# Modelling, Simulation & Optimisation (H9MSO)

1. Introduction to Linear Programming
   1. Dr Christian Horn

# Learning Outcomes

- On successful completion of this module, you will be able to:

1. Categorize different types of simulation modelling technologies
2. Implement and test a conceptual model using a simulation tool
3. Critically analyse output data produced by a model and test the validity of the model
4. Perform optimisation according to chosen criteria
5. Comprehend, apply and develop new (hybrid) methodologies of the most commonly used heuristics (Greedy, Simulated Annealing, Tabu Search, Evolutionary algorithms, Ant Colony optimisation)

# Today's Outline

- Introduction to Optimisation
- Introduction to Linear Programming
- LP examples
- Understanding LP solutions graphically
- Solving LPs:
  - Simplex method
  - Interior point methods
- Software for Solving LPs

# Introduction to Optimisation

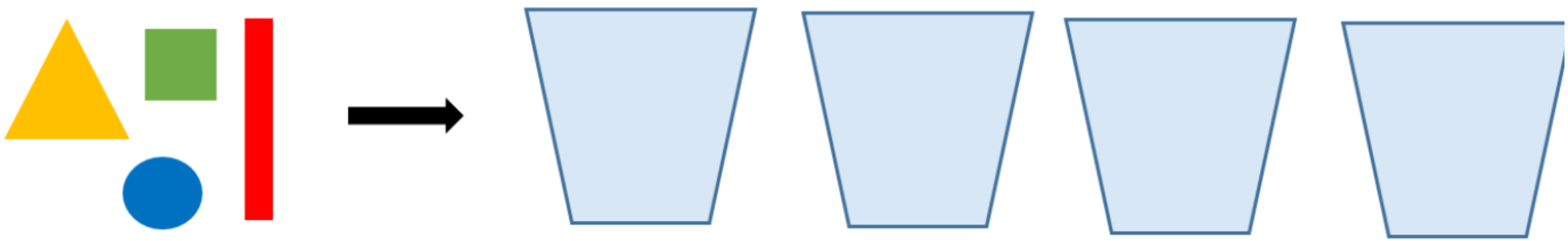
- What do we mean by "optimisation"?
  - It could mean a lot of things, for example speeding up:
    - computer programs
    - database queries
    - etc.

- This module: what we mean is *finding solutions to optimisation problems*.

- We are concerned with finding the optimal (or best) solution to a problem

# Introduction to Optimisation

- Example:
  - Use the minimum number of bins to pack the shapes below



# Introduction to Optimisation

- Both of the solutions below are valid solutions, but only figure 2 is optimal

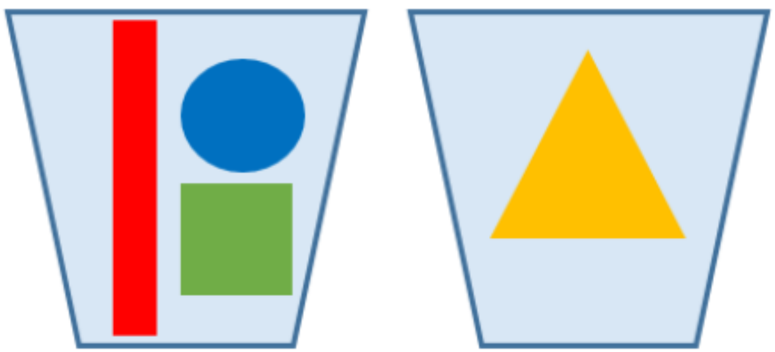


Fig 1: Solution A

Fig 2: Solution B (optimal)

# Introduction to Optimisation

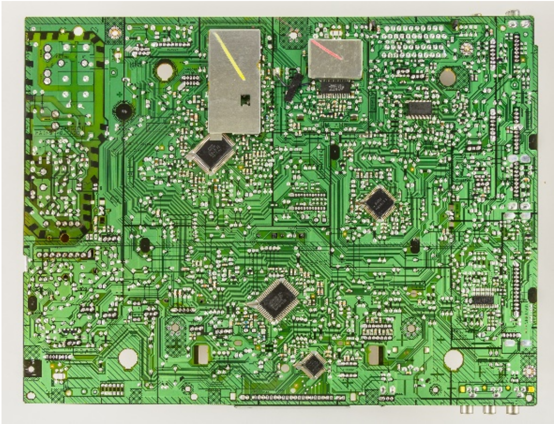
- As another example, suppose we want to timetable lectures for a university department. We want to assign each lecture to a room and a time.

- We have *constraints* such as
  - 2 lectures can't be given in the same room at the same time
  - a lecturer or student can't be in 2 rooms at the same time
  - every lecture must occur somewhere at some time
  - Dr Smith can't lecture on Tuesday mornings

- We often have something we want to optimise, usually called an *objective function* (it's our "objective" to optimise this function), for example
  - minimise the number of lectures given on Monday morning

# Introduction to Optimisation

- There are *many* problems like this in industry, business, science, etc. Some example are:
  - Organizing a factory production line
  - Designing a circuit board
  - Delivering packages
  - Managing inventories…



# Introduction to Optimisation

- We may want to minimise cost or risk, maximise profit, minimise delivery time, etc.

- All these are solved by optimisation techniques.

- Very many optimisation techniques developed by researchers in several fields

- This module:
  - We're only covering selected topics in Optimisation
  - Minimal mathematics
  - Not too much programming

- But at the end, you should be able to model a wide range of problems and solve them using downloadable software, or write your own software for some types of problem.

# Historical Perspectives

- The original optimisation field is Operations Research (OR).
- Began in the late 1930s in a systematic fashion
- World War 2
  - 1936: British RAF used OR to exploit large amount radar data being generated
  - 1939: Pre-war air defense exercise involving:
    - 33000 men,
    - 300 aircraft
    - 110 anti-aircraft guns
    - 700 searchlights
    - 100 barrage balloons
  - The contribution made by the OR team was so impressive that teams were set up at other RAF commands

# Historical Perspectives

- During WWII they continued to make great contributions, solving hard problems such as:
  - How many fighter squadrons should be sent to France
    - (none, which turned out to be the right decision as they were used instead to defend Britain)
  - Organizing flying and maintenance to make best use of squadron resources
    - 61% improvement by reorganizing the system
  - Improvement of "attack kill probability" on U-boats
    - Improvement from 2-3% to 40%, by considering the setting for depth charge explosion, the lethal radius of a depth charge, aiming errors etc)

# Historical Perspectives

- The successes during WW2 led to OR being adopted in many non-military domains such as business, manufacturing and economics.

  - https://en.wikipedia.org/wiki/Operations_research

- Parallel development in Russia application within planning economy1939-1949
  - Kantorovich

# Facility Location Problem

- https://www.wired.co.uk/article/kfc-chicken-crisis-shortage-supply-chain-logistics-experts

# Linear Programming

A linear program is a problem with a special form:

$$
\begin{aligned}
\text{maximise or minimise} \quad & \textstyle\sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \textstyle\sum_{j=1}^{n} a_{ij} x_j \{\leq, =, \geq\} b_i \\
& x_j \leq u_j \ (j = 1 \ldots n) \\
& x_j \geq 0 \ (j = 1 \ldots n)
\end{aligned}
$$

With

1. **Decision Variables:**
   - Variables $x_j$ to model the problem
2. **Objective:**
   - Goal statement (maximise profit or minimise cost)
3. **Constraints:**
   - Rules that we must obey (don't make more items than we can sell…)
4. **Non-negativity constraints:**
   - E.g. we can't make a negative amount of items

# Linear Programming

- We must choose values for the *decision variables* $x_j$ that *satisfy* (do not *violate*) the linear *constraints*, the *upper bounds*, and the *non-negativity restrictions*, so that we get the greatest or least (depending on the problem) value for the *objective function*.
- The objective function may represent cost, profit, yield, or something else that we're trying to optimise.
- The constraints are things that must be true in all solutions, and may be inequalities ($\geq, \leq$) or equalities ($=$)
- The non-negativity restrictions occur in many problems, but sometimes we might have unrestricted variables. Some variables might also have no upper bound, though again they're commonly used.
- The $a_{ij}$ , $b_j$ , $c_j$ , $u_j$ are called the *parameters* of the problem, and are given: we have no control over them.

- A large number of real problems can be expressed in this way. Also, a strength of this approach is its simplicity: highly specialized software has been developed that can be used to solve LPs: we just have to model them.

# An example

- A company produces 2 products P and Q. How much should they produce each week to maximise profits?

- Each product has different profit per unit sold, and only so much of each product can be sold in a given week:

| | P | Q |
|---|---|---|
| profit per unit | 45 | 60 |
| maximum sales | 100 | 40 |

# An example

- To make things more complicated, the company can't simply make as much of any product as they want as every product needs to run through each of the machines:

| | Processing time per unit | | Available time |
|---|---|---|---|
| Machine | P | Q | |
| A | 20 | 10 | 1800 |
| B | 12 | 28 | 1440 |
| C | 15 | 6 | 1440 |
| D | 10 | 15 | 2400 |

- The availability of the machine is specific through the machine type and process.

# An example

- One more piece of information: the plant costs 4000 euro a week to operate, which must be subtracted from our profits.

- The problem is to decide how much of each product to make during the week.

# Modelling the problem

- First, what are the variables?
  - All we want to know here is how much of each product to make, so let's use 2 variables and call them P, Q.
  - If we decide to make 30 units of product P this week then variable P = 30, and so on.

- Objective?
  - We want to maximise total profit
  - Total Profit = Gross profit - Expenses
  - $45P + 60Q - 4000$

# Modelling the problem

- So the (linear) objective is:

$$\text{maximising } 45P + 60Q$$

- Note that the $4000$ isn't used, as it's not a function of the decision variables:
  - maximising $45P + 60Q - 4000$

  gives the same values for P,Q as
  - maximising $45P + 60Q$.

- But we'll use it later to work out the actual profit (or loss).

# Modelling the problem

- Constraints
  - The market limitations (maximum sales above) tell us that we shouldn't make too much of any product, as we won't be able to sell them:

$$P \leq 100$$
$$Q \leq 40$$

  - Non-negativity restrictions?
    We can't make negative amounts of products

$$P \geq 0$$
$$Q \geq 0$$

# Modelling the problem

- Machine Limitations:
  - Product p requires 20 minutes machine time on machine A
  - $P$ units take $20\times P$ minutes to make on machine A

Each machine has a productive runtime per week due to machine maintenance requirements

$$
\begin{array}{lll}
\text{maximise} & Z = 45P + 60Q & \\
\text{subject to} & 20P + 10Q \leq 1800 & \text{(machine A)} \\
& 12P + 28Q \leq 1440 & \text{(machine B)} \\
& 15P + 6Q \leq 1440 & \text{(machine C)} \\
& 10P + 15Q \leq 2400 & \text{(machine D)} \\
& P \leq 100 \quad Q \leq 40 & \text{(market constraints)} \\
& P \geq 0 \quad Q \geq 0 & \text{(non-negativity constraints)}
\end{array}
$$

# Our Model

- Solving with standard optimisation software gives:

$$\begin{aligned} P &= 81.82 \\ Q &= 16.36 \end{aligned}$$

- What do these figures mean for our problem?
  If we feed these values into the objective function we get 4663.64€. The actual profit is this figure minus $4000$ (the cost for running the workshop) giving $663.64$€ profit. Assuming that all our assumptions were correct (about markets etc.) this is guaranteed to be the optimal solution.

# Interpreting our results

- We can also use the figures to check the machine usages.
  - For example machine A is used for $20P + 10Q$ units, giving exactly $1800$ minutes of usage (total availability)
  - Machine B is utilized for 1440 minutes (total availability)
  - But machine C is only used for $1326$ minutes, and machine D for $1064$ minutes.
  - So machines C and D are idle for some of the week.

# Interpreting our results

- Fewer units of products P and Q (81.82 and 16.36) are made than the market could stand (100 and 40).

- The availability for machines A and B are *only just* satisfied by the optimal solution: that is, the left hand side of each isn't just to the right hand side, it's actually equal.

- In other words, if any more of P or Q were manufactured then constraints would be violated, and we would no longer have a *feasible solution* (satisfying all constraints) to the problem. (In LP any assignment to all variables is a *solution*.)
  - We call these constraints *tight* (or *active*).

- In contrast, the other constraints are not tight: we could make more of P, for example, and still have a feasible solution (though it wouldn't be an optimal solution).

# Interpreting our results

- The tight constraints here represent bottlenecks for the manufacturing process.

- They tell us that, if the company could increase the availabilities of machines A and B then more profit might be made.

- On the other hand, increasing the availability of machines C and D, or the market for products P and Q, would not be worthwhile.

# Abstractions

To model this problem as an LP we made some simplifying assumptions ("abstractions"):

- Integrality:
  - We cant manufacture a fraction of a product  (e.g. 81.82 *units)*
  - Integrality constraints relaxed (more on this later)
- Linearity:
  - Real situations might be nonlinear (later)
  - Buying raw materials in bulk could reduce the amount spent
- Forecasting market demand with certainty:
  - Real-life market demands are uncertain and modelled using probability distributions, not exact figures.
  - Handling uncertainty is beyond the scope of this module
- Many Other Things:
  - Even this simple model has highlighted some interesting aspects of the problem: the bottlenecks on machines A and B.
  - All models are wrong, but some are useful - **George E. P.  Box**

# Understanding LP solutions graphically

- Remaining problem with only 2 variables $P$ and $Q$:

$$
\begin{array}{lll}
\text{maximise} & Z = 45P + 60Q & \\
\text{subject to} & 20P + 10Q \leq 1800 & \text{(machine A)} \\
 & 12P + 28Q \leq 1440 & \text{(machine B)} \\
 & 15P + 6Q \leq 1440 & \text{(machine C)} \\
 & 10P + 15Q \leq 2400 & \text{(machine D)} \\
 & P \leq 100 \quad Q \leq 40 & \text{(market constraints)} \\
 & P \geq 0 \quad Q \geq 0 & \text{(non-negativity constraints)}
\end{array}
$$

# Understanding LP solutions graphically

- Coordinates for Machine A line are calculated as follows:

- Machine A: $20P + 10Q \leq 1800$

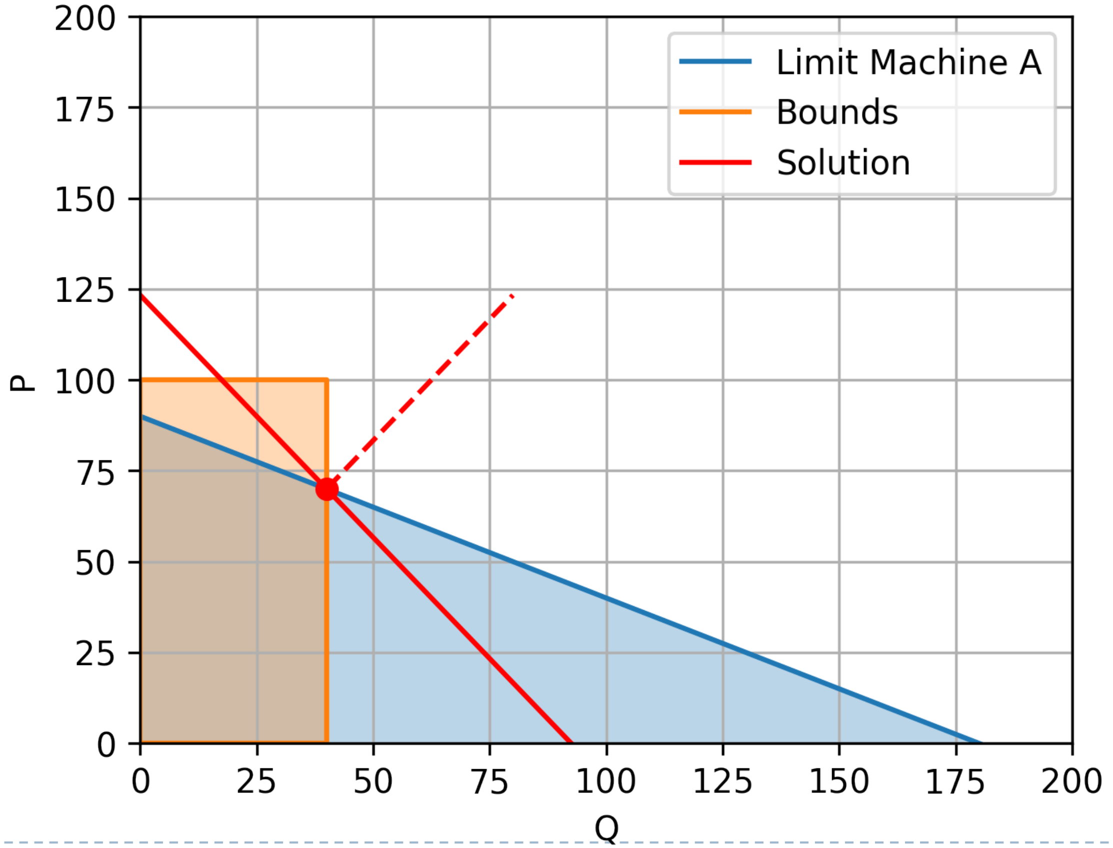
  - When $P = 0$: $Q = \frac{1800}{10} = 180$

  - When $Q = 0$: $P = \frac{1800}{20} = 90$

- Draw a straight line from $(P, Q) = (0, 180)$ to $(P, Q) = (90, 0)$

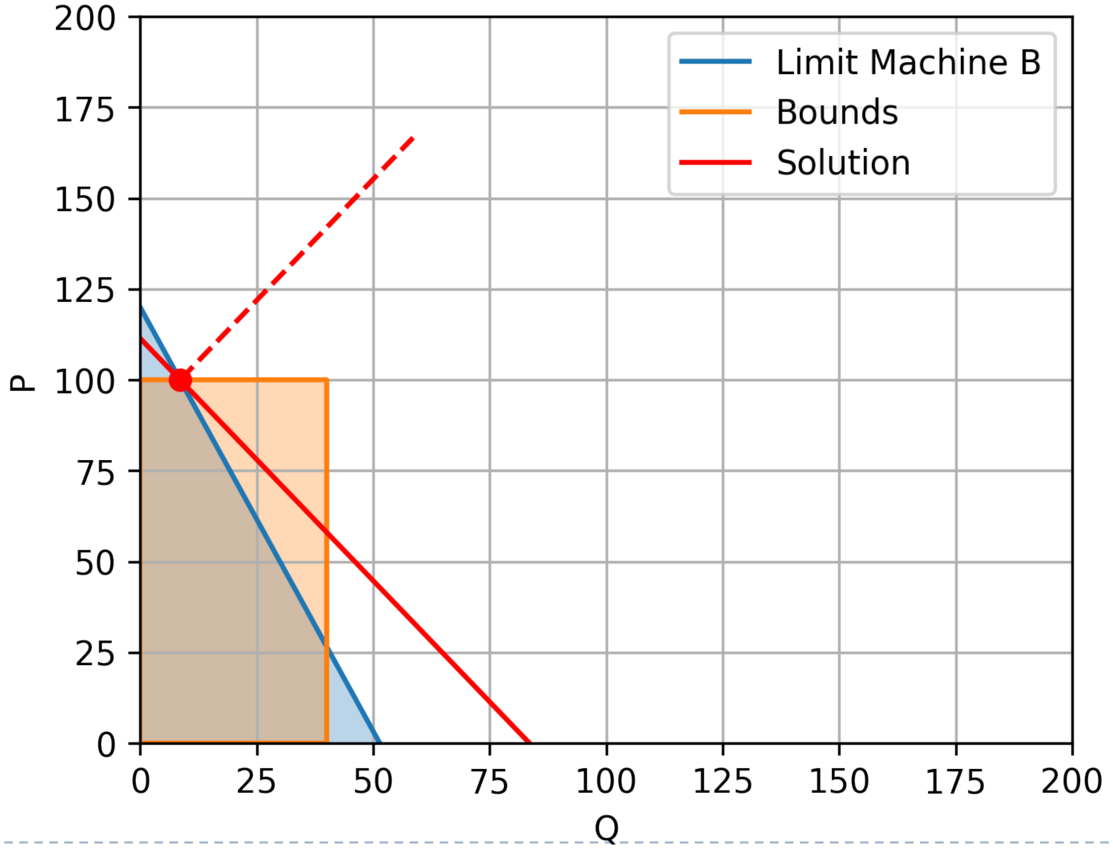
# Machine A

$$
\begin{array}{lll}
\text{maximise} & Z = 45P + 60Q & \\
\text{subject to} & 20P + 10Q \leq 1800 & \text{(machine A)} \\
& \boxed{\begin{array}{ll} 12P + 28Q \leq 1440 & \text{(machine B)} \\ 15P + 6Q \leq 1440 & \text{(machine C)} \\ 10P + 15Q \leq 2400 & \text{(machine D)} \end{array}} & \\
& P \leq 100 \quad Q \leq 40 & \text{(market constraints)} \\
& P \geq 0 \quad Q \geq 0 & \text{(non-negativity constraints)}
\end{array}
$$



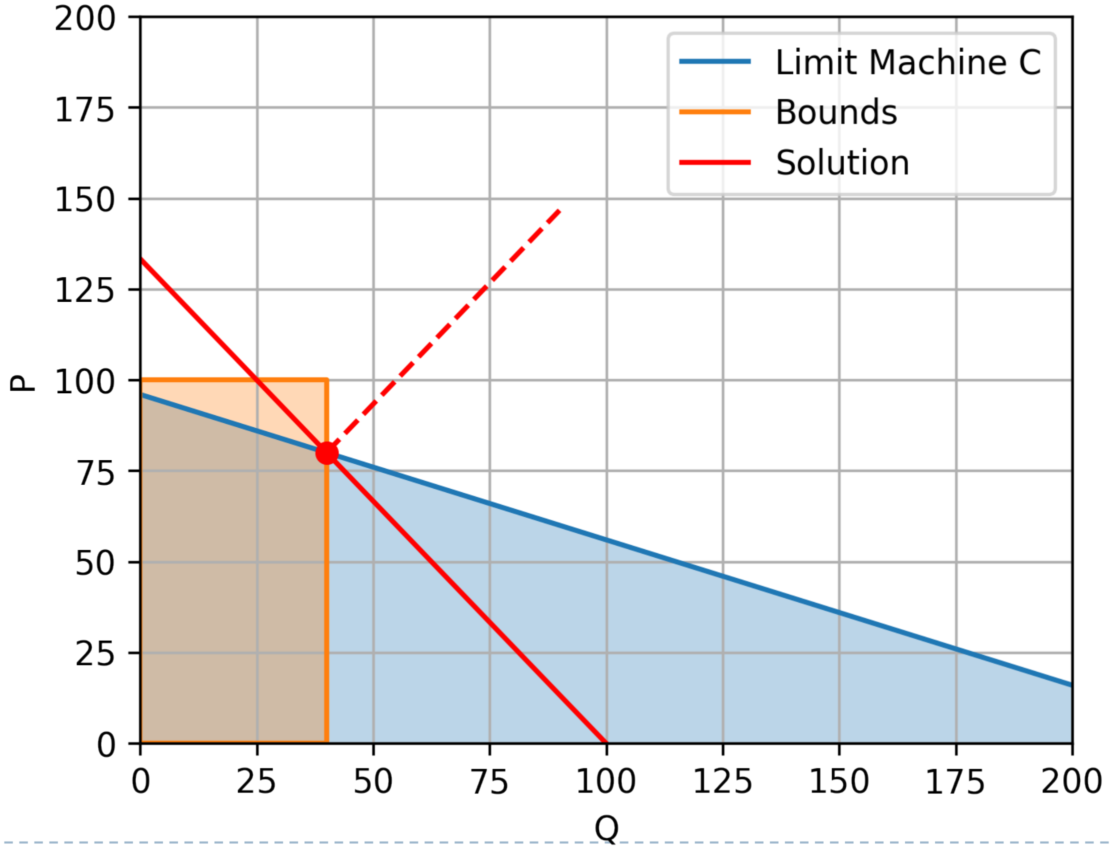
# Machine B

$$
\begin{aligned}
\text{maximise} \quad & Z = 45P + 60Q \\
\text{subject to} \quad & \boxed{20P + 10Q \leq 1800 \quad \text{(machine A)}} \\
& 12P + 28Q \leq 1440 \quad \text{(machine B)} \\
& \boxed{\begin{array}{ll} 15P + 6Q \leq 1440 & \text{(machine C)} \\ 10P + 15Q \leq 2400 & \text{(machine D)} \end{array}} \\
& P \leq 100 \quad Q \leq 40 \quad \text{(market constraints)} \\
& P \geq 0 \quad Q \geq 0 \quad \text{(non-negativity constraints)}
\end{aligned}
$$



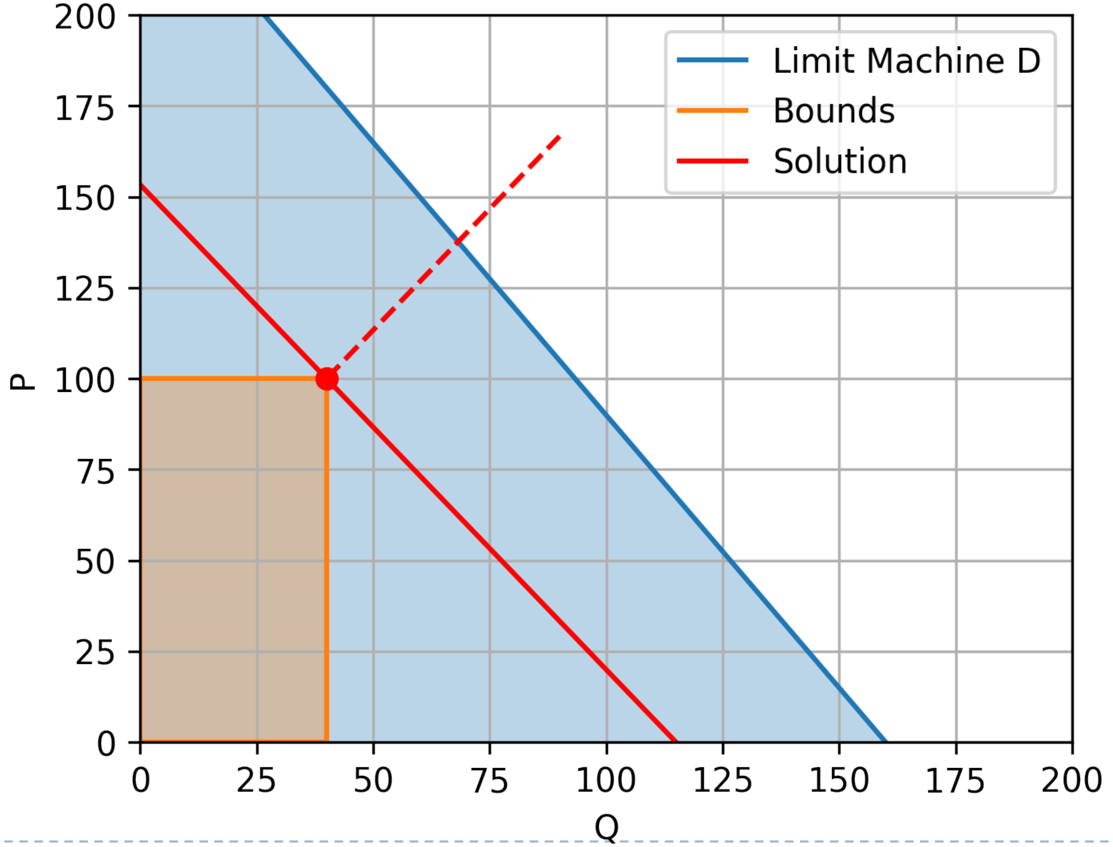
# Machine C

$$
\begin{array}{lll}
\text{maximise} & Z = 45P + 60Q & \\
\text{subject to} & \boxed{\begin{array}{ll} 20P + 10Q \leq 1800 & \text{(machine A)} \\ 12P + 28Q \leq 1440 & \text{(machine B)} \end{array}} & \\
& 15P + 6Q \leq 1440 & \text{(machine C)} \\
& \boxed{10P + 15Q \leq 2400 \quad \text{(machine D)}} & \\
& P \leq 100 \quad Q \leq 40 & \text{(market constraints)} \\
& P \geq 0 \quad Q \geq 0 & \text{(non-negativity constraints)}
\end{array}
$$



# Machine D

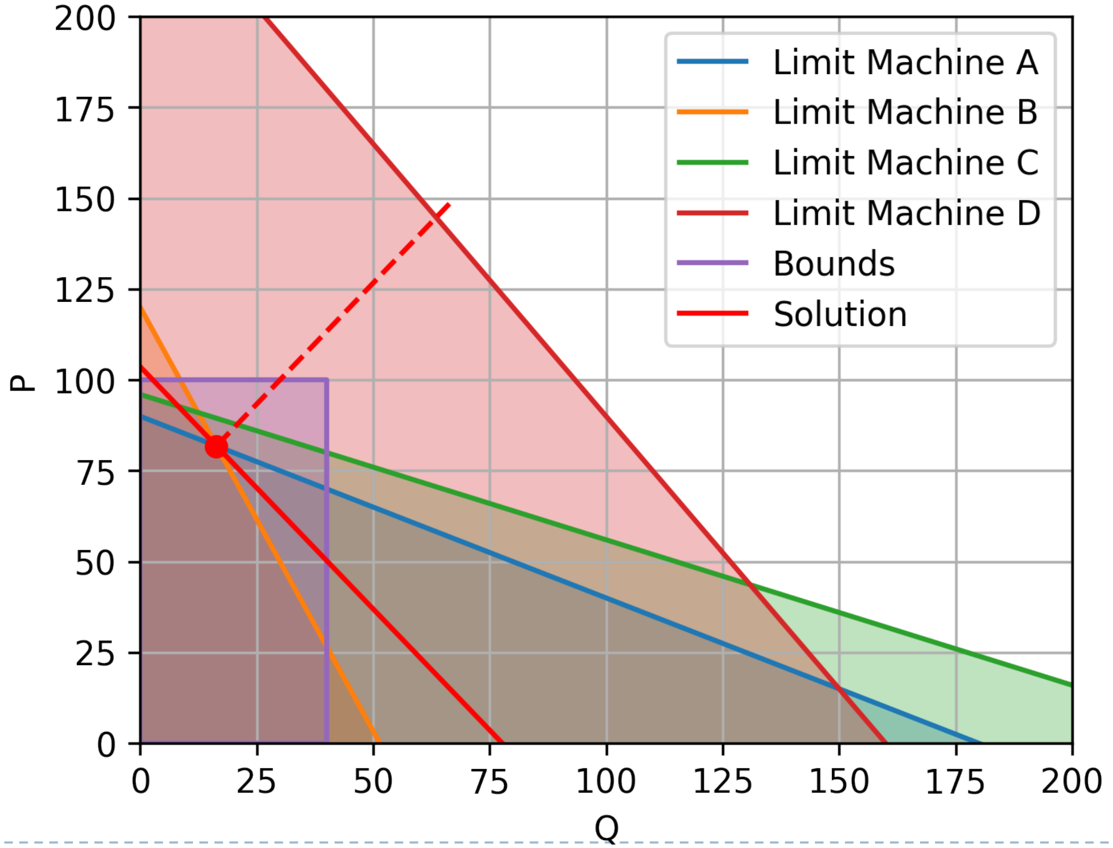
$$
\begin{array}{ll}
\text{maximise} & Z = 45P + 60Q \\
\text{subject to} & \boxed{\begin{array}{ll} 20P + 10Q \leq 1800 & \text{(machine A)} \\ 12P + 28Q \leq 1440 & \text{(machine B)} \\ 15P + 6Q \leq 1440 & \text{(machine C)} \end{array}} \\
& 10P + 15Q \leq 2400 \quad \text{(machine D)} \\
& P \leq 100 \quad Q \leq 40 \quad \text{(market constraints)} \\
& P \geq 0 \quad Q \geq 0 \quad \text{(non-negativity constraints)}
\end{array}
$$




Machine D is uncritical. The constraints for D have no impact on the solution, one could delete the constraint.

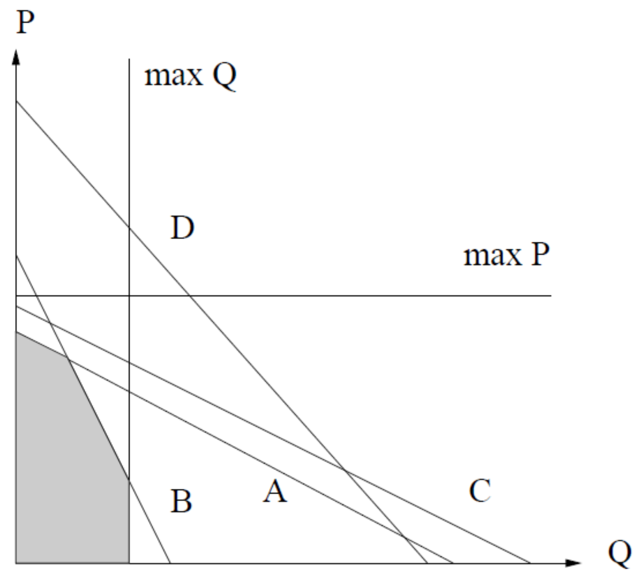
# Combined

$$
\begin{array}{lll}
\text{maximise} & Z = 45P + 60Q & \\
\text{subject to} & 20P + 10Q \leq 1800 & \text{(machine A)} \\
& 12P + 28Q \leq 1440 & \text{(machine B)} \\
& 15P + 6Q \leq 1440 & \text{(machine C)} \\
& 10P + 15Q \leq 2400 & \text{(machine D)} \\
& P \leq 100 \quad Q \leq 40 & \text{(market constraints)} \\
& P \geq 0 \quad Q \geq 0 & \text{(non-negativity constraints)}
\end{array}
$$



# Understanding LP solutions graphically

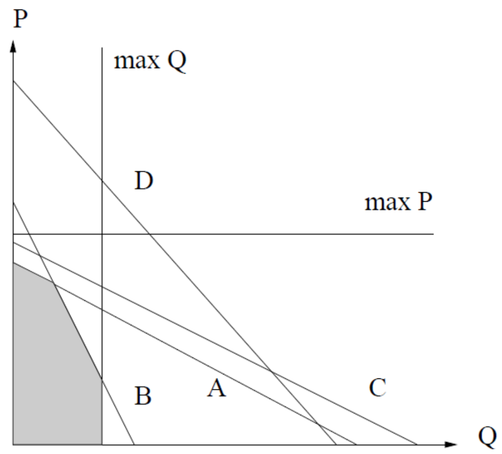
- All constraints can be graphed in this way
- Each constraint in the problem cuts out a different set of points, leaving a ***feasible*** region of the space called a *polyhedron*:




- We want to find a point in the polyhedron with the maximum objective function value, in other words an optimal feasible solution.

# Understanding LP solutions graphically

- Notice that only some of the constraints affect the shape of the polyhedron:
  - those for machines A and B,
  - non-negativity
  - the market limitations
  - Constraint for machine C is weaker than constraint for machine A
  - Constraint for machine D is weaker than market constraints (limits)



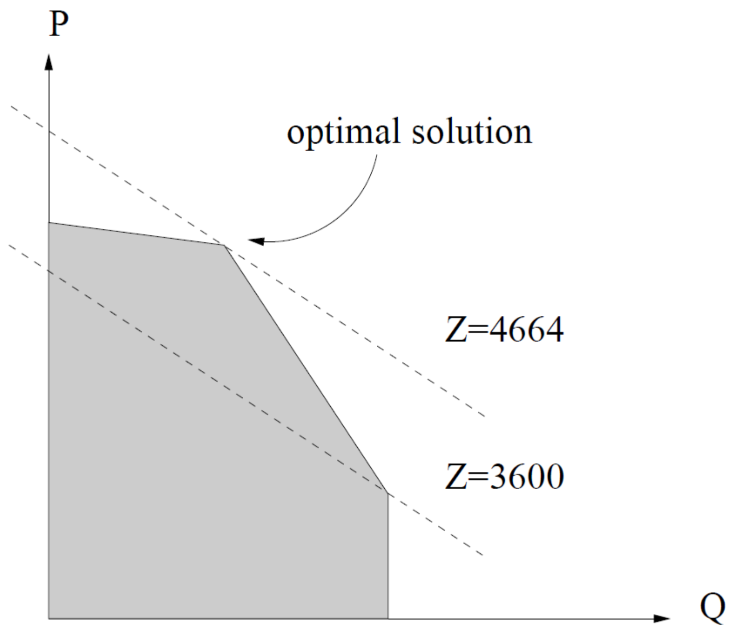

- If we removed the other constraints the polyhedron would be unchanged: we call these *redundant* constraints.

# Understanding LP solutions graphically

- We can now see the ***feasible*** solutions: but how do we find an ***optimal feasible*** solution?
- Notice that for any value of the function Z the values of P,Q lie on a straight line, because the function is linear:

$$Z = 45P + 60Q.$$

- Here are the lines for $Z = 3600$ and $Z = 4664$, for example:



# But there is a minor problem…

The optimal solution is:

- $P = 81.82$, $Q = 16.36$, $f = -4663.64$

But this is not feasible. We need to check the "near by" integer points:

- $P=80, Q=17, f= -4620$ $Q=18$ breaks constraint
- $P=81, Q=16, f= -4605$ $Q=17$ breaks constraint
- $P=82, Q=16, f= -4650$ $Q=17$ breaks constraint

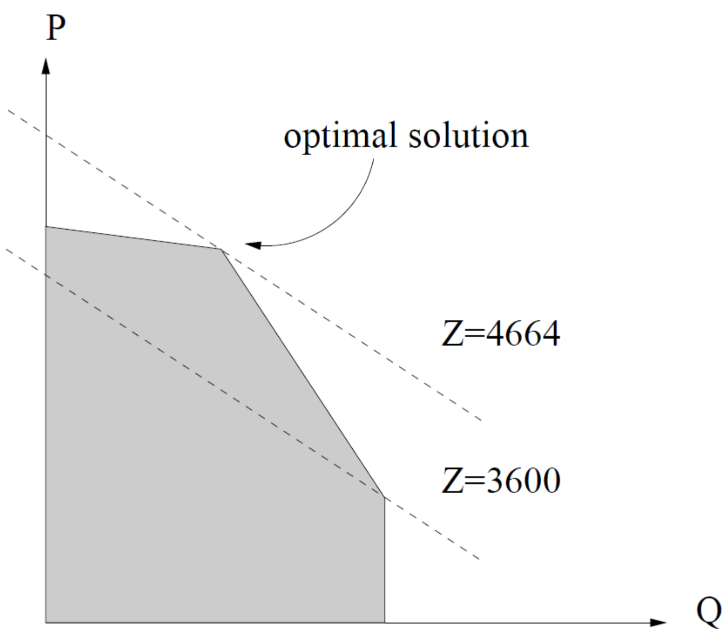
Optimisation over integer values is a complete different problem

# Understanding LP solutions graphically

- Notice that they're parallel: whatever Z we choose, we'll get a line parallel to these, called an ***isovalue contour***.
- Now we want a solution on one of these contours: unfortunately we don't know which one so there are an infinite number of possibilities.
- **But** the solution should have the greatest possible value of Z, so we can slide a contour downwards until it just touches the feasible solution: where it touches will be a feasible optimal solution.
- This fact is the key to solving LP problems: it can be shown that it touches will always touch *one of the vertices of the polyhedron*.
- There may be other optimal feasible solutions, but all we care about is finding one of them.

# Understanding LP solutions graphically

- This bit of geometry shows that we need only check the Z value at each vertex of the polyhedron (the *extreme points* of the feasible region):
  - The one with greatest value is feasible and optimal.
- In our example we have only 5 such points. From an infinite number of possibilities, we've reduced the problem to a relatively easy one. (Of course it's more complicated with many variables.)



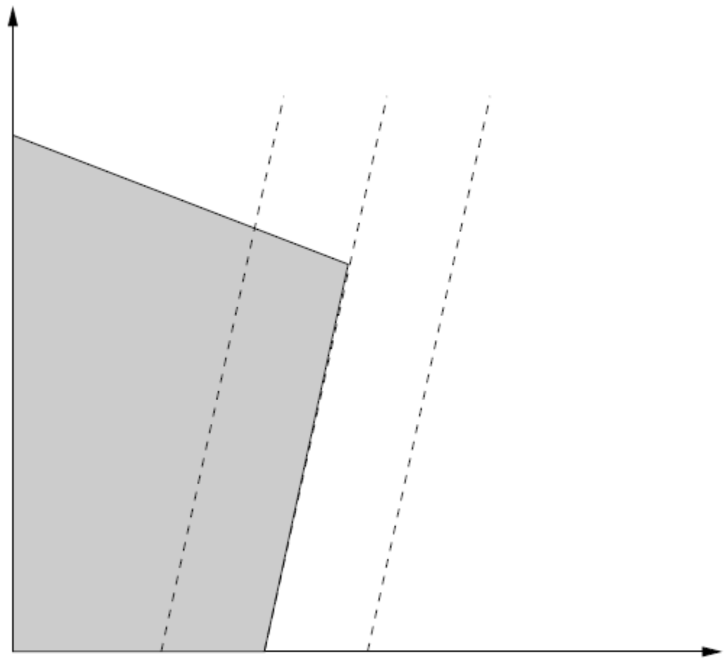
# More graphical examples

- There are several things that might happen when we try to find an optimal feasible solution.
- One of them is as above: there are a finite number of such solutions, at the extreme points.
- Here's another possibility: the isovalue contours are parallel to one of the polyhedron edges. This occurs in the LP:

$$
\begin{array}{ll}
\text{maximise} & Z = 3X_1 - X_2 \\
\text{subject to} & 15X_1 - 5X_2 \leq 30 \\
 & 10X_1 + 30X_2 \leq 120 \\
 & X_1 \geq 0 \quad X_2 \geq 0
\end{array}
$$

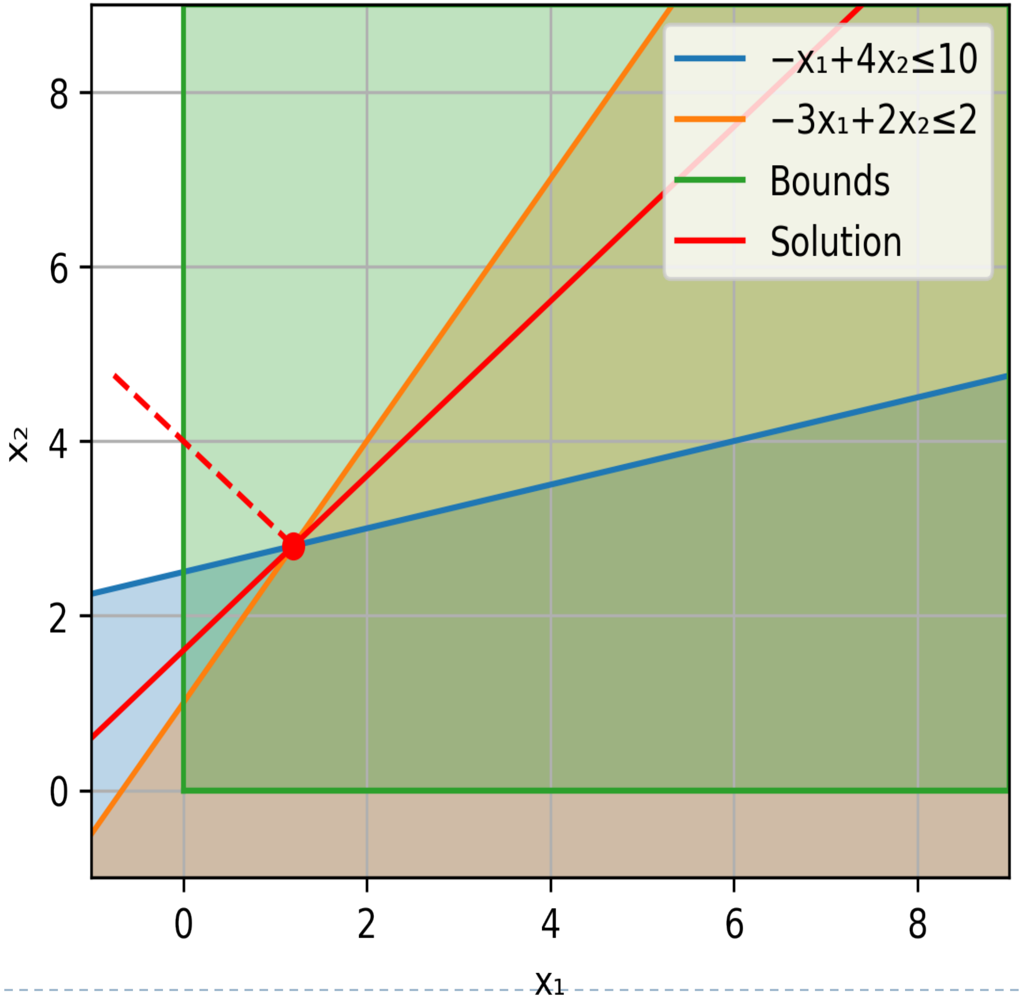
# More graphical examples

- Here there are an infinite number of feasible optima, lying along the constraint boundary. But two of these are at vertices, so we still need to check only the vertices.



# More examples

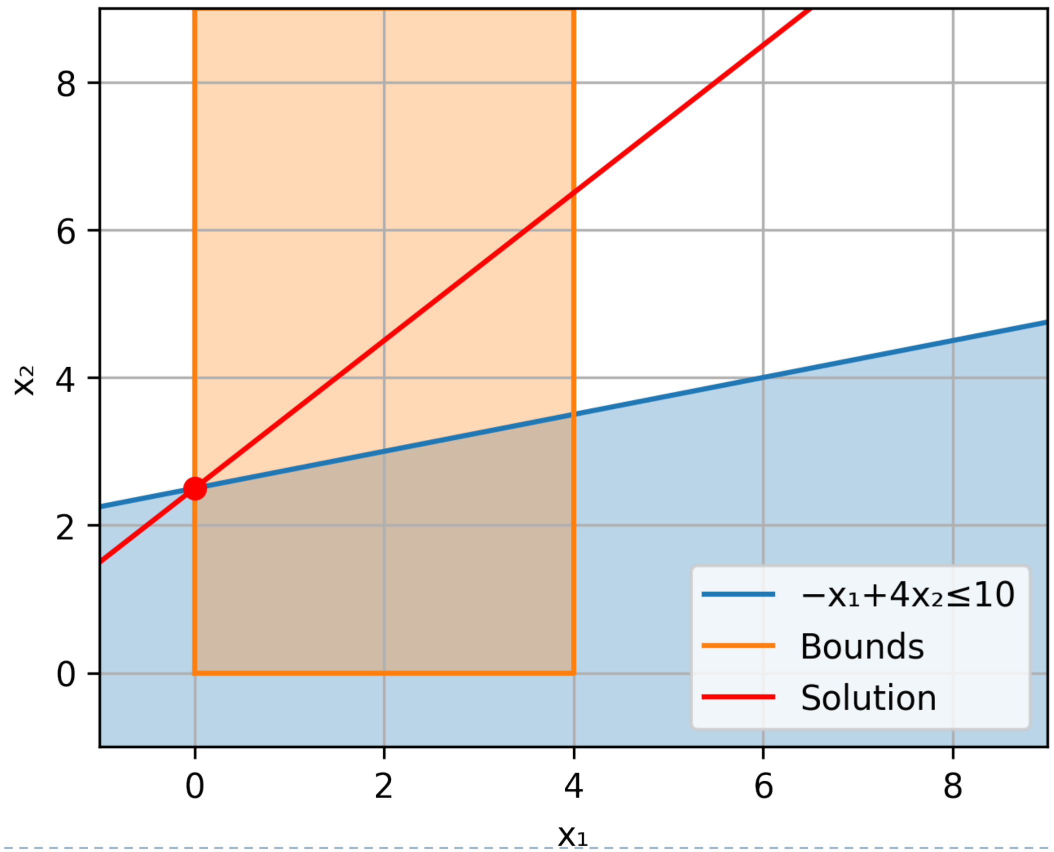
$$\begin{aligned} \text{maximise} \quad & Z = -X_1 + X_2 \\ \text{subject to} \quad & -X_1 + 4X_2 \leq 10 \\ & -3X_1 + 2X_2 \leq 2 \\ & X_1 \geq 0 \quad X_2 \geq 0 \end{aligned}$$



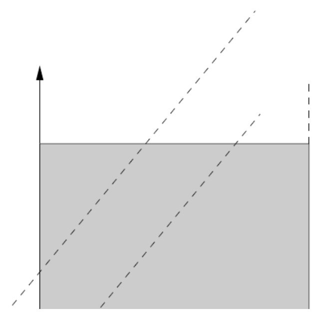

The feasible region is unbounded, but the optimal solution is finite.

# More examples

$$\begin{aligned} \text{maximise} \quad & Z = -X_1 + X_2 \\ \text{subject to} \quad & -X_1 + 4X_2 \leq 10 \\ & X_1 \leq 4 \\ & X_1 \geq 0 \quad X_2 \geq 0 \end{aligned}$$




The optimal solution is unbounded: X1 has an upper bound but X2 doesn't. (If all variables are bounded then the feasible solution is always finite.)

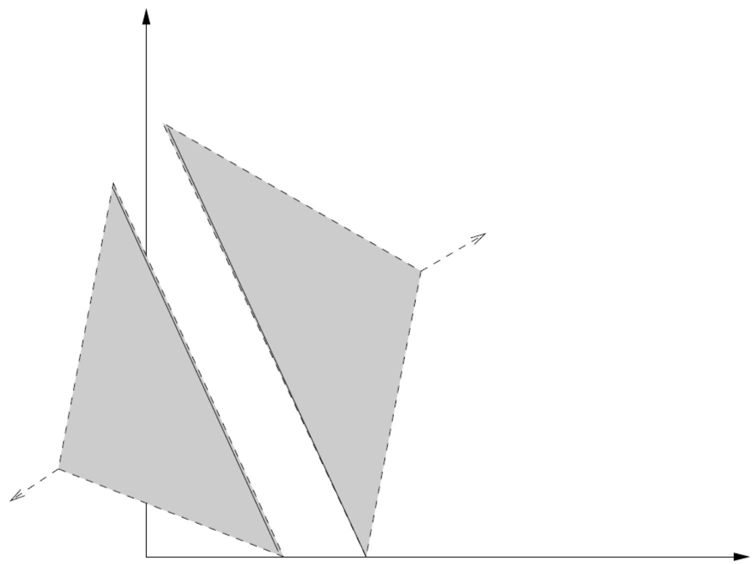


# More graphical examples

- Another case:

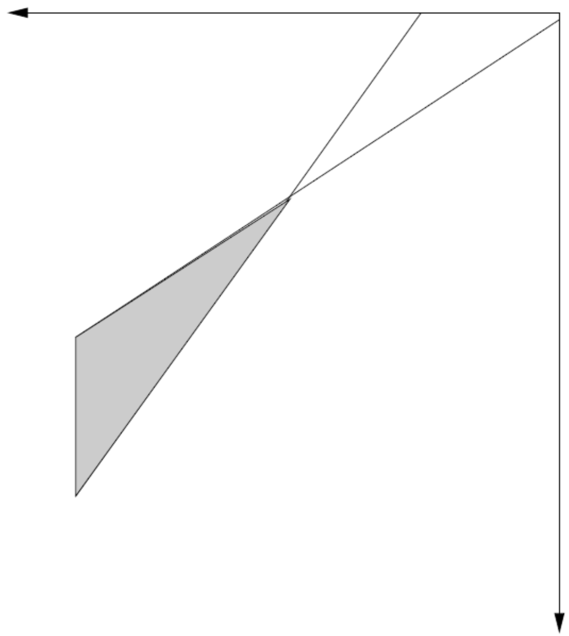
$$\begin{array}{ll} \text{maximise} & Z = -X_1 + X_2 \\ \text{subject to} & 3X_1 + X_2 \geq 6 \\ & 3X_1 + X_2 \leq 3 \\ & X_1 \geq 0 \quad X_2 \geq 0 \end{array}$$

- The constraints are inconsistent, and there's no feasible region



# More graphical examples

- Something else that can occur is that the constraints, apart from the non-negativity ones, are consistent but not in the positive quadrant: (X1,X2 positive).



# More graphical examples

- If these last 3 cases occur in practice, the reason is probably an error by the modeller: omitted constraints, typos, etc.

- Real problems are usually feasible and bounded by resource limits. But it can be hard to spot these cases when there are more than 2 variables, as we can't visualise them: the feasible region is an n-dimensional polyhedron, and its boundary is a hyperplane.

- But we still find optimal solutions at the extreme points of the feasible region, if they exist.

# Solving LPs

- There are several algorithms, but the best-known algorithm is the ***Simplex algorithm***. Its development in the 1940s was a landmark in OR and it's been greatly improved since.
- Simplex uses the method mentioned already: it searches the boundary of the feasible region to find an optimal feasible solution, which must occur at an extreme point.
- The latest algorithms don't do this: instead they follow a path through the feasible region to an optimum. These ***interior point methods*** have an important theoretical property: they always solve the LP in a time that's polynomial in the problem size.
- It's not known if Simplex is always polynomial, but in practice Simplex is often most efficient, and is used in almost all commercial optimisation packages.
- Interior point methods seem to be best on large, *sparse* problems (in which each constraint contains only a few of the variables, so that the matrix of coefficients is sparse).

# Software

MATLAB

Maple™

GUROBI OPTIMIZATION

AMPL

AIMMS

GAMS

PuLP

Next week we will be using PuLP

SciPy

Today we will be using SciPy

IBM ILOG CPLEX Optimization Studio

# Problem 1:

A family wants to pack a picknick basket with as many pieces of fruit as possible. Different kids of the family have a very strong opinion on how that should be packed:

- No more than 4 bananas must be packed;
- No more than 9 apples must be packed;
- The number of bananas must be no more than twice the number of apples.
- The number of bananas must be no less than half the number of apples.

Help the family to find a solution using linear programming!

# Problem 2:

- A small farm needs to make decision how much wheat and how much barley they should grow.
- The farm size is 40ha, the farm has 50,000€ cash reserves
- The production costs (ploughing, insecticides, fungicides, fertiliser, harvesting) add up to 1,150€/ha for barley and 1,360€/ha for wheat.
- In 2018 the yield for wheat was 10t/ha, for barley 9t/ha.
- The expected price for wheat is 170€/t and for barley 170€/t.
- The price for straw is 243€/ha for barley straw is 300€/ha.
- What would be the optimal breakdown for wheat and barley and the margin at the end of the year?