Jacob Snyder

ACM104 Set 5

2. Using the Gram-Schmidt Process, we can create an orthogonal basis for the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-\frac{x^2}{2}}\,dx \quad \text{on the elementary basis } 1, x, x^2, \dots, x^n$$

G-S Process

$$h_0(x) = \boxed{1}$$

$$h_1(x) = x - \frac{\langle x, 1\rangle}{\|1\|^2}(1) = x - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}}\,dx = \boxed{x}$$

$$h_2(x) = x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2}1 - \frac{\langle x^2, x\rangle}{\|x\|^2}x = \boxed{x^2 - 1}$$

$$h_3(x) = x^3 - \frac{\langle x^3, 1\rangle}{\|1\|^2} - \frac{\langle x^3, x\rangle}{\|x\|^2}x - \frac{\langle x^3, x^2-1\rangle}{\|x^2-1\|^2}(x^2-1) = \boxed{x^3 - 3x}$$

$$h_4(x) = x^4 - \frac{\langle x^4, 1\rangle}{\|1\|^2} - \frac{\langle x^4, x\rangle}{\|x\|^2}x - \frac{\langle x^4, x^2-1\rangle}{\|x^2-1\|^2}(x^2-1) - \frac{\langle x^4, x^3-3x\rangle}{\|x^3-3x\|^2}(x^3-3x)$$

$$= x^4 - 3 - 0 - \frac{12\sqrt{2\pi}}{2\sqrt{2\pi}}(x^2-1) - 0$$

$$= \boxed{x^4 - 6x^2 + 3}$$

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3. To calculate a basis for $W_1^\perp$:

$$\langle \vec{v}_1, (x_1, x_2, x_3) \rangle_1 = 0$$
$$\langle \vec{v}_2, (x_1, x_2, x_3) \rangle_1 = 0$$

$\rightarrow x_1 + 2x_2 + 3x_3 = 0 \rightarrow x_1 + 2x_2 - 6x_1 = 0 \rightarrow x_2 = \frac{5}{2}x_1$

$\rightarrow 2x_1 + x_3 = 0 \rightarrow x_3 = -2x_1$

$\Rightarrow$ $\boxed{\begin{bmatrix} 1 \\ 5/2 \\ -2 \end{bmatrix} \text{ is the basis for } W_1^\perp}$

For $W_2^\perp$:

$$\langle \vec{v}_1, (x_1, x_2, x_3) \rangle_2 = 0$$
$$\langle \vec{v}_2, (x_1, x_2, x_3) \rangle_2 = 0$$

$\rightarrow x_1 + 4x_2 + 9x_3 = 0 \rightarrow x_1 + 4x_2 - 6x_1 = 0 \rightarrow x_2 = \frac{5}{4}x_1$

$\rightarrow 2x_1 + 3x_3 = 0 \rightarrow x_3 = -\frac{2}{3}x_1$

$\Rightarrow$ $\boxed{\begin{bmatrix} 1 \\ 5/4 \\ -2/3 \end{bmatrix} \text{ is the basis for } W_2^\perp}$

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4. We must first find the eigenvalues of A:

$$\det \begin{bmatrix} -\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = 0$$

$$\rightarrow \quad (1-\lambda)(\lambda^2+1) = 0 \rightarrow \quad (1-\lambda)(\lambda+i)(\lambda-i) = 0$$

$\rightarrow$ eigenvalues are $1, \pm i$

Because the eigenvalues are distinct and complex, we can conclude that the associated eigenvectors form an eigenbasis of $\mathbb{C}^3$ and therefore that A is complete.

Eigenvectors:

i:

$$\begin{bmatrix} -i & 0 & -1 \\ 0 & 1-i & 0 \\ 1 & 0 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \rightarrow \begin{array}{l} -ix_1 - x_3 = 0 \\ x_2 - ix_2 = 0 \Rightarrow x_2 = 0 \\ x_1 - ix_3 = 0 \Rightarrow x_1 = ix_3 \end{array} \Rightarrow \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$$

Therefore, because we have a complex eigenvector, we cannot have an eigenbasis for $\mathbb{R}^3$.

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5. a) $\underline{D_1}$: $r_1 = \sum_{j \neq 1} |a_{1j}| = 1$

$z \in \mathbb{C}$: $|z - 0| \leq 1$

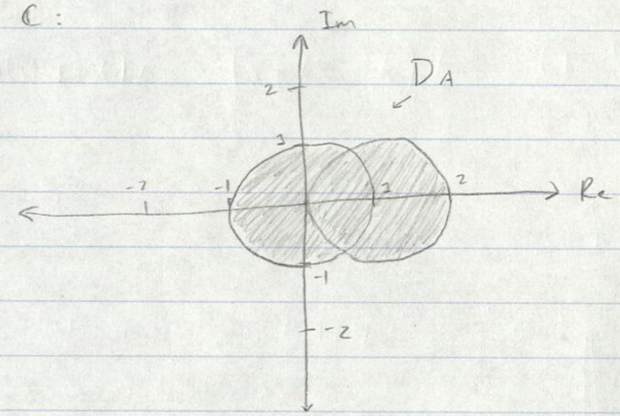
$\underline{D_2}$: $r_2 = \sum_{j \neq 2} |a_{2j}| = 1$

$z \in \mathbb{C}: |z - 1| \leq 1$

D_3: $r_3 = \sum_{j \neq 3} |a_{3j}| = 1$

$z \in \mathbb{C} : |z - 1| \leq 1$

On $\mathbb{C}$:



b) It is given that $\text{spec}(A) \subset D_A$.

Also, $\text{spec}(A^T) \subset D_{A^T}$.

We will now prove that $\text{spec}(A) = \text{spec}(A^T)$:

Eigenvalues are determined by the roots of $\det(A - \lambda I)$.

We know that $\det(A) = \det(A^T)$. Also, $(\alpha I)^T = \alpha I$ by definition.

Therefore we have: $\det((A - \lambda I)^T) = \det(A - \lambda I)$

$\rightarrow \det(A^T - \lambda I) = \det(A - \lambda I)$

Therefore, A and A^T have the same eigenvalues, so $\text{spec}(A) = \text{spec}(A^T)$.

Therefore, $\text{spec}(A) \subset D_A$ AND $\text{spec}(A) \subset D_{A^T}$

Therefore, $\text{spec}(A) \subset D_A \cap D_{A^T} \rightarrow \boxed{\text{spec}(A) \subset D_A^*}$

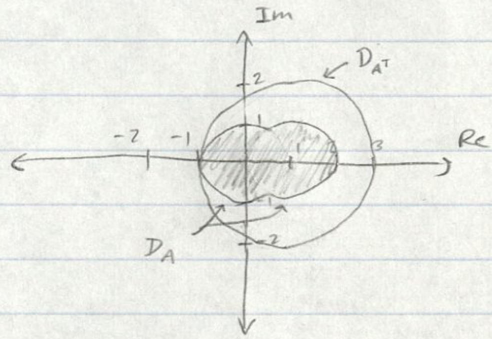
(cont'd on back)

c) $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$\underline{D_1}: r_1 = 0 \quad z \in \mathbb{C} : |z| \leq 0$

$\underline{D_2}: r_2 = 2 \quad z \in \mathbb{C} : |z-1| \leq 2$

$\underline{D_3}: r_3 = 1 \quad z \in \mathbb{C} : |z-1| \leq 1$

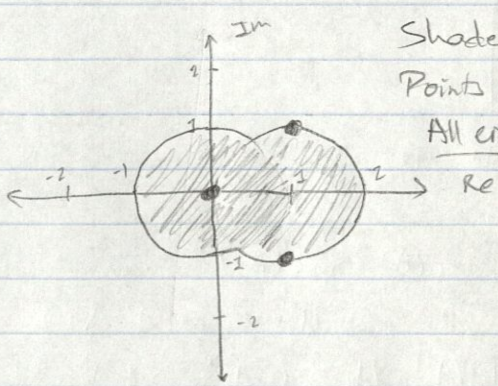


Shaded area is $D_A \cap D_{A^T} = D_A^*$

d) $\det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{bmatrix} = -\lambda\left((1-\lambda)^2+1\right) = -\lambda\left(\lambda^2 - 2\lambda + 2\right) = -\lambda(\lambda-1-i)(\lambda-1+i)$

Eigenvalues: $0, 1+i, 1-i$

On $\mathbb{C}$:



Shaded region is D_A^*

Points are eigenvalues.

<u>All eigenvalues of A are contained in D_A^*</u>