Jacob Snyder

ACM157 Set 1

1. $\bar{y} = \frac{1}{n}\sum_{i=1}^{n}(\alpha+\beta x_i) = \frac{n\alpha}{n} + \frac{\beta}{n}\sum_{i=1}^{n} x_i = \boxed{\alpha+\beta\bar{x}}$

$\tilde{y} = \begin{cases} y_{(\frac{n+1}{2})} & \text{if } n \text{ is odd} \\ \frac{1}{2}\left(y_{(\frac{n}{2})} + y_{(\frac{n}{2}+1)}\right) & \text{if } n \text{ is even} \end{cases} = \begin{cases} \alpha+\beta x_{(\frac{n+1}{2})} & \text{if } n \text{ is odd} \\ \frac{1}{2}\left(\alpha+\beta x_{(\frac{n}{2})}+\alpha+\beta x_{(\frac{n}{2}+1)}\right) & \text{if } n \text{ is even} \end{cases}$

$\rightarrow \tilde{y} = \begin{cases} \alpha+\beta x_{(\frac{n+1}{2})} & \text{if } n \text{ is odd} \\ \alpha+\frac{\beta}{2}\left(x_{(\frac{n}{2})}+x_{(\frac{n}{2}+1)}\right) & \text{if } n \text{ is even} \end{cases} = \boxed{\alpha+\beta\tilde{x}}$

$s_y = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\alpha+\beta x_i-\alpha-\beta\bar{x}\right)^2} = \boxed{|\beta| s_x}$ We put an absolute value because $s_y \geq 0$

$IQR_y = Q_{3_y} - Q_{1_y} = \alpha+\beta Q_{3_x} - \alpha - \beta Q_{1_x} = \boxed{|\beta|(IQR_x)}$ Again, $IQR \geq 0$ so $|\beta|$

2. $\arg\min_{\alpha} \sum_{i=1}^{n}(x_i-\alpha)^2$

$\rightarrow \frac{d}{d\alpha}\left[\sum_{i=1}^{n}(x_i-\alpha)^2\right] = -2\sum_{i=1}^{n}(x_i-\alpha)$

$\rightarrow 0 = -2\sum_{i=1}^{n}(x_i-\alpha) \rightarrow \sum_{i=1}^{n}\alpha = \sum_{i=1}^{n}x_i \rightarrow n\alpha = \sum_{i=1}^{n}x_i \rightarrow \alpha = \frac{1}{n}\sum_{i=1}^{n}x_i = \bar{x}$

$\arg\min_{\alpha}\sum_{i=1}^{n}|x_i-\alpha| = \arg\min_{\alpha}\sum_{i=1}^{a}\alpha-x_i + \sum_{i=a+1}^{n}x_i-\alpha$ where a is the largest integer s.t. $\alpha-x_a \geq 0$

$\frac{d}{d\alpha} \rightarrow a-(n-a) = 0 \rightarrow 2a = n \rightarrow a = \frac{n}{2}$

Therefore we want a to split the dataset in half, this happens when $\alpha - x_{\frac{n+1}{2}} = 0$ (if n odd)

or if $\alpha - \frac{1}{2}\left(x_{(\frac{n}{2})}+x_{(\frac{n}{2}+1)}\right) = 0 \rightarrow \begin{cases} x_{(\frac{n+1}{2})} & \text{if } n \text{ odd} \\ \frac{1}{2}\left(x_{(\frac{n}{2})}+x_{(\frac{n}{2}+1)}\right) & \text{if } n \text{ even} \end{cases}$

3. b is a shift in the mean. Because Standard Normal distribution has mean 0, if we have a QQ plot of $y=ax+b$ then we have mean b.
a is a scaling of the standard deviation. Because Standard Normal Distribution has standard deviation 1, if we have a QQ plot of $y=ax+b$ then we have standard deviation of a. Therefore, if we have a QQ plot of $y=ax+b$, we have a normal distribution with mean b and standard deviation a.

(cont'd on back)

6\. a) $\boxed{\frac{1}{N}}$ for all

b) $1 - \left[\frac{N-1}{N} \cdot \frac{N-2}{N-1} \cdots \frac{N-n}{N-n+1}\right] = 1 - \frac{N-n}{N} = \boxed{\frac{n}{N}}$

c) $\sum_{i=1}^{N} i(1/N) = \frac{N(N+1)}{2N} = \boxed{\frac{N+1}{2}}$

d) $\boxed{\frac{1}{N(N-1)}}$

e) $\frac{1}{N} \cdot \frac{1}{N-1} \cdot \frac{1}{N-2} \cdots \frac{1}{N-n+1} = \boxed{\prod_{i=1}^{n} \frac{1}{N-i+1}}$

7\. a) $\boxed{\sum_{i=1}^{n} w_i = 1}$ so that on average $\bar{X}_n^W = \mu$

b) $\underset{w_i}{\operatorname{Argmin}} \left(\bar{X}_n^w - \frac{1}{N}\sum_{i=1}^{N} x_i\right)^2 = \underset{w_i}{\operatorname{argmin}} \left(\sum_{i=1}^{n} w_i X_i - \frac{1}{N}\sum_{i=1}^{N} x_i\right)^2$

$\Rightarrow 0 = 2\left[\sum_{i=1}^{n} w_i X_i - \frac{1}{N}\sum_{i=1}^{N} x_i\right] \sum_{i=1}^{n} w_i$

$\Rightarrow \sum_{i=1}^{n} w_i X_i = \frac{1}{N}\sum_{i=1}^{N} x_i$

$\rightarrow \sum_{i=1}^{n} w_i X_i = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] \Rightarrow \boxed{w_i = \frac{1}{n}}$ for all i