PIECEWISE TESTABLE EVENTS*

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1. Introduction and definitions

The free monoid generated by Σ is denoted by Σ^* and has identity λ. $\Sigma^+ = \Sigma^* - \lambda$. For a word x in Σ^*, $|x|$ denotes its length. An event is a subset of Σ^*.

A word x is a *piecewise subword* of y, denoted by $x \leq y$, iff there exist $x_1, \ldots x_n, z_0, z_1, \ldots, z_n$ in Σ^* such that $x = x_1 \ldots x_n$ and $y = z_o x_1 z_1 \ldots x_n z_n$. For x and y in Σ^*, and a natural m, define $x \underset{m}{\sim} y$ iff for every s in Σ^*, $|s| \leq m$ implies that $s \leq x$ iff $s \leq y$. An event is *piecewise testable* iff there exists a natural m, such that for every x and y in Σ^*, $x \underset{m}{\sim} y$ implies that $x \in E$ iff $y \in E$.

Thus, an event E is piecewise testable iff there exists an m such that membership of x in E is determined by the set of piecewise subwords of length at most m, which occur in x. In its form, this definition is similar to that of locally testable events [1, 6, 7 and 11], the main difference being the substitution of length m subwords by piecewise subwords of length m. Piecewise testable events were introduced in the author's doctoral dissertation [9], where γ_1 denotes the family of piecewise testable events. It has been shown [1,9]

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* Part of this work has been done at the Department of Computer Science, University of Waterloo, Canada. It was supported by FAPESP (Brasil) under grants 70/400 and 73/1213 and by the NRC (Canada) under grant A-1617.

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that both locally and piecewise testable events constitute subfamilies of regular star-free events with dot-depth one. The dot-depth of a regular star-free event has been introduced in [3]. Indeed, combining properly these two testing concepts, one gets precisely the family of dot-depth one events [9]. Another related result is that an event whose syntactic semigroup is a monoid has dot-depth one iff it is piecewise testable [9].

As far as we know, piecewise subwords were introduced by Haines in [5] and he obtains a truly remarkable result, namely that every set of pairwise noncomparable elements (with respect to the partial order $\leq$ over Σ^*) is finite. Certain subclasses of piecewise testable events were also studied in [5] and [10].

Let a and b be elements of a monoid M. We say that $a \mathcal{J} b$ iff $MaM = MbM$. This is one of the well-known Green equivalence relations [2]. We say that M is *$\mathcal{J}$-trivial* iff for every a and b in M, $a \mathcal{J} b$ implies $a = b$.

Given an event $E \subseteq \Sigma^*$, we define $x \equiv y \pmod E$, for x and y in Σ^*, iff for every u and v in Σ^*, $uxv \in E$ iff $uyv \in E$. It is easy to see that $\equiv \pmod E$ is a congruence relation over Σ^*. The quotient monoid $\Sigma^*/\equiv \pmod E$ is called the *syntactic monoid* of E. It is well-known that E is regular iff its syntatic monoid is finite; see for instance [7].

The main result of this paper is that an event E is piecewise testable iff its syntactic monoid is finite and $\mathcal{J}$-trivial. This was first stated and proved in [9]; here we give a much improved version of that proof. A corollary to the main result is that it is decidable whether a given regular event is piecewise testable. Indeed, it is sufficient to verify, whether its syntactic monoid is $\mathcal{J}$-trivial.

We will use the well-known left-right duality for semigroups; see for instance [2].

2. Characterization of equivalent words

In this section we study the properties of $_m\!\sim$ and show that $x \,_m\!\sim y$ iff y can be obtained from x by a finite number of steps of a simple transformation (R_m or R_m^{-1}). Each step of this transformation consists of adding or deleting a single letter, whenever this preserves equivalence. A byproduct of the results in this section is that one can efficiently verify (in about $O((|x| + |y|)^3)$ steps), whether two given words are m-equivalent.

Lemma 1. Let x and y be in Σ^* and let m be a natural.

(a) ${}_m\sim$ is a congruence relation of finite index over Σ^*.

(b) $x \ {}_{m+1}\sim \ y$ implies $x \ {}_m\sim \ y$.

(c) If $x \le y$ then $x \ {}_m\sim \ y$ iff for every $s \le y$, $|s| \le m$ implies $s \le x$.

Proof. The proofs are left to the reader. □

Lemma 2. For every u in Σ^* and σ in Σ, there exists a natural p and a word s, such that $u \ {}_p\sim \ u\sigma$, $|s| = p$, $s \le u$ and $s\sigma \not\le u$.

Proof. Let p be the greatest natural, such that $u \ {}_p\sim \ u\sigma$. The existence of p follows from Lemma 1(b) and the facts that $u \ {}_0\sim \ u\sigma$ and $u \ {}_{|u|+1}\not\sim \ u\sigma$. Thus, $u \ {}_{p+1}\not\sim \ u\sigma$, hence there exists a word s, such that $|s| = p$, $s \le u$ and $s\sigma \not\le u$. □

The p and s refered to in Lemma 2 can be efficiently found by the next lemma, which will also be used in section 3. First, we have the notation: for u in Σ^*, $u\Sigma = \{\sigma \in \Sigma \mid \sigma \le u\}$.

Lemma 3. Let u and v be in Σ^+, and let $m > 0$. Then $u \ {}_m\sim \ uv$ iff there exist $u_1, u_2, \ldots, u_m$ in Σ^+, such that $u = u_1u_2\ldots u_m$ and $u_1\Sigma \supseteq u_2\Sigma \supseteq \ldots \supseteq u_m\Sigma \supseteq v\Sigma$.

Proof. Let us prove the only if part by induction on m. For $m = 1$, $u \ {}_1\sim \ uv$ implies that $u\Sigma = (uv)\Sigma$, hence $u\Sigma \supseteq v\Sigma$. Suppose the assertion holds for $m \ge 1$, and let u and v in Σ^+ be such that $u \ {}_{m+1}\sim \ uv$. Let u_0 be the shortest prefix of u, such that $u_0\Sigma = (uv)\Sigma$. Such a prefix exists, since $u \ {}_{m+1}\sim \ uv$ implies $u \ {}_1\sim \ uv$, hence $u\Sigma = (uv)\Sigma$. Since u is not empty, so is u_0; and being $u_0 = u_0'\sigma$, with σ in Σ, the choice of u_0 implies that $\sigma \not\le u_0'$. Let w be such that $u = u_0w$; we claim that $w \ {}_m\sim \ wv$. Indeed, let s in Σ^* be such that $|s| \le m$ and $s \le wv$, then $|\sigma s| \le m+1$ and $\sigma s \le u_0wv = uv$. Since $u \ {}_{m+1}\sim \ uv$, it follows that $\sigma s \le u = u_0'\sigma w$, and since $\sigma \not\le u_0'$, $s \le w$. Hence, in view of Lemma 1(c), $w \ {}_m\sim \ wv$. By the induction hypothesis, there exist $u_1, \ldots, u_m$ in Σ^+, such that $u_1 \ldots u_m = u$

and $u_1\Sigma \supseteq \ldots \supseteq u_m\Sigma \supseteq v\Sigma$. Since $u_0\Sigma = (uv)\Sigma \supseteq u_1\Sigma$, the assertion follows.

The if part is also proved by induction on m. For $m = 1$, $u_1 = u$, and $u\Sigma \supseteq v\Sigma$ implies $u\Sigma = (uv)\Sigma$; hence $u \ {}_1\!\sim uv$. Let $u_0, u_1, \ldots, u_m, v$ in Σ^+ be such that $u_0\Sigma \supseteq u_1\Sigma \supseteq \ldots \supseteq u_m\Sigma \supseteq v\Sigma$, and let $w = u_1 \ldots u_m$. Then $u_0\Sigma = (u_0wv)\Sigma$, and by the induction hypothesis, $w \ {}_m\!\sim wv$. We claim that $u_0w \ {}_{m+1}\!\sim u_0wv$. Let s in Σ^* be such that $0 < |s| \le m+1$ and $s \le u_0wv$. Let s' be the longest prefix of s, such that $s' \le u_0$, and let $s = s's''$. Since $u_0\Sigma = (u_0wv)\Sigma$, it follows that s' is not empty, hence $|s''| \le m$. On the other hand, the choice of s' and the fact that $s's'' \le u_0wv$, imply that $s'' \le wv$; hence $s'' \le w$, since $w \ {}_m\!\sim wv$. Thus, $s \le u_0w$, which in view of Lemma 1(c) proves the claim. □

<u>Corollary 3a</u>. For every x and y in Σ^* and $m \ge 0$, $(xy)^m \ {}_m\!\sim (xy)^m x$.

<u>Proof</u>. It is sufficient to take $u_1 = \ldots = u_m = xy$. □

<u>Lemma 4</u>. For u and v in Σ^* and σ in Σ, $u\sigma v \ {}_m\!\sim uv$ iff there exist p and p', such that $p+p' \ge m$, $u \ {}_p\!\sim u\sigma$ and $v \ {}_{p'}\!\sim \sigma v$.

<u>Proof</u>. To prove the if part, let p and p' be as in the statement of the lemma. In view of Lemma 1(c), it is sufficient to show that if $s\sigma s' \le u\sigma v$, with $s \le u$, $s' \le v$ and $|s\sigma s'| \le m$, then $s\sigma s' \le uv$. Indeed, since $p+p' \ge m$, and $|s\sigma s'| \le m$, it follows that either $|s| < p$ or $|s'| < p'$, hence either $s\sigma \le u$ or $\sigma s' \le v$. In any case, $s\sigma s' \le uv$.

Conversely, assume that $u\sigma v \ {}_m\!\sim uv$. By Lemma 2, there exist p and s, such that $u \ {}_p\!\sim u\sigma$, $|s| = p$, $s \le u$ and $s\sigma \not\le u$. By duality, there exist p' and s', such that $v \ {}_{p'}\!\sim \sigma v$, $|s'| = p'$, $s' \le v$ and $\sigma s' \not\le v$. It follows that $|s\sigma s'| = p+p'+1$, $s\sigma s' \le u\sigma v$ and $s\sigma s' \not\le uv$. Thus, if $p+p' < m$, then $u\sigma v \ {}_m\!\not\sim uv$, a contradiction, hence $p+p' \ge m$. □

<u>Lemma 5</u>. Let u, v and w in Σ^*, and σ and ξ in Σ, be such that $u\sigma v \ {}_m\!\sim u\xi w$, and $\sigma \ne \xi$. Then, either $u\sigma\xi w \ {}_m\!\sim u\sigma v$ or $u\xi\sigma v \ {}_m\!\sim u\xi w$.

<u>Proof</u>. By Lemma 2 there exist p, q, s and t, such that

$$u \;{}_p\!\sim u\xi, \quad |s| = p, \quad s \le u \quad \text{and} \quad s\xi \not\le u, \tag{1}$$
$$\text{and} \quad u \;{}_q\!\sim u\sigma, \quad |t| = q, \quad t \le u \quad \text{and} \quad t\sigma \not\le u. \tag{2}$$

By duality, there exist p', q' and t', such that

$$\sigma v \;{}_{p'}\!\sim \xi\sigma v, \quad |s'| = p', \quad s' \le \sigma v \quad \text{and} \quad \xi s' \not\le \sigma v, \tag{3}$$
$$\text{and} \quad \xi w \;{}_{q'}\!\sim \sigma\xi w, \quad |t'| = q', \quad t' \le \xi w \quad \text{and} \quad \sigma t' \not\le \xi w. \tag{4}$$

If $p+p' \ge m$, then by Lemma 4, (1) and (3), $u\xi\sigma v \;{}_m\!\sim u\sigma v$, and since $u\sigma v \;{}_m\!\sim u\xi w$ by hypothesis, we have $u\xi\sigma v \;{}_m\!\sim u\xi w$. Similarly, if $q+q' \ge m$, then $u\sigma\xi w \;{}_m\!\sim u\sigma v$. In either case the lemma holds.

Assume therefore that

$$p+p' < m \quad \text{and} \quad q+q' < m. \tag{5}$$

Assume further that $q' \le p'$. Now, we claim that $t' \le v$. Indeed, from (4), $t' \le \xi w$. Let t_2' be the longest suffix of t' such that $t_2' \le w$. Then $t' = t_1't_2'$ with $t_1' = \lambda$ or $t_1' = \xi$, and $|t_2'| \le q'$. Then, from (1), $s \le u$, hence $s\xi t_2' \le u\xi w$. On the other hand, since $|t_2'| \le q'$, $q' \le p'$ by assumption, $|s| = p$ from (1), and $p+p' < m$ from (5), it follows that $|s\xi t_2'| \le m$. This implies that $s\xi t_2' \le u\sigma v$, since $u\sigma v \;{}_m\!\sim u\xi w$ by hipothesis. Now, from (1), $s\xi \not\le u$, hence $\sigma \ne \xi$ implies that $\xi t_2' \le v$. Since either $t' = t_2'$ or $t' = \xi t_2'$, it follows now that $t' \le v$. But then, $t\sigma t' \le u\sigma v$, since $t \le u$ by (2). On the other hand, from (2), (4) and (5), $|t\sigma t'| = q+1+q' \le m$; since $u\sigma v \;{}_m\!\sim u\xi w$, it follows that $t\sigma t' \le u\xi w$. This is impossible, since $t\sigma \not\le u$ by (2), $\sigma t' \not\le \xi w$ by (4), and $\sigma \ne \xi$ by hipothesis. Thus $q' > p'$. By a similar argument, one proves that $p' > q'$, a contradiction which shows that (5) is untenable, which in turn establishes the lemma. □

Before proceeding, we need a definition. For x and y in Σ^*, define $x\, R_m\, y$ (x m-reduces to y) iff $x \;{}_m\!\sim y$, and there exist u and v in Σ^*, and σ in Σ, such that $x = u\sigma v$ and $y = uv$. Let R_m^* denote the reflexive and transitive closure of R_m, and let R_m^{-1} and R_m^{*-1} denote the inverse of R_m and R_m^*, respectively. In view of Lemma 1(c), it is easy to see, that $z\, R_m^*\, x$ iff $x \le z$ and $x \;{}_m\!\sim z$.

<u>Lemma 6</u>. For every x and y in Σ^*, $x \;{}_m\!\sim y$ iff there exists a z in Σ^*, such that $z\, R_m^*\, x$ and $z\, R_m^*\, y$.

<u>Proof</u>. We proceed by induction on $|x| + |y| - 2|u|$, where u is the longest common prefix of x and y. If $|x| + |y| - 2|u| = 0$, then $x = y = u$, and $z = u$ satisfies the proposition. Let then v' and w' be such that $x = uv'$ and $y = uw'$, with $v'w' \ne \lambda$.

If $v' = \lambda$, then $x \le y$, and since $x \;{}_m\!\!\sim y$, it follows that $y \; R_m^* \; x$. Thus $z = y$ satisfies the lemma. If $w' = \lambda$, a similar argument holds. Assume therefore that $v' \neq \lambda$ and $w' \neq \lambda$; then, from the choice of u, there exist σ and ξ in Σ, such that $\sigma \neq \xi$, and $x = u\sigma v$ and $y = u\xi w$, for some v and w in Σ^*. By Lemma 5, either $u\sigma\xi w \;{}_m\!\!\sim u\sigma v$, or $u\xi\sigma v \;{}_m\!\!\sim u\xi w$. If $u\sigma\xi w \;{}_m\!\!\sim u\sigma v$, then, since $u\sigma v \;{}_m\!\!\sim u\xi w$, $u\sigma\xi w \;{}_m\!\!\sim u\xi w$, hence $u\sigma\xi w \; R_m \; u\xi w = y$. On the other hand, letting u' be the longest common prefix of $u\sigma\xi w$ and $u\sigma v$, we have $|u\sigma\xi w| + |u\sigma v| - 2|u'| \le$ $\le |\xi w| + |v| < |\xi w| + |\sigma v| = |x| + |y| - 2|u|$. Thus, by the induction hypothesis there exists a z, such that $z \; R_m^* \; u\sigma v = x$ and $z \; R_m^* \; u\sigma\xi w$. Since $u\sigma\xi w \; R_m \; u\xi w = y$, it follows that $z \; R_m^* \; y$. A similar argument holds if $u\xi\sigma v \;{}_m\!\!\sim u\xi w$. □

<u>Corollary 6a (Characterization of ${}_m\!\!\sim$)</u>. For every x and y in Σ^*, $x \;{}_m\!\!\sim y$ iff $x \; (R_m^{*-1} \circ R_m^*) \; y$ iff $x \; (R_m \cup R_m^{-1})^* \; y$.

<u>Proof</u>. Follows immediately from Lemma 6. □

3. <u>The main result</u>

In this section, we derive the main result , using the lemmas in section 2.

<u>Lemma 7</u>. Let $E \subseteq \Sigma^*$ be a piecewise testable event, and let M be its syntactic monoid. Then M is a finite $\mathcal{J}$-trivial monoid.

<u>Proof</u>. Let m be a natural, such that, for every x and y in Σ^*, $x \;{}_m\!\!\sim y$ implies that $x \in E$ iff $y \in E$. Since ${}_m\!\!\sim$ is a congruence relation (Lemma 1(a)), it follows that $x \;{}_m\!\!\sim y$ implies $x \equiv y \pmod E$. Thus, since ${}_m\!\!\sim$ is of finite index (Lemma 1(a)), so is $\equiv \pmod E$, i.e. M is a finite monoid. Let $\gamma : \Sigma^* \to M$ be the natural epimorphism defined by $\equiv \pmod E$. Assume now, that for some a and b in M, $a \,\mathcal{J}\, b$, i.e. there exist c_1, d_1, c_2 and d_2 in M, such that $a = c_1 b d_1$ and $b = c_2 a d_2$. We claim that $a = b$. Indeed, $a = (c_1c_2)^m a (d_2d_1)^m$. Let y_1 and y_2 in Σ^* be such that $y_i\gamma = d_i$, then by Corollary 3a, $(y_2y_1)^m \;{}_m\!\!\sim (y_2y_1)^m y_2$, and since this implies that $(y_2y_1)^m \equiv$ $\equiv (y_2y_1)^m y_2 \pmod E$, it follows that $(d_2d_1)^m = (d_2d_1)^m d_2$, i.e. $a = ad_2$. By a dual argument, $a = c_2 a$, hence $b = c_2 a d_2 = a$. Thus, M is a $\mathcal{J}$-trivial monoid. □

<u>Lemma 8</u>. Let M be a finite $\mathcal{J}$-trivial monoid, and let $\gamma : \Sigma^* \to M$ be

an epimorphism. Then for every subset X of M, $X\gamma^{-1}$ is a piecewise testable event.

Proof. It is sufficient to prove that there exists an m, such that for all x and y in Σ^*, $x \underset{m}{\sim} y$ implies $x\gamma = y\gamma$. Let k be the cardinality of M, and let $m = 2k$. First we show that if u in Σ^+ and σ in Σ are such that $u \underset{k}{\sim} u\sigma$, then $u\gamma = (u\sigma)\gamma$. Indeed, by Lemma 3, there exist $u_1, u_2, \ldots, u_k$ in Σ^+, such that $u = u_1 u_2 \ldots u_k$ and $u_1\Sigma \supseteq u_2\Sigma \supseteq \ldots \supseteq u_k\Sigma \supseteq \{\sigma\}$. Let $w_0 = \lambda$, $w_1 = u_1$, $w_2 = u_1u_2, \ldots$, $w_k = u_1u_2\ldots u_k = u$. Since M has k elements only, there exist $i < j$ such that $w_i\gamma = w_j\gamma$. Now we claim that for all ξ in $u_{i+1}\Sigma$, $w_i\gamma = (w_i\xi)\gamma$. Indeed, if $\xi \in u_{i+1}\Sigma$, then $u_{i+1} = z_1\xi z_2$ for some z_1 and z_2 in Σ^*. Since each element in the sequence w_i, w_iz_1, $w_iz_1\xi$, w_j is a prefix of its successor, it follows that $M(w_j\gamma)M \subseteq M(w_iz_1\xi\gamma)M \subseteq$ $\subseteq M(w_iz_1\gamma)M \subseteq M(w_i\gamma)M$. Since $w_i\gamma = w_j\gamma$, it follows that all sets in the chain are equal, and since M is $\mathcal{J}$-trivial, this implies that $w_i\gamma = w_iz_1\gamma = w_iz_1\xi\gamma$. It follows that $w_i\gamma = w_i\xi\gamma$. Then, since $(u_{i+1}\ldots u_k\sigma)\Sigma = u_{i+1}\Sigma$, it follows that $u\gamma = (u\sigma)\gamma$. By a dual argument, if v in Σ^+ and σ in Σ are such that $v \underset{k}{\sim} \sigma v$, then $v\gamma = (\sigma v)\gamma$. Consider now u and v in Σ^* and σ in Σ, such that $u\sigma v \underset{m}{\sim} uv$. By Lemma 4, there exist p and p', such that $p+p' \geq m$, $u \underset{p}{\sim} u\sigma$ and $v \underset{p'}{\sim} \sigma v$. Since $m = 2k$, either $p \geq k$ or $p' \geq k$, hence by Lemma 1(b), either $u \underset{k}{\sim} u\sigma$ or $v \underset{k}{\sim} \sigma v$. Thus, either $u\gamma = (u\sigma)\gamma$ or $v\gamma = (\sigma v)\gamma$; in either case $(u\sigma v)\gamma = (uv)\gamma$. But this implies that for all x and y, $x \, R_m \, y$ implies $x\gamma = y\gamma$, hence by Lemma 6, for all x and y, $x \underset{m}{\sim} y$ implies $x\gamma = y\gamma$. This completes the proof. □

Thus we have:

Theorem. An event E is piecewise testable iff its syntactic monoid is finite and $\mathcal{J}$-trivial.

Proof. Immediate from Lemmas 7 and 8. □

4. Other characterizations of piecewise testable events

In this section we indicate other characterizations of piecewise testable events. Proofs and further details can be found in [9]. Our notation on automata follows [4].

First we need a few definitions. Let $\mathcal{C}$ be the smallest family of events which contains $\Sigma^{*}\sigma\Sigma^{*}$ for every σ in Σ, and is closed under concatenation. Let $\mathcal{D}$ be the smallest family of events which contains $\mathcal{C}$ and is closed under the Boolean operations.

Let $A = (Q,\Sigma,M)$ be a semiautomaton. A is a *chain-reset*, iff there exists a linear ordering $q_0,q_1,\ldots,q_m$ of Q, such that for all $q_i \in Q - \{q_m\}$, and for all $\sigma \in \Sigma$, $q_i\sigma^A$ is either q_i or q_{i+1}, and $q_m\sigma^A = q_m$ for all $\sigma \in \Sigma$. A is *partially ordered* iff for all q in Q and for all x and y in Σ^{*}, $q(xy)^A = q$ implies $qx^A = q$. A *component* of A is a minimal nonempty subset P of Q, such that for all $q \in Q$ and for all $\sigma \in \Sigma$, $q\sigma^A \in P$ iff $q \in P$. Let θ be a nonempty subset of Σ. The *restriction of* A *to* θ is the semiautomaton $A|\theta = (Q,\theta,N)$, where $\sigma^{A|\theta} = \sigma^A$ for all $\sigma \in \theta$. A *dead state* of A is a state $q \in Q$ such that for all $\sigma \in \Sigma$ $q\sigma^A = q$.

Now we have

<u>Theorem</u>. Let $E \subseteq \Sigma^{*}$ be a regular event, let E^{T} be the reverse of E, let $\hat{A}$ and $\hat{B}$ be the reduced automata accepting E and E^{T} respectively, and let M be the syntactic monoid of E. The following are equivalent:

(a) E is piecewise testable.

(b) E is in $\mathcal{D}$.

(c) A can be covered by a direct product of chain-resets.

(d) A and B are both partially ordered.

(e) A is partially ordered, and for all $q \in Q$ and for all $x,y \in \Sigma^{*}$, $qx^A = q(xx)^A = q(xy)^A$ and $qy^A = q(yy)^A = q(yx)^A$ imply $qx^A = qy^A$.

(f) A is partially ordered and for every nonempty subset θ of Σ, each component of $A|\theta$ contains exactly one dead state of $A|\theta$.

(g) M is $\mathcal{J}$-trivial.

It is relatively simple to show the equivalence of (a), (b) and (c), and that of (d), (e), (f) and (g). The most difficult part in the proof of this theorem is to show that one of (d) to (g) implies one of (a) to (c). In the previous section we proved that (g) implies (a). Another possibility would be to give a proof of (g) implies (b) (or even more interesting would be (f) implies (b)) by constructing regular expressions, of the form required to show that an event is in $\mathcal{D}$, which would denote each congruence class of $\equiv$ (mod E) (denote the event accepted by each state of A, respectively). Such a construction has been carried out by Schützenberger in [8], constructing star-free regular expressions for events whose syntactic monoid is group-free. Unfortunately, his proof, when applied to $\mathcal{J}$-trivial monoids, does not produce expressions in D. We have been unable to carry out such a proof, unless in the very simple case of idempotent and commutative monoids.

Acknowledgment. I am indebted to Professor J.A. Brzozowski for introducing me to the fascinating world of star-free regular events.

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São Paulo, April 28, 1975.