250 Homework #1

Sean O'Dea *

October 6, 2023

P2.1.1 [8 pts]

Let A be any set. What are the direct products $\emptyset \times A$ and $A \times \emptyset$? If x is any `thing`, what are the direct products $A \times \{x\}$ and $\{x\} \times A$? Justify your answers.

Solution:

If we let A be any set and take the cartesian product of A and the empty set in any order, the result will also be the empty set as there is nothing to relate the elements of A with.

$\emptyset \times A = \emptyset$, $A \times \emptyset = \emptyset$

If we take the cartesian product of A with a set that contains a single element x we'll get a set consisting of relations of every element in A and x. If x comes last in the product, it will be the thing related to, whereas if it comes first will relate to every element in A.

$A \times \{x\} = \{(A_1, x), (A_2, x), \ldots, (A_n, x)\}$
$\{x\} \times A = \{(x, A_1), (x, A_2), \ldots, (x, A_n)\}$

*Collaborated with Jonah Willers and Tomas Acuña

P2.1.5 [10 pts]

Let n be a natural and let $I(x)$ be a unary relation on the set $\{0, \ldots, n-1\}$. Let w be the binary string of length n that has 1 in position x whenever $I(x)$ is true and 0 in position x when $I(x)$ is false. (As in Java, we consider the positions of the letters in the string to be numbered starting from 0.) What is the string corresponding to the predicate $I(x)$ meaning "x is an even number" in the case where $n = 5$? The case where $n = 8$? If w is an arbitrary string and $I(x)$ the corresponding unary predicate, describe the set corresponding to the predicate in terms of w.

Solution:

$n = 5$ - The set is $\{0, 1, 2, 3, 4\}$. Constructing the string w over the set using $I(x)$, we get $w =$ "10101" because 0 is even, 1 is odd, 2 is even, etc up to 5.

$n = 8$ - The set is $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Constructing the string w over the set using $I(x)$, we get $w =$ "101010101" because 0 is even, 1 is odd, 2 is even, etc up to 8.

The set corresponding to $I(x)$ in terms of w will be only the indices of w where it equals 1.

P2.3.2 [12 pts]

Suppose that for *any* unary predicate P on a particular type T, you know that the proposition $(\exists x : P(x)) \leftrightarrow (\forall x : P(x))$ is true. What does this tell you about T? Justify your answer – state a property of T and explain why this proposition is always true if T has your property, and not always true if T does not have your property.

Solution:

From this proposition, we know that a property of the type T is that it has only one possible value. If this property is true, the proposition is *always* true because the predicates on either side of the equivalence always evaluate the same value, and therefore are always equal.

If this property is false, T is not always true because there could be values of the same type that don't both make $P(x)$ true. For example, if T is the natural numbers (which has more than one possible value), and $P(x)$ is true if the value is even, the proposition is not always true. There are even naturals, but not all naturals are even.

P2.5.6 [12 pts]

Suppose that A is a language such that $\lambda \notin A$. Let w be a string of length k. Show that there exists a natural i such that for every natural $j > i$, every string in A^j is longer than k. Explain how this fact can be used to decide whether w is in A^*.

Solution:

Supposing A is a language that does not contain λ, we know every string in A must have a length of at least 1. If we concatenate A j times, the minimum length of a string in the resulting set will be j because it had a length of 1 added j times. This minimum length string, which we can call v has a length of j. Let's assume $i = k$. We know $j > i$ so the string v is longer than w. Therefore, every string in A must be longer than k because this is true for the smallest possible string in A^j.

A^* is defined as $A^0 \cup A^1 \cup A^2 \cup \ldots$ Following from this, since every string in A^j is longer than w, we don't need to check if $w \in A^{k+1}$ and above since w is smaller than the elements of all of those sets. We only need to check if $w \in A^0 \cup A^1 \cup A^2 \cup \ldots \cup A^k$.

P2.6.3 [14 pts]

Heinlein's second puzzle has the same form. Here you get to figure out what the intended conclusion is to be[1], and prove it as above:

1. Everything, not absolutely ugly, may be kept in a drawing room;
2. Nothing, that is encrusted with salt, is ever quite dry;
3. Nothing should be kept in a drawing room, unless it is free from damp;
4. Time-traveling machines are always kept near the sea;
5. Nothing, that is what you expect it to be, can be absolutely ugly;
6. Whatever is kept near the sea gets encrusted with salt.[2]

Solution:

Definitions:

I. $\forall x : \neg AU(x) \implies DR(x)$

II. $\forall x : ES(x) \implies \neg D(x)$

III. $\forall x : DR(x) \implies D(x)$

IV. $\forall x : TM(x) \implies S(x)$

V. $\forall x : AU(x) \implies \neg WYE(x)$

VI. $\forall x : S(x) \implies ES(x)$

$AU(x)$ - is absolutely ugly, $DR(x)$ - is kept in a drawing room, $ES(x)$ - is encrusted with salt, $D(x)$ - is dry, $TM(x)$ - is a time machine, $S(x)$ - is kept near the sea, $WYE(x)$ - is what you expect.

[1] You must also translate the statements into formal predicate calculus — note for example the two different phrasings used for "is quite dry". In the novel, the solver of the puzzle concludes (correctly) that the nearby aircar is also a time-traveling machine, but strictly speaking this is not a valid conclusion from the given premises.

[2] You may want to look at P2.6.2 on page 108 for reference.

Proof:

1	$\forall x : AU(x) \implies \neg WYE(x)$	Premise V
2	$\forall x : WYE(x) \implies \neg AU(x)$	Contrapositive(V)
3	$\forall x : WYE(x) \implies DR(x)$	Transitivity(2, I)
4	$\forall x : WYE(x) \implies D(x)$	Transitivity(3, III)
5	$\forall x : D(x) \implies \neg ES(x)$	Contrapositive(II)
6	$\forall x : WYE(x) \implies \neg ES(x)$	Transitivity(4, 5)
7	$\forall x : \neg ES(x) \implies \neg S(x)$	Contrapositive(VI)
8	$\forall x : WYE(x) \implies \neg S(x)$	Transitivity(6, 7)
9	$\forall x : \neg S(x) \implies \neg TM(x)$	Contrapositive(IV)
10	$\forall x : WYE(x) \implies \neg TM(x)$	Transitivity(8, 9)

The conclusion $\forall x : WYE(x) \implies \neg TM(x)$ means that everything you expect is not a time machine.

P2.8.1 [10 pts]

Let $A = \{1, 2\}$ and $B = \{x, y\}$. There are exactly sixteen different possible relations from A to B. List them. How many are total? How many are well-defined? How many are functions? How many are neither well-defined nor total?

Solution:

$$
\begin{array}{rl}
\{(1,x),(1,y),(2,x),(2,y)\} & \text{Total} \\
\{(1,x),(1,y),(2,x)\} & \text{Total} \\
\{(1,x),(1,y),(2,y)\} & \text{Total} \\
\{(1,x)(2,x),(2,y)\} & \text{Total} \\
\{(1,y),(2,x),(2,y)\} & \text{Total} \\
\{(1,x),(1,y)\} & \text{Neither} \\
\{(2,x),(2,y)\} & \text{Neither} \\
\{(1,x),(2,y)\} & \text{Total, Well-Defined, Function} \\
\{(1,y),(2,x)\} & \text{Total, Well-Defined, Function} \\
\{(1,x),(2,x)\} & \text{Total, Well-Defined, Function} \\
\{(1,y),(2,y)\} & \text{Total, Well-Defined, Function} \\
\{(1,x)\} & \text{Well-Defined} \\
\{(1,y)\} & \text{Well-Defined} \\
\{(2,x)\} & \text{Well-Defined} \\
\{(2,y)\} & \text{Well-Defined} \\
\emptyset & \text{Well-Defined}
\end{array}
$$

9 Total, 9 Well-Defined, 4 Functions, and 2 are None

P2.9.3 [10 pts]

Let $f : A \to B$ and $g : B \to C$ be functions such that $g \circ f$ is a bijection. Prove that f must be one-to-one and that g must be onto. Give an example showing that it is possible for neither f nor g to be a bijection.

Solution:

$g \circ f$ maps $A \to C$ and is both one-to-one and onto. If the map to C is onto, all elements of C must have something mapped to them. This means $g(x)$ must be onto because it also maps to C. If g is not onto, the resulting composition could not be. Suppose g is not onto. The elements of B would not map to all of the elements in C and thus the elements of A would not map to all of the elements of C, so the composition could not be onto.

Similarly f must also be one-to-one because otherwise multiple elements in A would map to the same element in C. Suppose f is not one-to-one. Two or more of the inputs from A map to one output in B. This output in B then maps to an output in C. This would mean two inputs in A map to one output in C, and f could not be one-to-one.

An example of f and g not being bijections individually would be if the intermediary set B had elements that A does not map to, but still map to C This would mean f is not onto because there are elements in the output set not mapped to. It also means g is not one-to-one because more than one of the inputs in B would have to map to outputs in C.

P2.9.7 [12 pts]

Let A be a set and f a bijection from A to itself. We say that f fixes an element x of A if $f(x) = x$.

(a) Write a quantified statement, with variables ranging over A, that says "there is exactly one element of A that f does not fix."

(b) Prove that if A has more than one element, the statement of part (a) leads to a contradiction. That is, if f does not fix x, and there is another element in A besides x, then there is some other element that f does not fix.

Solution:

(a) $\exists x : \forall y : (x \neq y) \land (f(x) \neq x) \land \neg(f(y) \neq y)$

(b) Let's assume $|A| > 1$. Let's also assume only one element of A is not fixed, meaning all the other elements are fixed. If all the other elements only map to themself, and there does not exist a mapping to or from the non-fixed element we have a contradiction because f is not onto (nothing maps to the non-fixed element) and thus not bijective.

If we try to fix this by mapping one of the fixed elements to the non-fixed one, f is no longer well-defined (and thus not a bijective function) because one of its elements maps to more than one output. In both cases here we have a contradiction.

P3.1.7 [12 pts]

A **perfect number** is a natural that is the sum of all its proper divisors. For example, $6 = 1+2+3$ and $28 = 1+2+4+7+14$. Prove that if $2^n - 1$ is prime, then $(2^n - 1)2^{n-1}$ is a perfect number. (A prime of the form $2^n - 1$ is called a **Mersenne prime**. Every perfect number known is of the form given here, but it is unknown whether there are any others.)

Solution:

Let's assume $2^n - 1$ is prime. This means the divisors of $(2^n - 1)2^{n-1}$ are the powers of 2 up to 2^{n-1} inclusive, and the powers of 2 (up to 2^{n-1} exclusive) mulitplied by the Mersenne Prime. We can express the first grouping as a sum of $2^n - 1$ based on the axiom:

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

The second grouping can be expressed as the sum:

$$\sum_{i=0}^{n-2} 2^i(2^n - 1) = (2^n - 1)\sum_{i=0}^{n-2} 2^i = (2^n - 1)(2^{n-1} - 1)$$

This simplification follows from the previous axiom. If we add the two sums we get: $2^n - 1 + (2^n - 1)(2^{n-1} - 1) = (2^n - 1) + (2^{n-1} - 1)(2^n - 1) = (2^n - 1)(1 + (2^{n-1} - 1)) = (2^n - 1)2^{n-1}$. By summing the divisors, we've simplified to our first statement, thus a number expressed in the form $(2^n - 1)2^{n-1}$ where ($2^n - 1$ is a prime number) is a perfect number.

EC: P2.10.6 [10 pts]

There is only one partial order possible on the set $\{a\}$, because $R(a, a)$ must be true. On the set $\{a, b\}$, there are three possible partial orders, as $R(a, a)$ and $R(b, b)$ must both be true and either zero or one of $R(a, b)$ and $R(b, a)$ can be true. List all the possible partial orders on the set $\{a, b, c\}$. (Hint: There are nineteen of them.) How many are linear orders?

Solution: