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1. a) False. The covariance matrix is always symmetric.

b) True, since gaussian vectors are jointly normally distributed.

c) false. $\mathbb{E}[B_2 - B_1] = 0 \rightarrow \mathbb{E}[B_2 | B_1 = 1] = 1$

d) false. $S_x(f) = 1 + f + \sin(f)$ is not an even function.

e) true. $\Sigma_x = \begin{bmatrix} \mathbb{V}[N_1] & Cov(N_1,N_2) & Cov(N_1,N_3) \\ Cov(N_1,N_2) & \mathbb{V}[N_2] & Cov(N_2,N_3) \\ Cov(N_3,N_1) & Cov(N_3,N_2) & \mathbb{V}[N_3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

2. a) We can describe N as the sum of two poisson distributions, one for wins and one for losses. We then have that $W \sim Poisson(p\lambda)$

So $\mathbb{E}[W] = p\lambda$

b) Because $W \sim Poisson(p\lambda)$, $\mathbb{V}[W] = p\lambda$

3. Suppose $U = X + Y$, $V = X - Y$

$\rightarrow U + V = 2X$, $U - V = 2Y$

$\rightarrow \frac{U+V}{2} = X \quad \frac{U-V}{2} = Y$

Therefore $g(u,v) = f(\frac{u+v}{2}, \frac{u-v}{2})$

4. $Y \sim N(A\mu_x + b, A\Sigma_x A^T)$

$\Rightarrow A\mu_x + b = 0$, $A\Sigma_x A^T$ is diagonal $\Rightarrow A = [q_1 | q_2 | \cdots | q_n]$ where q_i are eigenvectors of Σ_x

$\Rightarrow [q_1 | q_2 | \cdots | q_n]\mu_x + b = 0 \rightarrow b = -[q_1 | q_2 | \cdots | q_n]\mu_x$

(we also note that $A = [0]_{n \times n}$, $b = [0]_{n \times 1}$ is a solution)

5. a) Waiting times are distributed according to Exp(2).

Therefore $\mathbb{E}[\text{Exp}(2)] = \boxed{\frac{1}{2}\text{ weeks}}$

b) $\mathbb{P}(N_2 = 0) = e^{-4}\frac{(4)^0}{0!} = \boxed{e^{-4}}$

c) $N_{52} \sim \text{Poisson}(52(2)) = \text{Poisson}(104)$

$\mathbb{E}[\text{Poisson}(104)] = \boxed{104}$

d) Given that the waiting time for the n^{th} event is distributed as Gamma(n, λ), and that the poisson process is memoryless, if at any time the waiting period for the 7^{th} event is less than a week, then 7 events have occurred during 1 week. Each week is independent of the last so we can essentially compute this as 52 separate trials each with a success rate of $F(1)$. The odds that we fail all 52 times is $(1-F(1))^{52}$ so the odds of at least one success (i.e. probability that during at least one week there are at least 7 attacks) is $\boxed{1-(1-F(1))^{52}}$

Recall that this is $x=1$ week

6. $R_{Y_t}(t_1, t_2) = \mathbb{E}[Y_{t_1} Y_{t_2}] = \mathbb{E}[e^{B_{t_1}+B_{t_2}}]$

We note that $B_{t_1} + B_{t_2} \sim N(0, t_1) + N(0, t_1) + N(0, t_2 - t_1)$ (we are assuming $t_2 > t_1$ without loss of generality)

$$\sim 2N(0, t_1) + N(0, t_2 - t_1)$$

$$\sim N(0, 4t_1 + t_2 - t_1) \sim N(0, 3t_1 + t_2)$$

$\Rightarrow \mathbb{E}[e^{B_{t_1}+B_{t_2}}] = M(\sqrt{3t_1+t_2}) = \boxed{\exp\left(\frac{3t_1+t_2}{2}\right)}$ where $t_2 > t_1$

7. a) $m_{X_t}(t) = \mathbb{E}[X_t] = \int_{-\infty}^{\infty} x \mathbb{P}(X_t = x)\,dx = (-1/2)\mathbb{P}(X_0(-1)^{N_t} = -1/2) + (1/2)\mathbb{P}(X_0(-1)^{N_t} = 1/2)$

$= (-1/2)\mathbb{P}(X_0 = 1/2)\mathbb{P}((-1)^{N_t} = -1) + (-1/2)\mathbb{P}(X_0 = -1/2)\mathbb{P}((-1)^{N_t} = 1) + (1/2)\mathbb{P}(X_0 = -1/2)\mathbb{P}((-1)^{N_t} = -1) + (1/2)\mathbb{P}(X_0 = 1/2)\mathbb{P}((-1)^{N_t} = 1)$

$= \frac{1}{2}\mathbb{P}((-1)^{N_t} = 1)\underbrace{[\mathbb{P}(X_0 = 1/2) - \mathbb{P}(X_0 = -1/2)]}_{0} + \frac{1}{2}\mathbb{P}((-1)^{N_t} = -1)\underbrace{[\mathbb{P}(X_0 = -1/2) - \mathbb{P}(X_0 = 1/2)]}_{0}$

$= 0 + 0$

$= \boxed{0}$

b) $R_{X_t}(t_1, t_2) = \mathbb{E}[X_{t_1} X_{t_2}] = \mathbb{E}[X_0^2 (-1)^{N_{t_1} + N_{t_2}}] = \mathbb{E}[1/4\,(-1)^{N_{t_1} + N_{t_2}}]$

We note that only the parity of $N_{t_1} + N_{t_2}$ is important; for this reason,

$\mathbb{E}[1/4\,(-1)^{N_{t_1} + N_{t_2}}] = \mathbb{E}[1/4\,(-1)^{N_{t_2} - N_{t_1}}]$ (we assume $t_2 > t_1$ without loss of generality)

$= \frac{1}{4}(\mathbb{P}((-1)^{N_{t_2} - N_{t_1}} = 1) - \mathbb{P}((-1)^{N_{t_2} - N_{t_1}} = -1))]$

$= \frac{1}{4}\left[\sum_{n=-\infty}^{\infty}[\mathbb{P}(N_{t_2} - N_{t_1} = 2n) - \mathbb{P}(N_{t_2} - N_{t_1} = 2n+1)]\right]$

$= \frac{1}{4}\left[\sum_{n=-\infty}^{\infty}\left[e^{-\lambda(t_2 - t_1)}\frac{(\lambda(t_2 - t_1))^{2n}}{(2n)!} - e^{-\lambda(t_2 - t_1)}\frac{(\lambda(t_2 - t_1))^{2n+1}}{(2n+1)!}\right]\right]$

c) Given that $R_{X_t}(t_1, t_2)$ is a function of $t_2 - t_1$, yes, $\{X_t, t \geq 0\}$ is WSS.

8. $\mathbb{E}[X_t^2] = \int_{-\infty}^{\infty} S_X(f)\,df = \int_{-\infty}^{\infty} e^{-f^2/2}\,df = \sqrt{2\pi}\,\Phi(\infty) = \boxed{\sqrt{2\pi}}$