PHYSICS APPLIED

Heart pacemaker

RC Circuits—Resistor and Capacitor in Series

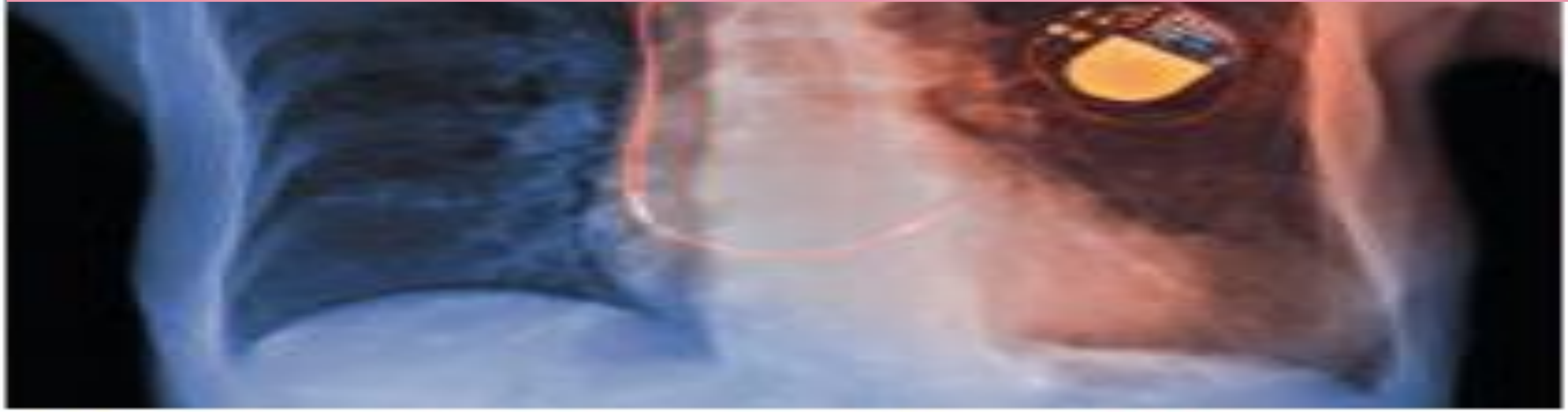
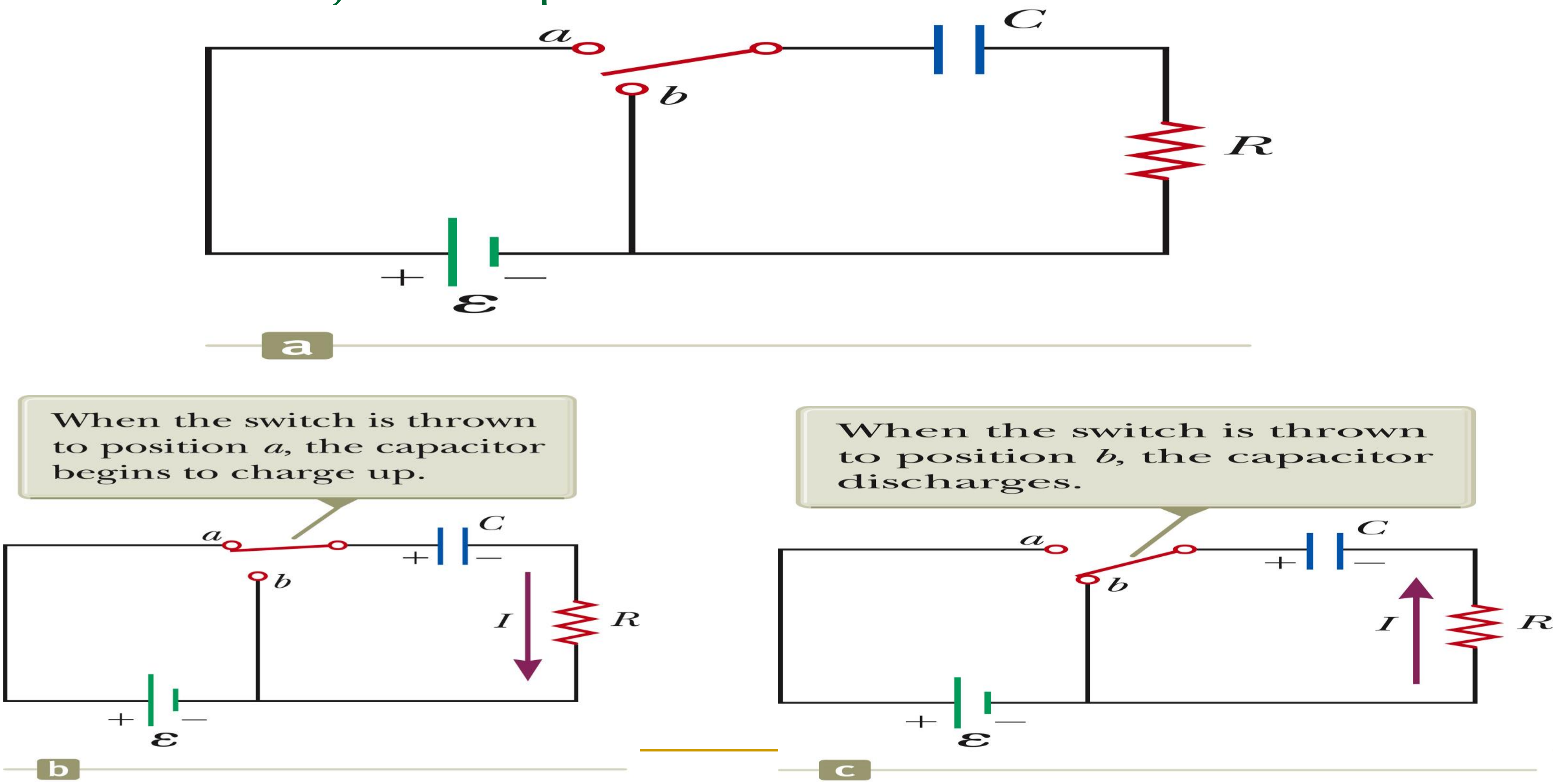


FIGURE 19–24 Electronic battery-powered pacemaker can be seen on the rib cage in this X-ray (color added).

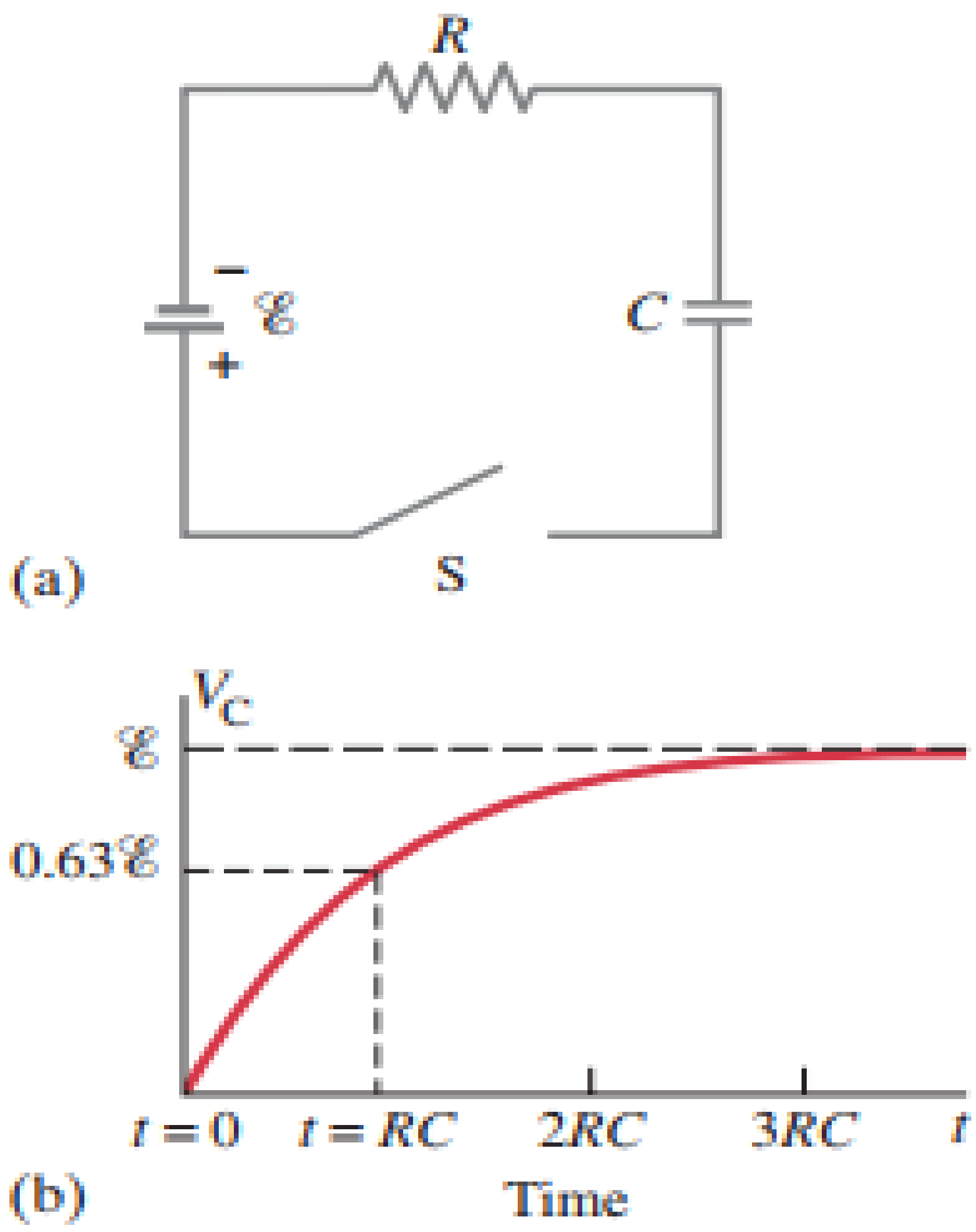
RC Circuits

- In direct current circuits containing capacitors, the current may vary with time.
  - The current is still in the same direction.
- An RC circuit will contain a series combination of a resistor and a capacitor.

RC Circuit, Example

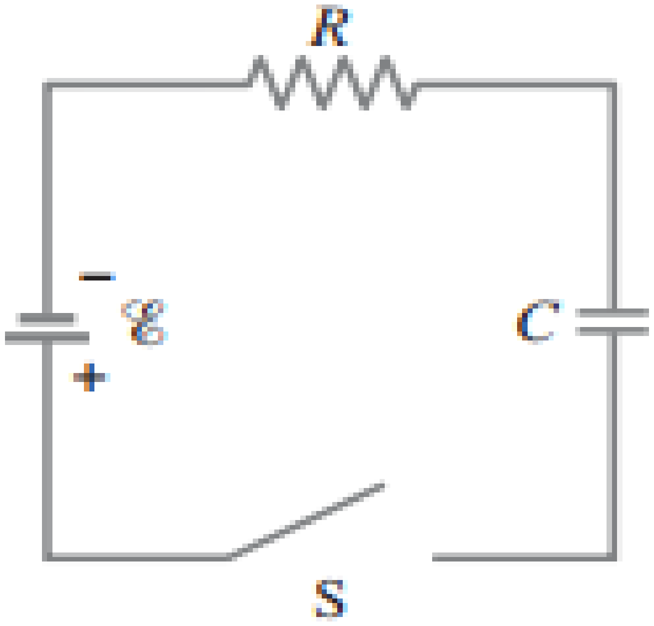


- Capacitors and resistors are often found together in a circuit. Such RC circuits are common in everyday life.
- They are *used to control the speed of a car's windshield wipers* and the *timing of traffic lights*; *they are used in camera flashes*, *in heart pacemakers, and in many other electronic devices.*
- In RC circuits, we are not so interested in the final "steady state" voltage and charge on the capacitor, but rather in how these variables change in time.

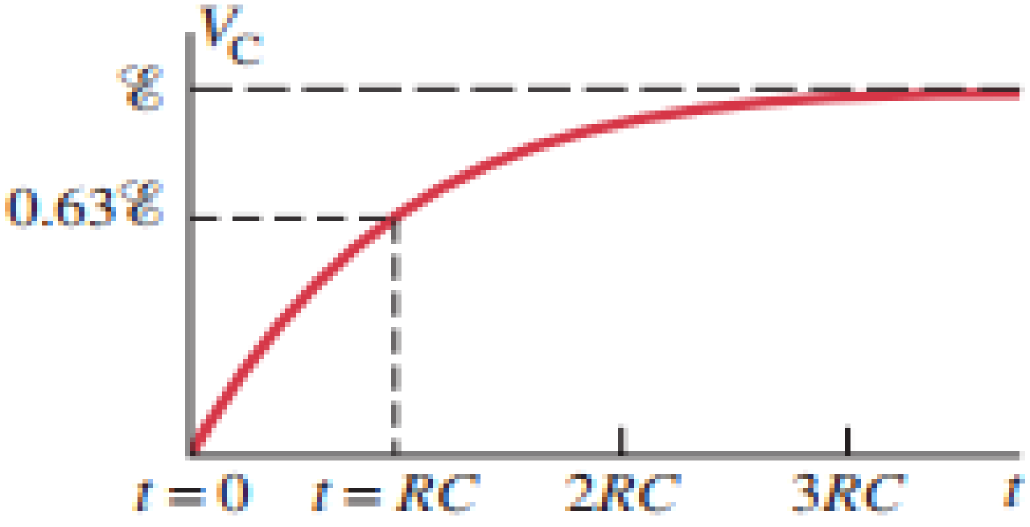


CAPACITOR CHARGING

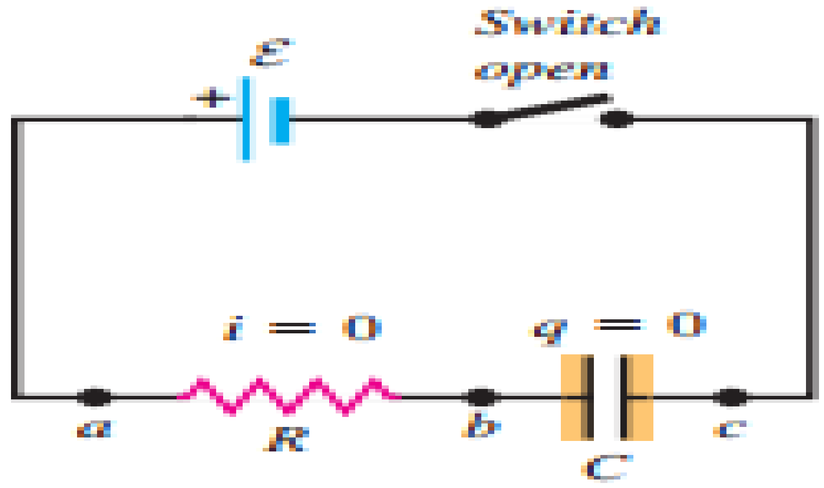
❖ The capacitor is initially uncharged; at some initial time t= 0, we close the switch, completing the circuit and permitting current around the loop to begin charging the capacitor.



(a)



(b)

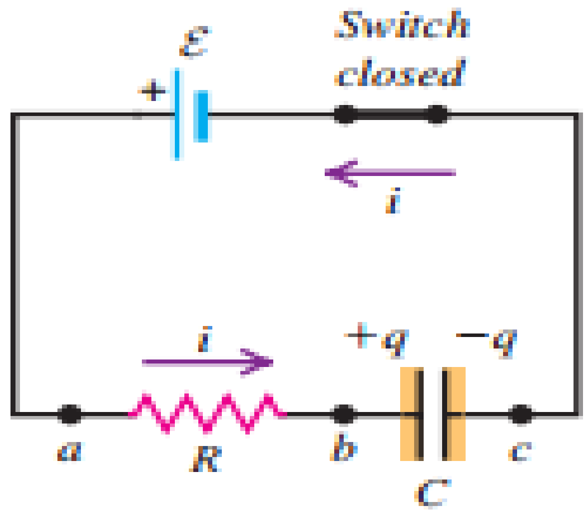


(a) Capacitor initially uncharged

CAPACITOR CHARGING

- When the switch S is closed, current immediately begins to flow through the circuit. Electrons will flow out from the negative terminal of the battery, through the resistor R, and accumulate on the upper plate of the capacitor.

- *We'll neglect the internal resistance of the battery, so its terminal voltage is constant and equal to the battery emf. The capacitor is initially uncharged; the E. potential difference across it is initially zero*



(b) Charging the capacitor

- *There is then no further current flow, and no potential difference across the resistor. The potential difference V_C across the capacitor, which is proportional to the charge on it ($V_C = Q/C$), thus increases in time. The shape of this curve is a type of exponential, and is given by the formula†*

- *As the capacitor charges, its voltage increases, so the potential difference across the resistor must decrease, corresponding to a decrease in current.* From Kirchhoff's loop rule, *the sum of these two is constant and equal to the battery emf E:*

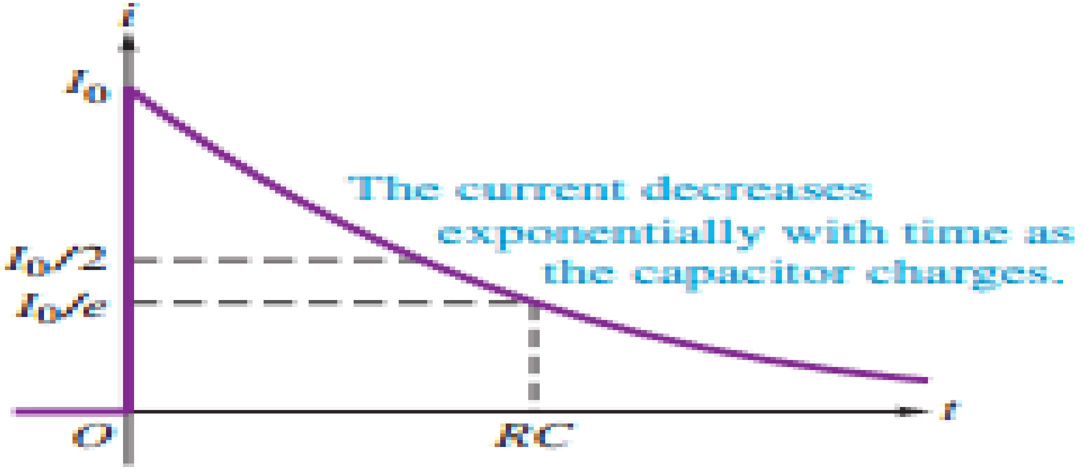
$$\mathcal{E} = iR + \frac{q}{C}.$$

We'll omit the details of these calculations; it turns out that if the switch is closed at time t = 0, the current i and charge q vary with time t according to the equations

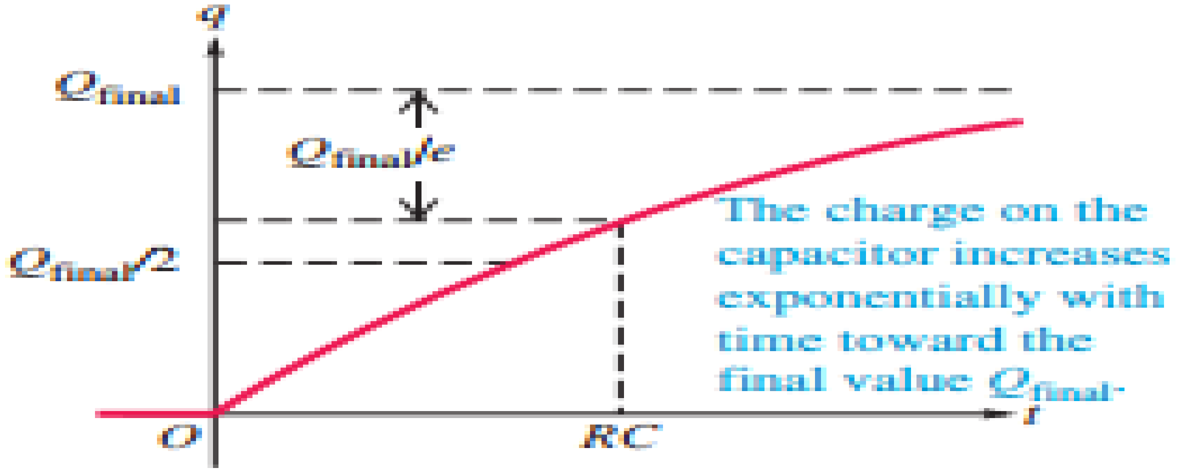
$$i = I_0 e^{-t/RC}, \qquad q = Q_{\text{final}}\left(1 - e^{-t/RC}\right).$$

- After a long time, the capacitor becomes fully charged,. The current decreases to zero, the potential difference across the resistor becomes zero, and the entire battery voltage appears across the capacitor. Thus, the capacitor charge and current vary with time as shown by the graphs

$$V_C = \mathcal{E}\left(1 - e^{-t/RC}\right),$$



(a) Graph of current versus time



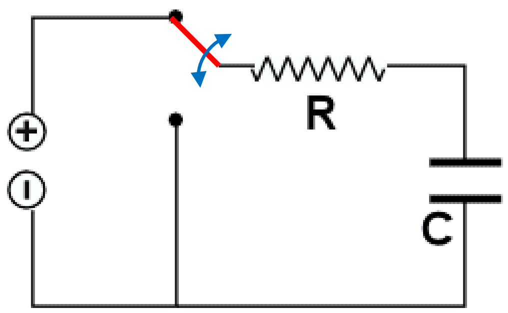
(b) Graph of capacitor charge versus time

▲ **FIGURE 19.33** Current i and capacitor charge q as functions of time for charging the capacitor in the circuit in Figure 19.32.

RC Circuit – Initial Conditions

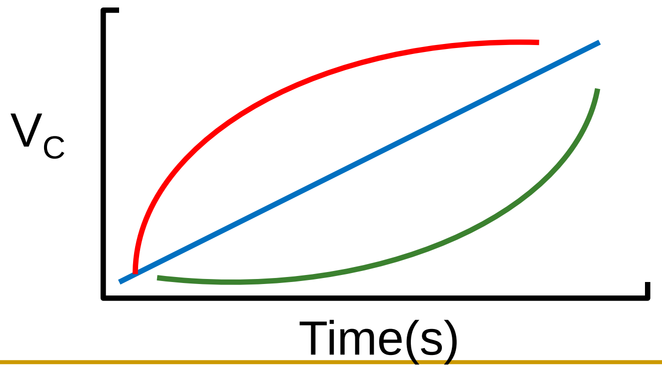
An RC circuit is one where you have a capacitor and resistor in the same circuit.

Suppose we have the following circuit:



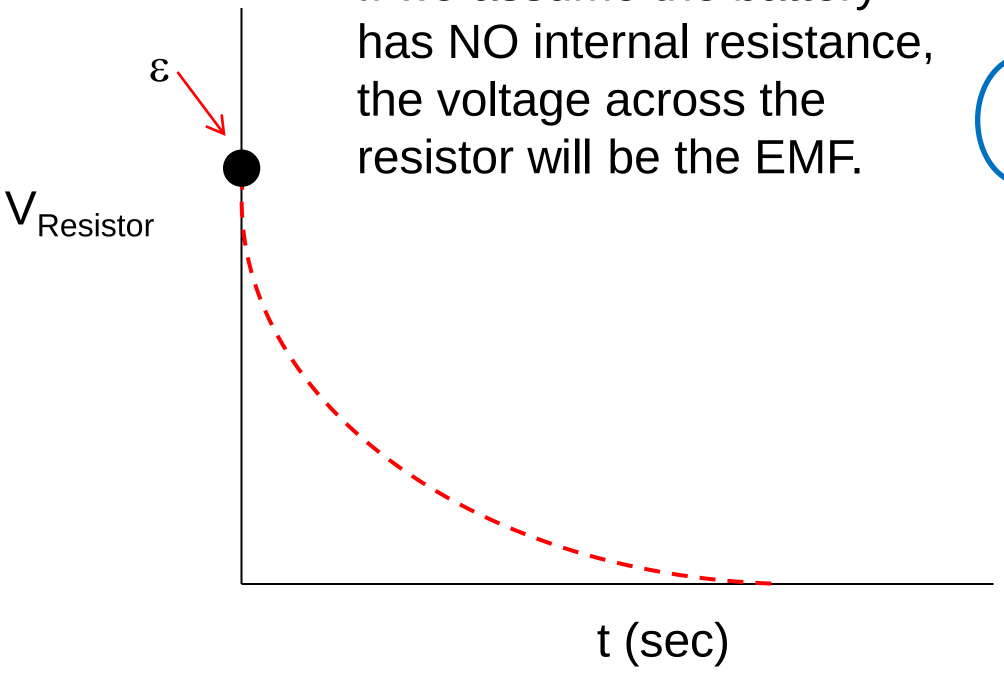
Initially, the capacitor is UNCHARGED (q = 0) and the current through the resistor is zero. A switch (in red) then closes the circuit by moving upwards.

The question is: ***What happens to the current and voltage across the resistor and capacitor as the capacitor begins to charge as a function of time?***

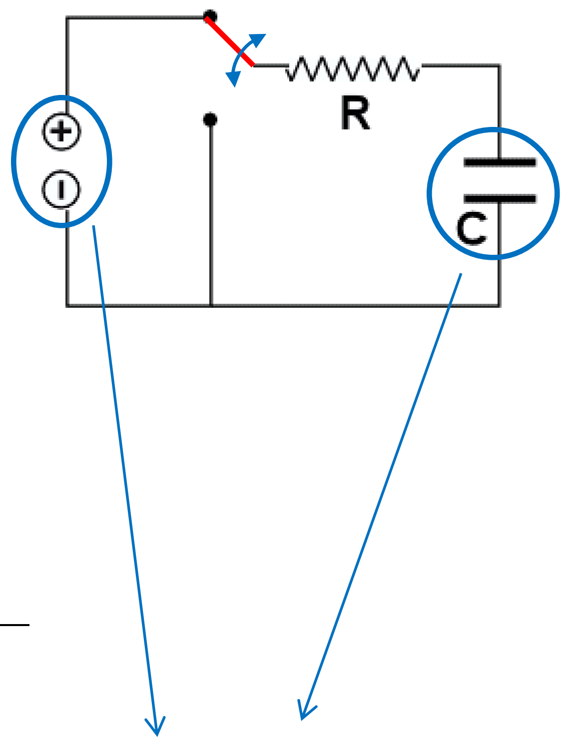


Which path do you think it takes?

Voltage Across the Resistor - Initially



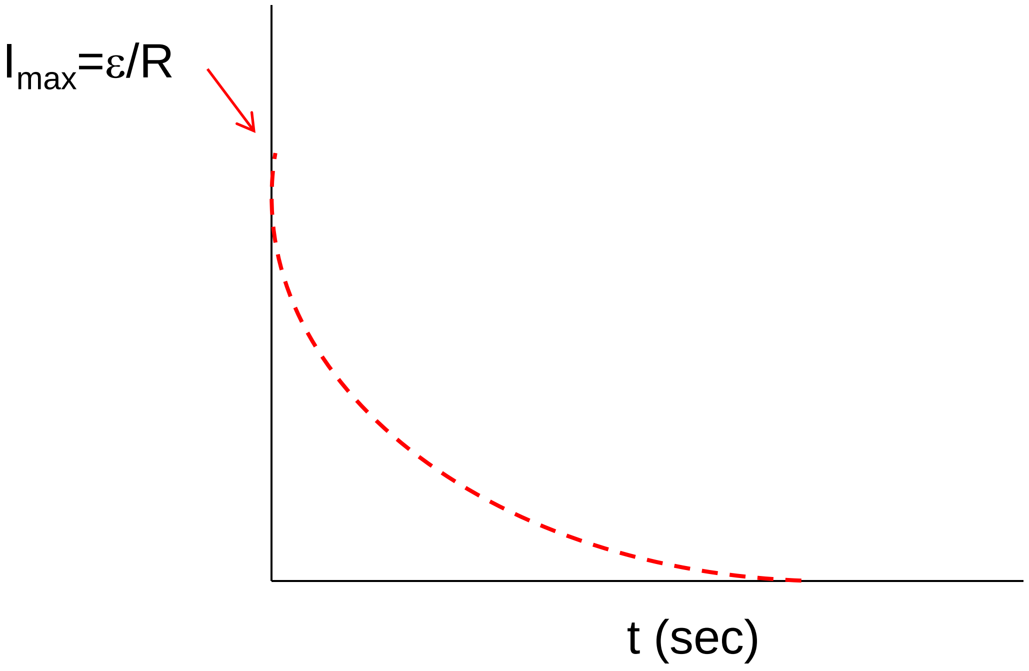
If we assume the battery has NO internal resistance, the voltage across the resistor will be the EMF.

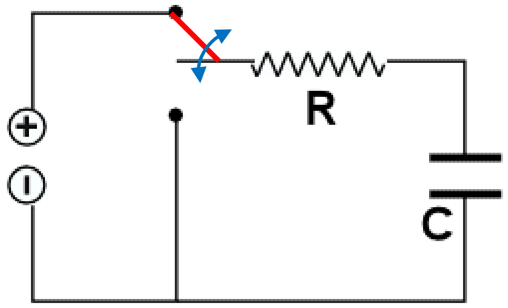


After a very long time, $V_{cap} = \varepsilon$, as a result the potential difference between these two points will be ZERO. Therefore, there will be NO voltage drop across the resistor after the capacitor charges.

Note: This is while the capacitor is CHARGING.

Current Across the Resistor - Initially

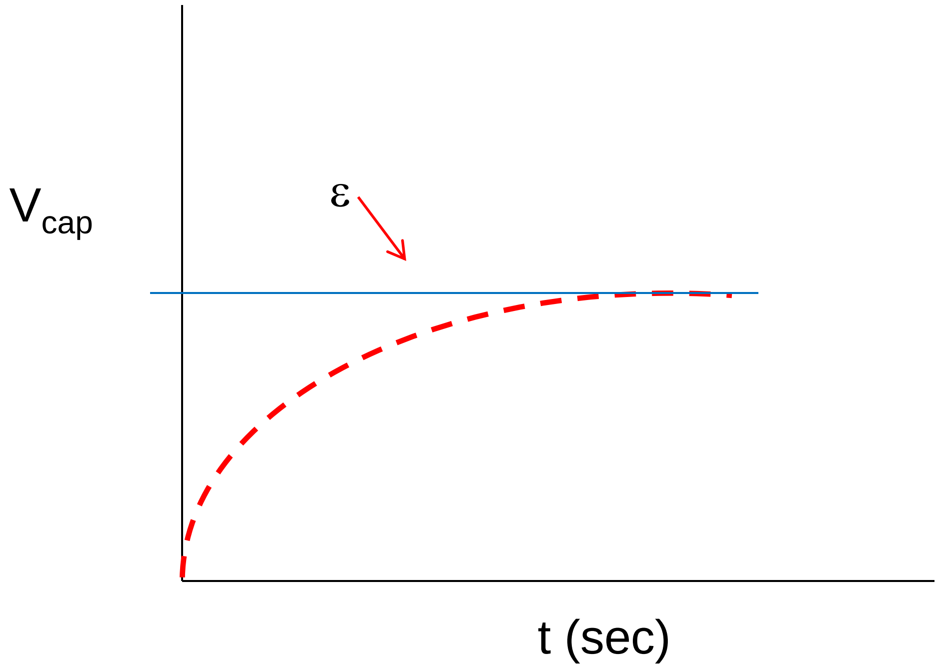


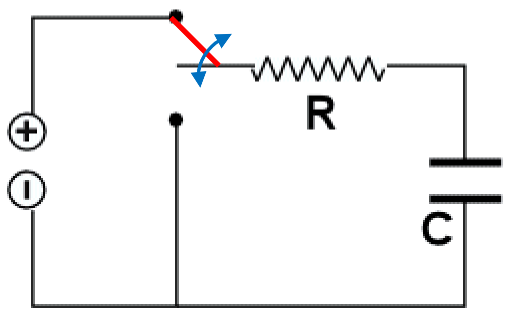


Since the voltage drop across the resistor decreases as the capacitor charges, the current across the resistor will reach ZERO after a very long time.

Note: This is while the capacitor is CHARGING.

Voltage Across the Capacitor - Initially

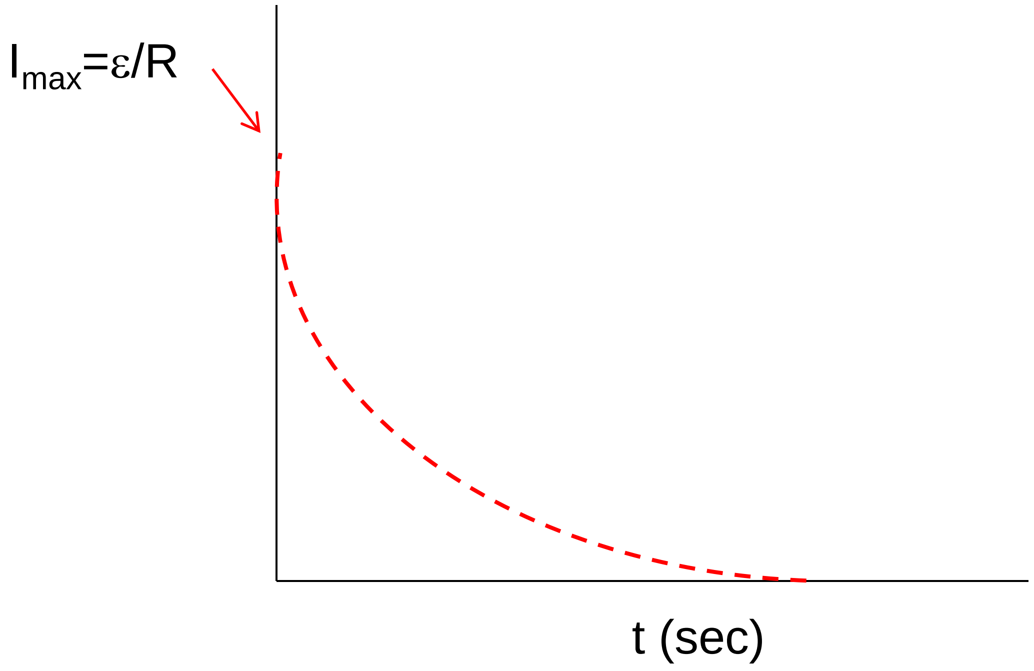


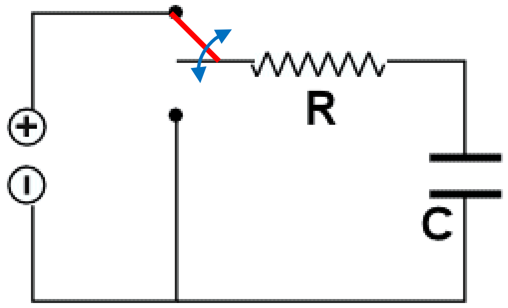


As the capacitor charges it eventually reaches the same voltage as the battery or the EMF in this case after a very long time. This increase DOES NOT happen linearly.

Note: This is while the capacitor is CHARGING.

Current Across the Capacitor - Initially





Since the capacitor is in SERIES with the resistor the current will decrease as the potential difference between it and the battery approaches zero. It is the potential difference which drives the value for the current.

Note: This is while the capacitor is CHARGING.

Time Domain Behavior

The graphs we have just seen show us that this process depends on the time. Let's look then at the **UNITS** of both the resistance and capacitance.

Unit for Resistance = Ω = Volts/Amps Unit for Capacitance = Farad = Coulombs/Volts

$$R\,x\,C \rightarrow \frac{Volts}{Amps}\,x\,\frac{Coulombs}{Volts} = \frac{Coulombs}{Amps}$$

$$1\,Amp = \frac{Coulomb}{Sec}$$

$$R\,x\,C = \frac{Coulombs}{Coulombs/Seconds} = SECONDS!$$

The "Time" Constant

It is clear, that for a GIVEN value of "**C", for any value of "R"** it effects the time rate at which the capacitor charges or discharges.

Thus the PRODUCT of R and C produce what is called the CIRCUIT Capacitive TIME CONSTANT.

We use the Greek letter, Tau, for this time constant.

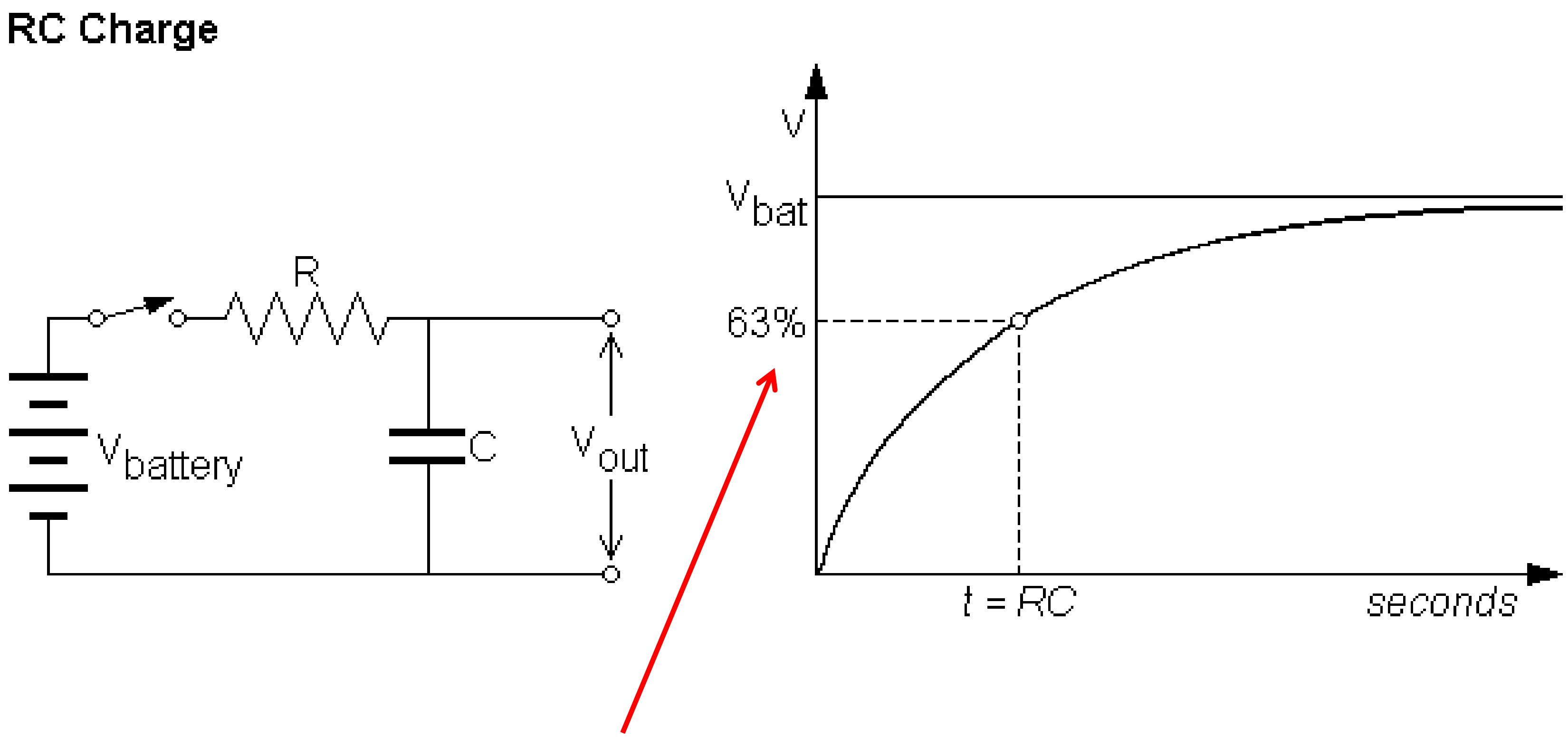
The question is: What exactly is the time constant?

$$\tau = RC = \text{time constant}$$

$$\tau = \Omega \bullet \mathrm{F}$$

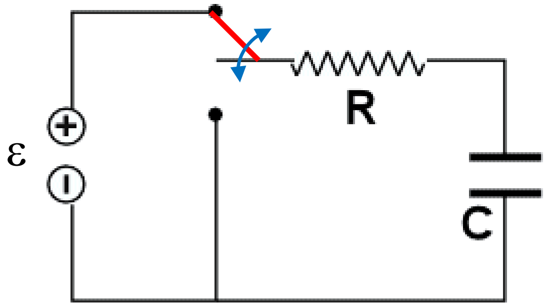
$$\tau = \frac{\mathrm{V}}{\mathrm{A}} \bullet \frac{C}{V} = \frac{C}{A} = \frac{C}{\frac{C}{s}} = \text{seconds}$$

The "Time" Constant



The **time constant** is the time that it takes for the capacitor to reach 63% of the EMF value during charging.

Charging Behavior



Is there a function that will allow us to calculate the voltage at any given time “t”?

Let’s begin by using KVL

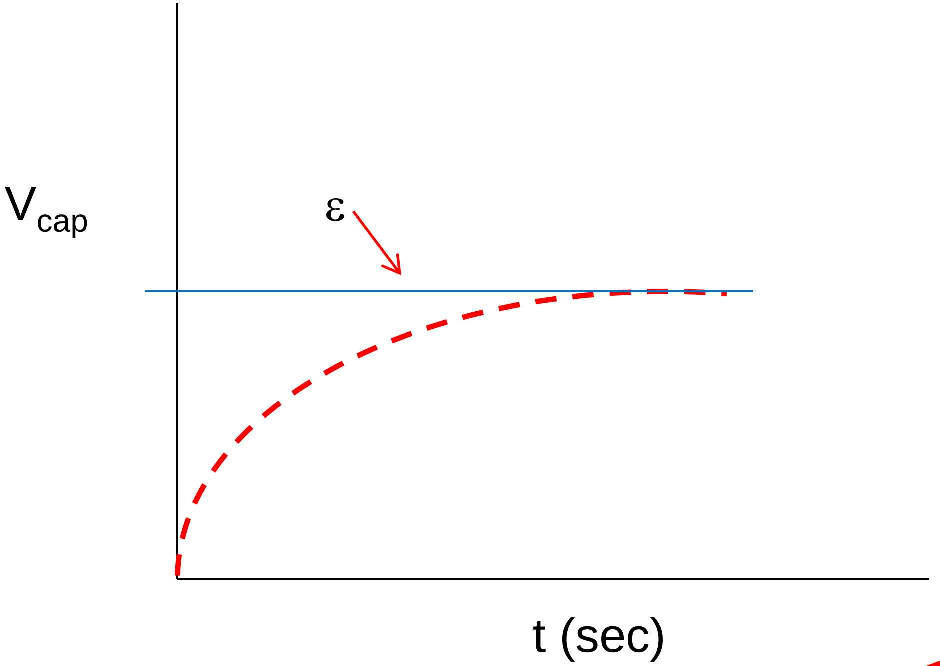
$$\varepsilon - IR - V_{capac} = 0$$

$$\varepsilon - IR - \frac{q}{C} = 0$$

$$\varepsilon - I(t)R - \frac{q(t)}{C} = 0,\ I = \frac{dq}{dt}$$

$$\varepsilon - \frac{dqR}{dt} - \frac{q(t)}{C} = 0$$

$$\frac{\varepsilon}{R} - \frac{dq}{dt} - \frac{q(t)}{RC} = 0$$



We now have a first order differential equation.

Charging function

$$\frac{\varepsilon}{R} - \frac{dq}{dt} - \frac{q(t)}{RC} = 0$$

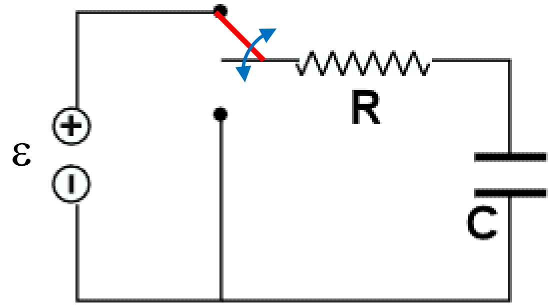
$$\frac{\varepsilon}{R} - \frac{q}{RC} = \frac{dq}{dt}$$

$$\frac{C\varepsilon - q}{RC} = \frac{dq}{dt}$$

$$-\frac{1}{RC}(q - C\varepsilon) = \frac{dq}{dt}$$

$$-\frac{dt}{RC} = \frac{dq}{q - C\varepsilon}$$

$$\int_0^t -\frac{dt}{RC} = \int_0^q \frac{dq}{q - C\varepsilon}$$

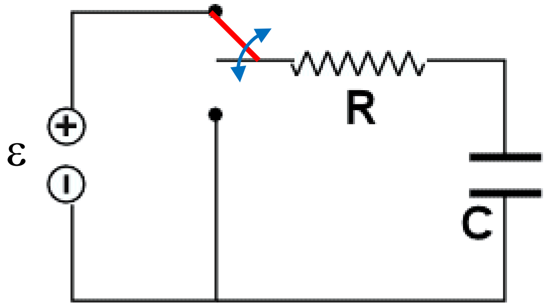


How do we solve this when we have 2 changing variables?

To get rid of the differential we must integrate. To make it easier we must get our two changing variables on different sides of the equation and integrate each side respectively.

•Re-arranging algebraically.
•Getting the common denominator
•Separating the numerator from the denominator,
•Cross multiplying.
•Since both changing variables are on opposite side we can now integrate.

Charging function



$$\int_{q=0}^{q} \frac{dq}{q - C\varepsilon} = -\frac{1}{RC}\int_{t=0}^{t} dt$$

$$\ln\left(\frac{C\varepsilon - q}{C\varepsilon}\right) = -\frac{t}{RC}$$

$$\frac{q - C\varepsilon}{-C\varepsilon} = e^{-t/RC}$$

$$q - C\varepsilon = -C\varepsilon\, e^{-t/RC}$$

$$q(t) = C\varepsilon - C\varepsilon\, e^{-t/RC} = C\varepsilon(1 - e^{-t/RC})$$

As it turns out we have derived a function that defines the CHARGE as a function of time.

However if we divide our function by a CONSTANT, in this case "C", we get our voltage function.

$$\frac{q(t)}{C} = \frac{C\varepsilon(1 - e^{-t/RC})}{C}$$

$$\boxed{V(t) = \varepsilon(1 - e^{-t/RC})}$$

Let’s test our function

$$V(t) = \varepsilon(1 - e^{-t/RC})$$

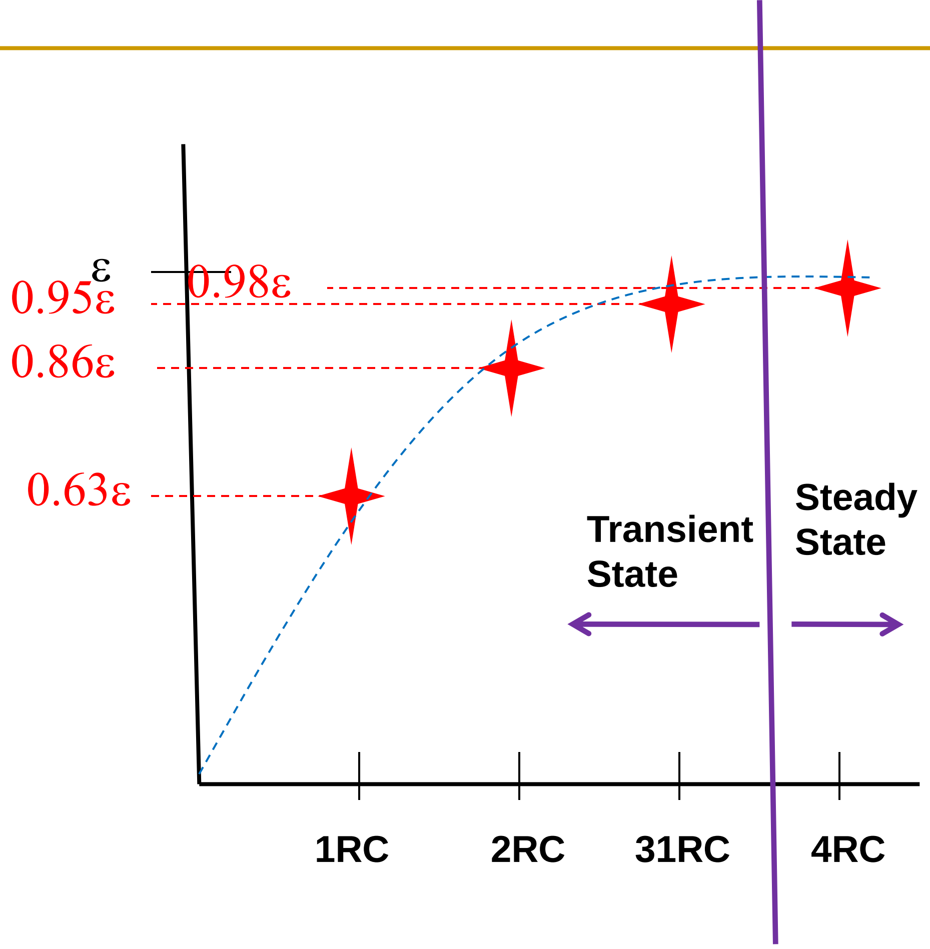
$$V(1RC) = \varepsilon(1 - e^{-RC/RC})$$

$$V(1RC) = \varepsilon(1 - e^{-1}) = \textbf{0.63}\varepsilon$$

$$V(2RC) = \textbf{0.86}\varepsilon$$

$$V(3RC) = \textbf{0.95}\varepsilon$$

$$V(4RC) = \textbf{0.98}\varepsilon$$



Applying each time constant produces the charging curve we see. For practical purposes the capacitor is considered fully charged after 4-5 time constants(steady state). Before that time, it is in a transient state.

Charging Functions

$$q(t) = C\varepsilon(1 - e^{-t/RC})$$

$$V(t) = \varepsilon(1 - e^{-t/RC}) \rightarrow \frac{V(t)}{R} = \frac{\varepsilon(1 - e^{-t/RC})}{R}$$

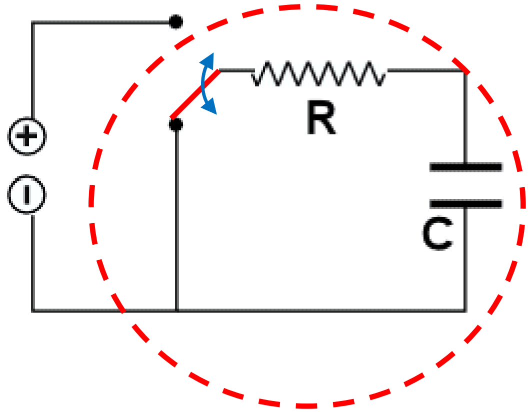
$$\boxed{I(t) = I_o(1 - e^{-t/RC})}$$

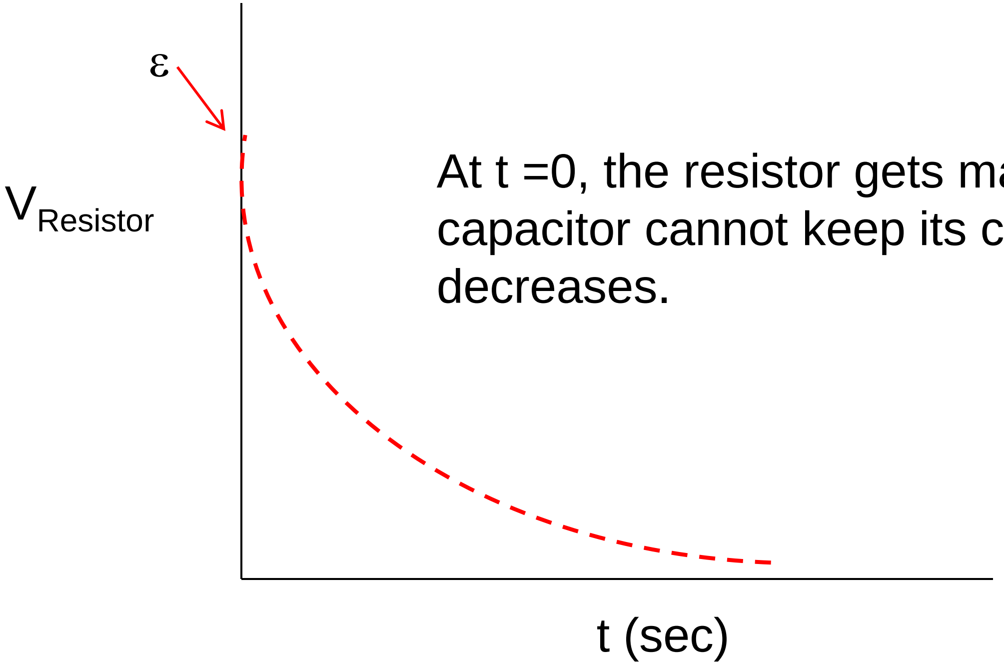
Likewise, the voltage function can be divided by another constant, in this case, “R”, to derive the current charging function.

Now we have 3 functions that allows us to calculate the Charge, Voltage, or Current at any given time “t” while the capacitor is charging.

Capacitor Discharge – Resistor's Voltage

Suppose now the switch moves downwards towards the other terminal. This prevents the original EMF source to be a part of the circuit.

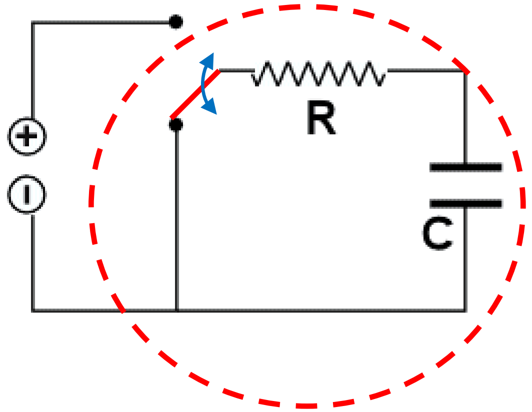


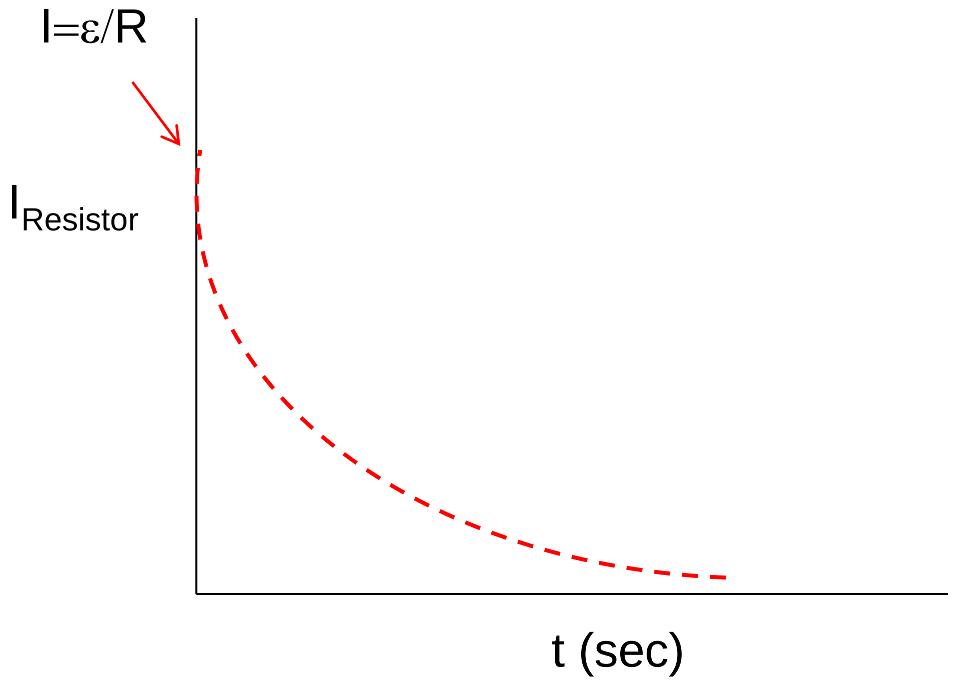


At t =0, the resistor gets maximum voltage but as the capacitor cannot keep its charge, the voltage drop decreases.

Capacitor Discharge – Resistor's Current

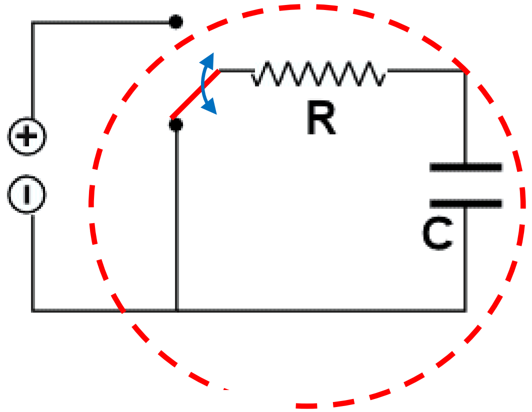
Similar to its charging graph, the current through the resistor must decrease as the voltage drop decreases due to the loss of charge on the capacitor.

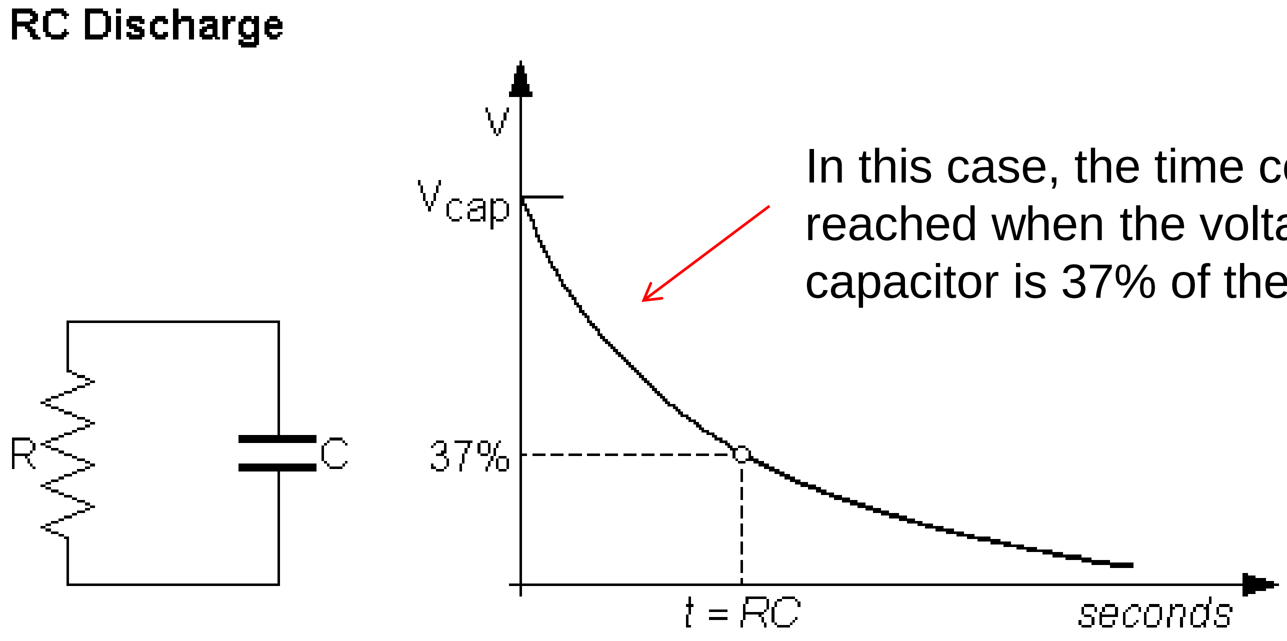




Capacitor Discharge – Capacitor's Voltage

The discharging graph for the capacitor is the same as that of the resistor. There WILL be a time delay due to the TIME CONSTANT of the circuit.

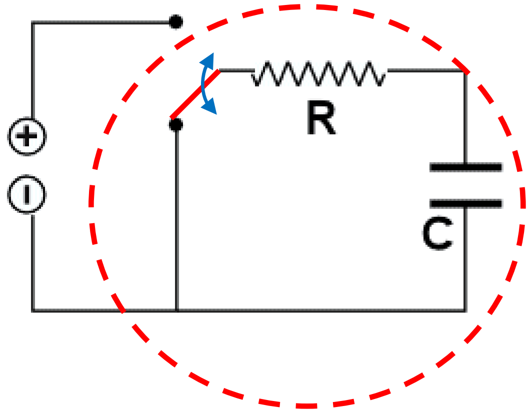


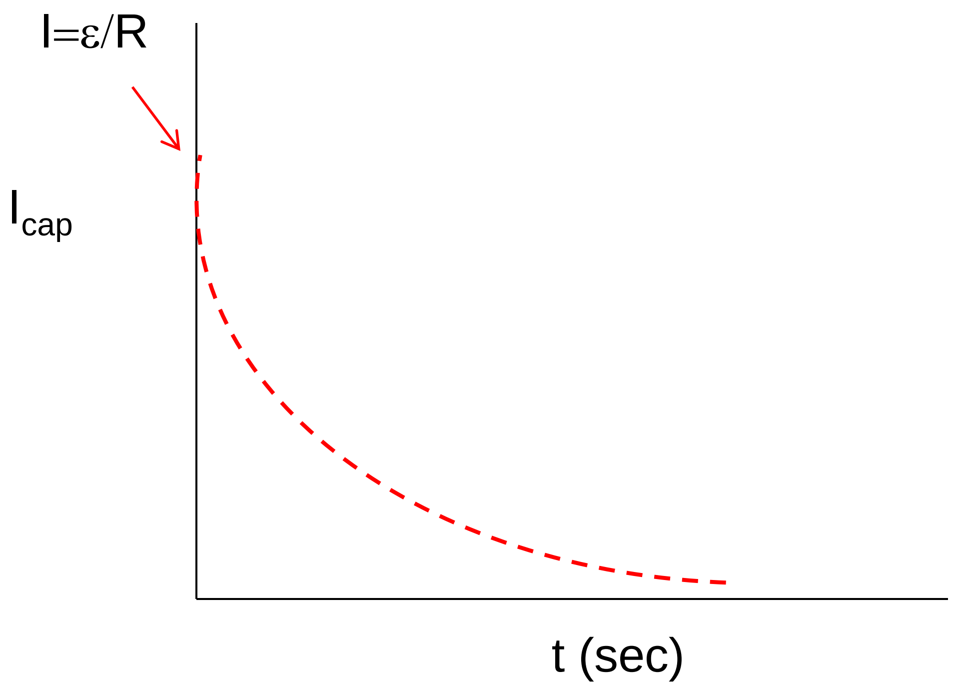


In this case, the time constant is reached when the voltage of the capacitor is 37% of the EMF.

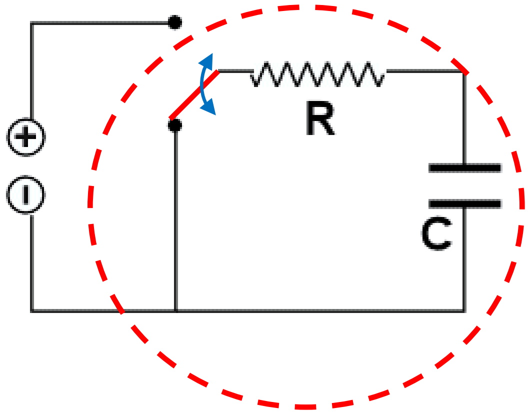
Capacitor Discharge – Capacitor’s Current

Similar to its charging graph, the current through the capacitor must decrease as the voltage drop decreases due to the loss of charge on the capacitor.





Discharging Functions



$$0 - IR - V_{cap} = 0$$

$$-\frac{dq}{dt}R - \frac{q}{C} = 0$$

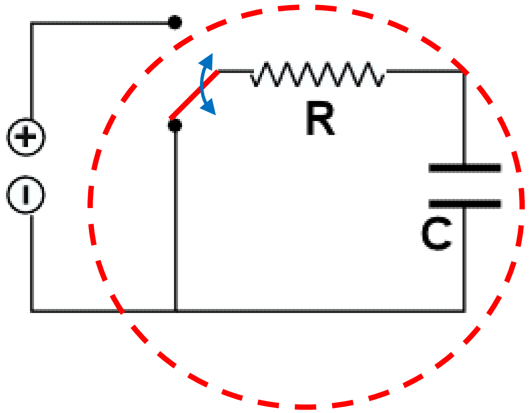
$$\frac{dq}{dt}R = -\frac{q}{C} \rightarrow \frac{dq}{dt} = -\frac{q}{RC}$$

$$\int_{q_o}^{q} \frac{1}{q}\,dq = -\frac{1}{RC}\int_{t=0}^{t} dt$$

Once again we start with KVL, however, the reason we start with ZERO is because the SOURCE is now gone from the circuit.

Discharging Functions

$$\int_{q_o}^{q} \frac{1}{q}\,dq = -\frac{1}{RC}\int_{t=0}^{t} dt \rightarrow \qquad \ln\left(\frac{q}{q_o}\right) = -\frac{t}{RC}$$



$$\frac{q}{q_o} = e^{-\frac{t}{RC}} \rightarrow \qquad q(t) = q_o e^{-\frac{t}{RC}}$$

Dividing by "C" then "R"

$$\boxed{\begin{array}{l} V(t) = \varepsilon_o e^{-\frac{t}{RC}} \\ \\ I(t) = I_o e^{-\frac{t}{RC}} \end{array}}$$

We now can calculate the charge, current, or voltage for any time "t" during the capacitors discharge.

SUMMARY: Charging a Capacitor

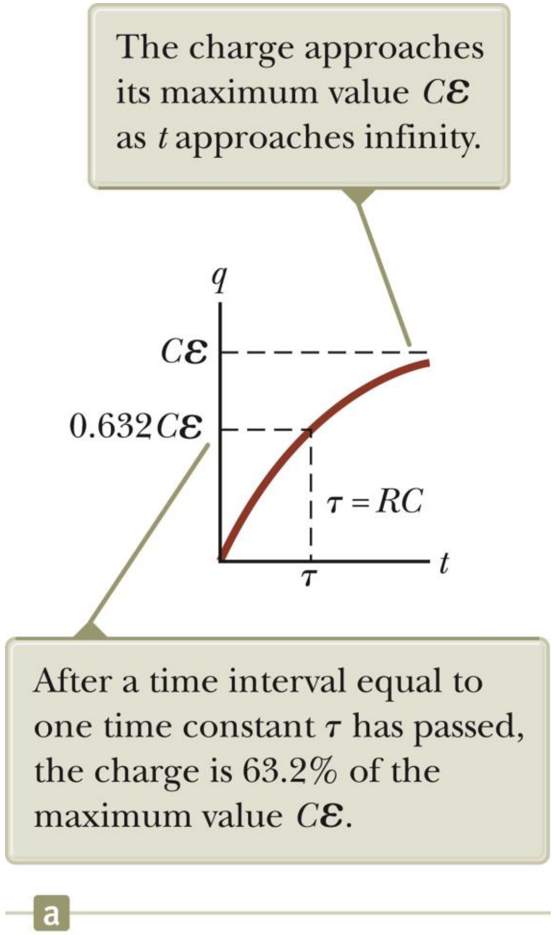
- When the circuit is completed, the capacitor starts to charge.
- The capacitor continues to charge until it reaches its maximum charge ($Q = C\varepsilon$).
- Once the capacitor is fully charged, the current in the circuit is zero.
- As the plates are being charged, the potential difference across the capacitor increases.
- At the instant the switch is closed, the charge on the capacitor is zero.
- Once the maximum charge is reached, the current in the circuit is zero.
  - The potential difference across the capacitor matches that supplied by the battery.

Charging a Capacitor in an RC Circuit

- The charge on the capacitor varies with time.
  - $q(t) = C\mathcal{E}(1 - e^{-t/RC})$
    $= Q(1 - e^{-t/RC})$
- The current can be found

$$I(t) = \frac{\varepsilon}{R} e^{-t/RC}$$

  - τ is the *time constant*
    - $\tau = RC$

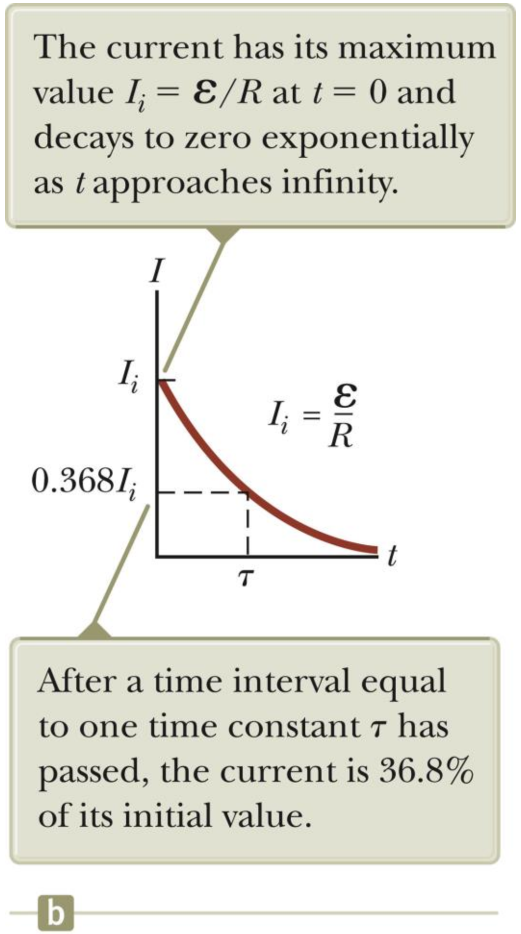


Time Constant, Charging

- The time constant represents the time required for the charge to increase from zero to 63.2% of its maximum.
- τ has units of time
- The energy stored in the charged capacitor is ½ $Q\varepsilon$ = ½ $C\varepsilon^2$.

Discharging a Capacitor in an RC Circuit

- When a charged capacitor is placed in the circuit, it can be discharged.
  - $q(t) = Qe^{-t/RC}$
- The charge decreases exponentially.



Discharging Capacitor

- At $t = \tau = RC$, the charge decreases to 0.368 Q_{max}
  - In other words, in one time constant, the capacitor loses 63.2% of its initial charge.
- The current can be found

$$I(t) = \frac{dq}{dt} = -\frac{Q}{RC} e^{-t/RC}$$

- Both charge and current decay exponentially at a rate characterized by $\tau = RC$.