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ACM106B Set 3 Problem 1

1. a) $I_h^{2h} w_{k}^{h}$

Given any j, we have:

$$(I_h^{2h} w_k^h)_j$$
$$= \frac{1}{4}\left[w_{k,2j-1}^h + 2w_{k,2j}^h + w_{k,2j+1}^h \right]$$
$$= \frac{1}{4}\left[\sin\left(\frac{k(2j-1)\pi}{n}\right) + 2\sin\left(\frac{2kj\pi}{n}\right) + \sin\left(\frac{k(2j+1)\pi}{n}\right)\right]$$

→ Recall that $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$$= \frac{1}{4}\left[\sin\left(\frac{2kj\pi}{n}\right)\cos\left(\frac{\pi k}{n}\right) - \cos\left(\frac{2kj\pi}{n}\right)\sin\left(\frac{\pi k}{n}\right) + 2\sin\left(\frac{2kj\pi}{n}\right) + \sin\left(\frac{2kj\pi}{n}\right)\cos\left(\frac{\pi k}{n}\right) + \cos\left(\frac{2kj\pi}{n}\right)\sin\left(\frac{\pi k}{n}\right)\right]$$
$$= \frac{1}{4}\left[2\sin\left(\frac{2kj\pi}{n}\right)\cos\left(\frac{\pi k}{n}\right) + 2\sin\left(\frac{2kj\pi}{n}\right)\right]$$
$$= \frac{1}{2}\left[\sin\left(\frac{2kj\pi}{n}\right)\left[1+\cos\left(\frac{\pi k}{n}\right)\right]\right]$$

→ Note that $\sin\left(\frac{2kj\pi}{n}\right) = w_{k,j}^{2h}$

$$= \frac{1}{2}\left[1+\cos\left(\frac{\pi k}{n}\right)\right] w_{k,j}^{2h}$$

→ Recall that $\cos(2x) = 2\cos^2(x) - 1$

$$= \frac{1}{2}\left[1 + 2\cos^2\left(\frac{\pi k}{2n}\right) - 1\right] w_{k,j}^{2h}$$
$$= \cos^2\left(\frac{\pi k}{2n}\right) w_{k,j}^{2h} \quad \text{as desired.}$$

$I_h^{2h} w_{n-k}^h$

Given any j, we have:

$$(I_h^{2h} w_{n-k}^h)_j$$
$$= \frac{1}{4}\left[\sin\left(\frac{(n-k)(2j-1)\pi}{n}\right) + 2\sin\left(\frac{(n-k)(2j)\pi}{n}\right) + \sin\left(\frac{(n-k)(2j+1)\pi}{n}\right)\right]$$
$$= \frac{1}{4}\left[2\sin\left(\frac{(n-k)(2j)\pi}{n}\right)\cos\left(\frac{(n-k)\pi}{n}\right) + 2\sin\left(\frac{(n-k)(2j)\pi}{n}\right)\right]$$
$$= \frac{1}{2}\sin\left(\frac{(n-k)(2j)\pi}{n}\right)\left[1+\cos\left(\frac{(n-k)\pi}{n}\right)\right]$$
$$= \frac{1}{2}\sin\left(2\pi j - \frac{2\pi kj}{n}\right)\left[1+\cos\left(\pi - \frac{\pi k}{n}\right)\right]$$
$$= \frac{1}{2}\sin\left(-\frac{2\pi kj}{n}\right)\left[1-\cos\left(\frac{\pi k}{n}\right)\right]$$

Recall that $\cos(2x) = 1 - 2\sin^2(x)$

$$= -\frac{1}{2}\sin\left(\frac{2\pi kj}{n}\right)\left[1 - 1 + 2\sin^2\left(\frac{\pi k}{2n}\right)\right]$$
$$= -\sin^2\left(\frac{\pi k}{2n}\right) w_{k,j}^{2h} \quad \text{as desired.}$$

1.4. The number of iterations required when using Simple Weighted Jacobi is much higher than that of the multigrid method. The max norm error is also higher for simple weighted Jacobi than multigrid, by several orders of magnitude.

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ACM106b Set 3 Problem 2

2.1) $f(x) = -(a(x)u'(x))'$

$\rightarrow \int_0^1 f(x)\,dx = \int_0^1 -(a(x)u'(x))'\,dx$

$= -[a(x)u'(x)]_0^1$

$= a(0)u'(0) - a(1)u'(1)$

note that $u'(0) = u'(1)$

$= u'(0)[a(0) - a(1)]$

We note that $a(0) = a(1)$ since $a(x)$ is periodic on $[0,1]$

$= u'(0)[a(0) - a(0)] = 0$ as desired.

2) For any given solution $u_i(x)$:

$-(a(x)u_i'(x))' = f(x)$

$\rightarrow a(x)u_i'(x) = -\int f(x)\,dx$

$\rightarrow u_i'(x) = \frac{-1}{a(x)}\int f(x)\,dx$

$\rightarrow u_i(x) = \int \frac{-1}{a(x)}\int f(x)\,dx\,dx + C$

So $u_i(x) - u_j(x)$, $i \neq j$

$= c_1 - c_2 = c$ as desired.

3) Since $u^*(x) \in S$, there exists at least one value in the set of possible arguments for minimization, so $u^+(x)$ must exist. Consider the following equality:

$$\int_0^1 |u^*(x)|^2\,dx = \int_0^1 |u^*(x) + c|^2\,dx$$

The equality is satisfied for pairs of c values surrounding the optimal c (by nature of L^2 energy). At the minimum, only one c satisfies this, and deviating in either direction increases L^2 energy. So we have

$$u^+(x) = u^* + c_{optimal}$$