LECTURE - 4

M/M/c QUEUING MODEL

Learning objective

- **To analyze multi-server M/M/c queues**

9.7 M|M|c Queuing Model

- In a M|M|c queue, there are c parallel servers, each serving customers, (c>1).
- The arrival process and service process follow Poisson distribution.
- All arriving customers after entering the service system join a single queue.
- If all c servers are already busy in serving customers, the first customer in the queue will be served by any of the c servers as soon as any server will be free from serving previous customer. The service rate in the case will be (μc) as shown in Figure 9.7
- Hence the utilization factor for the **M|M|c** service system will be

$$\rho = \frac{\lambda}{c \times \mu}$$

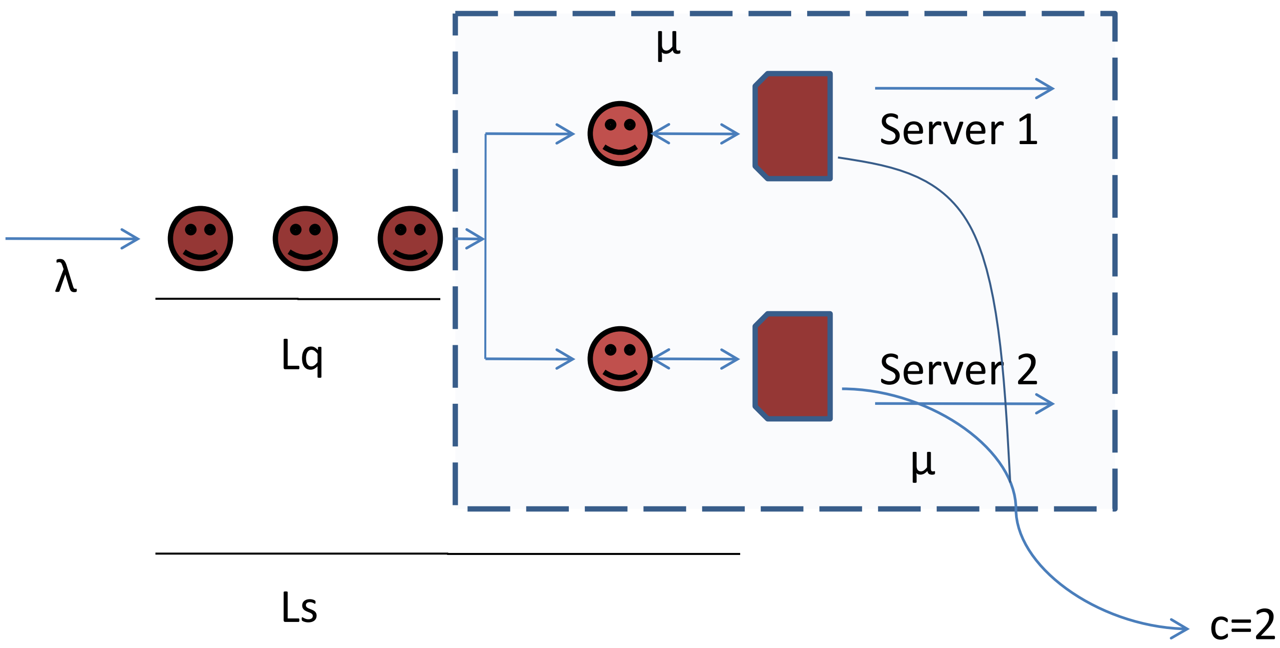

Figure 9.7: M|M|C Queuing Model with two servers

9.7.1 Transition of states in M|M|c queues and associated probabilities

In M|M|c queues, the arrival rate remains same as M|M|1 queues but the service rate will depend on the number of servers. The service rate will be $n\mu$ for n<=c. As soon as the number of customers exceeds c, the service rate becomes μc as shown in Figure 9.8.

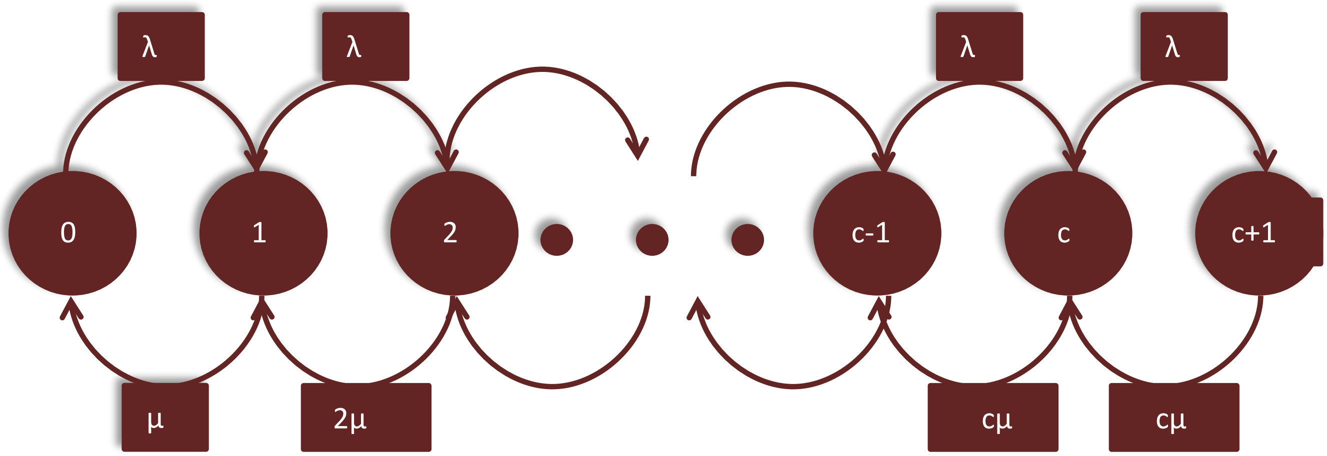

Figure 9.8: Transition of states in M|M|c Queuing Model

- The service rate, μ_c in this case will be,

$$\mu_c = \begin{cases} (n\mu) & n < c \text{ for } n = 1,2,\ldots c \\ (c\mu) & n > c \text{ for } n = c, c+1,\ldots \end{cases}$$

The probability of having n customers in the service system can be written in similar way as we wrote for M|M|1 model but with revised service rate.

$$P_n = \left(\frac{\lambda}{\mu_c}\right)^n \times P_0$$

Or $$P_n = \begin{cases} \left(\dfrac{\lambda^n}{\mu(2\mu)(3\mu)\ldots\ldots\ldots\ldots(n\mu)}\right)P_0 \text{ if } n < c \\ \left(\dfrac{\lambda^n}{\mu(2\mu)(3\mu)\ldots\ldots\ldots\ldots(c\mu)(c\mu)^{n-c}}\right)P_0 \text{ if } n \geq c \end{cases}$$

$$P_n = \begin{cases} \left(\dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n\right)P_0 \text{ if } n \leq c \\ \left(\dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c\left(\dfrac{\lambda}{c\mu}\right)^{n-c}\right)P_0 \text{ if } n > c \end{cases}$$

9.7.2 Performance measures of M|M|c Queuing model

First, we will determine the number of customers in the queue, *Lq*. In a system, there will be no queue formed till the number of customers are less than or equal to the number of servers. The customer will enter in the queue when on the arrival in the system a customer will find all the servers busy. Hence, *n-c* represents the number of customer in the queue. We can write L_q as given below.

$$L_q = \sum_{n=c}^{\infty}(n-c)P_n$$

To determine L_q, substitute $j=n-c$ or $n=c+j$ in the above expression as given below.

$$L_q = \sum_{j=0}^{\infty} jP_{c+j}$$

P_{c+j} can be written as

$$P_{c+j} = \left(\frac{\lambda}{\mu} + \frac{\lambda}{2\mu} + \ldots.. + \frac{\lambda}{c\mu}\right)\left(\frac{\lambda}{c\mu}\right)^j P_0$$

$$or\ P_{c+j} = \frac{(\rho)^c}{c!c^j}\rho^j P_0$$

$$Hence,\ L_q = \sum_{j=0}^{\infty}\left(\frac{j(\rho)^c}{c!c^j}\rho^j P_0\right)$$

$$= \left(\frac{\rho^{c+1}}{c!c}\right)P_0 \sum_{j=0}^{\infty} j\left(\frac{\rho}{c}\right)^{j-1}$$

which can be written as

$$= \left(\frac{\rho^{c+1}}{c!\times c^j}\right)\times P_0 \times \left(\frac{\partial\left(\sum_{j=0}^{\infty}\left(\frac{\rho}{c}\right)^j\right)}{\partial\left(\frac{\rho}{c}\right)}\right)$$

$$= \left(\frac{\rho^{c+1}}{c!\times c^j}\right)\times P_0 \times \left(\frac{\partial\left(\frac{1}{1-\frac{\rho}{c}}\right)}{\partial\left(\frac{\rho}{c}\right)}\right)$$

$$L_q = \left(\frac{\rho^{c+1}}{(c-1)!(c-\rho)^2}\right)P_0$$

After determining L_q , we can determine the waiting time in the queue W_q using Little's law as given below.

$$W_q = \frac{L_q}{\lambda}$$

Customers waiting in the service system will be addition of W_q and service time.

$$W = W_q + \frac{1}{\mu}$$

The number of customers in the service system,

$$L = \lambda W \text{ (using little's law)}$$

$$= \lambda W_q + \frac{\lambda}{\mu}$$

Example

- A small internet café has two computer terminals. The arrival rate of internet users in the café is 10 users per hour. Each user spends 10 minutes on the computer. The arrival and service process follow exponential distribution
- Find the following:
  - What is the probability that both computers are free?
  - What is the probability that a customer can use the computer after arriving?
  - What is the probability that a customer will find no queue on arrival?
  - Find the average number of customers in the system?

Solution

- The internet café has two computers, means c=2
- The arrival and the service time follow exponential distribution, hence it is a M|M|2 queuing model
- **(a)**

  Probability that both the computers will be free: We are interested to find the probability of no customers in the café i.e., P_0

$$P_0 = \left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \left(\frac{1}{1 - \frac{\rho}{c}} \right) \right]^{-1}$$

$$= \left[1 + \rho + \frac{\rho^2}{2\left(1 - \frac{\rho}{c}\right)} \right]^{-1}$$

c=2, λ=10 users per hour, μ=6 users per hour

ρ = 10/6 = 1.666

P_0 = 0.0909 or 9.1%

- (b)

  P(a customer can use a computer after arriving)

  = P(no customer in the café)+P(one customer in the café)

  = P_0+P_1

  = $P_0+\rho P_1$ = 0.2424 or 24.2%

- (c)

  P(No queue on arrival)

  = $P_0+P_1+P_2$

  = $P_0+ \rho\ P_0+ (\rho^2/2)\ P_0$

  = 0.3687 or 36.9%

- (d) Average number of customers in the system

$$L_s = \mathrm{Lq} + \rho$$

$$= \frac{\rho^{c+1}}{(c-1)! \times (c - \rho)^2} \times P_0$$

$$= 3.787 \text{ users}$$