Jacob Snyder
Section 8

ACM116 Set 5

1. a) $\mathbb{E}[W_2] = \int_{30}^{\infty} (x-30)(\lambda)(e^{-\lambda x})\,dx$

$= \int_{30}^{\infty} (x-30)(1/30)(e^{-x/30})\,dx$

$= 11.036$ minutes (by Mathematica)

b) Matlab

c) Matlab. Using the code from (b), it is worse to be 10^{th} patient with $\Delta t = 30$ min leading to an expected wait time of 54.77 min vs. 20^{th} patient with $\Delta t = 15$ min leading to expected wait time of 43.24 min.

2. a) $\mathbb{P}(N_{t_2} - N_{t_1} = 0) = e^{-\lambda(t_2-t_1)} \dfrac{(\lambda(t_2-t_1))}{0!} = \boxed{e^{-\lambda(t_2-t_1)}}$

b) Given that each interval is independent of the last (because the Exponential distribution is memoryless), the desired probability corresponds to

$[\mathbb{P}(N_1 = 1)]^n$

$= [\lambda e^{-\lambda}]^n$

$= \boxed{\lambda^n e^{-\lambda n}}$

c) Consider the interval $(1,2]$. We will consider 3 cases:

i) $N_2 - N_1 = 0$, ii) $N_2 - N_1 = 1$, iii) $N_2 - N_1 = 2$.

i) $P(N_2 - N_1 = 0)\,\mathbb{P}(N_1 = 2)\,\mathbb{P}(N_4 - N_2 = 3)$

$= [e^{-\lambda}][e^{-\lambda}(\frac{\lambda^2}{2})][e^{-2\lambda}(\frac{8\lambda^3}{6})]$

$= e^{-4\lambda}(\frac{8\lambda^5}{12}) = e^{-4\lambda}(\frac{2\lambda^5}{3})$

ii) $\mathbb{P}(N_2 - N_1 = 1)\,\mathbb{P}(N_1 = 1)\,\mathbb{P}(N_4 - N_2 = 2)$

$= [\lambda e^{-\lambda}][\lambda e^{-\lambda}][e^{-2\lambda}(\frac{4\lambda^2}{2})] = 2\lambda^4 e^{-4\lambda}$

iii) $\mathbb{P}(N_2 - N_1 = 2)\,\mathbb{P}(N_1 = 0)\,\mathbb{P}(N_4 - N_2 = 1)$

$= [e^{-\lambda}(\frac{\lambda^2}{2})][e^{-\lambda}][2\lambda e^{-2\lambda}] = \lambda^3 e^{-4\lambda}$

So all together, $\mathbb{P}(N_2 = 2, N_4 - N_1 = 3) = \boxed{e^{-4\lambda}[\frac{2}{3}\lambda^5 + 2\lambda^4 + \lambda^3]}$

3. $\mathbb{P}(T_1 = x \mid N_t = 1) = \dfrac{\mathbb{P}(T_1 = x)\,\mathbb{P}(N_t = 1 \mid T_1 = x)}{\mathbb{P}(N_t = 1)} = \begin{cases} 0 & x \le 0 \\ \dfrac{\mathbb{P}(T_1 = x)\,\mathbb{P}(N_t - N_x = 0)}{\mathbb{P}(N_t = 1)} & 0 < x \le t \\ 0 & t < x \end{cases}$

Note that $\mathbb{P}(T_1 = x) = \lambda e^{-\lambda x}$, $\mathbb{P}(N_t = 1) = e^{-\lambda t}(t\lambda)$, $\mathbb{P}(N_t - N_x = 0) = e^{-\lambda(t-x)}$

$\Rightarrow \boxed{\mathbb{P}(T_1 = x \mid N_t = 1) = \begin{cases} 0 & t < x \\ \frac{1}{t} & 0 < x \le t \\ 0 & x \le 0 \end{cases}}$

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4. a) $\mathbb{P}(N_\tau = 0) = e^{-\lambda\tau}\frac{(\lambda\tau)^0}{0!} = e^{-\lambda\tau}$

b) We can treat this as bernoulli trials with a success probability of $\mathbb{P}(N_\tau = 0) = e^{-\lambda\tau}$. This is because of the memoryless property of the exponential distribution, so if we fail we essentially treat it as a reset. By finding the expected number of "trials" before success and the expected wait per trial we can find the expected wait time.

Success probability: $e^{-\lambda\tau}$

$\varepsilon = 1(e^{-\lambda\tau}) + (1+\varepsilon)(1-e^{-\lambda\tau})$ where ε is expected number of trials

$\rightarrow \varepsilon(1-1+e^{-\lambda\tau}) = e^{-\lambda\tau}+1-e^{-\lambda\tau}$

$\rightarrow \varepsilon = e^{\lambda\tau}$ expected trials $\rightarrow e^{\lambda\tau}-1$ expected failed trials

Expected wait per trial $= \int_0^\tau x\,\mathbb{P}(T_1 = x \mid N_\tau = 1)\,dx$

$$= \int_0^\tau x\frac{\mathbb{P}(T_1=x)}{\mathbb{P}(N_\tau=1)}\,dx = \int_0^\tau x\frac{\lambda e^{-\lambda x}}{e^{-\lambda\tau}(\lambda\tau)}\,dx = \frac{1}{\tau e^{-\lambda\tau}}\int_0^\tau xe^{-\lambda x}\,dx$$

$$= \frac{1}{\tau e^{-\lambda\tau}}\left[\frac{1-e^{-\lambda\tau}(1+\lambda\tau)}{\lambda^2}\right] = \frac{1}{\tau\lambda^2 e^{-\lambda\tau}} - \frac{(1+\lambda\tau)}{\tau\lambda^2} = \frac{e^{\lambda\tau}-1-\lambda\tau}{\tau\lambda^2}$$

Therefore the total expected wait time is:

$$\boxed{\left[e^{\lambda\tau}-1\right]\left[\frac{e^{\lambda\tau}-1-\lambda\tau}{\lambda^2\tau}\right]}$$

5. a) Win probability $= \mathbb{P}(N_{\tau^*} - N_\tau = 1) = \boxed{e^{-\lambda(\tau^*-\tau)}(\lambda(\tau^*-\tau))}$

b) $\frac{d}{d\tau}\left[e^{-\lambda(\tau^*-\tau)}(\lambda(\tau^*-\tau))\right] = \lambda^2\tau^* e^{-\lambda(\tau^*-\tau)} - \lambda e^{-\lambda(\tau^*-\tau)} - \lambda^2\tau e^{-\lambda(\tau^*-\tau)}$

$= e^{-\lambda(\tau^*-\tau)}\left[\lambda^2\tau^* - \lambda - \lambda^2\tau\right] = 0$

$\rightarrow \lambda^2\tau^* - \lambda = \lambda^2\tau \rightarrow \boxed{\tau = \tau^* - 1/\lambda}$

c) $\mathbb{P}(N_{\tau^*} - N_{(\tau^*-1/\lambda)} = 1) = e^{-\lambda(1/\lambda)}(\lambda(1/\lambda)) = e^{-1} \approx \boxed{0.368}$

d) $\mathbb{E}[N_{\tau^*-1/\lambda}] = \left[\tau^*-1/\lambda\right]\lambda = \tau^*\lambda - 1$

$\mathbb{P}(N_{1/\lambda} = 0) = e^{-\lambda(1/\lambda)} = e^{-1} \approx 0.368$

So expected number of croissants under this optimal strategy is:

$\tau^*\lambda - 1 + (1-0.368)(1) = \boxed{\tau^*\lambda - 0.368}$

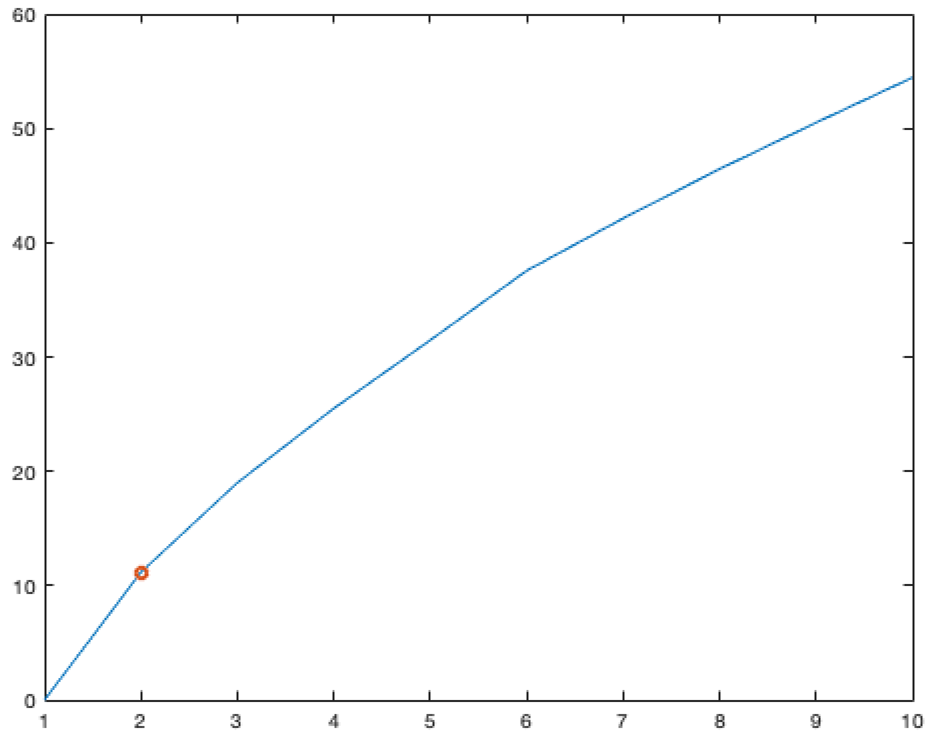
```
delt = 30;
W = zeros(10,10000);

for a=1:10000
    waittime = 0;
    for p=1:10
        W(p,a) = waittime;
        data = exprnd(delt);
        if data < delt
            waittime = max(0, waittime - delt + data);
        else
            waittime = waittime + data - delt;
        end
    end
end

W1 = mean(W(1,:));
W2 = mean(W(2,:));
W3 = mean(W(3,:));
W4 = mean(W(4,:));
W5 = mean(W(5,:));
W6 = mean(W(6,:));
W7 = mean(W(7,:));
W8 = mean(W(8,:));
W9 = mean(W(9,:));
W10 = mean(W(10,:));

Ws = [W1 W2 W3 W4 W5 W6 W7 W8 W9 W10];

plot(Ws);
hold on;
scatter([2],[11.036]);
```



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