1) (a) show that $E[(\bar{Y}-\mu)^2] = \sigma^2/N$ where $\bar{Y} = \frac{1}{N} \Sigma_i Y_i$

given $E[Y_i] = \mu$ & $E[(Y_i-\mu)^2] = \sigma^2$

⇒ rewrite: $E[(\bar{Y}-\mu)^2] = E[(Y_i-\mu)^2] \cdot \frac{1}{N}$

⇒ substitute: $E[(\frac{1}{N} \Sigma_i Y_i - \mu)^2] = E[(Y_i-\mu)^2] \cdot \frac{1}{N}$

⇒ split expectation: $E[(\frac{1}{N} \Sigma_i Y_i)^2] - [E(\mu)]^2 = \quad ''$

⇒ Pull out constant: $\frac{1}{N^2} E[\Sigma_i Y_i^2] - [E(\mu)]^2 = \quad ''$

⇒ simplify sum $\frac{1}{\cancel{N^2}} \cdot \cancel{N}\, E(Y_i^2) - [E(\mu)]^2 =$

$\frac{1}{N} \cdot E(Y_i^2) - [E(\mu)]^2 =$

⇒ use $E[(X-\mu)^2] = E(X^2) - [E(\mu)]^2$

$\frac{1}{N} \cdot E[(Y_i-\mu)^2] = \frac{1}{N} \cdot E[(Y_i-\mu)^2]$ $\blacksquare$

(b) $D = \frac{1}{N} \sum_i (Y_i - \bar{Y})^2$; using $(Y_i - \bar{Y}) = ((Y_i-\mu) + (\mu - \bar{Y}))$

show

$D = \frac{1}{N} \sum_i (Y_i-\mu)^2 - \frac{1}{N} \sum_i (\bar{Y}-\mu)^2$

Sofia Guo – Econ 142 PSET 2

$\Rightarrow$ substitute in given $(Y_i - \bar{Y})$: $D = \frac{1}{N} \sum_i [(Y_i - \mu) + (\mu - \bar{Y})]^2$

$\Rightarrow$ Expand the square : $D = \frac{1}{N} \sum_i [(Y_i - \mu)^2 + 2(Y_i - \mu)(\mu - \bar{Y}) + (\mu - \bar{Y})^2]$

$\Rightarrow$ Flip signs $D = \frac{1}{N} \sum_i [(Y_i - \mu)^2 + 2(-1)(\mu - Y_i)(\mu - \bar{Y}) + (\mu - \bar{Y})^2]$

$\Rightarrow$ Given $\bar{Y} = \frac{1}{N} \sum_i Y_i$: $= \frac{1}{N} \sum_i [(Y_i - \mu)^2 + (-2)(\mu - Y_i)(\mu - \frac{1}{N} \sum_i Y_i) + (\mu - \frac{1}{N} \sum_i Y_i)^2]$

$\Rightarrow$ Distribute sum : $= \frac{1}{N} \sum_i (Y_i - \mu)^2 + \frac{1}{N} \sum_i (-2)(\mu - Y_i)(\mu - \frac{1}{N} \sum_i Y_i) + \frac{1}{N} \sum_i (\mu - \frac{1}{N} \sum_i Y_i)^2$

$\Rightarrow$ Simplify. $= \frac{1}{N} \sum_i (Y_i - \mu)^2 + (-2) \frac{1}{N} \sum_i (\mu - \frac{1}{N} \sum_i Y_i)^2 + \frac{1}{N} \sum_i (\mu - \frac{1}{N} \sum_i Y_i)^2$

'' $= \frac{1}{N} \sum_i (Y_i - \mu)^2 - \frac{1}{N} \sum_i (\mu - \frac{1}{N} \sum_i Y_i)^2$

'' $= \frac{1}{N} \sum_i (Y_i - \mu)^2 - \frac{1}{N} \sum_i (\bar{Y} - \mu)^2$ ☐

(c) show $E[D] = \frac{N-1}{N}\sigma^2$ using (a)

$\Rightarrow$ $E[D] = E\left[\frac{1}{N}\Sigma(Y_i - \bar{Y})^2\right] = \frac{N-1}{N}\cdot\sigma^2$

$\Rightarrow$ from (a), substitute

$\sigma^2 = N\cdot E[(\bar{Y}-\mu)^2]$: $E\left[\frac{1}{N}\Sigma_i(Y_i - \bar{Y})^2\right] = \frac{N-1}{N}\cdot N\cdot E[(\bar{Y}-\mu)^2]$

$\Rightarrow$ simplify & plug in (b) : $E\left[\frac{1}{N}\Sigma_i(Y_i-\mu)^2\right] - E\left[\frac{1}{N}\Sigma_i(\bar{Y}-\mu)^2\right] = (N-1)E[(\bar{Y}-\mu)^2]$

$\Rightarrow$ distribute over $(N-1)$: $E\left[\frac{1}{N}\Sigma_i(Y_i-\mu)^2\right] - E[(\bar{Y}-\mu)^2] = N\cdot E[(\bar{Y}-\mu)^2] - E[(\bar{Y}-\mu)^2]$

$\Rightarrow$ simplify :

$E\left[\frac{1}{N}\Sigma_i(Y_i-\mu)^2\right] = \cancel{N}\cdot E[(Y_i-\mu)^2]/\cancel{N}$

$\frac{1}{N}\Sigma_i E[(Y_i-\mu)^2] = E[(Y_i-\mu)^2]$

$\Rightarrow$ use variance property : $\frac{1}{N}\left[E\,\Sigma_i(Y_i^2) - (E(\mu))^2\right] =$ "

$\Rightarrow$ simplify sum : $\cancel{\frac{1}{N}}\left[\cancel{N}E(Y_i^2) - (E(\mu))^2\right] =$ "

$E(Y_i^2) - (E(\mu))^2 =$ "

$\Rightarrow$ simplify :

$E[(Y_i-\mu)^2] = E[(Y_i-\mu)^2]$ ∎

(d) show $E\left[\frac{1}{N-1}\Sigma_i (Y_i - \bar{Y})^2\right] = \sigma^2$

$\Rightarrow$ use (d): $E[D] = \frac{N-1}{N}\sigma^2$

$\Rightarrow$ substitute D in and set equal to desired expression: $E\left[\frac{1}{N}\Sigma_i (Y_i - \bar{Y})^2\right] = \frac{N-1}{N} \cdot E\left[\frac{1}{N-1}\Sigma_i (Y_i - \bar{Y})^2\right]$

$\frac{1}{N} E\left[\Sigma_i (Y_i - \bar{Y})^2\right] = \frac{\cancel{N-1}}{N} \cdot \frac{1}{\cancel{N-1}} E\left[\Sigma_i (Y_i - \bar{Y})^2\right]$

$\Rightarrow$ simplify sums/constants: $\frac{1}{N} E\left[\Sigma_i (Y_i - \bar{Y})^2\right] = \frac{1}{N} E\left[\Sigma_i (Y_i - \bar{Y})^2\right]$ ∎

2) (a) $D_i = 1, 0$

$N = N_0 + N_1$

$\Rightarrow y_i = \alpha + \beta D_i + u_i$

show $\hat{\alpha} = \frac{1}{N_0}\sum_{i \in 0} y_i$

$\hat{\alpha} + \hat{\beta} = \frac{1}{N_1}\sum_{i \in 1} y_i$

use FOC: $\min_{\hat{\beta}} E\left[(y_i - \hat{\alpha} - D_i'\hat{\beta})^2\right] = 0$

$E\left[-2D_i (y_i - \hat{\alpha} - D_i'\hat{\beta})\right] = 0$

$E\left[D_i (y_i - \hat{\alpha} - D_i'\hat{\beta})\right] = 0$

$$\Rightarrow E[D_i y_i - \hat{\alpha} D_i - D_i D_i' \hat{\beta}] = 0$$

$$- E[D_i y_i] + E[\hat{\alpha} D_i] + E[D_i D_i' \hat{\beta}] = 0$$

$$\hat{\alpha} E[D_i] + \hat{\beta} E[D_i D_i'] = E[D_i y_i]$$

$\Rightarrow$ For $N_0 \Rightarrow D_i = 0$

$$\hat{\alpha} E[D_i] = E[D_i y_i] - \hat{\beta} E[D_i D_i']$$

$N_1 \Rightarrow D_i = 1$

$$\hat{\alpha} = \frac{1}{E[D_i]} \left[E[D_i y_i] - \hat{\beta} E[D_i D_i'] \right]$$

$\Rightarrow$ evaluate when N_1 $(D_i = 1)$

$$\hat{\alpha} = E[y_i] - \hat{\beta} D_i$$

$$\hat{\alpha} + \hat{\beta} = \frac{1}{N_1} \sum_{i \in 1} y_i$$

$\Rightarrow$ eval when N_0 $(D_i = 0)$

$$\hat{\alpha} = E[y_i] - \cancel{\hat{\beta} D_i}$$

$$\hat{\alpha} = \frac{1}{N_0} \sum_{i \in 0} y_i$$

(b) i)

$$\min_{\beta} E[(y_i - \alpha - \beta_1 D_{1i} - \beta_2 D_{2i})^2] \quad \text{where } u_i = y_i - \alpha - \beta_1 D_{1i} - \beta_2 D_{2i}$$

$$\Rightarrow FOC_{\hat{\beta}_1} \Rightarrow \min_{\hat{\beta}_1} E[(y_i - \hat{\alpha} - \hat{\beta}_1 D_{1i}' - \hat{\beta}_2 D_{2i}')^2] = 0$$

$$\Rightarrow E[2(y_i - \hat{\alpha} - \hat{\beta}_1 D_{1i}' - \hat{\beta}_2 D_{2i}')(-D_{1i})] = 0$$

$$\Rightarrow E[D_{1i} y_i - \hat{\alpha} D_{1i} - \hat{\beta}_1 D_{1i}' D_{1i} - \hat{\beta}_2 D_{2i}' D_{1i}] = 0$$

$$E[D_{1i} y_i] - \hat{\alpha} E[D_{1i}] - \hat{\beta}_1 E[D_{1i}' D_{1i}] - \hat{\beta}_2 E[D_{2i}' D_{1i}] = 0$$

$$\hat{\beta}_1 E[D_{1i}' D_{1i}] = E[D_{1i} y_i] - \hat{\alpha} E[D_{1i}] - \hat{\beta}_2 E[D_{2i}' D_{1i}]$$

$$\hat{\beta}_1 = \frac{1}{E(D_{1i}' D_{1i})} \left[E[D_{1i} y_i] - \hat{\alpha} E[D_{1i}] - \hat{\beta}_2 E[D_{2i}' D_{1i}] \right]$$

$$\Rightarrow FOC_{\hat{\beta}_2}: \quad \Rightarrow \min_{\hat{\beta}_2} E[(y_i - \hat{\alpha} - \hat{\beta}_1 D_{1i}' - \hat{\beta}_2 D_{2i}')^2] = 0$$

$$\Rightarrow E\left[2(y_i - \hat{\alpha} - \hat{\beta}_1 D_{1i}' - \hat{\beta}_2 D_{2i}')(-D_{2i})\right] = 0$$

$$\Rightarrow E\left[D_{2i} y_i - \hat{\alpha} D_{2i} - \hat{\beta}_1 D_{1i}' D_{2i} - \hat{\beta}_2 D_{2i}' D_{2i}\right] = 0$$

$$E[D_{2i} y_i] - \hat{\alpha} E[D_{2i}] - \hat{\beta}_1 E[D_{1i}' D_{2i}] - \hat{\beta}_2 E[D_{2i}' D_{2i}] = 0$$

$$\hat{\beta}_2 E[D_{2i}' D_{2i}] = E[D_{2i} y_i] - \hat{\alpha} E[D_{2i}] - \hat{\beta}_1 E[D_{1i}' D_{2i}]$$

$$\hat{\beta}_2 = \frac{1}{E(D_{2i}' D_{2i})}\left[E[D_{2i} y_i] - \hat{\alpha} E[D_{2i}] - \hat{\beta}_1 E[D_{1i}' D_{2i}]\right]$$

ii) $\Rightarrow$ eval. when N_1 $(D_{1i} = 1)$

$$\hat{\beta}_1 = \frac{1}{E(\cancel{D_{1i}'} D_{1i})}\left[\underbrace{E[D_{1i} y_i]}_{1} - \hat{\alpha} \underbrace{E[D_{1i}]}_{1} - \hat{\beta}_2 \underbrace{E[D_{2i}' D_{1i}]}_{0}\right]$$

$$\hat{\beta}_1 = E[y_i] - \hat{\alpha}$$

$$\hat{\alpha} + \hat{\beta}_1 = \frac{1}{N_1} \sum_{i \in 1} y_i$$

$\Rightarrow$ eval. when N_2 $(D_{2i} = 1)$

$$\hat{\beta}_2 = \frac{1}{\underbrace{E[D_{2i}' D_{2i}]}_{1}}\left[\underbrace{E[D_{2i} y_i]}_{1} - \hat{\alpha} \underbrace{E[D_{2i}]}_{1} - \hat{\beta}_1 \underbrace{E[D_{1i}' D_{2i}]}_{0}\right]$$

$$\hat{\beta}_2 = E[y_i] - \hat{\alpha}$$

$$\hat{\alpha} + \hat{\beta}_2 = \frac{1}{N_2} \sum_{i \in 2} y_i$$

$\Rightarrow$ when N_0:

all $\hat{\beta}$ terms go to 0

$$\hat{\alpha} = \frac{1}{N_0} \sum_{i \in 0} y_i$$

3) know: $y_i = x_i \beta^* + u_i$ $\rightarrow$ j^{th} row

residualize: $x_{ji} = x'_{(\sim j)i} \pi + \xi_i$

Then: $\beta_j^* = E[\xi_i^2]^{-1} E[\xi_i y_i]$

show: given $y_i = x'_{(\sim j)_i}\lambda + \phi_i \Rightarrow \phi_i = y_i - x'_{(\sim j)_i}\lambda$

$\beta_j^* = E[\xi_i^2]^{-1} E[\xi_i \phi_i]$

$\Rightarrow$ use FOC for β^* & λ

From lecture 3:

$FOC_\lambda = E[x_{(\sim j)i}\phi_i] = 0 \Rightarrow E[\phi_i x_{ni}] = 0$ unless $n = j$

$FOC_{\beta^*} = E[x_i u_i] = 0 \Rightarrow E[\phi_i u_i] = E[(y_i - x'_{(\sim j)_i}\lambda)u_i] = 0$

$\Rightarrow E[\phi_i y_i] = E[\phi_i(x'_{(\sim j)_i}\lambda + \phi_i)] = E[\phi_i^2]$ usin FOC_λ

$\Rightarrow E[\xi_i \phi_i] = \beta_j^* E[\phi_i^2]$

$\beta_j^* = (E[\phi_i^2])^{-1} E[\xi_i \phi_i] = (E[\xi_i^2])^{-1} E[\xi_i y_i]$

$$x_{ij} = x_{(\sim j)_i}\pi + \xi_i \Rightarrow x_{(\sim j)_i} = \frac{x_{ij} - \xi_i}{\pi}$$

$$y_i = x_{(\sim j)_i}\lambda + \phi_i \Rightarrow \phi_i = y_i - x_{(\sim j)_i}\lambda$$

want $\phi_i = y_i$ $\quad$ $\phi_i = y_i - \underbrace{\left(\frac{\lambda}{\pi}\right)(x_{ij} - \xi_i)}_{0}$ $\quad$ either $\lambda = 0$ or $x_{ij} = \xi_i$

$\Rightarrow$ likely that $\lambda = 0$ since $x_{ij} \neq \xi_i$

($\frac{\lambda}{\pi}$ undefined if $\pi = 0$)

Therefore $\rightarrow \phi_i = y_i$

$$\Rightarrow \beta_j^* = E[\xi_i^2]^{-1} E[\xi_i \phi_i]$$