FINITE DIFFERENCE TIME DOMAIN METHOD (FDTD)

The FDTD method, proposed by Yee, 1966, is another numerical method, used widely for the solution of EM problems. It is used to solve open-region scattering, radiation, diffusion, microwave circuit modelling, and biomedical etc. problems.

One of the most important concerns of the FDTD method is the requirement of the **artificial mesh truncation** (boundary) conditions. These conditions are used to truncate the solution domain and they are known as **absorbing boundary conditions** (ABCs), as they theoretically absorb fields. The space domain includes the object and it is terminated by Absorbing Boundary Conditions (ABCs).
Imperfect ABCs create reflections and the accuracy of the FDTD method depends on the accuracy of the ABCs.

The following advantages make the FDTD method popular:

- It's a direct solution of Maxwell's equations, no integral equations are required and no matrix inversions are necessary.
- Its implementation is easy and it is conceptually simple.
- It can be applied to the three-dimensional, arbitrary geometries.
- It can be applied to materials with any conductivity.

The FDTD method has also the following disadvantages:

- When the FDTD method is applied, the object and its surroundings must be defined.
- Since computational meshes are rectangular in shape it is difficult to apply the method to the curved scatterers.
- It has low order of accuracy and stability unless fine mesh is used.

Basic Finite-Difference Time-Domain Algorithm

In this method the coupled Maxwell's curl equations in the differential form are discretized, approximating the derivatives with two point centred difference approximations in both time and space domains.

The six scalar components of electric and magnetic fields are obtained in a time-stepped manner.

$$\nabla X\bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \nabla X\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

$$\nabla.\bar{D} = \rho_v \qquad \nabla.\bar{B} = 0$$

In linear, isotropic and homogeneous materials $\bar{D}$ and $\bar{B}$ are related to $\bar{E}$ and $\bar{H}$ with the following constitutive relations:

$$\bar{D} = \varepsilon\bar{E}$$
$$\bar{B} = \mu\bar{H}$$

Also $\bar{J}$ is related to $\bar{E}$ as,

$$\bar{J} = \sigma\bar{E}$$

Substituting the constitutive relations into the Maxwell's equations, we can write six scalar equations in the Cartesian coordinate system.

Electric Field Intensity:

$$\left.\begin{aligned}
\frac{\partial E_x}{\partial t} &= \frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}-\sigma E_x\right)\\
\frac{\partial E_y}{\partial t} &= \frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}-\sigma E_y\right)\\
\frac{\partial E_z}{\partial t} &= \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}-\sigma E_z\right)
\end{aligned}\right\}$$

Magnetic field intensity:

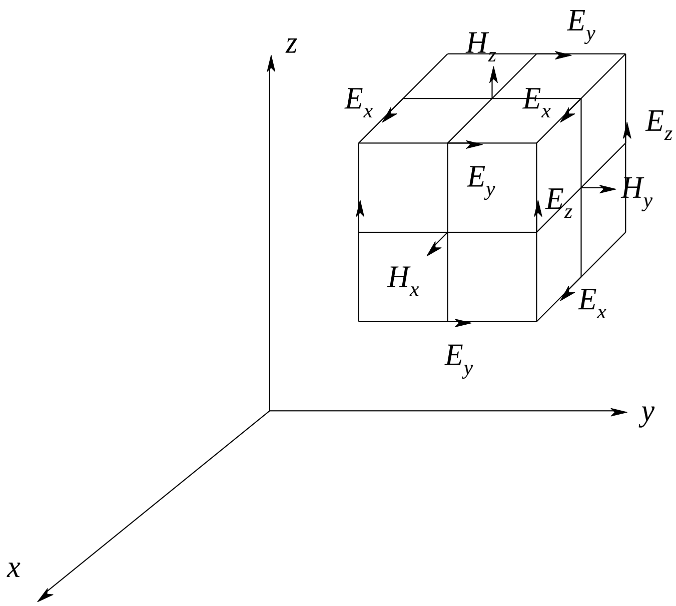
$$\left.\begin{aligned}
\frac{\partial H_x}{\partial t} &= \frac{1}{\mu}\left(\frac{\partial E_y}{\partial z}-\frac{\partial E_z}{\partial y}\right)\\
\frac{\partial H_y}{\partial t} &= \frac{1}{\mu}\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right)\\
\frac{\partial H_z}{\partial t} &= \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right)
\end{aligned}\right\}$$

Yee Algorithm

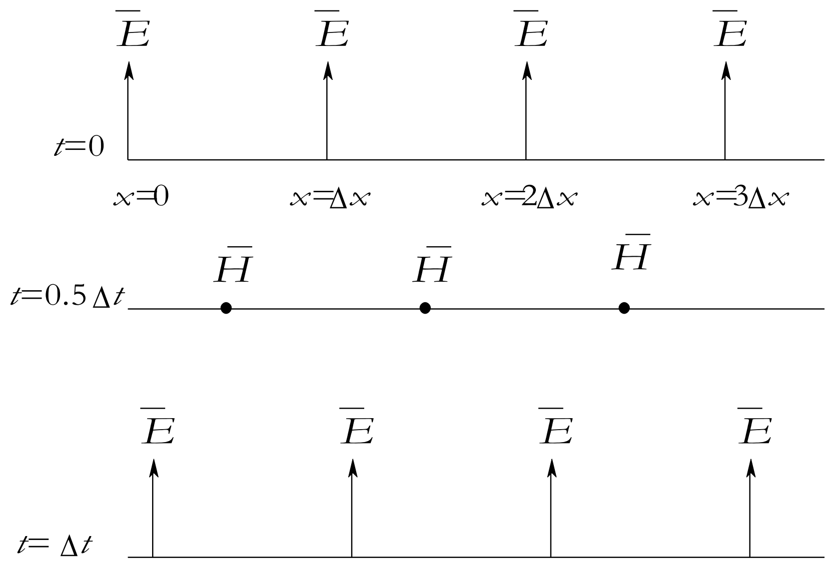
- Yee algorithm solves for both electric and magnetic fields in time and space using the coupled Maxwell's curl equations.

- The Yee algorithm centers its $\bar{E}$ and $\bar{H}$ field components in three-dimensional space so that every $\bar{E}$ component is surrounded by four circulating $\bar{H}$ components, and every $\bar{H}$ component is surrounded by four circulating $\bar{E}$ components.

Yee's Unit Space Lattice Cell:



The computational domain is divided into a number of rectangular unit cells. According to Yee algorithm $\bar{E}$ and $\bar{H}$ field components are separated $\frac{\Delta t}{2}$ by in time.



Finite Differences (Discretization)

A space point in a uniform rectangular lattice is denoted as:

$$(i, j, k) = (i\Delta x, j\Delta y, k\Delta z)$$

Here Δx , Δy and Δz are the lattice space increments in the *x*, *y*, and *z* coordinate directions respectively and *i*, *j*, and *k* are integers. If any scalar function of space and time evaluated at a discrete point in the grid and at a discrete point in time is denoted by *u,* then;

$$u(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = u^n_{i,j,k}$$

Using central, finite difference approximation in space, i.e. w.r.t. x:

$$\frac{\partial u}{\partial x}\left(i\Delta x, j\Delta y, k\Delta z, n\Delta t\right) = \frac{u^n_{i+1/2,j,k} - u^n_{i-1/2,j,k}}{\Delta x} + O\left(\left(\Delta x\right)^2\right)$$

Using central, finite difference approximation in time:

$$\frac{\partial u}{\partial t}(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = \frac{u_{i,j,k}^{n+1/2} - u_{i,j,k}^{n-1/2}}{\Delta t} + O\left[(\Delta t)^2\right]$$

Scalar equations are discretized as:

Electric Field Intensity:

$$E_{x(i,j,k)}^{n+1} = \left(\frac{2\varepsilon - \sigma\Delta t}{2\varepsilon + \sigma\Delta t}\right)E_{x(i,j,k)}^{n} + \frac{2\Delta t}{\Delta y(2\varepsilon + \sigma\Delta t)}\left(H_{z(i,j+1,k)}^{n+1/2} - H_{z(i,j,k)}^{n+1/2}\right) - \frac{2\Delta t}{\Delta z(2\varepsilon + \sigma\Delta t)}\left(H_{y(i,j,k+1)}^{n+1/2} - H_{y(i,j,k)}^{n+1/2}\right)$$

$$E_{y(i,j,k)}^{n+1} = \left(\frac{2\varepsilon - \sigma\Delta t}{2\varepsilon + \sigma\Delta t}\right)E_{y(i,j,k)}^{n} + \frac{2\Delta t}{\Delta z(2\varepsilon + \sigma\Delta t)}\left(H_{x(i,j,k+1)}^{n+1/2} - H_{x(i,j,k)}^{n+1/2}\right) - \frac{2\Delta t}{\Delta x(2\varepsilon + \sigma\Delta t)}\left(H_{z(i+1,j,k)}^{n+1/2} - H_{z(i,j,k)}^{n+1/2}\right)$$

$$E_{z(i,j,k)}^{n+1} = \left(\frac{2\varepsilon - \sigma\Delta t}{2\varepsilon + \sigma\Delta t}\right)E_{z(i,j,k)}^{n} + \frac{2\Delta t}{\Delta x(2\varepsilon + \sigma\Delta t)}\left(H_{y(i+1,j,k)}^{n+1/2} - H_{y(i,j,k)}^{n+1/2}\right) - \frac{2\Delta t}{\Delta y(2\varepsilon + \sigma\Delta t)}\left(H_{x(i,j+1,k)}^{n+1/2} - H_{x(i,j,k)}^{n+1/2}\right)$$

Magnetic Field Intensity:

$$H_{x(i,j,k)}^{n+1/2} = H_{x(i,j,k)}^{n-1/2} + \frac{\Delta t}{\mu\Delta z}\left(E_{y(i,j,k)}^{n} - E_{y(i,j,k-1)}^{n}\right) - \frac{\Delta t}{\mu\Delta y}\left(E_{z(i,j,k)}^{n} - E_{z(i,j-1,k)}^{n}\right)$$

$$H_{y(i,j,k)}^{n+1/2} = H_{y(i,j,k)}^{n-1/2} + \frac{\Delta t}{\mu\Delta x}\left(E_{z(i,j,k)}^{n} - E_{z(i-1,j,k)}^{n}\right) - \frac{\Delta t}{\mu\Delta z}\left(E_{x(i,j,k)}^{n} - E_{x(i,j,k-1)}^{n}\right)$$

$$H_{z(i,j,k)}^{n+1/2} = H_{z(i,j,k)}^{n-1/2} + \frac{\Delta t}{\mu\Delta y}\left(E_{x(i,j,k)}^{n} - E_{x(i,j-1,k)}^{n}\right) - \frac{\Delta t}{\mu\Delta x}\left(E_{y(i,j,k)}^{n} - E_{y(i-1,j,k)}^{n}\right)$$

One dimensional free space formulation:

Assume a plane wave with the electric filed intensity having E_x component, magnetic field intensity having H_y component and traveling in the *z* direction.

Maxwell's Equations become:

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0}\frac{\partial H_y}{\partial z}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_x}{\partial z}$$

Taking central difference approximation for both temporal and spectral derivatives:

$$\frac{E_x^{n+1}(k) - E_x^n(k)}{\Delta t} = -\frac{1}{\varepsilon_0}\frac{H_y^{n+1/2}(k+1) - H_y^{n+1/2}(k)}{\Delta z}$$

$$\frac{H_y^{n+1/2}(k) - H_y^{n-1/2}(k)}{\Delta t} = -\frac{1}{\mu_0}\frac{E_x^n(k) - E_x^n(k-1)}{\Delta z}$$

Update equations:

$$E_x^{n+1}(k) = E_x^n(k) - \frac{\Delta t}{\varepsilon_0 \Delta z}\left[H_y^{n+1/2}(k+1) - H_y^{n+1/2}(k)\right]$$

$$H_y^{n+1/2}(k) = H_y^{n-1/2}(k) - \frac{\Delta t}{\mu_0 \Delta z}\left[E_x^n(k) - E_x^n(k-1)\right]$$

Notice that:

- n means a time $t = n\Delta t$,
- The calculations are interleaved in both time and space. For example the new value of E_x is calculated from the previous value of E_x and the most recent values of H_y.

Writing the expressions of E_x and H_y in Matlab computer code:

- ex(k)=ex(k)-(dt/(eps0*dz))*(hy(k+1)-hy(k));
- hy(k)=hy(k)-(dt/(mu0*dz))*(ex(k)-ex(k-1));
- Note that n, n+1/2, n-1/2 superscripts are ignored.
- Also note that k+1/2 and k-1/2 are rounded off in order to specify a position in an array in the program.

Writing a Matlab program code:

- Calculate E_x field by using a loop.
- Calculate the source. i. e. the initial condition. Generate a Gaussian pulse in the center of the problem space.
- Apply the B.C. to find the E_x values at the boundaries of the problem space.
- Calculate the H_y fields by using a loop.

Generation of the Gaussian pulse:

- pulse=exp(-((n*dt-t0)^2)/(t1^2));

ex(ks)=pulse+ ex(ks);

The stability criteria:

$$\Delta t \le \frac{\Delta x}{\sqrt{n}c}$$

where n is the dimension number of the simulation.

For one dimensional simulation:

$$\Delta t \le \frac{\Delta x}{c}$$

```
clear all
KE=200;
c=3e8;
for k=1:KE
    ex(k)=0;
    hy(k)=0;
end
dz=0.1;
% Stability criteria

dt=dz/(c);

% Initialize the constants

eps0=8.85e-12; mu0=1.25663e-6;
nStop=250;
dtedz=dt/(eps0*dz);
dtmdz=dt/(mu0*dz);
%  time stepping

for n=1:nStop
```

```
% save the field at k=25 and k=50 for all time steps
    ex25(n)=ex(25);
    ex50(n)=ex(50);


        %update electric field

for k=1:KE-1
    ex(k)=ex(k)-dtedz*(hy(k+1)-hy(k));

% save the field at n=10 for the whole space domain

if(n==10)

        ext(k)=ex(k);
    end
    end

%source
%Apply sine wave excitataion
%ex(1)=sin(2*3.14*3*10^7*n*dt);
% Apply Gaussian pulse excitataion
ex(1)=exp(-(n-30)^2/100);

%update magnetic field
for k=2:KE
    hy(k)=hy(k)-dtmdz*(ex(k)-ex(k-1));
end


end
```

References

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[3] Taflove, S.C. Hagness, *same as above*, 2nd ed., Artech, 2000

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https://www.eeweb.com/finite-difference-time-domain-method-using-yees-algorithm/---------my suggesion