ANALYTICAL MODEL OF A CRANE BASED ON NEWTON'S DYNAMIC LAWS

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***Abstract*—The following report present the explanation about analytical model of a crane with pendulum. The model is explain step by step, since Newton's Model to the graphics. Those graphics shows the results of model using a step and pulse as input. Every equation that is showing in the report is a important part of the development, because the variables shows the behavior of every part of crane.**

***Keywords*—Crane, pendulum, Newton's Laws, Analytical model.**

I. INTRODUCTION

Automatic control is present on everyday lives, from the refrigerator to cellphones, and many other electronic appliances. But, it is not circumscribed to such commodities. It is also applied in industrial procedures, such as cranes on ports. This paper approaches this last application of control, developing an analytical model of such crane, based on Newton's dynamic laws, with the objective of determining the space state model of the system and later analysis of is response and stability.

II. NEWTON'S MODEL

The physical model of the crane is based on Newton's dynamic laws, that because it allows analyze and describe separately by each force interaction every component of the crane. [1]. The components are going to describe by the following equation (1):

$$\sum F = m \cdot a \tag{1}$$

In this case, the sum of forces is equal to the mass (m) and the acceleration (a). The mass is specific if the plant of crane and the acceleration is the variable that describe the movement of the crane. In other hand, for the pendulum uses a sum of torques (I) it is because that the movement describe is an angular and it is represent by the following equation (2):

$$\sum M = I\alpha \tag{2}$$

Those equations was the beginning of the development of mathematical model, those are going to be use to obtain the states variables, the transfer function and it allows show the behavior. After have that, we can use the software Matlab from Mathworks to simulate and obtain the response of the system to an impulse and a step function.

III. PHYSICAL MODEL

The correct way of show a physical model is using a free body diagram, of the system, that is show in Figure 1. It is important to explain that for this system, friction is taken as ideal.

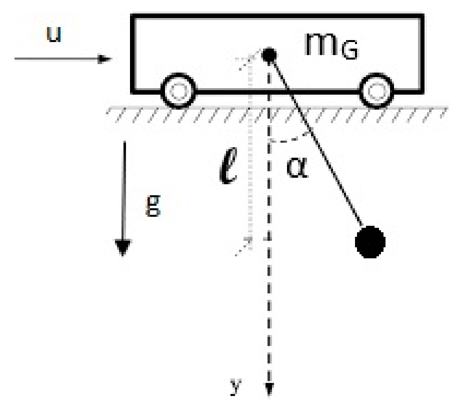

Figure 1: Physical free body diagram of the plant

As the first step of all is to define the state variables that are going to be used in the model. The force applied to begin and keep the movement (u) is assigned as an input, and the outputs are the angle of pendulum and the crane position. Therefore, the state variables are given by:

$$x_1 = y_c \quad ; \; Crane\ position \tag{3}$$

$$x_2 = \dot{y_c} \quad ; \; Velocity\ of\ crane \tag{4}$$

$$x_3 = \alpha \quad ; \; Pendulum\ angle \tag{5}$$

$$x_4 = \dot{\alpha} \; ; \; Angular\ vel.\ of\ the\ pendulum \tag{6}$$

The system has two elements that are the principals: the crane and the pendulum. Both of them will be analyzed individually. The crane has a linear movement in the **x** axis that is given by the action of **u**. Meanwhile, the pendulum action, defined as T from now on, is given by inertia of the movement from the crane. The equations that describe the physical behavior of crane (9) and the physical behavior of pendulum 12 are shown below:

$$\sum F_x = m \cdot a \tag{7}$$

$$m_G \cdot a_G = u \cdot T_x \tag{8}$$

$$m_G \cdot \dot{x_2} = U - T \cdot sen(x_3) \tag{9}$$

For pendulum:

$$\sum M = J \cdot \dot{x_4} \tag{10}$$

$$m_C \cdot l^2 \cdot \dot{x_4} = l \cdot x \cdot g \cdot m_C \tag{11}$$

$$\dot{x_4} = \frac{g}{l} \cdot sen(x_3) \tag{12}$$

It is important to be clear about the sum of torques for the pendulum, that sum is made at joint between crane and pendulum. Using the equation 12 is possible to obtain the angular acceleration of pendulum.

Now, the next step is to relate the principal systems, crane and pendulum, in order to find T. This can be done through analysis in forces that interact with the mass hanging from the pendulum. The behavior of the mass is described by two sources: a linear acceleration due to the displacement of the crane and one tangential-radial due to the movement of the pendulum.

To obtain the radial and tangential components, let introduce the next equations:

$$\sum F = m_C \cdot a_m \tag{13}$$

$$\sum F = m_C \cdot a_c + a_r + a_t \tag{14}$$

Due to the nature of the analysis, only the radial acceleration is used, and the forces that affect the mass are (T) from the pendulum and its weight on the y axis.

$$m_p \cdot a_r + a_c \cdot cos(x_3) = T - m_c \cdot g \cdot cos(x_3) \tag{15}$$

Using the components of the last equation is obtained the linear behavior of T, which conduces to the final equation:

$$T = m_C[((\dot{x_4})^2 \cdot l + \dot{x_2} \cdot cos(x_3)) + gcos(x_3)] \tag{16}$$

Replacing equation (16) in equation (9) comes out the equation that describes the velocity of the crane.

$$\dot{x_2} = \frac{1}{[(m_G + m_c \cdot sen(x_3) \cdot cos(x_3))]} \cdot [u - m_c \cdot sen(x_3) \cdot (x_4^2 \cdot l + g \cdot cos(x_3))] \tag{17}$$

As can be seen in the previous equations, those are not linearized. So, in order to work with linealized equations, is needed to apply this to the final equations to represent the system. Here are the linearized eq.:

$$\dot{x_1} = x_2 \tag{18}$$

$$\dot{x_2} = -m_c \cdot g \cdot x_3 + \frac{u}{m_G} \tag{19}$$

$$\dot{x_3} = x_4 \tag{20}$$

$$\dot{x_4} = g \cdot \frac{x_3}{l} \tag{21}$$

The linearization process works only for small angles in the resolution of the system, values that almost reaches 0.

IV. State equations

After linearization, in automatic control the state variables and their respective equations are taken to represent them in a matrix form. This is known as state equations. This equations for this system will be shown below:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & m_C g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/l & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m_G \\ 0 \\ 0 \end{bmatrix} \cdot U \tag{22}$$

$$y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \underline{x} \tag{23}$$

Where y is the output.

V. System responses

In order to verify the analysis made and the process followed for this plant, in the software Matlab, exactly in Simulink, the system is implemented virtually and parting from it is obtained the impulse and step responses. The step response is the behaviour of the output when the input changes from zero to one in a very short time.

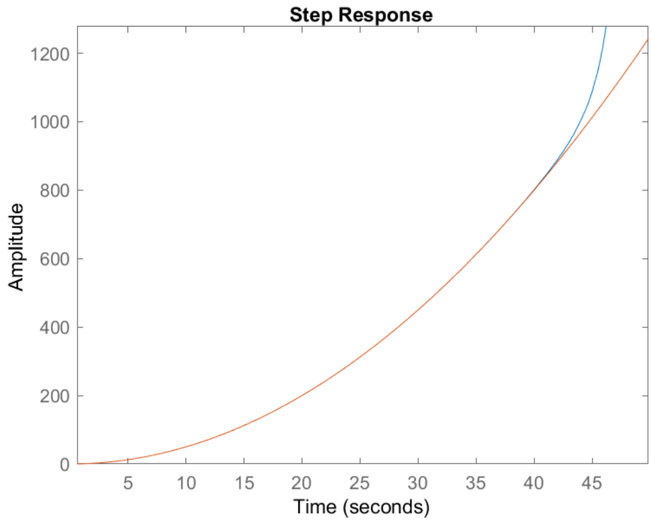

Figure 2: Step response of the plant. Source: Own elaboration

Also executing the impulse command in Matlab is obtained figure 3.

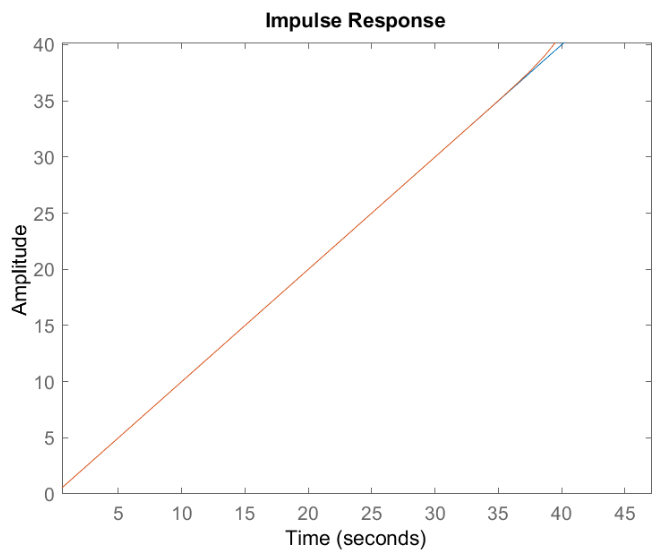

Figure 3: Impulse response of the system. Source: Own elaboration

VI. Conclusiones

It was possible to obtain a functional analytical model of the proposed plant, with response to the step and the impulse within the expected. When a step function is applied the crane is set in motion on the x axis, therefore having a constant acceleration and, product of this, a parabolic descriptive function when position is plotted versus time. On the other hand, when applied a impulse, the cart is set in motion but only for an instant, hence the linear plot of position versus time, since the speed of the crane is constant in this case.

It was also observed that the results obtained from the transfer function are congruent to those obtained from the space state model. For this reason it can be said those to models are equivalent, just having minimal differences once time tends to infinite.

References

[1] K. Ogata, *Ingeniería de Control Moderna*, 5th ed. Madrid: PEARSON EDUCACIÓN S.A, 2010.