Solutions to modelling exercises

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Solutions to exercises of List 5 - Modeling 1

Exercise 1. Solutions:
Let us denote

1. x_{ij}: number of units produced by factory U_i for client E_j $i = 1, 2$ and $j = 1, 2, 3$;
2. c_{ij}: cost of sending 1 unit from factory U_i for client E_j $i = 1, 2$ and $j = 1, 2, 3$ (see the table);

According to the statement of the problem we have the following set of restrictions:

1. demands of the clients should be fulfilled.
   (a) $x_{11} + x_{21} = 100$;
   (b) $x_{21} + x_{22} = 200$;
   (c) $x_{31} + x_{32} = 300$.
2. the stokes of the factories must be considered:
   (a) $x_{11} + x_{12} + x_{13} \leq 400$;
   (b) $x_{21} + x_{22} + x_{23} \leq 300$.

The total cost is

$$\sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} x_{ij} = x_{11} + 1.5x_{12} + 3.5x_{13} + 2x_{21} + x_{22} + 2x_{23}.$$

In summary the linear problem to consider is

$$
\begin{aligned}
\min \sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}x_{ij} &= x_{11} + 1.5x_{12} + 3.5x_{13} + 2x_{21} + x_{22} + 2x_{23} \\
\text{s.t} \quad & x_{i1} + x_{21} = 100 \\
& x_{i2} + x_{22} = 200; \\
& x_{i3} + x_{23} = 300; \\
& x_{11} + x_{12} + x_{13} \leq 400; \\
& x_{21} + x_{22} + x_{23} \leq 300; \\
& x_j \in \mathbb{Z}_+ \; i = 1, 2, \;\; j = 1, 2, 3.
\end{aligned}
$$

Exercise 2. Solutions:

Let us denote: p_1 the produced amount of product 1 and p_2 the produced amount of product 2; x_{11} amount of nitrates used for the production of product 1; x_{12} amount of nitrates used for the production of product 2; x_{21} amount of potasio salt used for the production of product 1; x_{22} amount of potasio salt used for the production of product 2; x_{32} amount of phosphates used for the production of product 2 (all non-negative variables that will be measured in kilograms).

According to the statement of the problem we have the following set of restrictions:

1. the stocks of the fertilizers should be considered
   (a) $x_{11} + x_{12} \leq 8$;
   (b) $x_{21} + x_{22} \leq 19$;
   (c) $x_{32} \leq 4$.
2. The required combination of fertilizers to obtain 1 unit of each product is predetermined. Hence we can model the situations as follows: let
   (a) $2x_{11} = x_{21} = p_1 \longrightarrow$ to obtain p_1 units of product 1;
   (b) $3x_{12} = x_{22} = 3x_{32} = p_2 \longrightarrow$ to obtain p_2 units of product 2.

Hence, for a produced amount of p_1 kg of product 1 and p_2 kg of product 2, the profit obtained by selling the production is given by

$$7p_1 + 9p_2 = 7x_{11} + 9x_{22}.$$

In summary, the LPP to be considered in the first question is:

$$\begin{aligned}
\max\ & 7x_{21} + 9x_{22} \\
\text{s.t}\quad & \frac{1}{2}x_{21} + \frac{1}{3}x_{22} \leq 8; \\
& x_{21} + x_{22} \leq 19; \\
& \frac{1}{3}x_{22} \leq 4; \\
& x_{21},\ x_{22} \geq 0.
\end{aligned}$$

Clearly the problem is feasible and bounded above, then it has an optimal solution (x_{21}^*, x_{22}^*) The dual of this problem is

$$\begin{aligned}
\min\ & 8\lambda_1 + 19\lambda_2 + 4\lambda_3 \\
\text{s.t}\quad & \frac{1}{2}\lambda_1 + \lambda_2 \geq 7; \\
& \frac{1}{3}\lambda_1 + \lambda_2 + \frac{1}{3}\lambda_3 \geq 9; \\
& \lambda_1,\ \lambda_2,\ \lambda_3 \geq 0.
\end{aligned}$$

For the second problem, let us denote by p_1 the selling price for the nitrate, p_2 the selling price for the phosphates and p_3 the selling price for the salt of potassio.

Exercise 3. Solutions:

Let us denote: x_{ij} is the number of engines delivered from the factory in city V_i to the factory in city U_j, and by c_{ij} the cost of transporting 1 engine from the factory in city V_i to the factory in city U_j (see the table), $i = 1, 2$ and $j = 1, 2, 3$;

According to the statement of the problem we have the following set of restrictions:

1. the stocks of engines in the factories of cities $V_!$, V_2 and V_3 should be considered:
   (a) $\sum_{j=1}^{3} x_{ij} \leq 6\ i = 1, 2$;
2. the demands of the factories in cities U_1, U_2 and U_3 should be fulfilled:
   (a) $x_{11} + x_{21} \geq 5$;
   (b) $x_{12} + x_{22} \geq 4$;
   (c) $x_{13} + x_{23} \geq 3$;

The total cost of delivering the engines is $\sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij}x_{ij}$

In summary, the LPP to be considered is:

$$\begin{aligned}
\min\ & \sum_{i=}^{2}\sum_{j=1}^{3} c_{ij}x_{ij};\\
\text{s.t}\ & \sum_{j=1}^{3} x_{1j} \le 6;\\
& \sum_{j=1}^{3} x_{2j} \le 6;\\
& x_{11}+x_{21} \ge 5;\\
& x_{12}+x_{22} \ge 4;\\
& x_{13}+x_{23} \ge 3;\\
& x_{ij} \in \mathbb{Z}_+\ i=1,2,\ \ j=1,2,3.
\end{aligned}$$

Exercise 4. Solutions:
Let us denote: s the number of shoes and c the number of belts produced in one day. According to the statement of the problem we have the following set of restrictions:

1. the upper bound on the number of hours dedicated to work should be considered:
   (a) $s/6 + c/5 \le 10$
2. the upper bound on the number of units of leather used in making the products (shoes and belts) should be considered:
   (a) $2s + c \le 78$

The total income due to selling the products is $5s + 4c$
In summary, the LPP to be considered is:

$$\begin{aligned}
\max\ & 5s + 4c.\\
\text{s.t}\ & s/6 + c/5 \le 10\\
& 2s + c \le 78\\
& s, c \in \mathbb{Z},
\end{aligned}$$

Exercise 5. Solutions:
Let us denote the diesels of type A and B by diesel 1 and 2 and by g_1 and g_2 the amounts of gasoline used in diesels 1 and 2, respectively; let x_{ij} be the amount of gasoline i used to produce diesel j $i = 1, \ldots, 3$, $j = 1, 2$.

According to the statement of the problem we have the following set of restrictions:

1. the stocks of the three types of gasoline should be considered:
   (a) $x_{11} + x_{12} \leq 500$;
   (b) $x_{21} + x_{22} \leq 200$;
   (c) $x_{31} + x_{32} \leq 200$;
2. the amounts of each gasoline on each diesel should be considered:
   (a) $x_{11} = .25g_1$;
   (b) $x_{21} = .25g_1$;
   (c) $x_{31} = 0.5g_1$;
   (d) $x_{22} = 0.5g_2$;
   (e) $x_{32} = 0.5g_2$;

In summary, the LPP to be considered is:

$$\begin{aligned} \max\ & 20g_1 + 30g_2; \\ \text{s.t}\quad & .25g_1 \leq 500; \\ & .25g_1 + 0.5g_2 \leq 2000; \\ & 0.5g_1 + 0.5g_2 \leq 4000; \\ & g_1, g_2 \geq 0. \end{aligned}$$

Exercise 6. Solutions:

Let us denote diesels of type A, B and C by diesel 1,2 and 3, respectively; x_{ij} the amount of oil i used to produce diesel j $i = 1, \ldots, 4$, $j = 1, \ldots, 3$; and d_i the amount of oil j produced $j = 1, \ldots, 3$.

According to the statement of the problem we have the following set of restrictions:

1. the stocks of the four type of oils should be considered:
   (a) $x_{21} + x_{22} \leq 2000$;
   (b) $x_{31} + x_{32} \leq 4000$;
   (c) $x_{41} + x_{42} \leq 1000$;
2. the limits on the amount of each oil on each diesel should be considered:
   (a) $x_{11} \leq 0.3d_1$;
   (b) $x_{31} \leq 0.5d_1$;
   (c) $x_{21} \geq 0.4d_2$;
   (d) $x_{22} \geq 0.1d_2$;

(e) $x_{12} \leq 0.5d_2$;

(f) $x_{13} \leq 0.7d_3$;

Other facts: the cost of the production process is $3(x_{11}+x_{12})+6(x_{21}+x_{22})+4(x_{31}+x_{32})+5(x_{41}+x_{42})$, and the total income due to the sale of the diesels is $5.5d_1+4.5d_2$

In summary, the LPP to be considered is:

$$\begin{aligned}
\max\ & 5.5d_1+4.5d_2-3(x_{11}+x_{12})-6(x_{21}+x_{22})-4(x_{31}+x_{32})-5(x_{41}+x_{42}) \\
\text{s.t}\quad & x_{13}+x_{11}+x_{12} \leq 3000; \\
& x_{21}+x_{22} \leq 2000; \\
& x_{31}+x_{32} \leq 4000; \\
& x_{41}+x_{42} \leq 1000; \\
& x_{11} \leq 0.3d_1; \\
& x_{31} \leq 0.5d_1; \\
& x_{21} \leq 0.4d_2; \\
& x_{31} \leq 0.7d_3; \\
& x_{22} \geq 0.1d_2; \\
& x_{12} \geq 0.5d_2; \\
& x_{ij} \geq 0,\ d_1, d_2 \geq 0.
\end{aligned}$$

Clearly we can eliminate variables x_{41} and x_{42} in the above formulation.

Solutions to exercises of List 6 - Modeling 2

Electricity production management. Let us denote by

- $\overline{u}_{i,j}$ the maximal capacity of energy production of thermal plant j in region i per day;
- $\overline{v}_i$ the maximal hydro production in region i per day;
- $\overline{x}_i$ and $\underline{x}_i$ the maximal and minimal levels of reservoir i;
- $x_{t,i}$ the reservoir level at the beginning of day t;
- T the number of days of the planning horizon;
- $u_{t,i,j}$ the production of thermal plant j in region i for day t;
- $v_{t,i}$ the hydro production in region i for day t;
- $s_{t,i}$ the energy bought on the spot market for day t and region i;
- p_t the unit spot price for day t;
- $D_{t,i}$ the demand for day t;
- $A_{t,i}$ the inflows for day t and reservoir i;
- $c_{i,j}$ the unit production cost for thermal plant j in region i;
- $\mathcal{I}$ the set of regions and $\mathcal{T}_i$ the set of thermal units of region i;
- $E_{t,i,j}$ the energy exchanged from region i to region j for day t and G the corresponding graph;
- $\overline{E}_{i,j}$ is the maximal energy exchange from i to j.

We obtain the linear program:

$$
\begin{array}{l}
\min \displaystyle\sum_{t=1}^{T}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{T}_i} c_{i,j}u_{t,i,j} + \sum_{t=1}^{T}\sum_{i\in\mathcal{I}} s_{t,i}p_t\\
x_{t,i} = x_{t-1,i} + 0.8A_{t,i} - v_{t,i},\ t = 2,\ldots,T+1, i\in\mathcal{I},\\
0.2A_{t,i} + \displaystyle\sum_{j\in\mathcal{T}_i} u_{t,i,j} + v_{t,i} + s_{t,i} + \sum_{j|(j,i)\in G} E_{t,j,i} - \sum_{j|(i,j)\in G} E_{t,i,j} \geq \mathcal{D}_{t,i},\\
s_{t,i} \geq 0,\ t = 1,\ldots,T, i\in\mathcal{I},\\
0 \leq u_{t,i,j} \leq \overline{u}_{i,j},\ t = 1,\ldots,T, i\in\mathcal{I},\\
0 \leq v_{t,i} \leq \overline{v}_i,\ t = 1,\ldots,T, i\in\mathcal{I},\\
0 \leq E_{t,i,j} \leq \overline{E}_{i,j},\ t = 1,\ldots,T, (i,j)|(i,j)\in G,\\
\underline{x}_i \leq x_{t,i} \leq \overline{x}_i,\ t = 1,\ldots,T, i\in\mathcal{I},
\end{array}
$$

for $x_{1,i}$, $i \in \mathcal{I}$ given.

Asset management.

Denote by x_i the money invested in asset i $i = 1, \ldots, n$. The model is

$$\begin{aligned} \max & \sum_{i=1}^{n} r_i x_i \\ \text{s.t} \quad & \sum_{i=1}^{n} x_i \leq M \\ & x_i \geq 0, \; i = 1, \ldots, n. \end{aligned}$$

The optimal solution is given by $x_i = M$ where $r_i = \max\{r_j, j = 1, \ldots, n\}$.

Exercise: production expansion.

Let us denote by:

1. d_i: demand of product i with $i = 1, \ldots, n$;
2. h_j and e_j: respectively, the number of regular and extra hours of usage of the machine j with $j = 1, \ldots, m$;
3. s_i: quantity of product i unsold;
4. $h_{i,j}$: number of hours machine j is used to produce product i;
5. x_i: quantity of product i produced.

The linear program is

$$\begin{aligned} & \min \sum_{i=1}^{n} \sum_{j=1}^{m} g_{i,j} h_{i,j} + \sum_{j=1}^{m} c_j e_j + \sum_{i=1}^{n} p_i s_i \\ & h_{i,j}, e_j, s_i \geq 0, \forall\, i = 1, \ldots, n, j = 1, \ldots, m, \\ & \sum_{j=1}^{m} t_j (e_j + h_j) \leq T, \\ & x_i + s_i = d_i, \; i = 1, \ldots, n, \\ & x_i = \sum_{j=1}^{m} h_{i,j} a_{i,j}, \; i = 1, \ldots, n, \\ & h_j + e_j \leq u_j, \; j = 1, \ldots, m, \\ & \sum_{i=1}^{n} h_{i,j} = h_j + e_j, \; j = 1, \ldots, m. \end{aligned}$$

LNG with cancellation options. Denote

- s_P^t the pipeline storage at the beginning of time t;

- S_t the spot price for month t;
- Q_t the quantity of contracted gas for t, one part of it, $u_P^{0,t}$, being injected in the pipeline and the rest $u_S^{1,t}$ being injected in the storage;
- u_n^t: quantity of gas from ship of contract n to storage at t;
- $u_n^{0,t}$: quantity of gas from ship of contract n to pipeline at t;
- u_P^t: quantity of gas transferred from the pipeline to clients at t;
- u_{NS}^t: quantity of unsatisfied demand at t;
- c_t: unit storage cost at t;
- p_t: cost of unsastisfied demand;
- s_n^t: stock in ship of contract n at $t \geq t_n - 1$;
- $\mathcal{D}_t$: demand at t,
- t_n: time ship of contract n will arrive;
- y_n^t is 1 if contract n is cancelled at t or before and 0 otherwise;
- Q_n: load of contract n;
- C_n^t: cancellation cost at t;
- F_n^t: ship load n cost at t;
- $f_{n,t}$: unit price for cancellation fee paid at t on contract n;

The model is

$$
\begin{array}{l}
\min \displaystyle\sum_{t=1}^{T}\left(c_t(s^t+s^t_P)+p_t u^t_{NS}-1.3S_t u^t_P+\sum_{n=1}^{N}C^t_n+F^t_n\right)\\
Q_t=u^{0,t}_P+u^{1,t}_S, \forall\, t=1.\dots,T,\\
\underline{s}_t\le s^t\le \overline{s}_t, \forall\, t=1.\dots,T,\\
s^t=s^{t-1}+\sum_{n=1}^{N}u^t_n+u^{1,t}_s-u^{2,t}_s, \forall\, t=1.\dots,T,\\
\underline{s}_{P,t}\le s^t_P\le \overline{s}_{P,t}, \forall\, t=1.\dots,T,\\
s^t_P=s^{t-1}_P+\sum_{n=1}^{N}u^{0,t}_n+u^{2,t}_s+u^{0,t}_P-u^t_P, \forall\, t=1.\dots,T,\\
u^t_P+u^t_{NS}=\mathcal{D}_t,\ t=1.\dots,T,\\
y^t_n\in\{0,1\},\\
y^t_n\ge y^{t-1}_n, 2\le t\le t_n-1,\\
s^t_n=(1-y^t_n)Q_n, t=t_n-1,\\
s^t_n=s^{t-1}_n-u^t_n u^{0,t}_n, t\ge t_n,\\
C^t_1=y^1_n Q_n f_{n,1},\\
C^t_n=(y^t_n-y^{t-1}_n)Q_n f_{n,t}, 2\le t\le t_n-1\\
F^t_n=(1-y^{t-1}_n S_t Q_n,\ t=t_n,\\
F^t_n=c_t s^t_n,\ t\ge t_n+1.
\end{array}
$$

Exercise: Problema da mochila

Let us denote, for $i=1,\dots,n$:

- x_i a binary variable that indicates whether Mickey is taking object i (value 1) or not (value 0) $i=1,\dots,n$;
- u_i is the utility of object i;
- p_i the weight of object i;
- v_i volume of object i.

The LPP problem to be considered is

$$
\begin{array}{ll}
\max & \displaystyle\sum_{i=1}^{n}u_i x_i\\
\text{s.t} & \displaystyle\sum_{i=1}^{n}p_i x_i\le P\\
& x_i\in\{0,1\}\ i=1,\dots,n.
\end{array}
$$

If we add the condition on the volumes the LPP to be consider is

$$\begin{aligned}
\min \sum_{i=1}^{n} u_i x_i \\
\text{s.t} \quad \sum_{i=1}^{n} p_i x_i \leq P \\
\sum_{i=1}^{n} v_i x_i \leq V \\
x_i \in \{0,1\}\ i = 1, \ldots, n.
\end{aligned}$$

“Unit commitment”

Let us denote

- $y_{t,i}$ is 1 if unit i is turned on at t and 0 otherwise;
- $u_{t,i}$: production of unit i at t;
- $\mathcal{D}_t$: demand at t;
- K_i: maximal production of unit i per step.

The model is

$$\begin{array}{l}
\min\ \sum_{t,i} c_i u_{t,i} + \sum_i p_i y_{1,i} + \sum_{t\geq 2,i} (y_{t,i} - y_{t-1}, i) p_i \\
y_{t,i} \geq y_{t-1,i}, t = 2, \ldots, T, \\
y_{t,i} \in \{0,1\}, \\
0 \leq u_{t,i} \leq y_{t,i} K_i, \\
\sum_i u_{t,i} = \mathcal{D}_t,
\end{array}$$