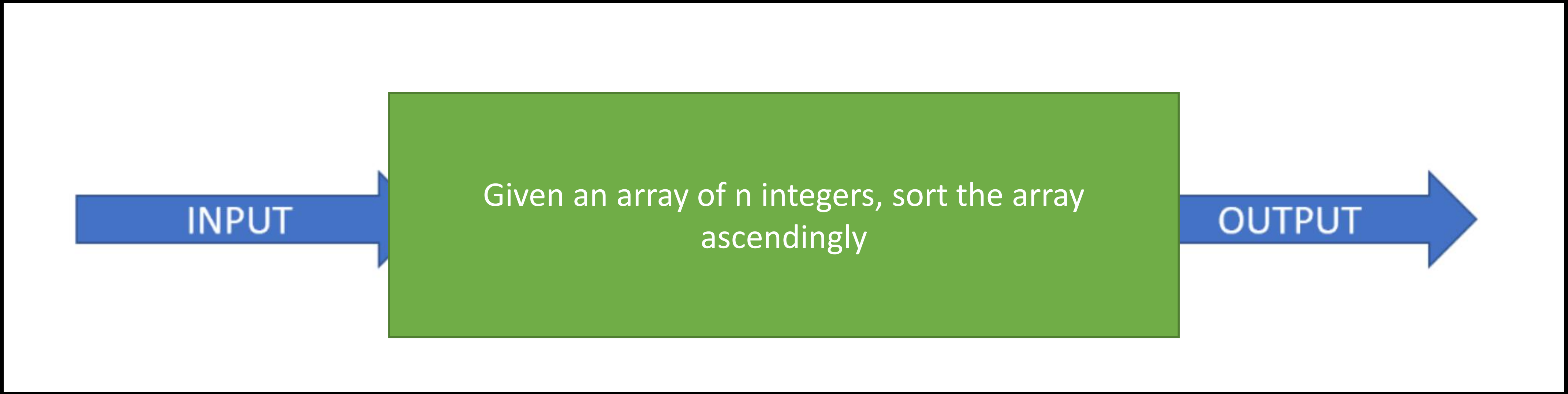
ALGORITHMS & DATA STRUCTURES

4th October 2021

TODAY’S PLAN

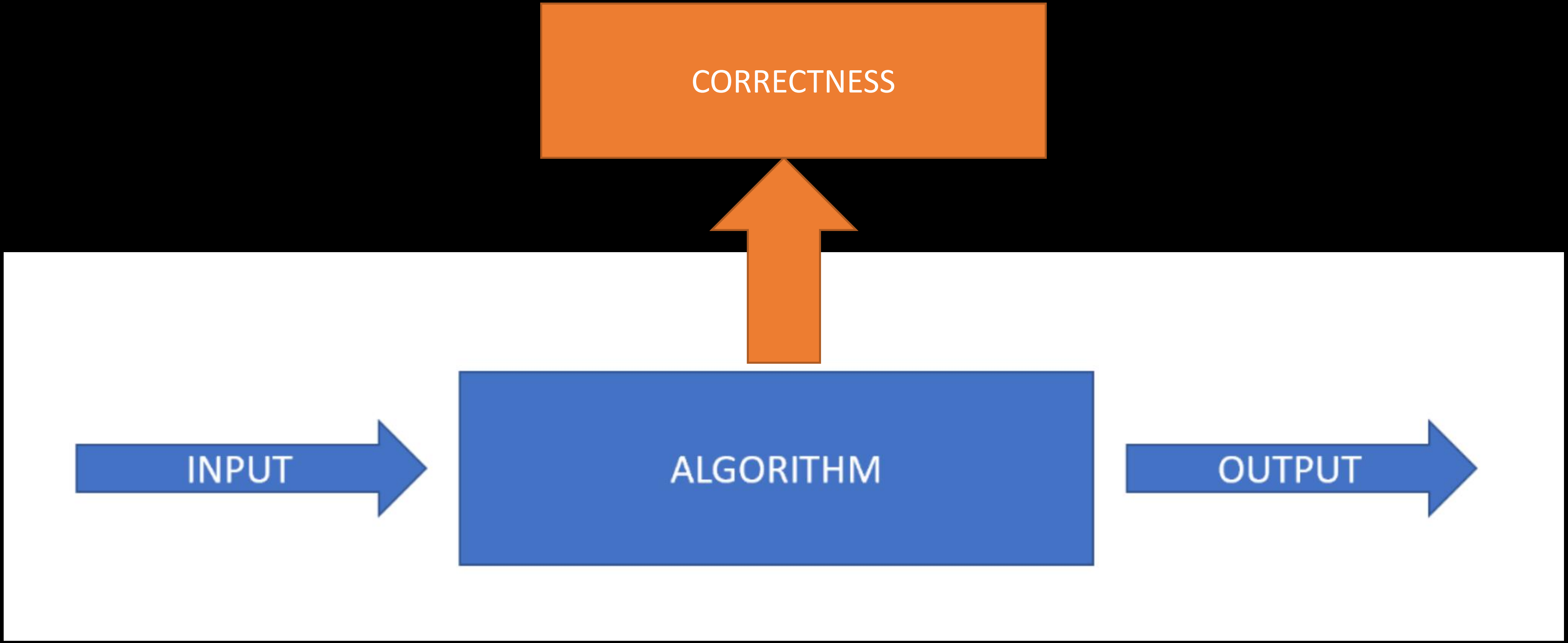
- Lecture Recap + Preview
- Recursion
- Induction
- Exercise 1.2
- Exercise 1.4

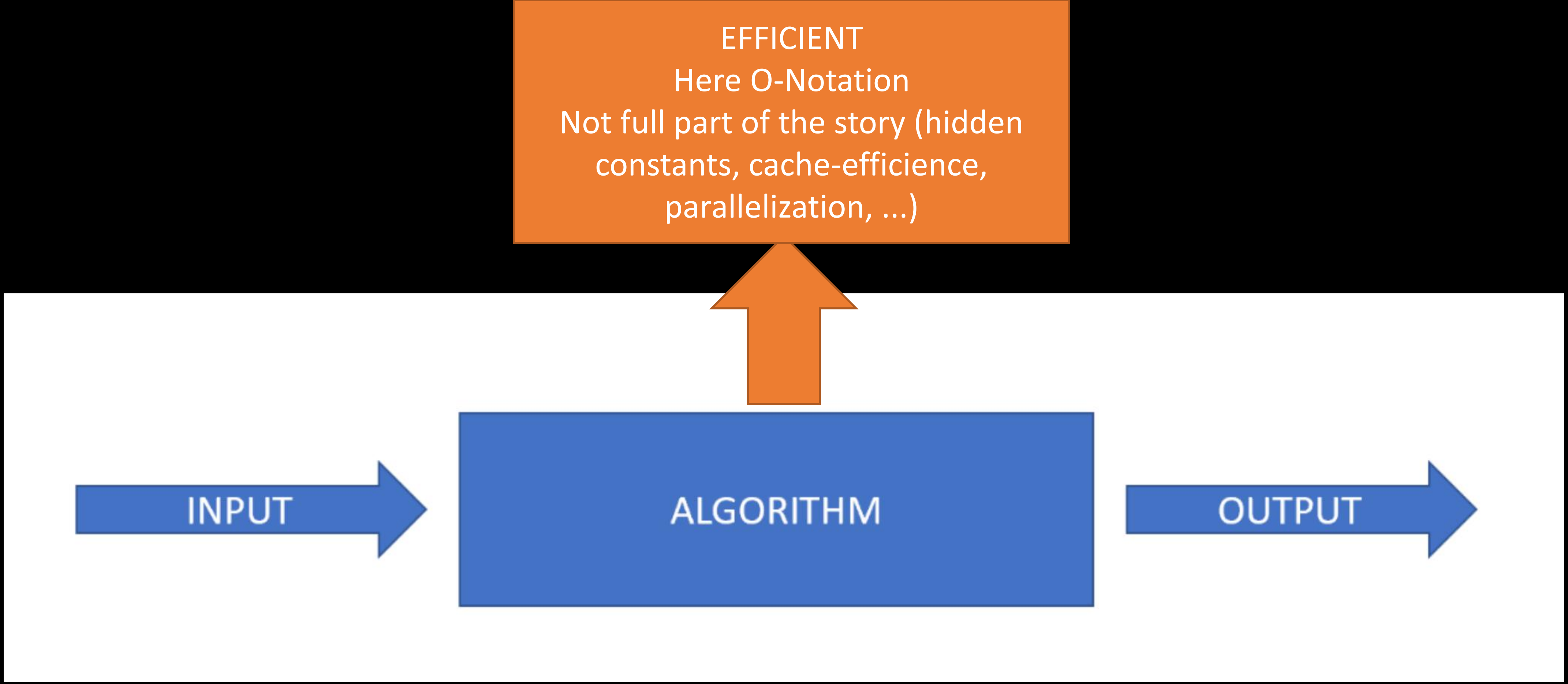

INPUT
ALGORITHM
OUTPUT


INPUT
Given an array of n integers, sort the array ascendingly
OUTPUT

ALGORITHMS AND DATA STRUCTURES

DATA STRUCTURES


CORRECTNESS
INPUT
ALGORITHM
OUTPUT


EFFICIENT
Here O-Notation
Not full part of the story (hidden constants, cache-efficience, parallelization, ...)
INPUT
ALGORITHM
OUTPUT

$$n^2+2n+1,\ \ln(n^n)\cdot\sum_{k=0}^{n}k,\ \frac{n}{\ln n},\ \sqrt{n}\ln(n),\ n\ln(n^2),\ \sum_{k=0}^{n}k^2,\ n\ln(n^n)$$

f(n) grows asymptotically slower than g(n)

if $\lim_{n\to\infty}\frac{f(n)}{g(n)}=0$

$$n^2 + 2n + 1,\ \ln(n^n) \cdot \sum_{k=0}^{n} k,\ \frac{n}{\ln n},\ \sqrt{n}\ln(n),\ n\ln(n^2),\ \sum_{k=0}^{n} k^2,\ n\ln(n^n)$$

$$\sqrt{n}\ln(n);\ \frac{n}{\ln n};\ n\ln(n^2);\ n^2+2n+1;\ n\ln(n^n);$$

$$\sum_{k=0}^{n} k^2;\ \ln(n^n)\sum_{k=0}^{n} k$$

RECURSION

- In class: Find a Star Problem

RECURSION

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- BASE CASE

RECURSION

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- BASE CASE

- RECURSION STEP

RECURSION

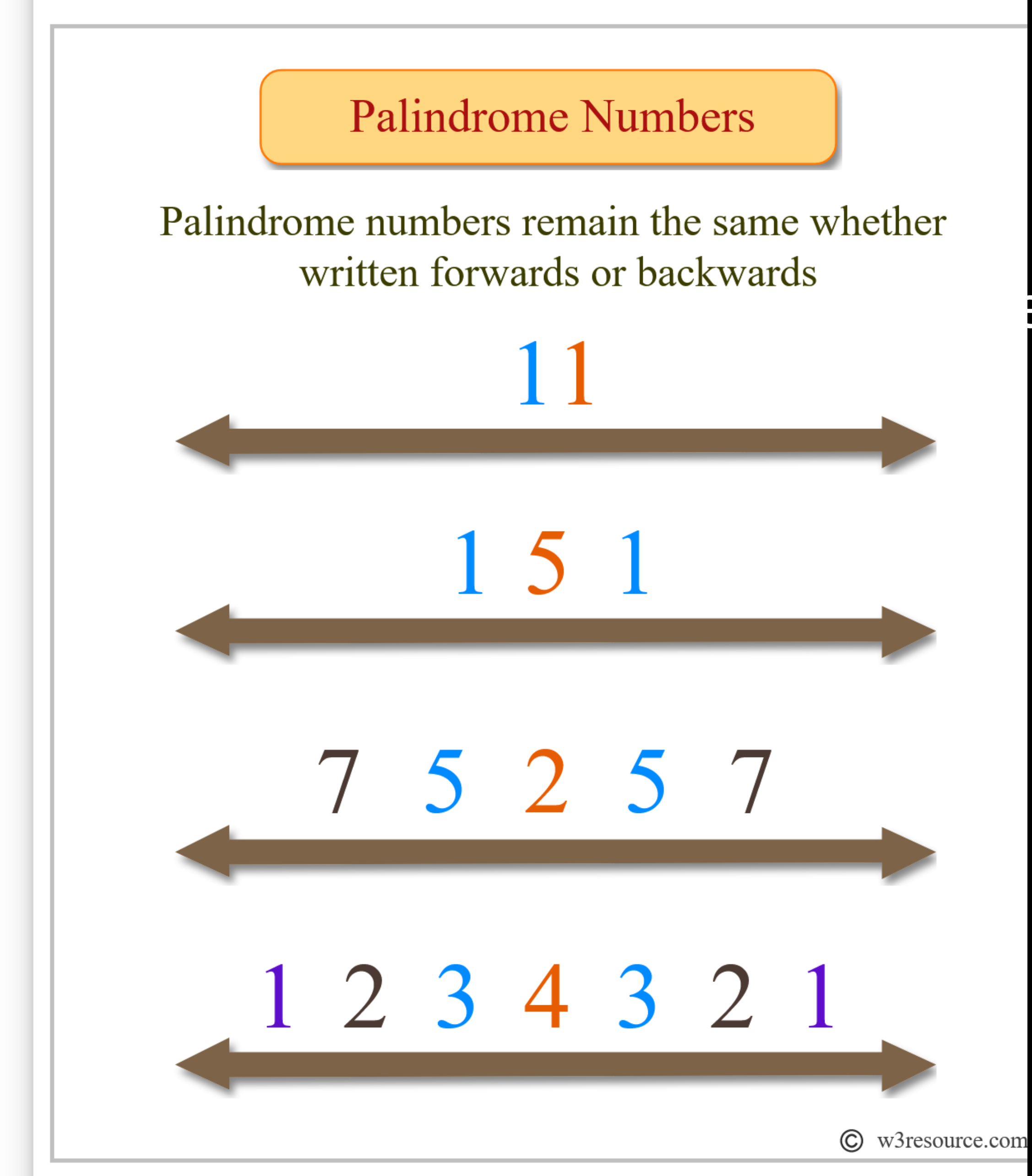
- In class: Find a Star Problem
- BASE CASE
- RECURSION STEP
- This principle is also known as Divide-And-Conquer, and will be discussed extensively in the next weeks

RECURSION: EXAMPLE

- Given a number x, determine whether it is palyndrome or not

RECURSION:

- Given a number x … e or not



RECURSION: EXAMPLE

$x = x_1 \dots x_n$

- Given a number x, determine whether it is palyndrome or not

BC

x single digit $\Rightarrow$ YES

x two digits:

$x_1 = x_2 \Rightarrow$ YES

$x_1 \neq x_2 \Rightarrow$ NO

RECURSION STEP

$x_1 \neq x_n \Rightarrow$ NO

$x_1 = x_n \Rightarrow$ CHECK WHETHER $x_2 \dots x_{n-1}$ IS PALINDROME

INDUCTION

- Typical scenario: prove P(n) for all natural numbers n.
- In other words prove P(0), P(1), P(2), ... for all natural numbers

- **Base case:** show that $P(0)$ holds.
- **Induction hypothesis:** we assume $P(n)$ for an **arbitrary** n.
- **Induction step:** prove $P(n+1)$ under the assumption $P(n)$ for an *arbitrary* n.

a) Let $a \neq 1$ be a real number. Prove by mathematical induction that for every non-negative integer n,

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}.$$

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BC $n = 0$

$$\sum_{i=0}^{0} a^i = a^0 = 1$$

$$\frac{a^{0+1} - 1}{a - 1} = 1 \checkmark$$

IH $\displaystyle\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}, a \neq 1$

holds for an n

IS $\displaystyle\sum_{i=0}^{n+1} a^i = \sum_{i=0}^{n} a^i + a^{n+1}$

$$\overset{IH}{=} \frac{a^{n+1} - 1}{a - 1} + a^{n+1}$$

$$= \frac{a^{n+1} - 1 + a^{n+2} - a^{n+1}}{a - 1}$$

$$= \frac{a^{n+2} - 1}{a - 1} = \frac{a^{(n+1)+1} - 1}{a - 1}$$

Let $T(x_{\text{hole}})$ be the time taken to finish the search assuming that the hole is at position x_{hole}. We define:

$$T_{\text{best}}(k) = \min\left\{T(-k), T(+k)\right\} \quad \text{and} \quad T_{\text{worst}}(k) = \max\left\{T(-k), T(+k)\right\}.$$

In class we analyzed $T_{\text{worst}}(k)$ for two choices of f: $f(n) = n$ and $f(n) = 2^{n-1}$. In this exercise, we will see what happens for other choices of f.

We extend the domain of definition of f to non-negative integers by adopting the convention that $f(0) = 0$. We define $n_k \geq 0$ as the unique non-negative integer satisfying $f(n_k) < k \leq f(n_k + 1)$.

a) Show that

$$T_{\text{best}}(k) = k + 2\sum_{i=1}^{n_k} f(i) \quad \text{and} \quad T_{\text{worst}}(k) = k + 2\sum_{i=1}^{n_k+1} f(i).$$

It suffices to provide an informal argument.

Let us see what happens if f "grows quadratically", i.e., $f(n) = n^2$.

$f(n_k) < k \leq f(n_{k+1})$

We need either n_{k+1} or n_{k+2} iterations. Fix $m \in \{n_{k+1}, n_{k+2}\}$

Time needed to reach k:

$f(1)$

$f(1) + f(2)$

$f(2) + f(3)$

$\vdots$

$f(m-1) + k$

$\Rightarrow 2\sum_{i=1}^{m-1} f(i) + k$

$f(n) = n^2$

b) Show that for every positive integer $k \geq 1$, we have

$$\frac{k\sqrt{k}}{100} \leq T_{\text{best}}(k) \leq T_{\text{worst}}(k) \leq 100k\sqrt{k}.$$

Hint: *Note that*[1] $n_k = \left\lceil \sqrt{k} \right\rceil - 1$ *and use the formula from Exercise 1.1. Also, you can use the fact that the function* $g(x) := \frac{x-1}{x}$ *is strictly increasing for all* $x > 0$ *without justification.*

Remark. Intuitively, this means that if $f(n) = n^2$, then the asymptotic growth of $T_{\text{best}}(k)$ and $T_{\text{worst}}(k)$ is similar to that of $k\sqrt{k}$. More generally, if $f(n) = n^\ell$ for some $\ell > 0$, then it is possible to show that the asymptotic growth of $T_{\text{best}}(k)$ and $T_{\text{worst}}(k)$ is similar to that of $k^{1+\frac{1}{\ell}}$. This means that for every function $f(n)$ that "grows polynomially in n", both $T_{\text{best}}(k)$ and $T_{\text{worst}}(k)$ grow asymptotically faster than k.

1.1 $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$T_{best}(k) = k + 2\sum_{i=1}^{n_k} i^2 = k + \frac{n_k(n_k+1)(2n_k+1)}{3}$

$n_k = \lceil \sqrt{k} \rceil - 1 \geq \sqrt{k} - 1$. Since $\frac{\sqrt{k}-1}{\sqrt{k}}$ has min. for $k=2$, $n_k \geq \sqrt{k} \cdot \frac{\sqrt{2}-1}{\sqrt{2}} \geq \frac{\sqrt{k}}{4}$

$\Rightarrow T_{best}(k) \geq \frac{n_k(n_k+1)(2n_k+1)}{3} \geq \frac{\sqrt{k} \cdot \sqrt{k} \cdot 2\sqrt{k}}{64 \cdot 3} = \frac{k\sqrt{k}}{96} \geq \frac{k\sqrt{k}}{100}$

$f(n) = n^2$

b) Show that for every positive integer $k \geq 1$, we have

$$\frac{k\sqrt{k}}{100} \leq T_{\text{best}}(k) \leq T_{\text{worst}}(k) \leq 100k\sqrt{k}.$$

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1.1 $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$$T_{\text{worst}}(k) = k + \sum_{i=1}^{n_k+1} i^2 = k + \frac{(n_k+1)(n_k+2)(2n_k+1)}{3} \leq k\sqrt{k} + \frac{2\sqrt{k}\cdot 3\sqrt{k}\cdot 5\sqrt{k}}{3}$$

$$= 11\, k\sqrt{k}$$

$$\leq 100\sqrt{k}$$

$n_k = \lceil \sqrt{k} \rceil - 1 \leq \sqrt{k}$

$f(n) = a^{n-1}$

$a \geq 2$

c) What is the worst-case search time $T_{\text{worst}}(k)$? (In terms of k and n_k).

Hint: *Use the formula from Exercise 1.2.*

$$T_{\text{worst}}(k) = k + 2\sum_{i=1}^{n_k+1} a^{i-1} = k + \sum_{i=1}^{n_k} a^{i} = k + \frac{2(a^{n_k+1} - 1)}{a-1}$$

1.2 $\quad \displaystyle\sum_{i=0}^{n} a^{i} = \frac{a^{n+1} - 1}{a-1}$

d) What is the value of $R_{\text{worst}}(N)$?

Hint: *For fixed* $n_k = N$, *the value of* $\frac{T_{\text{worst}}(k)}{k}$ *achieves its maximum when* $k = a^{N-1} + 1$.

For non-negative integer N, let $R_{\text{worst}}(N)$ be the maximum value of the worst-case relative delay $\frac{T_{\text{worst}}(k)}{k}$ over all k that satisfy $n_k = N$.

From (c) $T_{\text{worst}}(k) = \frac{a^{n_k+1} - 1}{a-1}$. Fix $n_k = N$

$$\frac{T_{\text{worst}}(k)}{k} = \frac{2(a^{N+1}-1)}{(a-1)k} + 1, \text{ using hint we get}$$

$$R_{\text{worst}}(N) = \frac{2(a^{N+1}-1)}{(a-1)(a^{N-1}+1)} + 1$$

The *asymptotic worst-case relative delay* is defined as $\lim_{N\to\infty} R_{\text{worst}}(N)$.

e) Compute the asymptotic worst-case relative delay.

Using (d)

$$\lim_{N\to\infty} \frac{2(a^{N+1}-1)}{(a-1)(a^{N-1}+1)} + 1 = 1 + \frac{2a^2}{a-1}$$

f) What is the parameter $a \geq 2$ that minimizes the asymptotic worst-case relative delay?

Hint: *You can use the fact that the function* $g(x) := \dfrac{x^2}{x-1}$ *is strictly increasing for* $x \geq 2$ *without justification.*

$$1 + \frac{2a^2}{a-1} \quad \text{strictly increasing}$$

$$\Rightarrow a = 2$$