Jacob Snyder
Section 8

ACM116 set 4

1. a) $\mathbb{E}[W] = \begin{bmatrix} \mathbb{E}[X] \\ \mathbb{E}[Y] \\ \mathbb{E}[Z] \end{bmatrix}$

$\mathbb{E}[X] = \int_1^2 x\,dx = [\frac{1}{2}x^2] = \frac{1}{2}(4-1) = 3/2$

$\mathbb{E}[Y|X=x] = [1/x]^{-1} = x$ by Lecture slides 2 ($\mathbb{E}[\text{Exp}(\lambda)] = 1/\lambda$)

$\mathbb{E}[\mathbb{E}[Y|X=x]] = 3/2 = \mathbb{E}[Y]$

$\mathbb{E}[Z|X=x, Y=y] = x$ by Lecture slides 2 ($\mathbb{E}[\mathcal{N}(\mu,\sigma^2)] = \mu$)

$\rightarrow \mathbb{E}[\mathbb{E}[Z|X=x, Y=y]] = 3/2 = \mathbb{E}[Z]$

$\rightarrow \mathbb{E}[W] = \begin{bmatrix} 3/2 \\ 3/2 \\ 3/2 \end{bmatrix}$

$\Sigma_w = \begin{bmatrix} \mathbb{V}[X] & \text{Cov}(X,Y) & \text{Cov}(X,Z) \\ \text{Cov}(Y,X) & \mathbb{V}[Y] & \text{Cov}(Y,Z) \\ \text{Cov}(Z,X) & \text{Cov}(Z,Y) & \mathbb{V}[Z] \end{bmatrix}$

$\mathbb{V}[X] = \frac{1}{12}$

$\mathbb{V}[Y] = \mathbb{E}[\mathbb{V}[Y|X]] + \mathbb{V}[\mathbb{E}[Y|X]] = \mathbb{E}[X^2] + \mathbb{V}[X]$

$= \int_1^2 x^2\,dx + \frac{1}{12} = [\frac{1}{3}x^3]_1^2 + \frac{1}{12} = \frac{7}{3} + \frac{1}{12} = \frac{29}{12}$

$\mathbb{V}[Z] = \mathbb{E}[\mathbb{V}[Z|X,Y]] + \mathbb{V}[\mathbb{E}[Z|X,Y]] = 1 + \mathbb{V}[X] = \frac{13}{12}$

$\text{Cov}(X,Y) = \text{Cov}(Y,X) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[XY|X=x]] - \frac{9}{4}$

$= \mathbb{E}[X \cdot \mathbb{E}[Y|X=x]] - \frac{9}{4} = \mathbb{E}[X^2] - \frac{9}{4} = \frac{7}{3} - \frac{9}{4} = \frac{1}{12}$

$\text{Cov}(X,Z) = \text{Cov}(Z,X) = \mathbb{E}[XZ] - \mathbb{E}[X]\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[XZ|X=x]] - \frac{9}{4}$

$= \mathbb{E}[X \cdot \mathbb{E}[Z|X=x]] - 9/4 = \mathbb{E}[X^2] - 9/4 = \frac{1}{12}$

$\text{Cov}(Y,Z) = \text{Cov}(Z,Y) = \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[YZ|X=x]] - 9/4$

$= \mathbb{E}[\mathbb{E}[Y|X=x]\mathbb{E}[Z|X=x]] - 9/4 = \mathbb{E}[X^2] - 9/4 = 1/12$

So $\Sigma_w = \begin{bmatrix} 1/12 & 1/12 & 1/12 \\ 1/12 & 29/12 & 1/12 \\ 1/12 & 1/12 & 13/12 \end{bmatrix}$

b) MATLAB

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2. a) We will first calculate the joint distribution $f_{X_1,X_2}(Y_1,Y_2)$ using the method described in the "Transformation of random vectors" section of the notes.

We are given:

$$Y_1 = (-2\log(X_1))^{1/2}\cos(2\pi X_2)$$
$$Y_2 = (-2\log(X_1))^{1/2}\sin(2\pi X_2)$$
$$\rightarrow Y_1^2 + Y_2^2 = (-2\log(X_1))\left[\cos^2(2\pi X_2) + \sin^2(2\pi X_2)\right]$$
$$\rightarrow Y_1^2 + Y_2^2 = -2\log(X_1)$$
$$\rightarrow X_1 = e^{-\frac{1}{2}[Y_1^2+Y_2^2]}$$
$$\rightarrow Y_1^2 = (Y_1^2+Y_2^2)\cos^2(2\pi X_2)$$
$$\rightarrow 2\pi X_2 = \cos^{-1}\left(\left[\frac{Y_1^2}{Y_1^2+Y_2^2}\right]^{1/2}\right)$$
$$\rightarrow X_2 = \frac{1}{2\pi}\cos^{-1}\left(\left[\frac{Y_1^2}{Y_1^2+Y_2^2}\right]^{1/2}\right)$$

So we define
$$H(Y_1,Y_2): \begin{cases} X_1 = e^{-1/2[Y_1^2+Y_2^2]} \\ X_2 = \frac{1}{2\pi}\cos^{-1}\left(\left[\frac{Y_1^2}{Y_1^2+Y_2^2}\right]^{1/2}\right)\end{cases}$$

$$\rightarrow J_H = \begin{bmatrix} \frac{d}{dY_1}\left[e^{-1/2[Y_1^2+Y_2^2]}\right] & \frac{d}{dY_2}\left[e^{-1/2[Y_1^2+Y_2^2]}\right] \\ \frac{d}{dY_1}\left[\frac{1}{2\pi}\cos^{-1}\left(\left[\frac{Y_1^2}{Y_1^2+Y_2^2}\right]^{1/2}\right)\right] & \frac{d}{dY_2}\left[\frac{1}{2\pi}\cos^{-1}\left(\left[\frac{Y_1^2}{Y_1^2+Y_2^2}\right]^{1/2}\right)\right]\end{bmatrix}$$

$$= \begin{bmatrix} -Y_1 e^{-1/2[Y_1^2+Y_2^2]} & -Y_2 e^{-1/2[Y_1^2+Y_2^2]} \\ -\frac{1}{2\pi Y_1}\left(\frac{Y_1^2Y_2^2}{(Y_1^2+Y_2^2)^2}\right)^{1/2} & \frac{1}{2\pi Y_2}\left(\frac{Y_1^2Y_2^2}{(Y_1^2+Y_2^2)^2}\right)^{1/2}\end{bmatrix}$$

So:
$$\det J_H = \frac{-Y_1}{2\pi Y_2}e^{-1/2[Y_1^2+Y_2^2]}\left(\frac{Y_1^2Y_2^2}{(Y_1^2+Y_2^2)^2}\right)^{1/2} - \frac{Y_2}{2\pi Y_1}\left(\frac{Y_1^2Y_2^2}{(Y_1^2+Y_2^2)^2}\right)^{1/2}e^{-1/2[Y_1^2+Y_2^2]}$$
$$= \frac{-1}{2\pi}e^{-1/2[Y_1^2+Y_2^2]}\left(\frac{Y_1^2Y_2^2}{(Y_1^2+Y_2^2)^2}\right)^{1/2}\left[\frac{Y_1^2+Y_2^2}{Y_1Y_2}\right] = \frac{-1}{2\pi}e^{-1/2[Y_1^2+Y_2^2]}$$

Also, since $X_1 \perp\!\!\!\perp X_2$: $f_{Y_1,X_2}(X_1,X_2) = f_{X_1}(X)f_{X_2}(X) = 1\cdot 1 = 1$

So we have:
$$f_{Y_1,Y_2}(Y_1,Y_2) = f_{X_1,X_2}(H(Y_1,Y_2))\,|\det J_H| = \frac{1}{2\pi}e^{-1/2[Y_1^2+Y_2^2]}$$

(cont'd on next page)

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2.a cont'd) We have $f_{Y_1,Y_2}(Y_1,Y_2) = \frac{1}{2\pi} e^{-\frac{1}{2}[Y_1^2+Y_2^2]}$

We will now first do part (b) of the problem then return to part (a):

b) Given $f_{Y_1,Y_2}(Y_1,Y_2) = \frac{1}{2\pi} e^{-\frac{1}{2}[Y_1^2+Y_2^2]}$, we have

$$f_{Y_1}(Y_1) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}[Y_1^2+Y_2^2]} \, dY_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[Y_1^2]}$$

by symmetry we have

$$f_{Y_2}(Y_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[Y_2^2]}$$

Now back to part (a):

2.a cont'd) $f_{Y_1}(Y_1) f_{Y_2}(Y_2) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-\frac{1}{2}[Y_1^2]} e^{-\frac{1}{2}[Y_2^2]} = \frac{1}{2\pi} e^{-\frac{1}{2}[Y_1^2+Y_2^2]} = f_{Y_1,Y_2}(Y_1,Y_2)$

Therefore the components of Y are independent.

3.a) We must define Σ_{XY} and Σ_Y in terms of known quantities.

First we find μ_Y:

$$\mu_Y = \mathbb{E}[Y] = \mathbb{E}[GX+W] = \mathbb{E}[GX] + \mathbb{E}[W] = \mathbb{E}[GX] = G\,\mathbb{E}[X] = G\mu_X$$

$$\Sigma_{XY} = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)^T] = \mathbb{E}[(X-\mu_X)(GX+W-G\mu_X)^T]$$

$$= \mathbb{E}[(X-\mu_X)(GX-G\mu_X)^T] + \mathbb{E}[(X-\mu_X)(W)^T]$$

$$= \mathbb{E}[(X-\mu_X)(G(X-\mu_X))^T] + \Sigma_{XW}$$

$$= \mathbb{E}[(X-\mu_X)(X-\mu_X)^T G^T] + 0$$

$$= \Sigma_X G^T$$

$$\Sigma_Y = \mathbb{E}[(Y-\mu_Y)(Y-\mu_Y)^T] = \mathbb{E}[(GX+W-G\mu_X)(GX+W-G\mu_X)^T]$$

$$= \mathbb{E}[(GX-G\mu_X)(GX+W-G\mu_X)^T] + \mathbb{E}[W(GX+W-G\mu_X)^T]$$

$$= \mathbb{E}[(GX-G\mu_X)(GX-G\mu_X)^T] + \mathbb{E}[W(GX+W-G\mu_X)^T] + \mathbb{E}[(GX-G\mu_X)(W)^T]$$

$$= G\Sigma_X G^T + \mathbb{E}[W(GX-G\mu_X)^T] + \mathbb{E}[W(W)^T] + G\Sigma_{XW}$$

$$= G\Sigma_X G^T + \Sigma_{WX} G^T + \Sigma_W + G\Sigma_{XW} = G\Sigma_X G^T + \Sigma_W$$

Therefore, by the notes, the Wiener filter is

$$g(Y) = \Sigma_{XY}\Sigma_Y^{-1}(Y-\mu_Y) + \mu_X = \Sigma_X G^T (G\Sigma_X G^T + \Sigma_W)^{-1} (GX+W-G\mu_X) + \mu_X$$

b) MATLAB

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4. a) Yes. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ where $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

Then $AZ = \begin{bmatrix} Z_1 \\ 3Z_2 \end{bmatrix} \sim X$

So X_1, X_2 are jointly normally distributed and thus X is Gaussian.

b) $\Sigma_x = \begin{bmatrix} \cancel{\mathbb{V}[X_1]} & \cancel{Cov(X_1,X_2)} \\ Cov(X_2,X_1) & \mathbb{V}[X_2] \end{bmatrix}$ $\quad$ $\mathbb{V}[X_1] = 1$ $\quad$ $\cancel{\mathbb{V}[X_2] = \mathbb{V}[3X_2] = 9\mathbb{V}[X_2] = 9}$

$Cov(X_1, X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] = \mathbb{E}[3X_1^2] - 0$

$= 3\mathbb{E}[X_1^2] = 3\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

$= 3 = Cov(X_2, X_1)$

$\Rightarrow \boxed{\Sigma_x = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}}$

c) No, X does not have a density. This is because if there did exist a density function for X: $f_{X_1,X_2}(X_1,X_2)$, then $f_{X_1,X_2}(X_1,X_2) \neq 0$ iff $X_2 = 3X_1$.
Therefore $f_{X_1,X_2}(X_1,X_2)$ is only nonzero at discrete points, so a density does not exist.
We instead would work with a PMF.

5. a) Suppose X is Gaussian. Then it can be written as $X = BZ + \mu$ where $Z_1, \ldots, Z_m \overset{iid}{\sim} N(0,1)$
$\rightarrow Y = AX = A(BZ+\mu) = (AB)Z + A\mu$ so Y can be written in the Gaussian form as well. Therefore Y is Gaussian.

b) Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ s.t. $X_1 \sim N(0,1)$, $X_2 \sim Exp(5)$, $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Note that X is NOT Gaussian.
Then $Y = AX = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_1 \end{bmatrix}$ so Y is Gaussian. Therefore, by counterexample, If Y is Gaussian X is not necessarily Gaussian.

Scanned with CamScanner

```
covw = [1/12 1/12 1/12; 1/12 29/12 1/12; 1/12 1/12 13/12];
[V,D] = eig(covw);
Qt = transpose(V);

disp("Decorrelating transformation for W: ");
disp(Qt);
```

```
Decorrelating transformation for W:
    0.9963   -0.0326   -0.0796
    0.0772   -0.0670    0.9948
    0.0378    0.9972    0.0643
```

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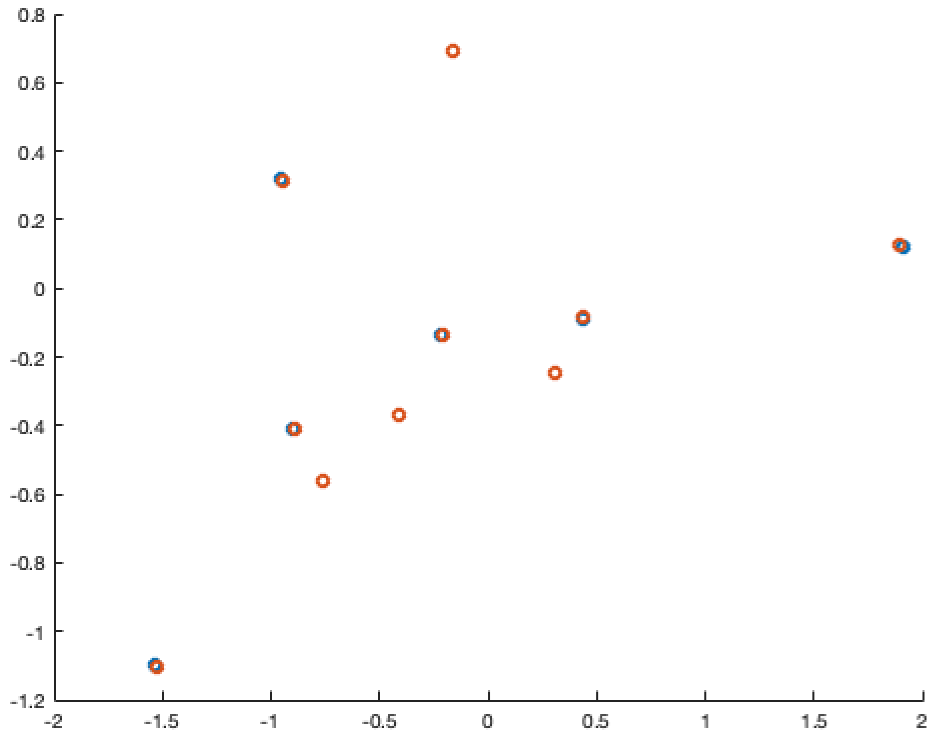
```matlab
noisevar = 0.03.^2;
noisevarmat = [noisevar 0; 0 noisevar];

G = [1 2;3 4];
inputs = [];
outputs = [];
predinputs = [];

for a=1:10
   X = normrnd(0,1,2,1);
   W = normrnd(0,noisevar,2,1);
   Y = G*X;

   pred = transpose(G)*inv(G*transpose(G)+noisevarmat)*(G*X+W);

   inputs = [inputs X];
   outputs = [outputs Y];
   predinputs = [predinputs pred];
end

inputs = transpose(inputs);
predinputs = transpose(predinputs);

scatter(inputs(1:10),inputs(11:20));
hold on;
scatter(predinputs(1:10),predinputs(11:20));
```



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