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Section 8

ACM116 Midterm

1. a) True. Given that $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ and $\mathbb{V}[X] \geq 0$, $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$

b) False. Suppose A is rolling a 6 on a die, B is rolling a 1. They're trivially disjoint but $P(AB) = 0 \neq P(A)P(B) = 1/36$ So they're not independent.

c) True. $\mathbb{V}[XY] = \mathbb{E}[X^2Y^2] - \underbrace{\mathbb{E}[XY]^2}_{=0} = \underbrace{\mathbb{E}[X^2]\mathbb{E}[Y^2]}_{\text{because independence}} - 0 = (\mathbb{E}[X^2] - \underbrace{\mathbb{E}[X]^2}_{=0})(\mathbb{E}[Y^2] - \underbrace{\mathbb{E}[Y]^2}_{=0})$

$= \mathbb{V}[X]\mathbb{V}[Y]$

d) False. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$ $\quad \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$

And $\mathbb{E}[Y] = \mathbb{E}[Y]$ does not hold for any two random variables X and Y.

e) True. $\mathbb{P}(X=a) = \int_a^a f(x)\,dx = 0$

2. $P(\text{spam}|\text{free}) = \dfrac{P(\text{spam, free})}{P(\text{free})} = \dfrac{(0.4)(0.9)}{P(\text{spam,free}) + P(\text{low prio, free}) + P(\text{high prio, Free})}$

$= \dfrac{0.4(0.9)}{(0.4)(0.9)+(0.5)(0.05)+(0.1)(0.05)} = \boxed{0.923}$

3. After 1 jump: $P(X_1 = -1) = p \quad P(X_1 = 1) = 1-p$

$\mathbb{E}[X_1] = p(-1) + (1-p)(1) = 1-2p$

After 1 jump has been completed, the same probabilities apply to the next jump, so we can conclude that $\mathbb{E}[X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_1] = 2(1-2p)$. In fact, extending this logic:

$\boxed{\mathbb{E}[X_n] = n(1-2p)}$

$\mathbb{V}[X_n] = \mathbb{V}[X_{1_1} + X_{1_2} + X_{1_3} + \dots + X_{1_n}] = \mathbb{V}[X_1] + \mathbb{V}[X_1] + \dots = n\mathbb{V}[X_1]$

note that $\mathbb{V}[X_1] = \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 = p(1) + (1-p)(1) - (1-2p)^2 = 1 - 1 + 4p - 4p^2$

$= 4p(1-p)$

So $\mathbb{V}[X_n] = n(4p(1-p)) = \boxed{4pn(1-p)}$

4. $\mathbb{E}[X|Y=y] = \int_0^\infty x \dfrac{f(x,y)}{f_Y(y)}\,dx$ where $f_Y(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty \dfrac{e^{-x/y}e^{-y}}{y}\,dx$

$= \dfrac{e^{-y}}{y}\left[-ye^{-x/y}\right]_0^\infty = \dfrac{e^{-y}}{y}(y) = e^{-y}$

$\rightarrow \mathbb{E}[X|Y=y] = \int_0^\infty x\dfrac{e^{-x/y}}{y}\,dx \qquad u = x \quad dv = e^{-x/y}dx \quad du = dx \quad v = -ye^{-x/y}$

$= \left(-xye^{-x/y}\right)_0^\infty + \int_0^\infty ye^{-x/y}\,dx = y\left[-ye^{-x/y}\right]_0^\infty = \boxed{y^2}$

5. $M_{X,Y}(s,t) = \exp(1 + s + s^2 - \cos t + s^{2019} t)$

$\Rightarrow M_{X,Y}(s,0) = \exp(s + s^2 + 1 - 1) = \exp(s + s^2) = M_X(s)$

Which corresponds to $\mu = 1$, $\sigma^2 = 2$, by the hint given.

So $X \sim \mathcal{N}(1, 2)$

6. We can safely assume that $\mathbb{P}(2 < X < 8)$ is large, likely in the range $0.75 \leq \mathbb{P}(2<X<8) \leq 1$, given that the range $(2,8)$ spans one variance away from the mean on either side. We can't make any conclusive results because we don't know the distribution of X but we can say that $P(2<X<8)$ is closer to 1 than it is to 0.

7. By CLT, for large n, $\overline{T}_n \to \mathcal{N}(\mu, \frac{\sigma^2}{n})$

We want $\mathbb{P}(\overline{T}_n \geq 2\mu) = \mathbb{P}\left(\frac{\overline{T}_n - \mu}{\sigma/\sqrt{n}} \geq \frac{\mu}{\sigma/\sqrt{n}}\right) = 1 - \Phi\left(\frac{\mu}{\sigma/\sqrt{n}}\right) = \Phi\left(\frac{-\mu}{\sigma/\sqrt{n}}\right)$

8. a1) Let X_r denote the poisson rainy day variable $(\lambda = 9)$ and let X_d denote the poisson dry day variable $(\lambda = 3)$.

$\mathbb{E}[X] = 0.1\,\mathbb{E}[X_r] + 0.9\,\mathbb{E}[X_d] = 0.1(9) + 0.9(3) = \boxed{3.6}$

a2) $\mathbb{P}[X=0] = 0.1\,\mathbb{P}[X_r = 0] + 0.9\,\mathbb{P}[X_d = 0] = \boxed{0.1e^{-9} + 0.9e^{-3}} \approx 0.0448$

a3) $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$= \sum_{n=0}^{\infty} (0.1)n^2\mathbb{P}(X_r = n) + (0.9)n^2\mathbb{P}(X_d = n) - 3.6^2 = 0.1\sum_{n=0}^{\infty} n^2\mathbb{P}(X_r = n) + 0.9\sum_{n=0}^{\infty} n^2\mathbb{P}(X_d = n) - 3.6^2$

$= 0.1\,\mathbb{E}[X_r^2] + 0.9\,\mathbb{E}[X_d^2] - 3.6^2$

$= 0.1(\mathbb{V}[X_r] + \mathbb{E}[X_r]^2) + 0.9(\mathbb{V}[X_d] + \mathbb{E}[X_d]^2) - 3.6^2$

$= 0.1(9 + 9^2) + 0.9(3 + 3^2) - 3.6^2$

$= 0.1(90) + 0.9(12) - 3.6^2$

$= \boxed{6.84}$