1.a) $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2$ by definition

$= \frac{1}{N}\sum_{i=1}^{N}[x_i^2 - 2x_i\mu + \mu^2]$

$= \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \frac{2\mu}{N}\sum_{i=1}^{N} x_i + \mu^2$

$= \mu - 2\mu^2 + \mu^2 = \mu - \mu^2 = \boxed{\mu(1-\mu)}$

b) $s^2 = \hat{\sigma}_n^2 \frac{Nn-n}{Nn-N}$ (from notes)

from part (a), we have that $\sigma^2 = \mu(1-\mu) \rightarrow \hat{\sigma}_n^2 = \bar{X}_n(1-\bar{X}_n)$

$\Rightarrow \boxed{\hat{s} = \bar{X}_n(1-\bar{X}_n)\frac{n(N-1)}{N(n-1)}}$

c) Under the assumption that $\bar{X}_n$ is approximately normally distributed:

$\bar{X}_n = \frac{70}{90} = 0.778$

margin of error $= z_{0.95}\frac{s}{\sqrt{90}}\sqrt{1-\frac{90-1}{300-1}}$ where $s = \left[\left[1-\frac{1}{300}\right]\frac{1}{90-1}\sum_{n=1}^{90}(X_i-\bar{X}_n)^2\right]^{1/2}$

$\rightarrow s = \left[\left[1-\frac{1}{300}\right]\left[\frac{1}{89}\right]\left[\sum_{i=1}^{70}(1-0.778)^2 + \sum_{i=1}^{20}(0-0.778)^2\right]\right]^{1/2} = 0.417$

$\rightarrow$ margin of error $= z_{0.95}\frac{0.417}{\sqrt{90}}\sqrt{1-\frac{89}{299}} = (0.0368)1.95996 = 0.0721$

Therefore we are 95% confident that the proportion of students in the class who like statistics falls between 0.706 and 0.850

2. margin of error $= 0.02 = z_{0.95}\frac{\sigma}{\sqrt{n}}\sqrt{1-\frac{n-1}{N-1}}$

Assuming the population is large enough, $\sqrt{1-\frac{n-1}{N-1}} \approx 1$ (ignore finite population correction)

The worst case scenario for σ is if $\mu = 0.5 \rightarrow \sigma^2 = 0.25 \rightarrow \sigma = 0.5$

$\rightarrow 0.02 = z_{0.95}\frac{0.5}{\sqrt{n}} \rightarrow n = \left[(1.95996)\frac{0.5}{0.02}\right]^2 = 2400.9$

$\Rightarrow \boxed{n \geq 2401}$

4. a) By definition of the uniform distribution, $\mu = \theta/2 \rightarrow \theta = 2\mu$

Therefore $\theta = \mathbb{E}[\hat{\theta}] = 2\mu$. $\boxed{\text{No bias}}$

By definition, $\hat{\sigma} = \frac{\hat{\theta}}{\sqrt{12}} \rightarrow se[\bar{X}_n] = \frac{\theta}{\sqrt{12n}} \rightarrow se[\hat{\theta}] = 2\frac{\theta}{\sqrt{12n}} = \boxed{\frac{\theta}{\sqrt{3n}}}$

$$MSE = 0^2 + \frac{\theta^2}{3n} = \boxed{\frac{\theta^2}{3n}}$$

b) $\mu = \theta/2 \rightarrow \theta = 2\mu \rightarrow \text{Bias} = 2\mu - \mathbb{E}[\hat{\theta}]$

$\mathbb{E}[\hat{\theta}] = \int_0^\theta \frac{x}{\theta}(n)\left(\frac{x}{\theta}\right)^{n-1} dx = \frac{\theta n}{n+1} \rightarrow \text{bias} = \theta - \frac{\theta n}{n+1} = \boxed{\theta\left(1 - \frac{n}{n+1}\right)}$

By definition of the uniform distribution, we have

$$\mathbb{E}[\hat{\theta}^2] = \int_0^\theta \frac{x^2}{\theta}(n)\left(\frac{x}{\theta}\right)^{n-1} dx = \frac{\theta^2 n}{n+2}$$

$$\mathbb{V}[\hat{\theta}] = \mathbb{E}[\hat{\theta}^2] - \mathbb{E}[\hat{\theta}]^2 = \frac{\theta^2 n}{n+2} - \frac{\theta^2 n^2}{(n+1)^2}$$

$$\rightarrow se[\hat{\theta}] = \sqrt{\mathbb{V}(\hat{\theta})} = \boxed{\sqrt{\frac{\theta^2 n}{n+2} - \frac{\theta^2 n^2}{(n+1)^2}}}$$

$$MSE = \theta^2\left(1 - \frac{n}{n+1}\right)^2 + \frac{\theta^2 n}{n+2} - \frac{\theta^2 n^2}{(n+1)^2}$$

c) We see that the MSE of our $\hat{\theta}$ in (b) has n^2 terms in the denominator, whereas $\hat{\theta}$ in (a) has only n. Therefore, for large n we can conclude that our estimate used in part (b) is more efficient.