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ACM 106a Set 4 Problem 1

1. 1) $u_{ij} \stackrel{?}{=} \tilde{a}_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}, \quad \forall i,j = 1,2,\dots,n$

We know $\tilde{A} = LU \rightarrow \tilde{a}_{ij} = \sum_{k=1}^{n} l_{ik}u_{kj}$

$$\rightarrow u_{ij} \stackrel{?}{=} \sum_{k=1}^{n} l_{ik}u_{kj} - \sum_{k=1}^{i-1} l_{ik}u_{kj}$$

$$\rightarrow u_{ij} = \sum_{k=i}^{n} l_{ik}u_{kj} = \underbrace{l_{ii}}_{\substack{=1 \text{ by} \\ \text{design}}} u_{ij} + \underbrace{l_{i(i+1)}u_{(i+1)j} + \dots + l_{in}u_{nj}}_{=0 \text{ since } L \text{ is lower-triangular so } l_{ab}=0 \text{ when } a<b}$$

$\rightarrow u_{ij} = 1(u_{ij}) + 0 = u_{ij}$ as desired. ■

2)
$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 1 \\ -1 & -1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \\ -1 & -1 & -1 & \cdots & 1 \end{bmatrix}$$

Apply LU factorization step-by-step (referencing Algorithm 3 in the notes):
After one iteration ($k=1$) we have:

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & & 1 \\ -1 & 1 & 0 & \cdots & & 2 \\ -1 & -1 & 1 & \cdots & & 2 \\ -1 & -1 & -1 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & & 1 & 2 \\ -1 & -1 & -1 & \cdots & -1 & 2 \end{bmatrix} \quad \text{After } (k=2): \quad A^{(2)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & & 1 \\ -1 & 1 & 0 & \cdots & & 2 \\ -1 & -1 & 1 & \cdots & & 4 \\ -1 & -1 & -1 & 1 & \cdots & 4 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 4 \end{bmatrix}$$

We can see the pattern. We have:

$$A^{(n-1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & & 1 \\ -1 & 1 & 0 & \cdots & & 2 \\ -1 & -1 & 1 & \cdots & & 4 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & 1 & 2^{n-2} \\ -1 & -1 & -1 & & -1 & 2^{n-1} \end{bmatrix}$$

So we have $\max_{i,j} |u_{ij}| = 2^{n-1}$
and $\max_{i,j} |a_{ij}| = 1$
Therefore $\rho = \frac{2^{n-1}}{1} = 2^{n-1}$ as desired.

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ACM106a Set 4 Problem 2.2

2.2) We can see that the growth factor appears to grow roughly linearly with n.