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ACM 106 B Final Problem 1

1.1) $$\frac{-u_{m+2}^{n+1} + 9u_{m+1}^{n+1} + 9u_m^{n+1} - u_{m-1}^{n+1}}{16k} + a\,\frac{-u_{m+2}^{n+1} + 27u_{m+1}^{n+1} - 27u_m^{n+1} + u_{m-1}^{n+1}}{48h}$$

$$= \frac{-u_{m+2}^{n} + 9u_{m+1}^{n} + 9u_m^{n} - u_{m-1}^{n}}{16k} - a\,\frac{-u_{m+2}^{n} + 27u_{m+1}^{n} - 27u_m^{n} + u_{m-1}^{n}}{48h}$$

$$+ \frac{1}{32}\left[-f_{m+2}^{n+1} + 9f_{m+1}^{n+1} + 9f_m^{n+1} - f_{m-1}^{n+1} - f_{m+2}^{n} + 9f_{m+1}^{n} + 9f_m^{n} - f_{m-1}^{n}\right]$$

$$\rightarrow \frac{1}{16k}\left[-\hat{u}^{n+1}e^{i\omega x_{m+2}} + 9\hat{u}^{n+1}e^{i\omega x_{m+1}} + 9\hat{u}^{n+1}e^{i\omega x_m} - \hat{u}^{n+1}e^{i\omega x_{m-1}}\right] + \left[\frac{a}{48h}\right]\left[\hat{u}^{n+1}\right]\left[-e^{i\omega x_{m+2}} + 27e^{i\omega x_{m+1}}\right.$$
$$\left. - 27e^{i\omega x_m} + e^{i\omega x_{m-1}}\right] = \frac{1}{16k}\left[\hat{u}^n\right]\left[-e^{i\omega x_{m+2}} + 9e^{i\omega x_{m+1}} + 9e^{i\omega x_m} - e^{i\omega x_{m-1}}\right] - \frac{a\hat{u}^n}{48}\left[-e^{i\omega x_{m+2}} + 27e^{i\omega x_{m+1}} - 27e^{i\omega x_m} + e^{i\omega x_{m-1}}\right]$$

$$\rightarrow \hat{u}^{n+1}\left[\frac{1}{16k}\left[-e^{2i\omega h} + 9e^{i\omega h} + 9 - e^{-i\omega h}\right] + \frac{a}{48h}\left[-e^{2i\omega h} + 27e^{i\omega h} - 27 + e^{-i\omega h}\right]\right]$$
$$= \hat{u}^n\left[\frac{1}{16k}\left[-e^{2i\omega h} + 9e^{i\omega h} + 9 - e^{-i\omega h}\right] - \frac{a}{48h}\left[-e^{2i\omega h} + 27e^{i\omega h} - 27 + e^{-i\omega h}\right]\right]$$

$$\rightarrow \hat{Q} = \left(\left[9 + 8\cos(\omega h) - \cos(2\omega h) + 10i\sin(\omega h) - i\sin(2\omega)\right] - \frac{a\lambda}{3}\left[4(\cos(\omega h) - 13)(\sin^2(\tfrac{\omega}{2})) + i(26\sin(\omega) - \sin(2\omega))\right]\right)$$
$$\cdot \left[\left[9 + 8\cos(\omega h) - \cos(2\omega h) + 10i\sin(\omega h) - i\sin(2\omega)\right] + \frac{a\lambda}{3}\left[4(\cos(\omega h) - 13)(\sin^2(\tfrac{\omega}{2})) + i(26\sin(\omega) - \sin(2\omega))\right]\right]^{-1}$$

$\Rightarrow |\hat{Q}| = 1$ (using Wolfram Alpha to simplify algebra) so the scheme is unconditionally stable.

1.2) In operator form: $P_t^{k,h}u + aP_x^{k,h}u = R^{k,h}f$

This can be shown by Taylor expanding the operators and applying $u_t + au_x = f$ to the expansions.

1.3) Define

$$A = \begin{bmatrix} 9 & 9 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 9 & 9 & -1 & & & & 0 \\ 0 & -1 & 9 & 9 & -1 & & & \vdots \\ 0 & & & & \ddots & \ddots & & 0 \\ \vdots & & & & -1 & 9 & 9 & -1 \\ -1 & 0 & & & & -1 & 9 & 9 \\ 9 & -1 & 0 & \cdots & & 0 & -1 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} -27 & 27 & -1 & & \cdots & & 0 & 1 \\ 1 & -27 & 27 & -1 & & & & 0 \\ 0 & 1 & -27 & 27 & -1 & & & \vdots \\ \vdots & & & & \ddots & \ddots & & \vdots \\ 0 & \cdots & & & 1 & -27 & 27 & -1 \\ -1 & & & & & 1 & -27 & 27 \\ 27 & -1 & & \cdots & & & 1 & -27 \end{bmatrix}$$

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ACM106B Final Problem 2

2.1) $u'''' = f$

$\rightarrow u''''v = fv$

$\rightarrow \int_0^1 u''''v\,dx = \int_0^1 fv\,dx$ $\quad u = v \quad dv = u'''' \quad du = v'dx \quad v = u'''$

$\rightarrow \underbrace{[vu''']_0^1}_{\text{set } v(0)=v(1)=0} - \int_0^1 u'''v'\,dx = (f, v)$

$\rightarrow -\int_0^1 u'''v'\,dx = (f, v)$ $\quad u = v' \quad dv = u''' \quad du = v''dx \quad v = u''$

$\rightarrow \underbrace{-[u''v']_0^1}_{\text{set } v'(1)=v'(0)=0} + \int_0^1 v''u''\,dx = (f, v)$

$\rightarrow (u'', v'') = (f, v)$ as desired. $\forall v \in W$, W as described in the problem.

2.2) $\varphi_1(x) = 2\left[\frac{x-a}{b-a}\right]^3 - 3\left[\frac{x-a}{b-a}\right]^2 + 1 \qquad \varphi_2(x) = -2\left[\frac{x-a}{b-a}\right]^3 + 3\left[\frac{x-a}{b-a}\right]^2$

$\psi_1(x) = (b-a)\left[\left[\frac{x-a}{b-a}\right]^3 - 2\left[\frac{x-a}{b-a}\right]^2 + \left[\frac{x-a}{b-a}\right]\right] \qquad \psi_2(x) = (b-a)\left[\left[\frac{x-a}{b-a}\right]^3 - \left[\frac{x-a}{b-a}\right]^2\right]$

$\varphi_1(a) = 0 - 0 + 1 = 1$ ✓ $\quad \varphi_1(b) = 2 - 3 + 1 = 0$ ✓

$\varphi_2(a) = 0 + 0 = 0$ ✓ $\quad \varphi_2(b) = -2 + 3 = 1$ ✓

$\psi_1(a) = 0$ ✓ $\quad \psi_1(b) = (b-a)0 = 0$ ✓

$\psi_2(a) = (b-a)0 = 0$ ✓ $\quad \psi_2(b) = (b-a)(1-1) = 0$ ✓

$\varphi_1'(a) = 0 - 0 = 0$ ✓ $\quad \varphi_1'(b) = \frac{6}{b-a} - \frac{6}{b-a} = 0$ ✓

$\varphi_2'(a) = 0 + 0 = 0$ ✓ $\quad \varphi_2'(b) = -\frac{6}{b-a} + \frac{6}{b-a} = 0$ ✓

$\psi_1'(a) = \frac{b-a}{b-a} = 1$ ✓ $\quad \psi_1'(b) = (4-4) = 0$ ✓

$\psi_2'(a) = 0 - 0 = 0$ ✓ $\quad \psi_2'(b) = \frac{b-a}{b-a} = 1$ ✓

$\rightarrow c_0(\varphi_1 + \varphi_2) = \boxed{c_0}$

$\rightarrow (b-a)\varphi_2 + \psi_1 + \psi_2 = (b-a)\left[\frac{x-a}{b-a}\right] = x - a \Rightarrow \underbrace{c_1[(b-a)\varphi_2 + \psi_1 + \psi_2 + a(\varphi_1 + \varphi_2)]}_{} = \boxed{c_1 x}$

$\rightarrow 2(b-a)^{-1}\psi_2 + \varphi_2 = \left[\frac{x-a}{b-a}\right]^2 \rightarrow (b-a)^2[2(b-a)^{-1}\psi_2 + \varphi_2] = x^2 - 2ax + a^2$

$\rightarrow c_2[(b-a)^2[2(b-a)^{-1}\psi_2 + \varphi_2] + 2a \quad - a^2(\varphi_1 + \varphi_2)] = \boxed{c_2 x^2}$

$\rightarrow \varphi_2 + \frac{3}{b-a}\psi_2 = \left[\frac{x-a}{b-a}\right]^3 \rightarrow (b-a)^3\left[\varphi_2 + \frac{3}{b-a}\psi_2\right] = [x-a]^3 = x^3 + \underbrace{ix^2 + jx + k}_{}$

Can remove these with earlier found linear combinations so we are left with $\boxed{c_3 x^3}$

So the cubic Hermite basis forms a linear basis for $P_3(I)$.

2.3) We define nodal basis functions:

$$\phi_i(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \quad i,j = 1:n \text{ (with } n \text{ grid points).}$$

Then, given a collection of cubic functions over the grid, we can define functions in W_h as such: $v(x) = \sum_{j=1}^{n} v_j \phi_j(x)$ where v_j are the cubic functions to be patched.

As shown in the previous part of the problem, we can use linear combinations of $\varphi_1, \varphi_2, \psi_1, \psi_2$ to ensure that the boundary conditions are met for all v_j.

2.4) From earlier: $(u'', v'') = (f, v)$

let $v = \phi_i$ for $i = 1, \dots, n$

$\rightarrow (u'', \phi_i'') = (f, \phi_i)$

Using our representation from earlier: $v(x) = \sum_{j=1}^{n} v_j \phi_j(x)$:

$$\sum_{j=1}^{n} v_j (\phi_j'', \phi_i'') = (f, \phi_i) \quad i = 1, \dots, n$$

Where each v_j is a cubic function constructed from $\varphi_1, \varphi_2, \psi_1, \psi_2$.

So we have a system of equations (one for each i) which is structured like so:

$$A\vec{v} = \vec{b}$$

Where $A_{ij} = (\phi''_j, \phi''_i) = \int_0^1 \phi''_j \phi''_i \, dx$

$b_i = (f, \phi_i) = \int_0^1 f \cdot \phi_i \, dx$

2.5) We will iterate through i and j (from 1 to n), on each pair of i,j we evaluate (ϕ_j'', ϕ_i'') and populate the stiffness matrix in this way. Every time i changes, we compute (f, ϕ_i) in order to populate the load vector. This will be done using Mathematica explicitly.

After having constructed the stiffness matrix and load vector, we can use A\b in MATLAB to solve for the cubic functions v_j.

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ACM106B Final Problem 3

3.1) $\lim_{r \to 0} u^r(x,t)$

$$= \lim_{r\to 0}\left(\bar{a} - c\tanh\left(\frac{c}{2r}(x - \bar{a}t - x_0)\right)\right)$$ Note that $\tanh(\infty) = 1$, $\tanh(-\infty) = -1$

$$= \begin{cases} \bar{a} - c & \text{if } (x - \bar{a}t - x_0) > 0 \\ \bar{a} + c & \text{if } (x - \bar{a}t - x_0) < 0 \end{cases}$$

$= a - \text{sign}(x - \bar{a}t - x_0)c$ as desired.

3.5) a) $U_j^{n+1} = U_j^n - \frac{k}{h}[F(U;j) - F(U;j-1)]$

Try $F(U;j) = f(U_j) + \frac{1}{2}(1 - \eta_j)(f(U_{j+1}) - f(U_j))$, $\eta_j = \frac{k}{h}\frac{f(U_{j+1}) - f(U_j)}{U_{j+1} - U_j}$

$\rightarrow U_j^{n+1} = U_j^n - \frac{k}{h}\left[f(U_j) + \frac{1}{2}(1-\eta_j)(f(U_{j+1}) - f(U_j)) - f(U_{j-1}) - \frac{1}{2}(1 - \eta_{j-1})(f(U_j) - f(U_{j-1}))\right]$

$\rightarrow U_j^{n+1} = U_j^n - \frac{k}{2h}\left[2f(U_j) - 2f(U_{j-1}) + f(U_{j+1}) - f(U_j) - f(U_j) + f(U_{j-1}) - \eta_j(f(U_{j+1}) - f(U_j)) + \eta_{j-1}(f(U_j) - f(U_{j-1}))\right]$

$\rightarrow U_j^{n+1} = U_j^n - \frac{k}{2h}\left[f(U_{j+1}) - f(U_{j-1}) - \frac{k}{h}\frac{(f(U_{j+1}) - f(U_j))^2}{U_{j+1} - U_j} + \frac{k}{h}\frac{(f(U_j) - f(U_{j-1}))^2}{U_j - U_{j-1}}\right]$ ...?