Question 1:

a) See Python notebook

b)

i) for N : even $\longrightarrow$ $\boxed{\ell_{(\text{levels})} = \frac{N}{2} + 1}$

for N : odd $\longrightarrow$ $\boxed{\ell_{(\text{levels})} = \lfloor N/2 \rfloor + 1}$

floor division

ii) the lowest energy will be $-N$, from then, every time we flip a spin the energy either:

$$\begin{cases} \text{increases by } 4 \\ \text{decreases by } 4 \\ \text{stays the same} \end{cases}$$

e.g.: $\downarrow\downarrow\downarrow$ $E = -3$ $\quad$ $\downarrow\uparrow\downarrow$ $E = 1$

So $\boxed{E_n = -N + 4 \cdot n}$ $\quad n = 0, 1, \dots, \ell - 1$

spin flip symmetry

degeneracy : $\boxed{D = 2 \times \binom{N}{2n} = 2 \frac{N!}{2n!(N-2n)!}}$

e.g.: $N = 4$, $n = 1$ should have $D = 12$; $D = 2 \cdot \binom{4}{2} = 2 \frac{4!}{2!2!} = 2 \cdot 3! = 12$

c)

i) $\boxed{\ell_{(\text{levels})} = N + 1}$

ii) $\boxed{E'_n = -N + 2n}$ where $n = 0, 1, \dots, \ell - 1$

$\boxed{D' = \binom{N}{n} = \frac{N!}{n!(N-n)!}}$

Question 2:

See Python notebook

Question 3:

a)

$$E_\alpha = A + \frac{B}{N}\left(\Delta_\alpha - c/12\right) + \mathcal{O}(1/N^2)$$

$$E_0^{PBC} \simeq A - \frac{B}{N} c$$

$$\begin{cases} E_\alpha - E_0^{PBC} = \frac{B}{N}\Delta_\alpha \\ E_T^{PBC} - E_0^{PBC} = \frac{B}{N} 2 \end{cases}$$

$$\longrightarrow \boxed{\Delta_\alpha \approx 2\,\frac{E_\alpha - E_0^{PBC}}{E_T^{PBC} - E_0^{PBC}}}$$

b) See Python notebook

Question 4:

See Python notebook