Jacob Snyder

ACM104 Set 4

1. $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 0 & 3 & 0 \\ 0 & 7 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Therefore $\ker(A) = \{0\}$ so we can apply $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$

Because all column vectors are linearly independent

Using MATLAB: (code attached)

$$x^* = (A^TA)^{-1}A^T\vec{b} = \boxed{\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}}$$

Using MATLAB for LSE: (code also attached)

$$LSE = \sqrt{\|\vec{b}\|^2 - \vec{b}^T A(A^TA)^{-1}A^T\vec{b}} = 1.1921 \times 10^{-7}$$

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2. a) $L_1(x) = \frac{(x-b)}{(a-b)}$ $L_2(x) = \frac{(x-a)}{(b-a)}$

$$P_1(x) = f(a)\frac{(x-b)}{(a-b)} + f(b)\frac{(x-a)}{(b-a)} = \frac{f(a)}{(a-b)}(x-b) + \frac{f(b)}{(b-a)}(x-a)$$

$$= \frac{f(a)x - f(a)b - f(b)x + f(b)a}{a-b}$$

$$\rightarrow \int_a^b \frac{f(a)x - f(a)b - f(b)x + f(b)a}{a-b}\,dx = \frac{1}{a-b}\left[\frac{f(a)-f(b)}{2}x^2 + f(b)ax - f(a)bx\right]_a^b$$

$$= \frac{1}{a-b}\left[\frac{f(a)-f(b)}{2}b^2 + f(b)ab - f(a)b^2 - \frac{f(a)-f(b)}{2}a^2 - f(b)a^2 + f(a)ab\right] = \boxed{\frac{b-a}{2}(f(a)+f(b))}$$

b) $L_1(x) = \frac{x - \frac{2}{3}(a+b)}{-\frac{1}{3}(a+b)}$ $L_2(x) = \frac{x - \frac{1}{3}(a+b)}{\frac{1}{3}(a+b)}$

$$P_1(x) = f\left(\tfrac{1}{3}(a+b)\right)\frac{x - \frac{2}{3}(a+b)}{-\frac{1}{3}(a+b)} + f\left(\tfrac{2}{3}(a+b)\right)\frac{x - \frac{1}{3}(a+b)}{\frac{1}{3}(a+b)}$$

$$\rightarrow \int_a^b f\left(\tfrac{1}{3}(a+b)\right)\frac{x - \frac{2}{3}(a+b)}{-\frac{1}{3}(a+b)} + f\left(\tfrac{2}{3}(a+b)\right)\frac{x - \frac{1}{3}(a+b)}{\frac{1}{3}(a+b)}\,dx$$

$$= \frac{1}{\frac{1}{3}(a+b)}\int_a^b f\left(\tfrac{1}{3}(a+b)\right)\left(\tfrac{2}{3}(a+b) - x\right) + f\left(\tfrac{2}{3}(a+b)\right)\left(x - \tfrac{1}{3}(a+b)\right)dx$$

$$= \frac{1}{\frac{1}{3}(a+b)}\left[f\left(\tfrac{1}{3}(a+b)\right)\left(\tfrac{2}{3}x(a+b) - \tfrac{1}{2}x^2\right) + f\left(\tfrac{2}{3}(a+b)\right)\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x(a+b)\right)\right]_a^b$$

$$= \frac{3}{a+b}\left[f\left(\tfrac{1}{3}(a+b)\right)\left(\tfrac{2}{3}b(a+b) - \tfrac{1}{2}b^2\right) + f\left(\tfrac{2}{3}(a+b)\right)\left(\tfrac{1}{2}b^2 - \tfrac{1}{3}b(a+b)\right) - f\left(\tfrac{1}{3}(a+b)\right)\left(\tfrac{2}{3}a(a+b) - \tfrac{1}{2}a^2\right) - f\left(\tfrac{2}{3}(a+b)\right)\left(\tfrac{1}{2}a^2 - \tfrac{1}{3}a(a+b)\right)\right]$$

$$= \frac{3}{a+b}\left[f\left(\tfrac{1}{3}(a+b)\right)\left(\tfrac{1}{6}b^2 - \tfrac{1}{6}a^2\right) + f\left(\tfrac{2}{3}(a+b)\right)\left(\tfrac{1}{6}b^2 - \tfrac{1}{6}a^2\right)\right] = \boxed{\frac{1}{2}(b-a)\left(f\left(\tfrac{1}{3}(a+b)\right) + f\left(\tfrac{2}{3}(a+b)\right)\right)}$$

(cont'd on back)

c) $\int_0^1 e^x\,dx = [e^x]_0^1 = e-1 \approx \boxed{1.718}$

Trapezoid rule: $\frac{b-a}{2}\left(f(a)+f(b)\right) = \frac{1}{2}(e+1) \approx \boxed{1.859}$

Open rule: $\frac{b-a}{2}\left(f\left(\frac{1}{3}(a+b)\right)+f\left(\frac{2}{3}(a+b)\right)\right) = \frac{1}{2}\left(e^{1/3}+e^{2/3}\right) \approx \boxed{1.674}$

$\int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = 1+1 = \boxed{2}$

Trapezoid rule: $\frac{b-a}{2}\left(f(a)+f(b)\right) = \frac{\pi}{2}\left(\sin(0)+\sin(\pi)\right) = \frac{\pi}{2}(0) = \boxed{0}$

Open rule: $\frac{b-a}{2}\left(f\left(\frac{1}{3}(a+b)\right)+f\left(\frac{2}{3}(a+b)\right)\right) = \frac{\pi}{2}\left(\sin\left(\frac{\pi}{3}\right)+\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\sqrt{3}}{2}\pi \approx \boxed{2.72}$

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3. a)

$$\begin{bmatrix} r_1 \\ \vdots \\ \vdots \\ r_m \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_m \end{bmatrix} - \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & x_1^{(1)}x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & x_1^{(2)}x_2^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1^{(m)} & x_2^{(m)} & x_1^{(m)}x_2^{(m)} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & x_1^{(1)}x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & x_1^{(2)}x_2^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1^{(m)} & x_2^{(m)} & x_1^{(m)}x_2^{(m)} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}_{\vec{b}}$$

Least squares solution $= \vec{x}^* = (A^TA)^{-1}A^T\vec{b}$

$$\rightarrow \vec{x}^* = \left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(m)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(m)} \\ x_1^{(1)}x_2^{(1)} & x_1^{(2)}x_2^{(2)} & \cdots & x_1^{(m)}x_2^{(m)} \end{bmatrix} \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & x_1^{(1)}x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & x_1^{(2)}x_2^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1^{(m)} & x_2^{(m)} & x_1^{(m)}x_2^{(m)} \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(m)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(m)} \\ x_1^{(1)}x_2^{(1)} & x_1^{(2)}x_2^{(2)} & \cdots & x_1^{(m)}x_2^{(m)} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

b) $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$

$$\vec{x}^* = \begin{bmatrix} 63.5579 \\ -0.0108 \\ -0.2508 \\ 0.0001 \end{bmatrix}$$ using MATLAB