Jacob Snyder

ACM104 Set 2

1. a) Not a subspace: let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A = \det B = 0$ so $A, B \in W$

$\det(A+B) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ so $A+B \notin W$

Because W isn't closed under addition, it is not a subspace of V.

b) W is a subspace. Proof:

Let A and B be matrices such that $A, B \in W$

$\rightarrow \operatorname{tr} A = \operatorname{tr} B = 0 \rightarrow \sum_{i=1}^{n} a_{ii} = 0, \quad \sum_{i=1}^{n} b_{ii} = 0$

$\operatorname{tr}(A+B) = \sum_{i=1}^{n} a_{ii} + b_{ii} = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = 0 + 0 = 0$

$\operatorname{tr}(\alpha A) = \sum_{i=1}^{n} \alpha a_{ii} = \alpha \sum_{i=1}^{n} a_{ii} = \alpha(0) = 0$

$\operatorname{tr}(0) = 0$

Therefore, because W is closed under addition and scalar multiplization and includes the zero matrix, W is a subspace of V

c) Not a subspace. Proof:

Let $f(x) = 1$

$f(0) f(1) = 1(1) = 1$ so $f \in W$

$(2f(0))(2f(1)) = 4$ so $2f \notin W$ so W is not a subspace of V

d) W is a subspace. Proof:

Let $f = 0$: $f(1/2) = \int_0^1 f(t)\,dt \rightarrow 0 = 0$ ✓

we can say $(f+g)(1/2) = f(1/2) + g(1/2)$ bc. f and g are continuous

$(f+g)(1/2) = \int_0^1 (f+g)(t)\,dt \rightarrow f(1/2) + g(1/2) = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt$ ✓

$(\alpha f)(1/2) = \int_0^1 \alpha f(t)\,dt \rightarrow \alpha f(1/2) = \alpha \int_0^1 \alpha f(t)\,dt$ ✓

Therefore because W is closed under addition, scalar multiplication, and includes 0, it is a subspace of V.

e) W is a subspace. Proof:

Let $v(x,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \nabla \cdot v = 0 + 0 = 0$ ✓

Suppose $a(x,y) = \begin{bmatrix} a_1(x,y) \\ a_2(x,y) \end{bmatrix}$ and $b(x,y) = \begin{bmatrix} b_1(x,y) \\ b_2(x,y) \end{bmatrix}$ s.t. $a, b \in W$

$(a+b)(x,y) = \begin{bmatrix} a_1(x,y) + b_1(x,y) \\ a_2(x,y) + b_2(x,y) \end{bmatrix} \rightarrow \nabla \cdot (a+b)(x,y) = \frac{\partial a_1}{\partial x} + \frac{\partial b_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial b_2}{\partial y} = \nabla \cdot a + \nabla \cdot b = 0$

$(\alpha a)(x,y) = \begin{bmatrix} \alpha a_1(x,y) \\ \alpha a_2(x,y) \end{bmatrix} \rightarrow \nabla \cdot (\alpha a)(x,y) = \alpha \frac{\partial a_1}{\partial x} + \alpha \frac{\partial a_2}{\partial y} = \alpha(0) = 0$ ✓

Therefore W is a subspace of V because it includes 0 and is closed under scalar multiplization, and closed under addition.

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ACM104 Set 2

2\. a) $\begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow x^2 \\ \leftarrow x \\ \leftarrow 1 \end{array}$

$$\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to$$ p_1, p_2, p_3 are linearly independent.

b) Because quadratic polynomials have 3 terms that define all of their characteristics, $\dim(P^{(2)}) = 3$.

Therefore because p_1, p_2, p_3 are linearly independent, they are sufficient to span $P^{(2)}$. So <u>yes</u>.

c) Because p_1, p_2, p_3 are linearly independent and span $P^{(2)}$, by definition they form a basis for $P^{(2)}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ -3 & 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & 4 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 2 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 1/2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/8 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/8 \end{array}\right]$$

So the coordinates of $q(x) = 1$ in this basis is $(-1/8, 1/4, 1/8)$

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ACM104

3. a) • Consider $f_0 = (0, 0, 0, \cdots, 0)$

For all x_n: $x_n = x_{n-1} + x_{n-2} \rightarrow 0 = 0 + 0 = 0$ ✓

Consider $f_1 = (x_1, x_2, \ldots)$, $f_2 = (y_1, y_2, \ldots)$

• $f_1 + f_2 = (x_1 + y_1, x_2 + y_2, \ldots) \rightarrow$ for all $(f_1+f_2)_n$: $(f_1+f_2)_n = (f_1+f_2)_{n-1} + (f_1+f_2)_{n-2}$

$= x_{n-1} + y_{n-1} + x_{n-2} + y_{n-2}$

$= x_n + y_n$ ✓

• $\alpha f_1 = (\alpha x_1, \alpha x_2, \ldots)$ for all $(\alpha f_1)_n$: $(\alpha f_1)_n = (\alpha f_1)_{n-1} + (\alpha f_1)_{n-2} = \alpha x_{n-1} + \alpha_{n-2} = \alpha x_n$ ✓

• $f_2 + f_1 = (x_1 + y_1, x_2 + y_2, \ldots) = f_1 + f_2$ ✓

• $f_3 = (z, z_2, \ldots)$

$f_3 + (f_1 + f_2) = (x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots)$, $(f_3 + f_1) + f_2 = (x_1 + y_1 + z_1, x_2 + y_2 + z_2 \ldots) \rightarrow f_3 + (f_1 + f_2) = (f_3 + f_1) + f_2$ ✓

• $-f_1 = (-x_1, -x_2, \ldots) \rightarrow f_1 + (-f) = (0, 0, \ldots)$ ✓

• $1 \cdot f_1 = (1 \cdot x_1, 1 \cdot x_2, \cdots) = (x_1, x_2, \ldots) = f_1$ ✓

Therefore, because all axioms are satisfied, F is a vector space.

b) Because all numbers beyond the first two are derived from the first two, the first two numbers completely describe any one fibonacci sequence. Therefore, because there are only 2 factors we can vary to differentiate sequences (numbers 1 and 2), the dimension of F is 2.

To get a basis we just need to get a basis of all possible 2 starting numbers and then derive the rest of the basis vectors from the 2 starting numbers:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ \vdots \end{bmatrix}$$

Therefore,

These span F. They're linearly independent by observation: (Looking at the first entry, the only time they'll be equal is if the left vector is multiplied by 0).

Therefore they form a basis of F.

c) (1,1) yields:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ 13 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ 13 \\ 21 \\ \vdots \end{bmatrix} = f^*$$

So the coordinates of f^* in this basis is (1,1).

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ACM104 Set 2

4.
$$A=\begin{bmatrix}1 & 2 & \cdots & n\\ n+1 & n+2 & \cdots & 2n\\ \vdots & \vdots & & \vdots\\ n^2-n+1 & n^2-n+2 & \cdots & n^2\end{bmatrix}\rightarrow\begin{bmatrix}1 & 2 & \cdots & n\\ n & n & \cdots & n\\ \vdots & \vdots & & \vdots\\ n^2-n & n^2-n & \cdots & n^2-n\end{bmatrix}\rightarrow\begin{bmatrix}1 & 2 & \cdots & n\\ n & n & \cdots & n\\ & 0 & & \end{bmatrix}$$

$$\rightarrow\begin{bmatrix}1 & 2 & \cdots & n\\ 1 & 1 & \cdots & 1\\ & 0 & & \end{bmatrix}\rightarrow\begin{bmatrix}1 & 2 & 3 & \cdots & n\\ 0 & -1 & -2 & \cdots & -n+1\\ & & 0 & & \end{bmatrix}\rightarrow\begin{bmatrix}1 & 0 & -1 & \cdots & -n+2\\ 0 & 1 & 2 & \cdots & n-1\\ & & 0 & & \end{bmatrix}$$

Therefore, because there are pivots in the first 2 columns, we can use the first 2 columns of A to get a basis for $\text{im}(A)$:

$$\underline{\text{im}(A)}=\text{span}\left(\begin{bmatrix}1\\ n+1\\ \vdots\\ n^2-n+1\end{bmatrix},\begin{bmatrix}2\\ n+2\\ \vdots\\ n^2-n+2\end{bmatrix}\right)$$

NOW from the reduced matrix we can get $\ker(A)$:

$$\begin{bmatrix}1 & 0 & -1 & \cdots & -n+2\\ 0 & 1 & 2 & \cdots & n+1\\ & & 0 & & \end{bmatrix}\rightarrow \begin{matrix}x_1=-x_3-2x_4-\cdots-(n-2)x_n\\ x_2=2x_3+3x_4+\cdots+(n-1)x_n\end{matrix}\rightarrow\underline{\ker(A)}=\text{span}\left(\begin{bmatrix}-1\\ 2\\ 1\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix},\begin{bmatrix}-2\\ 3\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix},\begin{bmatrix}-3\\ 4\\ 0\\ 0\\ 1\\ \vdots\\ 0\end{bmatrix},\cdots,\begin{bmatrix}-n+2\\ n-1\\ 0\\ 0\\ \vdots\\ 0\\ 1\end{bmatrix}\right)$$

$$A^T=\begin{bmatrix}1 & n+1 & \cdots & n^2-n+1\\ 2 & n+2 & \cdots & n^2-n+2\\ \vdots & \vdots & & \vdots\\ n & 2n & \cdots & n^2\end{bmatrix}\rightarrow$$ Subtract each row from the row beneath it $$\rightarrow\begin{bmatrix}1 & n+1 & \cdots & n^2-n+1\\ 1 & 1 & \cdots & 1\\ \vdots & \vdots & & \vdots\\ 1 & 1 & \cdots & 1\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}0 & n & \cdots & n^2-n\\ 1 & 1 & \cdots & 1\\ & 0 & & \end{bmatrix}\rightarrow\begin{bmatrix}0 & 1 & \cdots & n-1\\ 1 & 1 & \cdots & 1\\ & 0 & & \end{bmatrix}\rightarrow\begin{bmatrix}1 & 0 & -1 & \cdots & -n+2\\ 0 & 1 & 2 & \cdots & n-1\\ & & 0 & & \end{bmatrix}$$ $\rightarrow$ pivots in first 2 columns $\rightarrow$ $$\underline{\text{coIm}(A)}=\text{span}\left(\begin{bmatrix}1\\ 2\\ \vdots\\ n\end{bmatrix},\begin{bmatrix}n+1\\ n+2\\ \vdots\\ 2n\end{bmatrix}\right)$$

Now we use the reduced matrix to find $\text{coKer}(A)$:

$$\begin{bmatrix}1 & 0 & -1 & \cdots & -n+2\\ 0 & 1 & 2 & \cdots & n-1\\ & & 0 & & \end{bmatrix}\rightarrow \begin{matrix}x_1=-x_3-2x_4-\cdots-(n-2)x_n\\ x_2=2x_3+3x_4+\cdots+(n-1)x_n\end{matrix}\rightarrow\underline{\text{coKer}(A)}=\text{Span}\left(\begin{bmatrix}-1\\ 2\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix},\begin{bmatrix}-2\\ 3\\ 0\\ 1\\ \vdots\\ 0\end{bmatrix},\begin{bmatrix}-3\\ 4\\ 0\\ 0\\ 1\\ \vdots\\ 0\end{bmatrix},\cdots,\begin{bmatrix}-n+2\\ n-1\\ 0\\ 0\\ \vdots\\ 1\end{bmatrix}\right)$$