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ACM106B Set 8 Problem 1

1. a) We can use (i) and (ii) combined to get:

$$u^* = (1 + \tfrac{1}{2} k \delta_y^2) u^n + (1 - \tfrac{1}{2} k \delta_y^2) u^{n+1}$$

So then using the boundary condition $u(x,y,t) = g(x,y,t)$, we have:

$$u^* = (1 + \tfrac{1}{2} k \delta_y^2) g^n + (1 - \tfrac{1}{2} k \delta_y^2) g^{n+1}$$ for the boundary.

1. b) From u^n to u^*:

$$-\tfrac{1}{2}\mu u^*_{i-1,j} + (1+\mu) u^*_{i,j} - \tfrac{1}{2}\mu u^*_{i+1,j} = \tfrac{1}{2}\mu u_{i,j-1} + (1-\mu) u_{i,j} + \tfrac{1}{2}\mu u_{i,j+1}$$

where j is a fixed index ranging from 1 to $m-1$.

In this way, given a j, we can construct a tridiagonal linear system. Thus we have $m-1$ tridiagonal systems, each of the form

$$\begin{bmatrix} 1+\mu & -\frac{\mu}{2} & & & \\ -\frac{\mu}{2} & 1+\mu & -\frac{\mu}{2} & & \\ & -\frac{\mu}{2} & 1+\mu & & \\ & & & \ddots & -\frac{\mu}{2} \\ & & & -\frac{\mu}{2} & 1+\mu \end{bmatrix} u^*_j = \begin{bmatrix} \frac{1}{2}\mu u_{1,j-1} + (1-\mu) u_{1,j} + \frac{1}{2}\mu u_{1,j+1} \\ \vdots \\ \frac{1}{2}\mu u_{m-1,j-1} + (1-\mu) u_{m-1,j} + \frac{1}{2}\mu u_{m-1,j+1} \end{bmatrix}$$

Similarly, for u^* to u^{n+1} we have:

$$-\tfrac{1}{2}\mu u^{n+1}_{i,j-1} + (1+\mu) u^{n+1}_{i,j} - \tfrac{1}{2}\mu u^{n+1}_{i,j+1} = \tfrac{1}{2}\mu u^*_{i-1,j} + (1-\mu) u^*_{i,j} + \tfrac{1}{2}\mu u^*_{i+1,j}$$

where i is a fixed index ranging from 1 to $m-1$. So we have $m-1$ tridiagonal systems of the form:

$$\begin{bmatrix} 1+\mu & -\frac{\mu}{2} & \\ -\frac{\mu}{2} & 1+\mu & \\ & & \ddots \end{bmatrix} u^{n+1}_i = \begin{bmatrix} \frac{1}{2}\mu u^*_{i-1,1} + (1-\mu) u^*_{i,1} + \frac{1}{2}\mu u^*_{i+1,1} \\ \vdots \\ \frac{1}{2}\mu u^*_{i-1,m-1} + (1-\mu) u^*_{i,m-1} + \frac{1}{2}\mu u^*_{i+1,m-1} \end{bmatrix}$$

1.2 ab) See code

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ACM106B Set 8 Problem 2

2.1. BTCS

$$v_m^{n+1} + \frac{\lambda}{2}\left(v_{m+1}^{n+1} - v_{m-1}^{n+1}\right) = v_m^n$$

$$\rightarrow \hat{v}^{n+1} e^{i\omega x_m} + \frac{1}{2}\left(\hat{v}^{n+1} e^{i\omega x_{m+1}} - \hat{v}^{n+1} e^{i\omega x_{m-1}}\right) = \hat{v}^n e^{i\omega x_m}$$

$$\rightarrow \hat{v}^{n+1}\left[1 + \frac{1}{2}e^{i\omega h} - \frac{1}{2}e^{-i\omega h}\right] = \hat{v}^n$$

$$\rightarrow \hat{v}^{n+1} = \hat{v}^n\left[1 + i\sin(\omega h)\right]^{-1}$$

$$\hat{Q} = \left[1 + i\sin(\omega h)\right]^{-1}$$

$$\rightarrow |\hat{Q}|^2 = \frac{1}{1+\sin^2(\omega h)} \leq 1$$

So the dissipation relation is $|\hat{Q}|^2 = \frac{1}{1+\sin^2(\omega h)} \leq 1$

C-N

$$v_m^{n+1} + \frac{\lambda}{4}\left(v_{m+1}^{n+1} - v_{m-1}^{n+1}\right) = v_m^n - \frac{\lambda}{4}\left(v_{m+1}^n - v_{m-1}^n\right)$$

$$\rightarrow \hat{v}^{n+1} e^{i\omega x_m} + \frac{1}{4}\left[\hat{v}^{n+1} e^{i\omega x_{m+1}} - \hat{v}^{n+1} e^{i\omega x_{m-1}}\right] = \hat{v}^n e^{i\omega x_m} - \frac{1}{4}\left[\hat{v}^n e^{i\omega x_{m+1}} - \hat{v}^n e^{i\omega x_{m-1}}\right]$$

$$\rightarrow \hat{v}^{n+1}\left[1 + \frac{1}{4}e^{i\omega h} - \frac{1}{4}e^{-i\omega h}\right] = \hat{v}^n\left[1 - \frac{1}{4}e^{i\omega h} + \frac{1}{4}e^{-i\omega h}\right]$$

$$\rightarrow \hat{v}^{n+1} = \hat{v}^n\left[\frac{1 - \frac{1}{2}i\sin(\omega h)}{1 + \frac{1}{2}i\sin(\omega h)}\right]$$

$$\rightarrow \hat{Q} = \frac{1 - \frac{1}{2}i\sin(\omega h)}{1 + \frac{1}{2}i\sin(\omega h)}$$

So the dissipation factor $|\hat{Q}|^2 = 1$ (non-dissipative)

Leapfrog

$$v_m^{n+1} = v_m^{n-1} - \lambda\left(v_{m+1}^n - v_{m-1}^n\right)$$

$$\rightarrow \hat{v}^{n+1} e^{i\omega x_m} = \hat{v}^{n-1} e^{i\omega x_m} - \hat{v}^n e^{i\omega x_{m+1}} + \hat{v}^n e^{i\omega x_{m-1}}$$

$$\rightarrow \hat{v}^{n+1} = \hat{v}^{n-1} + \hat{v}^n\left[e^{-i\omega h} - e^{i\omega h}\right]$$

$$\rightarrow \hat{v}^{n+1} = \hat{v}^{n-1} - \hat{v}^n\left[2i\sin(\omega h)\right]$$

$\hat{Q} = -i\sin(\omega h) + \left[1 - \sin^2(\omega h)\right]^{1/2}$ (slide 116 lecture notes ACM106C 2)

$$\rightarrow |\hat{Q}|^2 = 1 \text{ (non dissipative)}$$

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ACM106B Set 8 Problem 2 (cont'd)

2.2. BTCS

$\hat{Q} = |\hat{Q}| e^{-i\alpha\lambda\theta}$

$\rightarrow \dfrac{1-\frac{1}{2}\lambda i\sin(\theta)}{1+\frac{1}{2}\lambda i\sin(\theta)} = e^{-i\alpha(\theta)\lambda\theta}$

$\rightarrow 1-\frac{1}{2}\lambda i\sin(\theta) = \left[\cos(\alpha\lambda\theta) - i\sin(\alpha\lambda\theta)\right]\left[1+\frac{1}{2}\lambda i\sin\theta\right]^{-1}$

$\rightarrow \tan[\alpha\lambda\theta] = \lambda\sin\theta$

Taylor expanding, we have

$\alpha(\theta) = 1 - \frac{\theta^2}{6}(1+2\lambda^2) + O(\theta^4)$

Leapfrog

$\hat{Q} = -ia\lambda\sin\theta + [1-a^2\lambda^2\sin^2\theta]^{1/2}$

$\rightarrow |\hat{Q}| = 1$

$\rightarrow \hat{Q} = |\hat{Q}| e^{-i\alpha\lambda\theta}$

$\rightarrow \hat{Q} = \cos\alpha\lambda\theta - i\sin\alpha\lambda\theta$

$\rightarrow \sin[\alpha(\theta)\theta\lambda] = a\lambda\sin\theta$

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