COMS 4771 Spring 2017 Homework 2

Jin Tack Lim, jl4312

Problem 3
(a) Exponential distribution
[Posterior]
The given prior is the gamma distribution with $\alpha = 4$ and $\beta = 1$

$$p(\alpha|X) = \frac{p(\alpha)p(X|\alpha)}{p(X)}$$

$$= \frac{\frac{\lambda^3 e^{-\lambda}}{6} \prod_{i=1}^{N} \lambda e^{-\lambda x_i}}{p(X)}$$

$$\propto \lambda^{N+3} e^{-\lambda} e^{\sum_{i=1}^{N} x_i}$$

$$= \lambda^{(N+4)-1} e^{-\lambda(1+\sum_{i=1}^{N} x_i)}$$

Therefore

$$p(\alpha|X) = Gamma(\lambda; N+4, 1+\sum_{i=1}^{N} x_i)$$

$$= \frac{1}{\Gamma(N+4)} \lambda^{N+3} e^{-\lambda(1+\sum_{i=1}^{N} x_i)}$$

[EAP]
Mean of the gamma distribution $E[Gamma(\lambda; \alpha, \beta)] = \frac{\alpha}{\beta}$
Therefore

$$EAP = \frac{N+4}{1+\sum_{i=1}^{N} x_i}$$

(b) Coin problem
[Posterior]
Let's say N_1 is the number of heads out of N trials.

$$p(\alpha|X) = \frac{p(\alpha)p(X|\alpha)}{p(X)}$$

$$= \frac{30\alpha^2(1-\alpha)^2 \alpha^{N_1}(1-\alpha)^{N-N_1}}{p(X)}$$

$$= \frac{30\alpha^{N_1+2}(1-\alpha)^{N-N_1+2}}{p(X)}$$

$$= \frac{\alpha^{N_1+2}(1-\alpha)^{N-N_1+2}}{c(N_1+2, N-N_1+2)}$$

where c(m, k) is defined as following

$$c(m, k) = \frac{m!k!}{(m + k + 1)!}$$

[EAP]

$$
\begin{aligned}
EAP &= E[Posterior] \\
&= E\left[\frac{\alpha^{N_1+2}(1 - \alpha)^{N-N_1+2}}{c(N_1 + 2, N - N_1 + 2)}\right] \\
&= \frac{\int_0^1 \alpha\alpha^{N_1+2}(1 - \alpha)^{N-N_1+2}d\alpha}{c(N_1 + 2, N - N_1 + 2)} \\
&= \frac{c(N_1 + 3, N - N_1 + 2)}{c(N_1 + 2, N - N_1 + 2)} \\
&= \frac{N_1 + 3}{N_1 + 5}
\end{aligned}
$$