Vibrations and Heat Diffusion on the Unit Disc

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Abstract

1 Introduction

Partial differential equations (PDEs) are a natural extension to ordinary differential equations, in that they describe systems that can vary with respect to multiple variables, traditionally position and time. Two fundamental PDEs are the heat and wave equations. Both of these involve the Laplace operator, defined in the following way:

$$\triangle f := \mathrm{div}(\mathrm{grad} f) \tag{1}$$

Note, that it is sometimes written as $\nabla^2 f$. Due to its nature as being the divergence of the gradient of some function, it is a natural generalization to the second derivative with respect to a position variable, and can be written in the following way:

$$\triangle f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} \tag{2}$$

where (x, y) represent a point in Cartesian coordinates and (r, ϕ) represents a point in polar coordinates. Solutions to Laplace's equation $\triangle u = 0$ are known as harmonic functions, and are of particular interest in both pure and applied mathematics.
This operator is used regularly in the description of many dynamical systems, including wave propagation, heat diffusion, and many other PDEs.
The wave equation, as the name suggests, describes how waves propagate through a given domain. Consider a string with fixed ends that is allowed to vibrate. The motion that we are interested in is transverse motion, so we consider the forces in a direction normal to the

line. Newton's second law tells us the following for the dynamics of a small bit of mass on a string:

$$F_{net} = dm \cdot a = \lambda dx \cdot a \tag{3}$$

where λ is the linear mass density, and a is the vertical acceleration of the string. If we raise part of the string near this point so that the angle the string makes between these two points is θ, then the force is:

$$F_{net} = T\sin(\theta + d\theta) - T\sin(\theta) = T\cos(\theta)d\theta) \tag{4}$$

where T is the tension in the rope. If we assume small enough displacements, we can approximated $\cos\theta \approx 1$ and write $d\theta = (\partial\theta/\partial x)dx$. Since we are assuming small displacements, we can say that the angle θ is the slope of the string. We will write the displacement of the string as $u = u(x,t)$, and say that this function must satisfy the following equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\lambda}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{5}$$

We are introducing the constant c which is the speed of propagation on the string. This can be absorbed into our time variable, by redefining our unit of time. For higher spacial dimensions, the second partial derivative with respect to x is replaced with the Laplacian, so the general wave equation is:

$$\boxed{\triangle u = \frac{\partial^2 u}{\partial t^2}} \tag{6}$$

We see that when the vibrating object is not accelerating, the Laplacian is zero, which means that the steady state behavior of the wave equation is a harmonic function. As we will see later, solutions to the wave equation oscillate about harmonic functions.

The heat equation has a similar form as the wave equation, but with very different dynamics. The general assumption behind the heat equation is conservation of energy. This conservation law is formulated in terms of a continuity equation:

$$\frac{\partial u}{\partial t} + \mathrm{div}\boldsymbol{J} = 0 \tag{7}$$

Here u is the energy density, and $\boldsymbol{J}$ is the energy flux. This equation states that if energy is flowing into a region, then the energy density must be increasing. In order to find the energy flux, we use Newton's Law of cooling which states that the energy flow is proportional to the temperature difference between two regions. This can be formulated in the following way:

$$\boldsymbol{J} = -\sigma \mathrm{grad} u \tag{8}$$

where σ is the thermal conductivity. The reason for using the energy gradient instead of the temperature gradient in this equation is that temperature is just a useful method for defining the energy. Using this, we may write the heat equation as:

$$\boxed{\triangle u = \frac{\partial u}{\partial t}} \tag{9}$$

Here we have absorbed the thermal conductivity into our definition of time. The interesting part of this equation is that it can be used to describe diffusive processes as well. In fact it can be used to describe many systems that do no violate the first and second law of

thermodynamics. The first of these states energy conservation, and the second states that the entropy of a system must increase. This second law is one of the few that does depend on the direction of time. This is why the heat equation is not symmetric in the spatial and temporal variables, when the wave equation is.
This system has harmonic solutions whenever the system is not changing in time. Therefore, we see that in time, the solution tends towards a harmonic solution. It is for this reason that solutions to Laplace's equations describe steady state temperature distributions.
In order to analyze the different types of solutions these equations provide, as well as some general methods for solving PDEs, we will solve both of these equations on the unit disc B^2 with the boundary condition that the function must vanish at the boundary. Similar to the case for ODEs, we must also provide initial conditions. Since the heat equation is first order in time, we only need to know the initial heat distribution $u(r, \phi, 0) = u_o(r, \phi)$. The wave equation is second order in time, so we must supply both the initial displacement of the circular membrane, $u(r, \phi, 0) = u_o(r, \phi)$, and the initial velocity of $\partial_t u(r, \phi, 0) = \dot{u}_o(r, \phi)$. Together with the boundary data, these conditions uniquely define a solution to the differential equation.

2 Separation of Variables

To begin solving these equations, we will employ a method known as separation of variables. In this we assume that the solution can be written as the product of single-variable functions. Since these are both functions of position and time, we assume a solution $u(r, \phi, t) = \psi(r, \phi)\chi(t)$. If we insert this into the heat equation, and divide by $u(r, \phi, t)$ we obtain the following:

$$\frac{\triangle\psi}{\psi} = \frac{1}{\chi(t)}\frac{d\chi}{dt} \tag{10}$$

The left-hand side is just a function of the spacial variables, while the right-hand side is just a function of time. This means that the two sides are independent of each other, and therefore must be equal to a constant. For reasons that will be clear later on, I will use $-k^2$ as the constant. This leaves us with two differential equations:

$$\begin{gathered} \triangle\psi + k^2\psi = 0 \\ \dot{\chi} = -k^2\chi \end{gathered} \tag{11}$$

We can easily solve the time equation to obtain:

$$\chi_{\text{heat}}(t) = \exp(-k^2 t) \tag{12}$$

Any constants of integration may be absorbed into $\psi(r, \phi)$. If we follow this same procedure for the wave equation, we are left with the two separated differential equations:

$$\begin{gathered} \triangle\psi + k^2\psi = 0 \\ \ddot{\chi} = -k^2\chi \end{gathered} \tag{13}$$

Again the time component can be easily solved, to give the following solution:

$$\chi_{\text{wave}} = A\cos(kt) + B\sin(kt) \tag{14}$$

It is important to notice that for both of these equations, the spacial part is the same. Given that we are using the same boundary conditions for both equations, solving this equation will provide us with solutions to both the wave and heat equation.

3 The Helmholtz Equation

The above differential equation:

$$\triangle\, \psi + k^2 \psi = 0 \tag{15}$$

is known as the Helmholtz equation, and it describes the eigenvalues and eigenfunctions of the Laplacian on the unit disc. Recall that an eigenvalue equation is one where:

$$\mathcal{L} v = \lambda v \tag{16}$$

where $\mathcal{L}$ is a linear operator, v is an eigenvector, and λ is an eigenvalue. In the context of differential operators, the eigenvectors are referred to as eigenfunctions.
Given our boundary data, the eigenfunctions will be ones that satisfy the following equations:

$$\boxed{\begin{array}{c} \triangle\psi + k^2\psi = 0 \\ \psi(1, \phi) = 0 \end{array}} \tag{17}$$

4 Inner Products and Orthonormal Basis

5 Fourier Series

6 Fourier-Bessel Series

7 The Normal Modes on a Disc

8 Matching Initial Conditions

9 A Particular Solution

10 References

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