BOSTON UNIVERSITY SOCIETY OF MATHEMATICS COLLOQUIUM PROCEEDINGS

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Abstract. To motivate this paper, I shall simply state two theorems, which we will endeavor to prove;
Existence: For any irreducible root system ¶, there exists a simple Lie algebra over $\mathbb{C}$ which has a root system equivalent to ¶.
Uniqueness: It is also the case that any two Lie algebras over $\mathbb{C}$ with equivalent root systems are isomorphic.

1. Root systems

Definition: *A Euclidean vector space is a real vector space V with a positive definite symmetric bilinear form which we will call the dot product, i.e a bilinear form B such that $B(v,w) = B(w,v)$ for all $v, w \in V$ and $B(v,v) > 0$ $\forall v \neq 0$.*
Definition: *Let Φ be a subset of a finite dimensional real vector space V which is equipped with the dot product. Φ is a root system if:*

- *Φ is a finite set of non-zero vectors*
- *Φ spans V.*
- $\alpha, \beta \in \Phi \implies \beta - \frac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha \in \Phi$

If the root system is crystalline, then we have a fourth condition:

- $\alpha, \beta \in \Phi \implies \frac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$

Definition: *A subset $\Delta \subset \Phi$ is a base if the following conditions are satisfied:*

- *Δ is a basis for V as s vector space, where $\Phi \subseteq V$*
- *Each root $\alpha \in \Phi$ can be expressed as a linear combination of elements in Δ with linear coefficients such that the coefficients are either all positive or all negative.*

A root in Δ is called a simple root.
Definition: *Let $\langle \beta, \alpha \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. Two root system (V_1, Φ_1) and (V_2, Φ_2) are isomorphic if there is an invertible linear map between V_1 and V_2 that preserves $\langle \alpha, \beta \rangle$.*
Definition: *For $\alpha \in V$, H_α denotes the hyperplane perpendicular to α, i.e $\beta \in V : \langle \alpha, \beta \rangle = 0$*

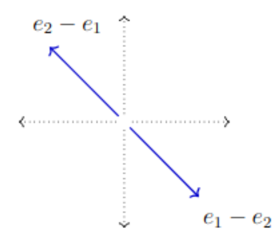
In any root system Φ the hyperplanes H_α for some α divide V into connected components, which are the Weyl chambers of V.
Definition: *Let Φ be a root system in a Euclidian space V. For each root $\alpha \in \Phi$, define $s_\alpha(\beta)$ as $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ where $(,)$ is the inner product on V. The Weyl group of Φ is the subgroup generated by the s_α*
It is a fact that every root is conjugate to a simple root under the Weyl group.

Key words and phrases. Lie Algebras, Root Systems, the Weyl Group, Abstract Algebra.

Definition: *A root system Φ which is non empty is said to be irreducible if it is not the direct sum of two nonempty root systems*
Definition: *A nonempty root system Φ is said to be reducible if it can be written as a disjoint union of nonempty root system Φ_1, Φ_2, i.e $\Phi = \Phi_1 \bigsqcup \Phi_2$*

Each root system can be written as the direct sum of irreducible root systems, and this summation is unique up to the ordering of the terms. Therefore, it suffices to only consider the irreducible root systems in our classification.

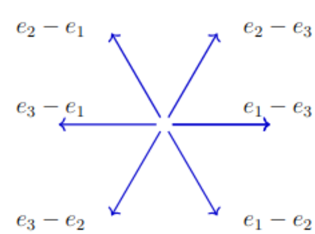
1.1. **Examples.** Take $V = \mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$. The A_1 root system $\Phi = \{e_1 - e_2, e_2 - e_1\}$ is pictured below:



We can check the integrality condition:

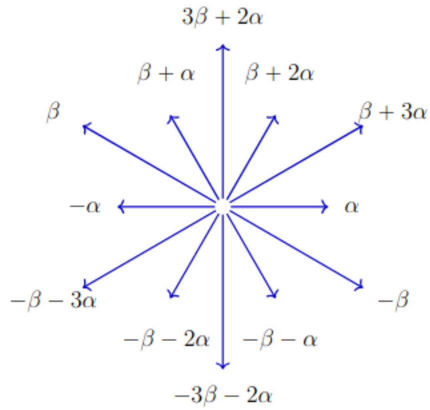
$$\frac{2(e_1 - e_2, e_2 - e_1)}{(e_2 - e_1)} = \frac{2(-1-1)}{(1+1)} = -2$$

Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. The A_2 root system $\Phi = \{e_1 - e_2, e_2 - e_1, e_1 - e_3, e_3 - e_1, e_2 - e_3, e_3 - e_2\}$ is a root system in the subspace $V = Span(\Phi)$, which is the plane with normal vector $e_1 + e_2 + e_3$. This root system is the A_2 root system, and fulfills the last integrality condition, and has base $\Delta = \{e_1 - e_2, e_3 - e_1\}$



In general, we can define the A_l root system as $\Phi = \{\pm(e_i, e_j) : 1 \leq i|j \leq l+1\}$ where $e_1, e_2, \ldots.., e_{l+1}$ is the standard basis of $\mathbb{R}^{l+1}$, and $V = Span(\Phi) \subset \mathbb{R}^{l+1}$ equipped with the dot product.

We now consider the more complex G_2 root system. Let e_1, e_2, e_3 and V be as before. Then, the G_2 root system is the set of vectors $\{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\} = A_2 \cup \{\pm(2e_1 - e_3 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}$. Let $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$. The base for G_2 is $\Delta = \{\alpha, \beta\}$.



1.2. **Classification.** It is an interesting consequence that the integrality condition yields some constraints on the possible angles between two roots. Consider the following:

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \frac{2(\alpha, \beta)}{(\beta, \beta)}$$

$$= 4\frac{(\alpha, \beta)^2}{|\alpha|^2|\beta|^2} = 4\cos^2(\theta) = (2\cos\theta)^2 \in \mathbb{Z}$$

Since $2\cos\theta \in [-2, 2]$, we see that the only possible values for $\cos\theta$ are $0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm1$.The corresponding angles are $60°, 120°, 90°, 45°, 135°, 30°, 150°, 0°, 180°$. Recall that if α is a root, the only multiples of the α in the root system are α and $-\alpha$. Therefore, $0°$ and $180°$ are not possible angles, since they correspond to 2α and -2α. We note that roots at an angle of $60°$ or $120°$ are of equal length, roots at an angle of $45°$ or $135°$ have a ratio of $\sqrt{2}$, and roots at an angle of $30°$ or $150°$ correspond to a length ratio of $\sqrt{3}$.

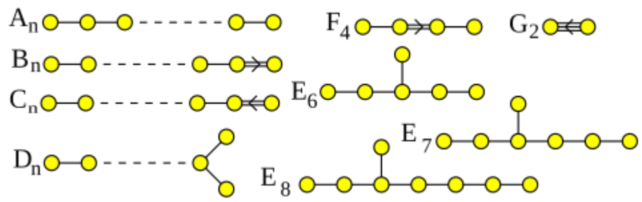
1.2.1. *Dynkin Diagrams.* Let Φ be a root system with base Δ. We can construct the associated Dynkin diagram by drawing a vertex for each root in Δ and drawing edges between these vertices according to the following rules:

- If the roots associated with two vertices is orthogonal, then there is no edge.
- If the two roots form an angle of $120°$, then there is an undirected single edge.
- If the vectors form an angle of $135°$, then there is a directed double edge.
- If the vectors form an angle of $150°$, there is a directed triple edge.

1.3. **Examples.** Recall the A_2 root system. The Dynkin diagram has two vertices α_1, α_2, with one undirected edge:

Let α_1, α_2 be vertices representing the two elements in the base of G_2. We see that they form an angle of $150°$, and so the Dynkin diagram is

Connected Dynkin diagrams can all be classified as one of 8 pictures: $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$.



2. Classification of Lie Algebras

Definition. *A Lie Algebra is a vector space $\mathfrak{g}$ over a field with a Lie bracket, which satisfies the following:*

- $[ax + by, z] = a[x, z] + b[y, z]$
- $[z, ax + by] = a[z, x] + b[z, y]$
- *Jacobi Identity:* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \forall x, y, z \in \mathfrak{g}$

A Lie algebra is semisimple if it is a direct sum of non-abelian Lie algebras with no non-zero proper ideals (simple Lie algebras).

Definition: *A Cartan Subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is an abelian, diagonalizable subalgebra which is maximal under set inclusion, with dimension equal to the rank of $\mathfrak{g}$.*

Cartan subalgebras always exist for finite dimensional complex Lie algebras, and are all conjugate to each other under automorphisms of the Lie algebra, meaning that they all have the same dimension. It is possible to classify semisimple Lie algebras defined over a algebraically closed field of characteristic zero by finding the root systems associated with their Cartan Subalgebras, which as we have discussed above, are classified according to their Dynkin diagrams. Let $\{H_1, ..., H_2\}$ be a basis for $\mathfrak{h}$. Extending this basis to a basis of $\mathfrak{g}$ will yield a basis with very nice commutator relations, since any Cartan subalgebra is abelian and so $[H_i, H_j] = 0$.

Definition: *The adoint operator of x for $x \in \mathfrak{g}$, denoted $ad_x : \mathfrak{g} \to \mathfrak{g}$ takes $x \mapsto [x, y]$.*

The adoint operators determine the linear mapping $ad : \mathfrak{g} \to gl(\mathfrak{g})$, the Lie algebra of all linear endomorphisms of $\mathfrak{g}$. Since we consider only finite dimensional Lie Algebras, $gl(\mathfrak{g})$ is the Lie algebra of square matrices under matrix multiplication. We see that ad is a representation of $\mathfrak{g}$ called the adoint representation.

We will now note some nice facts about linear operators:

- Pairwise commuting, diagonalizable linear operators share a common set of eigenvectors.
  Proof. Since we are working with matrices, we shall do a nice matrix proof. Note that if $Ax = \lambda x$. Then $ABx = BAx = B\lambda x = \lambda Bx$ since we have assumed that A, B are pairwise commuting. Then, x, Bx are eigenvectors of A,
- For $H_1, H_2 \in \mathfrak{h}$, ad_{H_1}, ad_{H_2} commute and are diagonalizable. By the first fact, they then share a common set of eigenvectors.
  Proof. First, we show that they commute; by the Jacobi identity, we have

that

$$[H_1,[H_2,X]] = -[H_2,[X,H_1]] - [X,[H_1,H_2]] = -[H_2,[X,H_1]] - [X,0] = [H_2,[H_1,X]]$$

Recall from linear algebra that if two linear transformations have the same eigenvectors, then they can be simultaneously diagonalized. Therefore, we have the desired result.

By the spectral theorem, we can decompose $\mathfrak{g}$ into shared eigenspaces g_α of the adoint operators:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \text{ where the } \alpha's \text{ are the eigenvalues of } ad_{H_i} \text{ on the eigenspace } \mathfrak{g}_\alpha$$

Therefore, for each eigenvector $E \in \mathfrak{g}_\alpha$, $[H_i, E] = \alpha_i E$. Each such α_i is called a root of $\mathfrak{g}$. Let Φ denote the set of roots. Φ forms a root system in $\mathbb{R}^r$, where r is the rank of $\mathfrak{g}$. In particular, each eigenspace $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ is one-dimensional. We can now direct our attention to proving the two theorems stated in the beginning.

2.1. **The Nice Stuff. Serre's Theorem:** Given a root system Φ in a Euclidean space with inner product $(,)$, $\langle \beta, \alpha \rangle$ defined as before and base $\{\alpha_1, \alpha_2, ...\alpha_n\}$, the Lie algebra $\mathfrak{g}$ defined by $3n$ generators e_i, f_i, h_i and the relations

$$[h_i, h_j] = 0$$
$$[e_i, f_i] = h_i,\ [e_i, f_j] = 0, i \neq j$$
$$[h_i, e_j] = \langle \alpha_i, \alpha_j \rangle e_j,\ [h_i, f_j] = -\langle \alpha_i, \alpha_j \rangle f_j$$
$$ad(e_i)^{-\langle \alpha_i, \alpha_j \rangle + 1}(e_j) = 0,\ i \neq j$$
$$ad(f_i)^{-\langle \alpha_i, \alpha_j \rangle + 1}(f_j) = 0,\ i \neq j$$

is a finite-dimensional semisimple Lie algebra with the Cartan subalgebra generated by the $h_i's$ and with the root system Φ.

Sketch: $L_0 = \bar{L}/\bar{K}$ where $\bar{K}$ is the ideal in $\bar{L}$ where $\bar{L}$ is a free Lie algebra generated on 3n elements by the following generators: $\{e_i, f_i, h_i | 1 \leq i \leq l\}$. Let $\bar{K}$ be generated by $[h_i, h_j], [e_i, f_i] - \delta_{ij} h_i, [h_i, e_i] - c_{ji}, [h_i, f_i] + c_{ji} f_i$ where c_{ij} is the Cartan integer $\langle \alpha_i, \alpha_j \rangle$. Let L_0 be decomposed into $E + F + H$ where E is generated by the e_i and F is generated by the f_i.

Now, let $L = L_0/K$ where K is the ideal generated by all e_{ij}, f_{ij} $i \neq j$.

We will first consider elements of L_0. Let I be the ideal of E generated by all the e_{ij} and J be the ideal of F generated by the f_{ij}. Note that this means that K includes I and J. We shall proceed from here in steps, to avoid any further confusion than that caused by these definitions.

(1) **I and J are ideals of L_0.** The argument for I and J will be roughly the same, so we consider only J. First, we see that y_{ij} is an eigenvector for $ad\ h_k$ (this is discussed above) with eigenvalue $-c_{jk} + (c_{ji} - 1)c_{ik}$. Since $ad\ h_k(F) \subset F$, we have that $ad\ h_k(J) \subset J$ by the Jacobi identity. However, it is also the case that $ad\ e_k(f_{ij}) = 0$. Then, e_k maps F into $F + H$, and so since $ad\ h_k(J) \subset J$, we have that $ad\ e_k(J) \subset J$ again by the Jacobi identity. Then we have also $ad\ L_0(J) \subset J$.

(2) $K = I + J$. Recall that $I + J \subset K$. But by 1), we have that $I + J$ is an ideal of L_0 which contains all e_{ij}, f_{ij}, and K is the smallest such ideal. Therefore, we have that $I + J = K$.

(3) **Let $N^- = E/F$, $N = E/I$. Then, $L = N^- + H + N$ where $+$ denotes the direct sum of subspaces.** Let H be identified with its image under the canonical map $L_0 \to L$. This follows fairly directly from 2) and the direct sum decomposition $L_0 = E + F + H$.
(4) **$E \oplus F \oplus H$ is isomorphic to L.** We won't thoroughly prove this, but it follows loosely from the relations detailed above, since we have already shown that H maps isomorphically into L by 3). As a consequence, we can identify e_i, f_i, h_i with elements of L, and in fact these generate L.
(5) **If $\lambda \in H^*$, then $L_\lambda = \{x \in L | [hx] = \lambda(h)(x) \ \forall h \in H\}$. Then, $H = L_0$ and $N = \sum_{\lambda > 0} L_\lambda$, $N^- = \sum_{\lambda < 0} L_\lambda$, and each L_λ is finite dimensional**. This remark follows from 3) and 4).
(6) **For $1 \leq i \leq n$, we have that $ad\ e_i$ and $ad\ f_i$ are locally nilpotent endomorphisms of L.** Again, we have that the arguments for the e_i is roughly the same as the argument for the f_i, so we consider only the e_i. Let M be the subspace of all elements of L that are killed by some power of ade_i. If $e \in M$ is killed by $(ade_i)^r$, and $f \in M$ is killed by $(ade_i)^r$, then $[e, f]$ is killed by $(ade_i)^{r+s}$. Then M is a subalgebra of L, but all $e_k \in M$ and all $f_k \in M$, and these elements generated L, so $M = L$.
(7) **Let $\tau_i = \exp(ade_i)\exp(ad(-f_i))\exp(ade_i)$ for $1 \leq i \leq n$. Then, τ_i is a well defined automorphism of L.** We also won't prove this fact rigorously, but it follows from 6).
(8) **If $\lambda, \mu \in H^*$, and $\sigma\lambda = \mu$ for σ in the Weyl group of Φ, then $\dim L_\lambda = \dim L_\mu$.** It suffices to consider only the generators of the Weyl group. The automorphism τ_i of L from 7) coincides on the finite dimensional space $L_\lambda + L_\mu$, and we see that τ_i interchanges L_λ and L_μ. In particular, we see that $\dim L_\lambda = \dim L_\mu$.
(9) **For $1 \leq i \leq n$, $\dim L_\alpha = 1$, while $L_{k\alpha_i} = 0$ for $k \neq -1, 0, 1$.** It follows from 4) that this holds for L_0, and consequently must hold for L.
(10) **If $\alpha \in \Phi$, then $\dim L_\alpha = 1$ and $L_{k\alpha} = 0$ for $k \neq -1, 0, 1$.** Recall that each root is conjugate to a simple root under the action of the Weyl group. Therefore, this follows 8), 9).
(11) **If $L_\lambda \neq 0$, then either $\lambda \in \Phi$ or $\lambda = 0$.** If this were not the case, then λ would be an integral combination of simple roots with coefficients that were either all positive or all negative. We see that by 10), λ is not a multiple of a root. Let $\sigma\lambda$ be a conjugate of λ under the Weyl group action. By various properties of this action, we see that $L_{\sigma\lambda} = 0$, which contradicts 8).
(12) **$\dim L = n + |\Phi| < \infty$.** Since by 5 we see that each L_λ is finite dimensional, this follows by 10) and 11).
(13) **L is semisimple.** Let A be an abelian ideal of L. We show that $A = 0$. Note that $ad\ H$ stabilizes A, and so $A = A \cap H + \sum_{\alpha \in \Phi}(A \cap L_\alpha)$ since $L = H + \sum_{\alpha \in \Phi} L_\alpha$. If $L_\alpha \in A$, then $[L_{-\alpha}, L_\alpha] \subset A$ where $L_{-\alpha} \subset A$ and $\mathfrak{sl}_2(F) \subset A$ where L is an algebra over F. This cannot be the case, and so $A = A \cap H \subset H$ where $[L_\alpha, A] = 0$ for $\alpha \in \Phi$ and $A \subset \bigcap_{\alpha \in \Phi} \ker \alpha = 0$.
(14) **H is a Cartan subalgebra of L and Φ is the root system.** H is abelian, and therefore nilpotent and, due to the direct sum decomposition self-normalizing. This is precisely the definition of a Cartan subalgebra, and it is immediete that Φ is the corresponding set of roots. □

This theorem implies existence. Let us restate the uniqueness theorem as follows: *Let L, L' be semisimple Lie algebras, with respective Cartan sub-algebras H, H' and root system Φ, Φ'. let an isomorphism $\Phi \to \Phi'$ be given, sending a given base Δ to a base Δ', and inducing the isomorphism $\pi : H \to H$. For each $\alpha \in \Delta$ (respectively $(\alpha' \in \Delta')$), select an arbitrary nonzero $x_\alpha \in L_\alpha$ (respectively $(x'_\alpha \in L'_\alpha)$). Then, there exists a unique isomorphism $\pi : L \to L'$ extending $\pi : H \to H'$ and sending x_α to $x_{\alpha'}$ for $\alpha \in \Delta$.*

Proof. It suffices to show the case where L is the lie algebra constructed according to Serre's theorem. Take e_α, f_α and $h_\alpha = [e_\alpha, f_\alpha]$ to be the specified generators with $\alpha \in \Delta$. Set $h'_\alpha = \pi(h_\alpha)$ and choose $f'_{\alpha'}$ uniquely satisfying $[x'_\alpha, y'_\alpha] = h'_{\alpha'}$ for each $\alpha' \in \Delta'$. Since $\Phi \cong \Phi/$, the chosen elements in L' satisfy the relations in Serre's theorem. Therefore, Serre's theorem provides a unique homomorphism $\pi : L \to L'$ sending $e_\alpha, f_\alpha, h_\alpha (\alpha \in \Delta)$ to $e'_\alpha, f'_\alpha, h'_\alpha$ respectively, extending the given isomorphism $\pi : H \to H'$. Since $\dim L = \dim H + |\Phi| = \dim H' + |\Phi'| = \dim L'$, we see that π is indeed an isomorphism. □

2.2. **Examples.** Consider the special linear Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, and let $\mathfrak{h}$ be the subalgebra of diagonal matrices with trace 0. Then, the root vectors are matrices $E_{i,j}$ where $i \neq j$, with a 1 in i, j spot and zeroes everywhere else. Then, $[H, E_{i,j}] = (\lambda_i - \lambda_j)E_{i,j}$ where H is the diagonal matrix with entries $\lambda_1, ..., \lambda_n$. Therefore, we can represent the roots as the linear functionals $\alpha_{i,j}(H) = \lambda_i - \lambda_j$. However, we can identify $\mathfrak{h}$ with its dual $\mathfrak{h}^*$, and so we can rewrite the roots as the vectors $\alpha_{i,j} = e_i - e_j$ in the subspace of $\mathbb{R}^n$ consisting of n-tuples that sum to 0. This can be identified as the A_{n-1} root system. For example, we see that the associated root system of $\mathfrak{sl}_2(\mathbb{C})$ is $\{e_1 - e_2, e_2 - e_1\}$ which is the A_1 root system. [**PAWS**] [**Mehrle**] [**Coelho**] [**Morgan**] [**Humphreys**]

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