ACM 116 Set 3

1. $\mathbb{E}[X \mid Y=y] = \int_{-y}^{y} \frac{y^2-x^2}{8} e^{-y} x \, dx$

$= e^{-y} \int_{-y}^{y} \frac{y^2-x^2}{8}(x)\,dx = \frac{e^{-y}}{8} \int_{-y}^{y} xy^2 - x^3 \, dx$

$= \frac{e^{-y}}{8}\left[\frac{1}{2}x^2y^2 - \frac{1}{4}x^4\right]_{-y}^{y} = \frac{e^{-y}}{8}\left[\left(\frac{1}{2}y^4 - \frac{1}{4}y^4\right) - \left(\frac{1}{2}y^4 - \frac{1}{4}y^4\right)\right]$

$= 0$

2. a) $f_X(x \mid \text{influenza}) = \frac{P(\text{influenza}, X=x)}{P(\text{influenza})} = \frac{x^{\gamma}(x^{\alpha-1})(1-x)^{\beta-1}}{\int_0^1 x^{\gamma}(x^{\alpha-1})(1-x)^{\beta-1}\,dx}$

$= \frac{x^{\gamma+\alpha-1}(1-x)^{\beta-1}}{\int_0^1 x^{\gamma+\alpha-1}(1-x)^{\beta-1}\,dx}$

b) $\mathbb{E}[X \mid \text{influenza}] = \int_0^1 f_X(x \mid \text{influenza})\, x \, dx$

$= \int_0^1 \frac{x^{\gamma+\alpha}(1-x)^{\beta-1}}{\int_0^1 x^{\gamma+\alpha-1}(1-x)^{\beta-1}\,dx}\,dx = \frac{\int_0^1 x^{\gamma+\alpha}(1-x)^{\beta-1}\,dx}{\int_0^1 x^{\gamma+\alpha-1}(1-x)^{\beta-1}\,dx}$

c) MATLAB. By Mathematica, if $\gamma=2, \alpha=2, \beta=6$, $\mathbb{E}[X \mid \text{influenza}] = 0.4$, which matches MATLAB's simulated value.

3. We first note that if starting from a state with no ends tied, if two ends not from the same lace are tied, then essentially both laces combine into a single lace (with 2 untied ends), effectively reducing the number of shoelaces by 1. If two ends from the same shoelace are tied, then a loop is formed and the number of shoelaces decreases by 1. Therefore, regardless of which ends are tied together, the number of shoelaces decreases by 1 every time we tie 2 ends.

If we have n shoelaces remaining, then there are $2n-1$ ends to select from once we've selected the first end (arbitrarily). The odds that we select the end that forms a loop is $\frac{1}{2n-1}$. Therefore the expected value of # of loops is as follows:

$\mathbb{E}[\#\text{ of Loops}] = \sum_{i=1}^{n} \frac{1}{2n-1}(1) = \boxed{\sum_{i=1}^{n} \frac{1}{2n-1}}$

ACM116 Set 3

4. a) We first note that getting an outcome different to the previous outcome essentially resets the game, resulting in an $\mathbb{E}[N_k]$ of $\mathcal{E}+1$ (where $\mathcal{E}$ is expected number of rolls after the very first roll. We omit the first roll and add it at the end since it's not recursive).
At this point, we have a $\frac{1}{m^{k-1}}$ chance of getting the same outcome $k-1$ times, and thus winning. Therefore this corresponds to a $\frac{1}{m^{k-1}}(k-1)$ term.
Suppose we get the same result a times (in addition to the initial roll). This has a $\frac{1}{m^a}$ chance of occuring and then leaves us with a $\frac{m-1}{m}$ chance of failure (reset).
So this will create terms of the form $\left(\frac{1}{m^a}\right)\left(\frac{m-1}{m}\right)\left(\underbrace{a+1}_{\text{number of wasted rolls.}}+\mathcal{E}\right)$ for $a\in\{0,1,2,\dots,k-2\}$

So the final expression for $\mathcal{E}$ looks like:

$$\mathcal{E}=\frac{1}{m^0}\left(\frac{m-1}{m}\right)(\mathcal{E}+1)+\frac{1}{m^1}\left(\frac{m-1}{m}\right)(\mathcal{E}+2)+\dots+\frac{1}{m^{k-2}}\left(\frac{m-1}{m}\right)(\mathcal{E}+k-1)$$
$$+\frac{1}{m^{k-1}}(k-1)$$

$$\rightarrow\quad \mathcal{E}=\frac{1}{m^{k-1}}(k-1)+\sum_{i=1}^{k-1}\left(\frac{m-1}{m^i}\right)(\mathcal{E}+i)$$

$$\rightarrow\quad \mathcal{E}=\frac{1}{m^{k-1}}(k-1)+\sum_{i=1}^{k-1}\frac{m-1}{m^i}\mathcal{E}+\sum_{i=1}^{k-1}\frac{m-1}{m^i}i$$

$$\rightarrow\quad \mathcal{E}\left(1-\sum_{i=1}^{k-1}\frac{m-1}{m^i}\right)=\frac{1}{m^{k-1}}(k-1)+\sum_{i=1}^{k-1}\frac{m-1}{m^i}i$$

$$=\mathcal{E}\left(1-\frac{m-1}{m}-\frac{m-1}{m^2}-\frac{m-1}{m^3}-\dots-\frac{m-1}{m^{k-1}}\right)$$
$$=\mathcal{E}\left(1-\frac{m^2-m+m-1}{m^2}-\frac{m-1}{m^3}-\dots\frac{m-1}{m^{k-1}}\right)$$
$$=\mathcal{E}\left(1-\frac{m^3-1}{m^3}-\dots-\frac{m-1}{m^{k-1}}\right)=\mathcal{E}\left(1-\frac{m^{k-1}-1}{m^{k-1}}\right)=\mathcal{E}\left(\frac{m^{k-1}-m^{k-1}+1}{m^{k-1}}\right)=\mathcal{E}\left(\frac{1}{m^{k-1}}\right)$$

$$\rightarrow\quad \mathcal{E}=\frac{m^{k-1}}{m^{k-1}}(k-1)+(m^{k-1})\sum_{i=1}^{k-1}\frac{m-1}{m^i}i$$
$$\mathcal{E}=k-1+\sum_{i=1}^{k-1}m^{(k-1-i)}(m-1)(i)$$
$$=k-1+m^{k-2}(m-1)+m^{k-3}(m-1)(2)+\dots+m^0(m-1)(k-1)$$
$$=k-1+m^{k-1}-m^{k-2}+2m^{k-2}-2m^{k-3}+3m^{k-3}-3m^{k-4}+\dots+m(k-1)-(k-1)$$
$$=m^{k-1}+m^{k-2}+m^{k-3}+\dots+m$$

And now we add the very first roll that we had previously set aside:

$$\mathbb{E}[N_k]=\mathcal{E}+1=m^{k-1}+m^{k-2}+\dots+m+1\quad\text{as desired.}$$

Jacob Snyder
Section 8

ACM116 Set 3 cont'd

4.b) $\mathbb{V}[N_k] \approx \mathbb{E}[N_k]^2$

$\rightarrow \sigma \approx \mathbb{E}[N_k]$

So for $k=9$, $m=10$, $\mathbb{E}[N_k] = 1 + 10 + 10^2 + 10^3 + 10^4 + 10^5 + 10^6 + 10^7 + 10^8 = 111111111$

so $\sigma \approx \mathbb{E}[N_k] = 111111111$

$n_9 = 24658609$ is within the range $[\mathbb{E}[N_9] - \sigma, \mathbb{E}[N_9] + \sigma]$ so this evidence is weak.

5. If $X|Q=q \sim Bin(n,q)$ then:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Q=q]] = \mathbb{E}[nq] = 0.5n$$

$$\begin{aligned}
\mathbb{V}[X] &= \mathbb{E}[\mathbb{V}[X|Q=q]] + \mathbb{V}[\mathbb{E}[X|Q=q]] \\
&= \mathbb{E}[nq(1-q)] + \mathbb{V}[nq] \\
&= \mathbb{E}[nq(1-q)] + \mathbb{E}[n^2q^2] - \mathbb{E}[nq]^2 \\
&= \mathbb{E}[nq] - \mathbb{E}[nq^2] + \mathbb{E}[n^2q^2] - 0.25n^2 \\
&= 0.5n - 0.25n^2 + \mathbb{E}[q^2](n^2-n) \\
&= 0.5n - 0.25n^2 + \int_0^1 x^2\,dx \;(n^2-n) \\
&= 0.5n - 0.25n^2 + (n^2-n)\left[\tfrac{1}{3}x^3\right]_0^1 \\
&= 0.5n - 0.25n^2 + \frac{n^2}{3} - \frac{n}{3}
\end{aligned}$$

$$\mathbb{V}[X] = \boxed{\frac{n^2}{12} + \frac{n}{6}}$$

Scanned with CamScanner

```
infct = 0;
xsum = 0;

for a = 1:10000
    data = betarnd(2,6);
    data2 = data.^2;
    data3 = rand();
    if data3 <= data2
        infct = infct + 1;
        xsum = xsum + data;
    end
end

expxinf = xsum./infct;

disp("Expected value of X given influenza: ");
disp(expxinf);
```

```
Expected value of X given influenza:
    0.4011
```

Published with MATLAB® R2018a