Jacob Snyder

ACM106B Set 5

1.1a)
$$\begin{bmatrix} x_a & y_a & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}}_{\vec{b}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We must solve for $\vec{b}$ as a function of $x_a, x_b, x_c, y_a, y_b, y_c$

Using row reduction:

$$\left[\begin{matrix} x_a & y_a & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{matrix}\right] : \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \rightarrow \left[\begin{matrix} x_a - x_c & y_a - y_c & 0 \\ x_b - x_c & y_b - y_c & 0 \\ x_c & y_c & 1 \end{matrix}\right] : \left[\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}\right]$$

$\frac{y_a - y_c - 1}{y_b - y_c}$

$$\rightarrow \left[\begin{matrix} x_a - x_c & y_a - y_c & 0 \\ \frac{x_b - x_c}{y_b - y_c} & 1 & 0 \\ x_c & y_c & 1 \end{matrix}\right] : \left[\begin{matrix} 1 & 0 & -1 \\ 0 & (y_b - y_c)^{-1} & -(y_b - y_c)^{-1} \\ 0 & 0 & 1 \end{matrix}\right] \rightarrow \left[\begin{matrix} x_a - x_c - \frac{(x_b - x_c)(y_a - y_c)}{y_b - y_c} & 0 & 0 \\ \frac{x_b - x_c}{y_b - y_c} & 1 & 0 \\ x_c - \frac{y_c(x_b - x_c)}{y_b - y_c} & 0 & 1 \end{matrix}\right] : \left[\begin{matrix} 1 & \frac{y_c - y_a}{y_b - y_c} & \\ 0 & (y_b - y_c)^{-1} & -(y_b - y_c)^{-1} \\ 0 & \frac{-y_c}{y_b - y_c} & 1 + \frac{y_c}{y_b - y_c} \end{matrix}\right]$$

$$\rightarrow \left[\begin{matrix} 1 & 0 & 0 \\ \frac{x_b - x_c}{y_b - y_c} & 1 & 0 \\ x_c - \frac{y_c(x_b - x_c)}{y_b - y_c} & 0 & 1 \end{matrix}\right] : \left[\begin{matrix} \frac{y_b - y_c}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} & \frac{(y_c - y_a)}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} \\ 0 & (y_b - y_c)^{-1} \\ 0 & -y_c(y_b - y_c)^{-1} \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{(y_a - y_c + y_c - y_b)}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} \\ -(y_b + y_c)^{-1} \\ 1 + (y_c)(y_b - y_c)^{-1} \end{matrix}\right]$$

$$\rightarrow \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] :$$

$$\underbrace{\left[\begin{matrix} \frac{y_b - y_c}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} & \frac{(y_c - y_a)}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} & \frac{y_a - y_b}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} \\ \frac{-(x_b - x_c)}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} & (y_b - y_c)^{-1} - \frac{+(x_b - x_c)(y_c - y_a)(y_b - y_c)^{-1}}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} & -(y_b - y_c)^{-1} - \frac{-(x_b - x_c)(y_b - y_c)^{-1}(y_a - y_b)}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} \\ \frac{-x_c(y_b - y_c) + y_c(x_b - x_c)}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} & -y_c(y_b - y_c)^{-1} - \frac{[y_c - y_a][x_c(y_b - y_c) - y_c(x_b - x_c)](y_b - y_c)^{-1}}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} & 1 + y_c(y_b - y_c)^{-1} - \frac{(y_a - y_b)(y_b - y_c)^{-1}[x_c(y_b - y_c) - y_c(x_b - x_c)]}{(x_a - x_c)(y_b - y_c) - (x_b - x_c)(y_a - y_c)} \end{matrix}\right]}_{\vec{b}\text{, containing the values of } \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3 \text{ as functions of } xyz}$$

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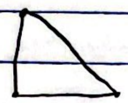
ACM106B Set 5 Problem 1 (cont'd)

1.1.b) We must consider the two types of triangles:

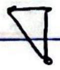
$\Delta z_{ij}, z_{i+1,j}, z_{i,j+1}$



Plugging into the local basis function parameter matrix, we have

$$\begin{bmatrix} \frac{-1}{1} & \frac{-1}{1} & 0 \\ \frac{-1}{1} & -1+\frac{+1}{1} & 1 \\ \frac{(i)+(j+1)}{1} & (j+1)+\frac{i(-1)-(j+1)(1)}{1} & 1-j-1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ i+j+1 & -i & -j \end{bmatrix}$$

And for the other triangle:  $\Delta z_{ij}, z_{i-1,j}, z_{i,j-1}$ we have:

$$\begin{bmatrix} \frac{1}{1} & \frac{-1}{1} & 0 \\ \frac{1}{1} & 1-1 & -1 \\ -i-j+1 & i & j \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -i-j+1 & i & j \end{bmatrix}$$

We construct P_{ij} as follows:

$$P_{ij}(z_{k,l}) = \begin{cases} 0 & (i,j) \text{ shares no triangle with } (k,l) \\ \prod_{n=1}^{6} \varphi_{i,j,n}(z_{k,l}) & (i,j) \text{ shares a triangle with } (k,l) \end{cases}$$

Where $\varphi_{i,j,n}$ denotes the local basis function that returns 1 for $x_{i,j}$, of which there are 6 for any i,j pair; for this reason the third parameter n describes which of the 6 functions it is (arbitrarily enumerated).

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ACM106B Set 5 Problem 1 (cont'd)

1.2. $a(u,v) = (f,v)$ where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$, $(f,v) = \int_\Omega fv$

$\rightarrow \sum_{1 \le i,j \le n-1} \tilde{u}_{ij}\, a(P_{i,j}, P_{k,\ell}) = (f, P_{k,\ell}) \qquad 1 \le k,\ell \le n-1$

So we have $Au = b$ where

$\cancel{a_{1,1:n-1} = a(P_{1,1}, P_{1:n-1,1})}$

$a_{1,n:2n-2} = a(P_{1,1}, P_{1:n-1,2})$

$\vdots$

$a_{2,1:n-1} = a(P_{2,1}, P_{1,n-1,1})$

$\vdots$

$a_{n-1,1:n-1} = a(P_{n-1,1}, P_{1:n-1,1})$

$a_{n,1:n-1} = a(P_{1,2}, P_{1:n-1,1})$

$\vdots$

$a_{(n-1)^2,1:n-1} = a(P_{n-1,n-1}, P_{1:n-1,1})$

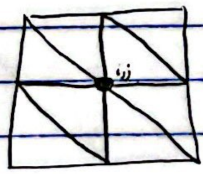
and $b_{1:n-1} = (f, P_{1:n-1,1})$

$b_{n:2n-2} = (f, P_{1:n-1,2})$

$\vdots$

$b_{(n-1)^2-n+1:(n-1)^2} = (f, P_{1:n-1,n-1})$

1.3. First we will write P_{ij} out explicitly, in the case that (i,j) shares a triangle with (k,ℓ)



Recall the parameter matrices from part 1:

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ i+j+1 & -i & -j \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -i-j+1 & i & j \end{bmatrix}$$ for the two types of triangles.

So $P_{ij}(z_{k,\ell}) = [i+j+1-k-\ell][k+\ell+1-i-j][k-i+1][\ell-j+1][-k+i+1][-\ell+j+1]$

We note that $a(P_{ij}, P_{kl}) = 0$ if (i,j) and (k,l) don't share a triangle. So we can calculate the possible overlaps of P_{ij} with $P_{i\pm1,j\pm1}$ to compute all of the possible $a(P_{ij}, P_{k\ell})$

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ACM106B Set 5 Problem 2

2.1. Forward time Backwards space

Order of accuracy: first-order

Amplification factor: $\hat{a} = 1 + a\lambda(\cos\xi - 1) - ia\lambda\sin\xi$

Lax-Friedrichs

Order of accuracy: first-order

Amplification factor: $\hat{Q} = \cos\xi - ia\lambda\sin\xi$, $\lambda = \frac{k}{h}$, $\xi = \omega h$

Lax-Wendroff

Order of accuracy: Second-order

Amplification factor: $\hat{Q} = 1 - ia\lambda\sin\xi - 2a^2\lambda^2\sin^2(\xi/2)$

2.2. The order of accuracy for FTBS and Lax-Friedrichs appears to be closer to 0 than to 1. Similarly, the Lax-Wendroff scheme appears to have a numerical order closer to 1 than to 2. This aligns with LW being 2nd order accurate vs FTBS/LF being first-order accurate.* It is also plausible that my implementations of these schemes are missing something, or incorrect, causing the numerical order to be lower that it should be optimally (1 for FTBS/LF, 2 for LW)

* In the sense that the numerical order of LW is roughly 1 greater than that of FTBS/LF.