Jacob Snyder

ACM104 Set 3

a) Consider:

$$\|u+v\| = \sqrt{\langle u+v, u+v\rangle} \quad \text{by definition}$$

$$\rightarrow \|u+v\| = \sqrt{\langle u, u+v\rangle + \langle v, u+v\rangle}$$

$$\rightarrow (\|u+v\|)^2 = \langle u, v\rangle + \langle u, u\rangle + \langle v, u\rangle + \langle v, v\rangle$$

$$\rightarrow (\|u+v\|)^2 - \|u\|^2 - \|v\|^2 = 2\langle u, v\rangle$$

$$\rightarrow \frac{(\|u+v\|)^2 - \|u\|^2 - \|v\|^2}{2} = \langle u, v\rangle$$

Because $\|u\|, \|v\|, \|u+v\|$ are known quantities, we can compute $\langle u, v\rangle$ for any $u, v \in V$.

b) Suppose there are two distinct inner products that induce the same norm: $\|a\| = \|b\|$ but $\langle a,a\rangle \neq \langle b,b\rangle$

$$\|a\| = \|b\| \rightarrow \sqrt{\langle a,a\rangle} = \sqrt{\langle b,b\rangle} \rightarrow \langle a,a\rangle = \langle b,b\rangle$$

CONTRADICTION - therefore there cannot exist two inner products that induce the same norm.

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2. a) $\langle f,g\rangle_2$ is the inner product. It satisfies all properties of an inner product while $\langle f,g\rangle_1$ violates Positive-definite:

Suppose $f(x)=1 \rightarrow f'(x)=0$

Then $\langle f,f\rangle_1 = \int_0^1 (0)(0)\,dx = 0$

So we have $\langle f,f\rangle_1 = 0$ BUT $f(x)\neq 0$ which violates the positive-definite requirement.

Therefore $\langle f,g\rangle_2$ is the inner product.

b) $|\langle f,g\rangle_2| \leq \|f\|\cdot\|g\|$

$\rightarrow \left|\int_0^1 (f(x)g(x)+f'(x)g'(x))\,dx\right| \leq \sqrt{\langle f,f\rangle}\sqrt{\langle g,g\rangle}$

Cauchy-Scharz:

$\rightarrow \left|\int_0^1 (f(x)g(x)+f'(x)g'(x))\,dx\right| \leq \sqrt{\int_0^1 (f(x)f(x)+f'(x)f'(x))\,dx \cdot \int_0^1 (g(x)g(x)+g'(x)g'(x))\,dx}$

$\|f+g\| \leq \|f\|+\|g\|$

$\rightarrow \sqrt{\langle f+g, f+g\rangle} \leq \sqrt{\langle f,f\rangle}+\sqrt{\langle g,g\rangle}$

Triangle Inequality:

$\rightarrow \sqrt{\int_0^1 ((f+g)(x))^2 + ((f+g)'(x))^2\,dx} \leq \sqrt{\int_0^1 (f(x))^2+(f'(x))^2\,dx} + \sqrt{\int_0^1 (g(x))^2+(g'(x))^2\,dx}$

c) $\cos\theta = \dfrac{\langle f,g\rangle}{\|f\|\cdot\|g\|} = \dfrac{\int_0^1 (e^x+0)\,dx}{\sqrt{\int_0^1 1\,dx}\cdot\sqrt{\int_0^1 (e^{2x}+e^{2x})\,dx}} = \dfrac{e-1}{\sqrt{[e^{2x}]_0^1}} = \dfrac{e-1}{\sqrt{e^2-1}}$

$\rightarrow \theta = \cos^{-1}\left(\dfrac{e-1}{\sqrt{e^2-1}}\right) = \boxed{47.17°}$

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4. a) $\langle 1, e^x \rangle = \int_0^1 e^x \, dx = e - 1$ $\qquad \langle 1, 1 \rangle = \int_0^1 1 \, dx = 1$

$\langle 1, e^{2x} \rangle = \int_0^1 e^{2x} dx = \frac{1}{2}(e^2 - 1)$ $\qquad \langle e^x, e^x \rangle = \int_0^1 e^{2x} dx = \frac{1}{2}(e^2-1)$

$\langle e^x, e^{2x} \rangle = \int_0^1 e^{3x} dx = \frac{1}{3}(e^3 - 1)$ $\qquad \langle e^{2x}, e^{2x} \rangle \int_0^1 e^{4x} dx = \frac{1}{4}(e^4 - 1)$

$$\rightarrow G = \begin{bmatrix} 1 & e-1 & \frac{1}{2}(e^2-1) \\ e-1 & \frac{1}{2}(e^2-1) & \frac{1}{3}(e^3-1) \\ \frac{1}{2}(e^2-1) & \frac{1}{3}(e^3-1) & \frac{1}{4}(e^4-1) \end{bmatrix}$$

b) We know that $1, e^x, e^{2x}$ are linearly independent, so yes $G$ must be positive definite.

c) $\langle 1, e^x \rangle = \int_0^1 e^x dx = e - 1$ $\qquad \langle 1, 1 \rangle = \int_0^1 1 \, dx = 1$

$\langle 1, e^{2x} \rangle = \int_0^1 e^{2x} dx = \frac{1}{2}(e^2 - 1)$ $\qquad \langle e^x, e^x \rangle = \int_0^1 e^{2x} + e^{2x} dx = e^2 - 1$

$\langle e^x, e^{2x} \rangle = \int_0^1 e^{3x} + 2e^{3x} dx = e^3 - 1$ $\qquad \langle e^{2x}, e^{2x} \rangle = \int_0^1 e^{4x} + 4e^{4x} dx = \frac{5}{4}(e^4 - 1)$

$$G = \begin{bmatrix} 1 & e-1 & \frac{1}{2}(e^2-1) \\ e-1 & e^2-1 & e^3-1 \\ \frac{1}{2}(e^2-1) & e^3-1 & \frac{5}{4}(e^4-1) \end{bmatrix}$$

Again, we know $G$ is positive definite because $1, e^x, e^{2x}$ are linearly independent.

d) No, because $1, e^x, e^{2x}$ are linearly independent so the Gram matrices will be positive definite regardless of the definition of $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$