Jacob Snyder

ACM106a Set 2 Problem 1

1. $f_0(x) = 1$

$$p_0(x) = \frac{f_0(x)}{\|f_0(x)\|} = \frac{1}{\left[\int_{-1}^{1} dx\right]^{1/2}} = \frac{1}{\sqrt{2}} \rightarrow \boxed{p_0(x) = \frac{1}{\sqrt{2}}}$$

$f_1(x) = x$

$$\hat{p}_1(x) = f_1(x) - \langle f_1(x), p_0(x) \rangle p_0(x)$$
$$= x - \left[\int_{-1}^{1} \frac{x}{\sqrt{2}} dx\right] \frac{1}{\sqrt{2}}$$
$$= x - 0 = x$$
$$\rightarrow p_1(x) = \frac{\hat{p}_1(x)}{\|\hat{p}_1(x)\|} = \frac{x}{\left[\int_{-1}^{1} x^2 dx\right]^{1/2}} = \frac{x}{\left[\frac{2}{3}\right]^{1/2}} = \frac{\sqrt{3}x}{\sqrt{2}} \rightarrow \boxed{p_1(x) = \frac{\sqrt{3}x}{\sqrt{2}}}$$

$f_2(x) = x^2$

$$\hat{p}_2(x) = f_2(x) - \langle f_2(x), p_0(x) \rangle p_0(x) - \langle f_2(x), p_1(x) \rangle p_1(x)$$
$$= x^2 - \left[\int_{-1}^{1} \frac{x^2}{\sqrt{2}} dx\right] \frac{1}{\sqrt{2}} - \left[\int_{-1}^{1} x^3 \sqrt{\frac{3}{2}} dx\right] \left[x \sqrt{\frac{3}{2}}\right]$$
$$= x^2 - \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{3}\right) - 0 = x^2 - \frac{1}{3}$$
$$p_2(x) = \frac{\hat{p}_2(x)}{\|\hat{p}_2(x)\|} = \frac{x^2 - \frac{1}{3}}{\left[\int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9} dx\right]^{1/2}} = \frac{x^2 - \frac{1}{3}}{\left[2\left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right)\right]^{1/2}} = \frac{3x^2 - 1}{\left[2\left(\frac{9}{5} - 1\right)\right]^{1/2}} = \frac{3x^2 - 1}{\left[8/5\right]^{1/2}}$$
$$\rightarrow \boxed{p_2(x) = (3x^2 - 1)\left[\frac{5}{8}\right]^{1/2}}$$

$f_3(x) = x^3$

$$\hat{p}_3(x) = f_3(x) - \langle f_3(x), p_0(x) \rangle p_0(x) - \langle f_3(x), p_1(x) \rangle p_1(x) - \langle f_3(x), p_2(x) \rangle p_2(x)$$
$$= x^3 - \int_{-1}^{1} \frac{x^3}{\sqrt{2}} dx \left(\frac{1}{\sqrt{2}}\right) - \int_{-1}^{1} x^4 \sqrt{\frac{3}{2}} dx \left(x \sqrt{\frac{3}{2}}\right) - \left[\frac{5}{8}\right](3x^2 - 1)\int_{-1}^{1} 3x^5 - x^3 dx$$
$$= x^3 - 0 - \frac{\sqrt{6}}{5}\left(x\sqrt{\frac{3}{2}}\right) - 0 = x^3 - \frac{3}{5}x$$
$$p_3(x) = \frac{\hat{p}_3(x)}{\|\hat{p}_3(x)\|} = \frac{x^3 - \frac{3}{5}x}{\left[\int_{-1}^{1}\left[x^3 - \frac{3}{5}x\right]^2 dx\right]^{1/2}} = \left(x^3 - \frac{3}{5}x\right)\left[\frac{\sqrt{175}}{\sqrt{8}}\right] = (5x^3 - 3x)\left[\frac{\sqrt{7}}{\sqrt{8}}\right] \rightarrow \boxed{p_3(x) = (5x^3 - 3x)\left[\frac{7}{8}\right]^{1/2}}$$

$f_4(x) = x^4$

$$\hat{p}_4(x) = f_4(x) - \langle f_4(x), p_0(x) \rangle p_0(x) - \langle f_4(x), p_1(x) \rangle p_1(x) - \langle f_4(x), p_2(x) \rangle p_2(x) - \langle f_4(x), p_3(x) \rangle p_3(x)$$
$$= x^4 - \left[\frac{1}{\sqrt{2}}\right]^2 \int_{-1}^{1} x^4 dx - \frac{3}{2}x \int_{1}^{-1} x^5 dx - \frac{5}{8}(3x^2 - 1)\int_{1}^{-1} 3x^6 - x^4 dx - \frac{7}{8}(5x^3 - 3x)\int_{1}^{-1} 5x^7 - 3x^5 dx$$
$$= x^4 - \frac{1}{5} - 0 - \frac{5}{8}(3x^2 - 1)(2)\left(\frac{3}{7} - \frac{1}{5}\right) - 0 = x^4 + \frac{2}{7}(3x^2 - 1) - \frac{1}{5}$$
$$p_4(x) = \frac{\hat{p}_4(x)}{\|\hat{p}_4(x)\|} = \frac{x^4 - 2/7(3x^2 - 1) - 1/5}{\left[\int_{-1}^{1}\left[x^4 - 2/7(3x^2 - 1) - 1/5\right]^2 dx\right]^{1/2}} = \left[x^4 - 2/7(3x^2 - 1) - 1/5\right]\left[\frac{105}{8\sqrt{2}}\right]$$
$$= \left[35x^4 - 30x^2 + 10 - 7\right]\left[\frac{3}{8\sqrt{2}}\right] \rightarrow \boxed{p_4(x) = \left[35x^4 - 30x^2 + 3\right]\left[\frac{3}{8\sqrt{2}}\right]}$$

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3. a) U and V $(n\times n)$ are upper triangular, i.e. $u_{ij}=0$, $v_{ij}=0$ for $i>j$.

(This generalizes to the lower triangular case by symmetry).

$$UV = \begin{bmatrix} \sum_{i=1}^{n} u_{1i}v_{i1} & \cdots & \sum_{i=1}^{n} u_{1i}v_{in} \\ \vdots & \ddots & \\ \sum_{i=1}^{n} u_{ni}v_{i1} & & \sum_{i=1}^{n} u_{ni}v_{in} \end{bmatrix}, \text{ i.e. } [UV]_{ab} = \sum_{i=1}^{n} u_{ai}v_{ib}$$

Consider $[UV]_{ab}$ for $a>b$.

$$\sum_{i=1}^{n} u_{ai}v_{ib} = \underbrace{u_{a1}v_{1b} + u_{a2}v_{2b} + \cdots + u_{a(a-1)}v_{(a-1)b}}_{u \text{ term}=0 \text{ since } a>1,\, a>2,\dots,\, a>a-1} + \underbrace{u_{aa}v_{ab} + \cdots + u_{an}v_{nb}}_{v \text{ term}=0 \text{ since } a>b,\, a+1>b,\dots,\, n>b}$$

$\Rightarrow \sum_{i=1}^{n} u_{ai}v_{ib} = 0$ for $a>b$.

Therefore, UV is upper triangular. (By symmetry, this also proves $U,V \in$ lower triangular $\Rightarrow$ $UV \in$ lower triangular.

b) U is upper triangular with non-zero diagonal entries. $(n\times n)$

We will prove that U^{-1} is upper triangular by contradiction. Suppose $\exists\, i,j,\ 1\le j < i \le n$, s.t. $U^{-1}_{ij} \neq 0$.

We know $UU^{-1}=I$. Consider $[UU^{-1}]_{ij}$:

$$[UU^{-1}]_{ij} = \sum_{a=1}^{n} [U_{ia}][U^{-1}_{aj}] = \underbrace{U_{i1}U^{-1}_{1j} + U_{i2}U^{-1}_{2j} + \cdots + U_{i(i-1)}U^{-1}_{(i-1)j}}_{U \text{ term}=0 \text{ since } i>1,\, i>2,\dots,\, i>i-1} + \underbrace{U_{ii}U^{-1}_{ij}}_{\text{nonzero}} + \underbrace{U_{i(i+1)}U^{-1}_{(i+1)j} + \cdots + U_{in}U^{-1}_{nj}}_{*}$$

$*$ If there exists a $k>i$ s.t. $U^{-1}_{kj} \neq 0$, then instead consider $[UU^{-1}]_{kj}$, in which case this section is equal to zero.

Therefore, $[UU^{-1}]_{ij} \neq 0$, but $[UU^{-1}]_{ij} = I$ so this is not possible. Therefore, by contradiction $U^{-1}_{ij}=0$ when $i>j$, so U^{-1} is upper triangular. (lower triangular follows by symmetry).

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3\. c) Given that R and $\hat{R}$ are upper triangular with positive diagonal, R^{-1} and $\hat{R}^{-1}$ are also upper triangular, and $R^T, \hat{R}^T, R^{-1^T}, \hat{R}^{-1^T}$ are all lower triangular. (by proof done in (b)).

- Then $R\hat{R}^{-1}$ is upper triangular (by (a) proof).
- $(R^T)^{-1}\hat{R}^T = [\hat{R}R^{-1}]^T$ which is lower triangular (by (a) proof).

But we have $R\hat{R}^{-1} = (R^T)^{-1}\hat{R}^T$, so it must be the case that $R\hat{R}^{-1} = (R^T)^{-1}\hat{R}^T = D$ must have nonzero entries only on the diagonal (since it is BOTH upper triangular and lower triangular).

$\rightarrow$ $R\hat{R}^{-1} = D \Rightarrow R = D\hat{R}$

$(R^T)^{-1}\hat{R}^T = D \Rightarrow [\hat{R}R^{-1}]^T = D \Rightarrow \hat{R}R^{-1} = \underbrace{D^T = D}_{\text{since}} \Rightarrow \hat{R} = DR$

Since we have $R = D\hat{R}$ and $\hat{R} = DR$, $D = I \Rightarrow \boxed{R = \hat{R}}$ as desired.

3.2) This conclusion follows naturally from Algorithm 3 and hardly needs a proof:

Values on the diagonal of $\hat{R}$ are defined as $r_{ii} := \|a_i\|_2$ following the column pivot.

The column is pivoted with the condition that $\|a_i\|_2 \geq \|a_j\|_2 \;\forall\, i \leq j \leq n$ post-pivot.

Then, for each of the remaining a_j columns ($j = i+1, \dots, n$), the projection onto the q vector is subtracted. We note that since the projection is being subtracted, it follows that

$$\|a_j\|_2 > \|a_j - \langle a_j, q_i \rangle q_i\|_2$$

Therefore, it holds from iteration to iteration that $\|a_i\|_2 \geq \|a_j\|_2 \;\forall\, i \leq j \leq n$.

$\Longrightarrow$ $\hat{r}_{11} \geq \hat{r}_{22} \geq \hat{r}_{33} \geq \dots \geq \hat{r}_{pp} > 0$. as desired.