Question 1:

a) See Python notebook

b)

i) for N : even $\longrightarrow$ $\boxed{\ell_{(levels)} = \frac{N}{2} + 1}$

for N : odd $\longrightarrow$ $\boxed{\ell_{(levels)} = \lfloor N/2 \rfloor + 1}$

floor division

ii) the lowest energy will be $-N$, from then, every time we flip a spin the energy either:

$$\begin{cases} \text{increases by } 4 \\ \text{decreases by } 4 \\ \text{stays the same} \end{cases}$$

e.g: $\downarrow\downarrow\downarrow$ $E = -3$ $\quad$ $\downarrow\uparrow\downarrow$ $E = 1$

So $\boxed{E_n = -N + 4 \cdot n}$ $\quad n = 0, 1, \dots, \ell - 1$

spin flip symmetry

degeneracy: $\boxed{D = 2 \times \binom{N}{2n} = 2 \frac{N!}{2n!(N-2n)!}}$

eg: $N = 4$, $n = 1$ should have $D = 12$; $D = 2 \cdot \binom{4}{2} = 2 \frac{4!}{2!2!} = 2 \cdot 3! = 12$

c)

i) $\boxed{\ell_{(levels)} = N + 1}$

ii) $\boxed{E'_n = -N + 2n}$ where $n = 0, 1, \dots, \ell - 1$

$$\boxed{D' = \binom{N}{n} = \frac{N!}{n!(N-n)!}}$$

Question 2:

See Python notebook

Question 3:

a)

$$E_\alpha = A + \frac{B}{N}\left(\Delta_\alpha - c/12\right) + \mathcal{O}(1/N^2)$$

$$E_0^{PBC} \simeq A - \frac{B}{N} c$$

$$\begin{cases} E_\alpha - E_0^{PBC} = \frac{B}{N}\Delta_\alpha \\ E_T^{PBC} - E_0^{PBC} = \frac{B}{N} 2 \end{cases}$$

$$\longrightarrow \boxed{\Delta_\alpha \approx 2\,\frac{E_\alpha - E_0^{PBC}}{E_T^{PBC} - E_0^{PBC}}}$$

b) See Python notebook

Question 4:

See Python notebook

$$H(\theta) = -\cos(\theta) \sum_\ell \sigma_\ell^x \sigma_{\ell+1}^x - \sin(\theta) \sum_\ell \sigma_\ell^z \qquad (\sigma_N^x \sigma_{N+1}^x = \sigma_N^x \sigma_1^x)$$

$H_A = H(\theta)$ with $\sigma_1^x \sigma_N^x$ interaction $\rightarrow + \cos(\theta)$

$T_A = \sigma_1^z \, T$ where $T|S_1 S_2 \cdots S_{N-1} S_N\rangle = |S_N S_1 S_2 \cdots S_{N-1}\rangle$

except for the $\sigma_1^x \sigma_N^x$ term the rest of the Hamiltonian is the same as H

$$H_A T_A = (H + 2\cos(\theta)\sigma_1^x \sigma_N^x)\sigma_1^z T = H\sigma_1^z T + 2\cos(\theta)\sigma_1^x \sigma_N^x \sigma_1^z T$$