Jacob Snyder
Section 8

ACM116 Set 6

1. Matlab

2. a) Because N is the number of trials until a failure with a fixed probability of failure, we can calculate the probability of $N=n$ by

$$(1-p)^{n-1}(p)$$

Because we require that $n-1$ trials end in success (probability $(1-p)^{n-1}$) and then that the system fails on the next trial (probability p).

Therefore $\mathbb{P}(N=n) = (1-p)^{n-1}(p)$

b) By splitting the poisson process into a process for successes and failures we can model the failures of the system as a poisson process with rate λp.

So the waiting time for the first event for this process is $T \sim \mathcal{E}xp(\lambda p) \rightarrow P(T=t) = e^{-\lambda p t}(\lambda p)$

c) $$\mathbb{P}(N=n \mid T=t) = \frac{\mathbb{P}(T=t \mid N=n)\,\mathbb{P}(N=n)}{P(T=t)}$$

We know $\mathbb{P}(N=n) = (1-p)^{n-1}(p)$, $\mathbb{P}(T=t) = \lambda p e^{-\lambda p t}$

We also know that $[T \mid N=n] \sim \text{Gamma}(n,\lambda)$ because this is the waiting time for the n^{th} event. Therefore $\mathbb{P}(T=t \mid N=n) = \dfrac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$

$$\Rightarrow \mathbb{P}(N=n \mid T=t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}(1-p)^{n-1}(p)}{\lambda p(e^{-\lambda t p})(n-1)!} = \boxed{\frac{(1-p)^{n-1}\lambda^{n-1} t^{n-1} e^{-\lambda t(1-p)}}{(n-1)!}}$$

d) As discussed in part (c), $\mathbb{P}(T=t \mid N=n) = \dfrac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$

e) Our estimated value from Problem 1 is 2.0079.

$$\mathbb{E}[T \mid N=10] = \int_0^\infty t\, \mathbb{P}(T=t \mid N=10)\,dt = \int_0^\infty \frac{\lambda^n t^n e^{-t\lambda}}{(n-1)!}\,dt$$

$$= \int_0^\infty \frac{5^{10} t^{10} e^{-5t}}{9!}\,dt = \frac{5^{10}}{9!}\int_0^\infty t^{10} e^{-5t}\,dt = 2$$ which is very close to our estimated value of 2.0079.

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3. a) We know $B_{t_2} = B_{t_1} + X$, $X \sim \mathcal{N}(0, t_2 - t_1)$

$\rightarrow B_{t_1} + B_{t_2} = 2B_{t_1} + X$

And because $B_{t_1} \sim \mathcal{N}(0, t_1)$, and $B_{t_1} \perp\!\!\!\perp X$, and $2B_{t_1} \sim \mathcal{N}(0, 4t_1)$

$2B_{t_1} + X \sim \mathcal{N}(0, 3t_1 + t_2)$

Therefore $\mathbb{P}(B_{t_1} + B_{t_2} > a) = \mathbb{P}(2B_{t_1} + X > a) = \mathbb{P}(\underbrace{Z}_{\sim\mathcal{N}(0,1)} > \frac{a}{\sqrt{3t_1+t_2}}) = \boxed{1 - \Phi\left(\frac{a}{\sqrt{3t_1+t_2}}\right)}$

b) $\mathbb{E}[X_t] = \int_{-\infty}^{\infty} e^x \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{x^2}{2}} \, dx = \boxed{\sqrt{e}}$ by Wolfram Alpha

c) $\mathbb{V}[X_t] = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 = \int_{-\infty}^{\infty} e^{2x} \left(\frac{1}{\sqrt{2\pi}}\right) e^{-x^2/2} \, dx - e = \boxed{e^2 - e}$ by Wolfram Alpha

4. a) $\mathbb{P}(T_a < \infty) \overset{\text{from bottom of slide 78}}{=} 2\left(1 - \Phi\left(\frac{|a|}{\infty}\right)\right) = 2\left(\frac{1}{2}\right) = \boxed{1}$

b) We note that:

$$\mathbb{P}(T_a = t) = \mathbb{P}(B_t = a)(1 - \mathbb{P}(T_a < t))$$

$$= \frac{1}{\sqrt{2\pi t}} \exp\left[-a^2/2t\right]\left[1 - 2\left(1 - \Phi\left(\frac{|a|}{\sqrt{t}}\right)\right)\right]$$

$$= \frac{e^{[-a^2/2t]}}{\sqrt{2\pi t}}\left[1 - 2 + 2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{|a|/\sqrt{t}} e^{-t^2/2} \, dt\right]$$

$$= \frac{e^{[-a^2/2t]}}{\sqrt{2\pi t}}\left[\operatorname{erf}\left[\frac{|a|}{\sqrt{2t}}\right]\right]$$

So $\mathbb{E}[T_a] = \int_0^{\infty} \mathbb{P}(T_a = t)\, t \, dt = \int_0^{\infty} \frac{t e^{-a^2/2t}}{\sqrt{2\pi t}} \operatorname{erf}\left[\frac{|a|}{\sqrt{2t}}\right] dt$

This integral does not converge, so we conclude that $\boxed{\mathbb{E}[T_a] = \infty}$

5. Matlab

```
M = 1e5;
lambda = 5;
p = 0.1;
critfail = 10;

failtimes = 0;
validfailures = 0;
for a = 1:M
    timewaited = 0;
    failroll = 1;
    count = 0;

    while failroll > 0.1
        timewaited = timewaited + exprnd(1/lambda);
        failroll = rand();
        count = count + 1;
        if count > critfail
           break;
        end
    end

    if count == critfail
       validfailures = validfailures + 1;
       failtimes = failtimes + timewaited;
    end
end

avgfailtime = failtimes./validfailures;

disp('Average waiting time for failure given that the system fails on
 the 10th shock: ');
disp(avgfailtime);
```

```
Average waiting time for failure given that the system fails on the
 10th shock:
    2.0079
```

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```
N = 1000;

T = 10;
delT = 0.01;
n = T/delT;

exceedct = 0;

for a = 1:N
    Zs = normrnd(0,1,n,1);
    A = sqrt(delT).*tril(ones(n,n));
    walk = A*Zs;
    maxval = max(walk);
    if maxval >= 4
       exceedct = exceedct + 1;
    end
end

disp('Proportion Brownian motion exceeded 4:');
disp(exceedct/N);

disp('Theoretical value: ');
disp(2*(1 - normcdf(4/sqrt(10))));

disp('We can see that the estimated and theoretical values are quite
 close to each other.');
```

```
Proportion Brownian motion exceeded 4:
    0.1990

Theoretical value:
    0.2059

We can see that the estimated and theoretical values are quite close
 to each other.
```

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