Master Thesis:
Heterogeneous Network Games: Conflicting Preferences

Manuel Muñoz-Herrera*

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Abstract

I propose a model of network games with heterogeneity introduced by endowing players with types that generate preferences among their choices. I study two classes of games: strategic complements (SC) or substitutes (SS) in payoffs. The payoff function depends on the network structure, and I ask how does heterogeneity shape players' decision making, what is its effect on equilibria, conditions of stability, and welfare. Heterogeneity in players' type and degree generate heterogeneous thresholds. Network configurations in equilibrium can be Unfrustrated is each player chooses the action corresponding to her type of Frustrated when at least one player is not. The Frustrated configurations can be either Specialized if all the players are choosing the same action (only in SC), or Hybrid when both actions coexist. A refinement of the Nash Equilibria through stochastic mutations of pairs of neighbors limits multiplicity to a subset of Stable Configurations (SSC). I find that the Nash networks are absorbing states from where it is possible to leave only through mutations and that such mutations in most cases will lead to a frustrated hybrid configuration which is the risk dominant equilibrium.

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*Master on Economics. ERI-CES, Universitat de Valencia. Email: edson.munoz@uv.es

Introduction

Agents in a network can be involved in different social and economic interactions, aiming to coordinate (anti-coordinate) choices with those connected to them, on account that their wellbeing depends on the behavior adopted by themselves and their neighbors. Examples such as acquiring a specific technology (input) between companies, getting involved in a riot, or job search, give account of this. However, the approach commonly used in social networks has considered players as homogeneous. In the sense that they have no preferences, but choose an action over the others if the number of their neighbors making that same choice is higher than a given threshold. Heterogeneity has been included mostly by allowing different degree distribution among players. This paper proposes a model of network games with individual-level heterogeneity introduced by endowing players with types that generate preferences over an action among their choices. I study two classes of network games: strategic complements (SC) or strategic substitutes (SS). This paper is motivated by the consideration that when endowing players with preferences over their choice set, the game played might be different depending on a player's and her opponents' type, even if they have the same degree. Moreover, I consider a payoff function that is characterized by a network structure. That is, the payoff of each agent depends on the actions of her neighbors. I ask how does heterogeneity in types and degree shape players' decision making and expected payoffs, what is its effect upon equilibrium in complete information, the conditions of stability for equilibrium, and how does this affect welfare.

For a review of network games see Jackson (2008), and Goyal (2007). Galeotti *et al* (2009) model network games with incomplete information, both SC and SS, where the type of a player is her degree. Bramoullé and Rogers (2010) use the same consideration of a player's type to model homophily, and Bramoullé and Kranton (2007) models games with SS. Analysis of equilibrium in network games are found in Bloch and Jackson (2006), Jackson and Yariv (2007). My contribution to the aforementioned literature is the modeling of heterogeneity in SC and SS games played on networks by endowing players with types that generate preferences over actions. I assume local information, so that each player knows her degree, her type and the actions chose by her neighbors. the structure of the network, all the connections, the distribution of types and the actions each player chooses. However, for my case, players only need to know local information and not the whole distribution of the network, they do not even need to know the type of neighbors they are connected with.

I illustrate the nature of my analysis by means of two examples of 2×2 games, which give an intuition on the two classes of games that can be played in my model. First, I consider games with strategic complements in payoffs (SC), which can also be considered as coordination games, and afterwards I follow to games with strategic substitutes (SS), or games of anti-coordination. For both cases I define how heterogeneity affects the structure and conditions of the game depending on the type of players interacting. The games can be played between two players of different or same type, generating multiple cases. This will lead to conflicting preferences when two players of different (the same) type interact in games with strategic complements (substitutes). It is possible to Pareto rank equilibria when there are no conflicting preferences. The two classes of games presented are the setting for my model on network games.

A games with strategic complements in payoffs can be considered as a coordination game, where each player faces a binary choice set (i.e., 1 or 0). When players are endowed with types, they will prefer one action rather than the other, so that even though they wish to coordinate, the payoff differs if the coordination occurs in the preferred choice or in the one a player dislikes, yet, payoffs will be higher than in the case of anti-coordination. There are many examples of strategic complements in the literature, one simple case is that of coauthors choosing an operative system or a specific technology to work with. When both coauthors coordinate in the same product, the higher the payoffs each will receive.

In a 2×2 game with strategic substitutes there is a higher payoff to a player when her neighbor decides for the opposite behavior she has adopted. That is, they are better-off when anti-coordinating. The case when both players coordinate gives them lower payoffs and a player receives the lowest payoff when she coordinates with her neighbor in the action she dislikes, that is, the action not corresponding to her type. If each one chooses the action most convenient to its type, both maximize their payoff. Even if each one chooses the disliked option, but still anti-coordinate, they will receive a higher payoff than in the case of coordination, where coordinating in the disliked choices gives the lowest payoff because they will compete for the same target population. This type of games are very common in examples such as differentiation of a product between two companies. Say for example, they can either produce in low quantities at high prices (high quality) focusing on a segment of the population with high income, or choose for high levels of production at low prices (low quality) targeting a wider range of population, that of a lower income. Each firm has a level of capital (type) that makes it more optimal for a particular choice.

For the network game, there is a fixed social geographic structure where each player i wants to adopt an action determined by her type, and interacts strategically with her neighbors. The more neighbors in the same (different) choice group of a given player for strategic complements (substitutes), the higher her payoff. A player's choice is affected by the size of her neighborhood selecting the action she likes in relation to a threshold specific to her degree. The payoff structure generates, as a result, two specific thresholds for each player according to her type. A player has incentives to adopt the behavior she likes if the number of her neighbors choosing the same (opposite) action exceeds a given threshold, for games with SC (SS). Literature on models of behavior with thresholds can be find in Granovetter (1978), who uses an example of collective behavior where each person decides to get involved in a riot or not conditioned to a given proportion of people they see doing it first. Chiang (2007) is a model of threshold heterogeneity in networks. Galeotti *et all* (2009) use thresholds to compute the Bayesian Nash equilibrium for network games. A relevant feature of my model is that heterogeneity in types generates heterogeneity in thresholds. A player who likes a specific action has a lower threshold of acceptance for such choice than a player who dislikes it, even with the same degree. A threshold of acceptance is the number of neighbors a player must have in order to choose what she likes, so that before it is crossed, she has incentives to adopt the disliked action. Heterogeneity in players' degree also provides a wider range of thresholds, because a specific player's threshold depends on her type and degree.

When the network game is in equilibrium, I find specific network configurations depending on the class of game being played. Denote a network as *Unfrustrated* in its configuration if all players choose the action they like, which is the action corresponding to their type. On the other case, a network is denoted as *Frustrated* when at least one chooses the action she dislikes (all players is possible). The frustrated configuration is subdivided into *Specialized Frustrated*, when every player chooses the same action, which is only possible in games with strategic complements, and *Hybrid Frustrated* when both actions coexist. This set of configurations of network games in equilibrium are refined by process of stochastic mutations.

The document is structured in 2 sections. In the first I introduce the 2×2 games with types of players and present the heterogeneous network games, where the relation between strategies and thresholds is exposed. I define a stable network and the stability and network configurations resulting from it. In section 3 I carry out a welfare analysis. The concluding remarks close the paper.

1 The Model

1.1 The 2x2 Games

Strategic Complements (Coordination): *Let SC be a 2-person game where each player has two types $\{0, 1\}$ and a set of actions $A = \{1, 0\}$. The payoff matrix depends on the action choices and the type of each player as follows:*

		0	
		1	0
1	1	a,b	c,c
	0	d,d	b,a

		1	
		1	0
1	1	a,a	c,d
	0	d,c	b,b

		0	
		1	0
0	1	b,b	d,c
	0	c,d	a,a

Table 1: Strategic Complements

I consider $a > b > c > d$.[1] The 2×2 coordination game has two Nash equilibria in pure strategies $(1, 1), (0, 0)$ and one in mixed. Players are denoted by their type and the game can be played between two players of equal or opposite types. The case of opposites where each one likes a different action and both rather coordinate, presents conflicting preferences and there is not possible to Pareto rank equilibria. In games between equally-typed players there is no conflict in preferences because each like the same action, and when both choose the action corresponding to their type the equilibrium is Pareto dominant in payoffs: $(1, 1)$

[1]Commonly, the relation of payoffs presented in coordination games is that $a > d$ and $b > c$, but for the case of players with types it is relevant to specify the combinations of actions that are not an equilibrium. In the case of anti-coordination in SC a player receives a lower payoff when being alone in the action she dislikes than being alone in the favorite action. If payoffs in anti-coordination where equal, a player is indifferent between being alone in either action, and the specification of types would admit some ambiguity.

Pareto dominates $(0,0)$ if two players of type 1 are playing, and the opposite for two players of type 0.[2]

Strategic Substitutes (Anti-Coordination): *Let SS be a 2-person game where each player has two types* $\{0,1\}$ *and a set of actions* $A = \{1,0\}$. *The payoff matrix depends on the action choices and the type of each player as follows:*

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\mathbf{0}} \\
 & & 1 & 0 \\ \cline{3-4}
\multirow{2}{*}{$\mathbf{1}$} & 1 & c,d & \underline{a,a} \\ \cline{3-4}
 & 0 & \underline{b,b} & d,c \\ \cline{3-4}
\end{array}
\qquad
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\mathbf{1}} \\
 & & 1 & 0 \\ \cline{3-4}
\multirow{2}{*}{$\mathbf{1}$} & 1 & c,c & \underline{a,b} \\ \cline{3-4}
 & 0 & \underline{b,a} & d,d \\ \cline{3-4}
\end{array}
\qquad
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\mathbf{0}} \\
 & & 1 & 0 \\ \cline{3-4}
\multirow{2}{*}{$\mathbf{0}$} & 1 & d,d & \underline{b,a} \\ \cline{3-4}
 & 0 & \underline{a,b} & c,c \\ \cline{3-4}
\end{array}
$$

Table 2: Strategic Substitutes

The pure Nash equilibria of the game are in the anti-coordination combinations $(1,0),(0,1)$. The payoffs follow the conditions: $a > d, b > c$ which determines the anti-coordination specification, and $c > d$ specifies the types (preferences) of each player. When two players of opposite type interact there are no conflicting preferences, because both players are better-off when choosing the action corresponding to each of their types, which is the Pareto dominant Nash equilibria of the game. For my example $(1,0)$ Pareto dominates $(0,1)$. Conflicting preferences arise when two players of the same type interact, because both of them like the same action but rather anti-coordinate, therefore they play a mixed strategy.

1.2 The Network Game

Define a network structure to modelize the agents interactions. This social network is denoted as Γ and is represented by $(\mathbf{N},\mathbf{g})$, where $\mathbf{N} = \{1,\ldots,N\}$ is a finite set of players, and $\mathbf{g}$ is the set of links among them (adjacency matrix). The relationship between two players i and j in the network $(\mathbf{N},\mathbf{g})$ is expressed by $g_{ij} \in \{0,1\}$. When there is an undirected link between them $g_{ij} = 1$, and I say they are neighbors. In case they are not connected $g_{ij} = 0$. The set of i's neighbors is $N_i(g) = \{j \in \{1,\ldots,N\} | g_{ij} = 1\}$, where $|N_i(g)| = n_i$ is the cardinality. A player's degree is $k_i(g) = n_i$.

Each player has a binary set of actions $A = \{1,0\}$ and a set of types $\Theta = \{1,0\}$, where a player i of type $\theta_i = 1$ likes action $a_i = 1$ and dislikes $a_i = 0$, which symmetrically holds for a player of type $\theta_i = 0$. I use linear payoff functions dependent on type and choice, where each player receives benefit from own and neighbor's actions. Denote $a_{N_i}(\Gamma)$ as the vector of actions taken by the neighbors of i where $I_{\{a_j=a_i\}}$ is the indicator function of a neighbors choosing the same action as player i, and $I_{\{a_j\neq a_i\}}$ indicates a neighbor choosing

[2] If analyzed in the perspective of cooperation (C: choosing your opponents preference) and defection (D: choosing your preference), neither (*D, D*) nor (*C, C*) are equilibria when opposite types interact, and the latter gives the worst payoffs to all players, see Szidarovszky *et all* (2007) for an application to the Battle of Sexes game. However, when the two players are of the same type, both (D,D) and (C,C) are the same Pareto dominant Nash equilibrium.

the opposite. The class of game being played, $m = \{sc, ss\}$, is either strategic complements (sc) or substitutes (ss). The payoff function of a player i given her degree and the class of game she is playing, is:

$$v_i(\theta_i, a_i, a_{N_i}(\Gamma)) = \lambda_{a_i}^{\theta_i}[1 + \delta_m \sum_{j=1}^{n_i} I_{\{a_j=a_i\}} + (1-\delta_m) \sum_{j=1}^{n_i} I_{\{a_j \neq a_i\}}]$$

where the parameter $\lambda_{a_i}^{\theta_i}$ is equal to α if $a_i = \theta_i$, and to β if $a_i \neq \theta_i$, where $\alpha > \beta$. The multiplier δ_m takes value $\delta_s c = 1$ in strategic complements, and $\delta_s s = 0$ in the substitutes cases. The network game can be represented by

$$\Gamma =< \{1, \dots, N\}, \{g_{ij}\}_{i,j \in \{1,\dots,N\}}, v_i, \Theta, A >$$

Local information is assumed, so that a player in the network knows her degree, her type θ_i, and the set of strategies $(\sigma_1, \dots, \sigma_{N_i}) \in \times_{i=1}^{N_i} A_i$ associated to her neighbors.

$$\sigma_i : \theta_i \times (a_{N_i}(\Gamma)) \to A_i, \quad i \in \{1, \dots, N\}$$

My model of heterogeneous network games gives account for games with strategic complements and strategic substitutes in payoffs, where there is a different payoff function for a player when choosing the action she likes than when not doing so. Payoff functions depend on a player's degree k_i and on her identity, that is, on her type θ_i. So, two players of the same degree $k_i = k_j$ and making the same choice $a_i = a_j$ do not need to have the same payoff function nor receive equal payoffs.

The equilibrium concept to focus on is Nash equilibrium given the local information structure. Notice that local and total information gives the same Nash equilibrium strategies in each classes of games.

Proposition 1 - ***Nash Equilibrium:*** *An action profile $(\sigma_1^*, \dots, \sigma_N^*)$ is a Nash equilibrium in the network game Γ, if and only if*

$$v_i(\theta_i, \sigma_1^*, \dots, \sigma_N^*) \geq v_i(\theta_i, \sigma_1^*, \dots, \hat{\sigma}_i, \dots, \sigma_N^*), \qquad \text{for all} \quad \sigma_i^* \neq \hat{\sigma}_i, \quad i \in \mathbf{N}, \quad \text{and} \quad \theta_i \in \Theta$$

1.2.1 Strategic Complements

Payoff Functions: For games with strategic complements, there are four payoff functions depending on a player's type and choice:

$$v_i(1, 1, (a_{j_1}, \dots, a_{j_{n_i}})) = \alpha(1 + \chi_i)$$
$$v_i(1, 0, (a_{j_1}, \dots, a_{j_{n_i}})) = \beta(1 + n_i - \chi_i)$$
$$v_i(0, 0, (a_{j_1}, \dots, a_{j_{n_i}})) = \alpha(1 + n_i - \chi_i)$$
$$v_i(0, 1, (a_{j_1}, \dots, a_{j_{n_i}})) = \beta(1 + \chi_i)$$

denoting $\chi_i = \sum_{j=1}^{n_i} I_{\{a_j=1\}}$, as the number of i's neighbors choosing 1, and $(n_i - \chi_i) = \sum_{j=1}^{n_i} I_{\{a_j=0\}}$, as those choosing action 0. The relation between the parameters is $\alpha > \beta > 0$.

Nash Equilibrium: A player is better off choosing the action she likes than not doing so with the same number of neighbors in the choice group, and $\beta(1+n_i) > \alpha$, being alone and making the choice one likes gives lower payoffs than the opposite action and having neighbors in that choice group.

Lets define two thresholds, $\underline{\tau}(n_i)$ and $\overline{\tau}(n_i)$ as functions of player i's degree for each type of player in the network game, independently of the class of game being played.

$$\underline{\tau}(n_i) = \tfrac{\beta}{\alpha+\beta} n_i - \tfrac{\alpha-\beta}{\alpha+\beta}$$

$$\overline{\tau}(n_i) = \tfrac{\alpha}{\alpha+\beta} n_i + \tfrac{\alpha-\beta}{\alpha+\beta}$$

It is deduced from the payoff structure that $\overline{\tau}(n_i) > \underline{\tau}(n_i)$, where $\overline{\tau}(n_i) - \underline{\tau}(n_i) = \frac{\alpha-\beta}{\alpha+\beta} n_i + 2(\frac{\alpha-\beta}{\alpha+\beta}) > 0$.

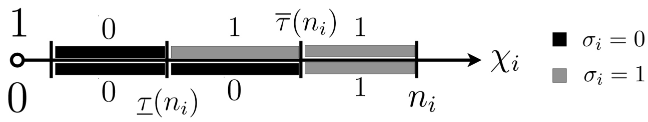

Figure 1: SC Thresholds

Proposition 2 *The Nash equilibrium strategies σ_i^* for network games with strategic complements are defined as:*

$$\begin{cases} \sigma_i^* = 1, & \text{if } \theta_i = 1 \text{ and } \chi_i \geq \underline{\tau}(n_i) \\ \sigma_i^* = 0, & \text{if } \theta_i = 1 \text{ and } \chi_i < \underline{\tau}(n_i) \\ \sigma_i^* = 0, & \text{if } \theta_i = 0 \text{ and } \chi_i \leq \overline{\tau}(n_i) \\ \sigma_i^* = 1, & \text{if } \theta_i = 0 \text{ and } \chi_i > \overline{\tau}(n_i) \end{cases}$$

Proof: A player $i \in \mathbf{N}$ of type $\theta_i = 1$ chooses her favorite action $a_i = 1$ instead of that she dislikes $a_i = 0$ if $v_i(1, 1, (a_{j_1}, \ldots, a_{j_{n_i}})) \geq v_i(1, 0, (a_{j_1}, \ldots, a_{j_{n_i}}))$. For this to happen, she needs to be connected to at least χ_i neighbors choosing action 1, where $\chi_i \geq \underline{\tau}(n_i)$. In case $\chi_i < \underline{\tau}(n_i)$ player i adopts her disliked behavior:

$$\alpha(1+\chi_i) \geq \beta(1+n_i-\chi_i)$$

$$\chi_i \geq \tfrac{\beta}{\alpha+\beta} n_i - \tfrac{\alpha-\beta}{\alpha+\beta} = \underline{\tau}(n_i)$$

A player $i \in \mathbf{N}$ of type $\theta_i = 0$ chooses her favorite action $a_i = 0$ instead of that she dislikes $a_i = 1$ if $v_i(0, 0, (a_{j_1}, \ldots, a_{j_{n_i}})) \geq v_i(0, 1, (a_{j_1}, \ldots, a_{j_{n_i}}))$. For this to happen, she needs to be connected to at most χ_i neighbors choosing action 1, where $\chi_i \leq \overline{\tau}(n_i)$. In case $\chi_i > \overline{\tau}(n_i)$ player i adopts her disliked behavior:

$$\alpha(1 + n_i - \chi_i) \geq \beta(1 + \chi_i)$$
$$\chi_i \leq \tfrac{\alpha}{\alpha+\beta} n_i + \tfrac{\alpha-\beta}{\alpha+\beta} = \overline{\tau}(n_i)$$

1.2.2 Strategic Substitutes

Payoff Functions: For games with strategic substitutes, there are four payoff functions depending on a player's type and choice:

$$v_i(1, 1, (a_1, \ldots, a_{j_{n_i}})) = \alpha(1 + n_i - \chi_i)$$
$$v_i(1, 0, (a_1, \ldots, a_{j_{n_i}})) = \beta(1 + \chi_i)$$
$$v_i(0, 0, (a_1, \ldots, a_{j_{n_i}})) = \alpha(1 + \chi_i)$$
$$v_i(0, 1, (a_1, \ldots, a_{j_{n_i}})) = \beta(1 + n_i - \chi_i)$$

where $\alpha > \beta > 0$, $\alpha(1 + n_i) > \beta(1 + n_i)$, and $\beta(1 + n_i) > \alpha$.

Nash Equilibrium: On the case of games with strategic substitutes, a player's payoff increases when choosing the action she likes given her type, $a_i = \theta_i$, and the rest of her neighbors adopt the opposite behavior, $a_j \neq \theta_i$. The threshold functions for both classes of games, strategic substitutes and strategic complements are the same, but the relation between thresholds and players' strategies are not. Independently of her type, a player chooses 1 if $\chi_i < \underline{\tau}(n_i)$, and if $\chi_i > \overline{\tau}(n_i)$ she chooses 0. The area between $\underline{\tau}(n_i)$ and $\overline{\tau}(n_i)$ allows any player to choose $a_i = \theta_i$, see Figure 2.

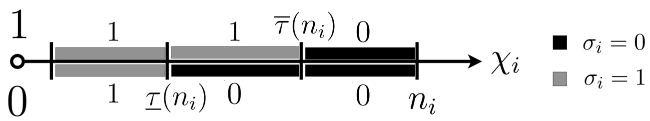

Figure 2: SS Thresholds

Proposition 3 *The Nash equilibrium strategies σ_i^* for network games with strategic substitutes are defined as:*

$$\begin{cases} \sigma_i^* = 1, & \text{if } \theta_i = 1 \text{ and } \chi_i \leq \overline{\tau}(n_i) \\ \sigma_i^* = 0, & \text{if } \theta_i = 1 \text{ and } \chi_i > \overline{\tau}(n_i) \\ \sigma_i^* = 0, & \text{if } \theta_i = 0 \text{ and } \chi_i \geq \underline{\tau}(n_i) \\ \sigma_i^* = 1, & \text{if } \theta_i = 0 \text{ and } \chi_i < \underline{\tau}(n_i) \end{cases}$$

Proof: A player $i \in \mathbf{N}$ of type $\theta_i = 1$ chooses her favorite action $a_i = 1$ instead of that she dislikes $a_i = 0$ if $v_i(1, 1, (a_{j_1}, \ldots, a_{j_{n_i}})) \geq v_i(1, 0, (a_{j_1}, \ldots, a_{j_{n_i}}))$. For this to happen, she needs to be connected to at most χ_i neighbors choosing action 1, where $\chi_i \leq \overline{\tau}(n_i)$. In case $\chi_i > \overline{\tau}(n_i)$ player i adopts her disliked behavior:

$$\alpha(1 + n_i - \chi_i) \geq \beta(1 + \chi_i)$$
$$\chi_i \leq \tfrac{\alpha}{\alpha+\beta} n_i + \tfrac{\alpha-\beta}{\alpha+\beta} = \overline{\tau}(n_i)$$

A player $i \in \mathbf{N}$ of type $\theta_i = 0$ chooses her favorite action $a_i = 0$ instead of that she dislikes $a_i = 1$ if $v_i(0, 0, (a_{j_1}, \ldots, a_{j_{n_i}})) \geq v_i(0, 1, (a_{j_1}, \ldots, a_{j_{n_i}}))$. For this to happen, she needs to be connected to at least χ_i neighbors choosing action 1, where $\chi_i \geq \underline{\tau}(n_i)$. In case $\chi_i < \underline{\tau}(n_i)$ player i adopts her disliked behavior:

$$\alpha(1 + \chi_i) \geq \beta(1 + n_i - \chi_i)$$
$$\chi_i \geq \tfrac{\beta}{\alpha+\beta} n_i - \tfrac{\alpha-\beta}{\alpha+\beta} = \underline{\tau}(n_i)$$

1.3 Equilibrium Configurations

Games in equilibrium portray two different overall network configurations: Unfrustrated and Frustrated. I say that an *Unfrustrated* network (U) is such in which all players choose the action corresponding to their type, and in the *Frustrated* network at least one player is choosing the disliked option. The frustrated configurations can be of two types: *Specialized Frustrated* (S) or *Hybrid Frustrated* (H). In the specialized frustrated all players, independently of their type are choosing the same action: 1 or 0. In the frustrated hybrid both actions coexist yet not all choices correspond to the players' type. In order to preserve the conflicting interactions, I assume that there is a distribution such that at most $N-1$ players are of one type and at least 1 player is of the other.

Proposition 4 *The configuration of a network $\Gamma(N, g)$ is* ***unfrustrated*** *in SC (SS) when each player chooses the action corresponding to her type, so that $a_i = \theta_i$, $\forall\ i \in \{1, \ldots, N\}$. A network is unfrustrated if and only if the following three conditions are jointly satisfied:*

1. *Players of type $\theta_i = 1$ have $\chi_i \geq \underline{\tau}(n_i)(\chi_i \leq \overline{\tau}(n_i))$ neighbors*
2. *Players of type $\theta_i = 0$ have $\chi_i \leq \overline{\tau}(n_i)(\chi_i \geq \underline{\tau}(n_i))$ neighbors*
3. *Two players i and j both of degree $k(g) = 1$, such that $g_{ij} = 1$, are of the same type: $\theta_i = \theta_j$*

Unfrustrated networks can result in games with SC or SS. This implies that there is multiplicity of equilibria.

Example 1 *The star network cannot shape unfrustrated profiles in SC, but it does in SS. In a circle where all players are of degree $k_i(g) = 2$, not all of them can have both of their neighbors of the same type, if not, the circle could specialize. For higher levels of connectivity, the threshold relation is the only relevant feature. The first three graphs of the example correspond to SC and the other three to SS games, although the complete graph works for both classes.*

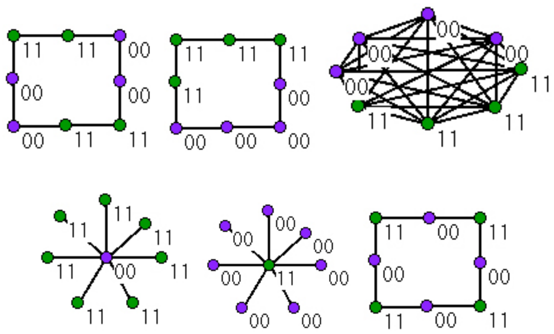

Figure 3: Unfrustrated Configurations

Proposition 5 *The configuration of a network* $\Gamma(N, g)$ *is* ***specialized frustrated*** *when all players choose one same action, so that* $a_i = a_j$, $\forall\ i, j \in \{1, \ldots, N\}$. *A network is specialized in action* $1(0)$ *if and only if the following three conditions are jointly satisfied:*

1. *Players of type* $\theta_i = 1$ *have* $\chi_i \geq \underline{\tau}(n_i)(\chi_i < \underline{\tau}(n_i))$ *neighbors*
2. *Players of type* $\theta_i = 0$ *have* $\chi_i > \overline{\tau}(n_i)(\chi_i \leq \overline{\tau}(n_i))$ *neighbors*
3. *All players of degree* $k_i(g) = 0$ *are* $\theta_i = 1(\theta_i = 0)$

Recall that $\chi_i = \sum_{j=1}^{n_i} I_{\{x_j=1\}}$ is the number of i's neighbors choosing 1. Specialized networks only exist in games with strategic complements, because the anti-coordination condition on payoffs for SS does not allow an equilibrium where all players make the same choice, regardless of their type.

Example 2 *If all players are of degree* $k_i(g) = 2$, *like in a circle, with just one neighbor of the same type, it is enough for a player to sustain the action she likes. To specialize a circle, it is necessary that a player who dislikes the specialized choice has both her neighbors of the opposite type. The same idea and opposite symmetric conditions hold for the case of a network specialized in action* 0.

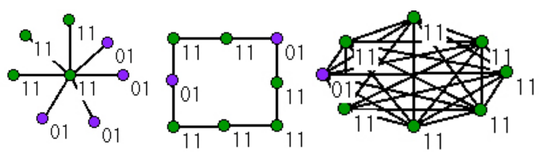

Figure 4: Specialized Configurations

Proposition 6 *The configuration of a network* $\Gamma(N, g)$ *is* ***hybrid frustrated*** *when both actions coexist, and there is at least one player who chooses the action not corresponding to her type, so that* $\exists\ i : a_i \neq \theta_i$.

Example 3 *In SC the star cannot portray a hybrid frustrated equilibrium, but in SS in occurs when a peripheral and the central node have the same type. Complete networks cannot either. A circle network is HF in SS when a player has two neighbors of her same type, but not all* *the circle has the same condition. The first row of graphs are examples of SC and the second shows SS graphs.*

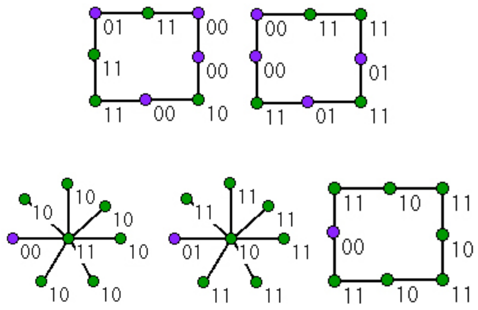

Figure 5: Unfrustrated Configurations

Finally, I illustrate the distribution of network configurations in the case in which all players of a given type are in the same range of X_i neighbors, either below, between or above the thresholds and face a player of the opposite type, in the two classes of games studied.

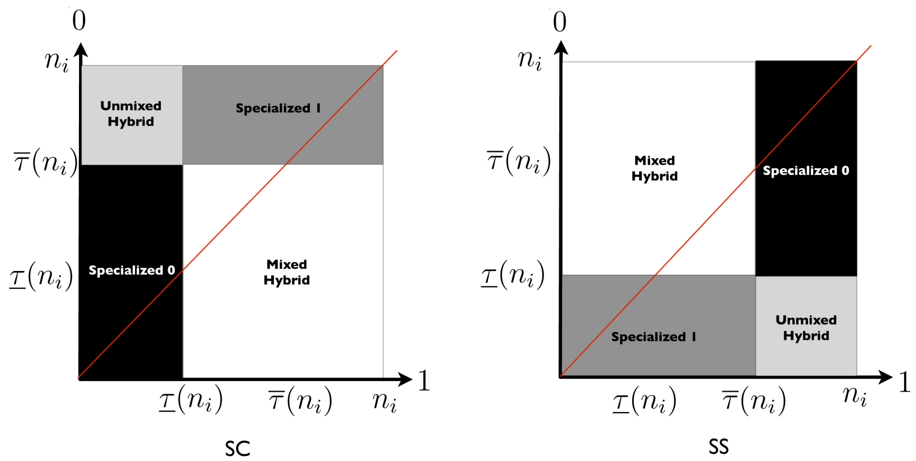

Figure 6: Network Configurations

2 The Dynamic Model

2.1 The Mutation Dynamics

I consider a discrete set of time $\{0, 1, 2, \ldots\}$. At period $t = 0$ the initial state is a network $\Gamma(\mathbf{N}, \mathbf{g})$ with a distribution of types from the set Θ and and action profile $(\sigma_1^*, \ldots, \sigma_N^*)$ that is a Nash equilibrium, so that the initial state belongs to the class of specialized, unfrustrated hybrid or frustrated hybrid configurations. At each period $t \geq 1$ there are two steps:

Step 1: With probability $p \in (0,1)$, two players $i, j \in \mathbf{N}$ independently receive an opportunity to experiment and change their action (with a given probability two players' choices mutate), where $g_{ij} = 1$.

Step 2: After the mutation has taken place, i and j receive a revision opportunity and each player uses a myopic best response to the mutated action profile. where each can choose to return to $x_i(t+1) = x_i(t)$ (the same for player j) or hold the mutated choice so that $x_i(t+1) \neq x_i(t)$. Players revise simultaneously.

Remark 1 If $g_{ij} = 0$ corresponds to the same case of a unilateral deviation, which is not optimal in a Nash equilibrium. The case when either i or j have degree $k = 0$ is the only case where the deviation would hold if $x_i(t+1) = \theta_i$, yet I assume that in case of an isolated component it shall be treated as a network itself, and therefore a network if only one node lacks any interest. If only one player mutates, even if they are connected, holds for the same argument. The resulting network after the dynamics does not need to be Nash

Stable Equlibrium: *A Nash-network $\Gamma(\mathbf{N}, \mathbf{g})$ is a stable equilibrium if in time $t+1$, for any pair of mutated neighbors $i, j \in \mathbf{N}$ their myopic best response coincides with the previous action: $x_i(t+1) = x_i(t)$ and $x_j(t+1) = x_j(t)$.*

A Stable Equilibrium is a network configuration that after the mutation of both players' choices, returns to its previous action configuration.

Proposition 7 *The set of Stable Equilibria are the network configurations in which all players, depending on their choice and type, hold for the conditions of the table in Figure 8.*

	$\theta_\mathbf{i} = \theta_\mathbf{j} = \mathbf{1}$	$\theta_\mathbf{i} = \theta_\mathbf{j} = \mathbf{0}$	$\theta_\mathbf{i} = \mathbf{1}, \theta_\mathbf{j} = \mathbf{0}$
$\mathbf{a_i} = \mathbf{a_j} = \mathbf{1}$	$\chi_i \geq \underline{\tau}_i + 1$ $\chi_j \geq \underline{\tau}_j + 1$	$\chi_i > \overline{\tau}_i + 1$ $\chi_j > \overline{\tau}_j + 1$	$\chi_i \geq \underline{\tau}_i + 1$ $\chi_j > \overline{\tau}_j + 1$
$\mathbf{a_i} = \mathbf{a_j} = \mathbf{0}$	$\chi_i < \underline{\tau}_i - 1$ $\chi_j < \underline{\tau}_j - 1$	$\chi_i \leq \overline{\tau}_i - 1$ $\chi_j \leq \overline{\tau}_j - 1$	$\chi_i < \underline{\tau}_i - 1$ $\chi_j \leq \overline{\tau}_j - 1$
$\mathbf{a_i} = \mathbf{1}, \mathbf{a_j} = \mathbf{0}$	$^*\ \chi_i \geq \underline{\tau}_i - 1$ $^*\ \chi_j < \underline{\tau}_j + 1$	$^*\ \chi_i > \overline{\tau}_i - 1$ $^*\ \chi_j \leq \overline{\tau}_j + 1$	$^*\ \chi_i \geq \underline{\tau}_i - 1$ $^*\ \chi_j \leq \overline{\tau}_j + 1$
$\mathbf{a_i} = \mathbf{0}, \mathbf{a_j} = \mathbf{1}$	$^*\ \chi_i < \underline{\tau}_i + 1$ $^*\ \chi_j \geq \underline{\tau}_j - 1$	$^*\ \chi_i \geq \overline{\tau}_i + 1$ $^*\ \chi_j > \overline{\tau}_j - 1$	$^*\ \chi_i < \underline{\tau}_i + 1$ $^*\ \chi_j > \overline{\tau}_j - 1$

Figure 7: Stability Conditions

Depending on the combination of actions an hybrid configuration rejects or accepts the mutations. When two players are making opposite choices, independently of their type, the mutation of such choices will not hold. This can be observed in rows 3 and 4 of the table of initial conditions. All the cells marked with a star $*$ are the cases in which mutations do not act because the choices are different. The table in Figure 9., shows the relation between a player's neighbors choosing action 1 and thresholds in order to accept the mutations. There are two conditions in each cell and it is enough that only one of them holds in order to accept a mutation. It is marked, as well, what kind of configuration can occur in each of the six cases belonging to the first two rows. S_1 is the configuration specialized in action 1, and S_0

the specialized in action 0, F the Frustrated Hybrid configuration and U the Unfrustrated Hybrid.

Remark 2. Any unfrustrated network that changes with one mutation turns into a frustrated configuration, wether it is hybrid or specialized. Figure 9., gives us the dimension of the set of Nash-networks that are not stable.

	$\theta_\mathbf{i} = \theta_\mathbf{j} = \mathbf{1}$	$\theta_\mathbf{i} = \theta_\mathbf{j} = \mathbf{0}$	$\theta_\mathbf{i} = \mathbf{1}, \theta_\mathbf{j} = \mathbf{0}$
$\mathbf{a_i = a_j = 1}$	$\chi_i = \underline{\tau}_i$ $\chi_j = \underline{\tau}_j$	$\chi_i = \overline{\tau}_i + 1$ $\chi_j = \overline{\tau}_j + 1$	$\chi_i = \underline{\tau}_i$ $\chi_j = \overline{\tau}_i + 1$
Configurations	S_1, F, U	S_1, F	S_1, F
$\mathbf{a_i = a_j = 0}$	$\chi_i = \underline{\tau}_i - 1$ $\chi_j = \underline{\tau}_j - 1$	$\chi_i = \overline{\tau}_i$ $\chi_j = \overline{\tau}_j$	$\chi_i = \underline{\tau}_i - 1$ $\chi_j = \overline{\tau}_i$
Configurations	S_0, F	S_0, F, U	S_0, F

Figure 8: Configurations and Mutation Acceptance Conditions

I can say that when starting at any Unfrustrated configuration, and it changes with at least one mutation, it turns into a Frustrated configuration, regardless of being specialized or hybrid. However, the configuration resulting after a mutation, independently of the initial state is a Hybrid Frustrated unless very specific condition hold (see Figure 10). In other for such cases to occur the following considerations are necessary:

2.1.1 Size of the neighborhood to allow mutations

2.2 Conditions

a.1 $\rightarrow \exists! i | \theta_i = 1$

a.2 $\rightarrow \exists! i, j | g_{i,j} = 1, \theta_i = \theta_j = 1$

Concluding Remarks

7) If a player's degree is $k_i(g) = 1$, so that she is a leaf (i.e. in a star), then she can play a mixed strategy, independently of the type of her only neighbor, just as in a 2×2 game. (ESTE ES PARA LAS CONCLUSIONS)

This paper analyzes how does endowing players with types in network games shape their behavior, equilibria of the games and conditions of stability in a context of complete information. I develop a model of heterogeneous network games in fixed structures. In particular I study games of strategic complements (SC) and games of strategic substitutes (SS) in payoffs and see how heterogeneity in types and degree may portray different network configurations when a network game is in equilibrium. I find that heterogeneity in types and degree result in heterogeneity in the acceptance thresholds each player faces as well as in diverse payoff

	Initial State	Resulting State
1	S_1	HF
	HF	HF
	U	HF S_0 : iff i, j are the only $\theta = 1$ and $\tilde{a}_i = \tilde{a}_j = 1$
2	S_1	HF U: iff i, j are the only $\theta = 0$ and $\tilde{a}_i = \tilde{a}_j = 0$
	HF	HF U: iff i, j are the only $\theta = 0$ and $\tilde{a}_i = \tilde{a}_j = 0$
3	S_1 or HF	HF: if $\tilde{a}_i = 0$ or $\tilde{a}_i = \tilde{a}_j = 0$ U: iff $\tilde{a}_j = 0$ and j is the only $\theta = 0$
4	S_0	HF U: iff $\tilde{a}_i = \tilde{a}_j = 0$ and i, j is the only $\theta = 1$
	HF	HF U: iff $\tilde{a}_i = \tilde{a}_j = 0$ and i, j is the only $\theta = 1$ or iff $\tilde{a}_i = \tilde{a}_j = 0$ and all $N \in \Theta = 1$ choose 1
5	S_0 or HF	HF
	U	HF S_1: iff $\tilde{a}_i = \tilde{a}_j = 1$ and i, j are the only $\theta = 0$
6	S_0	HF U: iff $\tilde{a}_i = 1, \tilde{a}_j = 0$ and i is the only $\theta = 1$
	HF	HF U: iff $\tilde{a}_i = 1, \tilde{a}_j = 0$ and i is the only $\theta = 1$ S_1: iff $\tilde{a}_i = \tilde{a}_j = 1$ and j is the only $\theta = 0$

Figure 9: Configurations and Mutation Acceptance Conditions

functions conditioned to each players type and strategy adopted. I have considered generation of types as a consequence of perturbing the payoffs of 2×2 games, and use as examples the Battle of Sexes and the Hawk and Dove games for it. The resulting games are equivalent in the sense that the Nash equilibria in pure strategies holds the same in the perturbed games. The extension of such 2×2 perturbed games into a network context generate heterogeneous network games where each player might be playing different games at the same type, because of her and her neighbors' type. I consider a specific payoff function with a network structure, so that each players payoff depends on her and her neighbors' choices as well as her type, and this results in two threshold functions that depend on the connectivity of each player. When a player likes action 1, because her type is $\theta_i = 1$, then the number of neighbors adopting action 1(0) necessary for her to choose what she likes in SC (SS) is lower than for a player of the opposite type. So, a player's strategy depends on her type and her neighbor's choices.

A network game in equilibrium can portray 3 different network configurations. Specialized, if all players are choosing one same action, Unmixed hybrid if both actions coexist and each player is adopting the behavior she likes, and Mixed Hybrid if both actions coexist and at least one player is making a choice opposite to her type. The assumption of complete

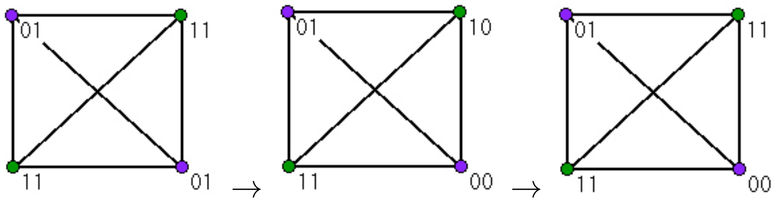

Figure 10: Example 1

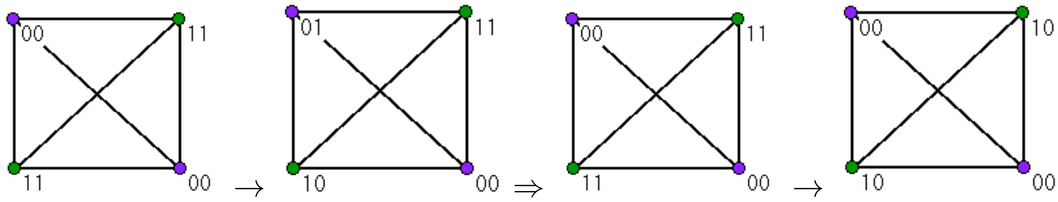

Figure 11: Example 2

information allows for multiple equilibria and I use an extension of the concept of *Stochastic Stability* developed by Jackson and Watts (2001). A Markov process of perturbing randomly chosen player's type is followed. The refinement of equilibria leads to a set of Stochastically Stable Network Configurations (SSC), $\sum$. When the set of players in the game $\mathbf{N}$, the adjacency matrix $\mathbf{g}$, and the action profile $(\sigma_1, \ldots, \sigma_N)$ that is Nash equilibrium are hold fixed, I find a subset of the SSC that form a *Stochastic*-Γ-tree, that is, an undirected graph where each network is a node and is connected to an adjacent network that is different from the previous by the perturbed type of one player. This subset of networks portray the same configuration, size and shape, however, the social welfare varies given each player's type. I develop a welfare analysis and Pareto rank the networks in each *Stochastic*-Γ-tree, given the configuration, and find a *Stochastic*-Core Stable network that Pareto dominates the others.

Further research to this model will be done by a variation in the levels of information. The inclusion of incomplete information in a game with types lead to a richer set of analysis, where I will develop a scenario of communication (messages) and punishments in order to reduce multiplicity of equilibria generated by the complete information assumption, into a Bayesian Nash equilibrium, similar to Galeotti *at al* (2010).

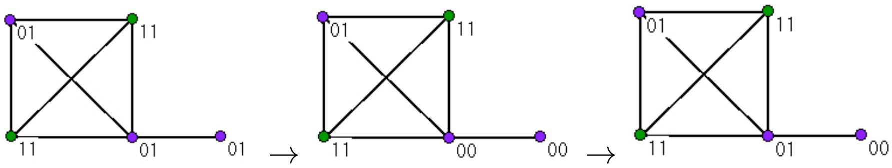

Figure 12: Example 2

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