Jacob Snyder

ACM106a Set 5 Problem 1

1\. 2) As one might expect by the complexities of the methods, we see that SOR performs best (approaches the solution fastest) ~~then~~, Gauss-Seidel is second-best, and Jacobi is slowest of the three (in terms of # of iterations).

3) It seems like the best ω is either $\omega = 1.2$ or $\omega = 1.4$.

4) The accuracy of the numerical solution does appear to ~~dec~~ increase as ε is lowered.

Jacob Snyder

ACM106a Set 5 Problem 2

2.1) We can consider the optimization $\min_{\mu \in \mathbb{C}} \|Ax - \mu x\|_2$ as projecting Ax onto x:
Therefore the proposed equality $\|Ax - R(x)x\|_2 = \min_{\mu \in \mathbb{C}} \|Ax - \mu x\|_2$ is true if and only if $R(x)x = \text{proj}_x(Ax)$

$$\rightarrow R(x)x \overset{?}{=} \frac{x \cdot Ax}{x \cdot x} x \rightarrow R(x)x = \underbrace{\frac{x^* Ax}{x^* x}}_{\text{note} = R(x)} x$$

which is trivially true, So we are done ■

2.2) i) We start from

$$q_k = \frac{Aq_{k-1}}{\|Aq_{k-1}\|_2}$$

$$\rightarrow = \frac{Aq_{k-1}}{\mu_{k-1}} = \frac{A}{\mu_{k-1}} \frac{Aq_{k-2}}{\|Aq_{k-2}\|_2} = \frac{1}{\mu_{k-1}\mu_{k-2}} AA q_{k-2} = \frac{1}{\mu_{k-1}\mu_{k-2}\mu_{k-3}} AAA q_{k-3}$$

$$= \frac{1}{\mu_{k-1}\mu_{k-2}\cdots\mu_0} A^k q_0 = \frac{1}{\mu_{k-1}\mu_{k-2}\cdots\mu_0} A^k (c_1 p_1 + c_2 p_2 + \dots + c_n p_n)$$

$$= \frac{1}{\mu_{k-1}\mu_{k-2}\cdots\mu_0} (c_1 A^k p_1 + c_2 A^k p_2 + \dots + c_n A^k p_n)$$

$$= \frac{1}{\mu_{k-1}\mu_{k-2}\cdots\mu_0} (c_1 \lambda_1^k p_1 + c_2 \lambda_2^k p_2 + \dots + c_n \lambda_n^k p_n)$$ as desired.

ii) From i), we have that $q_k = \frac{1}{\mu_{k-1}\mu_{k-2}\cdots\mu_0} (c_1 \lambda_1^k p_1 + c_2 \lambda_2^k p_2 + \dots + c_n \lambda_n^k p_n)$

$$\rightarrow \lim_{k\to\infty} \|q_k\|_2 = \lim_{k\to\infty} \left\| \frac{1}{\mu_{k-1}\mu_{k-2}\cdots\mu_0} (c_1 \lambda_1^k p_1 + c_2 \lambda_2^k p_2 + \dots + c_n \lambda_n^k p_n) \right\|$$

note that $\lim_{k\to\infty} \|(c_1 \lambda_1^k p_1 + c_2 \lambda_2^k p_2 + \dots + c_n \lambda_n^k p_n)\| = \lim_{k\to\infty} \|c_1 \lambda_1^k p_1\|$ since $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$

$$\rightarrow 1 = \lim_{k\to\infty} \frac{\|c_1 \lambda_1^k p_1\|}{\mu_{k-1}\mu_{k-2}\cdots\mu_0} \rightarrow 1 = \lim_{k\to\infty} \frac{|c_1 \lambda_1^k| \|p_1\|}{\mu_{k-1}\mu_{k-2}\cdots\mu_0} \rightarrow \lim_{k\to\infty} \frac{|c_1 \lambda_1^k|}{\mu_{k-1}\mu_{k-2}\cdots\mu_0} = 1$$ as desired.

(cont'd on next page).

Jacob Snyder

ACM106a Set 5 Problem 2 (cont'd)

2.2) iii) $q_k = \alpha_k p_1 + \alpha_k \frac{c_2}{c_1}\frac{\lambda_2^k}{\lambda_1^k} p_2 + \cdots + \alpha_k \frac{c_n \lambda_n^k}{c_1 \lambda_1^k} p_n$

$\rightarrow \|q_k\|_2 = |\alpha_k| \, \| p_1 + \frac{c_2 \lambda_2^k}{c_1 \lambda_1^k} p_2 + \cdots + \frac{c_n \lambda_n^k}{c_1 \lambda_1^k} p_n \|$

$\rightarrow 1 = |\alpha_k| \, \| p_1 + \frac{c_2 \lambda_2^k}{c_1 \lambda_1^k} p_2 + \cdots + \frac{c_n \lambda_n^k}{c_1 \lambda_1^k} p_n \|$

$\rightarrow \lim_{k \to \infty} 1 = \lim_{k \to \infty} |\alpha_k| \, \|p_1\| + |\alpha_k| \, \| \frac{c_2 \lambda_2^k}{c_1 \lambda_1^k} p_2 \| + \| \cdots \| + \| \frac{c_n \lambda_n^k}{c_1 \lambda_1^k} p_n \|$

can do this for sufficiently large k since $\|p_1\| >> \| \frac{c_2 \lambda_2^k}{c_1 \lambda_1^k} p_2 \|$ and so on

$\rightarrow 1 = \lim_{k \to \infty} |\alpha_k| \left(1 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) + O\left(\left|\frac{\lambda_3}{\lambda_1}\right|^k\right) + \cdots + O\left(\left|\frac{\lambda_n}{\lambda_1}\right|^k\right) \right)$

$\rightarrow |\alpha_k| - 1 = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$ as desired.

iv) $q_k^* A q_k = q_k^* \left[\alpha_k A p_1 + \alpha_k \frac{c_2 \lambda_2^k}{c_1 \lambda_1^k} p_2 + \cdots + \alpha_k \frac{c_n \lambda_n^k}{c_1 \lambda_1^k} p_n \right]$

$= q_k^* \left[\alpha_k \lambda_1 p_1 + \underbrace{\frac{c_2}{c_1} \frac{\lambda_2^{k+1}}{\lambda_1^k} p_2 + \cdots + \alpha_k \frac{c_n \lambda_n^{k+1}}{c_1 \lambda_1^k} p_n}_{O\left(\left(\frac{\lambda_2}{\lambda_1}\right)^k\right)} \right]$

$= \lambda_1 + O\left(\left(\frac{\lambda_2}{\lambda_1}\right)^k\right)$

$\Rightarrow |q_k^* A q_k - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$ as desired.

v) Since A is Hermitian, you instead get

$q_k^* A q_k = \lambda_1 + O\left(\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)$

$\rightarrow |q_k^* A q_k - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right)$ as desired.