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Carrera: **Ingeniería Mecatrónica**

Materia: **Robótica**

Reporte: **Manipulador RoRR: Modelado de Lagrange**

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Índice general

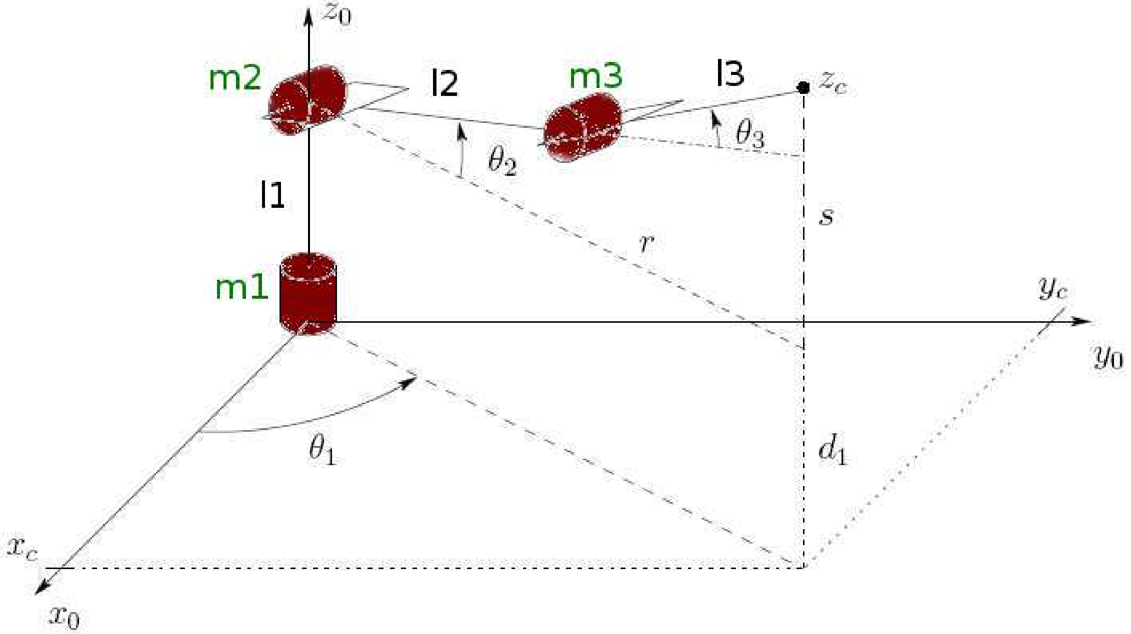
1 Modelado de Lagrange

1.1 Introducción

El modelado de Lagrange representa sistemas dinámicos descritos por un conjunto de ecuaciones diferenciales que representan la evolución en el tiempo del sistema está sujeto a limitaciones holonómicas, cuando las fuerzas de restricción satisfacen el principio del trabajo virtual (El trabajo hecho por fuerzas externas correspondientes a cualquier conjunto de desplazamientos virtuales es cero). La formulación de Lagrange puede definirse a grandes rasgos como una especie de "balance de energias" de sistemas dinámicos.

1.2 Manipulador RoRR

Realizar el modelado con ecuaciones de Lagrange del manipulador de 3 grados de libertad RoRR.



Para modelar con ecuaciones de Newton-Lagrange se considera que las masas de los eslabones estan concentradas en las uniones (m_1,m_2,m_3) y se realizan cambios de variable:

$$d_1 = l_1$$

$$a_2 = l_2$$

$$a_3 = l_3$$

*Se considerará la matriz de masas inerciales como nula ($I = 0$).

1.2.1 Enlace 1

1.2.1.1 Posición

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_1 \end{bmatrix}$$

1.2.1.2 Velócidad

No tiene jacobiano, por lo que

$$v_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1^T v_1 = 0$$

1.2.1.3 Energía Cinética

$$K_1(\theta_1, \dot{\theta}_1) = \frac{1}{2} m_1 v_1^T v_1 + \frac{1}{2} l_1 \dot{\theta}_1^2$$

$$K_1(\theta_1, \dot{\theta}_1) = \frac{1}{2} I_1 \dot{\theta}_1^2$$

$$K_1(\theta_1, \dot{\theta}_1) = 0$$

1.2.1.4 Energía Potencial

Si

$$U = -m_i^0 g^T P_{ci}^0 + U_{refi}$$

y el enlace 1 está en el suelo, lo que hace a $P_{ci}^0 = 0$

$$U_1 = 0 + U_{refi} = m_1 g l_1$$

1.2.2 Enlace 2

1.2.2.1 Posición

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} a_2C_1C_2 \\ a_2C_2S_1 \\ a_2S_2 + d_1 \end{bmatrix} = \begin{bmatrix} l_2C_1C_2 \\ l_2C_2S_1 \\ l_2S_2 + l_1 \end{bmatrix}$$

1.2.2.2 Velócidad

$$J_2 = \begin{bmatrix} \frac{\delta X_2}{\delta\theta_1} & \frac{\delta X_2}{\delta\theta_2} & \frac{\delta X_2}{\delta\theta_3} \\ \frac{\delta Y_2}{\delta\theta_1} & \frac{\delta Y_2}{\delta\theta_2} & \frac{\delta Y_2}{\delta\theta_3} \\ \frac{\delta Z_2}{\delta\theta_1} & \frac{\delta Z_2}{\delta\theta_2} & \frac{\delta Z_2}{\delta\theta_3} \end{bmatrix} = \begin{bmatrix} -l_2S_1C_2(\dot{\theta}_1) & -l_2C_1S_2(\dot{\theta}_2) & 0 \\ l_2C_1C_2(\dot{\theta}_1) & -l_2S_1S_2(\dot{\theta}_2) & 0 \\ 0 & l_2C_2(\dot{\theta}_2) & 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} l_2(-S_1C_2(\dot{\theta}_1) - C_1S_2(\dot{\theta}_2)) \\ l_2(C_1C_2(\dot{\theta}_1) - S_1S_2(\dot{\theta}_2)) \\ l_2C_2(\dot{\theta}_2) \end{bmatrix}$$

$$v_2^T v_2 = \begin{bmatrix} l_2(-S_1C_2(\dot{\theta}_1) - C_1S_2(\dot{\theta}_2)) \\ l_2(C_1C_2(\dot{\theta}_1) - S_1S_2(\dot{\theta}_2)) \\ l_2C_2(\dot{\theta}_2) \end{bmatrix}^T \begin{bmatrix} l_2(-S_1C_2(\dot{\theta}_1) - C_1S_2(\dot{\theta}_2)) \\ l_2(C_1C_2(\dot{\theta}_1) - S_1S_2(\dot{\theta}_2)) \\ l_2C_2(\dot{\theta}_2) \end{bmatrix}$$

$$v_2^T v_2 = \begin{bmatrix} l_2(-S_1C_2(\dot{\theta}_1) - C_1S_2(\dot{\theta}_2)) & l_2(C_1C_2(\dot{\theta}_1) - S_1S_2(\dot{\theta}_2)) & l_2C_2(\dot{\theta}_2) \end{bmatrix} \begin{bmatrix} l_2(-S_1C_2(\dot{\theta}_1) - C_1S_2(\dot{\theta}_2)) \\ l_2(C_1C_2(\dot{\theta}_1) - S_1S_2(\dot{\theta}_2)) \\ l_2C_2(\dot{\theta}_2) \end{bmatrix}$$

$$v_2^T v_2 = l_2^2[\dot{\theta}_1^2(S_1^2C_2^2 + C_1^2C_2^2) + \dot{\theta}_2^2(C_1^2S_1^2 + S_1^2S_2^2 + C_2^2)]$$

Aplicando Identidades trigonométricas

$$S_1^2C_2^2 + C_1^2C_2^2 = S_1^2C_2^2 + (1 - S_1^2)C_2^2 = S_1^2C_2^2 - S_1^2C_2^2 + C_2^2 = C_2^2$$

$$C_1^2S_1^2 + S_1^2S_2^2 + C_2^2 = C_1^2S_1^2 + (1 - C_1^2)S_2^2 + C_2^2 = C_1^2S_1^2 - C_1^2S_2^2 + C_2^2 + S_2^2 = 1 + C_1^2S_1^2 - C_1^2S_2^2 = 1 + C_1^2(S_1^2 - S_2^2)$$

$$v_2^T v_2 = l_2^2[\dot{\theta}_1^2C_2^2 + \dot{\theta}_2^2(1 + C_1^2(S_1^2 - S_2^2))]$$

1.2.2.3 Energía Cinética

$$K_2(\theta_2, \dot{\theta}_2) = \frac{1}{2}m_2v_2^Tv_2 + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$K_2(\theta_2, \dot{\theta}_2) = \frac{1}{2}m_2l_2^2[\dot{\theta}_1^2C_2^2 + \dot{\theta}_2^2(1 + C_1^2(S_1^2 - S_2^2))]$$

1.2.2.4 Energía Potencial

$$U_2 = m_2 l_2 g Sen\theta_2$$

1.2.3 Enlace 3

1.2.3.1 Posición

$$\begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} C_1\left[C_{23}l_3 + C_2 l_2\right] \\ S_1\left[S_{23}l_3 + S_2 l_2\right] \\ l_3 S_{23} + l_2 S_2 + d_1 \end{bmatrix} = \begin{bmatrix} l_3 C_1 C_{23} + l_2 C_1 C_2 \\ l_3 S_1 S_{23} + l_2 S_1 S_2 \\ l_3 S_{23} + l_2 S_2 + d_1 \end{bmatrix} = \begin{bmatrix} l_3 C_1 C_{(\theta_2+\theta_3)} + l_2 C_1 C_2 \\ l_3 S_1 S_{(\theta_2+\theta_3)} + l_2 S_1 S_2 \\ l_3 S_{(\theta_2+\theta_3)} + l_2 S_2 + d_1 \end{bmatrix}$$

1.2.3.2 Velócidad

$$J_3 = \begin{bmatrix} \frac{\delta X_3}{\delta\theta_1} & \frac{\delta X_3}{\delta\theta_2} & \frac{\delta X_3}{\delta\theta_3} \\ \frac{\delta Y_3}{\delta\theta_1} & \frac{\delta Y_3}{\delta\theta_2} & \frac{\delta Y_3}{\delta\theta_3} \\ \frac{\delta Z_3}{\delta\theta_1} & \frac{\delta Z_3}{\delta\theta_2} & \frac{\delta Z_3}{\delta\theta_3} \end{bmatrix}$$

$$J_3 = \begin{bmatrix} (-l_3 S_1 C_{23} - l_2 S_1 C_2)(\dot\theta_1) & (-l_3 C_1 S_{(\theta_2+\theta_3)})(\dot\theta_2 + \dot\theta_3) - (l_2 C_1 S_2)(\dot\theta_2) & (-l_2 C_1 S_{(\theta_2+\theta_3)}(\dot\theta_2 + \dot\theta_3)) \\ (l_3 C_1 S_{23} + l_2 C_1 S_2)(\dot\theta_1) & (l_3 S_1 C_{(\theta_2+\theta_3)})(\dot\theta_2 + \dot\theta_3) + (l_2 S_1 C_2)(\dot\theta_2) & (l_2 S_1 C_{(\theta_2+\theta_3)})(\dot\theta_2 + \dot\theta_3) \\ 0 & (l_3 C_{(\theta_2+\theta_3)}(\dot\theta_2 + \dot\theta_3) + (l_2 C_2)(\dot\theta_2) & (l_3 C_{(\theta_2+\theta_3)})(\dot\theta_2 + \dot\theta_3) \end{bmatrix}$$

$$v_3 = \begin{bmatrix} \dot X_3 \\ \dot Y_3 \\ \dot Z_3 \end{bmatrix} = \begin{bmatrix} (-l_3 S_1 C_{23} - l_2 S_1 C_2)(\dot\theta_1) + (-l_3 C_1 S_{(\theta_2+\theta_3)})(\dot\theta_2 + \dot\theta_3) - (l_2 C_1 S_2)(\dot\theta_2) + (-l_2 C_1 S_{(\theta_2+\theta_3)}(\dot\theta_2 + \dot\theta_3)) \\ (l_3 C_1 S_{23} + l_2 C_1 S_2)(\dot\theta_1) + (l_3 S_1 C_{(\theta_2+\theta_3)})(\dot\theta_2 + \dot\theta_3) + (l_2 S_1 C_2)(\dot\theta_2) + (l_2 S_1 C_{(\theta_2+\theta_3)})(\dot\theta_2 + \dot\theta_3) \\ (l_3 C_{(\theta_2+\theta_3)}(\dot\theta_2 + \dot\theta_3) + (l_2 C_2)(\dot\theta_2) + (l_3 C_{(\theta_2+\theta_3)})(\dot\theta_2 + \dot\theta_3) \end{bmatrix}$$

$$v_3^T = \begin{bmatrix} (-l_3S_1C_{23} - l_2S_1C_2)(\dot{\theta}_1) + (-l_3C_1S_{(\theta_2+\theta_3)})(\dot{\theta}_2 + \dot{\theta}_3) - (l_2C_1S_2)(\dot{\theta}_2) + (-l_2C_1S_{(\theta_2+\theta_3)}(\dot{\theta}_2 + \dot{\theta}_3)) \\ (l_3C_1S_{23} + l_2C_1S_2)(\dot{\theta}_1) + (l_3S_1C_{(\theta_2+\theta_3)})(\dot{\theta}_2 + \dot{\theta}_3) + (l_2S_1C_2)(\dot{\theta}_2) + (l_2S_1C_{(\theta_2+\theta_3)})(\dot{\theta}_2 + \dot{\theta}_3) \\ (l_3C_{(\theta_2+\theta_3)}(\dot{\theta}_2 + \dot{\theta}_3) + (l_2C_2)(\dot{\theta}_2) + (l_3C_{(\theta_2+\theta_3)})(\dot{\theta}_2 + \dot{\theta}_3) \end{bmatrix}^T$$

$$v_3^T = \begin{bmatrix} v_{3_1}^T & v_{3_2}^T & v_{3_3}^T \end{bmatrix}$$

$$v_{3_1}^T = (-l_3S_1C_{23} - l_2S_1C_2)(\dot{\theta}_1) + (-l_3C_1S_{(\theta_2+\theta_3)})(\dot{\theta}_2 + \dot{\theta}_3) - (l_2C_1S_2)(\dot{\theta}_2) + (-l_2C_1S_{(\theta_2+\theta_3)}(\dot{\theta}_2 + \dot{\theta}_3))$$

$$v_{3_2}^T = (l_3C_1S_{23} + l_2C_1S_2)(\dot{\theta}_1) + (l_3S_1C_{(\theta_2+\theta_3)})(\dot{\theta}_2 + \dot{\theta}_3) + (l_2S_1C_2)(\dot{\theta}_2) + (l_2S_1C_{(\theta_2+\theta_3)})(\dot{\theta}_2 + \dot{\theta}_3)$$

$$v_{3_3}^T = (2l_3C_{(\theta_2+\theta_3)}(\dot{\theta}_2 + \dot{\theta}_3) + (l_2C_2)(\dot{\theta}_2)$$

$$v_3^T = [(-l_3S_1C_{23} - l_2S_1C_2)(\dot{\theta}_1) + (-l_3C_1S_{(\theta_2+\theta_3)})(\dot{\theta}_2 + \dot{\theta}_3) - (l_2C_1S_2)(\dot{\theta}_2) + (-l_2C_1S_{(\theta_2+\theta_3)}(\dot{\theta}_2 + \dot{\theta}_3))$$

$$(l_3C_1S_{23}+l_2C_1S_2)(\dot{\theta}_1)+(l_3S_1C_{(\theta_2+\theta_3)})(\dot{\theta}_2+\dot{\theta}_3)+(l_2S_1C_2)(\dot{\theta}_2)+(l_2S_1C_{(\theta_2+\theta_3)})(\dot{\theta}_2+\dot{\theta}_3)\ (l_3C_{(\theta_2+\theta_3)}(\dot{\theta}_2+\dot{\theta}_3)+(l_2C_2)(\dot{\theta}_2)+(l_3C_{(\theta_2+\theta_3)})(\dot{\theta}_2+\dot{\theta}_3)]$$

Siendo así, el resultado es:

$$v_3^T v_3 = l_3^2(S_1^2 C_{23}^2 \dot{\theta}_1^2 + 2S_1C_1S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3) + C_1^2S_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2 + C_1^2S_{23}^2\dot{\theta}_1^2 + 2S_1C_1S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3) + S_1^2C_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2$$

$$+4C_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2) + l_2^2(S_1^2C_2^2\dot{\theta}_1^2 + 2S_1C_1S_2C_2\dot{\theta}_1\dot{\theta}_2 + 2C_1S_1C_2S_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3) + C_1^2S_2^2\dot{\theta}_2^2 + C_1^2S_2S_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3) + C_1^2S_{23}^2(\dot{\theta}_2+\dot{\theta}_2)^2$$

$$+C_1^2S_2^2\dot{\theta}_1^2 + S_1C_1C_2S_2\dot{\theta}_1\dot{\theta}_2 + 2C_1S_1S_2C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_2) + S_1C_1S_2C_2\dot{\theta}_1\dot{\theta}_2 + S_1^2C_2^2\dot{\theta}_2^2 + 2S_1^2C_2C_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3) + S_1^2C_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2$$

$$+C_2^2\dot{\theta}_2^2) + l_2l_3(2S_1^2C_2C_{23}\dot{\theta}_1^2 + 2S_1C_1S_2C_{23}\dot{\theta}_1\dot{\theta}_2 + 2C_1S_1S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3) + C_1S_1S_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3) + C_1S_1S_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3)$$

$$+3C_1^2S_2S_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3) + 2C_1^2S_{23}(\dot{\theta}_2+\dot{\theta}_3)^2 + 2C_1^2S_2S_{23}\dot{\theta}_1^2 + 2C_1S_1C_2S_{23}\dot{\theta}_1\dot{\theta}_2 + 2C_1S_2S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3)$$

$$+2C_1S_1S_2C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3) + 2S_1^2C_2C_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3) + 2S_1^2C_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2 + 4C_2C_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3))$$

1.2.3.3 Energía cinética

$$K_3(\theta_3,\dot{\theta}_3) = \frac{1}{2}m_3[l_3^2(S_1^2C_{23}^2\dot{\theta}_1^2 + 2S_1C_1S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3) + C_1^2S_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2 + C_1^2S_{23}^2\dot{\theta}_1^2 + 2S_1C_1S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3) + S_1^2C_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2$$

$$+4C_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2) + l_2^2(S_1^2C_2^2\dot{\theta}_1^2 + 2S_1C_1S_2C_2\dot{\theta}_1\dot{\theta}_2 + 2C_1S_1C_2S_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3) + C_1^2S_2^2\dot{\theta}_2^2 + C_1^2S_2S_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3) + C_1^2S_{23}^2(\dot{\theta}_2+\dot{\theta}_2)^2$$

$$+C_1^2S_2^2\dot{\theta}_1^2+S_1C_1C_2S_2\dot{\theta}_1\dot{\theta}_2+2C_1S_1S_2C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_2)+S_1C_1S_2C_2\dot{\theta}_1\dot{\theta}_2+S_1^2C_2^2\dot{\theta}_2^2+2S_1^2C_2C_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3)+S_1^2C_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2$$

$$+C_2^2\dot{\theta}_2^2)+l_2l_3(2S_1^2C_2C_{23}\dot{\theta}_1^2+2S_1C_1S_2C_{23}\dot{\theta}_1\dot{\theta}_2+2C_1S_1S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3)+C_1S_1S_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3)+C_1S_1S_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3)$$

$$+3C_1^2S_2S_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3)+2C_1^2S_{23}(\dot{\theta}_2+\dot{\theta}_3)^2+2C_1^2S_2S_{23}\dot{\theta}_1^2+2C_1S_1C_2S_{23}\dot{\theta}_1\dot{\theta}_2+2C_1S_2S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3)$$

$$+2C_1S_1S_2C_{23}\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_3)+2S_1^2C_2C_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3)+2S_1^2C_{23}^2(\dot{\theta}_2+\dot{\theta}_3)^2+4C_2C_{23}\dot{\theta}_2(\dot{\theta}_2+\dot{\theta}_3))]$$

1.2.3.4 Energía Potencial

$$U_3=m_3l_2gSen\theta_2+m_3l_3gSen(\theta_2+\theta_3)$$

1.2.4 Lagrangiano

Si el Lagrangiano de un sistema es

$$L = K - U$$

$$L = K(\theta, \dot{\theta}) - U(\theta)$$

$$L = K_1(\theta, \dot{\theta}) + K_2(\theta, \dot{\theta}) + K_3(\theta, \dot{\theta}) - U_1(\theta) - U_2(\theta) - U_3(\theta)$$

El Lagrangiano para éste manipulador (RoRR de 3 DOF) será como a continuación se muestra (Despreciando las masas inerciales).

$$L_{RoRR} = \frac{1}{2}(m_2 l_2^2[\dot{\theta}_1^2 C_2^2 + \dot{\theta}_2^2(1 + C_1^2(S_1^2 - S_2^2))] + m_3[l_3^2(S_1^2 C_{23}^2 \dot{\theta}_1^2 + 2S_1C_1S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_3) + C_1^2 S_{23}^2(\dot{\theta}_2 + \dot{\theta}_3)^2 + C_1^2 S_{23}^2 \dot{\theta}_1^2$$

$$+2S_1C_1S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_3) + S_1^2 C_{23}^2(\dot{\theta}_2 + \dot{\theta}_3)^2 + 4C_{23}^2(\dot{\theta}_2 + \dot{\theta}_3)^2) + l_2^2(S_1^2 C_2^2 \dot{\theta}_1^2 + 2S_1C_1S_2C_2\dot{\theta}_1\dot{\theta}_2$$

$$+2C_1S_1C_2S_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_3) + C_1^2 S_2^2 \dot{\theta}_2^2 + C_1^2 S_2 S_{23}\dot{\theta}_2(\dot{\theta}_2 + \dot{\theta}_3) + C_1^2 S_{23}^2(\dot{\theta}_2 + \dot{\theta}_2)^2 + C_1^2 S_2^2 \dot{\theta}_1^2 + S_1C_1C_2S_2\dot{\theta}_1\dot{\theta}_2$$

$$+2C_1S_1S_2C_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_2) + S_1C_1S_2C_2\dot{\theta}_1\dot{\theta}_2 + S_1^2 C_2^2 \dot{\theta}_2^2 + 2S_1^2 C_2 C_{23}\dot{\theta}_2(\dot{\theta}_2 + \dot{\theta}_3) + S_1^2 C_{23}^2(\dot{\theta}_2 + \dot{\theta}_3)^2$$

$$+C_2^2\dot{\theta}_2^2) + l_2 l_3(2S_1^2 C_2 C_{23}\dot{\theta}_1^2 + 2S_1 C_1 S_2 C_{23}\dot{\theta}_1\dot{\theta}_2 + 2C_1 S_1 S_{23} C_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_3) + C_1 S_1 S_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_3)$$

$$+C_1 S_1 S_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_3) + 3C_1^2 S_2 S_{23}\dot{\theta}_2(\dot{\theta}_2 + \dot{\theta}_3) + 2C_1^2 S_{23}(\dot{\theta}_2 + \dot{\theta}_3)^2 + 2C_1^2 S_2 S_{23}\dot{\theta}_1^2 + 2C_1 S_1 C_2 S_{23}\dot{\theta}_1\dot{\theta}_2$$

$$+2C_1 S_2 S_{23} C_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_3) + 2C_1 S_1 S_2 C_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_3) + 2S_1^2 C_2 C_{23}\dot{\theta}_2(\dot{\theta}_2 + \dot{\theta}_3) + 2S_1^2 C_{23}^2(\dot{\theta}_2 + \dot{\theta}_3)^2$$

$$+4C_2 C_{23}\dot{\theta}_2(\dot{\theta}_2 + \dot{\theta}_3))]) - m_1 g l_1 - m_2 l_2 g Sen\theta_2 - m_3 l_2 g Sen\theta_2 + m_3 l_3 g Sen(\theta_2 + \theta_3)$$

1.3 Conclusiones

Modelando con ecuaciones de Lagrange se puede representar cualquier sistema mecánico de una manera mucho más realista que como sería utilizando otros métodos, éstas ecuaciones consideran variables que muchas veces se desprecian, como son la inercia y la gravedad. Éste análisis es una herramienta importante para el diseño de controladores precisos que se usan para compensar éste tipo de sistemas que predeterminadamente son inestables.