Computational Astrophysics – HW #2

Answer the following questions. Hand in your (documented) code along with your solutions.

1. *Not so fast!* Use a perturbation analysis to show that the leapfrog scheme (time-centered difference):

$$\rho_j^{n+1} - \rho_j^{n-1} = -v_o \frac{\Delta t}{\Delta x}(\rho_{j+1}^n - \rho_{j-1}^n), \tag{1}$$

is conditionally stable, i.e. it is stable as long as $\Delta t \leq \Delta x / v_0$. Assume that v_0 is a constant velocity.

2. *Sod it!* Using the finite volume discretization given in class (see also notes) and a programming language of your choice, write your own 1D hydrodynamic solver.

(a) The quintessential shock benchmark problem is the "Sod Shock Tube" (first formulated by Sod in 1978). The problem involves setting up two discontinuous states at $t = 0$: a hot dense gas on the left and a cool, low-density gas on the right. The result is a shock wave that travels left to right into the cold gas, and a rarefaction wave that travels into the dense gas on the left.

Setup: Let the domain be $[0, 2]$, and set $\rho_1 = 1.0$, $p_1 = 1.0$ for $x \leq 0.75$ and $\rho_2 = 0.125$, $p_2 = 0.1$ for $x > 0.75$. Assume $\gamma = 1.4$ and use this to determine the specific energy. Set $v = 0$ everywhere initially. Use symmetry boundary conditions and $N = 200$ grid cells. Set the time step according to a CFL number of 0.5. implement an artificial viscosity of the form:

$$q_{i+1/2}^n = \left[q_0(u_{i+1}^n - u_i^n)^2 - q_1(u_{i+1}^n - u_i^n)\right] \frac{c_{s,i+1/2}}{\bar{\rho}} \qquad \left(\frac{u_{i+1}^n - u_i^n}{x_{i+1}^n - x_i^n}\right) < 0 \tag{2}$$

$$q_{i+1/2}^n = 0 \qquad \left(\frac{u_{i+1}^n - u_i^n}{x_{i+1}^n - x_i^n}\right) > 0 \tag{3}$$

where $\bar{\rho} = \frac{1}{2}\left(\frac{1}{\rho_{i+1/2}^{n+1}} + \frac{1}{\rho_{i+1/2}^n}\right)$, $q_0 = 4$ and $q_1 = 0.5$.

Compute the solution at $t = 0.245$ and make plots of the density, velocity, pressure and specific energy as a function of position. Compare the shock properties (speed, density jump, pressure jump, velocity) with that predicted for these conditions (e.g. see Stone & Norman 1992).

(b) Simple numerical schemes are prone to certain artifacts near the shock interface. What deviations from the analytic solution do you notice? What numerical improvements would you apply to address them?

(c) Check the shock solution for $N = 100, 200$ and 400 grid cells. How quickly is the solution converging? Is this consistent with your expectations?

(d) Decrease the CFL number to 0.25 and increase it to 1.1. What happens to the shock solution in each case?

(e) Evaluate how sensitive the solution is to magnitude of the coefficient of artificial viscosity. Increase the viscosity coefficients by a factor of 2 and decrease them by a factor of 2. How does the solution change? What impact do the two different viscous components have on the solution?