that these terms do cancel. But they definitely do. ☺

Jacob Snyder

ACM106A Set 8 Problem 1 (cont'd) And 2(iii) And 3(iii)

1.2. $\frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx$ where $g(x)$ is the approximating trig polynomial of degree n

Applying the trapezoidal rule:

$= \frac{2\pi}{(2\pi)2(n+1)}\left[g(0)+2\sum_{i=1}^{n} g(x_i)+g(2\pi)\right]$ (exact, by part (1.1))

← worst case, error $=\varepsilon$ everywhere

$= \frac{1}{2(n+1)}\Big[\underbrace{[g(0)+\varepsilon]+2\sum_{i=1}^{n}[g(x_i)+\varepsilon]+[g(2\pi)+\varepsilon]}_{\text{trapezoid rule on } f(x)} - \varepsilon(n+1)(2)\Big]$

We also know that

$$\left|\frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx\right| \le \varepsilon \quad \text{(this follows since } |f(x)-g(x)|\le\varepsilon\text{)}$$

$$\to \left|\frac{1}{2(n+1)}\left[[g(0)+\varepsilon]+2\sum_{i=1}^{n}[g(x_i)+\varepsilon]+[g(2\pi)+\varepsilon]-\varepsilon(n+1)(2)\right] - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx\right| \le \varepsilon$$

$$\to \left|\frac{1}{2(n+1)}\left[[g(0)+\varepsilon]+2\sum_{i=1}^{n}[g(x_i)+\varepsilon]+[g(2\pi)+\varepsilon]\right] - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx\right| - \varepsilon \le \varepsilon$$

$$\to \left|\underbrace{\frac{1}{2(n+1)}\left[[g(0)+\varepsilon]+2\sum_{i=1}^{n}[g(x_i)+\varepsilon]+[g(2\pi)+\varepsilon]\right]}_{\text{"worst case" trapezoid rule on } f(x) \text{ (error}=\varepsilon\text{ everywhere)}} - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx\right| \le 2\varepsilon \quad \text{as desired}$$

between g and f

1.3. The error improves drastically as you half h. The first halving (from k=0 to k=1) yields an error reduction by a factor of 100, the second halving by a factor of 100000, the third by ~ 50000000.

1.4. The function given, $e^{\frac{1}{\sqrt{2}}\sin(x)}$, is periodic and can therefore be closely approximated by trigonometric polynomials. Thus as we increase the degree of the approximating polynomial, we expect to see a reduction in the error of the approximation, and thus a reduction of the error of the trapezoidal method (by our result in part 1.2). So every time we increase k by 1 in part 1.3, we are effectively doubling the degree of said approximating polynomial.

2.iii) We can see that for each of the three ERK implementations, the error at t=2 decreases as we decrease h, as expected.

3.iii) The error is lower for smaller h, as expected.