Algorithm 2: CAVI for a Gaussian mixture model

Input: Data $x_{1:n}$, number of components K, prior variance of component means σ^2

Output: Variational densities $q(\mu_k; m_k, s_k^2)$ (Gaussian) and $q(z_i; \varphi_i)$ (K-categorical)

Initialize: Variational parameters $\mathbf{m} = m_{1:K}$, $\mathbf{s}^2 = s_{1:K}^2$, and $\boldsymbol{\varphi} = \varphi_{1:n}$

while *the ELBO has not converged* **do**

  for $i \in \{1, \ldots, n\}$ **do**

    Set $\varphi_{ik} \propto \exp\{\mathbb{E}[\mu_k; m_k, s_k^2]x_i - \mathbb{E}[\mu_k^2; m_k, s_k^2]/2\}$

  end

  for $k \in \{1, \ldots, K\}$ **do**

$$\text{Set } m_k \longleftarrow \frac{\sum_i \varphi_{ik} x_i}{1/\sigma^2 + \sum_i \varphi_{ik}}$$

$$\text{Set } s_k^2 \longleftarrow \frac{1}{1/\sigma^2 + \sum_i \varphi_{ik}}$$

  end

  Compute $\text{ELBO}(\mathbf{m}, \mathbf{s}^2, \boldsymbol{\varphi})$

end

return $q(\mathbf{m}, \mathbf{s}^2, \boldsymbol{\varphi})$

Compute ELOB:

$$\begin{aligned}\text{ELBO}(\mathbf{m}, \mathbf{s}^2, \boldsymbol{\varphi}) &= \sum_{k=1}^{K} \mathbb{E}\left[\log p(\mu_k); m_k, s_k^2\right] \\ &+ \sum_{i=1}^{n} \left(\mathbb{E}\left[\log p(c_i); \varphi_i\right] + \mathbb{E}\left[\log p(x_i \mid c_i, \boldsymbol{\mu}); \varphi_i, \mathbf{m}, \mathbf{s}^2\right]\right) \\ &- \sum_{i=1}^{n} \mathbb{E}\left[\log q(c_i; \varphi_i)\right] - \sum_{k=1}^{K} \mathbb{E}\left[\log q(\mu_k; m_k, s_k^2)\right]\end{aligned}$$

Where

$$\begin{aligned}\text{ELOB}(\mathbf{m},\mathbf{s}^2,\boldsymbol{\varphi}) \propto &\sum_{k=1}^{K} \mathbb{E}\left[-\frac{\mu_k^2}{2\sigma^2}\right] + \sum_{i=1}^{n} \mathbb{E}\left[-logK - \sum_{k=1}^{K} c_{ik} \frac{(x_i - \mu_k)^2}{2}\right] - \sum_{i=1}^{n}\sum_{k=1}^{K} log\varphi_{ik} \\ &- \sum_{k=1}^{K} \mathbb{E}\left[log\left(\frac{1}{\sqrt{2\pi s_k^2}} exp\left(-\frac{(\mu_k - m_k)^2}{2s_k^2}\right)\right)\right]\end{aligned}$$

$$\propto -\sum_{k=1}^{K} \frac{m_k^2 + s_k^2}{2\sigma^2} - \sum_{i=1}^{n}\sum_{k=1}^{K} \varphi_{ik} \frac{x_i^2 - 2x_i m_k + m_k^2 + s_k^2}{2} - \sum_{i=1}^{n}\sum_{k=1}^{K} log\varphi_{ik} + \frac{1}{2}\sum_{k=1}^{K} logs_k^2$$

$$\mathbb{E}\left[\frac{(\mu_k - m_k)^2}{2s_k^2}\right] = \frac{1}{2}$$

Update φ_{ik}:

$$\varphi_{ik} \propto \exp\left\{\mathbb{E}\left[\mu_k; m_k, s_k^2\right] x_i - \mathbb{E}\left[\mu_k^2; m_k, s_k^2\right]/2\right\}.$$

The factor $q(\mu_k; m_k, s_k^2)$ is a Gaussian distribution on the kth mixture component's mean parameter; its mean is m_k and its variance is s_k^2. So we can compute:

$$\mathbb{E}[\mu_k; m_k, s_k^2] = m_k$$
$$\mathbb{E}[\mu_k^2; m_k, s_k^2] = m_k^2 + s_k^2$$

Where $\mathbb{E}(X^2) = \left(\mathbb{E}(X)\right)^2 + D(X)$

$$\varphi_{ik} \propto exp\left(m_k x_i - \frac{m_k^2 + s_k^2}{2}\right)$$

Update m_k and s_k^2:

$$m_k = \frac{\sum_i \varphi_{ik} x_i}{1/\sigma^2 + \sum_i \varphi_{ik}}, \quad s_k^2 = \frac{1}{1/\sigma^2 + \sum_i \varphi_{ik}}.$$