Question 1 : Free fermions in 1D

a) we aim to rewrite:

$$H = -\sum_{n=1}^{N-1} \left(a_n^+ a_{n+1} + a_{n+1}^+ a_n \right) - \underbrace{\left(a_N^+ a_1 + a_1^+ a_N \right)}_{\text{PBCs}}$$

using
$$\begin{cases} a_n = \frac{1}{\sqrt{N}} \sum_k a_k \, e^{i \frac{2\pi}{N} kn} \\ a_n^+ = \frac{1}{\sqrt{N}} \sum_k a_k^+ \, e^{-i \frac{2\pi}{N} kn} \end{cases}$$

we rewrite the original Hamiltonian terms:

$$\begin{cases} a_n^+ a_{n+1} = \frac{1}{N} \sum_k \hat{a}_k^+ \hat{a}_k \, e^{i \frac{2\pi}{N} (-kn + k(n+1))} = \frac{1}{N} \sum_k \hat{a}_k^+ \hat{a}_k \, e^{i \frac{2\pi}{N} k} \\ a_{n+1}^+ a_n = \frac{1}{N} \sum_k \hat{a}_k^+ \hat{a}_k \, e^{i \frac{2\pi}{N} k (-k(n+1) + kn)} = \frac{1}{N} \sum_k \hat{a}_k^+ \hat{a}_k \, e^{-i \frac{2\pi}{N} k} \end{cases}$$

plugging these into our original H:

$$H = -\sum_{n=1}^{N-1} \frac{1}{N} \sum_k \hat{a}_k^+ \hat{a}_k \underbrace{\left(e^{i \frac{2\pi}{N} k} + e^{-i \frac{2\pi}{N} k} \right)}_{\cos(z) = \frac{e^{iz} + e^{-iz}}{2}} - \left(\hat{a}_N^+ a_1 + \hat{a}_1^+ a_N \right)$$

$$H = -\sum_k \hat{a}_k^+ \hat{a}_k \, 2 \cos\left(\frac{2\pi}{N} k \right)$$

$$\boxed{\rightarrow \quad H = \sum_k e_k \, \hat{a}_k^+ a_k \qquad \text{where} \qquad e_k = -2\cos\left(\frac{2\pi}{N} k \right)}$$

as expected, Hamiltonian is the sum over N single decoupled modes

b) $$\delta_{m,0} = \frac{1}{N} \sum_k e^{i \frac{2\pi}{N} mk} \longrightarrow 0 = \delta_{m,0} - \frac{1}{N} \sum_k e^{i \frac{2\pi}{N} mk}$$

$$f(m) = \frac{1}{N} \sum_k \operatorname{sign}(e_k) \, e^{i \frac{2\pi}{N} mk}$$

$$= \frac{1}{N} \left[-\sum_{|k| < N/4} e^{\frac{2\pi i}{N} mk} + \sum_{|k| > N/4} e^{\frac{2\pi i}{N} mk} \right] + \delta_{m,0} - \frac{1}{N} \sum_k e^{i \frac{2\pi}{N} mk}$$

$$\boxed{\rightarrow f(m) = \delta_{m,0} - \frac{2}{N} \sum_{|k|<N/4} e^{\frac{2\pi i}{N} mk}}$$

c) See attached notebook

d) $M_{nn'} = \delta_{n,n'} - 2 \int_{-\pi/2}^{\pi/2} \frac{dq}{2\pi} e^{imq}$

$$\int_{-\pi/2}^{\pi/2} \frac{dq}{2\pi} e^{imq} = \frac{1}{2\pi} \left[\frac{1}{im} e^{imq} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi i m} \left(e^{i\frac{\pi}{2}m} - e^{-i\frac{\pi}{2}m} \right)$$

$$= \frac{1}{2\pi i m} 2i \sin\left(\frac{\pi}{2}m\right) = \frac{1}{m\pi} \sin\left(\frac{\pi}{2}m\right)$$

$$f(m) = \delta_{m,0} - \frac{2}{m\pi} \sin\left(\frac{\pi}{2}m\right)$$

$$\boxed{\rightarrow f(m) = \begin{cases} 0 & \text{for } m: \text{even} \\ \frac{2}{m\pi}(-1)^{\frac{m+1}{2}} & \text{for } m: \text{odd} \end{cases}}$$

e) $S_N(L) = \frac{c}{3} \log_2 \left(\frac{N}{\pi} \sin \frac{L\pi}{N} \right) + c_1$

for $L \to \infty$: $\sin \frac{L\pi}{N} \simeq \frac{L\pi}{N}$

recall Taylor exp:

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

so:

$$S_N(L) \underset{L\to\infty}{\simeq} \frac{c}{3} \log_2 \left(\frac{N}{\pi} \frac{L\pi}{N} \right) + c_1 \longrightarrow \boxed{S_N(L) = \frac{c}{3} \log_2 (L) + c_1}$$

f) See attached notebook

Question 2: Free fermions in 2D

a) The same procedure as 1.a) applies:

using
$$\begin{cases} a_{\underline{n}} = \frac{1}{\sqrt{N}} \sum_{\underline{k}} a_{\underline{k}} \, e^{i\frac{2\pi}{N}\underline{k}\underline{n}} = \frac{1}{\sqrt{N}} \sum_{\underline{k}} a_{\underline{k}} \, e^{i\frac{2\pi}{N}(k_x n_x + k_y n_y)} \\ a_{\underline{n}}^{+} = \frac{1}{\sqrt{N}} \sum_{\underline{k}} a_{\underline{k}}^{+} \, e^{-i\frac{2\pi}{N}\underline{k}\underline{n}} = \frac{1}{\sqrt{N}} \sum_{\underline{k}} a_{\underline{k}}^{+} \, e^{-i\frac{2\pi}{N}(k_x n_x + k_y n_y)} \end{cases}$$

we rewrite the original Hamiltonian terms:

$$\begin{cases} a_{\underline{n}}^{+} a_{\underline{n}+x} = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}}^{+} \hat{a}_{\underline{k}} \, e^{i\frac{2\pi}{N}(-k_x n_x - k_y n_y + k_x(n_x+1) + k_y n_y)} \\ \qquad\quad = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}}^{+} \hat{a}_{\underline{k}} \, e^{i\frac{2\pi}{N}k_x} \\ a_{\underline{n}+x} a_{\underline{n}}^{+} = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}} \hat{a}_{\underline{k}}^{+} \, e^{-i\frac{2\pi}{N}k_x} \\ a_{\underline{n}}^{+} a_{\underline{n}+y} = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}}^{+} \hat{a}_{\underline{k}} \, e^{i\frac{2\pi}{N}k_y} \\ a_{\underline{n}+y} a_{\underline{n}}^{+} = \frac{1}{N} \sum_{\underline{k}} \hat{a}_{\underline{k}} \hat{a}_{\underline{k}}^{+} \, e^{-i\frac{2\pi}{N}k_y} \end{cases}$$

plugging these into our original H and rearranging:

$$H = -\sum_{\underline{k}} \hat{a}_{\underline{k}}^{+} \hat{a}_{\underline{k}} \left[\left(e^{i\frac{2\pi}{N}k_x} + e^{-i\frac{2\pi}{N}k_x} \right) - \left(e^{i\frac{2\pi}{N}k_y} + e^{-i\frac{2\pi}{N}k_y} \right) \right.$$

$$H = -\sum_{\underline{k}} \hat{a}_{\underline{k}}^{+} \hat{a}_{\underline{k}} \, 2\left(\cos\left(\frac{2\pi}{N}k_x\right) + \cos\left(\frac{2\pi}{N}k_y\right) \right)$$

$$\boxed{\rightarrow \quad H = \sum_{\underline{k}} e_{\underline{k}} \, \hat{a}_{\underline{k}}^{+} a_{\underline{k}} \quad \text{where} \quad e_{\underline{k}} = -2\cos\left(\frac{2\pi}{N}k_x\right) - 2\cos\left(\frac{2\pi}{N}k_y\right)}$$

b) This is easy to see by looking at the sign of $e_{\underline{k}}$

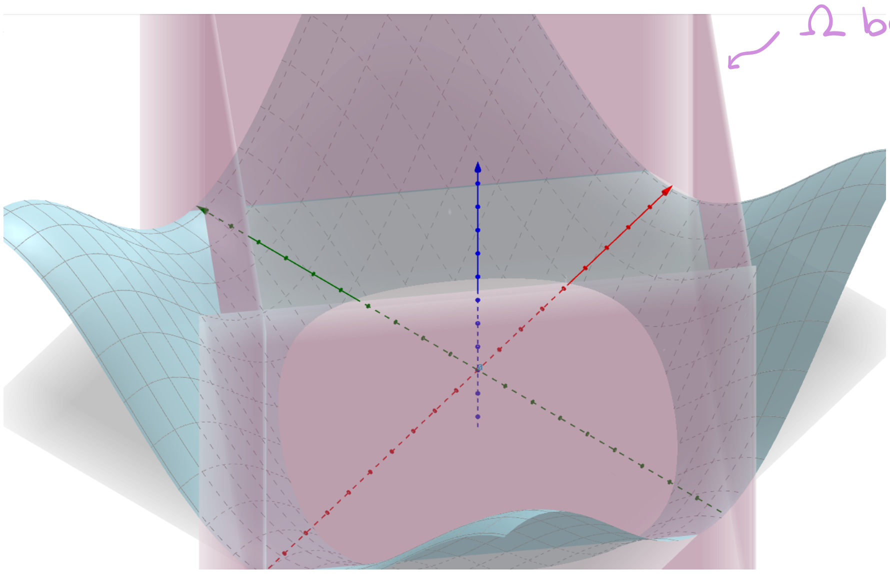
where $e_{\underline{k}} = -2\cos\left(\frac{2\pi}{N}k_x\right) - 2\cos\left(\frac{2\pi}{N}k_y\right)$

for $\underline{k} \in \Omega \longrightarrow e_{\underline{k}} < 0$

for $\underbrace{\underline{k} \notin \Omega}_{\text{note the bounds of } |\underline{k}|} \longrightarrow e_{\underline{k}} > 0$

$$\boxed{\longrightarrow \tilde{M}_{\underline{K},\underline{K}'} = \delta_{\underline{K},\underline{K}'} f(\underline{K}) \quad \text{where} \quad f(K) = \begin{cases} -1 & \underline{K} \in \Omega \\ 1 & \underline{K} \notin \Omega \end{cases}}$$

3D plot of $e_{\underline{K}}$



c) Following the same logic as 1.b :

$$\delta_{\underline{m},0} = \frac{1}{N^2} \sum_{\underline{K}} e^{i \frac{2\pi}{N} \underline{m} \underline{K}} \longrightarrow 0 = \delta_{\underline{m},0} - \frac{1}{N^2} \sum_{\underline{K}} e^{i \frac{2\pi}{N} \underline{m} \underline{K}}$$

$$f(\underline{m}) = \frac{1}{N^2} \sum_{K} f(\underline{K})\, e^{i \frac{2\pi}{N} \underline{m} \cdot \underline{K}}$$

$$= \frac{1}{N^2} \left[- \sum_{\underline{K} \in \tilde{\Omega}} e^{\frac{2\pi i}{N} \underline{m} \underline{K}} + \sum_{\underline{K} \notin \tilde{\Omega}} e^{\frac{2\pi i}{N} \underline{m} \underline{K}} \right] + \delta_{\underline{m},0} - \frac{1}{N^2} \sum_{\underline{K}} e^{i \frac{2\pi}{N} \underline{m} \underline{K}}$$

$$\boxed{\longrightarrow f(\underline{m}) = \delta_{\underline{m},0} - \frac{2}{N} \sum_{\underline{K} \in \tilde{\Omega}} e^{\frac{2\pi i}{N} \underline{m} \underline{K}}}$$

d) $f(\underline{m}) = \delta_{\underline{m},0} - 2 \underbrace{\int_{\underline{q} \in \tilde{\Omega}} \frac{d^2 \underline{q}}{(2\pi)^2} e^{i \underline{m} \cdot \underline{q}}}_{\mathrm{I}}$

I: $\frac{1}{(2\pi)^2} \int_{\underline{q} \in \tilde{\Omega}} dq_x \, dq_y \, e^{i(m_x q_x + m_y q_y)}$

change of variables: $\begin{cases} p_1 = q_x + q_y \\ p_2 = -q_x + q_y \end{cases}$

$\begin{cases} q_y = \frac{1}{2}(p_1 + p_2) \\ q_x = \frac{1}{2}(p_1 - p_2) \end{cases}$

$$dq_x dq_y = \begin{vmatrix} \frac{\partial q_x}{\partial p_1} & \frac{\partial q_x}{\partial p_2} \\ \frac{\partial q_y}{\partial p_1} & \frac{\partial q_y}{\partial p_2} \end{vmatrix} dp_1 dp_2 = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} dp_1 dp_2 = \frac{1}{2} dp_1 dp_2$$

Limits:

$q_y = q_x \pm \pi \longrightarrow \frac{1}{2}(p_1 + p_2) = \frac{1}{2}(p_1 - p_2) \pm \pi$; $p_2 = \pm \pi$

$q_y = -q_x \pm \pi \longrightarrow \frac{1}{2}(p_1 + p_2) = \frac{1}{2}(p_2 - p_1) \pm \pi$; $p_1 = \pm \pi$

$$\frac{1}{(2\pi)^2} \int \frac{1}{2} dp_1 dp_2 \, e^{\frac{i}{2}(m_x(p_1 - p_2) + m_y(p_1 + p_2))} =$$

$$= \frac{1}{(2\pi)^2} \int \frac{1}{2} dp_1 dp_2 \, e^{\frac{i}{2}(p_1(m_x + m_y) - p_2(m_x - m_y))}$$

$$= \frac{1}{8\pi^2} \left[\frac{-2}{i(m_x - m_y)} e^{-i/2 \, p_2 (m_x - m_y)} \right]_{-\pi}^{\pi} \left[\frac{2}{i(m_x + m_y)} e^{i/2 \, p_1 (m_x + m_y)} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi^2(m_x^2 - m_y^2)} \left(e^{-i\frac{\pi}{2}(m_x - m_y)} - e^{i\frac{\pi}{2}(m_x - m_y)} \right) \left(e^{i\frac{\pi}{2}(m_x + m_y)} - e^{-i\frac{\pi}{2}(m_x + m_y)} \right)$$

$$= \frac{-2}{\pi^2(m_x^2 - m_y^2)} \left(\sin\left(\frac{\pi}{2}(m_x - m_y)\right) \cdot \sin\left(\frac{\pi}{2}(m_x + m_y)\right) \right)$$

$$f(\underline{m}) = \delta_{\underline{m},0} - 2 \left[\frac{-2}{\ldots} \left(\sin\left(\frac{\pi}{2}(m_x - m_y)\right) \cdot \sin\left(\frac{\pi}{2}(m_x + m_y)\right) \right) \right]$$

$$f(\underline{m}) = \delta_{\underline{m},0} - 2\left[\frac{\quad}{\pi^2(m_x^2-m_y^2)}\left(\sin\left(\tfrac{\pi}{2}(m_x \;\; m_y)\right) - \sin\left(\tfrac{\pi}{2}(m_x \;\; m_y)\right)\right)\right]$$

$$\begin{cases} \sin\left(\frac{\pi}{2}(m_x+m_y)\right) = 0 & \text{if} \quad m_x+m_y : \text{even} \\ \sin\left(\frac{\pi}{2}(m_x-m_y)\right) = 0 & \text{if} \quad m_x-m_y : \text{even} \end{cases} \rightarrow$$

this happens if m_x, m_y even or m_x, m_y odd

$$\boxed{\rightarrow f(\underline{m}) = \begin{cases} 0 & \text{for } m_x, m_y \text{ either both even or both odd} \\ \dfrac{4(-1)^{m_x}}{\pi^2(m_x^2-m_y^2)} & \text{otherwise} \end{cases}}$$