COMS 4771 Spring 2017 Homework 2

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Problem 3
(a) Exponential distribution
[Posterior]
The given prior is the gamma distribution with $\alpha = 4$ and $\beta = 1$

$$\begin{aligned}
p(\alpha|X) &= \frac{p(\alpha)p(X|\alpha)}{p(X)} \\
&= \frac{\frac{\lambda^3 e^{-\lambda}}{6}\prod_{i=1}^{N}\lambda e^{-\lambda x_i}}{p(X)} \\
&\propto \lambda^{N+3}e^{-\lambda}e^{\sum_{i=1}^{N}x_i} \\
&= \lambda^{(N+4)-1}e^{-\lambda(1+\sum_{i=1}^{N}x_i)}
\end{aligned}$$

Therefore

$$\begin{aligned}
p(\alpha|X) &= Gamma(\lambda; N+4, 1+\sum_{i=1}^{N}x_i) \\
&= \frac{(1+\sum_{i=1}^{n}x_i)^{N+4}}{\Gamma(N+4)}\lambda^{N+3}e^{-\lambda(1+\sum_{i=1}^{n}x_i)}
\end{aligned}$$

[EAP]
Mean of the gamma distribution $E[Gamma(\lambda;\alpha,\beta)] = \frac{\alpha}{\beta}$
Therefore

$$EAP = \frac{N+4}{1+\sum_{i=1}^{N}x_i}$$

(b) Coin problem
[Posterior]
Let's say N_1 is the number of heads out of N trials.

$$\begin{aligned}
p(\alpha|X) &= \frac{p(\alpha)p(X|\alpha)}{p(X)} \\
&= \frac{30\alpha^2(1-\alpha)^2\alpha^{N_1}(1-\alpha)^{N-N_1}}{p(X)} \\
&= \frac{30\alpha^{N_1+2}(1-\alpha)^{N-N_1+2}}{p(X)} \\
&= \frac{\alpha^{N_1+2}(1-\alpha)^{N-N_1+2}}{c(N_1+2, N-N_1+2)}
\end{aligned}$$

where c(m, k) is defined as following

$$c(m,k) = \frac{m!k!}{(m+k+1)!}$$

[EAP]

$$\begin{aligned} EAP &= E[Posterior] \\ &= E[\frac{\alpha^{N_1+2}(1-\alpha)^{N-N_1+2}}{c(N_1+2, N-N_1+2)}] \\ &= \frac{\int_0^1 \alpha\alpha^{N_1+2}(1-\alpha)^{N-N_1+2}d\alpha}{c(N_1+2, N-N_1+2)} \\ &= \frac{c(N_1+3, N-N_1+2)}{c(N_1+2, N-N_1+2)} \\ &= \frac{N_1+3}{N+6} \end{aligned}$$

References

[1] https://www.probabilitycourse.com/chapter9/9_1_1_prior_and_posterior.php
[2] Bishop 2.3
[3] https://en.wikipedia.org/wiki/Gamma_function
[4] http://web.cse.ohio-state.edu/ kulis/teaching/788_sp12/scribe_notes/lecture3.pdf