Elementary Mathematics II (Differential Equations and Dynamics) (MTH 102)

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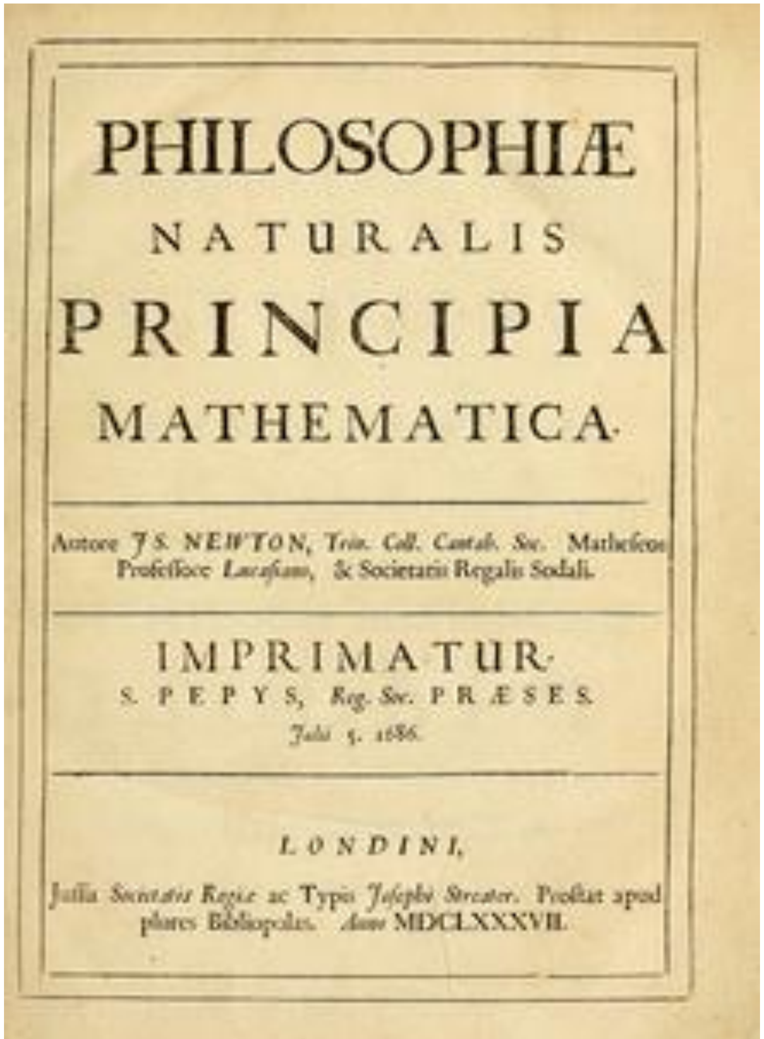
Newton’s Law of motion, Newton 1687


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore JS. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

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Newton’s Law of Motion
Laws of Universal Gravitation

Classical Mechanics
Kepler’s laws of planetary motion

Calculus
Geometry

Newton’s Law of Motion

1. A body remains at rest or in a state of uniform motion (non-accelerating) uncles acted on by an external force

2. Force= time rate of change of momentum. i.e.

$$F = \frac{dp}{dt} = m\frac{dv}{dt} = ma$$

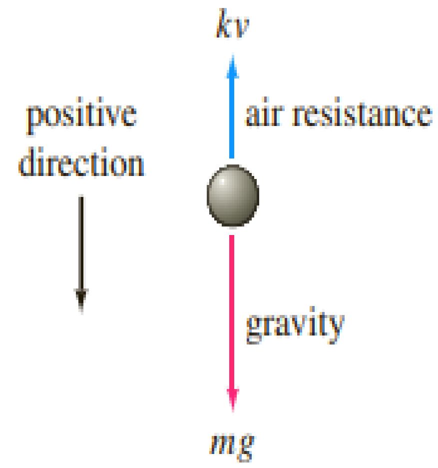
3. To every force (action) there is an equal but opposite reaction

Free Falling Problems ($a = -g$)

- Consider a vertically falling body of mass m that is being influenced only by gravity g and an air resistance that is proportional to the velocity of the body. Assume that both gravity and mass remain constant

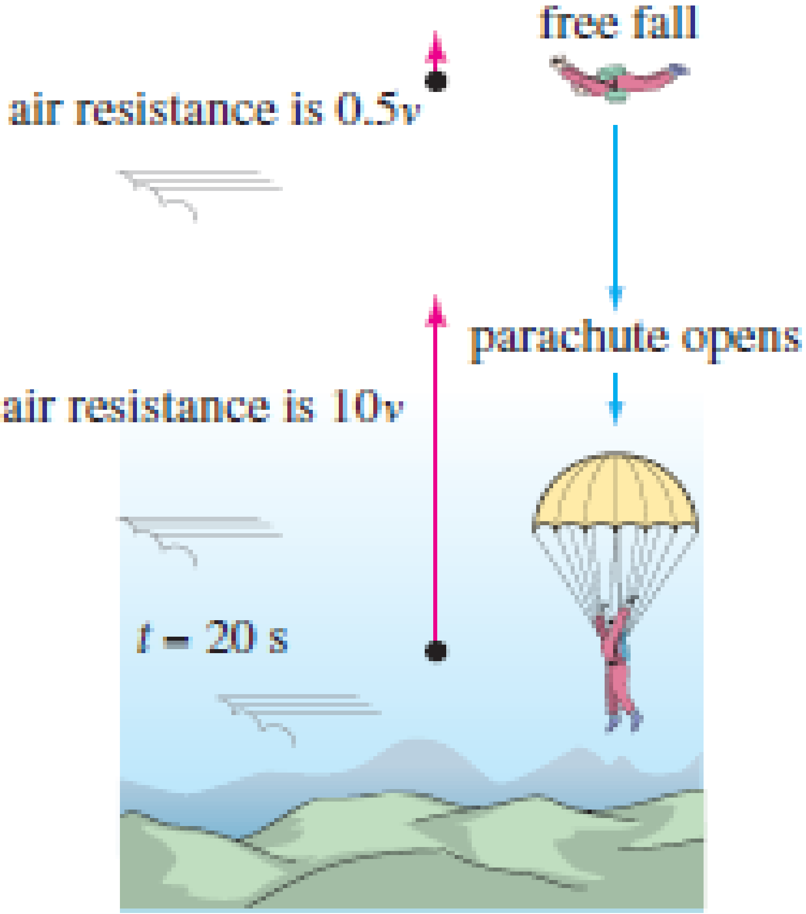
- We apply Newton's second law

$$F = ma = -mg$$



- Drag Force (Resistive force)

Skydiving



free fall
air resistance is 0.5v
parachute opens
air resistance is 10v
t = 20 s

Drag force $\propto$ Velocity

$$m\frac{dv}{dt} = -mg - kv$$

- k depends on the properties of the free falling object
- If the object is moving upwards then the drag force is pointing downwards $\therefore v > 0$
- If the object is moving downwards then the drag force is pointing downwards $\therefore v < 0$

$$\frac{dv}{dt} + \frac{k}{m}v = -g$$

Linear Equation with I.F.

$$\mu = e^{\int \frac{k}{m} dt} = e^{\frac{k}{m}t}$$

$$\therefore \mu \cdot v = \int \mu \cdot q \, dt$$

$$\therefore v e^{\frac{k}{m}t} = -g \int e^{\frac{k}{m}t} dt = -\frac{mg}{k} e^{\frac{k}{m}t} + c$$

$$\therefore v(t) = -\frac{mg}{k} + c e^{-\frac{k}{m}t}$$

Suppose the initial velocity is v_0

Velocity formulation

At $t = 0$, $v = v_0$

$$\therefore v(0) = -\frac{mg}{k} + c = v_0 \Rightarrow c = v_0 + \frac{mg}{k}$$

Consequently, the velocity

$$\therefore v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t}$$

- The terminal velocity is the equilibrium (steady state) solution, when $\frac{dv}{dt} = 0$

$$\Rightarrow v(t) = -\frac{mg}{k}$$

Time to hit the ground

$$v + \frac{mg}{k} = \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t}$$

$$e^{-\frac{k}{m}t} = \frac{\left(v + \frac{mg}{k}\right)}{\left(v_0 + \frac{mg}{k}\right)} \Rightarrow e^{\frac{k}{m}t} = \frac{\left(v_0 + \frac{mg}{k}\right)}{\left(v + \frac{mg}{k}\right)}$$

$$\frac{k}{m}t = \ln\left(\frac{kv_0 + mg}{kv + mg}\right)$$

$$\Rightarrow t = \frac{m}{k}\ln\left(\frac{kv_0 + mg}{kv + mg}\right)$$

Example

An object of mass $5kg$ is released from rest $1000m$ above the ground and allowed to fall freely under gravity. Assume that the force due to air resistance is proportional to the velocity of the object with proportionality constant $k = 50Nm/s$. Determine the equation of motion of the object. When will the object strike the ground?

Parameters:

$v_0 = 0, m = 5, g = 9.8$, and $k = 50$

Equation of motion $r'(t) = v(t)$

$$r'(t) = v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t}$$

$$r'(t) = -0.981 + 0.981e^{-10t}$$

Integrating w.r.t. t

$$r(t) = -0.981t - 0.0981e^{-10t} + c$$

But $r(0) = 1000$ so that $c = 1000.0981$. Hence

$$r(t) = -0.981t - 0.0981e^{-10t} + 1000.0981$$

Time to hit the ground

Time to hit the ground will be, calculate T,

$$r(T) = 0$$

That is

$$r(T) = -0.981t + 0.0981e^{-10t} + 1000.0981 = 0$$

We have that

$$t = 1019.467s$$

Classwork / Assignment

A particle of mass m is thrown vertically upwards with initial velocity u, the particle experienced a force mg vertically downwards (the force due to gravity), where g is a constant given that v is the velocity involved in the displacement. Find the greatest height reached by the particle under a resistive force mkv^2.

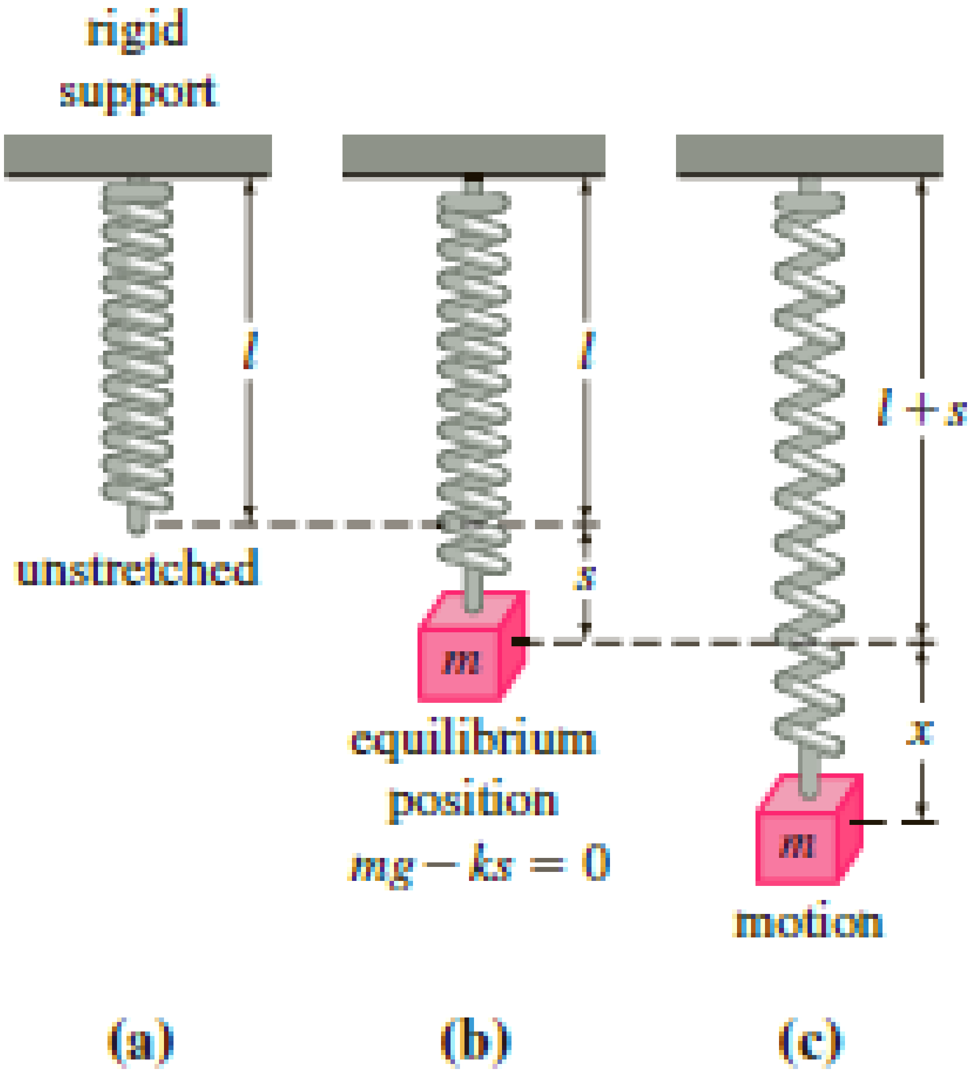
Second-order ODEs

In this section, we will consider a dynamical system represented by a second-order ODE with constant coefficients with initial conditions:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x), \qquad y(0) = y_0, y'(0) = y_1$$

- The function g is called the driving or forcing function or input of the system
- The solution $y(x)$ is called response or output of the system

Spring/Mass Systems: Free Undamped Motion



Hooke's law

The spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation s. Simply stated, $F = -ks$, where k is a constant of proportionality called the spring constant.

Newton's Law applied

The net or resultant force on a moving body of mass m is given by $\sum F_k = ma$, where $a = \frac{d^2y}{dx^2}$ is acceleration. Assume the mass vibrates free of all external forces - Free motion- then Newton's law gives

$$m\frac{d^2x}{dt^2} = -k(x+s) + mg = -kx + mg - ks = -kx$$

- $-k(x+s)$ is the restoring force
- mg is the weight of the mass

Differential Equation of Free Undamped Motion

Let $\omega^2 = \frac{k}{m}$, we have

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad x(0) = x_0, x'(0) = x'_0$$

- simple harmonic motion or free undamped motion
- The characteristic equation is given by

$$m^2 + \omega^2 = 0 \Rightarrow m = \pm i\omega$$

Therefore

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

General Solution

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

$$x'(t) = -c_1\omega \sin \omega t + c_2\omega \cos \omega t$$

Apply initial conditions: $\begin{cases} x(0) = c_1 = x_0 & \Rightarrow c_1 = x_0 \\ x'(0) = c_2\omega = x'_0 & \Rightarrow c_2 = \dfrac{x'_0}{\omega} \end{cases}$

Therefore, the general solution that describe the system Is given by:

$$x(t) = x_0 \cos \omega t + \frac{x'_0}{\omega} \sin \omega t$$

Remarks

- The period of motion is $T = \frac{2\pi}{\omega}$
- T is the time the mass makes one cycle of motion
- The frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$ is the number of cycles completed each seconds
- $\omega = \sqrt{\frac{k}{m}}$ measured in radians per second is called the circular (angular) frequency
- $x(t)$ after obtaining constants c_1 and c_2 is the equation of motion

Example

A mass weighing $2lb$ stretches a spring $6inches$.
At $t = 0$ the mass is released from a point 8 $inches$
Below the equilibrium position with an upward velocity
of $\frac{4}{3}ft/s$. Determine the equation of motion. ($g = 32ft/s^2$)

Remarks

Displacement : $6inches = \frac{1}{2}ft$; $8inches = \frac{2}{3}ft$

Velocity (v) : $\frac{4}{3}ft/s$

Mass (m) : $m = \frac{W}{g} = \frac{2}{32} = \frac{1}{16}$

Hooke's law (k): $2 = k\left(\frac{1}{2}\right) \Rightarrow k = 4$

$$\frac{1}{16}\frac{d^2x}{dt^2} = -4x \Rightarrow \frac{d^2x}{dt^2} + 8^2 x = 0, \quad x(0) = \frac{2}{3}, x'(0) = -\frac{4}{3}$$

The complementary solution is given by

$$x(t) = c_1 \cos 8t + c_2 \sin 8t$$

$$x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$$

Apply initial conditions: $\begin{cases} x(0) = c_1 = \dfrac{2}{3} & \Rightarrow c_1 = \dfrac{2}{3} \\ x'(0) = 8c_2 = -\dfrac{4}{3} & \Rightarrow c_2 = -\dfrac{1}{6} \end{cases}$

The equation of motion is given by

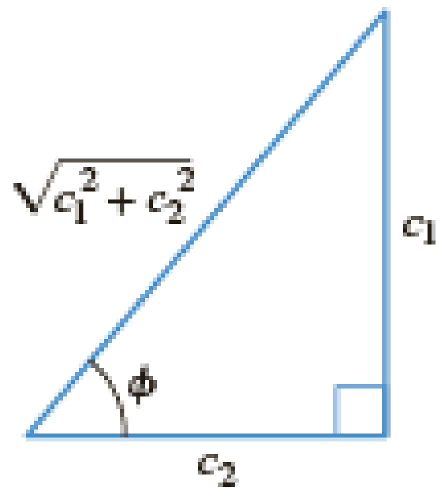
$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$$

Alternative form for $x(t)$

The complementary solution $x(t)$ can be represented alternatively in the form

$$x(t) = A\sin(\omega t + \varphi)$$

Where, $A = \sqrt{c_1^2 + c_2^2}$



The phase angle φ is defined by:

$$\left.\begin{array}{l} \sin\varphi = \dfrac{c_1}{A} \\ \cos\varphi = \dfrac{c_2}{A} \end{array}\right\} \varphi = \tan^{-1}\left(\frac{c_1}{c_2}\right)$$

- **Alternative formulation of $x(t) = A\cos(\omega t + \varphi)$???**

Solution in the form $x(t) = A\sin(\omega t + \varphi)$

The solution takes the form:

$$x(t) = A\sin(8t + \varphi)$$

Computation of the Amplitude

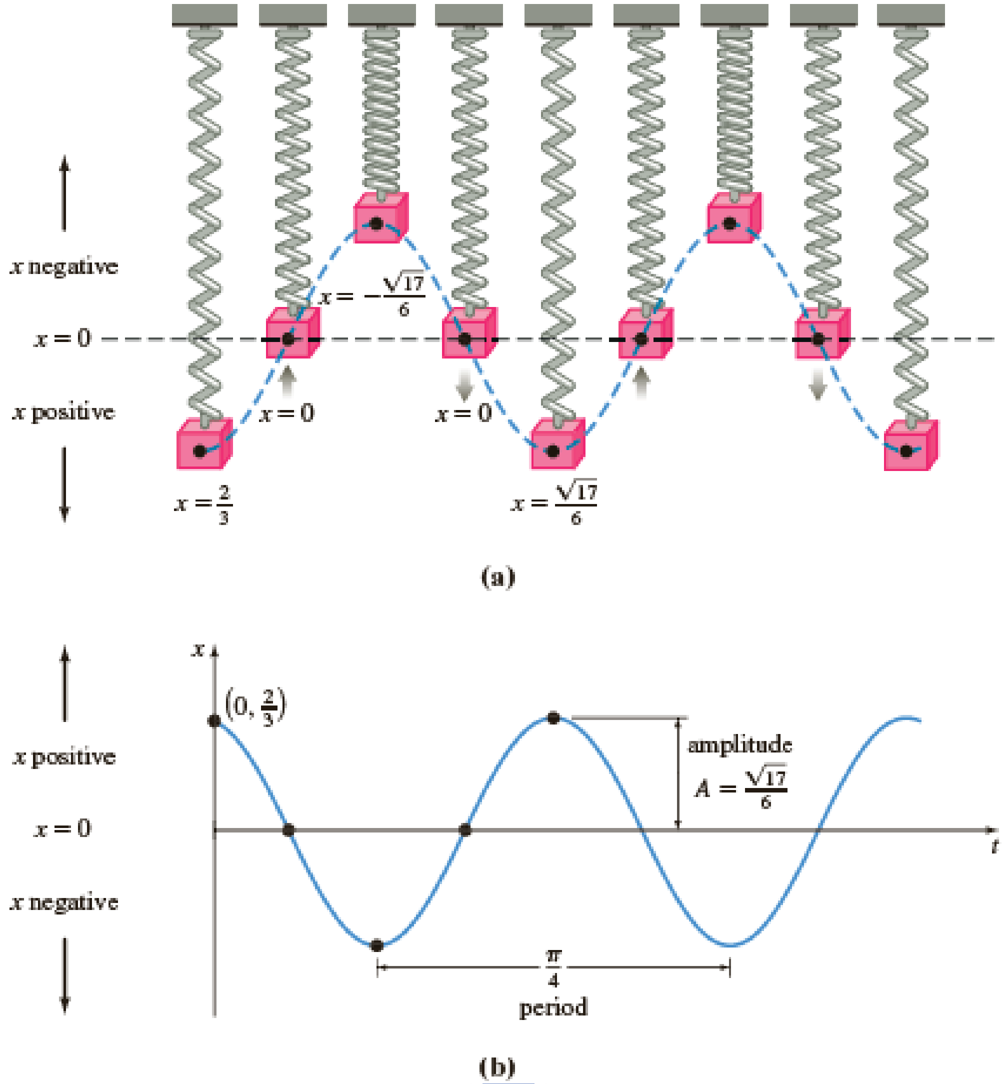
$$.A = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} = \frac{\sqrt{17}}{6} ft$$

$$\varphi = \tan(-4) = -1.326\, rad$$ **??? 4th Quadrant**

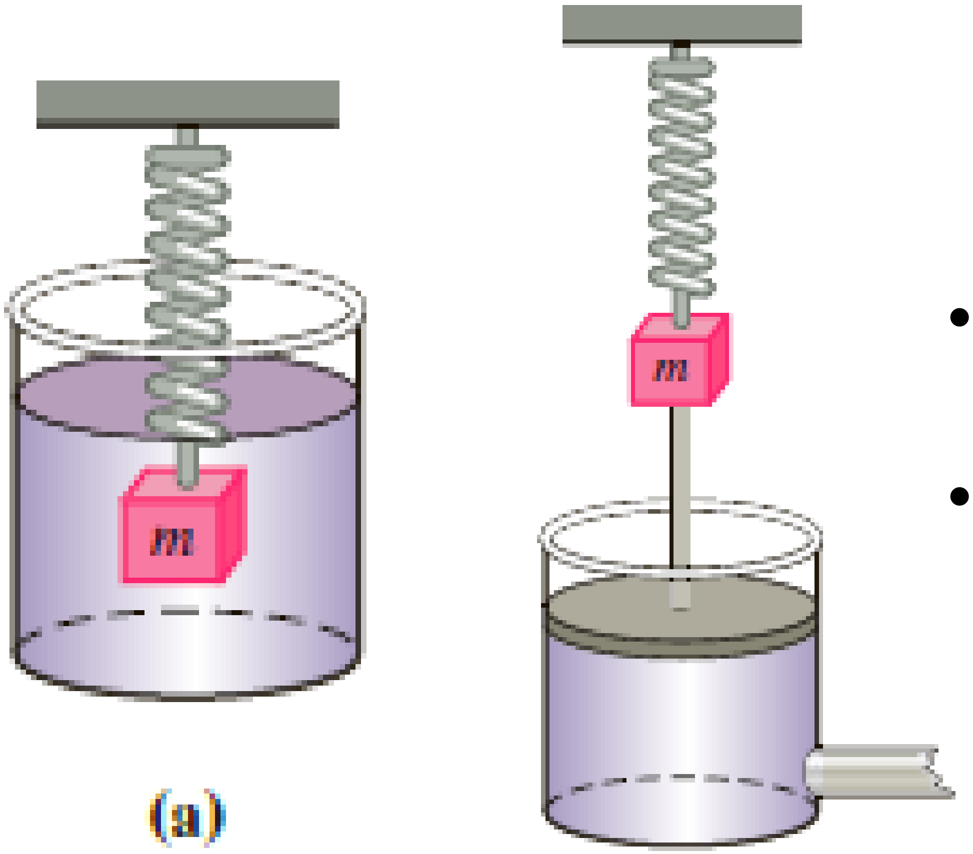
$$\varphi = \pi + (-1.326) = 1.816 rad$$

$$x(t) = \frac{\sqrt{17}}{6}\sin(8t + 1.816)$$

Graphical Illustration $\left(T = \frac{2\pi}{8} = \frac{\pi}{4}\right)$



Spring/Mass Systems: Free Damped Motion



(a) (b)

Assumptions

- Resistive Forces
- Forces ∝ Instataneous velocity

Governing Equation

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$$

- β is a positive damping constant opposing the motion

The free damped motion equation becomes

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = 0$$

Where $2\gamma = \frac{\beta}{m}$ and $\omega^2 = \frac{k}{m}$

Characteristic Equation and Solution

The Characteristic equation is given by

$$m^2 + 2\gamma + \omega^2 = 0$$

Roots: $$m_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

The roots obtained calls for three cases:

Case I: $\gamma^2 - \omega^2 > 0$

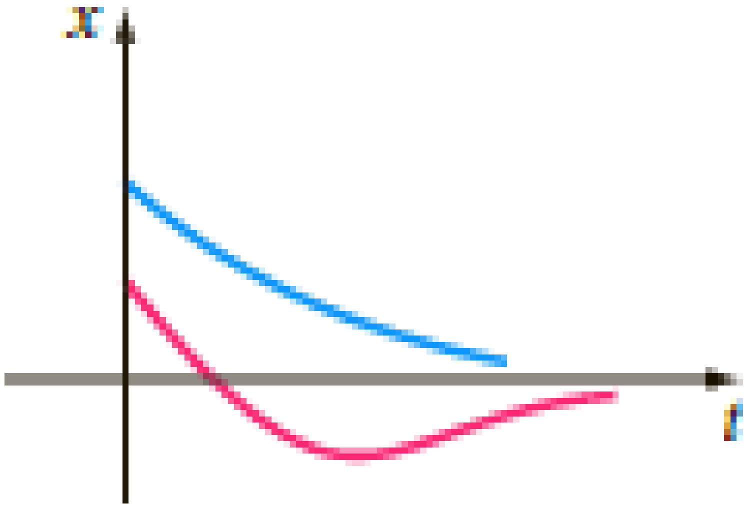
Case II: $\gamma^2 - \omega^2 = 0$

Case III: $\gamma^2 - \omega^2 < 0$

Complementary solution (Overdamped)

Case I: $\gamma^2 - \omega^2 > 0$

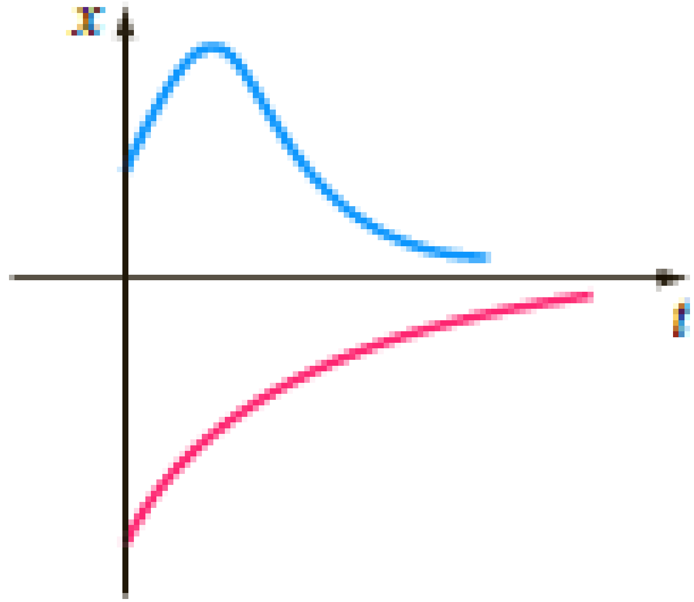
$$x(t) = c_1 e^{\left(-\gamma+\sqrt{\gamma^2-\omega^2}\right)t} + c_2 e^{\left(-\gamma-\sqrt{\gamma^2-\omega^2}\right)t}$$



Complementary solution (Critically damped)

Case II: $\boldsymbol{\gamma^2 - \omega^2 = 0}$

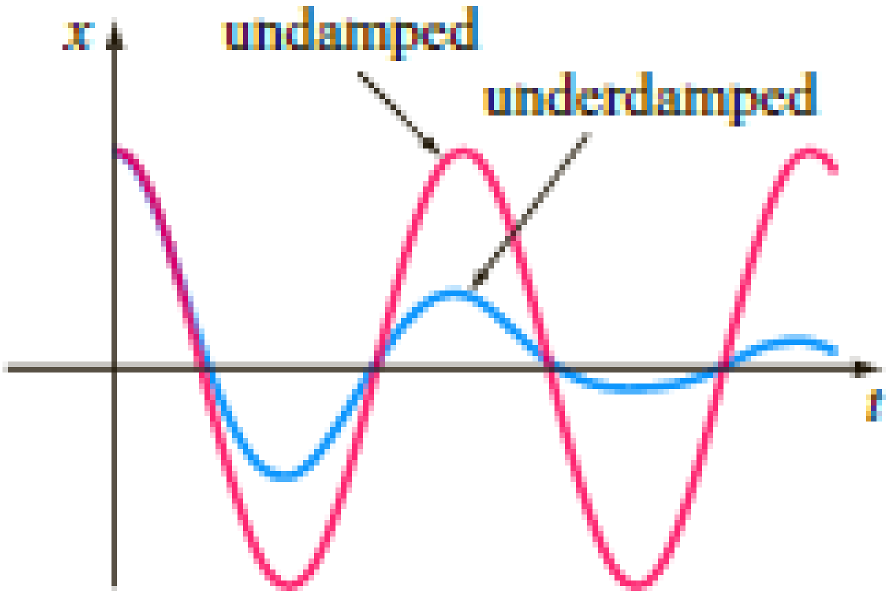
$$x(t) = e^{-\gamma t}(c_1 + c_2 t)$$



Complementary solution (Underdamped)

Case III: $\gamma^2 - \omega^2 < 0$

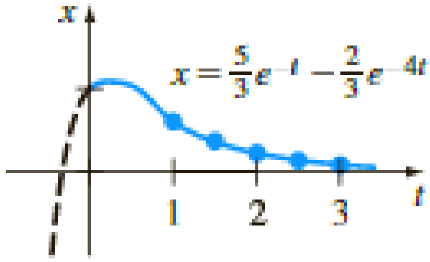
$$x(t) = e^{-\gamma t}\left(c_1 \cos\left(\sqrt{\omega^2 - \gamma^2}\right)t + c_2 \sin\left(\sqrt{\omega^2 - \gamma^2}\right)t\right)$$



Example: Overdamped

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0, \qquad x(0) = 1, x'(0) = 1$$

Solution: $$x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}$$



(a)

Over dampled motion of mass on a spring release from $1unit$ position with velocity $1\text{ft}/s$.

t	$x(t)$
1	0.601
1.5	0.370
2	0.225
2.5	0.137
3	0.083

(b)

How to calculate the extremum displacement and time the mass passes through equilibrium points?

Example: Critically Damped

A mass weighing $8lb$ stretches a spring $2ft$. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from the equilibrium position with an upward velocity of $\frac{3ft}{s}$.

Remarks

Damping force (β) : $\beta = 2$

Mass (m) : $m = \frac{W}{g} = \frac{8}{32} = \frac{1}{4}$

Hooke's law (k): $8 = k(2) \Rightarrow k = 4$

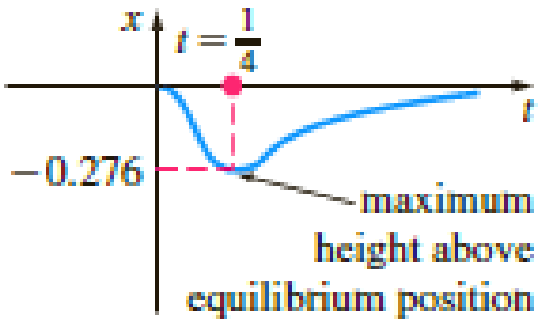
Governing Equation

$$\frac{1}{4}\frac{d^2x}{dt^2} = -4x - 2\frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$$

Solution: $x(t) = (c_1 + c_2 t)e^{-4t}$

Applying the initial values yield

$$x(t) = -3te^{-4t}$$



How to calculate the extremum displacement and time the mass passes through equilibrium points?