

Question 2: Free fermions in 1D

a) we aim to rewrite:

$$H = - \sum_{n=1}^{N-1} (a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n) - \underbrace{(a_N^\dagger a_1 + a_1^\dagger a_N)}_{\text{PBCs}}$$

using
$$\begin{cases} a_n = \frac{1}{\sqrt{N}} \sum_k a_k e^{i \frac{2\pi}{N} kn} \\ a_n^\dagger = \frac{1}{\sqrt{N}} \sum_k a_k^\dagger e^{-i \frac{2\pi}{N} kn} \end{cases}$$

we rewrite the original Hamiltonian terms:

$$\begin{cases} a_n^\dagger a_{n+1} = \frac{1}{N} \sum_k \hat{a}_k^\dagger \hat{a}_k e^{i \frac{2\pi}{N} (-kn + k(n+1))} = \frac{1}{N} \sum_k \hat{a}_k^\dagger \hat{a}_k e^{i \frac{2\pi}{N} k} \\ a_{n+1}^\dagger a_n = \frac{1}{N} \sum_k \hat{a}_k^\dagger \hat{a}_k e^{i \frac{2\pi}{N} k (-k(n+1) + kn)} = \frac{1}{N} \sum_k \hat{a}_k^\dagger \hat{a}_k e^{-i \frac{2\pi}{N} k} \end{cases}$$

plugging these into our original H:

$$H = - \sum_{n=1}^{N-1} \frac{1}{N} \sum_k \hat{a}_k^\dagger \hat{a}_k \left(e^{i \frac{2\pi}{N} k} + e^{-i \frac{2\pi}{N} k} \right) - (\hat{a}_N^\dagger a_1 + \hat{a}_1^\dagger a_N)$$

$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

$$H = - \sum_k \hat{a}_k^\dagger \hat{a}_k 2 \cos\left(\frac{2\pi}{N} k\right)$$

$$\rightarrow H = \sum_k e_k \hat{a}_k^\dagger a_k \quad \text{where} \quad e_k = -2 \cos\left(\frac{2\pi}{N} k\right)$$

as expected, Hamiltonian is the sum over N single decoupled modes

b)
$$\delta_{m,0} = \frac{1}{N} \sum_k e^{i \frac{2\pi}{N} m k} \longrightarrow 0 = \delta_{m,0} - \frac{1}{N} \sum_k e^{i \frac{2\pi}{N} m k}$$

$$\begin{aligned} f(m) &= \frac{1}{N} \sum_k \text{sign}(e_k) e^{i \frac{2\pi}{N} m k} \\ &= \frac{1}{N} \left[- \sum_{|k| < N/4} e^{\frac{2\pi i}{N} m k} + \sum_{|k| > N/4} e^{\frac{2\pi i}{N} m k} \right] + \delta_{m,0} - \frac{1}{N} \sum_k e^{i \frac{2\pi}{N} m k} \end{aligned}$$

$$\rightarrow f(m) = \delta_{m,0} - \frac{2}{N} \sum_{|K| < N/4} e^{\frac{2\pi i}{N} mK}$$

c) See attached notebook

d) $M_{nn'} = \delta_{n,n'} - 2 \int_{-\pi/2}^{\pi/2} \frac{dq}{2\pi} e^{imq}$

$$\int_{-\pi/2}^{\pi/2} \frac{dq}{2\pi} e^{imq} = \frac{1}{2\pi} \left[\frac{1}{im} e^{imq} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi im} (e^{i\frac{\pi}{2}m} - e^{-i\frac{\pi}{2}m})$$

$$= \frac{1}{2\pi im} 2i \sin\left(\frac{\pi}{2}m\right) = \frac{1}{m\pi} \sin\left(\frac{\pi}{2}m\right)$$

$$f(m) = \delta_{m,0} - \frac{2}{m\pi} \sin\left(\frac{\pi}{2}m\right)$$

$$\rightarrow f(m) = \begin{cases} 0 & \text{for } m: \text{even} \\ \frac{2}{m\pi} (-1)^{\frac{m+1}{2}} & \text{for } m: \text{odd} \end{cases}$$

e) $S_N(L) = \frac{c}{3} \log_2 \left(\frac{N}{\pi} \sin \frac{L\pi}{N} \right) + C_1$

for $L \rightarrow \infty$: $\sin \frac{L\pi}{N} \simeq \frac{L\pi}{N}$

recall Taylor exp:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

so:

$$S_N(L) \underset{L \rightarrow \infty}{\simeq} \frac{c}{3} \log_2 \left(\frac{N}{\pi} \frac{L\pi}{N} \right) + C_1 \rightarrow$$

$$S_N(L) = \frac{c}{3} \log_2(L) + C_1$$

f) See attached notebook

Question 2: Free fermions in 2D

a) The same procedure as 1.a) applies:

$$\text{using } \begin{cases} a_{\underline{n}} = \frac{1}{\sqrt{N}} \sum_{\underline{k}} a_{\underline{k}} e^{i \frac{2\pi}{N} \underline{k} \underline{n}} = \frac{1}{\sqrt{N}} \sum_{\underline{k}} a_{\underline{k}} e^{i \frac{2\pi}{N} (k_x n_x + k_y n_y)} \\ a_{\underline{n}}^\dagger = \frac{1}{\sqrt{N}} \sum_{\underline{k}} a_{\underline{k}}^\dagger e^{-i \frac{2\pi}{N} \underline{k} \underline{n}} = \frac{1}{\sqrt{N}} \sum_{\underline{k}} a_{\underline{k}}^\dagger e^{-i \frac{2\pi}{N} (k_x n_x + k_y n_y)} \end{cases}$$

we rewrite the original Hamiltonian terms:

$$\begin{cases} a_{\underline{n}}^\dagger a_{\underline{n}+x} = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}}^\dagger \hat{a}_{\underline{k}} e^{i \frac{2\pi}{N} (-k_x n_x - k_y n_y + k_x (n_x+1) + k_y n_y)} \\ \quad = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}}^\dagger \hat{a}_{\underline{k}} e^{i \frac{2\pi}{N} k_x} \\ a_{\underline{n}+x} a_{\underline{n}}^\dagger = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}} \hat{a}_{\underline{k}}^\dagger e^{-i \frac{2\pi}{N} k_x} \\ a_{\underline{n}}^\dagger a_{\underline{n}+y} = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}}^\dagger \hat{a}_{\underline{k}} e^{i \frac{2\pi}{N} k_y} \\ a_{\underline{n}+y} a_{\underline{n}}^\dagger = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}} \hat{a}_{\underline{k}}^\dagger e^{-i \frac{2\pi}{N} k_y} \end{cases}$$

plugging these into our original H and rearranging:

$$H = - \sum_{\underline{k}} \hat{a}_{\underline{k}}^\dagger \hat{a}_{\underline{k}} \left[\left(e^{i \frac{2\pi}{N} k_x} + e^{-i \frac{2\pi}{N} k_x} \right) - \left(e^{i \frac{2\pi}{N} k_y} + e^{-i \frac{2\pi}{N} k_y} \right) \right]$$

$$H = - \sum_{\underline{k}} \hat{a}_{\underline{k}}^\dagger \hat{a}_{\underline{k}} 2 \left(\cos\left(\frac{2\pi}{N} k_x\right) + \cos\left(\frac{2\pi}{N} k_y\right) \right)$$

$$\rightarrow H = \sum_{\underline{k}} e_{\underline{k}} \hat{a}_{\underline{k}}^\dagger a_{\underline{k}} \quad \text{where} \quad e_{\underline{k}} = -2 \cos\left(\frac{2\pi}{N} k_x\right) - 2 \cos\left(\frac{2\pi}{N} k_y\right)$$

b) This is easy to see by looking at the sign of $e_{\underline{k}}$

$$\text{where } e_{\underline{k}} = -2 \cos\left(\frac{2\pi}{N} k_x\right) - 2 \cos\left(\frac{2\pi}{N} k_y\right)$$

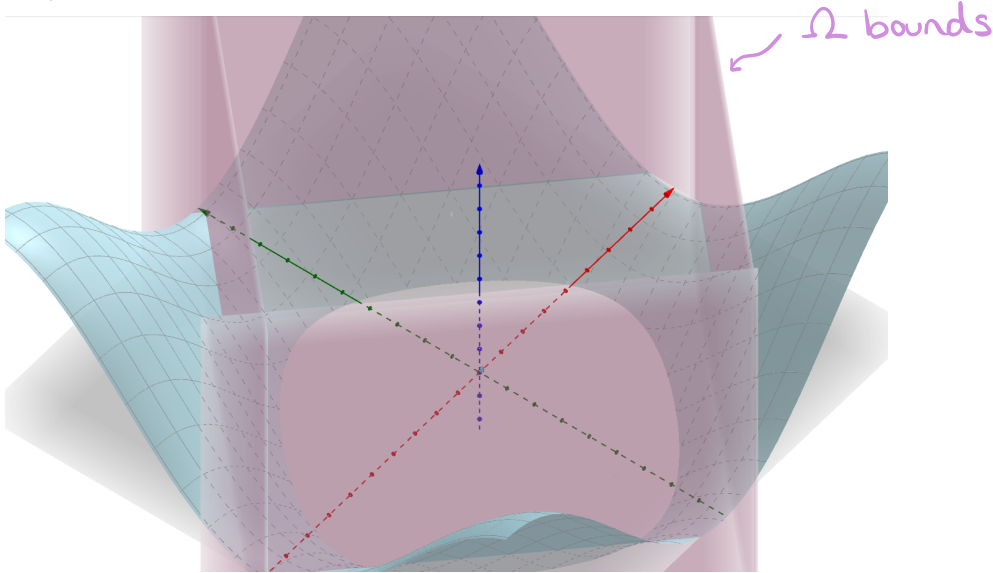
$$\text{for } \underline{k} \in \Omega \rightarrow e_{\underline{k}} < 0$$

$$\text{for } \underline{k} \notin \Omega \rightarrow e_{\underline{k}} > 0$$

note the bounds of $|\underline{k}|$

$$\rightarrow \tilde{M}_{\underline{k}, \underline{k}'} = \delta_{\underline{k}, \underline{k}'} f(\underline{k}) \quad \text{where} \quad f(\underline{k}) = \begin{cases} -1 & \underline{k} \in \Omega \\ 1 & \underline{k} \notin \Omega \end{cases}$$

3D plot of $e_{\underline{k}}$



c) Following the same logic as 1.b:

$$\delta_{\underline{m}, 0} = \frac{1}{N^2} \sum_{\underline{k}} e^{i \frac{2\pi}{N} \underline{m} \cdot \underline{k}} \longrightarrow 0 = \delta_{\underline{m}, 0} - \frac{1}{N^2} \sum_{\underline{k}} e^{i \frac{2\pi}{N} \underline{m} \cdot \underline{k}}$$

$$f(\underline{m}) = \frac{1}{N^2} \sum_{\underline{k}} f(\underline{k}) e^{i \frac{2\pi}{N} \underline{m} \cdot \underline{k}}$$

$$= \frac{1}{N^2} \left[- \sum_{\underline{k} \in \tilde{\Omega}} e^{\frac{2\pi i}{N} \underline{m} \cdot \underline{k}} + \sum_{\underline{k} \notin \tilde{\Omega}} e^{\frac{2\pi i}{N} \underline{m} \cdot \underline{k}} \right] + \delta_{\underline{m}, 0} - \frac{1}{N^2} \sum_{\underline{k}} e^{i \frac{2\pi}{N} \underline{m} \cdot \underline{k}}$$

$$\rightarrow f(\underline{m}) = \delta_{\underline{m}, 0} - \frac{2}{N} \sum_{\underline{k} \in \tilde{\Omega}} e^{\frac{2\pi i}{N} \underline{m} \cdot \underline{k}}$$

$$d) \quad f(\underline{m}) = \delta_{\underline{m}, 0} - 2 \underbrace{\int_{\underline{q} \in \tilde{\Omega}} \frac{d^2 \underline{q}}{(2\pi)^2} e^{i \underline{m} \cdot \underline{q}}}_{\textcircled{I}}$$

$$\textcircled{I}: \frac{1}{(2\pi)^2} \int_{\underline{q} \in \tilde{\Omega}} d q_x d q_y e^{i(m_x q_x + m_y q_y)}$$

change of variables:
$$\begin{cases} p_1 = q_x + q_y \\ p_2 = -q_x + q_y \end{cases}$$

$$\begin{cases} q_y = \frac{1}{2}(p_1 + p_2) \\ q_x = \frac{1}{2}(p_1 - p_2) \end{cases}$$

$$d q_x d q_y = \begin{vmatrix} \frac{\partial q_x}{\partial p_1} & \frac{\partial q_x}{\partial p_2} \\ \frac{\partial q_y}{\partial p_1} & \frac{\partial q_y}{\partial p_2} \end{vmatrix} d p_1 d p_2 = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} d p_1 d p_2 = \frac{1}{2} d p_1 d p_2$$

limits:

$$q_y = q_x \pm \pi \rightarrow \frac{1}{2}(p_1 + p_2) = \frac{1}{2}(p_1 - p_2) \pm \pi ; p_2 = \pm \pi$$

$$q_y = -q_x \pm \pi \rightarrow \frac{1}{2}(p_1 + p_2) = \frac{1}{2}(p_2 - p_1) \pm \pi ; p_1 = \pm \pi$$

$$\frac{1}{(2\pi)^2} \int \frac{1}{2} d p_1 d p_2 e^{\frac{i}{2}(m_x(p_1 - p_2) + m_y(p_1 + p_2))} =$$

$$= \frac{1}{(2\pi)^2} \int \frac{1}{2} d p_1 d p_2 e^{\frac{i}{2}(p_1(m_x + m_y) - p_2(m_x - m_y))}$$

$$= \frac{1}{8\pi^2} \left[\frac{-2}{i(m_x - m_y)} e^{-i/2 p_2(m_x - m_y)} \right]_{-\pi}^{\pi} \left[\frac{2}{i(m_x + m_y)} e^{i/2 p_1(m_x + m_y)} \right]_{-\pi}^{\pi}$$

$$= \frac{-2}{2\pi^2(m_x^2 - m_y^2)} \left(e^{-i\frac{\pi}{2}(m_x - m_y)} - e^{i\frac{\pi}{2}(m_x - m_y)} \right) \left(e^{i\frac{\pi}{2}(m_x + m_y)} - e^{-i\frac{\pi}{2}(m_x + m_y)} \right)$$

$$= \frac{-2}{\pi^2(m_x^2 - m_y^2)} \left(\sin\left(\frac{\pi}{2}(m_x - m_y)\right) \cdot \sin\left(\frac{\pi}{2}(m_x + m_y)\right) \right)$$

$$f(\underline{m}) = \delta_{\underline{m}, 0} - 2 \left[\frac{-2}{\pi^2(m_x^2 - m_y^2)} \left(\sin\left(\frac{\pi}{2}(m_x - m_y)\right) \cdot \sin\left(\frac{\pi}{2}(m_x + m_y)\right) \right) \right]$$

$$f(\underline{m}) = d_{\underline{m},0}^{-2} \left[\frac{1}{\pi^2(m_x^2 - m_y^2)} \left(\sin\left(\frac{\pi}{2}(m_x - m_y)\right) - \sin\left(\frac{\pi}{2}(m_x + m_y)\right) \right) \right]$$

$$\begin{cases} \sin\left(\frac{\pi}{2}(m_x + m_y)\right) = 0 & \text{if } m_x + m_y : \text{even} \\ \sin\left(\frac{\pi}{2}(m_x - m_y)\right) = 0 & \text{if } m_x - m_y : \text{even} \end{cases} \rightarrow \begin{array}{l} \text{this happens} \\ \text{if } m_x, m_y \\ \text{even or } m_x, m_y \\ \text{odd} \end{array}$$

$$\rightarrow f(\underline{m}) = \begin{cases} 0 & \text{for } m_x, m_y \text{ either both even or both odd} \\ \frac{4(-1)^{m_x}}{\pi^2(m_x^2 - m_y^2)} & \text{otherwise} \end{cases}$$