PHYS516 ASSIGNMENT 1—MONTE CARLO BASICS
Due January 25 (Mon), 2016, 11:59 pm

Email your submission to our TA (Zhengzhi Ma, zhengzhm@usc.edu) by 11:59 pm, or hand it in at the class. For email, please create a single file (*e.g.*, in PDF format) that has all materials (source code, plots, and explanation) and your name in it.

Part I—Theoretical Foundation

Submit the answers to the following two questions. Include all the algebra and proof steps, and explain what they mean *in your own words* (see the Phys 516 assignment guidelines).

1. (**Monte Carlo sampling**) Consider an integration, $F = \int_0^1 dx f(x)$, and its unbiased estimator,

$$\overline{f} = \frac{1}{M}\sum_{n=1}^{M} f(r_n),$$

where $\{r_n \mid n = 1, ..., M\}$ is a set of uniform random numbers in the range [0, 1] and M is the number of random numbers. Prove that the standard deviation (*i.e.*, error bar) of the finite sample mean $\overline{f}$ is estimated by

$$\sqrt{\frac{\overline{f^2} - \left(\overline{f}\right)^2}{M-1}},$$

where

$$\overline{f^2} = \frac{1}{M}\sum_{n=1}^{M} f^2(r_n).$$

2. **(Nonuniform random number generation: transformation method)** Prove that the Box-Muller algorithm below generates a normally distributed random number, following the lecture slides on "Monte Carlo Basics".

Box-Muller algorithm: Generates a normally distributed random number ζ with unit variance,

$$\rho(\zeta) = \sqrt{\frac{1}{2\pi}} \exp\left(-\frac{\zeta^2}{2}\right).$$ [1]

(a) Generate uniform random numbers r_1 and r_2 in the range (0, 1)
(b) Calculate $\zeta_1 = (-2\ln r_1)^{1/2}\cos(2\pi r_2)$ and $\zeta_2 = (-2\ln r_1)^{1/2}\sin(2\pi r_2)$
Then both ζ_1 and ζ_2 are the desired normally distributed random number, and either can be used as ζ.

[1] For the normalization of the normal distribution function, see Appendix A on p. 21 of the lecture note on "Monte Carlo basics". For its variance, see the Problem in p. 23 of the same lecture note.

Part II—Numerical Test

In this assignment, you will numerically test the dependence of the Monte Carlo (MC) error,

$$Std\{\overline{f(x)}\} = \frac{Std\{f(x)\}}{\sqrt{M}},$$

on the sample size M in the sample mean integral of π, `mean.c` (see §1 in the lecture note on "Monte Carlo Basics"):

$$\frac{1}{M}\sum_{n=1}^{M}\frac{4}{1+r_n^2} = \overline{\frac{4}{1+r_n^2}} \approx \pi \quad \left(r_n \in [0,1]\right) \quad .$$

(Assignment)

1. (**Monte Carlo estimate**) Plot your MC estimate of π *along with an error bar* using the unbiased estimate of its standard deviation,

$$\sqrt{\frac{\overline{f^2} - \left(\overline{f}\right)^2}{M-1}}$$

(where $f(r_n) = 4/(r_n^2+1)$) as a function of $\log_{10}M$ for $M = 10, 10^2, ..., 10^6$. Submit the source code and the plot.

2. (**Monte Carlo error**) We next perform a numerical experiment to directly measure the standard deviation of the MC estimate. To do so, for each of the above M values, estimate π for N_{seed} times using N_{seed} different random-number seeds (use $N_{\text{seed}} = 100$). Calculate the standard deviation σ_M of these N_{seed} estimates, $\pi_1, \pi_2, ..., \pi_{N\text{seed}}$:

$$\sigma_M = \sqrt{\frac{1}{N_{\text{seed}}}\sum_{i=1}^{N_{\text{seed}}}\pi_i^2 - \left[\frac{1}{N_{\text{seed}}}\sum_{i=1}^{N_{\text{seed}}}\pi_i\right]^2} \quad .$$

Plot the measured values of $\log_{10}\sigma_M$ as a function of $\log_{10}M$ for $M = 10, 10^2, ..., 10^6$, along with its unbiased estimate from question 1 above (are they similar?). If the MC error decreases as $\sigma_M = C/\sqrt{M}$ (C is the standard deviation of the same quantity in the underlying population), then

$$\log_{10}\sigma_M = \log_{10}C - \frac{1}{2}\log_{10}M$$

so that you can fit your data to a line with slope –0.5. Use the least square fit (see the lecture note on "Least square fit of a line") to obtain the power in your plot of σ_M measurement. Submit the source code, the plot, and the estimated value of the power.