Julio Snyder

ACM101b Set 9

1. $$k(x,y) = \frac{-(x-y)\cdot\nu(y)}{2\pi|x-y|^2}$$

We parametrize as follows: $x=(s,f(s))$, $y=(t,f(t))$

$$\to k(x,y) = \frac{(t-s, f(t)-f(s))\cdot\nu(y(s))}{2\pi|(s-t, f(s)-f(t))|^2}$$

Using a taylor expansion of $f(t)$ about $t=s$ we have

$$= \frac{(t-s, f(s)+f'(s)(t-s)+\frac{1}{2}f''(s)(t-s)^2+E_2(t)-f(s))\cdot\nu(y(s))}{2\pi|(s-t, f(s)-f(s)-f'(s)(t-s)-E_1(t))|^2}$$

$$= \frac{(t-s, f'(s)(t-s)+\frac{1}{2}f''(s)(t-s)^2+E_2(t))\cdot\nu(y(s))}{2\pi|-(t-s, f'(s)(t-s)+E_1(t))|^2}$$

$$= \frac{(t-s)\,(1, f'(s)+\frac{1}{2}f''(s)(t-s)+\frac{E_2(t)}{t-s})\cdot\nu(y(s))}{(t-s)^2\,2\pi|(1, f'(s)+\frac{E_1(t)}{t-s})|^2}$$

$$= \frac{(1,f'(s))\cdot\nu(y(s)) + (0,\frac{1}{2}f''(s)(t-s))\cdot\nu(y(s)) + (0,\frac{E_2(t)}{t-s})\cdot\nu(y(s))}{(t-s)\,2\pi|(1,f'(s))+(0,\frac{E_1(t)}{t-s})|^2}$$

$$= \frac{x'(s)\cdot\nu(y(s)) + \frac{t-s}{2}x''(s)\cdot\nu(y(s)) + (0,\frac{E_2(t)}{t-s})\cdot\nu(y(s))}{(t-s)\,2\pi|x'(s)+(0,\frac{E_1(t)}{t-s})|^2}$$

$$= \frac{\frac{1}{2}x''(s)\cdot\nu(y(s)) + (0,\frac{E_2(t)}{(t-s)^2})\cdot\nu(y(s))}{2\pi|x'(s)+(0,\frac{E_1(t)}{t-s})|^2}$$

So as we take the limit as $t\to s$, we have

$$= \frac{\frac{1}{2}x''(s)\cdot\nu(y(s))}{2\pi|x'(s)|^2}$$

Therefore, so long as S is twice continuously differentiable, we have that k is a continuous function of x,y for $x,y\in S$. In fact, we can conclude from this that if S is infinitely differentiable, then k is infinitely differentiable, as desired. ∴

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2\. a) $\frac{d}{dt}\left[t^{-m} J_m(t)\right]$

$$= \frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{(-1)^k (1/2)^{m+2k} (t)^{m+2k-m}}{k!\,(m+k)!}\right]$$

$$= \sum_{k=0}^{\infty} \frac{2k(-1)^k (1/2)^{m+2k} t^{2k-1}}{k!(m+k)!} = \sum_{k=1}^{\infty} \frac{2k(-1)^k (1/2)^{m+2k} t^{2k-1}}{k!(m+k)!}$$

$$= \sum_{k=0}^{\infty} \frac{2(k+1)(-1)^{k+1} (1/2)^{m+2k+2} t^{2k+1}}{(k+1)!(m+k+1)!} = -t^{-m} \sum_{k=0}^{\infty} \frac{(-1)^k (1/2)^{m+1+2k} t^{m+1+2k}}{k!(m+1+k)!} = -t^{-m} J_{m+1}(t)$$

as desired.

$$\frac{d}{dt}\left[t^m J_m(t)\right] = \frac{d}{dt}\left[t^m \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}t)^{m+2k}}{k!(m+k)!}\right]$$

$$= \frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2})^{m+2k} (t)^{2m+2k}}{k!(m+k)!}\right] = \sum_{k=0}^{\infty} (2m+2k) \frac{(-1)^k (\frac{1}{2})^{m+2k} t^{(2m+2k-1)}}{k!(m+k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2})^{m-1+2k} t^{m-1+2k+m}}{k!(m+k-1)!} = t^m \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2})^{m-1+2k} t^{m-1+2k}}{k!(m-1+k)!} = t^m J_{m-1}(t)$$

as desired.

b) $\frac{d}{dt}\left[t^2 \{J_m(t)^2 - J_{m+1}(t) J_{m-1}(t)\}\right]$

$$= \frac{d}{dt}\left[t^2 (t^m J_m)(t^{-m} J_m) - (t^{m+1} J_{m+1})(t^{-(m-1)} J_{m-1})\right]$$

$$= 2t J_m^2 + t^2 \left[\frac{d}{dt}[t^m J_m](t^{-m} J_m) + (t^m J_m)\frac{d}{dt}[t^{-m} J_m]\right] - \frac{d}{dt}[t^{m+1} J_{m+1}](t^{-(m-1)} J_{m-1})$$

$$- (t^{m+1} J_{m+1})\frac{d}{dt}[t^{-(m-1)} J_{m-1}]$$

$$= 2t J_m^2 + t^2 [J_m J_{m-1} - J_m J_{m+1}] - J_m J_{m-1} t^2 + J_{m+1} J_m t^2$$

$$= 2t J_m^2 \text{ as desired.}$$

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2. c) $\int_0^1 x J_m^2(j_{m,k} x)\,dx$

$= \frac{1}{2}\int_0^1 2x J_m^2(j_{m,k}x)\,dx$

Using (4):

$= \frac{1}{2}\left[x^2\{J_m(j_{m,k}x)^2 - J_{m+1}(j_{m,k}x)J_{m-1}(j_{m,k}x)\}\right]_0^1$

$= \frac{1}{2}\left[J_m(j_{m,k})^2 - J_{m+1}(j_{m,k})J_{m-1}(j_{m,k})\right]$

$= \frac{1}{2}\left[-J_{m+1}(j_{m,k})J_{m-1}(j_{m,k})\right]$

From (3): $\frac{d}{dt}[t^m J_m(t)] = t^m J_{m-1}(t)$

$\rightarrow J'_m(t)t^m + mt^{m-1}J_m(t) = t^m J_{m-1}(t)$

$\rightarrow J'_m(j_{m,k}) + 0 = J_{m-1}(j_{m,k})$

From (2): $\frac{d}{dt}[t^{-m}J_m(t)] = -t^{-m}J_{m+1}(t)$

$\rightarrow -mt^{-m-1}J_m(t) + t^{-m}J'_m(t) = -t^{-m}J_{m+1}(t)$

$\rightarrow 0 + J'_m(j_{m,k}) = -J_{m+1}(j_{m,k})$

$\Rightarrow J_{m-1}(j_{m,k}) = -J_{m+1}(j_{m,k})$

$= \frac{1}{2}\left[J^2_{m+1}(j_{m,k})\right]$ as desired.

d) $a_n = \frac{\int_0^1 x f_\gamma(x) J_0(j_{0,n}x)\,dx}{\int_0^1 x (J_0(j_{0,n}x))^2\,dx} \overset{\text{from (5)}}{=} \frac{\int_0^\gamma x J_0(j_{0,n}x)\,dx}{\frac{1}{2}J_1^2(j_{0,n})}$ let $u = j_{0,n}x \rightarrow du = j_{0,n}\,dx$

from (3)

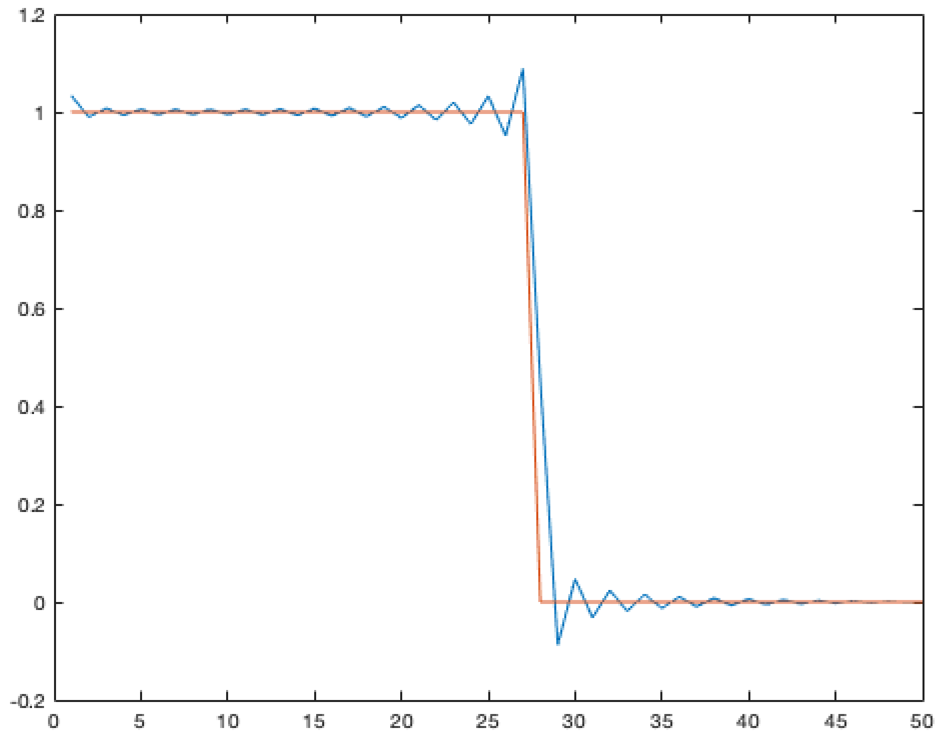
$= \frac{\left(\frac{1}{j_{0,n}}\right)^2 [uJ_1(u)]_0^{j_{0,n}\gamma}}{\frac{1}{2}J_1^2(j_{0,n})} = \boxed{\frac{2\gamma J_1(j_{0,n}\gamma)}{j_{0,n}J_1^2(j_{0,n})}} \rightarrow f_\gamma(x) = \sum_{n=1}^{\infty} a_n J_0(j_{0,n}x)$

We can see in the MATLAB plots attached that the Gibbs phenomenon occurs both at $x=0$ and $x=\gamma$. This makes sense given that discontinuities occur at both points. The overshoot at $x=\gamma$ appears to be approximately 8.95%, but it appears to be less at $x=0$.

We note that if $\gamma=1$, then the overshoot at $x=\gamma$ is significantly larger, approx 18%. This is likely because the approximation goes to -1 at $x=1$ in order to be 1-periodic.

```
N = 50;
gamma = 0.55;
xs = linspace(0,1,N);
joon = besselzero(0,N);

approx = zeros(1,N);
for a=1:N
    an = 2.*gamma.*besselj(1,joon(a).*gamma)./
(joon(a).*besselj(1,joon(a)).^2);
    approx = approx + an.*besselj(0,joon(a).*xs);
end

actual = zeros(1,N);
actual(xs<=gamma) = 1;
actual(xs>gamma) = 0;

plot(approx);
hold on;
plot(actual);

overshoot_at_gamma = abs(approx(floor(gamma.*N))-1) %approx 8.95
 percent, as expected
overshoot_at_0 = abs(approx(1)-1)
```

```
overshoot_at_gamma =

    0.0884


overshoot_at_0 =

    0.0336
```



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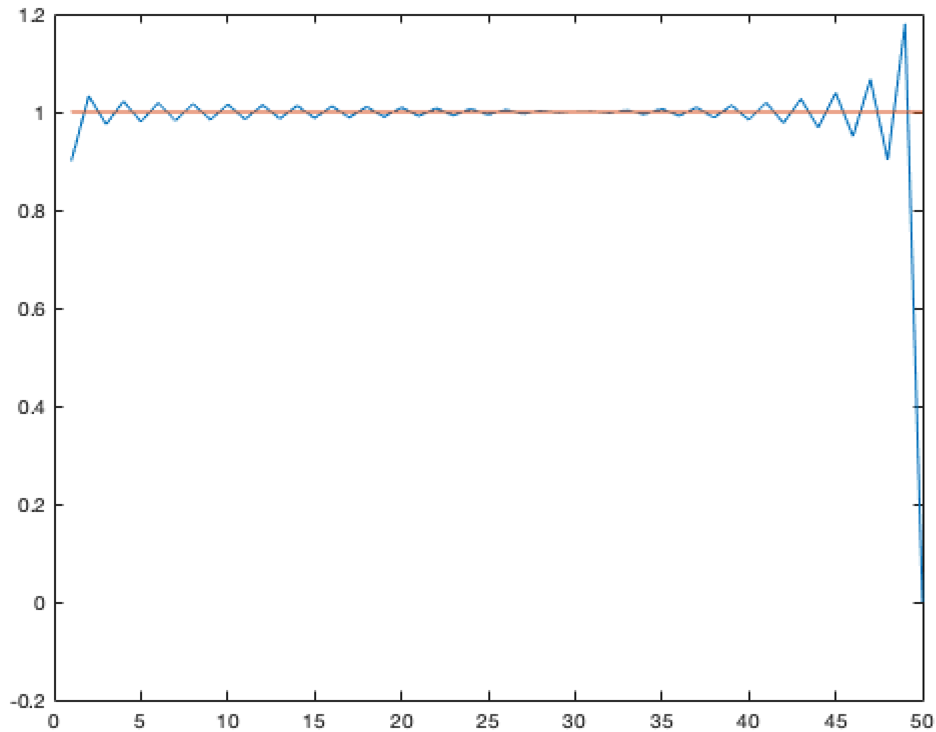
```
N = 50;
gamma = 1;
xs = linspace(0,1,N);
joon = besselzero(0,N);

approx = zeros(1,N);
for a=1:N
    an = 2.*gamma.*besselj(1,joon(a).*gamma)./
(joon(a).*besselj(1,joon(a)).^2);
    approx = approx + an.*besselj(0,joon(a).*xs);
end

actual = zeros(1,N);
actual(xs<=gamma) = 1;
actual(xs>gamma) = 0;

plot(approx);
hold on;
plot(actual);

overshoot_at_gamma = abs(approx(floor(gamma.*N)-1)-1) %approx 8.95
 percent, as expected
overshoot_at_0 = abs(approx(1)-1)
```

```
overshoot_at_gamma =

    0.1802


overshoot_at_0 =

    0.0997
```



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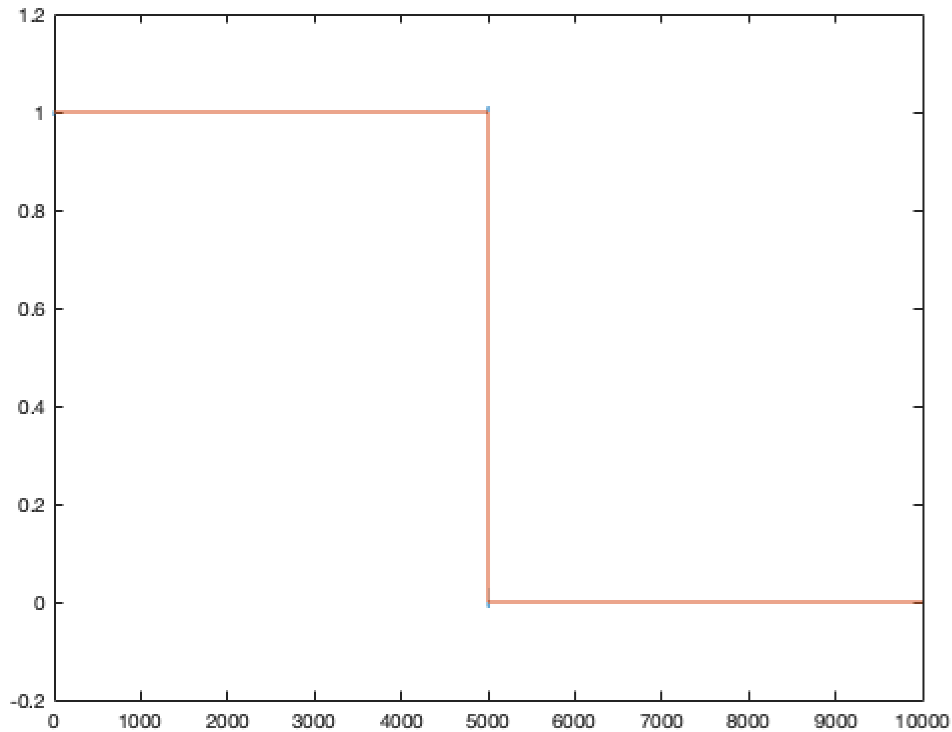
```
N = 10000;
gamma = 0.50;
xs = linspace(0,1,N);
joon = besselzero(0,N);

approx = zeros(1,N);
for a=1:N
    an = 2.*gamma.*besselj(1,joon(a).*gamma)./
(joon(a).*besselj(1,joon(a)).^2);
    approx = approx + an.*besselj(0,joon(a).*xs);
end

actual = zeros(1,N);
actual(xs<=gamma) = 1;
actual(xs>gamma) = 0;

plot(approx);
hold on;
plot(actual);

overshoot_at_gamma = abs(approx(floor(gamma.*N))-1) %approx 8.95
 percent, as expected
overshoot_at_0 = abs(approx(1)-1)
```

```
overshoot_at_gamma =

    0.0636


overshoot_at_0 =

    0.0065
```



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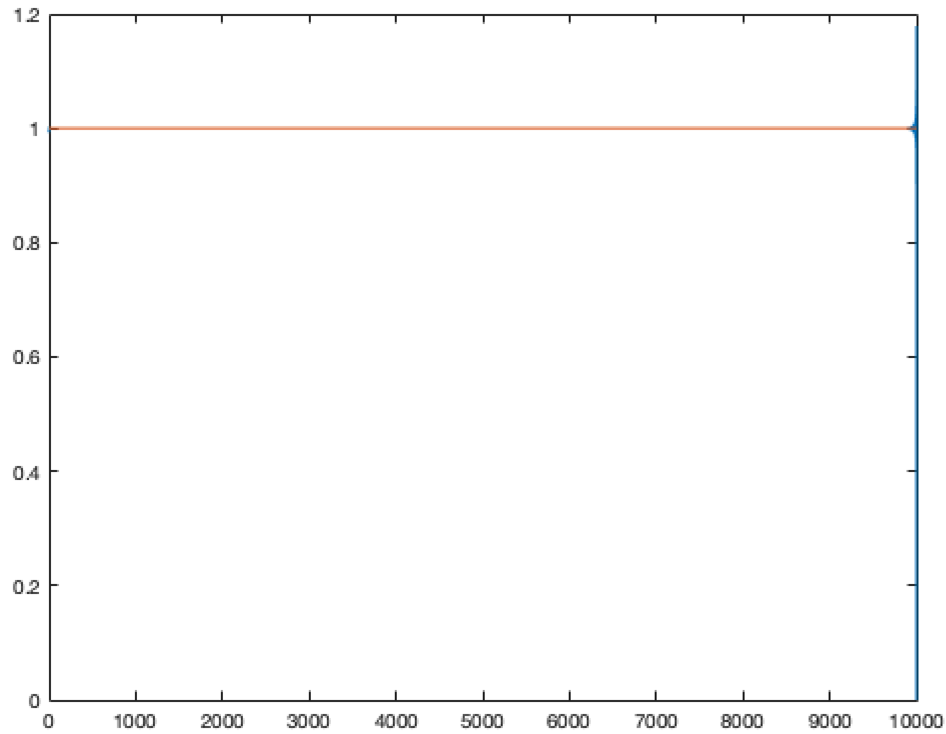
```
N = 10000;
gamma = 1;
xs = linspace(0,1,N);
joon = besselzero(0,N);

approx = zeros(1,N);
for a=1:N
    an = 2.*gamma.*besselj(1,joon(a).*gamma)./
(joon(a).*besselj(1,joon(a)).^2);
    approx = approx + an.*besselj(0,joon(a).*xs);
end

actual = zeros(1,N);
actual(xs<=gamma) = 1;
actual(xs>gamma) = 0;

plot(approx);
hold on;
plot(actual);

overshoot_at_gamma = abs(approx(floor(gamma.*N)-1)-1) %approx 8.95
 percent, as expected
overshoot_at_0 = abs(approx(1)-1)
```

```
overshoot_at_gamma =

    0.1790


overshoot_at_0 =

    0.0071
```



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