R03 Probability Concepts

Version 1.0

1. Introduction, Probability Concepts and Odds Ratios

Since many investment decisions are made in an environment of uncertainty, it is essential for portfolio managers and investment managers to have a fundamental grasp of probability concepts. In this reading, we will focus on:

- Definitions and rules related to probability
- Expected value and variance
- Covariance and correlation

1.1 Probability, Expected Value, and Variance

Fundamental Concepts

A *random variable* is an uncertain quantity/number. For example, when you roll a die, the result is a random variable.

An *outcome* is the observed value of a random variable. For example, if you roll a 2, it is an outcome.

An *event* can be a single outcome or a set of outcomes. For example, you can define an event as rolling a 2 or rolling an even number.

Mutually exclusive events are events that cannot happen at the same time. For example, rolling a 2 and rolling a 3 are examples of mutually exclusive events. They cannot happen at the same time.

Exhaustive events are those that cover all possible outcomes. For example, 'rolling an even number' or' rolling an odd number' are exhaustive events. They cover all possible outcomes.

The *two defining properties of probability* are:

- The probability of any event has to be between 0 and 1.
- The sum of the probabilities of mutually exclusive and exhaustive events is equal to 1.

Ways of Estimating Probability

The methods of estimating probabilities are:

- *Empirical probability*: Based on analyzing the frequency of an event's occurrence in the past.
- *A priori probability*: Based on formal reasoning and inspection rather than personal judgment.
- *Subjective probability*: Informed guess based on personal judgment.

Empirical and a priori probabilities are often grouped as objective probabilities because they do not vary from person to person.

Probability Stated as Odds

Odds for an event are defined as the probability of the event occurring to the probability of the event not occurring. Odds for E = P(E) / [1 – P(E)].

Given odds for E of "a to b", the implied probability of E is a / (a + b).

Example

If the probability of an event is 0.2, what are the odds of it occurring? Alternatively, if the odds are 1 to 4, what is the probability of this event?

Solution:

The odds of the event occurring are = $\frac{0.2}{0.8}$ = 1/4. This is stated as odds of 1 to 4.

Given the odds, the probability of the event occurring is $\frac{1}{1+4} = \frac{1}{5}$ = 0.20.

Odds against an event are defined as the probability of the event not occurring to the probability of the event occurring. Odds against E = [1 – P(E)] / P(E).

Give odds against E of "a to b", the implied probability of E is b / (a + b).

Example

If P(E) = 0.2, what are the odds against the event occurring? If the odds against an event are 4 to 1, what is the probability of the event?

Solution:

P(E) = $\frac{0.8}{0.2} = \frac{4}{1}$. Hence the odds against E are 4 to 1.

Given the odds against an event, the probability of the event is $\frac{1}{4+1}$ = 0.2

2. Conditional and Joint Probability

Conditional v/s Unconditional probabilities

Unconditional probability is the probability of an event occurring irrespective of the occurrence of other events. It is denoted as P(A). Unconditional probability is also called 'marginal' probability.

Conditional probability is the probability of an event occurring given that another event has occurred. It is denoted as P(A|B), which is the probability of event A given that event B has occurred.

Joint Probability and Multiplication Rule

Multiplication rule is used to determine the joint probability of two events. It is expressed as:

P(AB) = P(A|B) P(B)

Rearranging the equation, we get the formula for computing conditional probabilities:

P(A|B) = P(AB) / P(B)

Example

P(interest rates will decrease) = P(D) = 40%
P(stock price increases) = P(S)
P(stock price will increase given interest rates decrease) = P(S|D) = 70%

Compute probability of a stock price increase **and** an interest rate decrease.

Solution:

P(SD) = P(S|D) x P(D) = 0.7 x 0.4 = 0.28 = 28%

Addition Rule for Probabilities

Addition rule is used to determine the probability that at least one of the events will occur. It is expressed as:

P(A or B) = P(A) + P(B) – P(AB)

P(AB) represents the joint probability that both A and B will occur. It is subtracted from the sum of the unconditional probabilities: P(A) + P(B), to avoid double counting.

If the two events are mutually exclusive, the joint probability: P(AB) is zero and the probability that either A or B will occur is simply the sum of the unconditional probabilities for each event:

P(A or B) = P(A) + P(B)

Example

P(price of A increases) = P(A) = 0.5
P(price of B increases) = P(B) = 0.7
P(price of A and B increases) = P(AB) = 0.3

Compute the probability that the price of stock A **or** the price of stock B increases.

Solution

P(A or B) = 0.5 + 0.7 – 0.3 = 0.9

Independent and Dependent Events

If the occurrence of one event does not influence the occurrence of the other event, then the two events are called *independent events.*

i.e. P(A|B) = P(A) or P(B|A) = P(B)

Multiplication rule for independent events: P(AB) = P(A) P(B)

Addition rule for independent events: P(A or B) = P(A) + P(B) – P(AB). (The addition rule does not change.)

If the probability of an event is affected by the occurrence of another event, then it is called a *dependent event.*

Total Probability Rule

The total probability rule is used to calculate the unconditional probability of an event, given conditional probabilities.

In investment analysis, we often formulate a set of mutually exclusive and exhaustive scenarios and then estimate the probability of a particular event. For example, let's say that we have two scenarios S and non-S that are mutually exclusive and exhaustive.

According to the total probability rule, the probability of any event P(A) can be expressed as:

$$P(A) = P(AS) + P(AS^C)$$

Using the multiplication rule we get,

$$P(A) = P(A|S)\,P(S) + P(A|S^C)\,P(S^C)$$

If we have more than two scenarios, we can generalize this equation to:

$$P(A) = P(AS_1) + P(AS_2) + \ldots + P(AS_n) = P(A|S_1)\,P(S_1) + P(A|S_2)\,P(S_2) + \ldots + P(A|S_n)\,P(S_n)$$

3. Expected Value (Mean), Variance, and Conditional Measures of Expected Value and Variance

Expected Value of a Random Variable

The expected value of a random variable can be defined as the probability-weighted average of the possible outcomes of the random variable. For a random variable X, the expected value of X is denoted as E(X) and is calculated as:

$$E(X) = \sum_{i=1}^{n} P(X_i)\,X_i$$

where:
X_i = One of n possible outcomes of the random variable X
$P(X_i)$ = Probability of X_i

Variance of a Random Variable

The expected value is our forecast, but we cannot count on the individual forecast being realized. This is why we need to measure the risk we face. Variance and standard deviation are examples of how we can measure this risk. The variance of a random variable is the probability-weighted sum of the squared differences between each possible outcome and the expected value of the random variable. It is expressed as:

$$\sigma^2(X) = \sum_{i=1}^{n} P(X)\,[X - E(X)]^2$$

Variance is a number greater than or equal to 0 because it is the sum of squared terms. If variance is 0, there is no dispersion or risk. The outcome is certain and the quantity X is not random at all. Standard deviation is the positive square root of variance.

We can calculate the expected value and variance of a random variable using a financial calculator as shown below:

Example

A project's cash flow for the upcoming year depends on the state of the economy, as shown in the table below. What is the variance of the cash flow? What is the standard deviation?

State of Economy	**Probability**	**Cash Flow**
Good	0.3	50
Average	0.5	40
Weak	0.2	20

Solution:

Using a financial calculator:

Keystrokes	***Explanation***	***Display***
[2nd] [DATA]	Enters data entry mode	
[2nd] [CLR WRK]	Clears data register	X01
50 [ENTER]	1st possible value of random variable	X01 = 50
[↓] 30 [ENTER]	Probability of 30% for X01	Y01 = 30
[↓] 40 [ENTER]	2nd possible value of random variable	X02 = 40
[↓] 50 [ENTER]	Probability of 50% for X02	Y02 = 50
[↓] 20 [ENTER]	3rd possible value of random variable	X03 = 20
[↓] 20 [ENTER]	Probability of 20% for X03	Y03 = 20
[2nd] [STAT]	Puts calculator into stats mode	
[2nd] [SET]	Press repeatedly till you see →	1-V
[↓]	Total number of entries	N = 100
[↓]	Expected value of random variable	X = 39
[↓]	Sample standard deviation	Sx = 10.49
[↓]	Population standard deviation	σx = 10.44

We can then square the population standard deviation of 10.44 to get the variance i.e. 10.44^2 = 109.00

Total Probability Rule for Expected Value

Just like the total probability rule states unconditional probabilities in terms of conditional probabilities, the total probability rule for expected values states unconditional expected values in terms of conditional expected values.

$$E(X|S) = P(X_1|S)\, X_1 + P(X_2|S)\, X_2 + \ldots + P(X_n|S)\, X_n$$

Instructor's Note

Notice that this formula is exactly similar to the total probability rule formula.

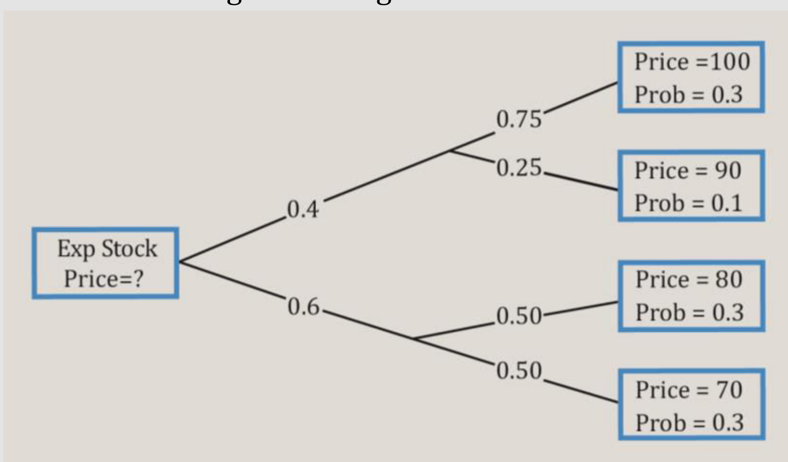
$$P(A) = P(A|S_1)\, P(S_1) + P(A|S_2)\, P(S_2) + \ldots + P(A|S_n)\, P(S_n)$$

Example

What is the expected price of a stock at the end of the current period given the following information: probability that interest rates will decline = 0.4. If interest rates decline there is a 75% chance that stock price will be \$100 versus a 25% chance that the stock price will be \$90. If interest rates do not decline there is a 50% chance that the stock price will be \$80 versus a 50% chance that stock price will be \$70.

Solution:

We can plot the probabilities using a tree diagram.



Consider the first node (top right). It refers to the probability that the stock price will be \$100 given a decline in interest rates. We can calculate the probability of that happening by multiplying the probability of a decline in interest rates (0.4) by the probability of the stock price being \$100 if that happens (0.75). This gives us a conditional probability of 0.30. In short, it is the joint probability of the stock price being \$100 given a decline in interest rates. Similarly, probabilities are calculated for each of the other three nodes. We can then calculate:

E(Price | decline in interest rates) = 0.75 (\$100) + 0.25 (\$90) = \$97.50

E(Price | no decline in interest rates) = 0.50 (\$80) + 0.50 (\$70) = \$75.00

Now we use the total probability rule for expected value of stock price at the end of the current period:

E(Price) = E(Price | decline in interest rates) P(decline in interest rates) + E(Price | no decline in interest rates) P(no decline in interest rates)

E(Price) = $97.50 (0.40) + $75.00 (0.60)
E(Price) = $84.00

4. Expected Value, Variance, Standard Deviation, Covariances, and Correlations of Portfolio Returns

Expected Return

A portfolio's expected return can be calculated as:

$$E(R_P) = w_1 E(R_1) + w_2 E(R_2) + \ldots + w_n E(R_n)$$

where:
w_n = portfolio weight of n^{th} security in the portfolio
R_n = expected return of n^{th} security in the portfolio
n = number of securities in the portfolio

We will discuss portfolio expected return and variance of return using a two-stock portfolio.

Example

40% of the portfolio is invested in Stock A and 60% is invested in Stock B. As shown in the table below, the expected return of each stock depends on the economic scenario.

Scenario	**P(Scenario)**	**Expected returns of A**	**Expected returns of B**
Recession	0.25	2%	4%
Normal	0.50	8%	10%
Boom	0.25	12%	16%

This information can also be presented as a joint probability function of A's and B's returns:

	R_B = 4%	**R_B = 10%**	**R_B = 16%**
R_A = 2%	0.25	0	0
R_A = 8%	0	0.50	0
R_A = 12%	0	0	0.25

Row 1 and Column 1 represent the returns of A and B respectively. The other cells contain probabilities. Calculate the expected return of A and B.

Solution:

Given the data presented above:

The expected return of A is: 0.25 x 2 + 0.50 x 8 + 0.25 x 12 = 7.5%.

The expected return of B is: 0.25 x 4 + 0.50 x 10 + 0.25 x 16 = 10%.

Expected return of the portfolio = weight of A in the portfolio x expected return of A + weight of stock B in the portfolio x expected return of B = 0.4 x 7.5 + 0.6 x 10 = 9%

The expected portfolio return is 9%. As the term implies, this is the expected return. The actual return will vary around 9%. The amount of variability is measured by the variance. In order to determine the variance of return, we must first calculate the covariance.

Covariance

Covariance tells us how movements in a random variable vary with movements in another random variable, whereas variance tells us how a random variable varies with itself. Assume there are two random variables R_i and R_j. The forward-looking, population covariance between R_i and R_j (used to measure how they move together) is given by:

$$\text{Cov}\left(R_i, R_j\right) = E\left([R_i - ER_i][R_j - ER_j]\right)$$

where:
ER_i = expected return for variable R_i
ER_j = expected return for variable R_j

If the random variables are returns, the units of both forward-looking covariance and historical variance would be returns squared.

The sample covariance between two random variables Ri and Rj is the average value of the product of the deviations of observations on two random variables from their sample means. It is calculated as follows:

$$\text{Cov}\left(R_i, R_j\right) = \sum_{i=1}^{n}\left(R_{i,t} - \bar{R}_i\right)\left(R_{j,t} - \bar{R}_j\right)\Big/(n-1)$$

Unlike the population covariance, the sample covariance is based on historical data set. This reading focuses on covariance in a forward-looking sense.

Example

Continuing with our previous example, calculate the covariance of returns between A and B.

Solution:

Say R_i represents the return on A and R_j represents the return on B, we have already calculated the expected returns of A and B as 7.5% and 10% respectively. The covariance of returns is:

E [(Ri – 7.5) (Rj – 10)]

= 0.25(2% - 7.5%)(4% - 10%) + 0.5(8% - 7.5%)(10% - 10%) + 0.25(12% - 7.5%)(16% - 10%)

= 0.000825 + 0 + 0.000675 = 0.0015

Correlation

The problem with covariance is that it can vary from negative infinity to positive infinity which makes it difficult to interpret. To address this problem, we use another measure called correlation. Correlation is a standardized measure of the linear relationship between two variables with values ranging between -1 and +1.

- A correlation of 0 (uncorrelated variables) indicates an absence of any linear (straight-line) relationship between the variables.
- A correlation of +1 indicates a perfect positive relationship.
- A correlation of -1 indicates a perfect negative relationship.

It is computed as:

$$\rho(R_i, R_j) = \mathrm{Cov}(R_i, R_j)/\sigma(R_i)\,\sigma(R_j)$$

The above correlation measure is forward-looking as it is calculated by dividing the forward-looking covariance by the product of forward-looking standard deviations.

Historical or sample correlation = Historical or sample covariance between two variables / (Sample standard deviation of variable Ri * Sample standard deviation of variable Rj)

We will now apply this formula to calculate the correlation between the returns of A and B from our example. We have already shown that the covariance of returns is 0.0015. In order to calculate the correlation, we need the standard deviation of A and B. Using a financial calculator, we can determine that the standard deviation of A is 0.0357 and the standard deviation of B is 0.0424. The correlation, $\rho(A, B) = \frac{\mathrm{Cov}(A,B)}{\sigma(A)\sigma(B)}$ = 0.0015/(0.0357 x 0.0424) = 0.99.

The correlation of 0.99 (almost 1) implies a very strong positive relationship between the returns of A and B. This is more meaningful than the covariance number of 0.0015 which tells us that there is a positive relationship between the returns of A and B but does not give a sense for the strength of the relationship.

5. Covariance Given a Joint Probability Function

Given two random variables R_i and R_j, the covariance between R_i and R_j is given by:

$$\mathrm{Cov}(R_i, R_j) = E[(R_i - ER_i)(R_j - ER_j)]$$

where:
ER_i = expected return for variable R_i
ER_j = expected return for variable R_j

Definition of Independence for Random Variables. Two random variables X and Y are independent if and only if P(X,Y) = P(X)P(Y).

For example, given independence, P(5,6) = P(5)P(6). Joint probabilities are obtained by multiplying the individual probabilities. Independence is a stronger property than uncorrelatedness because correlation addresses only linear relationships.

The following condition holds for independent random variables and, therefore, also holds for uncorrelated random variables.

Multiplication Rule for Expected Value of the Product of Uncorrelated Random Variables.

The expected value of the product of uncorrelated random variables is the product of their expected values.

E(XY) = E(X)E(Y) if X and Y are uncorrelated.

Many financial variables, such as revenue (price times quantity), are the product of random quantities.

Variance of returns

Once we know the covariance, we can calculate the variance of a portfolio using this formula:

$$\sigma^2(R_P) = w_1^2\sigma_1^2\,(R_1) + w_2^2\sigma_2^2\,(R_2) + 2w_1w_2\text{Cov}\,(R_1R_2)$$

Example

Continuing with our example, the variance of the portfolio is

- Weight of the first asset, w_1 = 0.40
- Weight of the second asset, w_2 = 0.60
- Standard deviation of first asset = 0.0357
- Standard deviation of second asset = 0.0424
- Covariance between the two assets = 0.0015

Variance of the portfolio = 0.4^2 x 0.0357^2 + 0.6^2 x 0.0424^2 + 2 x 0.4 x 0.6 x 0.0015 = 0.00157

Standard deviation of the portfolio = $\sqrt{0.00157} = 0.0396$

6. Bayes' Formula

Bayes' formula is a rational method for updating or adjusting the probability of an event based on new information. According to Bayes' formula, the updated probability of an event given new information is:

$$\text{P(Event}\,|\,\text{Information)} = \frac{\text{P(Information}\,|\,\text{Event)}}{\text{P(Information)}} \times \text{P(Event)}$$

Example

Consider a factory that has three assembly lines. The percentage of output produced at each assembly line is as follows: Line A = 45%, Line B = 35%, Line C = 20%. The output defective from each line is estimated to be 3%, 5%, and 4%, respectively. Given that the product is defective, what is the probability that it came from Line C?

Solution:

When dealing with questions related to Bayes' formula, the first step is to reproduce the information in probability notation:

P(Line A) = 0.45; P(Not Line A) = 0.55

P(Line B) = 0.35; P(Not Line B) = 0.65

P(Line C) = 0.20; P(Not Line C) = 0.80

P(Defective | Line A) = 0.03, P(Defective | Line B) = 0.05, P(Defective | Line C) = 0.04

P(Defective) = 0.45 x 0.03 + 0.35 x 0.05 + 0.20 x 0.04 = 0.039

Next write down the Bayes formula:

$$\text{P(Event}\,|\,\text{Information)} = \frac{\text{P(Information}\,|\,\text{Event)}}{\text{P(Information)}} \text{ x P(Event)}$$

We then have to distinguish between the event and the information and plug the relevant values into the formula. In this case, the information is that the product is defective. Hence, the formula can be written as:

$$\text{P(Line C | Defective)} = \frac{[\text{P(Defective | Line C)} * \text{P(Line C)}]}{\text{P(Defective)}} = \frac{0.04 * 0.20}{0.039} = 20.51\%$$

7. Principles of Counting

In counting, enumeration (counting the outcomes one by one) is the most basic resource. This process is difficult and is prone to error. We will discuss shortcuts and principles of counting, which make the process easier.

Multiplication Rule of Counting

The first of these principles is the multiplication rule**.** It states that 'if one task can be done in n_1 ways, and a second task, given the first, can be done in n_2 ways, and a third task, given the first two tasks, can be done in n_3 ways and so on for k tasks, then the number of ways the k tasks can be done is (n_1) (n_2) (n_3)… (n_k). So the multiplication rule for counting can be expressed as:

Number of ways of doing k tasks = n_1 x n_2 x n_3 … n_k

where:
n_1 = number of ways of doing the first task,
n_2 = number of ways of doing the second task and so on

Example

Consider a simple example. Suppose we have three steps in an investment decision process.

The first step can be done in 2 ways, the second step can be done in 4 ways and the third in 3 ways. In how many ways can the investment decision be made?

Solution:

Following the multiplication rule, there are (2) (4) (3) = 24 ways of making the investment decision.

Notice that there are three groupings in this problem. From each group, only one step can be selected.

Factorial

Another counting principle relates to the assignment of members of a group to an equal number of positions. The number of ways we can assign every member of a group of size n to n slots is n! (read as *n factorial*) = n (n - 1) (n - 2) (n - 3) … 1. By convention, 0! = 1. The difference between the multiplication rule and factorial is that there is only one group in a factorial. It involves arranging the set of items within the group and the order in which the items are arranged matters. The formula is:

Number of ways of assigning group of size *n* to *n* tasks = n!

Example

There are five equity analysts covering five emerging countries. In how many ways can the countries be assigned to the analysts?

Solution:

The total number of ways the assignments can be made = 5! = 120

Labeling

Labeling refers to the number of ways that *n* items can be labelled with *k* different labels with *n*, of the first type, *n* of the second type, and so on. This can be expressed as:

Number of ways in which *n* items can be labelled using *k* labels =

$$\frac{n!}{[(n_1!)(n_2!)\ldots(n_k!)]}$$

Example

A portfolio consists of eight stocks. The goal is to designate four of the stocks as "long-term holds," three of the stocks as "short-term holds," and one stock a "sell." How many ways can these labels be assigned to the eight stocks?

Solution:

Notice that there are eight items (stocks) that are to be labelled in three different ways.

$$\frac{8!}{4! \times 3! \times 1!} = 280$$

Combination

A special case of the labelling is the combination formula. It is the number of ways to choose *r* objects from a total of *n* objects, when the order in which the *r* objects are listed does not matter. This can be expressed as:

$${}_{n}C_{r} = n! / [(n - r)!\, r!]$$

where:
n = number of objects
r = number of objects chosen from n objects
${}_{n}C_{r}$ = number of ways to choose r objects from n objects where order does not matter

Example

A portfolio manager wants to eliminate four stocks from a portfolio that consists of six stocks. How many ways can the four stocks be sold when the order of the sale is NOT important?

Solution:

Using the formula for combination, we get the number of ways the four stocks can be sold

= 6! / [(6 – 4)! 4!] = 15.

Permutation

Permutation is the number of ways to choose *r* objects from a total of *n* objects, when the order in which the r objects are listed does matter. It is expressed as:

$${}_{n}P_{r} = \frac{n!}{(n - r)!}$$

where:
n = number of objects
r = number of objects chosen from n objects

Example

Assume that in a portfolio of eight stocks, we decide to sell three stocks. How many ways can we choose three of the eight to sell if the order of sale does matter?

Solution:

Using the formula for permutation, we can find the number of ways to sell three of the eight stocks where order matters:

$$\frac{8!}{(8-3)!} = 336$$

Instructor's Note: given a problem, use the following pointers to identify the correct counting method to apply.

- Factorial: if there is **one** group of size n and n items/tasks to be assigned, number of ways = n!
- Labeling: used when there are three or more labels. Each item/member of a group must be applied a label.
- Combination: used when there are two groups of a certain size, say n and r. Use combination when the order of choosing r objects from n objects does **NOT** matter.
- Permutation: used when there are two groups of a certain size, say n and r. Use permutation when the order of choosing r objects from n objects does matter.
- Combination and permutation functions are available on the financial calculator and should be used rather than the formula.

Summary

LO.a: Define a random variable, an outcome, and an event.

- A random variable is an uncertain quantity/number.
- An outcome is the observed value of a random variable.
- An event can be a single outcome or a set of outcomes.

LO.b: Identify the two defining properties of probability, including mutually exclusive and exhaustive events, and compare and contrast empirical, subjective, and a priori probabilities;.

The two defining properties of a probability are:

- The probability of any event E is a number between 0 and 1: $0 \leq P(E) \leq 1$.
- The sum of the probabilities of any set of mutually exclusive and exhaustive events equals 1.

Mutually exclusive events are events that cannot happen at the same time. Exhaustive events are those that include all possible outcomes.

The methods of estimating probabilities are:

- Empirical probability: Based on analyzing the frequency of an event's occurrence in the past.
- A priori probability: Based on formal reasoning and inspection rather than personal judgment.
- Subjective probability: Informed guess based on personal judgment.

LO.c: Describe the probability of an event in terms of odds for and against the event.

Odds for E = P(E) / [1 – P(E)].

Odds against E = [1 – P(E)] / P(E).

LO.d: Calculate and interpret conditional probabilities.

Unconditional probability (marginal probability) is the probability of an event occurring irrespective of the occurrence of other events. It is denoted as P(A).

Conditional probability is the probability of an event occurring given that another event has occurred. It is denoted as P(A|B), which is the probability of event A given that event B has occurred.

LO.e: Demonstrate the application of the multiplication and addition rules for probability.

Multiplication rule is used to determine the joint probability of two events. It is expressed as:

$$P(AB) = P(A|B)\ P(B)$$

Addition rule is used to determine the probability that at least one of the events will occur. It is expressed as:

$P(A \text{ or } B) = P(A) + P(B) - P(AB)$

The total probability rule is used to calculate the unconditional probability of an event, given conditional probabilities. It is expressed as:

$P(A) = P(A|S_1)\,P(S_1) + P(A|S_2)\,P(S_2) + \dots + P(A|S_n)\,P(S_n)$

LO.f: Compare and contrast dependent and independent events.

If the occurrence of one event does not influence the occurrence of the other event, then the events are called independent events.

i.e. $P(A|B) = P(A)$ or $P(B|A) = P(B)$

If the probability of an event is affected by the occurrence of another event then it is called a dependent event.

LO.g: Calculate and interpret an unconditional probability using the total probability rule.

Using the total probability rule the unconditional probability of A can be computed as:

$P(A) = P(A|S_1)\,P(S_1) + P(A|S_2)\,P(S_2) + \dots + P(A|S_n)\,P(S_n)$

Where S_1, S_2 .. S_n are mutually exclusive and exhaustive events.

LO.h: Calculate and interpret the expected value, variance, and standard deviation of random variables.

The expected value of a random variable can be defined as the probability-weighted average of the possible outcomes of the random variable.

$$E(X) = \sum_{i=1}^{n} P(X_i)\,X_i$$

The variance of a random variable is the probability-weighted sum of the squared differences between each possible outcome and the expected value of the random variable.

$$\sigma^2(X) = \sum_{i=1}^{n} P(X)\,[X - E(X)]^2$$

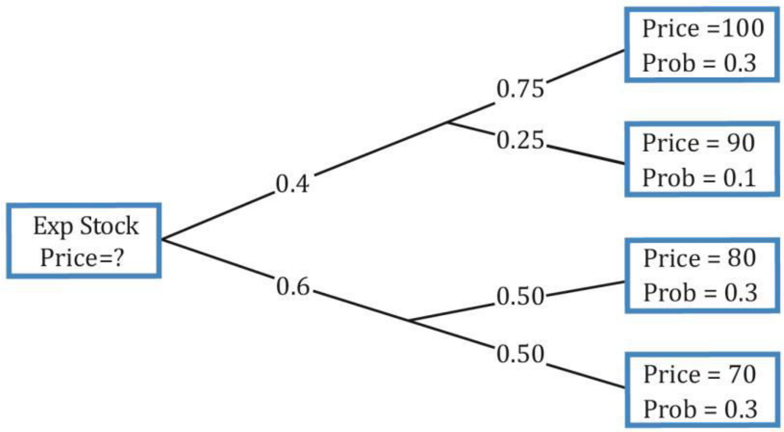
The square root of variance will give us the standard deviation.

LO.i: Explain the use of conditional expectation in investment applications.

The expected value is heavily used in investment applications e.g. forecasting EPS of a company, estimating rewards of alternative investments, etc. When you refine your expectations or forecasts, you are typically making adjustments based on new information or events; this is actually the use of conditional expected values.

LO.j: Interpret a probability tree and demonstrate its application to investment problems.

The tree diagram in an investment problem can help plot the probabilities of various outcomes and depict expected values based on the paths chosen and selection made at each node. For example:



LO.k: Calculate and interpret the expected value, variance, standard deviation, covariances, and correlations of portfolio returns.

The formula for calculating the expected portfolio return is:

$$E(R_P) = w_1 E(R_1) + w_2 E(R_2) + \ldots + w_n E(R_n)$$

Variance can be computed as:

$$\sigma^2(R_P) = w_1^2\sigma_1^2 (R_1) + w_2^2\sigma_2^2 (R_2) + 2w_1w_2Cov(R_1R_2)$$

LO.l: Calculate and interpret the covariances of portfolio returns using the joint probability function.

Given two random variables R_i and R_j, the covariance between R_i and R_j is given by:

$$Cov(R_i, R_j) = E[(R_i - ER_i) (R_j - ER_j)]$$

where:
ER_i = expected return for variable R_i
ER_j = expected return for variable R_j

LO.m: Calculate and interpret an updated probability using Bayes' formula.

Bayes' formula is a rational method for updating or adjusting the probability of an event based on new information. According to Bayes' formula, the updated probability of an event given new information is:

$$\text{P(Event} \mid \text{Information)} = \frac{\text{P(Information} \mid \text{Event)}}{\text{P(Information)}} \times \text{P(Event)}$$

LO.n: Identify the most appropriate method to solve a particular counting problem and analyze counting problems using factorial, combination, and permutation concepts.

The number of ways we can assign every member of a group of size n to n slots is n!

Number of ways in which *n* items can be labelled using *k* labels = $\frac{n!}{[(n_1!)(n_2!)\ldots(n_k!)]}$

The combination formula gives the number of ways to choose r objects from a total of n objects, when the order in which the r objects are listed does <u>not</u> matter.

$${}^{n}C_{r} = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

The permutation formula gives the number of ways to choose r objects from a total of n objects, when the order in which the r objects are listed does matter.

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

