Some Application of Variational Iteration Method for Solving Differential Equations

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SOME APPLICATIONS OF THE VARIATIONAL ITERATION METHOD IN SOLVING DIFFERENTIAL EQUATIONS

By

Salah A.A. Abu As'd

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The undersigned hereby certify that they have read and recommend to the Faculty of for acceptance a thesis entitled "**SOME APPLICATIONS OF THE VARIATIONAL ITERATION METHOD IN SOLVING DIFFERENTIAL EQUATIONS**" by **Salah A.A. Abu As'd** in partial fulfillment of the requirements for the degree of .

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Signature of Author

DEDICATION

To My Wife Asma ,

My Daughter Leen .

Salah Abu As'd

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Abstract in English

SOME APPLICATIONS OF THE VARIATIONAL ITERATION METHOD IN SOLVING DIFFERENTIAL EQUATIONS

Salah A.A. Abu As'd

Mutah University, 2005

In this thesis, we implement a new analytical technique, the He's variational iteration method for solving two types of differential equations:

*A nonlinear ordinary boundary value problem.

*partial differential equation: (Fisher's equation, the general Fisher's equation, a nonlinear diffusion equation of the Fisher type, parabolic partial differential equation with boundary and initial conditions).

The numerical and analytical solutions of the above equations have been obtained in terms of a convergent power series with easily computable components.

Numerical examples are given and comparisons are made with the Adomian decomposition method, The comparison shows that the variational iteration method is more effective and overcome the difficulty arising in calculating Adomian polynomials.

Abstract in Arabic

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Chapter 1

INTRODUCTION

1.1 Statement of the Problem

The variational iteration method is proposed to solve the generalized normalized diode equation, by suitable choice of the initial trial-function, one-step iteration leads to an high accurate solution, which is valid for the whole solution domain (He, 2004).

Drăgănescu and Căpălnăsan (Drăgănescu, 2003) applied the variational iteration method to non-linear anelastic model describing the acceleration of the relaxation process in the presence of the vibrations.

The combination of a perturbation method, variational iteration method, method of variation of constants and averaging method to establish an approximate solution of one degree of freedom weakly nonlinear systems was proposed in (Mariinca, 2002).

The application of the variational iteration method to nonlinear fractional differential equations can be found in details in (He, 1998). Moreover, the method was successfully applied to delay differential equations in (He, 1997), to autonomous ordinary differential systems(He, 2000), and other fields (He, 1998), (He, 1999).

In this method, general Lagrange multipliers are introduced to construct correction functionals for the problems. The multipliers in the functionals can be identified optimally via the variational theory. The initial approximations can be freely chosen with possible unknown constants, which can be determined by imposing the boundary/initial conditions.

In this thesis we use the variational iteration method for solving nonlinear boundary value problems of the type

$$-u'' = \beta F(u),$$

with the boundary conditions

$$u(0) = \alpha, u(1) = \gamma,$$

where $\beta > 0$ and the nonlinear function $F(u)$ is assumed to have a power series.

In chapter four, we extend the application of the variational iteration method to construct a numerical solution of Fisher's equation, the general Fisher's equation, nonlinear diffusion equation of the Fisher type, and parabolic partial differential equation with boundary and initial conditions.

1.2 aims and objectives of the Thesis

The purpose of this thesis is manifold. Firstly, we introduce a new kind of analytical method of nonlinear problem called the variationl iteration method. Second, we consider procedures for solving a nonlinear boundary value problems. Third, we consider procedures to approximate the numerical and analytical solutions of the Fisher's equation, the general Fisher's equation, a nonlinear diffusion equation of the Fisher type, and to find the exact solution of parabolic partial differential equation with boundary and initial conditions. fourth, we compare our results with Adomian decomposition method.

1.3 Motivation

Differential equations (He, 2000) which arise in real-world physical problems area, as Driver (Driver, 1977) says, often too complicated to solve exactly, and even if an exact solution is obtainable, the required calculation may be too complicated to be practical.

It is very difficult to solve a nonlinear problem(He, 1998), either numerically or theoretically, and even more difficult to establish a real model for the non linear problem. Much assumption has to be made artificially or unnecessarily to make the practical engineering problems solvable, leading to loss of most important information.

The variational iteration method can be implement directly in a straightforward manner without using restrictive assumption or linearization of the problem.

1.4 Synopsis of the Thesis

We introduce the variational iteration method and give the basic idea of calculating the Lagrange multipliers in chapter two.

In chapter three, we implement the variational iteration method to solve a non-linear boundary value problem.

In chapter four, the He's variational iteration method is proposed to approximate the numerical and analytical solutions of Fisher's equation, general Fisher's equation, a nonlinear diffusion equation of the Fisher type, and to find the exact solution of parabolic partial differential equation with boundary and initial conditions.

Finally. Chapter five, contains discussions and conclusion, summarized of the work.

Chapter 2

THE VARIARTIONAL ITERATION METHOD

This chapter is divided into two sections. In section 1, we introduce the preliminaries and review of earlier studies, which contains two subsections, the basic idea of general Lagrange multiplier and the basic idea of the variational iteration method. In section 2, we give a historical background of the applications of variationl iteration method.

2.1 Preliminaries and Review of Earlier Studies

2.1.1 General Lagrange Multiplier

To give a clear overview of the Lagrange multipliers, consider the following two examples (Inokuti, 1978).

Example 1.

Suppose we try to find the root x_0 of an (algebraic) equation

$$f(x) = 0,$$

where $f(x)$ is a function of real variable x having a continuous nonvanishing derivative in a neighborhood of x_0. We write down an expression for an estimate

$$(x_0)_{est} = x - \lambda f(x), \tag{2.1.1}$$

where λ is to be determined later. Suppose one knows an approximate root x, which differs from x_0 by small amount, i.e., $x = x_0 + \delta x$. Then $f(x) \neq 0$ by definition, and the second term $-\lambda f(x)$ in Eq.(2.1.1) represents a correction to x so that $(x_0)_{est}$ is closer to x_0.

We now want to choose λ so that this correction is optimal. To find a criterion for this purpose, we insert $x = x_0 + \delta x$ into Eq.(2.1.1) and obtain

$$\begin{aligned} (x_0)_{est} &= x_0 + \delta x - \lambda[f(x_0) + \delta x f'(x_0)] + O(\delta x^2), \qquad (2.1.2) \\ &= x_0 + [1 - \lambda f'(x_0)]\delta x + O(\delta x^2). \end{aligned}$$

The choice $\lambda = [f'(x_0)]^{-1}$ is the optimal in the sense that it makes $(x_0)_{est}$ differ from x_0 only by $O(\delta x^2)$.

The truly optimal λ is unfortunately difficult to calculate because it involves the exact root x_0. But this is no real problem. The quantity λ is a multiplier to $f(x)$, which is a small quantity of $O(\delta x)$ by assumption, and therefore we only need an approximate λ. Thus we can replace λ in Eq.(2.1.1)by $[f'(x_0)]^{-1}$ without causing an error not exceeding $O(\delta x^2)$.

Consequently, we arrive at the variational estimate

$$(x_0)_{est} = x - [f'(x_0)]^{-1} f(x). \tag{2.1.3}$$

Example 2.

Suppose we try to compute $y(1)$, i.e., the value of the function $y(x)$ defined by

$$y' + y^2 = 0, y(0) = 1. \tag{2.1.4}$$

The exact solution is $y_0 = (1 + x)^{-1}$ and therefore $y_0(1) = 1/2$. Pretending that we do not know this exact solution, we write as a variational estimate

$$(y(1))_{est} = y(1) - \int_0^1 \lambda(x)[y'(x) + y^2(x)]dx. \tag{2.1.5}$$

Now the second term is an integral. In other words, we regard the differential equation Eq.(2.1.4) as a condition on $y(x)$ to be satisfied at every point in the interval $0 \leq x < 1$. The Lagrange multiplier $\lambda(x)$ is a function rather than a number .

If we insert a trial function $y(x) = y_0(x) + \delta y(x)$ with $y(0) = 1$ so that $\delta y(0) = 0$, then we have

$$(y(1))_{est} = y_0(1)+\delta y(1)-\int_0^1 \lambda(x)[\delta y(x)]'dx-2\int_0^1 \lambda(x)y(x)\delta y(x)dx+O(\delta y^2). \tag{2.1.6}$$

Now we apply the partial integration to the first integral term, note the boundary term at $x = 1$, and obtain

$$(y(1))_{est} = y_0(1)+[1-\lambda(1)]\delta y(1)+\int_0^1 [\lambda'(x)-2\lambda(x)y(x)]\delta y(x)dx+O(\delta y^2). \tag{2.1.7}$$

This we see that the optimal $\lambda(x)$ is given by the condition that

$$\lambda'(x) = 2\lambda(x)y(x), \lambda(1) = 1, \tag{2.1.8}$$

or, alternatively

$$\lambda = \exp[2\int_1^x y(t)dt]. \tag{2.1.9}$$

Thus we arrive at the variatioal estimate

$$(y(1))_{est} = y(1) - \int_0^1 \exp[2\int_1^x y(t)dt][y'(x) + y^2(x)]dx. \tag{2.1.10}$$

Any trial function $y(x)$ with $y(0) = 1$ should give $(y(1))_{est}$ which is superior to $y(1)$ and is the best within the flexibility of the trial function. For instance, we may use

$y(x) = 1$ for $0 \leq x \leq 1$,and then obtain $(y(1))_{est} = \frac{1}{2}(1 + \exp[-2]) = \frac{1}{2} \times 1.135$. The 13.5% accuracy is remarkably good in view of the crudeness of this trial function.

One can obtain a much better result by using a trial function involving a parameter c, e.g., $y(x) = 1 - cx$, and by choosing c so that the value $(y(1))_{est}$ is stationary with respect to small changes in c.

2.1.2 Basic Idea of Variationl Iteration Method

In 1978, Inokuti et al. proposed a general Lagrange multiplier method to solve non-linear problems, which was first proposed to solve problems in quantum mechanics (see Ref.(Inokuti, 1978) and the references cited therein).

The main feature of the method is as follows: the solution of a mathematical problem with linearization assumption is used as initial approximation or trial-function, then a more highly precise approximation at some special point can be obtained.

Considering the following general non-linear system

$$Lu(t) + Nu(t) = g(t), \tag{2.1.11}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a known analytical function. Ji-Huan He has modified the above method into an iteration method [(He, 1998), (He, 1998), (He, 1999), (He, 2000), (He, 2004), (He, 1997)] in the following way

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\Big(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\Big)d\xi, \tag{2.1.12}$$

where λ is a general Lagrange multiplier(Inokuti,1978), which can be identified optimally via the variational theory [(Inkouti, 1978), (He, 1997), (He, 2001), (He, 2003), (He, 2003), (He, 2004), (Liu, 2004), (Liu, 2004), (Hao, 2005)], the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation [(He, 1998), (He, 1998), (He, 1999), (He, 2000), (He, 2004), (He, 2003)], i.e. $\delta\tilde{u}_n = 0$.

Equation (2.1.12) is called a correction functional.

To illustrate the basic idea of variational iteration method consider the following two examples:

Example 1.

(First Order Non-homogeneous Linear Differential Equation(He, 1997))

$$y' + y = \sin t + t, y(0) = A, \tag{2.1.13}$$

its correction functional can be written down as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(y_n'(\tau) + y_n(\tau) - \sin\tau - \tau)d\tau. \tag{2.1.14}$$

Making the above correction functional stationary, i.e.

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_0^t \lambda(y_n'(\tau) + y_n(\tau) - \sin\tau - \tau)d\tau, \tag{2.1.15}$$

$$\delta y_n(t) + \lambda(\tau)\delta y_n(\tau)|_{\tau=t} + \int_0^t (-\lambda' + \lambda)\delta y_n d\tau = 0$$

yields following stationary conditions:

$$-\lambda'(\tau)|_{\tau=t} + \lambda(\tau)|_{\tau=t} = 0, \tag{2.1.16}$$

$$1 + \lambda(\tau)|_{\tau=t} = 0. \tag{2.1.17}$$

Herein, the prime denotes the derivative with respect to τ. The general solution of (2.1.16) is $\lambda(\tau) = Ce^{\tau}$ with a suitable constant C, which can be determined by imposing the' boundary condition' (2.1.17): $C = -e^{-t}$, so the following iteration formula can be obtained

$$y_{n+1}(\tau) = y_n(\tau) - \int_0^t e^{(\tau-t)}(y_n'(\tau) + y_n(\tau) - \sin\tau - \tau)d\tau. \tag{2.1.18}$$

We begin with $y_0(t) = y(0) = A$, by the above iteration formula (2.1.18), we have

$$y_1(t) = A - \int_0^t e^{(\tau-t)}(A - \sin\tau - \tau)d\tau),$$

$$= A - Ae^{(\tau-t)}|_{\tau=0}^{\tau=T} + \frac{1}{2}(e^{(\tau-t)}\sin\tau - e^{(\tau-t)}\cos\tau)|_{\tau=0}^{\tau=T} + e^{(\tau-t)}(\tau-1)|_{\tau=0}^{\tau=T}$$

$$= (A + \frac{3}{2})e^{-t} + \frac{1}{2}(sint - cost) + t - 1$$

which is the exact solution .

Example 2.

(Second Order Non-homogeneous Linear Differential Equation(He, 1998))

$$y'' + \omega^2 y = f(t), with f(t) = A \sin \omega t + B \sin t. \tag{2.1.19}$$

Its correction functional can be written down as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda \big(y_n''(\tau) + \omega^2 y_n(\tau) - f(\tau)\big) d\tau. \tag{2.1.20}$$

Making the above correction functional stationary, and noticing that $\delta y(0) = 0$,

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_0^t \lambda \big(y_n''(\tau) + \omega^2 y_n(\tau) - f(\tau)\big) d\tau. \tag{2.1.21}$$

$$= \delta y_n(t) + \lambda(\tau)\delta y_n'(\tau)|_{\tau=t} - \lambda'(\tau)\delta y_n(\tau)|_{\tau=t} + \int_0^t (\lambda'' + \omega^2 \lambda)\delta y_n d\tau = 0$$

.

Yields the following stationary conditions:

$$\delta y_n : \lambda''(\tau)|_{\tau=t} + \omega^2 \lambda(\tau)|_{\tau=t} = 0,$$

$$\delta y_n' : \lambda(\tau)|_{\tau=t} = 0,$$

$$\delta y_n : 1 - \lambda'(\tau)|_{\tau=t} = 0.$$

The Lagrange multiplier, therefore, can be readily identified,

$$\lambda = \frac{1}{\omega}\sin\omega(\tau - t) \tag{2.1.22}$$

as a result, we obtain the following iteration formula

$$y_{n+1}(t) = y_n(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t) \times \big(y_n''(\tau) + \omega^2 y_n(\tau) - f(\tau)\big)d\tau. \tag{2.1.23}$$

If we use its complementary solution $y_0 = C_1\cos\omega t + C_2\sin\omega t$ as initial approximation, by the iteration formula (2.1.23), we get

$$y_1(t) = y_0(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t) \times (-A\sin\omega\tau - B\sin\tau)d\tau$$

$$= C_1\cos\omega t + C_2\sin\omega t - \frac{A}{2\omega}t\cos\omega t + \frac{B}{\omega^2 - 1}(\sin t + \sin\omega t), \tag{2.1.24}$$

which is the general solution Eq.(2.1.19).

However, if we apply restricted variations to the correction function(2.1.20), then its exact solution can be arrived at only by successive iterations. Considering homogeneous Eq. (2.1.19), i.e. f(x)=0, we re-write the correction functional of Eq.(2.1.20) as follows :

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda\big(y_n''(\tau) + \omega^2\tilde{y}_n(\tau)\big)d\tau, \tag{2.1.25}$$

herein $\tilde{y}_n$ is considered as restricted variation, under this condition, the stationary conditions of the above correction functional (2.1.25) can be expressed as follows: (noticing that $\delta\tilde{y}_n = 0$)

$$\lambda''(\tau)|_{\tau=t} = 0,$$

$$\lambda(\tau)|_{\tau=t} = 0,$$

$$1 - \lambda'(\tau)|_{\tau=t} = 0.$$

The Lagrange multiplier, therefore, can be easily identified as follows:

$$\lambda = \tau - t, \tag{2.1.26}$$

leading to the following iteration formula

$$y_{n+1}(t) = y_n(t) + \int_0^t (\tau - t)\big(y_n''(\tau) + \omega^2 y_n(\tau)\big)d\tau. \tag{2.1.27}$$

If for example, the initial conditions are $y(0) = 1$ and $y'(0) = 0$, we begin with $y_0 = y(0) = 1$, by the above iteration formula (2.1.27) we have the following approximate solutions:

$$y_1(t) = 1 + \omega^2 \int_0^t (\tau - t)dt = 1 - \frac{1}{2!}\omega^2 t^2, \tag{2.1.28}$$

$$\begin{aligned} y_2(t) &= 1 - \frac{1}{2!}\omega^2 t^2 + \int_0^t (\tau - t)\big(-\omega^2 + \omega^2 - \frac{1}{2!}\omega^4\tau^4\big)d\tau \\ &= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4, \end{aligned} \tag{2.1.29}$$

$$y_n(t) = 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 + ... + (-1)^n \frac{1}{(2n)!}\omega^{2n} t^{2n}. \tag{2.1.30}$$

Thus we have

$$\lim_{n\to\infty} y_n(t) = \cos\omega t, \tag{2.1.31}$$

which is the exact solution.

From the above solution process, we can see clearly that the approximate solutions converge to its exact solution relatively slowly due to the approximate identification of the multiplier.

It should be specially pointed out that the more accurate the identification of the multiplier, the faster the approximation converge to its exact solution.

The approximately identified multiplier (2.1.26) is actually the first-order approximation of its exact one (2.1.22), to get a closer approximation than Eq. (2.1.26), we expand Eq.(2.1.22) as

$$\lambda = \frac{1}{\omega}\sin\omega(\tau - t) \approx \tau - t - \frac{1}{3!}\omega^2(\tau - t)^3. \tag{2.1.32}$$

Substitution of Eq.(2.1.32) in Eq.(2.1.25) leads to the following iteration formula:

$$y_{n+1}(t) = y_n(t) + \int_0^t \left(\tau - t - \frac{1}{3!}\omega^2(\tau - t)^3\right) \times \left(y_n''(\tau) + \omega^2 y_n(\tau)\right) d\tau. \tag{2.1.33}$$

We also begin with $y_0 = 1$, by the same manipulation, we have

$$\begin{aligned} y_1(t) &= 1 + \int_0^t \left(\tau - t - \frac{1}{3!}\omega^2(\tau - t)^3\right)\omega^2 d\tau \\ &= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4, \end{aligned} \tag{2.1.34}$$

$$\begin{aligned} y_2(t) &= y_1 + \int_0^t \left(\tau - t - \frac{1}{3}\omega^2(\tau - t)^3\right) \times \left(\frac{1}{4!}\omega^6 t^4\right) d\tau \\ &= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 - \frac{1}{6!}\omega^6 t^6 + \frac{1}{8!}\omega^8 t^8. \end{aligned} \tag{2.1.35}$$

So, it can be seen clearly that the approximations obtained from Eq. (2.1.33) converge to its exact solution faster than those obtained from the iteration formula (2.1.25).

2.2 Historical Background of the Variationl Iteration Method

In this section we review some applications of the variationl iteration method.

Variationl Iteration Method for Delay Differential Equations(He, 1997).

In this paper The variationl iteration method is shown to be applicable to delay differential equations for analytical approximate solutions. And used the method to solve the following population growth model

$$x'(t) + c\theta(t-1)x(t) + c\theta(t-1) = 0.x(0) = \theta(0). -1 \leq t \leq 0. \tag{2.2.1}$$

Then the approximation will converge to its exact solution

$$x(t) = (\theta(0)+1)\exp(-\int_0^t c\theta(\xi-1)d\xi) - 1.$$

Approximate analytical solution for seepage flow with fractional derivatives in porous media (He, 1998) .

In this paper, the author proposes variationl iteration method which is very effective and convenient method to solve (fractional) differential equations, and presents a generalized Darcy law with fractional derivatives and a more general fractional differential equation for seepage flow without the assumption of continuity.

Approximate Analytical Solution of Blasius' Equation (He, 1998).

The Blasius'Equation

$$f''' + ff''/2 = 0, \tag{2.2.2}$$

with boundary conditions

$$f(0) = f'(0) = 0, f'(+\infty) = 1, \tag{2.2.3}$$

is studied in this paper. An approximate analytical solution is obtained via the variationl iteration method. The comparison with Howarth's numerical solution reveals that the proposed method is of high accuracy.

Approximate Solution of nonlinear differential equations with convolution product nonlinearities(He, 1998).

In this paper variationl iteration method is proposed to solve nonlinear problems. Consider the equation,

$$Ly(t) + Ny(t) = g(t). \tag{2.2.4}$$

Where L is a linear operator, N is a nonlinear operator and g(x) is a known analytical function. Consider non linear differential equations with convolution product nonlinearity, i.e. Ny in the non linear equation(2.2.4) can be expressed as

$$Ny = y * y = \int_0^t y(t-\tau)y(\tau)d\tau. \tag{2.2.5}$$

The results reveal the approximations obtained by the proposed method are uniformly valid for both small and large parameters in nonlinear problems.

Variationl iteration method - a kind of non-linear analytical technique: some examples.(He, 1999).

In this paper, variationl iteration method is described and used to give approximate solutions for some well-known non-linear problems, as: Duffing equation, mathematical pendulum, vibrations of the eardrum, and finally partial differential equation.

Comparison with Adomian's decomposition method reveals that the approximations obtained by the proposed method converge to its exact solution faster than those of Adomian's.

Variationl iteration method for autonomous ordinary differential systems(He, 2000).

In this paper, the author proposes variationl iteration method to autonomous ordinary differential systems. He consider the following population growth model:

$$x_1'(t) = -x_1(t) - 8x_2(t), x_2'(t) = 8x_1(t) - x_2(t), \tag{2.2.6}$$

with

$$X_1(0) = 1$$

$$X_2(0) = 0$$

.

Then the approximation will converge to its exact solutions:

$$X_1(t) = e^{-t}\cos 8t,$$

$$X_2(t) = e^{-t}\sin 8t.$$

Variationl iteration method for solving Burger's and coupled Burger's equations(Abdou, 2004).

In this paper variationl iteration method proposed to solve one-dimensional Burger's equation of the form

$$u_t + uu_x - vu_{xx} = 0, \tag{2.2.7}$$

with an initial condition

$$u(x,0) = \frac{\alpha + \beta + (\beta - \alpha)\exp(\gamma)}{1 + \exp(\gamma)}, t \geq 0, \tag{2.2.8}$$

where $\gamma = (\alpha/v)(x - \lambda)$ and the parameters α, β, λ, and v are arbitrary constants.

Also the method is proposed to solve the homogeneous coupled Burger's equations of the form

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0, \tag{2.2.9}$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0, \tag{2.2.10}$$

subject to the initial conditions

$$u(x,0) = \sin x, v(x,0) = \sin x. \tag{2.2.11}$$

The solution obtained by the variationl iteration method is an infinite power series for appropriate initial condition, which can, in turn, be expressed in a closed form, the exact solution.

Application of He's variational iteration method to Helmholtz equation(Momani and Abu As'd, 2005)(in press).

In this article, He's variational iteration method proposed to find the exact solution of the Helmholtz equation of the form:

$$\nabla^2 u + f(x,y)u = g(x,y), \tag{2.2.12}$$

with the boundary and initial conditions:

$$u(0,y) = \psi_1(y), \quad u_x(0,y) = \psi_2(y), \tag{2.2.13}$$

$$u(x,0) = \psi_3(x), \quad u_y(x,0) = \psi_4(x), \tag{2.2.14}$$

where $\psi_1(y)$, $\psi_2(y)$, $\psi_3(x)$, and $\psi_4(x)$ are given functions. And the method proposed to solve two special case of Helmholtz equation:

CASE(I):

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} - u(x,y) = 0, \tag{2.2.15}$$

with the initial conditions

$$u(0,y) = y, \quad u_x(0,y) = y + \cosh(y). \tag{2.2.16}$$

CASE(II):
$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + 5u(x,y) = 0, \tag{2.2.17}$$

with the initial conditions

$$u(0,y) = 0, \quad u_x(0,y) = 3\sinh(2y). \tag{2.2.18}$$

Chapter 3

VARIATIONAL ITERATION METHOD FOR SOLVING BOUNDARY VALUE PROBLEMS

3.1 Linear Boundary Value Problem

In this chapter, we implement a new analytical technique, the He's variational iteration method to approximate the numerical and analytical solutions for solving boundary value problems. Numerical examples are given and comparisons are made with the Adomian decomposition method.

The variational iteration method proposed by Ji-Huan He has been shown to solve effectively, easily, and accurately a large class of non-linear problems with approximations converging rapidly to accurate solutions. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

Consider this second order linear boundary value problem :(heat condition problem(He, 2000))

Example 1.

$$\frac{d^2T}{dx^2} + T + x = 0.0 < x < 1 \tag{3.1.1}$$

with boundary conditions

$$T(0) = T(1) = 0, \tag{3.1.2}$$

where T is temperature.

Its correction functional can be written down as follows:

$$T_{n+1}(x) = T_n(x) + \int_0^x \lambda(T_n''(s) + T_n(s) + s)ds. \tag{3.1.3}$$

Taking variation with respect to the independent variable T_n, noticing that $\delta T_n(0) = 0$,

$$\delta T_{n+1}(x) = \delta T_n(x) + \delta \int_0^x \lambda(T_n''(s) + T_n(s) + s)ds \tag{3.1.4}$$

$$= \delta T_n(x) + \lambda \delta T_n'(s)|_{s=x} - \lambda' \delta T_n(s)|_{s=x} + \int_0^x (\lambda'' + \lambda)\delta T_n ds = 0,$$

for all variations δT_n and $\delta T_n'$, implying following stationary conditions:

$$\delta T_n : \lambda''(s)|_{s=x} + \lambda(s)|_{s=x} = 0,$$

$$\delta T_n' : \lambda(s)|_{s=x} = 0,$$

$$\delta T_n : 1 - \lambda'(s)|_{s=x} = 0.$$

The general Lagrange multiplier, therefore, can be readily identified

$$\lambda = \sin(s - x),$$

as a result, we obtain following iteration formula

$$T_{n+1}(x) = T_n(x) + \int_0^x \sin(s - x)(T_n''(s) + T_n(s) + s)ds.$$

If we use its complementary solution $T_0 = A\cos x + B\sin x$ as initial approximation, by the iteration formula, we get

$$T_1 = A\cos x + B\sin x + \int_0^x s\sin(s-x)ds$$

$$= A\cos x + B\sin x - x - \sin x.$$

By imposing the boundary conditions (3.1.2) yields $A = 1$ and $B = \frac{1}{\sin 1} - 1$, as a result, we have $T_1(x) = \frac{\sin x}{\sin 1} - x$ which is an exact solution .

3.2 Applying the Variational Iteration Method to Boundary Value Problems

In this section we consider boundary value problems of the form

$$-u'' = \beta F(u), \tag{3.2.1}$$

subject to the boundary conditions

$$u(0) = \alpha, u(1) = \gamma,$$

where $\beta > 0$ and the nonlinear function F(u) is assumed to have a power series representation.

To solve (3.2.1) by means of the variational iteration method, we construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\Big(u_n''(\xi) + \beta F(\tilde{u}_n(\xi))\Big)d\xi, \tag{3.2.2}$$

where λ is a general Lagrange multiplier, u_0 is an initial approximation or trial-function with possible unknowns, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$

Making the correction functional (3.2.2) stationary, noticing that $\delta\tilde{u}_n = 0$,

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda\Big(u_n''(\xi) + \beta F(\tilde{u}_n(\xi))\Big)d\xi, \qquad (3.2.3)$$

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda\Big(u_n''(\xi))d\xi, \qquad (3.2.4)$$

yields the following stationary conditions:

$$\delta u_n(\xi) : 1 - \lambda'(\xi) = 0,$$

$$\delta u_n'(\xi) : \lambda(\xi) = 0,$$

$$\delta u_n(\xi) : \lambda''(\xi) = 0.$$

The Lagrange multiplier, therefore, can be readily identified as

$$\lambda(\xi) = \xi - x,$$

as a result, we obtain the following iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\Big(u_n''(\xi) + \beta F(u_n(\xi))\Big)d\xi \qquad (3.2.5)$$

To illustrate the scheme (Deeba, 2000) let the nonlinear operator $F(u_n)$ be a nonlinear function of u_n, say $g(u_n)$. Assume that the Taylor expansion of $g(u_n)$ around u_0 is

$$g(u) = g(u_0) + g^{(1)}(u_0)(u - u_0) + \frac{1}{2!}g^{(2)}(u_0)(u - u_0)^2 + \cdots$$

We begin with Boundary conditions given by Eq. (3.2.1), and by the above iteration formula we can obtain the result.

3.3 Numerical Application

Example 1. *Troesch's problem*

In this example we apply the algorithm described in the previous section to approximate the nonlinear boundary value problem, Troesch's problem.

$$u''(x) = \beta \ \sinh(\beta \ u(x)), 0 \leq x \leq 1, \tag{3.3.1}$$

with the boundary conditions

$$u(0) = 0, u(1) = 1.$$

Troesch's problem (Deeba, 2000) was described and solved by Weibel (Weibel, 1958). It arises from a system of nonlinear ordinary differential equations which occur in an investigation of the confinement of a plasma column by radiation pressure. The problem has been studied extensively.

Troesch found its numerical solution by the shooting method (see (Troesch, 1976)). The closed form solution to this problem in terms of the Jacobian elliptic function has been given in (Robets, 1976) as

$$u(x) = \frac{2}{\lambda}\sinh^{-1}\Big(\frac{\dot{u}(0)}{2}sc(\lambda x|1-\frac{1}{4}\dot{u}^2(0))\Big), \tag{3.3.2}$$

where $\dot{u}(0)$ the derivative of u at 0, is given by the expression $\dot{u}(0) = 2(1-m)^{1/2}$, with m being the solution of the transcendental equation

$$\frac{\sinh(\lambda/2)}{(1-m)^{1/2}} = sc(\lambda|m), \tag{3.3.3}$$

where $sc(\lambda|m)$ is the Jacobi elliptic function (see, for example, (Abramowitz, 1972), (Erdelyi, 1953). From (3.3.2), it was noted in (Robets, 1976) that the pole occurs at

$$x \approx \frac{1}{2\lambda} \ln(\frac{16}{1-m}). \tag{3.3.4}$$

Here we will show how to apply the variational iteration method to approximate the solution of Troesch's problem.

Thus the correction functional for (3.3.1)can be expressed as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\Big(u_n''(\xi) - \beta \sinh(\beta \tilde{u}_n(\xi))\Big) d\xi. \tag{3.3.5}$$

where λ is a general Lagrange multiplier(Inokuti,1978), which can be identified optimally via the variational theory [(Inkouti, 1978), (He, 1997), (He, 2001), (He, 2003), (He, 2003), (He, 2004), (Liu, 2004), (Liu, 2004), (Hao, 2005)], the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation [(He, 1998), (He, 1998), (He, 1999), (He, 2000), (He, 2004), (He, 2003)], i.e. $\delta\tilde{u}_n = 0$.

Making the correction functional (3.3.5) stationary, noticing that $\delta\tilde{u}_n = 0$,

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda\Big(u_n''(\xi) - \beta \sinh(\beta \tilde{u}_n(\xi))\Big) d\xi, \tag{3.3.6}$$

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda\Big(u_n''(\xi)\Big) d\xi, \tag{3.3.7}$$

$$\delta u_{n+1}(x) = \delta u_n(x) + \lambda \delta u_n'(x) - \lambda' \delta u_n(x) - \int_0^x \delta u_n(\xi) \lambda'' d\xi, \tag{3.3.8}$$

yields the following stationary conditions:

$$\delta u_n(\xi) : 1 - \lambda'(\xi) = 0,$$

$$\delta u_n'(\xi) : \lambda(\xi) = 0,$$

$$\delta u_n(\xi) : \lambda''(\xi) = 0.$$

The Lagrange multiplier, therefore, can be readily identified,

$$\lambda(\xi) = \xi - x.$$

As a result, we obtain the following iteration formulae,

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\Big(u_n''(\xi) - \beta \sinh(\beta u_n(\xi))\Big) d\xi. \tag{3.3.9}$$

Now we begin with an arbitrary initial approximation: $u_0 = A + Bx$, where A and B are constants to be determined by using the boundary conditions $u(0) = 0, u(1) = 1$, thus we have

$$u_0(x) = x.$$

Then the Taylor expansion of $\sinh(\beta u_n(\xi))$ around $u_0(x) = x$ is

$$\begin{aligned}\sinh(\beta u_n(\xi)) &= \sinh(\beta u_0(\xi)) + \beta \cosh(\beta u_0(\xi))(u_n - u_0(\xi)) \\ &+ \frac{\beta^2 \sinh(\beta u_0(\xi))}{2!}(u_n - u_0(\xi))^2 + \dots\end{aligned}$$

thus we have

$$\begin{aligned}\sinh(\beta u_n(\xi)) &\approx \sinh(\beta u_0(\xi)) + \beta \cosh(\beta u_0(\xi))(u_n - u_0(\xi)) \\ &+ \frac{\beta^2 \sinh(\beta u_0(\xi))}{2!}(u_n - u_0(\xi))^2,\end{aligned} \tag{3.3.10}$$

substituting Eq.(3.3.10) into Eq.(3.3.9), we get

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\Big(u_n''(\xi) - \beta[\sinh(\beta u_0(\xi)) + \tag{3.3.11}$$

$$\beta\cosh(\beta u_0(\xi))(u_n(\xi) - u_0(\xi)) + \frac{\beta^2 \sinh(\beta u_0(\xi))}{2!}(u_n(\xi) - u_0(\xi))^2]\Big)d\xi.$$

By the above variational iteration formula (3.3.11), we can obtain following result:

$$u_1(x) = u_0(x) + \int_0^x (\xi - x)\Big(u_0''(\xi) - \beta\sinh(\beta u_0(\xi))\Big)d\xi. \tag{3.3.12}$$

$$u_1(x) = x - \beta\Big(\frac{x}{\beta} - \frac{\sinh(x\beta)}{\beta^2}\Big)$$

Continuing in this manner, we obtain the first few components of $u_n(x)$

$$\begin{aligned}
u_0(x) &= x \\
u_1(x) &= x - \beta\Big(\frac{x}{\beta} - \frac{\sinh(x\beta)}{\beta^2}\Big) \\
u_2(x) &= \frac{1}{72\beta}\Big(- 147x\beta + 6x^3\beta^3 - 216x\beta\cosh(x\beta) - 9x\beta\cosh(2x\beta) \\
&+ 405\sinh(x\beta) + 36x^2\beta^2\sinh(x\beta) + 18\sinh(2x\beta) \\
&+ \sinh(3x\beta)\Big) \\
&\vdots
\end{aligned}$$

and so on, in the same manner the rest of components of the iteration formula (3.3.11) were obtained using the Mathematica Package.

Example 2.

In this example we consider the nonlinear boundary value problem(Wazwaz, 2001)

$$u''(x) + 2(u'(x))^2 + 8u(x) = 0; 0 < x < 1; u(0) = u(1) = 0. \tag{3.3.13}$$

Its correction variational functional can be expressed, as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\Big(u_n''(\xi) + 2(\tilde{u}_n'(\xi))^2 + 8u_n(\xi)\Big)d\xi \tag{3.3.14}$$

Making the correction functional (3.3.14) stationary, noticing that $\delta\tilde{u}_n = 0$,

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\int_0^x \lambda\Big(u_n''(\xi) + 2(\tilde{u}_n'(\xi))^2 + 8u_n(\xi)\Big)d\xi, \tag{3.3.15}$$

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\int_0^x \lambda\Big(u_n''(\xi) + 8u_n(\xi)\Big)d\xi, \tag{3.3.16}$$

$$\delta u_{n+1}(x) = \delta u_n(x) + \lambda\delta u_n'(\xi)|_{\xi=x} - \lambda'\delta u_n(\xi)|_{\xi=x} + \int_0^x \delta u_n(\xi)(8 - \lambda'')d\xi, \tag{3.3.17}$$

yields the following stationary conditions:

$$\delta u_n : 1 - \lambda'(\xi)|_{\xi=x} = 0,$$

$$\delta u_n' : \lambda(\xi)|_{\xi=x} = 0,$$

$$\delta u_n : 8 - \lambda''(\xi)|_{\xi=x} = 0.$$

Now to find the Lagrange multiplier we solve $8 - \lambda''(\xi)|_{\xi=x} = 0$ which is second-order homogeneous differential equation with constant coefficients, then the general solution of this equation is given by:

$$\lambda(\xi) = c_1 \exp[\sqrt{8}\xi] + c_2 \exp[-\sqrt{8}\xi], \tag{3.3.18}$$

by differentiate (3.3.18)with respect to ξ we have,

$$\lambda'(\xi) = c_1\sqrt{8}\exp[\sqrt{8}\xi] - c_2\sqrt{8}\exp[-\sqrt{8}\xi], \tag{3.3.19}$$

substituting the stationary conditions as $\xi = x$ yields:

$$0 = c_1 \exp[\sqrt{8}x] + c_2 \exp[-\sqrt{8}x], \tag{3.3.20}$$

$$1 = c_1\sqrt{8}\exp[\sqrt{8}x] - c_2\sqrt{8}\exp[-\sqrt{8}x], \tag{3.3.21}$$

then by multiply (3.3.20) by $-\sqrt{8}$ and adding the result to (3.3.21) we have,

$$c_2 = \frac{-1}{4\sqrt{2}}\exp[\sqrt{8}x]$$

$$c_1 = \frac{1}{4\sqrt{2}}\exp[-\sqrt{8}x]$$

then the the Lagrange multiplier becomes,

$$\lambda(\xi) = \frac{1}{4\sqrt{2}}\exp[2\sqrt{2}(\xi - x)] - \frac{1}{4\sqrt{2}}\exp[-2\sqrt{2}(\xi - x)]. \tag{3.3.22}$$

As a result, we obtain the following iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x \Big(\frac{1}{4\sqrt{2}}\exp[2\sqrt{2}(\xi - x)] - \frac{1}{4\sqrt{2}}\exp[-2\sqrt{2}(\xi - x)]\Big) \tag{3.3.23}$$

$$\Big(u_n''(\xi) + 2(u_n'(\xi))^2 + 8u_n(\xi)\Big)d\xi$$

Now we begin with initial approximation: $u_0(x) = x$.

By the above variational iteration formula (3.3.23), we can obtain following result:

$$u_1(x) = u_0(x) + \int_0^x \Big(\frac{1}{4\sqrt{2}}\exp[2\sqrt{2}(\xi - x)] - \frac{1}{4\sqrt{2}}\exp[-2\sqrt{2}(\xi - x)]\Big) \tag{3.3.24}$$

$$\Big(u_0''(\xi) + 2(u_0'(\xi))^2 + 8u_0(\xi)\Big)d\xi,$$

$$u_1(x) = x + \frac{1}{4\sqrt{2}}\int_0^x \Big(\exp(2\sqrt{2}(\xi - x)) - \exp[-2\sqrt{2}(\xi - x)]\Big)d\xi, \tag{3.3.25}$$

$$(0 + 2 + 8\xi)d\xi,$$

$$u_1(x) = x + \frac{1}{4\sqrt{2}}\Big(-\frac{1}{2}\exp[-2\sqrt{2}x](-2+\sqrt{2}+2\exp[4\sqrt{2}x]+\sqrt{2}\exp[4\sqrt{2}x]) \tag{3.3.26}$$

$$+\frac{1}{2}\exp[-4\sqrt{2}x](2\sqrt{2}\exp[4\sqrt{2}x] + 8\sqrt{2}\exp[4\sqrt{2}x])\Big).$$

Continuing in this manner, we obtain the first few components of $u_n(x)$

$$\begin{aligned}
u_0(x) &= x, \\
u_1(x) &= x + \frac{1}{4\sqrt{2}}\Big(-\frac{1}{2}\exp[-2\sqrt{2}x](-2+\sqrt{2}+2\exp[4\sqrt{2}x] \\
&+ \sqrt{2}\exp[4\sqrt{2}x]) + \frac{1}{2}\exp[-4\sqrt{2}x](2\sqrt{2}\exp[4\sqrt{2}x] \\
&+ 8\sqrt{2}\exp[4\sqrt{2}x])\Big), \\
u_2(x) &= \frac{1}{96}\exp[-4\sqrt{2}x]\Big(-3+2\sqrt{2}-(3+2\sqrt{2})\exp[8\sqrt{2}x] \\
&+ 6\exp[4\sqrt{2}x](25+64x) - 8\exp[2\sqrt{2}x](9-7\sqrt{2}+3(-4+3\sqrt{2})x) \\
&+ 8\exp[6\sqrt{2}x](-9-7\sqrt{2}+3(4+3\sqrt{2})x)\Big),
\end{aligned}$$

and so on, in the same manner the rest of components of the iteration formula (3.3.23) were obtained using the Mathematica Package.

3.4 Numerical Evaluations

In order to verify numerically whether the proposed method leads to higher accuracy, we can evaluate the approximate solution using the nth-order approximation $u_n(x)$.

Tables 3.1, 3.2, and 3.3 show the exact solution, the approximate solution obtained by the variational iteration method and the Adomian decomposition method.

It is to be noted that only the second-order approximate were used in evaluating the approximate solution using the variational iteration method for Tables 3.1 and 3.2 and third-order approximate for Table 3.3, two terms of the decomposition series for Tables 3.1 and 3.2, three terms of the decomposition series for Table 3.3 .

It is evident that the overall error can be made smaller by computing new terms using the variational iteration method .

We can observed from the tables, 3.1, 3.2, and 3.3 the decomposition method is more effective in solving Troesch's problem(example 1), rather than the variational iteration method, while in example 2 the variationl iteration method is more effective than the other method .

Many of the results obtained in this chapter can be extended to significantly more general classes of linear and nonlinear differential equations.

Table 3.1: Numerical results for Troesch's problem ($\beta = 0.5$)

x	u_{exact}	$u_{var.}$	$u_{decom.}$
0.1	0.095176902	0.100042	0.0959478
0.2	0.1906338691	0.200334	0.192135
0.3	0.286653403	0.301128	0.288803
0.4	0.3835229288	0.402677	0.386196
0.5	0.4815373854	0.505241	0.484559
0.6	0.5810019749	0.609082	0.584144
0.7	0.6822351326	0.71447	0.685211
0.8	0.7855717867	0.821682	0.788023
0.9	0.8913669875	0.931008	0.892858
1.0	0.9999999999	1.04274	1

Table 3.2: Numerical results for Troesch's problem ($\beta = 1.$)

x	u_{exact}	$u_{var.}$	$u_{decom.}$
0.1	0.0817969966	0.100167	0.0849253
0.2	0.1645308709	0.201339	0.170679
0.3	0.2491673608	0.304541	0.258105
0.4	0.3367322092	0.410841	0.348078
0.5	0.428347161	0.521373	0.441523
0.6	0.5252740296	0.637362	0.539438
0.7	0.6289711434	0.760162	0.642918
0.8	0.7411683782	0.891287	0.753195
0.9	0.8639700206	1.03246	0.871676
1.0	1.00000000020	1.18565	1.

Table 3.3: Numerical results for BVP (3.3.13)

x	u_{exact}	$u_{var.}$	$u_{decom.}$
0.1	0.09	0.0899901	0.089966
0.2	0.16	0.159705	0.15885
0.3	0.21	0.207928	0.200776
0.4	0.24	0.231962	0.199056
0.5	0.25	0.227756	0.118651
0.6	0.24	0.191372	-0.102914
0.7	0.21	0.115792	-0.566259
0.8	0.16	-0.0902297	-1.42252
0.9	0.09	-1.23397	-2.88742

Chapter 4

VARIATIONAL ITERATION METHOD FOR SOLVING FISHER'S EQUATION

4.1 Partial Differential Equations

Nonlinear partial differential equations arise in almost all areas of the natural and engineering sciences (Edelstein, 1987), (Murray, 1989), (Smoller, 1983). A particular class of equations are those modelling nonlinear reaction and/or diffusion phenomena (Smoller, 1983).

The one-dimensional Fisher equation (Edelstein, 1987), (Murray, 1989) provides an example for which the diffusion is linear and the reaction term is quadratic in the dependent variable.

However, many cases occur in which the diffusion term is either nonlinear or the diffusion coefficients is a function of the dependent variable (Murray, 1989), (Smoller, 1983).

The general properties of the solutions to such equations often depends on the given initial/boundary values selected consequently a variety of possible behaviours may exist, including wave-like shock solutions (Smoller, 1983).

Several numerical and analytical methods have been developed for solving Fisher's equation such as Laurent series (Wang, 1988), the tanh method (Malfliet, 1992), the finite difference method (Mickens, 2003), and the Adomian decomposition method (Wazwaz, 2004). For more details about these investigations, the reader is advised to see Refs. (Wang, 1988), (Malfliet, 1992), (Mickens, 2003), (Wazwaz, 2004) and the references therein.

In this chapter, we will implement the variational iteration method [(He, 1997), (He, 1998), (He, 1998), (He, 1999), (He, 2000), (He, 2004)] to find approximate solutions to Fisher's equation, the general Fisher's equation, and nonlinear diffusion equation of the Fisher type.

The variational iteration method proposed by Ji-Huan He has been shown to solve effectively, easily, and accurately a large class of non-linear problems with approximations converging rapidly to accurate solutions.

For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

The main objective of this contribution is to extend the variational iteration method proposed by He [(He, 1997), (He, 1998), (He, 1998), (He, 1999), (He, 2000), (He, 2004)] to solve three different types of Fisher's equation and comparison with that obtained previously by the Adomian decomposition method (Wazwaz, 2004).

This chapter is organized as follows. In section 2 we implement the variational iteration method to find approximate solutions to Fisher's equation, the general Fisher's equation, nonlinear diffusion equation of the Fisher type, and the Parabolic partial differential equation. Then in section 3 the numerical evaluations of these methods and comparisons will be given.

4.2 Applications

4.2.1 The Fisher's Equation

In this section we consider the following two important cases of nonlinear diffusion which are also solved by Adomian decomposition method in the study of Wazwaz (Wazwaz, 2004).

Case I: In this case we consider the Fisher's equation

$$u_t(x,t) - u_{xx}(x,t) - u(x,t)\Big(1 - u(x,t)\Big) = 0, \tag{4.2.1}$$

subject to a constant initial condition

$$u(x,0) = \beta. \tag{4.2.2}$$

To solve (4.2.1) by means of the variational iteration method, we construct a correction functional in t-direction as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} - \tilde{u}_n(x,\tau)(1-u_n(x,\tau))\Big)d\tau, \tag{4.2.3}$$

where λ is a general Lagrange multiplier, u_0 is an initial approximation or trial-function with possible unknowns, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$

Making the correction functional (4.2.3) stationary, noticing that $\delta\tilde{u}_n = 0$,

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda\Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} - \tilde{u}_n(x,\tau)(1-u_n(x,\tau))\Big)d\tau, \tag{4.2.4}$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda\Big(\frac{\partial u_n(x,\tau)}{\partial \tau}\Big)d\tau, \tag{4.2.5}$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t)(1+\lambda(\tau)) - \delta\int_0^t \lambda'(\tau)u_n(x,\tau)d\tau,$$

yields the following stationary conditions:

$$\delta u_n : 1+\lambda(\tau) = 0,$$

$$\delta u_n : \lambda'(\tau) = 0.$$

The Lagrange multiplier, therefore, can be readily identified as

$$\lambda(\tau) = -1,$$

as a result, we obtain the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - u_n(x,\tau)(1-u_n(x,\tau))\Big)d\tau. \tag{4.2.6}$$

Now we begin with initial condition given by Eq. (4.2.2) and by the above iteration formula we can obtain the following results:

$$\begin{aligned}
u_0(x,y) &= \beta,\\
u_1(x,y) &= \beta+\beta(1-\beta)t,\\
u_2(x,y) &= \beta+\beta(1-\beta)t+\frac{t^2}{2!}\beta(1-3\beta+2\beta^2)-\frac{t^3}{3}\beta^2(-1+\beta)^2,\\
u_3(x,y) &= \beta+\beta(1-\beta)t+\frac{t^2}{2!}\beta(1-\beta)(1-2\beta)\\
&+ \frac{t^3}{3!}\beta(1-\beta)(1-6\beta+6\beta^2)+\frac{t^4}{3}(-1+\beta)^2\beta^2(-1+2\beta)\\
&- \frac{t^5}{60}(-1+\beta)^2\beta^2(3-20\beta+20\beta^2)+\frac{t^6}{18}(-1+\beta)^3\beta^3(-1+2\beta)\\
&- \frac{t^7}{63}(-1+\beta)^4\beta^4,
\end{aligned}$$

and so on, in the same manner the rest of components of the iteration formula (4.2.6) were obtained using the Mathematica Package.

The solution $u(x,t)$ in a closed form is

$$u(x,t) = \frac{\beta \exp t}{1 - \beta + \beta \exp t}, \tag{4.2.7}$$

which is exactly the same as that obtained by Adomian decomposition method (Wazwaz, 2004).

Case II: In this case we will examine the Fisher's equation

$$u_t - u_{xx} - 6u(1-u) = 0, \tag{4.2.8}$$

subject to the initial condition

$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{4.2.9}$$

To solve (4.2.8) by means of the variational iteration method, we construct the following correction functional:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} \tag{4.2.10}$$
$$-6\tilde{u}_n(x,\tau)(1-u_n(x,\tau))\Big)d\tau,$$

where $\tilde{u}_n$ is considered as a restricted variation.

Noticing that $\delta\tilde{u}_n = 0$, by the same manipulation as before, the multiplier can be determined as $\lambda = -1$, as a result, we have the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} \qquad (4.2.11)$$
$$-6u_n(x,\tau)(1-u_n(x,\tau))\Big)d\tau.$$

We start with initial approximation(4.2.9). By the above iteration formula, we can obtain following results:

$$\begin{aligned}
u_0(x,t) &= \frac{1}{(1+e^x)^2},\\
u_1(x,t) &= \frac{1+e^x(1+10t)}{(1+e^x)^3},\\
u_2(x,t) &= \frac{1}{(1+e^x)^6}\Big(25e^x(1+e^x)^2(-1+2e^x)t^2 - 200e^{2x}t^3\\
&+ (1+e^x)^3(1+e^x(1+10t))\Big)\\
u_3(x,t) &= \frac{25e^x}{3(1+e^x)^6}\Big(5-6e^x-15e^{2x}+20e^{3x})t^3\Big)\\
&- \frac{50e^{2x}}{(1+e^x)^8}\Big(-17+5e^x+52e^{2x}t^4\Big) + \frac{150e^{2x}}{(1+e^x)^9}\Big(5-47e^x+20e3x)t^5\Big)\\
&+ \frac{10000e^{3x}(-1+2e^x)t^6}{(1+e^x)^{10}} - \frac{240000e^{4x}t^7}{7(1+e^x)^{12}}\\
&+ \frac{1}{(1+e^x)^6}\Big(25e^x(1+e^x(1+e^x)^2(-1+2e^x)t^2\\
&- 200e^{2x}t^3 + (1+e^x)^3(1+e^x(1+10t))\Big),
\end{aligned}$$

and so on.

Proceeding as before the rest of components of the iteration formula (4.2.11) were obtained using the Mathematica Package.

The solution $u(x,t)$ in the closed form is readily found to be

$$u(x,t) = \frac{1}{(1+\exp(x-5t))^2}. \qquad (4.2.12)$$

Again the result that we obtained is in full agreement with (Wazwaz, 2004).

4.2.2 The Generalized Fisher's Equation

In this section we examine the generalized Fisher's equation

$$u_t = u_{xx} + u(1 - u^6), \tag{4.2.13}$$

subject to the initial condition

$$u(x,0) = \frac{1}{(1 + e^{\frac{3}{2}x})^{\frac{1}{3}}}. \tag{4.2.14}$$

Its correction functional can be expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} - \tilde{u}_n(x,\tau)(1 - u_n^6(x,\tau))\Big)d\tau, \tag{4.2.15}$$

where $\tilde{u}_n$ is considered as a restricted variation.

Noticing that $\delta \tilde{u}_n = 0$, by the same manipulation as before, the multiplier can be determined as $\lambda = -1$, as a result, we have the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - u_n(x,\tau)(1 - u_n^6(x,\tau))\Big)d\tau. \tag{4.2.16}$$

We start with initial approximation (4.2.14). By the above iteration formula, we can obtain following results:

$$
\begin{aligned}
u_0(x,t) &= \frac{1}{(1+e^{\frac{3}{2}x})^{\frac{1}{3}}},\\
u_1(x,t) &= \frac{4+e^{3x}(4+3t)+e^{3x/2}(8+11t)}{4(1+e^{3x/2})^{7/3}},\\
u_2(x,t) &= \frac{1}{131072(1+e^{3x/2})^{49/3}}\Big(4096e^{3x/2}(1+e^{3x/2})^{12}\Big(-363+403e^{3x/2}\\
&- 9e^{3x}+9e^{9x/2}\Big)t^2-57344e^{3x}(1+e^{3x/2})^{10}(11+3e^{3x/2})^2t^3\\
&- 17920e^{9x/2}(1+e^{3x/2})^8(11+3e^{3x/2})^3t^4\\
&- 3584e^{6x}(1+e^{3x/2})^6(11+3e^{3x/2})^4t^5\\
&- 448e^{15x/2}(1+e^3x/2)^4(11+3e^{3x/2})^5t^6\\
&- 32e^{9x}(1+e^{3x/2})^2(11+3e3x/2)^6t^7\\
&- e^{21x/2}(11+3e^{3x/2})^7t^8\\
&+ 32768(1+e^{3x/2})^{14}(4+e^{3x}(4+3t)+e^{3x/2}(8+11t))\Big),
\end{aligned}
$$

while its exact solution (Wazwaz, 2004) is

$$u(x,t)=\Big(1/2\tanh[-3/4(x-5/2t)]+1/2\Big)^{1/3}. \qquad (4.2.17)$$

So, we can see clearly that we can obtain an ideal approximation by its second approximation, and its nth approximation converges to its exact solution.

4.2.3 Nonlinear Diffusion Equation of the Fisher Type

In this section we examine the nonlinear diffusion equation of the Fisher type

$$u_t = u_{xx} + u(1-u)(u-a), 0 < a < 1, \tag{4.2.18}$$

subject to the initial condition

$$u(x,0) = \frac{1}{1+e^{-x/\sqrt{2}}}. \tag{4.2.19}$$

This equation has three constant solutions; $u = 0, 1$, and a. The case with $0 < a < 1$ is what the genetics (Kawahara, 1983) refer to as the heterozygote inferiority. In (Kawahara, 1983) an exact solution of (4.2.18) which describes the coalescence of two traveling fronts of the same sense into a front connecting two stable constant states is found.

Its correction variational functional in t-direction can be expressed as follows:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\Big(&\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} \\ &-\tilde{u}_n(x,\tau)(1-u_n(x,\tau))(u_n(x,\tau)-a)\Big)d\tau, \end{aligned} \tag{4.2.20}$$

where $\tilde{u}_n$ is considered as a restricted variation.

The Lagrange multiplier, therefore, can be identified, $\lambda(\tau) = -1$, and the following variational iteration formula in t- direction can be obtained:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Big(&\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} \\ &-u_n(x,\tau)(1-u_n(x,\tau))(u_n(x,\tau)-a)\Big)d\tau. \end{aligned} \tag{4.2.21}$$

If we use the initial condition (4.2.19) as initial approximation, by the above iteration formula (4.2.21), we get

$$u_1(x,t) = u_0(x,t) - \int_0^t \Big(\frac{\partial u_0(x,\tau)}{\partial \tau} - \frac{\partial^2 u_0(x,\tau)}{\partial x^2} \qquad (4.2.22)$$
$$-u_0(x,\tau)(1-u_0(x,\tau))(u_0(x,\tau)-a)\Big)d\tau,$$

$$u_1(x,t) = \frac{e^{x/\sqrt{2}}\Big(2+2e^{x/\sqrt{2}}+t-2at\Big)}{2\Big(1+e^{x/\sqrt{2}}\Big)^2}. \qquad (4.2.23)$$

Continuing in this manner, we obtain $u_2(x,t)$

$$\begin{aligned} u_2(x,t) &= \frac{1}{96\Big(1+e^{x/\sqrt{2}}\Big)^6}\Big(e^{x/\sqrt{2}}\Big(-12(1-2a)^2(-1+e^{x/\sqrt{2}})(1+e^{x/\sqrt{2}})^3t^2 \\ &+ 8(1-2a)^2e^{x/\sqrt{2}}(1+e^{x/\sqrt{2}})(1+a-2e^{x/\sqrt{2}}+ae^{x/\sqrt{2}})t^3 \\ &+ 3(-1+2a)^3e^{\sqrt{2}x}t^4 + 48(1+e^{x/\sqrt{2}})^4(2+2e^{x/\sqrt{2}}+t-2at)\Big)\Big), \end{aligned}$$

and after some algebra, the solution in a closed form is given by

$$u(x,t) = \frac{1}{1+\exp[-\xi/\sqrt{2}]}, \qquad (4.2.24)$$

which is in full agreement with the results in (Wazwaz, 2004), where $\xi = x + ct$ and $c = \sqrt{2}(1/2 - a)$.

4.2.4 Parabolic Partial Differential Equations

In this section, we implement variational iteration method to find the exact solution of the parabolic partial differential equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < \ell, t \geq 0, \qquad (4.2.25)$$

$$u(0,t) = u(\ell,t) = 0, t > 0, \qquad (4.2.26)$$

$$u(x,0) = g(x), 0 \leq x \leq \ell. \tag{4.2.27}$$

This equation (Al-Khaled, 2003) is the basic mathematical model for the heat or diffusion equation. The physical problem considered here concerns the flow of heat along a rod of length ℓ. The constant α is determined by the heat-conductive properties of the material [see(Burden, 1993), (Gerald, 1994)].

Several numerical and analytical method have been developed for solving (4.2.25), such as finite-difference method, see[(Burden, 1993), (Gerald, 1994)], Sinc method, see(Stengr, 1992), and decomposition method .

In order to assess the advantages and the accuracy of He's variatonal iteration method for solving the parabolic partial differential equation, we will consider the following example.

Example 1.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < 1, t \geq 0, \tag{4.2.28}$$

with boundary and initial conditions

$$u(0,t) = u(1,t) = 0, t > 0, u(x,0) = \sin(\pi x), 0 \leq x \leq 1.$$

Its correction variational iteration in t-direction can be expressed, as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2}\Big)d\tau, \tag{4.2.29}$$

where λ is a general Lagrange multipliers , u_0 is an initial approximation or trial-function with possible unknowns, $\tilde{u}_n$ are considered as restricted variations, i.e. $\delta\tilde{u}_n = 0$.

Making the correction functional (4.2.29) stationary, noticing that $\delta\tilde{u}_n = 0$,

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda\Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2}\Big) d\tau, \qquad (4.2.30)$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda\Big(\frac{\partial u_n(x,\tau)}{\partial \tau}\Big) d\tau, \qquad (4.2.31)$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(\tau)\delta u_n(x,t) - \int_0^t \lambda' \delta u_n(x,\tau) d\tau = 0, \qquad (4.2.32)$$

yields the following stationary conditions:

$$1 + \lambda(\tau)|_{\tau=t} = 0$$

$$\lambda'(\tau)|_{\tau=t} = 0.$$

The Lagrange multiplier, therefore, can be readily identified,

$$\lambda(\tau) = -1,$$

as a result, we obtain the following iteration formulae in t- -direction.

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Big(\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2}\Big) d\tau. \qquad (4.2.33)$$

Now we begin with initial condition $u(x,0) = u_0(x) = \sin(\pi x)$.

By the above variational iteration formula in t-direction (4.2.33), we can obtain following result:

$$u_1(x,t) = u_0(x,t) - \int_0^t \Big(\frac{\partial u_0(x,\tau)}{\partial \tau} - \frac{\partial^2 u_0(x,\tau)}{\partial x^2}\Big) d\tau. \qquad (4.2.34)$$

$$u_1(x,t) = \sin(\pi x) - \int_0^t \Big(\frac{\partial \sin(\pi x)}{\partial \tau} - \frac{\partial^2 \sin(\pi x)}{\partial x^2}\Big) d\tau. \tag{4.2.35}$$

Thus we have

$$u_1(x,t) = \sin(\pi x) - \pi^2 t \sin(\pi x).$$

Continuing in this manner, we obtain the first few components of $u_n(x,t)$

$$\begin{aligned}
u_0(x,t) &= \sin(\pi x), \\
u_1(x,t) &= \sin(\pi x) - \pi^2 t \sin(\pi x), \\
u_2(x,t) &= \sin(\pi x) - \pi^2 t \sin(\pi x) + \frac{\pi^4 t^2}{2} \sin(\pi x), \\
u_3(x,t) &= \sin(\pi x) - \pi^2 t \sin(\pi x) + \frac{\pi^4 t^2}{2} \sin(\pi x) - \frac{\pi^6 t^3}{6} \sin(\pi x), \\
&\vdots
\end{aligned}$$

and so on, in the same manner the rest of components of the iteration formula (4.2.33) were obtained using the Mathematica Package.

The solution $u(x,t)$ in a closed form is

$$u(x,t) = \exp(-\pi^2 t) \sin(\pi x).$$

Which is exactly the same as obtained by Adomian decomposition method (Al-Khaled, 2003).

4.3 Numerical Evaluations

To validate the application of He's variational iteration method to the Fisher's equation, we compare the solution developed in section 2 with numerical results from Adomian decomposition method (Wazwaz, 2004).

Tables 4.1, 4.2, 4.3, 4.4, show the difference of the analytical solution and numerical solution of the absolute errors for the Fisher's equation (case I and II), the general Fisher's equation, and nonlinear diffusion equation of the Fisher type, respectively, using the two methods.

It is to be noted that the approximate solutions are derived using the third-order approximation $u_{var.} = u_3(x, t)$ for Tables 4.1 and 4.2 and the second-order approximation for Tables 4.3 and 4.4 and compared with the numerical solution obtained by the Adomian decomposition method, $u_{decom.}$, using three terms only of the decomposition series (Wazwaz, 2004).

As expected the numerical solutions in the tables are clearly indicated that how the proposed scheme obtains efficient results closer to the actual solution and easier to use than the Adomian decomposition method.

Also, the approximations obtained by this method are valid not only for small times, but also for large times.

A clear conclusion can be drawn from the numerical results that the variational iteration method provides highly accurate numerical solutions without spatial discretizations for nonlinear partial differential equations.

It is also worth noting that the variational iteration method has many merits and has much advantages over the Adomian method. It can be introduced to overcome the difficulty arising in calculating Adomian polynomials (Abdou, in press),(Abassy, 2004).

The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. Many of the results obtained in this chapter can be extended to significantly more general classes of linear and nonlinear differential equations.

Table 4.1: The difference of the analytical solution and numerical solution of the absolute errors for the Fisher's equation (case I)

	$\beta = 0.2$		$\beta = 0.8$	
t	$\lvert u_{exact} - u_{decom.}\rvert$	$\lvert u_{exact} - u_{var.}\rvert$	$\lvert u_{exact} - u_{decom.}\rvert$	$\lvert u_{exact} - u_{var.}\rvert$
0	0	0	0	0
0.2	6.19201E-06	1.96407E-06	5.57339E-06	2.63705E-06
0.4	1.03635E-04	2.60211E-05	1.13137E-04	4.7283E-05
0.6	5.45505E-04	1.05333E-04	4.00147E-04	2.63379E-04
0.8	1.78050E-03	2.54998E-04	1.18584E-03	9.01304E-04
1	4.45699E-03	4.515414E-04	2.34887E-03	2.70952E-03

Table 4.2: The difference of the analytical solution and numerical solution of the absolute errors for the Fisher's equation (case II)

	$t = 0.2$		$t = 0.4$	
x	$\lvert u_{exact} - u_{decom.}\rvert$	$\lvert u_{exact} - u_{var.}\rvert$	$\lvert u_{exact} - u_{decom.}\rvert$	$\lvert u_{exact} - u_{var.}\rvert$
0	7.22002E-03	2.69962E-03	5.75298E-02	5.01844E-02
0.2	9.89049E-03	2.15566E-03	1.6115E-01	5.27127E-02
0.4	1.09765E-02	1.136097E-03	1.39113E-01	4.12072E-02
0.6	1.04039E-02	7.38299E-04	1.51579E-01	2.25459E-02
0.8	8.50732E-03	5.73101E-04	1.43529E-01	5.28693E-03
1	5.87222E-03	9.07727E-04	1.19333E-01	4.23672E-03

Table 4.3: The difference of the analytical solution and numerical solution of the absolute errors for the generalized Fisher's equation

	$t = 0.2$		$t = 0.6$	
x	$\lvert u_{exact} - u_{decom.}\rvert$	$\lvert u_{exact} - u_{var.}\rvert$	$\lvert u_{exact} - u_{decom.}\rvert$	$\lvert u_{exact} - u_{var.}\rvert$
0	5.24926E-02	4.54137E-02	1.21845E-01	1.97465E-01
0.2	7.79547E-02	4.1746E-02	2.17494E-01	8.39974E-02
0.4	1.10805E-01	3.23276E-02	3.4171E-01	9.22231E-04
0.6	1.51375E-01	1.91936E-02	4.94354E-01	4.10631E-02
0.8	1.99601E-01	5.03821E-03	6.74017E-01	4.10631E-02
1	2.55137E-01	7.85833E-03	8.78892E-01	1.46625E-02

Table 4.4: The difference of the analytical solution and numerical solution of the absolute errors for the Fisher type

	$t = 0.2$		$t = 0.8$	
x	$\lvert u_{exact} - u_{decom.} \rvert$	$\lvert u_{exact} - u_{var.} \rvert$	$\lvert u_{exact} - u_{decom.} \rvert$	$\lvert u_{exact} - u_{var.} \rvert$
0	7.71519E-07	2.90209E-08	4.92951E-05	7.49504E-06
0.2	7.54407E-07	5.23388E-07	4.78602E-05	3.93916E-05
0.4	7.08084E-07	9.98088E-07	4.45906E-05	6.98374E-05
0.6	6.36337E-07	1.42203E-06	3.9752E-05	9.68423E-05
0.8	5.44789E-07	1.77086E-06	3.37194E-04	1.18866E-04
1	2.02937E-06	4.40168E-07	2.69298E-05	1.34968E-04

Chapter 5

CONCLUSIONS AND SUMMARY

In conclusion, the variational iteration method was used for solving nonlinear boundary value problem, exact and approximate solution of the Fisher's equation, the general Fisher's equation, , nonlinear diffusion equation of the Fisher type,and to find the exact solution of the parabolic partial differential equation .

The method can be also extended to other nonlinear evaluation equations, with aid of the powerful Computer Algebra Package, Mathematica (or Matlab, Maple, etc.), the course of solving nonlinear evaluation equations can be carried out in computer.

The results in chapter 3 and chapter 4 show that

(1) A correction functional can be easily constructed by a general Lagrange multiplier, and the multiplier can optimally identified by variational theory. The application of restricted variations in correction functional makes it much easier to determine the multiplier.

(2) The approximations obtained by this method are valid not only for small times, but also for large times.

(3)It should be specially pointed out that the more accurate the identification of the multiplier ,the faster the approximation converge to its exact solution .

(4) The numerical solutions in the Tables clearly indicate how the variational iteration method obtains efficient results closer to the actual solution and easier to use than those of Adomian's method. Also, the method can be introduced to overcome the difficulty arising in calculating Adomian polynomials.

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