Elementary Mathematics II (Differential Equations and Dynamics) (MTH 102)

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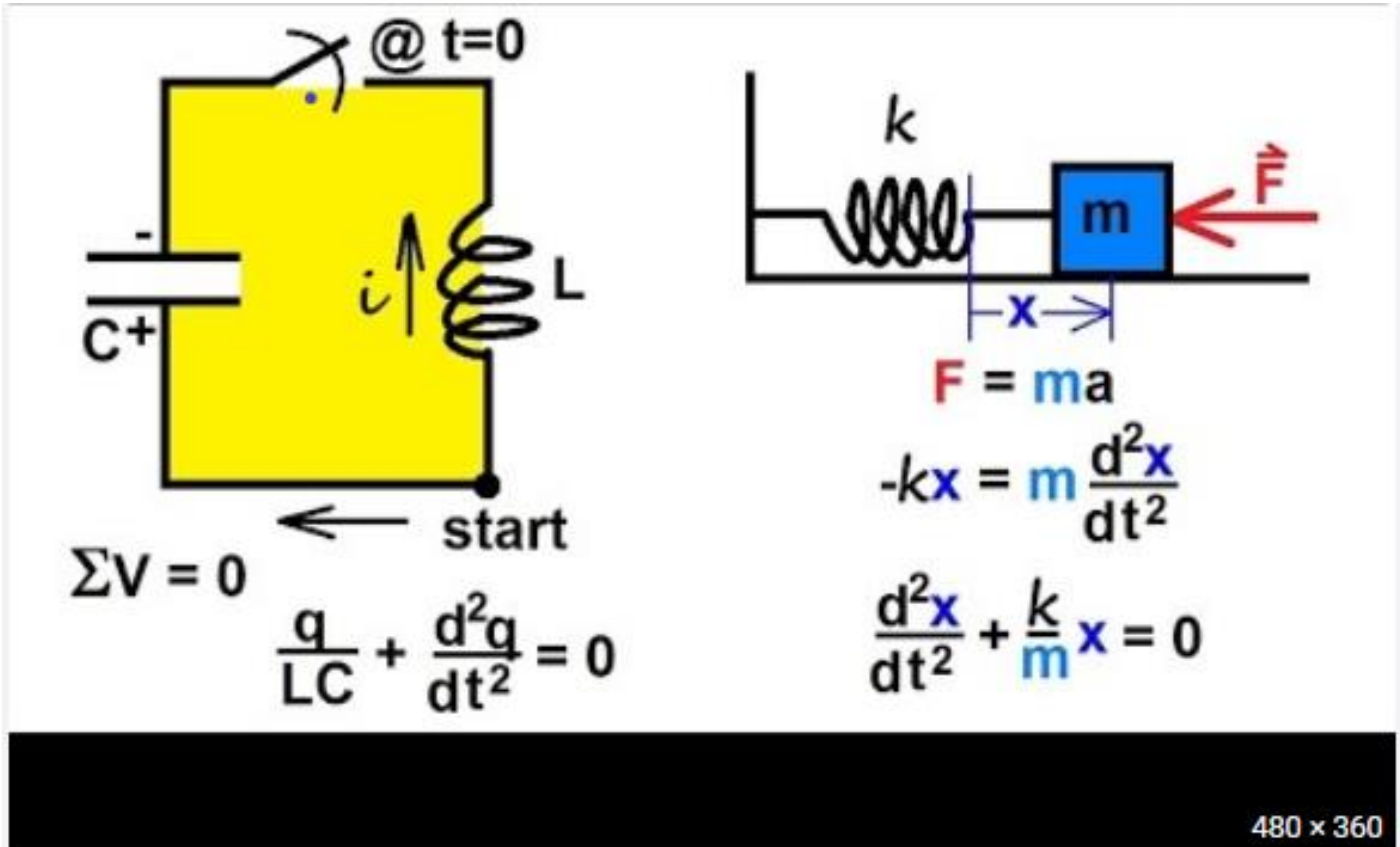
Application of Second Order ODEs



General Linear Ordinary Differential Equation

The general linear differential equation is given by:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x)$$

- Where $a_0(x)$, $a_1(x)$, $\cdots$, $a_n(x)$ and $f(x)$ are given functions of x or sometimes constants.

- Where $f(x) = 0$, It is said to be **Homogeneous**

- Where $f(x) \neq 0$, It is said to be **Inhomogeneous**

Second Order ODE

The general linear second order ODE is given by:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x)$$

- If $a_0(x)$, $a_1(x)$, $a_2(x)$ are constants, it is called a second order ODE with constant coefficients.

  - Where $f(x) = 0$, It is called a **Homogeneous Second order ODE with constant coefficients**

  - Where $f(x) \neq 0$, It is called a **Inhomogeneous Second order ODE with constant coefficients**

Homogeneous Second Order ODE with Constant Coefficients

The homogeneous second order ODE with constant coefficients is given by:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

The **complementary solution** $(\boldsymbol{y_c})$ is obtained by using the trial solution

$$y = e^{mx}$$

Allowing the trial solution satisfy the second order differential equation we have the following:

$$\frac{dy}{dx} = me^{mx} \qquad \frac{d^2y}{dx^2} = m^2e^{mx}$$

Auxillary / Characteristic Equation

Substituting in the second order ODE, we obtain

$$a_0 m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$$

$$e^{mx}(a_0 m^2 + a_1 m + a_2) = 0$$

Implies the following:

$$e^{mx} \neq 0 \qquad\qquad a_0 m^2 + a_1 m + a_2 = 0$$

- The resulting quadratic equation is known as the **Characteristic equation.**

- The form of the solution is dependent on the nature of solution of the characteristic equation.

Solution of Quadratic Equations

Given the quadratic equation

$$ax^2 + bx + c = 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The following are the types of roots and the condition where it occurs:

- Real and distinct Roots: $b^2 - 4ac > 0$
- Equal Roots: $b^2 - 4ac = 0$
- Complex Roots: $b^2 - 4ac < 0$

Solution of Second Order ODEs

Given the Characteristic equations

$$a_0m^2 + a_1m + a_2 = 0$$

- Obtain the roots and write the solution in the form:

- Real and distinct Roots $(m_1 \neq m_2)$

$$y = Ae^{m_1x} + Be^{m_2x}$$

- Equal Roots $(m_1 = m_2 = m)$

$$y = (A + Bx)e^{mx}$$

- Complex Roots $(m = \alpha \pm i\beta)$

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Example 1: Solve the ODE $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

$$y = e^{mx} \quad \Rightarrow \quad \frac{dy}{dx} = me^{mx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = m^2e^{mx}$$

Substitute in the ODE to obtain:

$$m^2e^{mx} + 3me^{mx} + 2e^{mx} = 0$$

$$e^{mx}(m^2 + 3m + 2) = 0$$

We have that $e^{mx} \neq 0$, therefore we obtain the characteristic equation

$$m^2 + 3m + 2 = 0$$

Solve by Factorization method

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$\Rightarrow m = -1 \text{ or } -2$$

The roots are real and distinct, hence the solution takes the form

$$y = Ae^{-x} + Be^{-2x}$$

Example 2: Solve the ODE $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

$$y = e^{mx} \quad \Rightarrow \quad \frac{dy}{dx} = me^{mx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = m^2e^{mx}$$

Substitute in the ODE to obtain:

$$m^2e^{mx} - 6me^{mx} + 9e^{mx} = 0$$

$$e^{mx}(m^2 - 6m + 9) = 0$$

We have that $e^{mx} \neq 0$, therefore we obtain the characteristic equation

$$m^2 - 6m + 9 = 0$$

Solve by Factorization method

$$m^2 - 6m + 9 = 0$$

$$(m - 3)^2 = 0$$

$$\Rightarrow m = 3 \text{ twice}$$

The roots are equal, hence the solution takes the form

$$y = (A + Bx)e^{3x}$$

Example 3: Solve the ODE $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

$$y = e^{mx} \quad \Rightarrow \quad \frac{dy}{dx} = me^{mx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = m^2e^{mx}$$

Substitute in the ODE to obtain:

$$m^2e^{mx} + 4me^{mx} + 5e^{mx} = 0$$

$$e^{mx}(m^2 + 4m + 5) = 0$$

We have that $e^{mx} \neq 0$, therefore we obtain the characteristic equation

$$m^2 + 4m + 5 = 0$$

Solve by Quadratic Formula

$$m = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4}{2} \pm \frac{\sqrt{16 - 20}}{2}$$

$$m = -2 \pm \frac{\sqrt{-4}}{2} = -2 \pm i\frac{\sqrt{4}}{2}$$

$$m = -2 \pm i$$

The roots are complex, hence the solution takes the form

$$y = e^{-2x}(A\cos x + B\sin x)$$

Solve the following Second Order ODEs

1. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

2. $\frac{d^2y}{dx^2} + 9y = 0$

3. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$

4. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$

5. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 37y = 0$

Inhomogeneous Second Order ODE with Constant Coefficients

The homogeneous second order ODE with constant coefficients is given by:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f(x).$$

The general solution:

$y =$ **complementary solution** + **particular solution**

- The **complementary solution** is obtained from the characteristics equation of homogeneous equation.

- The **particular solution** is determined by RHS $f(x)$

Particular Solution (Method of Undetermined Coefficients)

- We shall give the rules for obtaining the particular solution of the second order ODE using the method of undetermined coefficients.

- In the following slides, we give rules for the form of the particular solution for:

  - Exponential functions $\left(ae^{bx}\right)$
  - Trigonometric functions $(a\sin bx\ \&\ a\cos bx)$
  - Polynomials $\left(ax^{b}\right)$

Rule (Exponential): If $\boldsymbol{f(x) = ae^{bx}}$

- If $f(x) = ae^{bx}$, and if the characteristic equation has $m = b$ as a root that occurs k times, then the particular solution is of the form:

$$y_p = Ax^k e^{bx}$$

is used, where A is a constant to be determined different from A in the complementary solution.

Rule (Trigonometry): If $\boldsymbol{f(x) = a\cos bx\,/a\sin bx}$

- If $f(x) = a\sin bx$ or $a\cos bx$, and if $m^2 + b^2$ is a factor of the characteristic equation in which this factor may occur k times, then the particular solution is of the form:

$$y_p = x^k(A\sin bx + B\cos bx)$$

where constants a and b are to determined.

Rule (Polynomials): If $\boldsymbol{f(x) = ax^b}$

- If $f(x) = ax^b$, where a and b are constants, and if the characteristic equation has $m = 0$ as a root for which it occur k times, then the particular solution is of the form:

$$y_p = x^k\left(A_b x^b + A_{b-1}x^{b-1} + \cdots + A_1 x + A_0\right)$$

where $A_i, i = 1,2,\cdots,n$ are constants.

Example 1. Solve the ODE $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$

Characteristics equation: $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0 \Rightarrow m = 1$ or 2

Complementary solution $\Rightarrow y_c = Ae^x + Be^{2x}$

Particular solution with $f(x) = 1e^{2x}$

$m = 2$ appear $k = 1$ times

Therefore, the trial solution y_p is

$$y_p = Ax^1e^{2x} = Axe^{2x}$$

- We seek coefficient A such that y_p satisfy the ODE

$$y'_p = Ae^{2x} + 2Axe^{2x} \qquad y''_p = 4Ae^{2x} + 4Axe^{2x}$$

The ODE becomes

$$4Ae^{2x} + 4Axe^{2x} - 3(Ae^{2x} + 2Axe^{2x}) + 2(Axe^{2x}) = e^{2x}$$

Which simplifies to

$$Ae^{2x} = e^{2x} \Rightarrow A = 1$$

Therefore

$$y_p = xe^{2x}$$

$$y = y_c + y_p$$

Hence,

$$\boxed{y = Ae^{x} + Be^{2x} + xe^{2x}}$$

Example 2. Solve the ODE $\frac{d^2y}{dx^2} + 4y = 3\sin 2x$

Characteristics equation: $m^2 + 4 = 0 \quad \Rightarrow m = \pm 2i$

Complex root $(\alpha \pm i\beta) \Rightarrow e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Complementary solution $\Rightarrow y_c = A\cos 2x + B\sin 2x$

Particular solution with $f(x) = 3\sin 2x$

$m^2 + 2^2$ appear $k = 1$ times

Therefore, the trial solution y_p is

$$y_p = x^1(A\sin 2x + B\cos 2x) = x(A\sin 2x + B\cos 2x)$$

- We seek coefficient A & B such that y_p satisfy the ODE

$$y'_p = x(2A\cos 2x - 2B\sin 2x) + (A\sin 2x + B\cos 2x)$$

$$y''_p = -4Ax\sin 2x - 4Bx\cos 2x + 4A\cos 2x - 4B\sin 2x$$

The ODE simplifies to

$$4A\cos 2x - 4B\sin 2x = 3\sin 2x$$

$$\Rightarrow 4A = 0 \qquad \Rightarrow -4B = 3$$

$$\Rightarrow A = 0 \qquad \Rightarrow B = -\frac{3}{4}$$

Therefore

$$y_p = x\left(0\sin 2x - \frac{3}{4}\cos 2x\right) = -\frac{3}{4}x\cos 2x$$

$$y = y_c + y_p$$

Hence,

$$\boxed{y = A\cos 2x + B\sin 2x - \frac{3}{4}x\cos 2x}$$

Example 3. Solve the ODE $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 9x^2 + 1$

Characteristics equation: $m^2 - 2m - 3 = 0$

$$(m+1)(m-3) = 0 \Rightarrow m = -1 \text{ or } 3$$

Complementary solution $\Rightarrow y_c = Ae^{-x} + Be^{3x}$

Particular solution with $f(x) = 9x^2 + 1$

$m = 0$ appear $k = 0$ times

Therefore, the trial solution y_p is

$$y_p = x^0(A_2x^2 + A_1x + A_0) = A_2x^2 + A_1x + A_0$$

- We seek A_0, A_1, & A_2 such that y_p satisfy the ODE

$$y'_p = 2A_2x + A_1 \qquad y''_p = 2A_2$$

The ODE simplifies to

$$[-3A_0 - 2A_1 + 2A_2] - [3A_1 + 4A_0]x - [3A_2]x^2 = 1 + 9x^2$$

$$\Rightarrow \begin{cases} -3A_0 - 2A_1 + 2A_2 = 1 \\ -3A_1 - 4A_0 = 0 \\ -3A_2 = 9 \end{cases} \qquad \Rightarrow \begin{cases} A_0 = -5 \\ A_1 = 4 \\ A_2 = -3 \end{cases}$$

Therefore

$$y_p = -3x^2 + 4x - 5$$

$$y = y_c + y_p$$

Hence,

$$\boxed{y = Ae^{-x} + Be^{3x} - 3x^2 + 4x - 5}$$

Assignment / Classwork

Solve the following ODE:

1. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{4x}$

2. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2$

3. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = \cos x$

4. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 5x + e^{-2x}$

5. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 4\sin x + 2x^2$