Jacob Snyder

ACM101b Set 8

1. $I(x) = \frac{1}{2\pi}\int_{-1}^{1} \frac{\partial}{\partial \nu_y} \log(|x-(s,0)|)\,ds$

We will def $\nu_y = \binom{0}{-1}$ since it's a normal vector at y, this value is appropriate for crossing this boundary

$\rightarrow I(x) = \frac{1}{2\pi}\int_{-1}^{1} \binom{0}{-1}\cdot\nabla_y \log(|x-(s,0)|)\,ds$

$= \frac{1}{2\pi}\int_{-1}^{1} \frac{d}{dx_2}\log\left(\sqrt{(x_1-s)^2+x_2^2}\right)ds$

$= \frac{1}{2\pi}\int_{-1}^{1} \frac{x_2}{x_2^2+(x_1-s)^2}\,ds$

$= \frac{1}{2\pi}\left[\arctan\frac{s-x_1}{x_2}\right]_{-1}^{1} = \frac{1}{2\pi}\left[\arctan\left(\frac{1-x_1}{x_2}\right) - \arctan\left(\frac{-1-x_1}{x_2}\right)\right]$

$= \frac{1}{2\pi}\begin{cases} 0 & , x_2 = 0 \\ \arctan\left[\frac{x_1+1}{x_2}\right] - \arctan\left[\frac{x_1-1}{x_2}\right] & , x_2 \neq 0 \end{cases}$

since we are on the interval $-1 < x_1 < 1$, (x_1+1) and (x_1-1) must have opposite signs.

When x_2 crosses the indicated boundary, we go from $\frac{1}{2\pi}(\arctan(\infty) - \arctan(-\infty)) = \frac{1}{2}$

to $\frac{1}{2\pi}(\arctan(-\infty) - \arctan(\infty)) = -\frac{1}{2}$

Therefore the jump is 1.

From the result in class:

$\text{jump} = 2\left(\frac{1}{2\pi}\left(\arctan\left(\frac{x_1+1}{0}\right) - \arctan\left(\frac{x_1-1}{0}\right)\right)\right) = 1$

So our result is consistent with that of class, as desired.

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2. $Tf = \int_0^{2\pi} K(x-y) f(y)\, dy$

$\rightarrow |Tf|^2 = \left| \int_0^{2\pi} K(x-y) f(y)\, dy \right|^2$

By Cauchy-Schwarz:

since K is 2π-periodic

$|Tf|^2 \le \int_0^{2\pi} |K(x-y)|^2 dy \int_0^{2\pi} |f(y)|^2 dy = \int_0^{2\pi} |K(y)|^2 dy \int_0^{2\pi} |f(y)|^2 dy < \infty \quad \forall y \in [0, 2\pi]$

$\Rightarrow \int_0^{2\pi} |Tf|^2 dy < 2\pi \left[\max(\{|Tf(y)| : y \in [0,2\pi]\}) \right] < \infty$

Therefore $Tf \in L^2[0,2\pi]$

$\|Tf\|_{L_2} = \left[\int_0^{2\pi} \left| \int_0^{2\pi} K(x-y) f(y)\, dy \right|^2 dx \right]^{1/2}$ By Cauchy-Schwarz:

$\le \left[\int_0^{2\pi} \int_0^{2\pi} |K(x-y)|^2 dy \int_0^{2\pi} |f(y)|^2 dy \, dx \right]^{1/2}$

$= \left[\int_0^{2\pi} |f(y)|^2 dy \int_0^{2\pi} \int_0^{2\pi} |K(x-y)|^2 dy dx \right]^{1/2}$ since K is 2π-periodic

$= \left[\int_0^{2\pi} |f(y)|^2 dy \right]^{1/2} \left[2\pi \int_0^{2\pi} |K(y)|^2 dy \right]^{1/2}$

$= \sqrt{2\pi}\, \|K\|_{L_2} \|f\|_{L_2}$ and we know that $\|K\|_{L_2} < \infty$ so let $\sqrt{2\pi}\, \|K\|_{L_2} = C$

$\rightarrow \|Tf\|_{L_2} \le C \|f\|_{L_2}$ therefore, T is a continuous linear operator from $L^2[0,2\pi]$ to $L^2[0,2\pi]$

a) Suppose A is invertible, and suppose $Ax = Ay$.

$\rightarrow Ax = Ay$

$\rightarrow A^{-1}Ax = A^{-1}Ay \Rightarrow x = y \Rightarrow$ A is injective, as desired.

b) Suppose $K(z) = \sum_{j=-\infty}^{\infty} k_j e^{ijz}$ $\quad k_j \neq 0 \ \forall j$

$T_K f = \int_0^{2\pi} K(x-y) f(y) dy = \int_0^{2\pi} \sum_{j=-\infty}^{\infty} k_j e^{ij(x-y)} f(y) dy$ note: $f(y) = \sum_{j=-\infty}^{\infty} f_j e^{ijy}$ since $f \in L^2$

$= \sum_{j=-\infty}^{\infty} k_j e^{ijx} f_j$ (the other terms = 0 by orthogonality)

Suppose $T_K f = 0$. then:

$\sum_{j=-\infty}^{\infty} k_j f_j e^{ijx} = 0$ we know $k_j \neq 0 \ \forall j$, and that f_j are linearly independent.

Therefore, $f = 0$ so T_K must be injective (since $T_K f = 0 \Rightarrow f = 0$).

(cont'd on back)

2b cont'd) Now suppose T_k is injective: $T_k f = T_k g \Rightarrow f = g$

Suppose $K(z) = \sum_{j=-\infty}^{\infty} k_j e^{ijz}$, where $k_j = 0$ for some $j = q$

def $f(z) = e^{izq}$, $g(z) = 0$

$T_k f = \int_0^{2\pi} \sum_{-\infty}^{\infty} k_j e^{ij(x-y)} e^{iqy} dy = 0$ by orthogonality.

$T_k g = \int_0^{2\pi} 0 \, dy = 0$

So $T_k g = T_k f$ but $f \neq g \rightarrow$ contradiction $\rightarrow k_j \neq 0 \ \forall j$.

For equation (3), the corresponding condition is as follows:

$(I + T_k) f = (I + T_k) g$

$\rightarrow f + \int_0^{2\pi} \sum k_j e^{ij(x-y)} f(y) dy = g + \int_0^{2\pi} \sum k_j e^{ij(x-y)} g(y) dy$

$\rightarrow (f - g) = \int_0^{2\pi} \sum k_j e^{ij(x-y)} [g(y) - f(y)] dy$

$\rightarrow \sum (f_j - g_j) e^{ijx} = \int_0^{2\pi} \sum k_j e^{ij(x-y)} [\sum (g_j - f_j) e^{ijy}] dy$

$\rightarrow \sum (f_j - g_j) e^{ijx} = \sum k_j e^{ijx} (g_j - f_j)$, by orthogonality

$\rightarrow \sum (f_j - g_j) e^{ijx} = \sum k_j (g_j - f_j) e^{ijx}$

If $f \neq g$, then we need the above relation to not be satisfied for injectiveness

Therefore $\boxed{k_j \neq -1 \ \forall j}$

c) $T_k[\varphi] = f$

Suppose $f = \sum k_j e^{ijx}$

$\rightarrow \int_0^{2\pi} \sum k_j e^{ij(x-y)} \varphi(y) dy = \sum k_j e^{ijx}$

$\Rightarrow \varphi(y) = \sum \varphi_j e^{ijy} \Rightarrow \int_0^{2\pi} \sum k_j e^{ij(x-y)} (\sum \varphi_j e^{ijy}) dy = \sum k_j e^{ijx} \varphi_j$ by orthogonality

$\Rightarrow \varphi_j = 1 \rightarrow \sum_{j=-\infty}^{\infty} \varphi_j^2 = +\infty \Rightarrow \varphi \notin L^2[0, 2\pi]$ So no, not every $f \in L^2[0, 2\pi]$ admits a soln $\varphi \in L^2[0, 2\pi]$.

However, for some $f \in L^2[0, 2\pi]$ a solution $\varphi \in L^2[0, 2\pi]$ is admitted:

def $f = k_1 e^{ix} \rightarrow \varphi = e^{ix}$

$\Rightarrow \int_0^{2\pi} \sum k_j e^{ij(x-y)} e^{iy} dy = k_1 e^{ix}$

The uniqueness of the fourier series ensures uniqueness of the soln:

$\int_0^{2\pi} \sum k_j e^{ij(x-y)} (\sum \varphi_j e^{ijy}) dy = \sum f_j e^{ijx}$

$\rightarrow \sum k_j \varphi_j e^{ijx} = \sum f_j e^{ijx} \rightarrow f_j = k_j \varphi_j$. So solns are unique.

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2.d) $(I+T_k)[\varphi] = f$

$\to \sum \varphi_j e^{ijx} + \int_0^{2\pi} \sum k_j e^{ij(x-y)} (\sum \varphi_j e^{ijy})\, dy = \sum f_j e^{ijx}$

$\to \sum \varphi_j e^{ijx} + \sum k_j \varphi_j e^{ijx} = \sum f_j e^{ijx}$

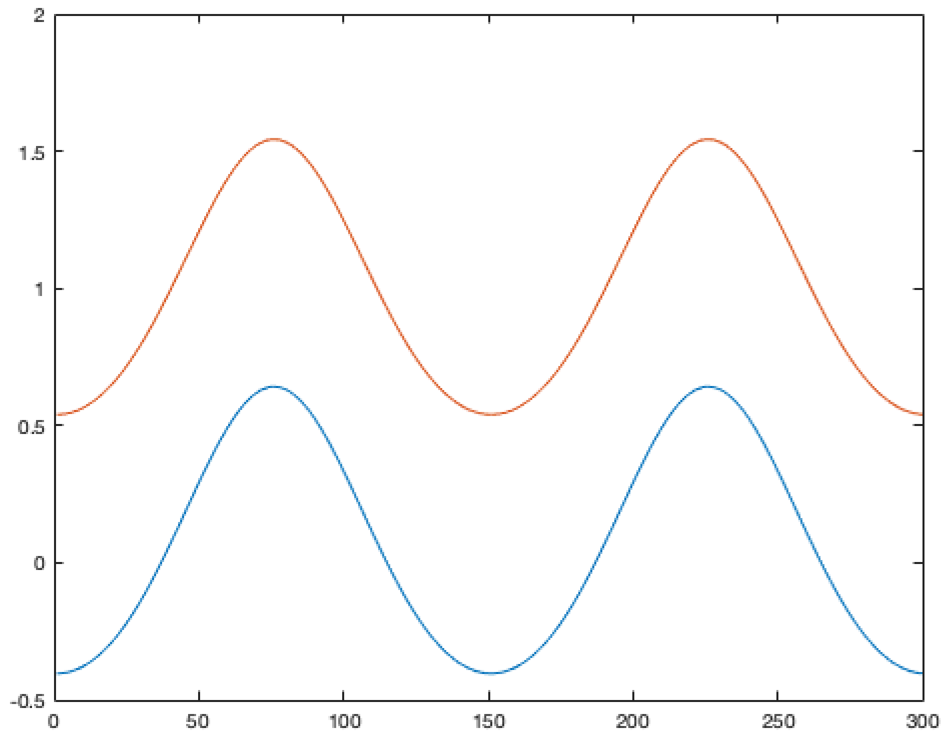
$\to \sum \varphi_j (k_j+1) e^{ijx} = \sum f_j e^{ijx}$

$\to \varphi_j = f_j/(k_j+1)$ again, the uniqueness of the fourier series guarantees uniqueness of a soln.

$\sum_{j=-\infty}^{\infty} \varphi_j^2 = \sum_{j=-\infty}^{\infty} \frac{f_j^2}{(1+k_j)^2} \lesssim \sum_{j=-\infty}^{\infty} f_j^2$ (since $k \in L^2[0,2\pi]$ so $1+k_j = O(1)$)

$< \infty \Rightarrow \varphi \in L^2[0,2\pi]$ so a unique soln always exists.

```
N = 300;
gamma = @(t) [cos(t); sin(t)]; %parametrization
gammadot = @(t) [-sin(t); cos(t)];
gammadotdot = @(t) [-cos(t); -sin(t)];

%1 Discretization
ts = linspace(0,2.*pi-2.*pi/N,N);

%2 construct D
D = zeros(N,N);
for i=1:N
   for j=1:N
       a = gammadotdot(ts(j));
       b = gammadot(ts(j));
       b = [-b(2) b(1)];
       vy = -dot(a,b)./dot(b,b).*b;
       if i==j
           D(i,i) = (vy*gamma(ts(i)))./(2.*N.*norm(gammadot(ts)));
       end
       if i~=j
           D(i,j) = vy*((gamma(ts(j))-gamma(ts(i)))./
(norm(gamma(ts(i))-gamma(ts(j))).^2)).*norm(gammadot(ts))./N;
       end
   end
end

%3 construct f
f = cos(cos(ts)).*cosh(sin(ts));

%4 solve for phi
phi = (eye(N)+2.*D)\f';

%5 plot results
figure
plot(phi')
hold on;
plot(f)

xs = gamma(ts);
figure
plot3(xs(1,:),xs(2,:),f)
figure
plot3(xs(1,:),xs(2,:),phi')
```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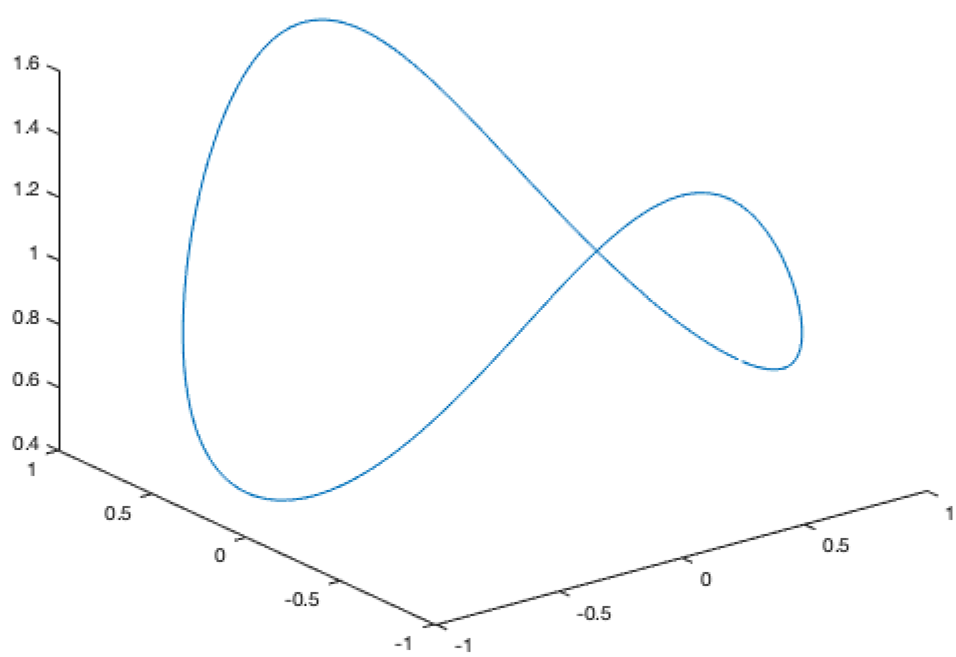
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0.4
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0.5
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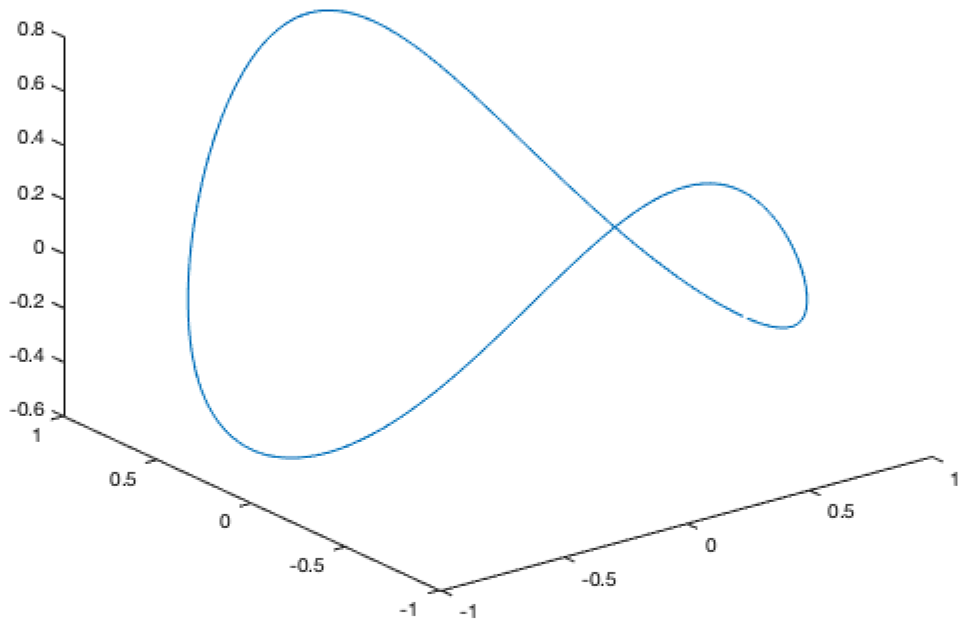
-1
-1
-0.5
0
0.5
1



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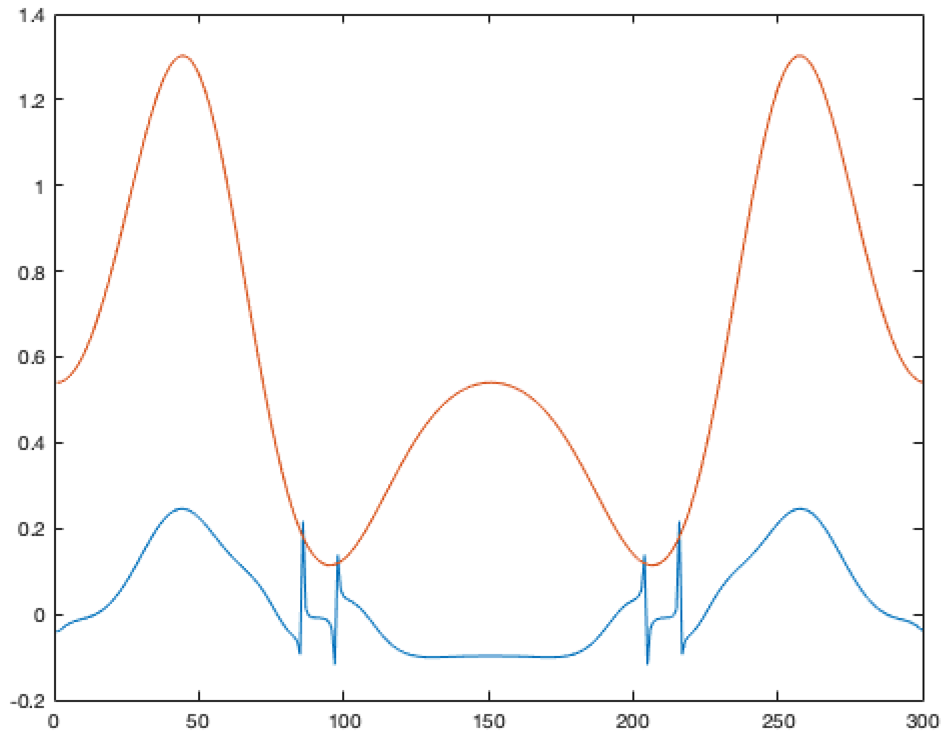
```
N = 300;
gamma = @(t) [cos(t) + 0.65.*cos(2.*t) - 0.65;
 sin(t)]; %parametrization
gammadot = @(t) [-sin(t)-1.3.*sin(2.*t); cos(t)];
gammadotdot = @(t) [-cos(t)-2.6.*cos(2.*t); -sin(t)];

%1 Discretization
ts = linspace(0,2.*pi-2.*pi/N,N);

%2 construct D
D = zeros(N,N);
for i=1:N
   for j=1:N
       a = gammadotdot(ts(j));
       b = gammadot(ts(j));
       b = [-b(2) b(1)];
       vy = -dot(a,b)./dot(b,b).*b;
       if i==j
           D(i,i) = (vy*gamma(ts(i)))./(2.*N.*norm(gammadot(ts)));
       end
       if i~=j
           D(i,j) = vy*((gamma(ts(j))-gamma(ts(i)))./
(norm(gamma(ts(i))-gamma(ts(j))).^2)).*norm(gammadot(ts))./N;
       end
   end
end

%3 construct f
f = cos(cos(ts) + 0.65.*cos(2.*ts) - 0.65).*cosh(sin(ts));

%4 solve for phi
phi = (eye(N)+2.*D)\f';

%5 plot results
figure
plot(phi')
hold on;
plot(f)

xs = gamma(ts);
figure
plot3(xs(1,:),xs(2,:),f)
figure
plot3(xs(1,:),xs(2,:),phi')
```



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0.8
0.6
0.4
0.2
0
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0
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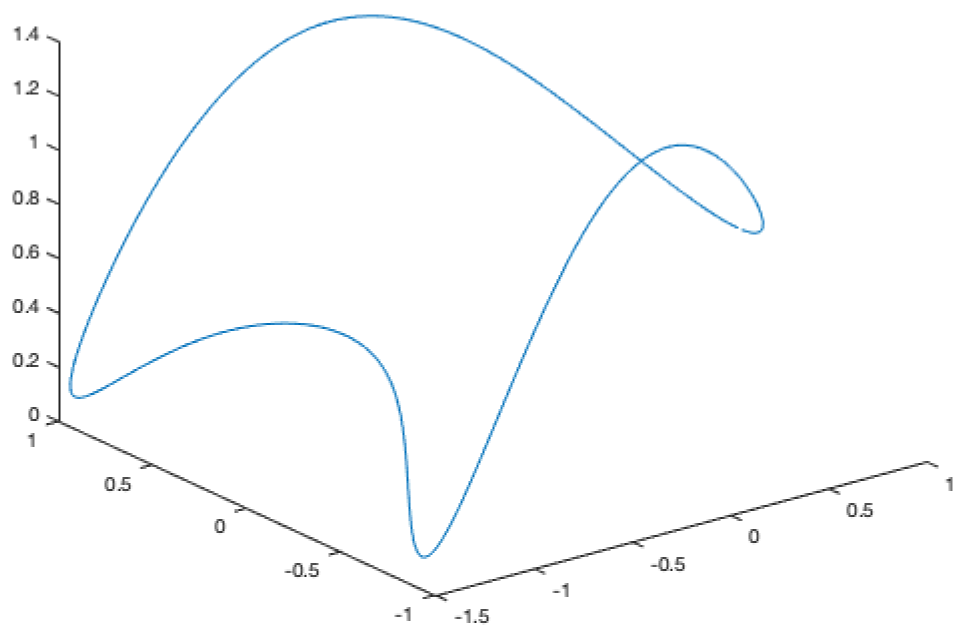
100
150
200
250
300



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1.2
1
0.8
0.6
0.4
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0
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0.5
0
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-1
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0
0.5
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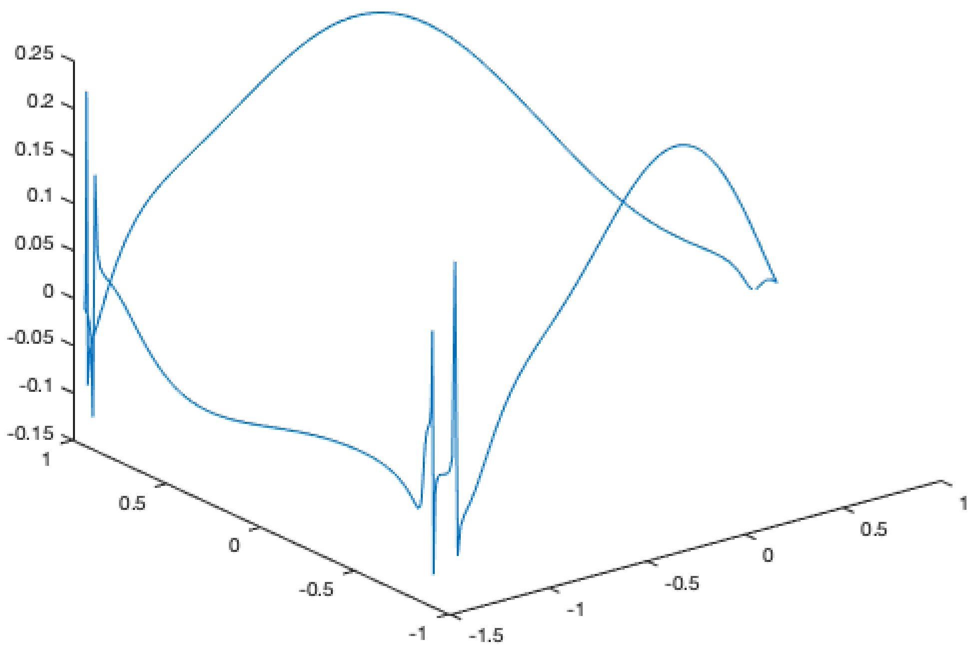

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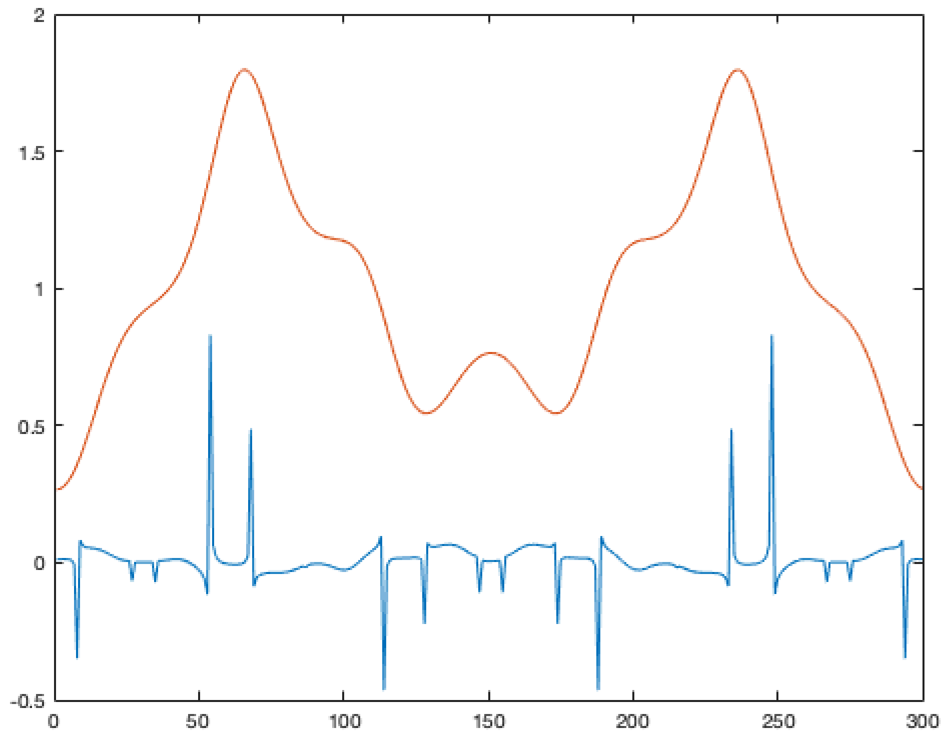
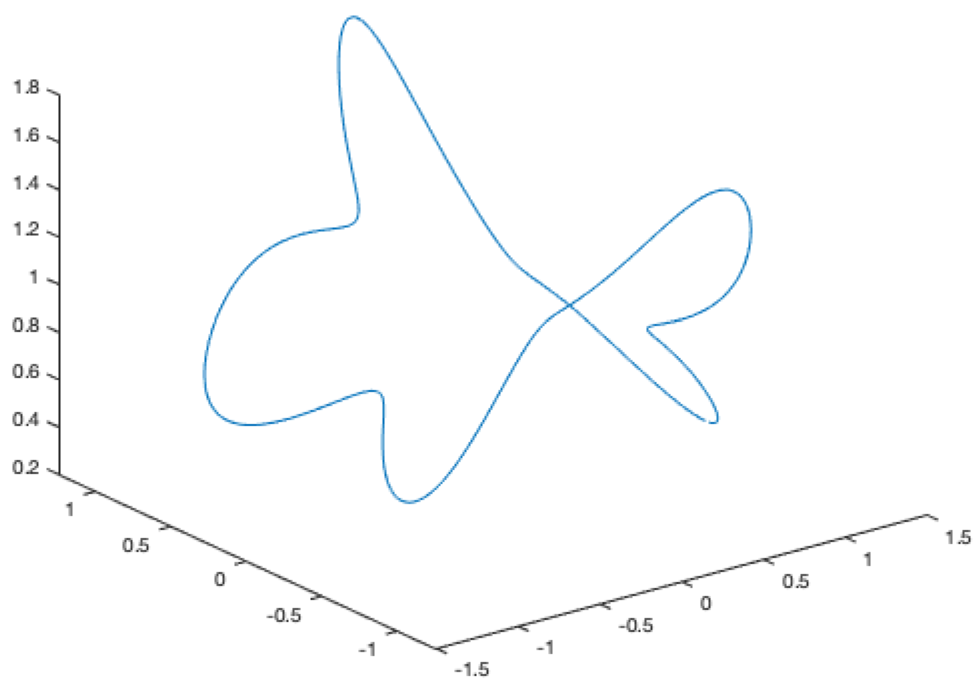
```
N = 300;
gamma = @(t) [(1+0.3.*cos(5.*t)).*cos(t);
 (1+0.3.*cos(5.*t)).*sin(t)]; %parametrization
gammadot = @(t) [(-1-0.3.*cos(5.*t)).*sin(t)-1.5.*cos(t).*sin(5.*t);
 cos(t)+0.3.*cos(t).*cos(5.*t)-1.5.*sin(t).*sin(5.*t)];
gammadotdot = @(t) [cos(t).*(-1-7.8.*cos(5.*t))+3.*sin(t).*sin(5.*t);
 (-1-7.8.*cos(5.*t)).*sin(t)-3.*cos(t).*sin(5.*t)];

%1 Discretization
ts = linspace(0,2.*pi-2.*pi/N,N);

%2 construct D
D = zeros(N,N);
for i=1:N
   for j=1:N
       a = gammadotdot(ts(j));
       b = gammadot(ts(j));
       b = [-b(2) b(1)];
       vy = -dot(a,b)./dot(b,b).*b;
       if i==j
           D(i,i) = (vy*gamma(ts(i)))./(2.*N.*norm(gammadot(ts)));
       end
       if i~=j
           D(i,j) = vy*((gamma(ts(j))-gamma(ts(i)))./
(norm(gamma(ts(i))-gamma(ts(j))).^2)).*norm(gammadot(ts))./N;
       end
   end
end

%3 construct f
f = cos(cos(ts).*(1 + 0.3.*cos(5.*ts))).*cosh((1 +
 0.3.*cos(5.*ts)).*sin(ts));

%4 solve for phi
phi = (eye(N)+2.*D)\f';

%5 plot results
figure
plot(phi')
hold on;
plot(f)

xs = gamma(ts);
figure
plot3(xs(1,:),xs(2,:),f)
figure
plot3(xs(1,:),xs(2,:),phi')
```


2
1.5
1
0.5
0
-0.5
0
50
100
150
200
250
300


1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
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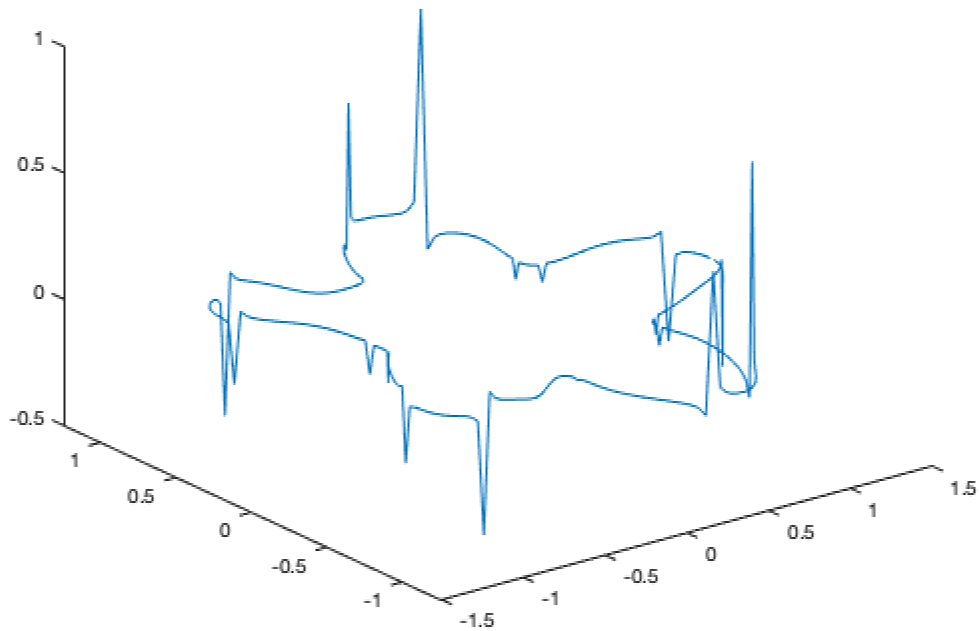
-1
-0.5
0
0.5
1
1.5



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