Jacob Snyder

ACM106a Final

1. a) $\varphi(x) = \frac{1}{2}\|Ax-b\|_2^2 + \frac{\beta}{2}\|x\|_2^2$

$\frac{d\varphi}{dx} = \sum_{i=1}^{n}(a_i x^* - b)(a_i) + \beta x^*$

$\rightarrow \quad 0 = \sum_{i=1}^{n}(a_i x^* - b)(a_i) + \beta x^* \rightarrow \sum_{i=1}^{n} a_i a_i^T x^* + \beta x^* = \sum_{i=1}^{n} b a_i$

$\rightarrow \quad \left(\sum_{i=1}^{n} a_i a_i^T + \beta\right) x^* = \sum_{i=1}^{n} b a_i$

$\rightarrow \quad x^* = \sum_{j=1}^{m} \frac{\sigma_j (u_j^T b)}{\sigma_j^2 + \beta}$

b) We have from page 51 of Lecture 1 that

$x^{**} = V\Sigma^{-1}U^T b = \sum_{j=1}^{m} \frac{u_j^T b}{\sigma_j} v_j$

c) $\|\tilde{x}^* - x^*\|_2 = \left\|\sum_{j=1}^{m} \frac{\sigma_j (u_j^T(b+\varepsilon))}{\sigma_j^2+\beta} v_j - \sum_{j=1}^{m} \frac{\sigma_j (u_j^T b)}{\sigma_j^2+\beta} v_j\right\|_2 = \left\|\varepsilon \sum_{j=1}^{m} \frac{\sigma_j u_j^T v_j}{\sigma_j^2+\beta}\right\|_2 = \left\|\varepsilon \sum_{j=1}^{m} \frac{u_j^T v_j}{\sigma_j} \cdot \underbrace{\frac{\sigma_j}{\sigma_j + \frac{\beta}{\sigma_j}}}_{\leq 1 \text{ since } \beta \geq 0}\right\|_2$

$\|\tilde{x}^{**} - x^{**}\|_2 = \left\|\sum_{j=1}^{m} \frac{u_j^T(b+\varepsilon)}{\sigma_j} v_j - \sum_{j=1}^{m} \frac{u_j^T b}{\sigma_j} v_j\right\|_2 = \left\|\varepsilon \sum_{j=1}^{m} \frac{u_j^T v_j}{\sigma_j}\right\|_2$

So $\|\tilde{x}^* - x^*\|_2 \leq \|\tilde{x}^{**} - x^{**}\|_2$ as desired.

3. a) $A = \begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & 0 & & & -a_2 \\ & & \ddots & \ddots & & \vdots \\ & & & 1 & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{bmatrix} \rightarrow xI_n - A = \begin{bmatrix} x & & & & & a_0 \\ -1 & x & & & & a_1 \\ & -1 & x & & & a_2 \\ & & \ddots & \ddots & & \vdots \\ & & & -1 & x & a_{n-2} \\ & & & & -1 & a_{n-1} \end{bmatrix}$

$\rightarrow$ Base case: $\det\left(\begin{bmatrix} x & a_{n-2} \\ -1 & a_{n-1} \end{bmatrix}\right) = x a_{n-1} + a_{n-2}$

$\rightarrow$ Induction step: $\det\begin{bmatrix} x & & a_i \\ -1 & x & a_{i+1} \\ & \ddots & \vdots \\ & & a_{n-1} \end{bmatrix} = x \det\begin{bmatrix} x & & a_{i+1} \\ -1 & x & a_{i+2} \\ & \ddots & \vdots \\ & & a_{n-1} \end{bmatrix} + a_i \underbrace{\det\begin{bmatrix} -1 & x & & \\ & -1 & x & \\ & & \ddots & x \\ & & & -1 \end{bmatrix}(-1)^{(n-i-2)}}_{= -1^{(n-i-2)}}$

$= x \det\begin{bmatrix} x & & a_{i+1} \\ -1 & x & a_{i+2} \\ & \ddots & \\ & & a_{n-1} \end{bmatrix} + a_i$

So by induction: $\det(xI_n - A) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ as desired

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4. 1) $u'' + \mu(u^2-1)u' + u = 0$

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}$$

$$\vec{f}(\vec{y}) = \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix}$$

$$\vec{y}'(t) = \vec{f}(\vec{y}(t))$$

$$\rightarrow \begin{bmatrix} u'(t) \\ u''(t) \end{bmatrix} = \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix} = \begin{bmatrix} y_2(t) \\ y_1(t)[1 - y_1^2(t)]y_2(t) - y_1(t) \end{bmatrix}$$