```
from __future__ import division
import numpy as np
import numpy.linalg as LA
from time import time
from itertools import count
from numpy.linalg import norm as LA_norm
import matplotlib.pyplot as plt
import builtins
import seaborn as sns


# Define proximal operators
def prox_norm_1(x, eps, u=0):
    """
    Find proximal operator of function eps*||x-u||_1
    """
    x1 = x + np.clip(u - x, -eps, eps)
    return x1

def f_conj(y):
    return 0.5*(y+b).dot(y+b)

def g(x):
    return la*LA.norm(x,1)

def prox_g(x, rho):
    return prox_norm_1(x,la*rho)

def prox_f_conj(y, rho):
    return (y - rho*b)/(1+rho)

def prox_f(y, rho):
    return (y + rho*b)/(1+rho)

# define energy
def J(x,y, min_val):
    t = A.dot(x)-b
    return np.abs(0.5* t.dot(t) + la* LA.norm(x,1) - min_val)

def J1(x,y):
    return J(x,y,0)

##################################################################################################################################

# Define the algorithms
def VI_adaptive_Golden(J, prox_g, prox_f_conj, K, x0, y0, tau, theta, rho,min_val, tau_max, phi,numb_iter=100):
    begin = time()
    x, y = x0, y0
    x1, y1 = x0, y0
    z_x, z_y = x0, y0
    values = [J(x0, y0, min_val)]
    func_value = [J(x0,y0,0)]
    iters = []
    stop_iter = numb_iter
    L_n = None
    tau_prev = tau
    theta_prev = theta
    tt = [0]

    for i in range(numb_iter):
        z0 = np.concatenate((x, y))
        z1 = np.concatenate((x1, y1))
        Fz_0 = np.concatenate((K.T.dot(y), -K.dot(x)))
        Fz_1 = np.concatenate((K.T.dot(y1), -K.dot(x1)))
        z_norm = LA.norm(z0 - z1)
        Fz_norm = LA.norm(Fz_0 - Fz_1)
        if Fz_norm != 0:
            L_n = (phi * theta_prev / (4 * tau_prev)) * ((z_norm / Fz_norm) ** 2)
            tau_n = builtins.min(rho * tau_prev, L_n, tau_max)
        else:
            tau_n = builtins.min(rho * tau_prev, tau_max)
        z_x = x1 - (1 / phi) * (x1 - z_x)
        z_y = y1 - (1 / phi) * (y1 - z_y)

        x2 = prox_g(z_x - tau_n * K.T.dot(y1), tau_n)
        y2 = prox_f_conj(z_y + tau_n * K.dot(x1), tau_n)
```

```
        theta_n = phi * (tau_n / tau_prev)
        tau_prev = tau_n
        theta_prev = theta_n
        values.append(J(x2, y2, min_val))
        func_value.append(J(x2,y2,0))
        print(f"Iteration {i}: J_gap = {J(x2, y2, min_val)}, tau_n = {tau_n}, z_norm = {z_norm}, F_z_norm = {Fz_norm}, L_n = {L_n
        x, y = x1, y1
        x1, y1 = x2, y2

        tt.append(time() - begin)
        err = values[-1]
        end = time()
        if err <= 1e-9:
            stop_iter = i+1
            print ("Iter:", i+1)
            print(f"Converged in {stop_iter} iters, {tt[-1]:.3f}s")
            # print ("----- Adaptive Golden-Ratio Algorithm for VI -----")
            print ("Time execution:", round(end - begin,2))
            break

    if err > 1e-9:
     print ("Adaptive Golden-Ratio Algorithm for VI does not terminate after", round(end - begin,2), "seconds")
    return [values,func_value, x, y,iters, stop_iter, tt]



def pd_adaptive_Golden(J, prox_g, prox_f_conj, K, x0, y0, tau, theta, beta, tau_max, min_val, rho_val, phi, numb_iter=100):
    begin = time()

    x, y, z = x0, y0, x0
    values = [J(x0, y0, min_val)]
    func_value = [J(x0,y0,0)]
    tau_prev = tau
    theta_prev = theta
    tt = [0]
    tau_values = []
    sigma_values = []

    for i in range(numb_iter):
        z = x - (1 / phi) * (x - z)
        x1 = prox_g(z - tau_prev * K.T.dot(y), tau_prev)
        Kx1 = K.dot(x1)
        Kx = K.dot(x)

        Kx_norm = LA.norm(Kx - Kx1)
        x_norm = LA.norm(x - x1)
        if Kx_norm != 0:
            L_n = (theta_prev / (4 * beta * tau_prev)) * (x_norm / Kx_norm) ** 2
            tau_n = builtins.min(rho_val * tau_prev, L_n, tau_max)
        else:
            tau_n = builtins.min(rho_val * tau_prev, tau_max)

        sigma_n = beta * tau_n
        y1 = prox_f_conj(y + sigma_n * Kx1, sigma_n)
        print(f"Iteration {i}: J_gap = {J(x1, y1, min_val)}, tau_n = {tau_n}, x_norm = {x_norm}, Kx_norm = {Kx_norm}, L_n = {L_n}
        theta_n = phi * (tau_n / tau_prev)
        tau_prev = tau_n
        theta_prev = theta_n
        values.append(J(x1, y1, min_val))
        func_value.append(J(x1,y1,0))
        x, y = x1, y1
        tau_values.append(tau_n)  # Store tau_n value
        sigma_values.append(sigma_n)  # Store sigma_n value
        tt.append(time() - begin)
        err = values[-1]
        end = time()
        if err <= 1e-9:
            print ("Iter:", i+1)
            print ("----- Adaptive Golden-Ratio PDA -----")
            print ("Time execution:", round(end - begin,2))
            break
    if err > 1e-9:
      print ("Adaptive Golden-Ratio PDA does not terminate after", round(end - begin,2), "seconds")
    return [values, func_value, x, y, tt, sigma_values, tau_values]



def pd_Golden_Partial_adaptive(J, prox_g, prox_f_conj, K, x0, y0, mu, beta, phi, min_val,tau, numb_iter=100):
```

```
def pd_Golden_Partial_adaptive(J, prox_g, prox_f_conj, K, x0, y0, mu, beta, phi, min_val,tau, numb_iter=100):
    begin = time()
    x, y, z = x0, y0, x0
    values = [J(x0, y0, min_val)]
    func_value = [J(x0,y0,0)]
    tt = [0]
    tau_prev = tau

    for i in range(numb_iter):

        z = x - (1 / phi) * (x - z)
        x1 = prox_g(z - tau_prev * K.T.dot(y), tau_prev)
        x_norm = LA.norm(x - x1)
        Kx_norm = LA.norm(K.dot(x) - K.dot(x1))

        if Kx_norm != 0:
            L_n =  (mu * x_norm )/(np.sqrt(beta) * Kx_norm)
            tau_n = builtins.min(tau_prev, L_n)
        else:
            tau_n = tau_prev

        sigma_n = beta * tau_n
        y1 = prox_f_conj(y + sigma_n * K.dot(x1), sigma_n)
        print(f"Iteration {i}:J_gap = {J(x1, y1, min_val)}, tau_n = {tau_n}, x_norm = {x_norm}, Kx_norm = {Kx_norm}, L_n = {L_n}"
        tau_prev = tau_n
        values.append(J(x1, y1, min_val))
        func_value.append(J(x1,y1,0))
        x, y = x1, y1

        tt.append(time() - begin)
        err = values[-1]
        end = time()
        if err <= 1e-9:
            print ("Iter:", i+1)
            print ("----- Golden-Ratio PDA with partial adaptive step size-----")
            print ("Time execution:", round(end - begin,2))
            break
    if err > 1e-9:
      print ("Golden-Ratio PDA with partial adaptive step size does not terminate after", round(end - begin,2), "seconds")

    return [values,func_value, x, y,tt]



def pd(J, prox_g, prox_f_conj, K,  x0, y0, sigma, tau, min_val, numb_iter=100):
    """
    Primal-dual algorithm of Pock and Chambolle for the problem min_x
    max_y [<Kx,y> + g(x) - f*(y)]
    J denotes some function which we compute in every iteration to
    study perfomance. It may be energy, primal-dual gap, etc.
    """
    begin = time()
    theta = 1.0
    x, y, z = x0, y0, x0
    values = [J(x0, y0, min_val)]
    func_value = [J(x0,y0,0)]
    tt = [0]

    for i in range(numb_iter):
        x1 = prox_g(x - tau * K.T.dot(y), tau)
        z = x1 + theta * (x1 - x)
        y = prox_f_conj(y + sigma * K.dot(z), sigma)
        values.append(J(x, y, min_val))
        func_value.append(J(x,y,0))
        print(f"Iteration {i}: J_gap = {J(x, y, min_val)}, tau = {tau}, theta = {theta}")
        x = x1

        tt.append(time() - begin)
        err = values[-1]
        end = time()
        if err <= 1e-9:
            print ("Iter:", i+1)
            print ("----- Primal-dual method -----")
            print ("Time execution:", round(end - begin,2))
            break
    if err > 1e-9:
      print ("Primal-dual method does not terminate after", round(end - begin,2), "seconds")
```

```
    return [values, func_value, x, y,tt]



def pd_Golden(J, prox_g, prox_f_conj, K,  x0, y0, sigma, tau, phi, min_val, numb_iter=100):
    """
    Golden-Ratio Primal-dual algorithm for the problem min_x
    max_y [<Kx,y> + g(x) - f*(y)]
    J denotes some function which we compute in every iteration to
    study perfomance. It may be energy, primal-dual gap, etc.
    """
    begin = time()

    x, y, z = x0, y0, x0
    values = [J(x0, y0, min_val)]
    func_value = [J(x0,y0,0)]
    tt = [0]

    for i in range(numb_iter):
        z = x-(1/phi)*(x - z)
        x = prox_g(z - tau * K.T.dot(y), tau)

        y = prox_f_conj(y + sigma * K.dot(x), sigma)

        values.append(J(x, y, min_val))
        func_value.append(J(x,y,0))
        print(f"Iteration {i}: J_gap = {J(x, y, min_val)}, tau = {tau}, phi = {phi}")
        tt.append(time() - begin)
        err = values[-1]
        end = time()
        if err <= 1e-9:
            print ("Iter:", i+1)
            print ("----- Golden-Ratio Primal-dual method -----")
            print ("Time execution:", round(end - begin,2))
            break
    if err > 1e-9:
      print ("Golden-Ratio Primal-dual method does not terminate after", round(end - begin,2), "seconds")
    return [values, func_value, x, y,tt]


def pd_Golden_linesearch(J, prox_g, prox_f_conj, K,  x0, y1, tau, beta, min_val, numb_iter=100, tol = 1e-8):
    """
    Golden-Ratio PDA with linesearch for problem min_x max_y [
    <Kx,y> + g(x) - f*(y)].
    beta denotes sigma/tau from a classical primal-dual algorithm.
    """
    begin = time()
    values = [J(x0, y1, min_val)]
    func_value = [J(x0,y1,0)]
    tt = [0]

    phi = 1.5
    rho = 1/phi+1/phi**2
    mu = 0.7
    delta = 0.99
    iterates = [values, func_value, x0, x0, y1, tau] + [tt, 0]
    sqrt_b = np.sqrt(beta)

    # function T is an operator that makes one iteration of the algorithm:
    # (x1, y1) = T(x,y, history)

    def T(values, func_value, z, x_old, y, tau_old, tt,l):
        z = ((phi-1)*x_old + z)/phi
        x = prox_g(z - tau_old * K.T.dot(y), tau_old)
        Kx = K.dot(x)
        tau = tau_old * rho
        for j in count(0):
            th = tau/tau_old
            y1 = prox_f_conj(y + tau * beta * Kx, tau * beta)
            if sqrt_b * tau * LA.norm(K.T.dot(y1 - y)) <= delta*np.sqrt(phi*th) * LA.norm(y1 - y):
                break
            else:
                tau *= mu
                l +=1

        #print(phi)
```

```
        values.append(J(x, y1, min_val))
        func_value.append(J(x,y1,0))
        #steps.append(l)
        tt.append(time() - begin)
        res = [values, func_value, z, x, y1, tau, tt,l]
        return res

    for i in range(numb_iter):
        iterates = T(*iterates)
        err = iterates[0][-1]
        x = iterates[3]
        y1 = iterates[4]
        tau = iterates[5]
        print(f"Iteration {i}:J_gap = {J(x, y1, min_val)}, tau = {tau}, phi = {phi}")
        end = time()
        if err <= tol:
            print ("Iter:", i+1)

            print ("----- Golden-Ratio Primal-dual method with linesearch-----")
            print ("Time execution:", round(end - begin,2))
            break

    if err > tol:
        print ("Golden-Ratio Primal-dual method with linesearch does not terminate after", round(end - begin,2), "seconds")
    print ("Number of linesearch:", iterates[-1])
    return [iterates[i] for i in [0, 1, -2,-1]]



##########################################################################################
n = 1000
m = 300
s = 10
la = 0.1
gen = 100
np.random.seed(gen)
A = np.random.normal(0,1,(m,n))

np.random.seed(gen)
w = np.random.uniform(-10,10, n)
w[s:] = 0
np.random.seed(gen)
w = np.random.permutation(w)

np.random.seed(gen)
nu = np.random.normal(0,0.1, m)
b = A.dot(w) + nu
###############################################################################################
# s = 10
# la = 0.1
# gen = 100

# np.random.seed(gen)
# B = np.random.normal(0,1, (m,n))
# p = 0.5
# A = np.zeros((m,n))
# A[:,0] = B[:,0]/np.sqrt(1-p**2)
# for j in np.arange(1,n):
#     A[:,j] = p*A[:,j-1] + B[:,j]

# np.random.seed(gen)
# w = np.random.uniform(-10,10, n)
# w[s:] = 0
# np.random.seed(gen)
# w = np.random.permutation(w)
# np.random.seed(gen)
# nu = np.random.normal(0, 0.1, m)
# b = A.dot(w) + nu

# Starting points
x0 = np.zeros(n)
y0 = - b

L = np.sqrt(np.max(LA.eigh(A.dot(A.T))[0]))
beta = .05
beta1 = 1 #.5
phi_1 = 1.618
phi2 = 1.76
```

```
beta2 = .01
phi = 1.6
s_p = np.sqrt(phi)
rho_val = 1./ phi + 1./ phi **2
theta = 2
tau_2 = 1
mu = .81
tau_max = 100000000
tau = 1.0 /  L
sigma = 1.0 / L




N = 20000
c_= 0

ans2 = pd_adaptive_Golden(J, prox_g, prox_f_conj, A, x0, y0, tau_2, theta, beta, tau_max, c_ , rho_val, phi, numb_iter=N)
ans29 = pd_Golden_linesearch(J, prox_g, prox_f_conj, A,  x0, y0, tau_2, beta1,c_, numb_iter=N, tol = 1e-9)
ans1 = VI_adaptive_Golden(J, prox_g, prox_f_conj,  A, x0, y0, tau_2, theta, rho_val,c_,tau_max, phi, numb_iter=N)
ans17 = pd(J, prox_g, prox_f_conj, A,  x0, y0, .04*sigma, 25*tau,c_, numb_iter=N)

ans3 =  pd_Golden(J, prox_g, prox_f_conj, A,  x0, y0,  s_p*sigma , s_p*tau , phi_1,c_, numb_iter=N)
ans4 = pd_Golden_Partial_adaptive(J, prox_g, prox_f_conj, A, x0, y0, mu, beta, phi2,c_, tau_2, numb_iter=N)
# mu = .30
# ans23 = pd_Golden_Partial_adaptive(J, prox_g, prox_f, dist_sub, A, x0, y0, mu, beta, phi, la, b, tau_2, numb_iter=N)
# mu = .15
# ans24 = pd_Golden_Partial_adaptive(J, prox_g, prox_f, dist_sub, A, x0, y0, mu, beta, phi, la, b, tau_2, numb_iter=N)
# phi = 2
# rho1 = 1.49
# ans19 = pd_Golden_relaxed(J, prox_g, prox_f_conj, A,  x0, y0, .05*sigma, 20*tau, phi, rho1, numb_iter=N)

# min_val = builtins.min(ans29[0]+ans2[0]+ans1[0])
# print(min_val)
```

Streaming output truncated to the last 5000 lines.

```
Iteration 15001:J_gap = 5.109370164468624, tau_n = 0.04233466887984343, x_norm = 1.8921464009288593e-15, Kx_norm = 5.92843
Iteration 15002:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 1.901880045962852e-15, Kx_norm = 6.326712
Iteration 15003:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 1.889596467802366e-15, Kx_norm = 6.178827
Iteration 15004:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 1.8867629992116287e-15, Kx_norm = 5.98336
Iteration 15005:J_gap = 5.109370164468624, tau_n = 0.04233466887984343, x_norm = 6.340705000665784e-16, Kx_norm = 5.119705
Iteration 15006:J_gap = 5.109370164468624, tau_n = 0.04233466887984343, x_norm = 1.8876378179634318e-15, Kx_norm = 6.22787
Iteration 15007:J_gap = 5.109370164468624, tau_n = 0.04233466887984343, x_norm = 1.888856187964957e-15, Kx_norm = 5.930503
Iteration 15008:J_gap = 5.109370164468624, tau_n = 0.04233466887984343, x_norm = 2.0881771220554627e-15, Kx_norm = 5.98554
Iteration 15009:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 6.437179936365987e-16, Kx_norm = 5.012457
Iteration 15010:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 2.0889168141568215e-15, Kx_norm = 6.56696
Iteration 15011:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.8911583070788454e-15, Kx_norm = 5.50679
Iteration 15012:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.8885578436652368e-15, Kx_norm = 5.67232
Iteration 15013:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 1.8891604585765246e-15, Kx_norm = 6.18007
Iteration 15014:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 6.392367742245266e-16, Kx_norm = 5.015013
Iteration 15015:J_gap = 5.109370164468621, tau_n = 0.04233466887984343, x_norm = 1.8896639506498326e-15, Kx_norm = 5.57111
Iteration 15016:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 2.0882565610100235e-15, Kx_norm = 6.27993
Iteration 15017:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 6.429789478835889e-16, Kx_norm = 4.346689
Iteration 15018:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 6.430146333645472e-16, Kx_norm = 4.630894
Iteration 15019:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 2.0857153583113816e-15, Kx_norm = 6.42090
Iteration 15020:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 2.0862960036040305e-15, Kx_norm = 6.55005
Iteration 15021:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 1.0947129149690507e-15, Kx_norm = 5.74404
Iteration 15022:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 1.0895699968089658e-15, Kx_norm = 5.02836
Iteration 15023:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 2.084259969298936e-15, Kx_norm = 5.983319
Iteration 15024:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.883619953043873e-15, Kx_norm = 5.721172
Iteration 15025:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 6.299811371593603e-16, Kx_norm = 5.261812
Iteration 15026:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 6.243534644258738e-16, Kx_norm = 4.863739
Iteration 15027:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 1.881885562134672e-15, Kx_norm = 6.018566
Iteration 15028:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 2.082087353807083e-15, Kx_norm = 6.554050
Iteration 15029:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.0848195582615314e-15, Kx_norm = 4.67771
Iteration 15030:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 2.080521130653346e-15, Kx_norm = 5.728664
Iteration 15031:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.8868598890635575e-15, Kx_norm = 6.14495
Iteration 15032:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.884744924474346e-15, Kx_norm = 5.403091
Iteration 15033:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.888938632617985e-15, Kx_norm = 6.207814
Iteration 15034:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 2.085078987398647e-15, Kx_norm = 6.043408
Iteration 15035:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.090443437650723e-15, Kx_norm = 5.092100
Iteration 15036:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.888116813227971e-15, Kx_norm = 6.022977
Iteration 15037:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 1.8985817374115563e-15, Kx_norm = 6.08087
Iteration 15038:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 6.362611235656658e-16, Kx_norm = 4.977465
Iteration 15039:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 6.393556298576347e-16, Kx_norm = 4.446328
Iteration 15040:J_gap = 5.109370164468621, tau_n = 0.04233466887984343, x_norm = 6.407332136201325e-16, Kx_norm = 4.482772
Iteration 15041:J_gap = 5.109370164468621, tau_n = 0.04233466887984343, x_norm = 1.0938466660876331e-15, Kx_norm = 4.95990
Iteration 15042:J_gap = 5.109370164468621, tau_n = 0.04233466887984343, x_norm = 1.094745557769734e-15, Kx_norm = 4.931890
Iteration 15043:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 6.420656616434381e-16, Kx_norm = 4.691674
Iteration 15044:J_gap = 5.109370164468624, tau_n = 0.04233466887984343, x_norm = 6.364443694779981e-16, Kx_norm = 4.293277
Iteration 15045:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 6.350628131786475e-16, Kx_norm = 4.615217
```

```
Iteration 15046:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 6.370864959722711e-16, Kx_norm = 5.012358
Iteration 15047:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 6.372098849473324e-16, Kx_norm = 4.894975
Iteration 15048:J_gap = 5.109370164468623, tau_n = 0.04233466887984343, x_norm = 6.244727433481201e-16, Kx_norm = 4.489860
Iteration 15049:J_gap = 5.109370164468621, tau_n = 0.04233466887984343, x_norm = 6.261192355527623e-16, Kx_norm = 4.658824
Iteration 15050:J_gap = 5.109370164468621, tau_n = 0.04233466887984343, x_norm = 6.217562090195472e-16, Kx_norm = 4.830286
Iteration 15051:J_gap = 5.109370164468621, tau_n = 0.04233466887984343, x_norm = 6.137122159569311e-16, Kx_norm = 4.526402
Iteration 15052:J_gap = 5.109370164468621, tau_n = 0.04233466887984343, x_norm = 6.094356637030326e-16, Kx_norm = 4.763206
Iteration 15053:J_gap = 5.109370164468621, tau_n = 0.04233466887984343, x_norm = 6.045063305463673e-16, Kx_norm = 4.284080
Iteration 15054:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 6.066679196140054e-16, Kx_norm = 4.812122
Iteration 15055:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 5.919519868512838e-16, Kx_norm = 4.826598
Iteration 15056:J_gap = 5.109370164468622, tau_n = 0.04233466887984343, x_norm = 5.894876507010693e-16, Kx_norm = 4.935699
```

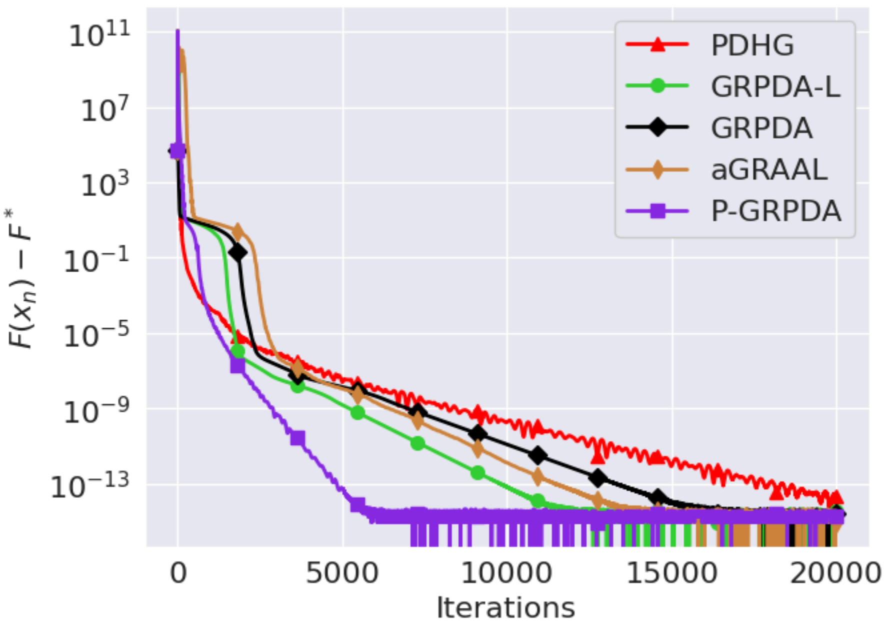
```python
min_values = []  # Create a list to store minimum values from each list
for v in [ans29[0], ans3[0], ans4[0], ans17[0]]:
    # Get the minimum value of the list 'v' and append it to min_values
    min_values.append(builtins.min(v))

# Now, find the overall minimum from the collected minimum values
min_val = min(min_values)

print(f'{min_val:.30f}')  # Print with 30 decimal places
```

```
5.109370164468620245656893530395
```

```python
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
from google.colab import files

# ——— 1) Set seaborn style ———
sns.set_style('darkgrid')
plt.rcParams['legend.facecolor'] = plt.rcParams['axes.facecolor']

# ——— 2) Update Matplotlib parameters ———
plt.rcParams.update({
    'font.size':        13,
    'figure.figsize':   (7, 5),
    'legend.fontsize':  16,
    'lines.linewidth':  2,
    'axes.labelsize':   16,
    'xtick.labelsize':  16,
    'ytick.labelsize':  16,
    'xtick.major.pad':  4,
    'ytick.major.pad':  4,
})

# ——— 3) Compute the common minimum for vertical shift ———
min_val = min(np.min(ans[0]) for ans in [ans1, ans3, ans4, ans17, ans29])

# ——— 4) Define your plots: (iters, error, label, marker, color) ———
plots = [
    (np.arange(len(ans17[0])), np.array(ans17[0]) - min_val, 'PDHG',    '^', 'red'),
    (np.arange(len(ans29[0])), np.array(ans29[0]) - min_val, 'GRPDA-L', 'o', 'limegreen'),
    (np.arange(len(ans3[0])),  np.array(ans3[0])  - min_val, 'GRPDA',   'D', 'black'),
    (np.arange(len(ans1[0])),  np.array(ans1[0])  - min_val, 'aGRAAL',  'd', '#CD853F'),
    (np.arange(len(ans4[0])),  np.array(ans4[0])  - min_val, 'P-GRPDA', 's', '#8A2BE2'),
]

# ——— 5) Plot each with markers every ~10% of iterations ———
for x, y, lbl, mk, col in plots:
    me = max(1, len(x)//11)
    plt.plot(x, y,
             color=col,
             marker=mk,
             markersize=7,
             markevery=me,
             label=lbl)

plt.yscale('log')
plt.xlabel('Iterations')
plt.ylabel('$F(x_n)-F^*$')

# ——— 6) Semi-transparent legend ———
plt.legend(framealpha=1.0, loc='upper right')
plt.tight_layout()
plt.savefig('log_scale_plot_comparison1.eps', dpi=600)
files.download('log_scale_plot_comparison1.eps')
```

```
plt.show()
```

```
################################################################################################################################
```


PDHG
GRPDA-L
GRPDA
aGRAAL
P-GRPDA
10^{11}
10^{7}
10^{3}
10^{-1}
10^{-5}
10^{-9}
10^{-13}
$F(x_n) - F^*$
0
5000
10000
15000
20000
Iterations