Transformation Properties

This document focuses on the invariance properties of equations and quantities. The methods for invariance property analysis are provided. A summary of the invariance properties of equations and quantities is provided along with examples.

1. Rotation and reflection

Any equations written in Cartesian tensor notation ensures invariance under rotations and reflections of coordinate axes. Thus, NS and Reynolds stress transport equations have the invariance. Note that the alternating symbol ε_{ijk} is not a tensor, and vorticity is a pseudovector. Hence, equations containing ε_{ijk} or vorticity **MAY** not be rotationally and reflectionally invariant. The method for the analysis of rotational and reflectional invariance is

$$f_{ij\ldots k} = a_{il} a_{jm} \ldots a_{kn} f_{lm\ldots n} \tag{1.1}$$

This applies to any first-order tensors f_i, such as velocity, position, and pressure gradient, and second-order tensors f_{ij}, such as Reynolds stress and velocity gradients, and any third-order tensors f_{ijk}, such as second-order velocity derivatives and Reynolds stress gradients.

2. Galilean invariance

The behavior of **(in)compressible** fluid flows is the same in all inertial frame. Hence, NS and Reynolds stress transport equations have the property. The method to verify Galilean invariance is

$$x_i = \overline{x}_i + V_i t \tag{2.1}$$

$$U_i = \overline{U}_i + V_i \tag{2.2}$$

Note that Eq. (2.2) can be regarded as $U_i\left(x,t\right) = f\left(\overline{x},t\right)$, so

$$\frac{\partial U_i}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \overline{x}_j}\frac{\partial \overline{x}_j}{\partial t} \tag{2.3}$$

$$\frac{\partial U_i}{\partial x_j}=\frac{\partial f}{\partial \overline{x}_k}\frac{\partial \overline{x}_k}{\partial x_j} \tag{2.4}$$

Instead of

$$\frac{\partial U_i}{\partial t}=\frac{\partial f}{\partial t} \tag{2.5}$$

Special attention should be given to Eq. (2.3), where $\overline{x}$ is a function of t when x is fixed, which can be easily seen from Eq.(2.1). Eq. (2.3) also applies to the analysis of extended Galilean invariance and material-frame invariance.

3. Extended Galilean invariance

The behavior of constant-density fluid flows is the same under rectilinear accelerations of the frame. The incompressible NS equations and Reynolds stress transport equations have the property.

$$x_i=\overline{x}_i+\int_0^t V_i\left(t\right)\mathrm{d}t \tag{3.1}$$

$$U_i=\overline{U}_i+V_i\left(t\right) \tag{3.2}$$

4. Rotating frame

A quantity that is the same in rotating and non-rotating frames is said to possess material-frame indifference.

$$x_i=a_{ij}\overline{x}_j+\int_0^t V_i\left(t'\right)\mathrm{d}t' \tag{4.1}$$

$$U_i=a_{ik}\Omega_{jk}\overline{x}_j+a_{ij}\overline{U}_j+V_i \tag{4.2}$$

$$\frac{DU_i}{Dt}=a_{il}\left(\frac{D\overline{U}_l}{Dt}+\overline{x}_j\Omega_{jk}\Omega_{kl}+2\overline{U}_j\Omega_{jl}+\overline{x}_j\frac{\partial \Omega_{jl}}{\partial t}\right)+\frac{\partial V_i}{\partial t} \tag{4.3}$$

$$\frac{\partial U_i}{\partial x_j}=a_{ik}a_{jl}\Omega_{lk}+a_{ik}a_{jl}\frac{\partial \overline{U}_k}{\partial \overline{x}_l} \tag{4.4}$$

$$\begin{aligned}\frac{\partial U_i}{\partial t}=&\ \Omega_{jk}\overline{x}_j a_{il}\Omega_{kl}+a_{ik}\overline{x}_j\frac{\partial \Omega_{jk}}{\partial t}+a_{ik}\Omega_{jk}\Omega_{jl}\overline{x}_l-a_{ik}\Omega_{jk}a_{nj}V_n+\overline{U}_j a_{im}\Omega_{jm}\\&+a_{ij}\frac{\partial \overline{U}_j}{\partial t}+a_{ij}\frac{\partial \overline{U}_j}{\partial \overline{x}_k}\Omega_{kl}\overline{x}_l-a_{ij}a_{nk}\frac{\partial \overline{U}_j}{\partial \overline{x}_k}V_n+\frac{\partial V_i}{\partial t}\end{aligned} \tag{4.5}$$

where rate of frame rotation $\Omega_{ij}=-\Omega_{ji}$ and $\frac{\mathrm{d}a_{ij}}{\mathrm{d}t}=a_{ik}\Omega_{jk}$. Given angular rotation velocity ω, the expression of Ω_{ij} is

$$\Omega_{ij}=(\boldsymbol{\omega}\times\overline{\mathbf{e}}_i)\cdot\overline{\mathbf{e}}_j=\begin{bmatrix}0 & \omega_3 & -\omega_2\\ -\omega_3 & 0 & \omega_1\\ \omega_2 & -\omega_1 & 0\end{bmatrix} \tag{4.6}$$

Refer to https://www.youtube.com/watch?v=NnkHfo3BNr8 for Eq. (4.6). Only 2D incompressible NS equations possess this property.

5. Summary

Table 1 Transformation properties of equations

Equations	Rotation/reflection	Galilean	Extended Galilean	Rotating frame
NS	Y	Y	P_eff	P_eff+ Coriolis force
RANS	Y	Y	P_eff	P_eff+ Coriolis force
$\langle u_i u_j \rangle$	Y	Y	P_eff	P_eff+ Coriolis force
k	Y	Y	P_eff	P_eff
ε	Y	Y	P_eff	P_eff
ω	Y	Y	P_eff	P_eff
Boussinesq	Y	Y	Y	Y

Table 2 Transformation properties of quantitities

Equations	Rotation/reflection	Galilean	Extended Galilean	Rotating frame
$\left\lvert\frac{\partial U_i}{\partial x_j}\right\rvert$	Y	Y	Y	Ω
$\left\lvert\frac{\partial \langle U_i\rangle}{\partial x_j}\right\rvert$	Y	Y	Y	Ω
$\sqrt{U_i U_i}$	Y	N	N	Ω
$\lvert u_i\rvert$	Y	Y	Y	Y
$\lvert S_{ij}\rvert$	Y	Y	Y	Y

$\left\|\Omega_{ij}\right\|$	Y	Y	Y	Ω
$\left\|\frac{\partial u_i}{\partial x_j}\right\|$	Y	Y	Y	Y
$\left\|\langle u_i u_j \rangle\right\|$	Y	Y	Y	Y
TKE	Y	Y	Y	Y
$\left\|\frac{\partial \langle u_i u_j \rangle}{\partial x_k}\right\|$	Y	Y	Y	Y

Note that $|\ |$ denotes modulus of a tensor.

6. Examples

(1) Material-frame indifference of NS equations

$$\frac{D\overline{U}_j}{Dt} = \nu \frac{\partial^2 \overline{U}_j}{\partial \overline{x}_i \partial \overline{x}_i} + \frac{\partial P}{\partial \overline{x}_j} - \overline{x}_i \Omega_{ik} \Omega_{kj} - 2\overline{U}_i \Omega_{ij} - \overline{x}_i \frac{\partial \Omega_{ij}}{\partial t} - a_{ij} \frac{\partial V_i}{\partial t} \tag{6.1}$$

The definition of Ω_{ij} is Eq. (4.6). Hence, if a reference frame is rotating at a constant angular velocity $\boldsymbol{\omega} = (0, 0, \omega_3)$ with its origin fixed, the fictitious force would be

$$F_1 = \overline{x}_1 \omega_3 \omega_3 + 2\omega_3 \overline{U}_2 \tag{6.2}$$

$$F_2 = \overline{x}_2 \omega_3 \omega_3 - 2\omega_3 \overline{U}_1 \tag{6.3}$$

$$F_3 = 0 \tag{6.4}$$

(2) Material-frame indifference of Reynolds stress transport equations