Implementing Turbulence Models into the Compressible RANS Equations

There are many technical papers and texts that derive and/or describe the compressible Reynolds-averaged Navier-Stokes equations (also termed the Favre-averaged Navier-Stokes equations). See, for example, (1) Gatski, T. B. and Bonnet, J.-P., "Compressibility, Turbulence and High Speed Flow," 2009, Elsevier, Amsterdam, (2) Wilcox, D. C., "Turbulence Modeling for CFD," 2006, DCW Industries, La Canada, CA, or (3) Hirsch, C., "Numerical Computation of Internal and External Flows, Vol. 2," 1990, John Wiley & Sons, Chichester. The equations can be written as follows:

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_j}(\overline{\rho}\hat{u}_j) = 0$$
$$\frac{\partial(\overline{\rho}\hat{u}_i)}{\partial t} + \frac{\partial}{\partial x_j}(\hat{u}_j\overline{\rho}\hat{u}_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial \overline{\sigma}_{ij}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j}$$
$$\frac{\partial(\overline{\rho}\hat{E})}{\partial t} + \frac{\partial}{\partial x_j}(\hat{u}_j\overline{\rho}\hat{H}) = \frac{\partial}{\partial x_j}(\overline{\sigma}_{ij}\hat{u}_i + \overline{\sigma_{ij}u_i''}) - \frac{\partial}{\partial x_j}(\overline{q}_j + c_p\overline{\rho u_j'' T''} - \hat{u}_i\tau_{ij} + \frac{1}{2}\overline{\rho u_i'' u_i'' u_j''})$$

where

$$\hat{H} = \hat{E} + \overline{p}/\overline{\rho}$$
$$\overline{q}_j = -\overline{k_T \partial T/\partial x_j} \approx -\frac{c_p\hat{\mu}}{Pr}\frac{\partial \hat{T}}{\partial x_j}$$

and the viscous stress tensor is:

$$\overline{\sigma}_{ij} \approx 2\hat{\mu}\left(\hat{S}_{ij} - \frac{1}{3}\frac{\partial \hat{u}_k}{\partial x_k}\delta_{ij}\right)$$

（没有加包含 **k** 的项）

Note that the Reynolds stress term $\tau_{ij} \equiv -\overline{\rho u_i'' u_j''}$ is defined in the literature both as shown here, as well as with the opposite sign, and sometimes without the density included in the definition. (This different terminology does not matter, as long as consistency is maintained throughout the derivation.) The term c_p is the heat capacity at constant pressure, and Pr is the Prandtl number (e.g., around 0.72 for air). On this page the overbar indicates conventional time-average mean, with the averaging time scale assumed to be long compared to turbulent fluctuations, and short compared to unsteadiness in the mean flow. The hat here represents the Favre (density-weighted) average: $\hat{f} = \overline{\rho f}/\overline{\rho}$. Note that $f = \overline{f} + f' = \hat{f} + f''$.

The dynamic viscosity, $\hat{\mu}$, is often computed using Sutherland's Law, which gives a relationship between the dynamic viscosity and the temperature of an ideal gas (See White, F. M., "Viscous Fluid Flow," McGraw Hill, New York, 1974, p. 28). In Sutherland's Law, the local value of dynamic viscosity is determined by plugging the local value of temperature (*T*) into the following formula:

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \left(\frac{T_0 + S}{T + S}\right)$$

where $\mu_0 = 1.716 \times 10^{-5} kg/(ms)$, $T_0 = 491.6R$, and $S = 198.6R$. The same formula can be found online (with temperature constants given in degrees K and some small conversion differences).

The equation of state is:

$$\overline{p} = (\gamma - 1)[\overline{\rho}\hat{E} - \frac{1}{2}\overline{\rho}(\hat{u}^2 + \hat{v}^2 + \hat{w}^2) - \overline{\rho}k]$$

where k is the local turbulent kinetic energy (the kinetic energy of the fluctuating field): $k = [(\hat{u_i''})^2 + (\hat{v_i''})^2 + (\hat{w_i''})^2]/2$. The heat capacity ratio (γ) is typically taken as constant at 1.4 for air. The following terms in the Favre-averaged equations need to be modeled:

$$\tau_{ij}$$

$$c_p\overline{\rho u_j'' T''}$$

$$\overline{\sigma_{ij} u_i''}$$

$$\frac{1}{2}\overline{\rho u_i'' u_i'' u_j''}$$

Most turbulence modeling focuses on the Reynolds stress terms (τ_{ij}). These are either solved directly (as in full second-moment Reynolds stress models) or defined via a constitutive relation for simpler models. For example, the common Boussinesq approximation is:

$$\tau_{ij} = 2\hat{\mu}_t \left(\hat{S}_{ij} - \frac{1}{3}\frac{\partial \hat{u}_k}{\partial x_k}\delta_{ij} \right) - \frac{2}{3}\overline{\rho} k \delta_{ij}$$

where $\hat{S}_{ij} = (\partial \hat{u}_i/\partial x_j + \partial \hat{u}_j/\partial x_i)/2$, and $\hat{\mu}_t$ is the eddy viscosity obtained by the turbulence model. (In the equation above, the $(2/3)\overline{\rho} k \delta_{ij}$ term is sometimes ignored for non-supersonic speed flows, and the second term in parentheses is identically zero for incompressible flows.)

Less attention is typically given to the other terms that need to be modeled. Most commonly, a Reynolds analogy is used to model the turbulent heat flux:

$$c_p \overline{\rho u_j'' T''} \approx -\frac{c_p \hat{\mu}_t}{Pr_t}\frac{\partial \hat{T}}{\partial x_j}$$

where Pr_t is a "turbulent Prandtl number," often taken to be constant (e.g., around 0.9 for air). Ideal gas relations are typically used to resolve the heat capacity at constant pressure (c_p); if required, the specific gas constant for air is usually taken as 287.058 J/(kgK). The terms associated with molecular diffusion and turbulent transport in the energy equation are modeled different ways (often lumped together). For example, one model is:

$$\overline{\sigma_{ij} u_i''} - \frac{1}{2}\overline{\rho u_i'' u_i'' u_j''} \approx \left(\hat{\mu} + \frac{\hat{\mu}_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j}$$

where σ_k (which is sometimes shown like this, in the denominator, and sometimes in the numerator with its value adjusted accordingly) is a coefficient associated with the modeling equation for k. This expression in the energy equation is also sometimes neglected.