Assignment 1

Hand in by April 14 to Søren (soren.wengel_mogensen@control.lth.se). Q&A via Zoom on April 12, find link online.

In this assignment, we will work with a certain characterization of d-separation. We first need to introduce some more terminology. An *undirected graph*, $\mathcal{U} = (V, E_U)$, is a pair of *nodes*, V, and a set of edges, E_U, all of which are *undirected*, $i - j$ (see the example graphs in Figure 1).

Let $\mathcal{U} = (V, E_U)$ be an undirected graph. For $S \subseteq V$, we let $\mathcal{U}_{V \setminus S}$ denote the undirected graph obtained from $\mathcal{U}$ by removing every node in S and every edge $k - l$ such that $k \in S$ or $l \in S$. Similarly, when $\mathcal{D} = (V, E_D)$ is a DAG, we define $\mathcal{D}_{V \setminus S}$ as the DAG obtained by removing every node in S and every edge $k \to l$ such that $k \in S$ or $l \in S$. In an undirected graph, $\mathcal{U} = (V, E_U)$, we say that two nodes $i, j \in V$ are *separated* by S, if there is no path between i and j in the graph $\mathcal{U}_{V \setminus S}$.

In this assignment, we will explain how one can use undirected graphs to decide d-separation in DAGs. For this purpose, we need the concept of a *moral graph*. We say that two nodes, i and j, are adjacent if there exists an edge between them. In a DAG, i and j are adjacent if $i \to j$ or if $i \leftarrow j$ and in an undirected graph i and j are adjacent if $i - j$. Given a DAG, $\mathcal{D} = (V, E_D)$, we say that $i \in V$ and $j \in V$ are *unshielded colliders* if they are not adjacent and there exists $k \in V$ such that $i \to k \leftarrow j$.

Definition. *Let $\mathcal{D} = (V, E_D)$ be a DAG. Its* moral graph *is the undirected graph obtained by adding the edge $i - j$ whenever i and j are unshielded colliders and then replacing all directed edges with undirected edges. We denote the moral graph of $\mathcal{D}$ by $\mathcal{D}^m$.*

Theorem. *Let $\mathcal{D} = (V, E_D)$ be a DAG and let $i, j \in V$, $C \subseteq V \setminus \{i, j\}$. We have that i and j are d-separated by C in $\mathcal{D}$ if and only if i and j are separated by C in the undirected graph $(\mathcal{D}_{an(\{i,j\} \cup C)})^m$.*

Figure 2 gives an example of how to apply this theorem.

After this lengthy introduction, we are ready to state your task. You need to solve Problem A or Problem B. You do not need to solve both. Hints are provided at the end of this document.

Problem A In this problem, we will prove the theorem. Let $\mathcal{D} = (V, E_D)$ be a DAG, let $i, j \in V$, and $C \subseteq V \setminus \{i, j\}$.

a. Assume that i and j are not separated by C in $(\mathcal{D}_{\mathrm{an}(A \cup B \cup C)})^m$. In this case, there is a path between i and j in $(\mathcal{D}_{\mathrm{an}(A \cup B \cup C)})^m$ such that no node on this path is in C. Construct from this a d-connecting path between i and j given C in $\mathcal{D}$.

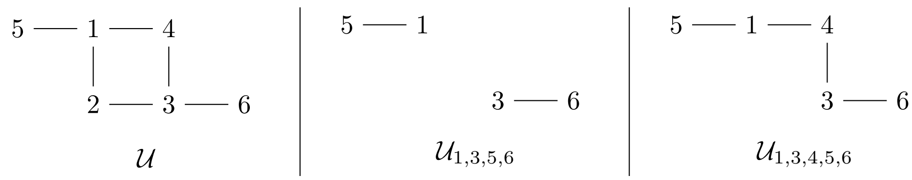

Figure 1: Example of an undirected graph, $\mathcal{U}$. If we want to decide if 1 and 2 are separated by $\{2,4\}$ in this graph, we consider $\mathcal{U}_{1,3,5,6}$, the graph obtained by removing 2 and 4 (and their edges). In this smaller graph, there is no path between 1 and 3, and we say that 1 and 3 are separated by $\{2,4\}$ in $\mathcal{U}$. On the other hand, if we want to decide if 1 and 3 are separated by 2 in $\mathcal{U}$, we remove 2 (and its edges) to obtain $\mathcal{U}_{1,3,4,5,6}$. In this graph, $1-4-3$ is a path between 1 and 3 and therefore 1 and 3 are not separated by $\{2\}$ in $\mathcal{U}$.

b. Assume that i and j are not d-separated by C in $\mathcal{D}$. This means that there exists a d-connecting path between i and j given C in $\mathcal{D}$. Construct from this a path between i and j in $(\mathcal{D}_{\mathrm{an}(A\cup B\cup C)})^m$ such that no node is in C.

c. Use the above to conclude that the theorem is true.

Problem B In this problem, we will use the theorem to implement d-separation in a programming language of your choice. The definition of d-separation is in terms of paths and it can be tedious to check every path between two nodes. The theorem allows a simpler approach. The goal is to make a function which takes a DAG, $\mathcal{D} = (V, E_D)$, two nodes $i, j \in V$, and a set $C \subseteq V \setminus \{i, j\}$ as input and outputs TRUE if i and j are d-separated given C in $\mathcal{D}$, and FALSE otherwise.

Both DAGs and undirected graphs can be represented using *adjacency matrices*. If $\mathcal{D} = (V, E_D)$ is a DAG and $|V| = n$, then its adjacency matrix is the $n \times n$ matrix, M, such that $M_{ji} = 1$ if and only if $i \to j$ is in $\mathcal{D}$. If $\mathcal{U} = (V, E_U)$ is an undirected graph and $|V| = n$, then its adjacency matrix is the $n \times n$ matrix such that $M_{ji} = M_{ij} = 1$ if and only if $i - j$ is in $\mathcal{U}$.

The following is a breakdown of how to implement a function that decides if i and j are d-separated given C in $\mathcal{D}$. Let $\mathcal{D} = (V, E_D)$ be a DAG, and let $i, j \in V$, and $C \subseteq V \setminus \{i, j\}$.

a. Construct a function to output $\mathcal{D}_{\mathrm{an}(\{i,j\}\cup C)}$.

b. Construct the moral graph of $\mathcal{D}_{\mathrm{an}(\{i,j\}\cup C)}$.

c. Decide if i and j is separated by C in the moral graph from above.

d. Subtasks a, b, and c defines the function we need. Now choose a DAG and i, j, and C and apply your function to decide d-separation of i and j given C in the DAG. Repeat this five times.

$$\begin{tikzcd} 5 & 1 \arrow[l] \arrow[r] \arrow[d] & 4 \\ & 2 \arrow[r] & 3 \arrow[u] \end{tikzcd}$$

$\mathcal{D}$

$$\begin{tikzcd} 5 \arrow[r, dash] & 1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & 4 \arrow[d, dash] \\ & 2 \arrow[r, dash] & 3 \end{tikzcd}$$

$\mathcal{D}^m$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 4 \\ 2 \arrow[r] & 3 \arrow[u] \end{tikzcd}$$

$\mathcal{D}_{\mathrm{an}_{\mathcal{D}}(\{1,2,3,4\})}$

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & 4 \arrow[d, dash] \\ 2 \arrow[r, dash] & 3 \end{tikzcd}$$

$(\mathcal{D}_{\mathrm{an}_{\mathcal{D}}(\{1,2,3,4\})})^m$

Figure 2: Example DAG, $\mathcal{D}$. The graph $\mathcal{D}^m$ is its *moral graph*. We see that $1 \to 4 \leftarrow 3$ is an unshielded collider in $\mathcal{D}$ and therefore $1 - 3$ is in $\mathcal{D}^m$. Say we wish to decide if 1 and 3 are d-separated by $\{2,4\}$ in $\mathcal{D}$ using the theorem. That is, we wish to decide if i and j are d-separated given C in $\mathcal{D}$, for $i = 1, j = 3, C = \{2,4\}$. We first need to find $\mathcal{D}_{\mathrm{an}(\{i,j\}\cup C)}$. By convention, a node is an ancestor of itself, so $\mathrm{an}(\{i,j\} \cup C))$ must contain $1, 2, 3$, and 4. On the other hand, 5 is not an ancestor of any of the nodes in $\{i,j\} \cup C$, and therefore $\mathrm{an}_{\mathcal{D}}(\{i,j\} \cup C) = \{1,2,3,4\}$ and we obtain the graph $\mathcal{D}_{\mathrm{an}_{\mathcal{D}}(\{i,j\}\cup C)}$ by removing 5 and the edge $5 \leftarrow 1$. Note that $1 \to 4 \leftarrow 3$ is the only unshielded collider in this graph, and we add $1 - 3$ and make the other edges undirected to obtain its moral graph. Finally, we see that 1 and 3 are not separated by $\{2,4\}$ in $(\mathcal{D}_{\mathrm{an}_{\mathcal{D}}(\{1,2,3,4\})})^m$ and this means that 1 and 3 are not d-separated by $\{2,4\}$ in $\mathcal{D}$. On the other hand, we can go through the same procedure to see that 1 and 3 are d-separated given $\{2\}$ in $\mathcal{D}$.

Hints

Problem A: The theorem is in Lauritzen, S. L., Dawid, A. P., Larsen, B. N., and Leimer, H. G. (1990). Independence properties of directed Markov fields. Networks, 20(5), 491-505, see Proposition 3, link:
`https://www.stats.ox.ac.uk/~steffen/papers/ldll.pdf`
Look at its proof for inspiration. Note that we have stated the theorem with singleton A and B, i.e., $A = \{i\}$ and $B = \{j\}$ in our formulation.

Problem B: a: Let M be the adjacency matrix of $\mathcal{D} = (V, E_D)$ and let $\tilde{M}$ denote the matrix obtained from M by replacing every diagonal element with a 1. Consider the n'th power of $\tilde{M}$, $n = |V|$. It holds that $(\tilde{M}^n)_{kl} > 0$ if and only if k is an ancestor of l in $\mathcal{D}$. Find the set $\mathrm{an}(\{i,j\} \cup C)$ using this. b: Nodes k and l are colliders if and only if there exists an m such that $M_{mk} = 1$ and $M_{ml} = 1$. Make another adjacency matrix, N, such that $N_{kl} = N_{lk} = 1$ if and only if $M_{kl} = 1$, $M_{lk} = 1$, or k and l are colliders in $\mathcal{D}$. Note that N is symmetric in the sense that $N_{kl} = N_{lk}$ for all k, l. The matrix, N, is the adjacency matrix of the moral graph of $\mathcal{D}_{\mathrm{an}(A\cup B\cup C)}$. c: Construct a new matrix, O, from N, by letting $O_{kl} = N_{kl}$ if $k, l \notin C$ and $O_{kl} = 0$ if $k \in C$ or $l \in C$. Add 1's to the diagonal of O. Nodes i and j are separated by C in $(\mathcal{D}_{\mathrm{an}(A\cup B\cup C)})^m$ if and only if $(O^n)_{ij} = 0$.