Jacob Snyder

ACM157 Set 4

2. a) $\int_\alpha^\beta x f(x;\alpha,\beta)\,dx = \frac{1}{n}\sum_{i=1}^n X_i \rightarrow \frac{\alpha+\beta}{2} = \bar{X}_n \rightarrow \alpha = 2\bar{X}_n - \beta$

$\int_\alpha^\beta x^2 f(x;\alpha,\beta)\,dx = \frac{1}{n}\sum_{i=1}^n [X_i]^2 \rightarrow \frac{1}{3}[\beta^3-\alpha^3]\left[\frac{1}{\beta-\alpha}\right] = \frac{1}{n}\sum_{i=1}^n [X_i]^2$

$\rightarrow \frac{1}{3}[\beta^2+\alpha\beta+\alpha^2] = \frac{1}{n}\sum_{i=1}^n [X_i]^2$

$\rightarrow \frac{1}{3}[\beta^2+\beta(2\bar{X}_n-\beta)+(2\bar{X}_n-\beta)^2] = \frac{1}{n}\sum_{i=1}^n [X_i]^2$

$\rightarrow \frac{1}{3}[\beta^2+2\beta\bar{X}_n-\beta^2+4\bar{X}_n^2-4\beta\bar{X}_n+\beta^2] = \frac{1}{n}\sum_{i=1}^n [X_i]^2$

$\rightarrow \frac{1}{3}[\beta^2-2\beta\bar{X}_n+4\bar{X}_n^2] = \frac{1}{n}\sum_{i=1}^n [X_i]^2$

$\rightarrow \beta^2-2\beta\bar{X}_n+\left[4\bar{X}_n^2-\frac{3}{n}\sum_{i=1}^n [X_i]^2\right] = 0$

$\rightarrow$ Solving (this is a quadratic): $\beta = \bar{X}_n \pm \sqrt{3}\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2 - \left[\frac{1}{n}\sum_{i=1}^n X_i\right]^2}$

Therefore: $$\boxed{\begin{aligned}\hat{\beta}_{MOM} &= \bar{X}_n + \sqrt{3}\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2 - \left[\frac{1}{n}\sum_{i=1}^n X_i\right]^2} \\ \hat{\alpha}_{MOM} &= \bar{X}_n - \sqrt{3}\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2 - \left[\frac{1}{n}\sum_{i=1}^n X_i\right]^2}\end{aligned}}$$

(Because $\alpha < \beta$)

b) Given our sample $X_1,\dots,X_n$, $\mathcal{L}(\theta \mid X_1,\dots,X_n) = 0$ for θ where

$\alpha > \min\{X_1,\dots,X_n\}$ or $\beta < \max\{X_1,\dots,X_n\}$ (intuitively, this is impossible)

so $\arg\max_{\theta_{MLE}} \mathcal{L}(\theta_{MLE}) = \arg\max_{\theta_{MLE}} \prod_{i=1}^n f(X_i;\theta_{MLE}) = \arg\max_{\theta_{MLE}} \frac{1}{\beta_{MLE}-\alpha_{MLE}}$

$$\rightarrow \boxed{\hat{\beta}_{MLE} = \max\{X_1,\dots,X_n\} \qquad \hat{\alpha}_{MLE} = \min\{X_1,\dots,X_n\}}$$

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4. a) $\mathbb{E}[Y_1] = P(X_1 > 0)(1) + P(X_1 \leq 0)(0) = P(X_1 > 0)$

$= 1 - \frac{1}{2}\operatorname{erfc}\left(\frac{\theta}{\sqrt{2}}\right) = \boxed{1 - \frac{1}{\sqrt{\pi}}\int_{\theta/\sqrt{2}}^{\infty} e^{-t^2}\,dt}$

$\hat{\psi}_{MLE} = \underset{\psi}{\operatorname{argmax}} \prod_{i=1}^{n} P(X_i ; \psi) = \boxed{1 - \frac{1}{\sqrt{\pi}}\int_{\bar{X}_n/\sqrt{2}}^{\infty} e^{-t^2}\,dt}$

Because this value of ψ corresponds to a normal distribution with mean $\bar{X}_n$, which is trivially the most likely normal distribution to yield the sample $X_1, \ldots, X_n$

b) First we find a confidence interval for θ based on $X_1, \ldots, X_n$:

$\bar{X}_n \pm \frac{1.96}{\sqrt{n}}$ because the distribution is normally distributed with standard deviation 1

So therefore our confidence interval is

$$\boxed{\left(1 - \frac{1}{\sqrt{\pi}}\int_{\frac{\bar{X}_n - \frac{1.96}{\sqrt{n}}}{\sqrt{2}}}^{\infty} e^{-t^2}\,dt,\ 1 - \frac{1}{\sqrt{\pi}}\int_{\frac{\bar{X}_n + \frac{1.96}{\sqrt{n}}}{\sqrt{2}}}^{\infty} e^{-t^2}\,dt\right)}$$

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3. a) Note that by definition, for a uniform distribution on $[\alpha, \beta]$,

$\mu = \frac{\alpha+\beta}{2}$. In problem 2 we found the MLE estimates of α and β:

$\hat{\alpha}_{MLE} = \min\{X_1, \dots, X_n\}$, $\hat{\beta}_{MLE} = \max\{X_1, \dots, X_n\}$.

Therefore we have that $\boxed{\hat{\mu}_{MLE} = \frac{\min\{X_1, \dots, X_2\} + \max\{X_1, \dots, X_2\}}{2}}$

b) $\mathbb{E}[\hat{\mu}_n] = \mu \rightarrow \text{bias}[\hat{\mu}_n] = 0$

$\mathbb{V}[\hat{\mu}_n] = \frac{\sigma^2}{n} = \frac{(3-1)^2}{12(10)} = \frac{1}{30} \rightarrow se[\hat{\mu}_n] = \frac{1}{\sqrt{30}}$

$\rightarrow MSE[\hat{\mu}_n] = 0 + \left(\frac{1}{\sqrt{30}}\right)^2 = \boxed{\frac{1}{30}}$

Monte carlo done in MATLAB

Scanned with CamScanner

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5. a) $cdf(x) = \prod_{i=1}^{n} P(X_i \le x) = P(X_i \le X)^n = \left(\frac{x}{\theta}\right)^n$

$\rightarrow pdf(x) = \frac{d}{dx}\left[\left(\frac{x}{\theta}\right)^n\right] = \boxed{\frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1}}$

b) done in MATLAB

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1.b) $$\hat{\theta}_n = \frac{\frac{1}{n}\Sigma(x_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\left[\frac{1}{n}\Sigma(X_i - \bar{X}_n)^2 \cdot \frac{1}{n}\Sigma(Y_i - \bar{Y}_n)^2\right]^{1/2}}$$