R43 Understanding Fixed-Income Risk and Return

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Version 1.0

1. Introduction

This reading covers:

- Sources of return on a bond: reinvestment of coupon payments and receipt of principal.
- How investors are exposed to different interest rate risk if the same bond is held for different time periods.
- Duration and convexity as measures of interest rate risk.
- How duration and convexity can be used to predict a change in a bond's price, and assess credit and liquidity risks.

2. Sources of Return

The total return is the future value of reinvested coupon interest payments and the sale price (or redemption of principal if the bond is held to maturity). The horizon yield (or holding period rate of return) is the internal rate of return between the total return and the purchase price of the bond.

Total return on a bond = reinvested coupon interest payments + sale/redemption of principal at maturity

A bond investor has three sources of return:

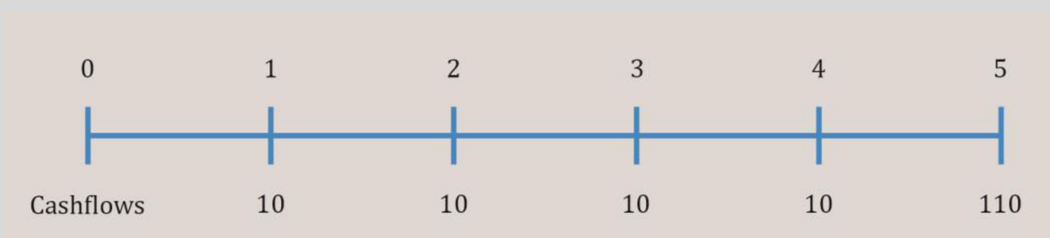
- Receiving the full coupon and principal payments on the scheduled dates.
- Reinvesting the interest payments. This is also known as interest-on-interest.
- Potential capital gain or loss on sale of the bond, if the bond is sold before maturity date.

Now, we will look at a series of examples that demonstrate the effect on two investors' realized rate of returns when one of these variable changes: time horizon, interest rate at which the coupons are reinvested, and market discount rates at the time of purchase and at the time of sale.

In Examples 1 and 2, we look at the realized rate of return for two investors with different time horizons for the same bond.

Example 1: Calculating the total return on a bond that is held until maturity

Investor 1: A "buy-and-hold" investor purchases a 5-year, 10% annual coupon payment bond at 92.79 per 100 of par value and holds it until maturity. Calculate the total return on the bond.



Solution:

$$92.79 = \frac{10}{(1+r)^1} + \frac{10}{(1+r)^2} + \frac{10}{(1+r)^3} + \frac{10}{(1+r)^4} + \frac{110}{(1+r)^5}$$

Use the following keystrokes to calculate the bond's yield to maturity:
N = 5; PV = -92.79; PMT = 10; FV = 100; CPT I/Y;
Hence, r = 12%.

This is the yield to maturity at the time of purchase. However, this holds good only if all of the following three conditions are true:

- The bond is held to maturity.
- The coupon and final principal payments are made on time (no default or delay).
- The coupon payments are reinvested at the same rate of interest.

To calculate the total return on the bond, we first need to calculate the interest earned when coupon payments are reinvested.

Coupon reinvestment:

- The investor receives 5 coupon payments of 10 (per 100 of par value) for a total of 50, plus the redemption of principal (100) at maturity. The investor has the opportunity to reinvest the cash flows. If the coupon payments are reinvested at 12% (i.e., yield to maturity), the future value of the coupons on the bond's maturity date is 63.53 per 100 of par value.
  $[10 \text{ x } (1.12)^4] + [10 \text{ x } (1.12)^3] + [10 \text{ x } (1.12)^2] + [10 \text{ x } (1.12)^1] + 10 = 63.53$
- The first coupon payment of 10 is reinvested at 12% for 4 years until maturity, the second is reinvested for 3 years, and so on. The future value of the annuity is obtained easily using a financial calculator: N = 5; PV = 0; PMT = 10; I/Y = 12; CPT FV. FV = -63.53
- The amount in excess of the coupons, 13.53 (= 63.53 – 50), is the 'interest-on-interest' gain from compounding.
- The investor's total return is 163.53, the sum of the reinvested coupons (63.53) and the redemption of principal at maturity (100). The realized rate of return is 12%.
  $92.79 = \frac{163.53}{(1+r)^5}$, $r = 12\%$

Example 2: Calculating the total return on a bond that is sold before maturity

Now let us consider investor 2 who buys the same 5-year, 10% annual coupon payment bond but sells the bond after three years. Assuming that the coupon payments are reinvested at 12% for three years, calculate the total return on the bond.

Solution:

The future value of the reinvested coupons is 33.74 per 100 of par value.

$[10 * (1.12)^2] + [10 * (1.12)^1] + 10 = 33.74$

The interest-on-interest gain from compounding is 3.74 (= 33.74 – 30). After three years, when the bond is sold, it has two years remaining until maturity. If the yield to maturity remains 12%, then the sale price of the bond is 96.62.

$$\frac{10}{(1.12)^1} + \frac{110}{(1.12)^2} = 96.62$$

The total return is 130.36 (= 33.74 + 96.62) and the realized rate of return is 12%.

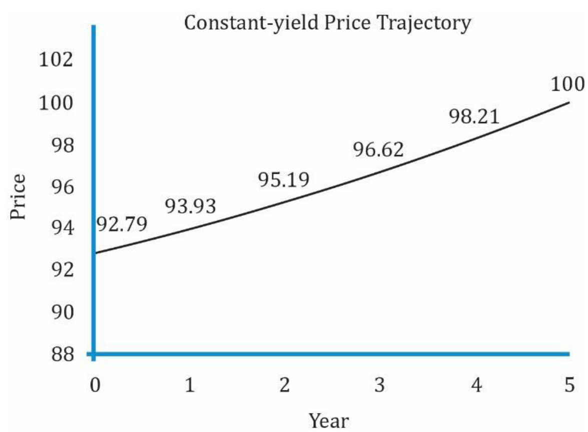
$$92.79 = \frac{130.36}{(1 + r)^3}, r = 12\%$$

This '*r*' is called the horizon yield**,** the internal rate of return between the total return and the purchase price of the bond. Horizon yield is equal to the original yield to maturity if:

- Coupon payments are reinvested at the same yield to maturity calculated at the time of purchase of the bond. In our case, it is 12% as calculated in Example 1.
- There are no capital gains or losses when the bond is sold. It is sold at a price on the constant-yield price trajectory. We arrive at the price 96.62 by taking 12% as the constant yield for the remaining two years. If the yield is more than 12%, then losses occur. Similarly, if the yield is less than 12%, then capital gains occur. This concept is elaborated below:

Constant-yield price trajectory for a 5-year, 10% annual payment bond

The price of the bond at different time periods for a yield of 12% is plotted in the graph below:



The calculations for determining the price are shown below:

Year	Price	Calculation
0	92.79	N = 5 ; I/Y = 12 ; PMT = 10 ; FV = 100 ; CPT PV
1	93.93	N = 4 ; I/Y = 12 ; PMT = 10 ; FV = 100 ; CPT PV
2	95.19	N = 3 ; I/Y = 12 ; PMT = 10 ; FV = 100 ; CPT PV
3	96.62	N = 2 ; I/Y = 12 ; PMT = 10 ; FV = 100 ; CPT PV
4	98.21	N = 1 ; I/Y = 12 ; PMT = 10 ; FV = 100 ; CPT PV

5	100	

<u>Observation</u>**:**

There will be a capital gain if the bond is sold at a price above the trajectory at any point in time during the bond's life. This will happen if the yield is below 12%. Remember a bond's price and interest rates are inversely related. Similarly, there will be a capital loss if the bond is sold at a price below the trajectory at any point in time during the bond's life. This will happen if the yield is above 12%. Any point on the trajectory represents the **carrying value** of the bond at that time.

Next, we analyze what happens to the investors' realized rate of return if interest rates go up after the bond is purchased.

Example 3: Calculating an investor's realized return when interest rates go up and bond is held until maturity

The buy-and-hold investor purchases the same 5-year, 10% annual payment bond at 92.79. After the bond is purchased and before the first coupon is received, interest rates go up to 15%.

Calculate the investor's realized rate of return.

Solution:

Use the following keystrokes to calculate the future value of the reinvested coupons at 15% for 5 years:

$$[10 \text{ x } (1.15)^4] + [10 \text{ x } (1.15)^3] + [10 \text{ x } (1.15)^2] + [10 \text{ x } (1.15)^1] + 10 = 67.42$$

The total return is 167.42 (= 67.42 + 100).

$$92.79 = \frac{167.42}{(1+\text{r})^5} \text{ ; r} = 12.53\%$$

The investor's realized rate of return is 12.53%.

<u>Observation</u>**:**

Compared to the investor in Example 1 (12%), the realized return of this investor is higher because the coupons are reinvested at a higher rate. There is no capital gain or loss because the bond is held to maturity and the principal of 100 is redeemed.

Example 4: Calculating an investor's realized return when interest rates go up and bond is sold before maturity

The second investor buys the same 5-year, 10% annual payment bond at 92.79 and sells it in three years. After the bond is purchased, interest rates go up to 15%. Calculate the investor's realized gain.

Solution:

The future value of the reinvested coupons at 15% after three years is:

$[10 \text{ x } (1.15)^2] + [10 \text{ x } (1.15)^1] + 10 = 34.73$
N = 3; PV = 0; PMT = 10; I/Y = 15; CPT FV; FV = -34.73

The sale price of the bond after three years is 91.87.

$$\frac{10}{(1.15)^1} + \frac{110}{(1.15)^2} = 91.87$$

The total return is 126.60 (= 34.73 + 91.87), resulting in a realized three-year horizon yield of 10.91%.

$$92.79 = \frac{126.60}{(1 + r)^3}, r = 10.91\%$$

N = 3; PV = 92.79; FV = 126.60; CPT I/Y
Hence, r = 10.91%.

Observation**:**

Compared to the investor in Example 2 with a similar time horizon (12%), the realized return to this investor is lower at 10.91% because there is a capital loss. Even though the coupons are reinvested at a higher rate, the capital loss is greater than the gain from reinvesting coupons.

Increase in the value of reinvested coupons = 34.73 − 33.74 = 0.99

Capital loss = 91.87 - 96.62 = -4.75. Capital gain or loss is always calculated relative to the carrying value at that point in time.

Next, we look at the effect of realized returns when interest rates go down after the bond is issued.

Example 5: Calculating an investor's realized return when interest rates go down

The buy-and-hold investor purchases the same 5-year, 10% annual payment bond at 92.79 and holds the security until it matures. After the bond is purchased, and before the first coupon is received, interest rates go down to 8%. Calculate the investor's realized return.

Solution:

The future value of reinvesting the coupon payments at 8% for 5 years is 58.67 per 100 of par value.

$$[10 * (1.08)^4] + [10 * (1.08)^3] + [10 * (1.08)^2] + [10 * (1.08)^1] + 10 = 58.67$$

The total return is 158.67 (= 58.67 + 100), the sum of the future value of reinvested coupons and the redemption of par value.

$$92.79 = \frac{158.67}{(1 + r)^5}, r = 11.33\%$$

The investor's realized rate of return is 11.33%.

Observation**:**

The realized return is lower than that in Example 1 (12%) because the coupons are reinvested at a lower rate of return. Since the bond is held to maturity, there is no capital gain or loss.

Decrease in the value of reinvested coupons = 58.67 - 63.53 = - 4.86

Example 6: Calculating an investor's realized return when interest rates go down

The second investor buys the same 5-year, 10% annual payment bond at 92.79 and sells it after three years. After the bond is purchased, interest rates go down to 8%. Calculate the investor's realized return.

Solution:

The future value of the reinvested coupons at 8% after three years is:
$[10 * (1.08)^2] + [10 * (1.08)^1] + 10 = 32.46$

This reduction in the future value of coupon reinvestments is offset by the higher sale price of the bond, which is 103.57 per 100 of par value.

$$\frac{10}{(1.08)^1} + \frac{110}{(1.08)^2} = 103.57$$

The total return is 136.03 (= 32.46 + 103.57), resulting in a realized three-year horizon yield of 13.60%.

$$92.79 = \frac{136.03}{(1+r)^3}, \qquad r = 13.60\%$$

<u>Observation</u>**:**

The realized return is greater than that of the investors in Examples 2 and 4 with a similar time horizon. It is primarily due to the capital gains.

Capital gain = 103.57 - 96.62 = 6.95

Decrease in the value of reinvested coupons = 32.46 - 33.74 = -1.28

As you can see, the capital gain is far greater than the decrease in the value of reinvested coupons.

Interest rate risk affects the realized rate of return for any bond investor in two ways: <u>coupon reinvestment risk</u> and <u>market price risk</u>. But, what is interesting is that these are offsetting types of risk. Two investors with different time horizons will have different exposures to interest rate risk. From the examples above, let us sum up what happens when interest rates go up or down:

<u>When interest rates go up or down</u>:

- Reinvestment income is directly proportional to interest rate movements. The value of reinvested coupons increases when the interest rate goes up.
- Bond price is inversely proportional to interest rate movements. Bond price decreases when the interest rate goes up.

<u>When does coupon reinvestment risk matter?</u>

- Coupon reinvestment risk matters when an investor has a long-term horizon. If the investor buys a bond and sells it before the first coupon payment, then this risk is irrelevant. But a buy-and-hold investor as in Examples 1, 3, and 5 has only coupon reinvestment risk.

<u>When does market price risk matter?</u>

- Market price risk matters when an investor has a short-term horizon relative to the time to maturity. If the investor buys a bond and sells it before the first coupon payment, then he is exposed to market price risk only. But a buy-and-hold investor such as those in Examples 1, 3, and 5 has only coupon reinvestment risk, and no market price risk.

Example 7: Calculating the purchase price of the bond for various YTM

An investor buys a five-year, 10% annual coupon payment bond priced to yield 8%. The investor plans to sell the bond in three years once the third coupon payment is received. Calculate the purchase price for the bond and the horizon yield assuming that the coupon reinvestment rate after the bond purchase and the yield to maturity at the time of sale are (1) 7%, (2) 8%, and (3) 9%.

Solution:

The purchase price is:

$$\frac{10}{(1.08)^1} + \frac{10}{(1.08)^2} + \frac{10}{(1.08)^3} + \frac{10}{(1.08)^4} + \frac{110}{(1.08)^5} = 107.99$$

N = 5; PMT = 10; FV = 100; I/Y = 8; CPT PV, PV = -107.99.

Solution to 1:

With YTM at sale = 7%:
N = 3; PV = 0; PMT = 10; I/Y = 7; CPT FV, FV = -32.15.
The future value of the reinvested coupons after three years is 32.15.

$$\frac{10}{(1.07)^1} + \frac{110}{(1.07)^2} = 105.42.$$

N = 2; I/Y = 7; PMT = 10; FV = 100; CPT PV, PV = 105.42.
The sale price of the bond after three years is 105.42.

The total return after three years is 137.57 (= 32.15 + 105.42).

The realized return calculated using the keystrokes: N = 3; PMT = 0; PV = 107.99; FV = 137.57; CPT I/Y,
r = 8. 4%.
$107.99 = \frac{137.57}{(1+r)^3}$, r = 8.40%.

If interest rates go down from 8% to 7%, then the realized rate of return over the three-year investment horizon is 8.40%, higher than the original yield to maturity of 8%.

Solution to 2:

With YTM at sale = 8%, the future value of the reinvested coupons after three years is:
N = 3; PV = 0; PMT = 10; I/Y = 8; CPT FV, FV = -32.46.

The sale price of the bond is calculated as:
N = 2, I/Y = 8, PMT = 10, FV = 100; CPT PV

The sale price of the bond after three years is 103.57.

The total return after three years is 136.03 (= 32.46 + 103.57).

The realized return calculated using the keystrokes: N = 3; PMT = 0; PV = -107.99; FV = 136.03; CPT I/Y.
r = 8%.

$$107.99 = \frac{136.03}{(1+\mathrm{r})^3}, \mathrm{r} = 8.00\%$$

If interest rates remain 8% for reinvested coupons and for the required yield on the bond, the realized rate of return over the three-year investment horizon is equal to the yield to maturity of 8%.

Solution to 3:

With YTM at sale = 9%: the future value of the reinvested coupons after three years is:
N = 3; PV = 0; PMT = 10; I/Y = 9; CPT FV, FV = -32.78.

The sale price of the bond is calculated as:
N = 2; I/Y = 9; PMT = 10; FV = 100; CPT PV

The sale price of the bond after three years is 101.76.

The total return after three years is 134.54 (= 32.78 + 101.76).

The realized return calculated using the keystrokes: N = 3; PV = -107.99; FV = 134.54; CPT I/Y

$$107.99 = \frac{134.54}{(1+\mathrm{r})^3}, \ \mathrm{r} = 7.60\%;$$

If interest rates go up from 8% to 9%, the realized rate of return over the three-year investment horizon is 7.60%; lower than the yield to maturity of 8%.

3. Macaulay and Modified Duration

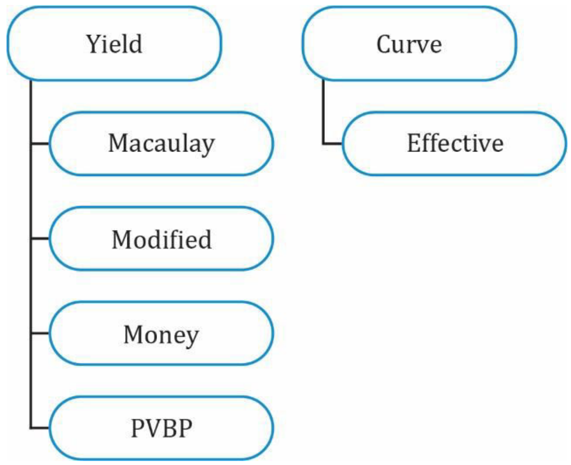
In this section, we look at two measures of interest rate risk: duration and convexity.

3.1 Macaulay, Modified, and Approximate Duration

The duration of a bond measures the sensitivity of the bond's full price (including accrued interest) to changes in interest rates. In other words, duration indicates the percentage change in the price of a bond for a 1% change in interest rates. The higher the duration, the more sensitive the bond is to change in interest rates. Duration is expressed in years.

There are two categories of duration: yield duration and curve duration.

- Yield duration is the sensitivity of the bond price with respect to the bond's own yield to maturity.
- Curve duration is the sensitivity of the bond price with respect to a benchmark yield curve such as a government yield curve on coupon bonds, the spot curve, or the forward curve.



As indicated in the diagram above, there are several types of yield duration.

Macaulay duration is a weighted average of the time to receipt of the bond's promised payments, where the weights are the shares of the full price that correspond to each of the bond's promised future payments. Let us consider a 10-year, 8% annual payment bond. To determine the Macaulay duration, we calculate the present value of each cash flow, multiply by weight and add, as shown in the below exhibit

Exhibit 2: Macaulay Duration of a 10 – Year, 8% Annual Payment Bond				
Period	**Cash flow**	**Present Value**	**Weight**	**Period x Weight**
1	8	7.246377	0.08475	0.0847
2	8	6.563747	0.07677	0.1535
3	8	5.945423	0.06953	0.2086
4	8	5.385347	0.06298	0.2519
5	8	4.878032	0.05705	0.2853
6	8	4.418507	0.05168	0.3101
7	8	4.002271	0.04681	0.3277
8	8	3.625245	0.04240	0.3392
9	8	3.283737	0.03840	0.3456
10	108	40.154389	0.46963	4.6963
		85.503075	**1.00000**	**7.0029**

We can also use the following formula to calculate Macaulay Duration:

$$\text{MacDur} = \left(\frac{1 + r}{r} - \frac{1 + r + [N * (c - r)]}{c * [(1 + r)^N - 1] + r}\right) - \frac{t}{T}$$

where:

r = yield to maturity
t = number of days from the last coupon payment date to the settlement date
T = number of days in the coupon period
c = coupon rate per period
N = number of periods to maturity

Instructor's Note:
Understanding how the Macaulay duration works is more important than memorizing the formula.

Modified duration provides an estimate of the percentage price change for a bond given a change in its yield to maturity. It represents a simple adjustment to Macaulay duration as shown in the equation below:

$$\text{Modified duration} = \frac{\text{Macaulay duration}}{1 + r}$$

where: r is the yield per period.

Therefore, percentage price change for a bond given a change in its YTM can be calculated as:

$$\%\Delta PV^{FULL} \approx -\text{AnnModDur x } \Delta\text{Yield}$$

The AnnModDur term is the *annual* modified duration, and the ΔYield term is the change in the *annual* yield to maturity. The ≈ sign indicates that this calculation is estimation. The minus sign indicates that bond prices and yields to maturity move inversely.

Example 8: Calculating the modified duration of a bond

A 2-year, annual payment, $100 bond has a Macaulay duration of 1.87 years. The YTM is 5%. Calculate the modified duration of the bond.

Solution:

Modified duration = 1.87/(1 + 0.05) = 1.78 years

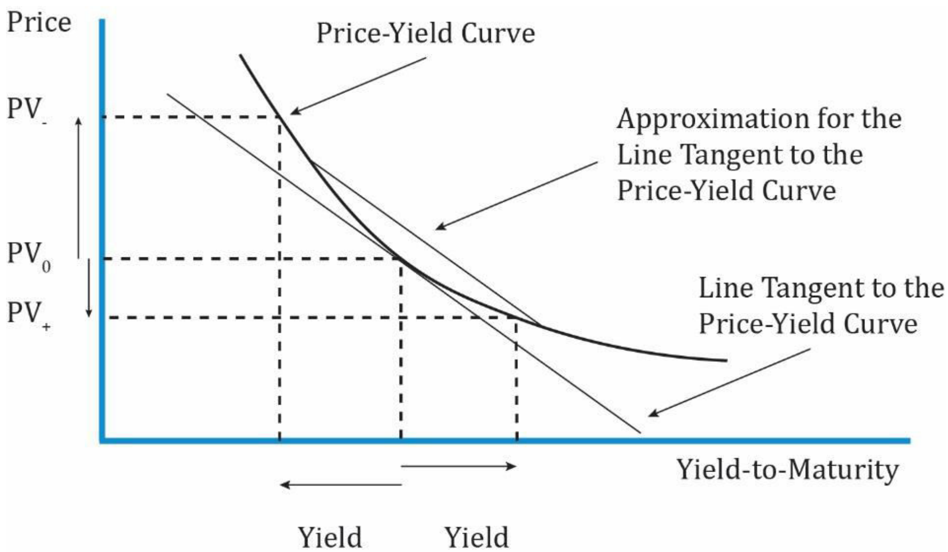
The percentage change in the price of the bond for a 1% increase in YTM will be:
-1.78 * 0.01 * 100 = -1.78%.

4. Approximate Modified and Macaulay Duration

Modified duration is calculated if the Macaulay duration is known. But there is another way of calculating an approximate value of modified duration: estimate the slope of the line tangent to the price-yield curve. This can be done by using the equation below:

$$\text{Approximate Modified Duration} = \frac{(PV_{-}) - (PV_{+})}{2 * \Delta\text{yield} * PV_0}$$

where:
PV_{-} = price of the bond when yield is decreased;
PV_0 = initial price of the bond
PV_{+} = price of the bond when yield is increased



Source: CFA Program Curriculum, Understanding Fixed-Income Risk and Return

<u>Interpretation of the diagram</u>**:**

- PV_0 denotes the original price of the bond.
- PV_{+} denotes the price of the bond when YTM is increased and PV_{-} denotes the bond price when YTM is decreased.
- Change in yield (up and down) is denoted by Δ yield.
- For slope calculation: vertical distance = $PV_{-} - PV_{+}$ and horizontal distance = 2 x Δ yield.

How to calculate approximate modified duration (or estimate the slope of the price-yield curve):

- The yield to maturity is changed (increased/decreased) by the same amount.
- Calculate the bond price for a decrease in yield (PV_{-}).
- Calculate the bond price for an increase in yield (PV_{+}).
- Use these values to calculate the approximate modified duration.

Once the approximate modified duration is known, the approximate Macaulay duration can be calculated using the formula below:

Approximate Macaulay Duration = Approximate Modified Duration x (1 + r)

Example 9: Calculating the approximate modified duration and approximate Macaulay duration

Assume that the 6% U.S. Treasury bond matures on 15 August 2017 is priced to yield 10% for settlement on 15 November 2014. Coupons are paid semiannually on 15 February and 15 August. The yield to maturity is stated on a street-convention semiannual bond basis. This settlement date is 92 days into a 184-day coupon period, using the actual/actual day-count convention. Compute the approximate modified duration and the approximate Macaulay duration for this Treasury bond assuming a 50bps change in the yield to maturity.

Solution:

The yield to maturity per semiannual period is 5% (=10/2). The coupon payment per period is 3% (= 6/2). When the bond is purchased, there are 3 years (6 semiannual periods) to maturity. The fraction of the period that has passed is 0.5 (=92/184).

The full price (including accrued interest) at an YTM of 5% is 92.07 per 100 of par value.

$$PV_0 = \left[\frac{3}{(1.05)^1} + \frac{3}{(1.05)^2} + \frac{3}{(1.05)^3} + \frac{3}{(1.05)^4} + \frac{3}{(1.05)^5} + \frac{103}{(1.05)^6}\right] * (1.05)^{\left(\frac{92}{184}\right)}$$

$$\Rightarrow 89.85 \times (1.05)^{0.5} = 92.07$$

Increase the yield to maturity from 10% to 10.5% - therefore, from 5% to 5.25% per semiannual period, and the price becomes 90.97 per 100 of par value.

$$PV_+ = \left[\frac{3}{(1.0525)^1} + \frac{3}{(1.0525)^2} + \frac{3}{(1.0525)^3} + \frac{3}{(1.0525)^4} + \frac{3}{(1.0525)^5} + \frac{103}{(1.0525)^6}\right] * (1.0525)^{\left(\frac{92}{184}\right)}$$

$$\Rightarrow 88.67 \times (1.0525)^{0.5} = 90.97$$

Decrease the yield to maturity from 10% to 9.5% - therefore, from 5% to 4.75% per semiannual period, and the price becomes 93.19 per 100 of par value.

$$PV_- = \left[\frac{3}{(1.0475)^1} + \frac{3}{(1.0475)^2} + \frac{3}{(1.0475)^3} + \frac{3}{(1.0475)^4} + \frac{3}{(1.0475)^5} + \frac{103}{(1.0475)^6}\right] * (1.0475)^{\left(\frac{92}{184}\right)}$$

$$\Rightarrow 91.05 \times (1.0475)^{0.5} = 93.19$$

The approximate annualized modified duration for the Treasury bond is 2.41

$$\text{ApproxModDur} = \frac{93.19 - 90.97}{2 * 0.005 * 92.07} = 2.41$$

The approximate annualized Macaulay Duration is 2.53

$$\text{ApproxMacDur} = 2.41 * 1.05 = 2.53$$

Therefore, from these statistics, the investor knows that the weighted average time to receipt of interest and principal payments is 2.53 years (the Macaulay Duration) and that the estimated loss in the bond's market value is 2.41% (the Modified Duration) if the market discount rate were to suddenly go up by 1.0%.

5. Effective and Key Rate Duration

Bonds with embedded options and mortgage-backed securities do not have a well-defined YTM. These securities may be prepaid well before the maturity date. Hence, yield duration statistics are not suitable for these instruments. For such instruments, the best measure of interest rate sensitivity is the effective duration, which measures the sensitivity of the bond's price to a change in a benchmark yield curve (instead of its own YTM).

$$\text{Effective Duration} = \frac{(PV_{-}) - (PV_{+})}{2 * \Delta \text{ curve} * PV_0}$$

Difference between approx. modified duration and effective duration:

The denominator for approx. modified duration has the bond's own yield to maturity. It measures the bond's price change to changes in its own YTM. But, the denominator for effective duration has the change in the benchmark yield curve. It measures the interest rate risk in terms of change in the benchmark yield curve.

Example 10: Calculating the effective duration

A Pakistani defined-benefit pension scheme seeks to measure the sensitivity of its retirement obligations to market interest rate changes. The pension scheme manager hires an actuarial consultant to model the present value of its liabilities under three interest rate scenarios

1. a base rate of 10%
1. a 50 bps drop in rates, down to 9.5%
2. a 50 bps increase in rates to 10.5%.

The following chart shows the results of the analysis:

Interest Rate Assumption	Present Value of Liabilities
9.5%	PKR 10.5 million
10% 10.5%	PKR 10 million PKR 9 million

Compute the effective duration of pension liabilities. PV_0 = 10, PV_+ = 9, PV_- = 10.5, and Δ curve = 0.005. The effective duration of the pension liabilities is 15.

$$\frac{10.5 - 9}{2 * 0.005 * 10} = 15$$

This effective duration statistic for the pension scheme's liabilities might be used in asset allocation decisions to decide the mix of equity, fixed income, and alternative assets.

5.1 Key Rate Duration

Key rate duration is a measure of a bond's sensitivity to a change in the benchmark yield curve at a specific maturity. This topic is covered in detail at Level II.

The duration measures we have seen so for assume a parallel shift in the yield curve. But what if the shift in the yield curve is not parallel? Here it is appropriate to use 'Key rate duration'.

Key rate durations are used to identify "shaping risk" of a bond, which is a bond's sensitivity to changes in the shape of the benchmark yield curve. For instance, analysts can analyze the interest rate sensitivity if the yield curve flattens or if the yield curve steepens.

6. Properties of Bond Duration

The input variables for determining Macaulay and modified yield duration of fixed-rate bonds are:

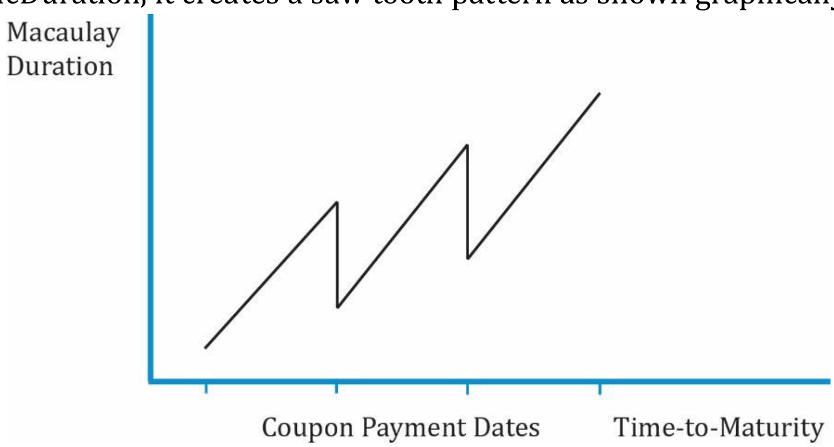
- Coupon rate or payment per period
- Yield to maturity per period
- Number of periods to maturity
- Fraction of the period that has gone by

By changing one of the above variables while holding others constant, we can analyze the properties of bond duration, which, in turn, helps us assess the interest rate risk. We will use the formula for Macaulay duration to understand the relationship between each variable and duration:

$$\text{MacDur} = \left\{\frac{1+r}{r} - \frac{1+r+[N*(c-r)]}{c*[(1+r)^N - 1]+r}\right\} - (t/T)$$

<u>The fraction of the coupon period that has gone by (t/T)</u>

First, let us consider the relationship between fraction of time that has gone by (t/T) and duration. There is no change to the expression in braces {} as time passes by. Fraction of time (t/T) increases as time passes by. Assume T is 180. If 50 days have passed, then t/T = 0.277. If 90 days have passed, then t/T = 0.5. If 150 days have passed, then t/T = 0.83. As t/T increases from t = 0 to t = T with passing time, MacDuration decreases in value. Once the coupon is paid, t/T becomes zero and MacDuration jumps in value. When time to maturity is plotted against MacDuration, it creates a saw tooth pattern as shown graphically below:



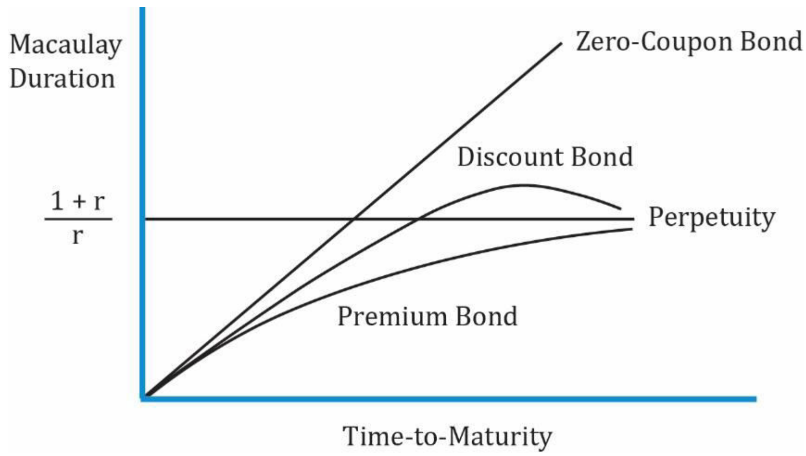
Source: CFA Program Curriculum, Understanding Fixed-Income Risk and Return

Interpretation of the Macaulay duration between coupon payments with a constant yield to *maturity*:

- Note: Read the diagram from right to left. As time passes between coupon periods, duration decreases in value.
- Once the coupon is paid, it jumps back up creating a saw tooth pattern.

Now, we look at the relationship between Macaulay duration and change in the coupon rate, yield to maturity, and the time to maturity. This is depicted in the exhibit below:

Properties of the Macaulay Yield Duration



Source: CFA Program Curriculum, Understanding Fixed-Income Risk and Return

Time to maturity

The relationship between Macaulay Duration and time to maturity for four types of bonds (zero-coupon, discount, premium, and perpetuity) at t/T = 0 are shown.

- Zero-coupon bond: Pays no coupon so c = 0. Plugging in c = 0 and t/T = 0 in MacDur formula, we get Macaulay Duration = N. For a zero-coupon bond, duration is its time to maturity.
- Perpetuity: A perpetual bond is one that does not mature. There is no principal to redeem. It makes a fixed coupon payment forever. If N is a large number in the above equation, then the second part of the expression in the braces becomes zero. Denominator: N is an exponent. $(1 + r)^N$ is a very large number. So $\frac{1}{(1+r)^N}$ must be zero and the value in the numerator will not matter here. Macaulay duration of a perpetual bond is $\frac{1+r}{r}$ as N approaches infinity.
- Premium bond: Bonds are trading at a premium above par or at par. The coupon rate is greater than or equal to yield to maturity (r). The numerator of the second expression in braces is always positive because c-r is positive. The denominator of the second expression in braces is always positive. Second expression as a whole is always positive. MacDuration = less than $\frac{1+r}{r}$ because the second expression in braces is positive. As time passes, it approaches $\frac{1+r}{r}$.

- Discount bond: For a discount bond, the coupon rate is below yield to maturity. The Macaulay Duration increases for a longer time to maturity. The numerator of the second expression in braces is negative because c-r is negative. Put together, the duration at some point exceeds $\frac{1+r}{r}$, reaches a maximum, and approaches $\frac{1+r}{r}$ (the threshold) from above. This happens when N is large and coupon rate (c) is below the yield to maturity (r). As a result, for a long-term discount bond, interest rate risk can be lesser than a shorter-term bond.

The above points are summarized below:

Relationship between bond duration and other input parameters	
Bond parameter	**Effect on Duration**
Higher coupon rate	Lower
Higher yield to maturity	Lower
Longer time to maturity	Higher for a premium bond. Usually, holds true for a discount bond, but there can be exceptions. Exception: low coupon (relative to YTM) bond with long maturity.

Example 11: Calculating the approximate modified duration

A mutual fund specializes in investments in sovereign debt. The mutual fund plans to take a position on one of these available bonds.

Bond	**Time to maturity**	**Coupon Rate**	**Price**	**Yield to maturity**
(A)	5 years	10%	70.093879	20%
(B)	10 years	10%	58.075279	20%
(C)	15 years	10%	53.245274	20%

The coupon payments are annual. The yields to maturity are effective annual rates. The prices are per 100 of par value.

1. Compute the approximate modified duration of each of the three bonds using a 5 bps change in the yield to maturity and keeping precision to six decimals (because approximate duration statistics are very sensitive to rounding).
2. Which of the three bonds is expected to have the highest percentage price increase if the yield to maturity on each decreases by the same amount – for instance, by 10 bps from 20% to 19.90%?

Solution to 1:

Calculate PV_+ and PV_-; then calculate modified duration.

Bond A:

$PV_0 = 70.093879$;

$$PV_+ = \frac{10}{(1.2005)^1} + \frac{10}{(1.2005)^2} + \frac{10}{(1.2005)^3} + \frac{10}{(1.2005)^4} + \frac{110}{(1.2005)^5} = 69.977386$$

$PV_+ = 69.977386$

$$PV_- = \frac{10}{(1.1995)^1} + \frac{10}{(1.1995)^2} + \frac{10}{(1.1995)^3} + \frac{10}{(1.1995)^4} + \frac{110}{(1.1995)^5} = 70.210641$$

$PV_- = 70.210641$

ApproxModDur = $\frac{70.210641 - 69.977386}{2 * 0.0005 * 70.093879}$ = 3.33.

The approximate modified duration of Bond A is 3.33.

Bond B:

$PV_0 = 58.075279$

$$PV_+ = \frac{10}{(1.2005)^1} + \frac{10}{(1.2005)^2} + \frac{10}{(1.2005)^3} + \frac{10}{(1.2005)^4} \ldots. + \frac{110}{(1.2005)^{10}} = 57.937075$$

$PV_+ = 57.937075$

$$PV_- = \frac{10}{(1.1995)^1} + \frac{10}{(1.1995)^2} + \frac{10}{(1.1995)^3} + \frac{10}{(1.1995)^4} \ldots. + \frac{110}{(1.1995)^{10}} = 58.213993$$

$PV_- = 58.213993$

ApproxModDur = $\frac{58.213993 - 57.937075}{2 * 0.0005 * 58.075279}$ = 4.77

The approximate modified duration of Bond B is 4.77.

Bond C:

$PV_0 = 53.245274$

$$PV_+ = \frac{10}{(1.2005)^1} + \frac{10}{(1.2005)^2} + \frac{10}{(1.2005)^3} + \frac{10}{(1.2005)^4} \ldots + \frac{110}{(1.2005)^{15}} = 53.108412$$

$PV_+ = 53.108412$

$$\frac{10}{(1.1995)^1} + \frac{10}{(1.1995)^2} + \frac{10}{(1.1995)^3} + \frac{10}{(1.1995)^4} \ldots + \frac{110}{(1.1995)^{15}} = 53.382753$$

$PV_- = 53.382753$

ApproxModDur = $\frac{53.382753 - 53.108412}{2 * 0.0005 * 53.245274}$ = 5.15.

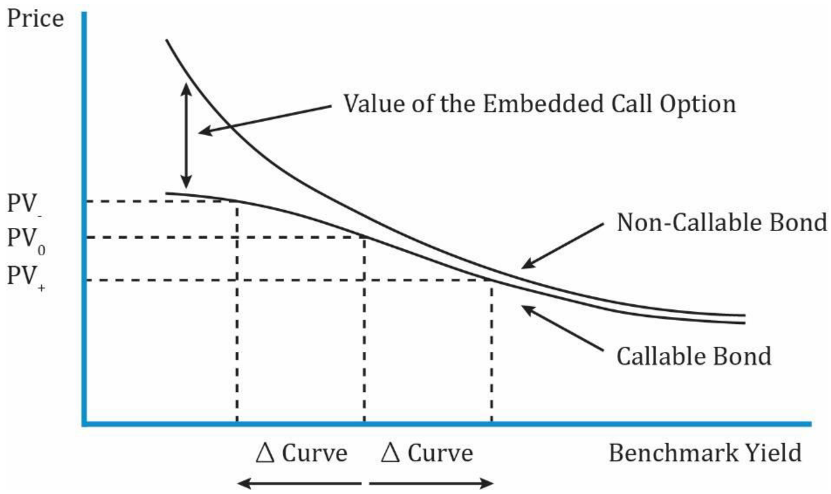
The approximate modified duration of Bond C is 5.15.

Solution to 2:

Bond C with 15 years-to-maturity has the highest modified duration. If the yield to maturity on each is decreased by the same amount – for instance, by 10bps, from 20% to 19.90% - Bond C would be expected to have the highest percentage price increase because it has the highest modified duration.

Interest Rate Risk Characteristics of a Callable Bond

A callable bond is one that might be called by the issuer before maturity. This makes the cash flows uncertain, so the YTM cannot be determined with certainty. The exhibit below plots the price-yield curve for a non-callable/straight bond and a callable bond. It also plots the change in price for a change in the benchmark yield curve. Bond price is plotted on the y-axis and the benchmark yield on the horizontal axis.

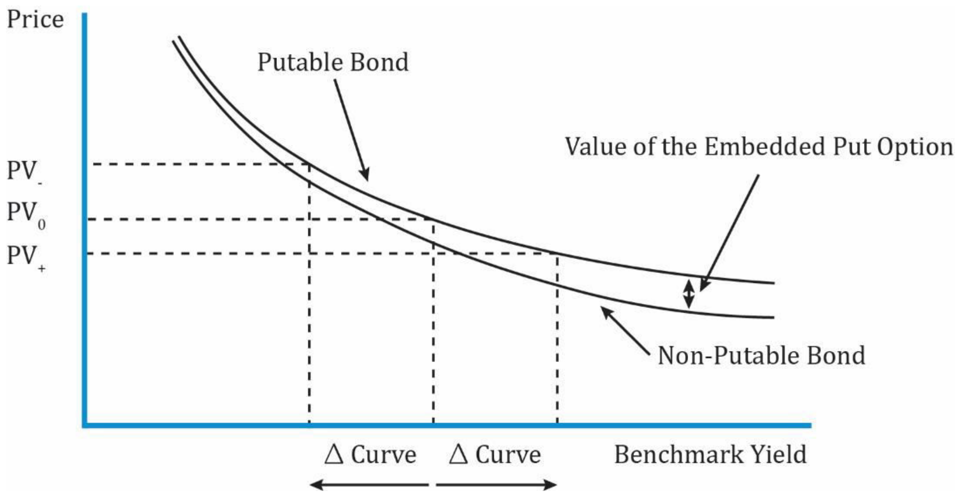


Source: CFA Program Curriculum, Understanding Fixed-Income Risk and Return

<u>Interpretation of the diagram</u>:

- The two bonds share the same features: credit risk, coupon rate, payment frequency, and time to maturity, but their prices are different when interest rates are low.
- When interest rates are low (left side of the graph), the price of the non-callable bond is always greater than the callable bond.
- The difference between the non-callable bond and callable bond is the value of the embedded call option.
- The call option is more valuable at low interest rates because the issuer has an option to refinance debt at lower interest rates than paying a higher interest to existing investors. For instance, if a bond is callable at 102% of par, then its price cannot increase beyond 102 of par even if interest rates decrease. This is the reason behind the callable bond's negative convexity.
- From an investor's perspective a call option is risky. At low interest rates if the issuer calls the bond (meaning the issuer pays back the money borrowed), then investors must reinvest the proceeds at lower rates. Notice that at higher interest rates, there is not much of a difference in price because the probability of calling the bond is less.
- At relatively low interest rates, the effective duration of a callable bond is lower than that of non-callable bond, i.e., interest rate sensitivity is low.

Interest Rate Risk Characteristics of Putable Bond



Source: CFA Program Curriculum, Understanding Fixed-Income Risk and Return

Interpretation of the diagram:

- The diagram above shows the price-yield curve for a putable bond and a straight bond.
- A putable bond allows the investor to sell the bond back to the issuer; usually at par.
- At low interest rates, there is not much of a difference between a straight bond and putable bond.
- At high interest rates, the difference between the putable bond and straight bond is the value of the put option.
- When interest rates rise, the price of the bond decreases. Investors buy putable bonds to protect against falling prices as rates rise.
- The value of the put option increases as rates rise. This also limits the sensitivity of the bond price to changes in benchmark rates.

7. Duration of a Bond Portfolio

In the previous section, we saw how to calculate the duration for an individual bond. What if a portfolio consists of a number of bonds, how will its duration be calculated?

There are two ways to calculate the duration of a bond portfolio:

- Weighted average of time to receipt of the aggregate cash flows.
- Weighted average of the individual bond durations that comprise the portfolio.

The key points related to each method are outlined in the table below.

Weighted Average Time to Receipt of Aggregate Cash Flows	**Weighted Average of Individual Bond Durations in Portfolio**
Theoretically correct, but difficult to use in practice.	Commonly used in practice.

Cash flow yield not commonly used. Cash flow yield is the IRR on a series of cash flows.	Easy to use as a measure of interest rate risk.
Amount and timing of cash flows might not be known because some of these bonds may be MBS, or with call options.	More accurate as difference in YTMs of bonds in portfolio become smaller.
Interest rate risk is usually expressed as a change in benchmark interest rates, not as a change in the cash flow yield.	Assumes parallel shifts in the yield curve, i.e., all rates change by the same amount in the same direction. That seldom happens in reality.
Change in the cash flow yield is not necessarily the same amount as the change in yields to maturity on the individual bonds.	

Example 12: Calculating portfolio duration

An investment fund owns the following portfolio of three fixed-rate government bonds:

	Bond A	Bond B	Bond C
Par value	15,000,000	20,000,000	40,000,000
Market value	14,769,542	25,650,379	43,796,854
Modified duration	4.328	5.643	7.210

The total market value of the portfolio is 84,216,775. Each bond is on a coupon date so that there is no accrued interest. The market values are the full prices given the par value.

1. Calculate the average modified duration for the portfolio using the shares of market value as the weights.
2. Estimate the percentage loss in the portfolio's market value if the yield to maturity on each bond goes up by 10 bps.

Solution to 1:

The average modified duration of the portfolio is 6.23.
$(14{,}769{,}542/84{,}216{,}775) \times 4.328 + (25{,}650{,}379/84{,}216{,}775) \times 5.643 + (43{,}796{,}854/84{,}216{,}775) \times 7.210 = 6.23$.

Solution to 2:

The estimated decline in market value if each yield rises by 10 bps is 0.623%. $-6.23 \times 0.001 = -0.00623$.

8. Money Duration and the Price Value of a Basis Point

The money duration of a bond is a measure of the price change in units of the currency in which the bond is denominated, given a change in annual yield to maturity.

$$\text{Money Duration} = \text{AnnModDur} \times PV_{FULL}$$

$\Delta PV_{FULL} \approx$ -MoneyDur x Δyield

Consider a bond with a par value of $100 million. The current yield to maturity (YTM) is 5% and the full price is $102 per $100 par value. The annual modified duration of this bond is 3. The money duration can be calculated as the annual modified duration (3) multiplied by the full price ($102 million): 3 x $102 million = $306 million. If the YTM rises by 1% (100 bps) from 5% to 6%, the decrease in value will be approximately $306 million x 1% = $3.06 million. If the YTM rises by 0.1% (10 bps), the decrease in value will be $306 million x 0.1% = $0.306 million.

An important measure which is related to money duration is the **price value of a basis point** (**PVBP**). The PVBP is an estimate of the change in the full price given a 1 bp change in the yield to maturity. The formal equation is given below.

$$PVBP = \frac{PV_{-} - PV_{+}}{2}$$

where PV_{-} and PV_{+} are full prices calculated by decreasing and increasing the YTM by 1 basis point.

A quick way of calculating the price value of a basis point is to take the money duration and multiply by 0.0001. For example, if the money duration of a portfolio is $200,000 the price value of a basis point is $200,000 x 0.0001 = $20. (1 bp = 0.01% = 0.0001)

Example 13: Calculating money duration of a bond

A life insurance company holds a USD 1 million (par value) position in a bond that has a modified duration of 6.38. The full price of the bond is 102.32 per 100 of face value.
1. Calculate the money duration for the bond.
2. Using the money duration, estimate the loss for each 10 bps increase in the yield to maturity.

Solution:

1. First calculate the full price of the bond: $1,000,000 x 102.32% = $1,023,200. The money duration for the bond is: 6.38 × $1,023,200 = $6,528,000.

2. 10 bps corresponds to 0.10% = 0.0010. For each 10 bps increase in the yield to maturity, the loss is estimated to be: $6,528,000 × 0.0010 = $6,528.02.

Example 14: Calculating PVBP for a bond

Consider a $100, five-year bond that pays coupons at a rate of 10% semi-annually. The YTM is 10% and it is priced at par. The modified duration of the bond is 3.81. Calculate the PVBP for the bond.

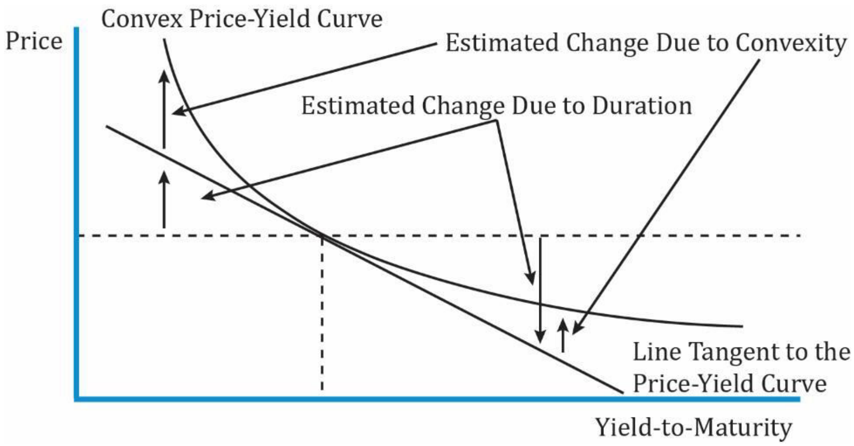
Solution:

Money duration = $100 x 3.81 = $381.00

PVBP = $381 x 0.0001 = $0.0381

9. Bond Convexity

The graph below shows the relationship between bond price and YTM. It shows the convexity for a traditional fixed-rate bond.



Source: CFA Program Curriculum, Understanding Fixed-Income Risk and Return

Interpretation of the diagram:

- Duration assumes there is a linear relationship between the change in a bond's price and change in YTM. For instance, assume the YTM of a bond is 10% and it is priced at par (100). According to the duration measure, if the YTM increases to 11% the price moves down to a point on the straight line.
- Similarly, the price moves up to a point on the straight line if the YTM decreases.
- The curved line in the above exhibit plots the actual bond prices against YTM. So in reality, the bond prices do not move along a straight line but exhibit a convex relationship.
- For small changes in YTM, the linear approximation is a good representation for change in bond price. That is, the difference between the straight and curved line is not significant.
- In other words, modified duration is a good measure of the price volatility.
- However, for large changes in YTM or when the rate volatility is high, a linear approximation is not accurate and a convexity adjustment is needed.

Here we need to factor in the convexity. The percentage change in the bond's full price with convexity-adjustment is given by the following equation:

Change in the price of a full bond:

$\% \, \Delta PV^{FULL} = (-\text{AnnModDur} * \Delta\text{yield}) + [\frac{1}{2} * \text{AnnConvexity} * (\Delta\text{yield})^2]$

Expression in first braces: duration adjustment

Expression in second braces: convexity adjustment

Approximate convexity can be calculated using this formula:

$$\text{Approx. Convexity} = \frac{PV_{-} + PV_{+} - 2 * PV_0}{(\Delta\text{yield})^2 * PV_0}$$

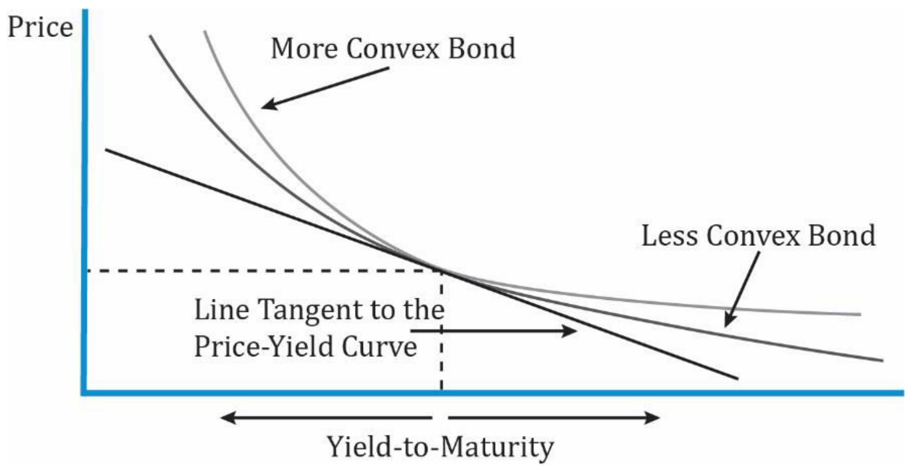
where:
PV_{-} and PV_{+} = new full price when YTM is decreased and increased by the same amount
PV_0 = original full price

The change in the full price of the bond in units of currency, given a change in YTM, can be calculated using this formula:

$$\Delta PV_{FULL} \approx -(\text{MoneyDur x } \Delta\text{yield}) + \left[\frac{1}{2} \text{ x MoneyCon x } (\Delta\text{yield})^2\right]$$

Convexity is good

The following exhibit shows the price-yield curves for two bonds with the same YTM, price, and modified duration, and why greater convexity is good for an investor.



Interpretation of the diagram:

- Both the bonds have the same tangential line to their price-yield curves.
- When YTM decreases by the same amount, the more convex bond appreciates more in price.
- When YTM increases by the same amount, the more convex bond depreciates less in price than the less convex bond.
- The bond with greater convexity outperforms when interest rates go up/down.

The relationship between various bond parameters with convexity is the same as with duration.

For a fixed-rate bond,

- The lower the coupon rate, the greater the convexity.
- The lower the yield to maturity, the greater the convexity.
- The longer the time to maturity, the greater the convexity.
- The greater the dispersion of cash flow or cash payments spread over time, the greater the convexity.

Effective Convexity

For bonds whose cash flows were unpredictable, we used effective duration as a measure of interest rate risk. Similarly, we use effective convexity to measure the change in price for a change in benchmark yield curve for securities with uncertain cash flows. The effective convexity of a bond is a curve convexity statistic that measures the secondary effect of a change in a benchmark yield curve. It is used for bonds with embedded options.

$$\text{Effective Convexity} = \frac{PV_- + PV_+ - 2 * PV_0}{(\Delta \text{curve})^2 * PV_0}$$

Here is a summary of some important points related to bonds with embedded options:

- Option-free bonds always have positive convexity.
- Callable bonds have positive convexity at high yields but they can have negative convexity at low yields. This is because at low yields, the call option becomes valuable and puts a limit on how much the bond price can appreciate.
- Putable bonds always have positive convexity.

Example 15: Calculating the full price and convexity-adjusted percentage price change of a bond

A German bank holds a large position in a 6.50% annual coupon payment corporate bond that matures on 4 April 2029. The bond's yield to maturity is 6.74% for settlement on 27 June 2014, stated as an effective annual rate. That settlement date is 83 days into the 360-day year using the 30/360 method of counting days.

1. Calculate the full price of the bond per 100 of par value.
2. Calculate the approximate modified duration and approximate convexity using a 1 bp increase and decrease in the yield to maturity.
3. Calculate the estimated convexity-adjusted percentage price change resulting from a 100 bp increase in the yield to maturity.
4. Compare the estimated percentage price change with the actual change, assuming the yield to maturity jumps to 7.74% on that settlement date.

Solution:

There are 15 years from the beginning of the current period on 4 April 2014 to maturity on 4 April 2029.

1. The full price of the bond is 99.2592 per 100 of par value.

FV = 100, I/Y = 6.74, PMT = 6.50, N = 15, CPT PV; PV = -97.777.

Full Price = $97.777 \times 1.0674^{\frac{83}{360}}$ = 99.2592.

2. $PV_+ = 99.1689$.

FV = 100, PMT = 6.5, I/Y = 6.75, N = 15, CPT PV; PV = -97.687.

$PV_+ = 97.687 \times 1.0675^{\frac{83}{360}}$ = 99.1689.

PV_ = 99.3497.

FV = 100, I/Y = 6.73, PMT = 6.5, N = 15, CPT PV; PV = -97.869.

PV_ = $97.869 \times 1.0673^{\frac{83}{360}}$ = 99.3497.

ApproxModDur = $\left(\frac{99.3497 - 99.1689}{2 * 99.2592 * .0001}\right)$ = 9.1075.

The approximate modified duration is 9.1075.

ApproxCon = $\frac{99.1689 + 99.3497 - (2 \times 99.2592)}{.0001^2 * 99.2592}$ = 201.493

The approximate convexity is 201.493.

3. The convexity-adjusted percentage price drop resulting from a 100 bp increase in the yield to maturity is estimated to be -8.1% (-9.1075 + 1.00746).

Modified duration alone estimates the percentage drop to be 9.1075%. The convexity adjustment adds 100.746 bps (0.5 × 201.493 × $.01^2$ = 1.00746%).

4. The new full price if the yield to maturity goes from 6.74% to 7.74% on that settlement date is 90.7623. The actual percentage change in the bond price is -8.5603%. The convexity-adjusted estimate is -8.1%.

Example 16: Calculating the approximate modified duration and approximate convexity

The investment manager for a US defined-benefit pension scheme is considering two bonds about to be issued by a large life insurance company. The first is a 25-year, 5% semiannual coupon payment bond. The second is a 75-year, 5% semiannual coupon payment bond. Both bonds are expected to trade at par value at issuance.

Calculate the approximate modified duration and approximate convexity for each bond using a 5 bp increase and decrease in the annual yield to maturity.

Solution:

In the calculations, the yield per semiannual period goes up by 2.5 bps to 2.525% and down by 2.5 bps to 2.475%.

The 25-year bond has an approximate modified duration of 14.18.

PV_+: FV = 100, I/Y = 2.525, PMT = 2.5, N = 50, CPT PV, PV = -99.2945.

PV__: FV = 100, I/Y = 2.475, PMT = 2.5, N = 50, CPT PV, PV = -100.7126.

ApproxModDur = $\frac{100.7126 - 99.2945}{2 * 100 * 0.0005}$ = 14.18 and an approximate convexity of 284.

ApproxCon = $\left(\frac{100.7126 + 99.2945 - (2 * 100)}{100 * 0.0005^2}\right)$ = 284. Similarly, the 75-year bond has an approximate modified duration of 19.51 and an approximate convexity of 708.

10. Investment Horizon, Macaulay Duration and Interest Rate Risk

In this section, we look at how yield volatility affects the investment horizon. We also study the interaction between the investment horizon, market price risk, and coupon reinvestment risk.

10.1 Yield Volatility

If there is a change in a bond's YTM, there will be a corresponding change in the price of a bond. The change in the price can be explained as the product of two factors:
1. Price value of a basis point (PVBP): Impact on bond price of a one basis point change in YTM. This factor is based on the duration and convexity of the bond.
2. Number of basis points: This is the change in yield measured in basis points.

The percentage change in the price of a bond for a given change in yield can also be determined using this equation:

$$\%\Delta PV^{FULL} = (-\text{AnnModDur} * \Delta\text{yield}) + [\frac{1}{2} * \text{AnnConvexity} * (\Delta\text{yield})^2]$$

Example 17: Ranking bonds in terms of interest rate risk

A fixed-income analyst is asked to rank three bonds in terms of interest rate risk. The increases in the yields to maturity represent the "worst case" for the scenario being considered.

Bond	**Modified Duration**	**Convexity**	**ΔYield**
A	3.65	14.8	50 bps
B	5.75	38.7	25 bps
C	12.28	146	15 bps

The modified duration and convexity statistics are annualized. ΔYield is the increase in the annual yield to maturity. Rank the bonds in terms of interest rate risk.

Solution:

Calculate the estimated price change for each bond:

Bond A:

The duration effect is -3.65 × 0.005 = -1.825%.

The convexity effect is $0.5 \times 14.8 \times 0.005^2$ = 0.0185%.

The expected change in bond price is -1.825% + 0.0185% = -1.8065%.

Bond B:

The duration effect is -5.75 × 0.0025 = -1.4375%.

The convexity effect is $0.5 \times 38.7 \times 0.0025^2$ = 0.0121%.

The expected change in bond price is -1.4375% + 0.0121% = -1.4254%.

Bond C:

The duration effect is -12.28 × 0.0015 = -1.842%.

The convexity effect is $0.5 \times 146 \times 0.0015^2$ = 0.0164%.

The expected change in bond price is -1.842% + 0.0164% = -1.8256%.

Bond C has the highest degree of interest rate risk (a potential loss of 1.8256%), followed by Bond A (a potential loss of 1.8065%) and Bond B (a potential loss of 1.4254%).

10.2 Investment Horizon, Macaulay Duration, and Interest Rate Risk

The impact of a sudden change in yield on the price of a bond is of particular concern to short-term investors (price risk). Long-term investors will also be concerned about the impact of a change in yield on the reinvestment income (reinvestment risk). An investor who plans to hold the bond to maturity will only be concerned about reinvestment risk.

Consider another 10-year, 8% annual coupon bond priced at 85.5 and YTM of 10.4%. If the investment horizon is 10 years, the only concern is reinvestment risk. Consider a scenario where interest rates go down. In this case, reinvestment income goes down. When the price of the bond goes up, it does not matter to the investor because at maturity he will simply receive par value.

When interest rates go up, the reinvestment income goes up. If the investment horizon is 4 years, then the major concern is price risk. In this case, the price effect dominates relative to the gain/loss from reinvestment of coupons. If the investment horizon is 7 years, the reinvestment risk and price risk offset each other. For this particular bond the Macaulay duration is 7 years.

Macaulay duration indicates the investment horizon for which coupon reinvestment risk and market price risk offset each other. The assumption is a one-time parallel shift in the yield curve.

Note: Investment horizon is different from the bond's maturity. In this case, the maturity is 10 years while the horizon is 7 years.

The **duration gap** of a bond a bond is defined as the Macaulay duration – investment horizon.

Duration gap = Macaulay duration – Investment horizon

- If Macaulay duration < investment horizon, the duration gap is negative: coupon reinvestment risk dominates.
- If investment horizon = Macaulay duration, the duration gap is zero: coupon reinvestment risk offsets market price risk.
- If Macaulay duration > investment horizon, the duration gap is positive: market price

risk dominates.

Example 18: Calculating duration gap and assessing interest rate risk

An investor plans to retire in 8 years. As part of the retirement portfolio, the investor buys a newly issued, 10-year, 6% annual coupon payment bond. The bond is purchased at par value, so its yield to maturity is 6.00% stated as an effective annual rate.

1. Calculate the approximate Macaulay duration for the bond, using a 1 bp increase and decrease in the yield to maturity, and calculating the new prices per 100 of par value.

2. Calculate the duration gap at the time of purchase.

3. Does this bond at purchase entail the risk of higher or lower interest rates? Interest rate risk here means an immediate, one-time, parallel yield curve shift.

Solution to 1:

The approximate modified duration of the bond is 7.36 = $\left(\frac{100.0736 - 99.9264}{2 \times 100 \times 0.0001}\right)$.

PV_0 = 100, PV_+ = 99.9264

FV = 100, PMT = 6, I/Y = 6.01, N = 10, CPT PV, PV = -99.9264

PV_- = 100.0736; FV = 100, PMT = 6, I/Y = 5.99, N = 10, CPT PV, PV = -100.0736.

The approximate Macaulay duration = 7.36 × 1.06 = 7.8016.

Solution to 2:

Given an investment horizon of 8 years, the duration gap for this bond at purchase is negative: 7.8016 - 8 = -0.1984

Solution to 3:

A negative duration gap entails the risk of lower interest rates. To be precise, the risk is an immediate, one-time, parallel, downward yield curve shift because the coupon reinvestment risk dominates market price risk. The loss from reinvesting coupons at a rate lower than 6% is larger than the gain from selling the bond at a price above the constant-yield price trajectory.

11. Credit and Liquidity Risk

So far in this reading, we focused on how to use duration and convexity to measure the price change given a change in YTM. Now, we will look at what causes the YTM to change.

- The YTM on a corporate bond consists of two parts: government benchmark yield and a spread over the benchmark. Change in YTM can be due to either of these or both.
- The change in the spread can result from a change in credit risk or liquidity risk.
- Credit risk involves the probability of default and degree of recovery if default occurs.
- Liquidity risk refers to the transaction costs associated with selling a bond.

- For a traditional (option-free) fixed rate bond, the same duration and convexity statistics apply if a change occurs in the benchmark yield or a change occurs in the spread.
- A change in benchmark yield could be because of a change in the expected rate of inflation or expected real rate of interest.
- In practice, there is often the interaction between changes in benchmark yields and in the spread over the benchmark, and between credit and liquidity risk. It is rare for any individual component of the YTM to change.

Example 19: Calculating modified duration

Consider a 4-year, 9% coupon paying semi-annual bond with an YTM of 9%. The duration of the bond is 6.89 periods. Calculate the modified duration.

Solution:

Modified duration = $\frac{\text{MacDuration}}{1 + \text{r}} = \frac{6.89}{1 + \frac{.09}{2}}$ = 6.593 periods or 3.297 years.

For a one percent change in the annual YTM, the percentage change in the bond price is 3.297%.

Example 20: Calculating the change in the credit spread on a corporate bond

The (flat) price on a fixed-rate corporate bond falls one day from 96.55 to 95.40 per 100 of par value because of poor earnings and an unexpected ratings downgrade of the issuer. The (annual) modified duration for the bond is 5.32. What is the estimated change in the credit spread on the corporate bond, assuming benchmark yields are unchanged?

Solution:

Given that the price falls from 96.55 to 95.40, the percentage price decrease is 1.191%.
-1.191% ≈ -5.32 × ΔYield,
ΔYield = 0.2239%

Given an annual modified duration of 5.32, the change in the yield to maturity is 22.39 bps.

12. Empirical Duration

The approach used in this reading to estimate duration and convexity with mathematical formulas is called **analytical duration**. This approach implicitly assumes that benchmark yields and credit spreads are uncorrelated with one another.

However, in practice, changes in benchmark yields and credit spreads are often correlated. So, many fixed income analysts use an alternate approach – **Empirical duration**. This approach uses statistical methods and historical bond prices to derive the price-yield relationship for specific bonds or bond portfolios.

For a government bond with little or no credit risk, the analytical and empirical duration would be similar because bond prices are largely driven by changes in the benchmark yield.

However, for a high-yield bonds with significant credit risk, the analytical and empirical duration will be different. In a market stress scenario, many investors switch to high quality government bonds due to which their yields (i.e., benchmark yields) fall. But at the same time the credit spreads on high-yield bonds will widen (i.e., credit spreads and benchmark yields are negatively correlated). The wider credit spreads will fully or partially offset the decline in government benchmark yields. Thus, the empirical duration for high yield bonds will be lower than their analytical duration.

Summary

LO.a: Calculate and interpret the sources of return from investing in a fixed-rate bond.

A bond investor has three sources of return:

- Receiving the full coupon and principal payments on the scheduled dates.
- Reinvesting the interest payments. This is also known as interest-on-interest.
- Potential capital gain or loss on sale of the bond, if the bond is sold before maturity date.

Interest rate risk affects the realized rate of return for any bond investor in two ways: coupon reinvestment risk and market price risk.

- Reinvestment income is directly proportional to interest rate movements. The value of reinvested coupons increases when the interest rate goes up.
- Bond price is inversely proportional to interest rate movements. Bond price decreases when the interest rate goes up.
- Coupon reinvestment risk matters when an investor has a long time horizon. If the investor buys a bond and sells it before the first coupon payment, then this risk is irrelevant.
- Market price risk matters when an investor has a short time horizon.

LO.b: Define, calculate, and interpret Macaulay, modified, and effective durations.

<u>Macaulay duration</u> is the weighted average of the time to receipt of coupon interest and principal payments.

<u>Modified duration</u> is a linear estimate of the percentage price change in a bond for a 100 basis points change in its yield to maturity.

$$\text{Modified duration} = \frac{\text{Macaulay Duration}}{(1 + \text{r})}$$

$$\%\,\Delta\,\text{PV}^{\text{FULL}} \approx\ -\text{AnnModDur} * \Delta\text{Yield}$$

$$\text{Approximate Modified Duration} = \frac{(\text{PV}_{-}) - (\text{PV}_{+})}{2 * \Delta\,\text{yield} * \text{PV}_0}$$

<u>Effective duration</u> is a linear estimate of the percentage change in a bond's price that would result from a 100 basis points change in the benchmark yield curve.

$$\text{Effective Duration} = \frac{(\text{PV}_{-}) - (\text{PV}_{+})}{2 * \Delta\text{curve} * \text{PV}_0}$$

LO.c: Explain why effective duration is the most appropriate measure of interest rate risk for bonds with embedded options.

The difference between modified duration and effective duration is that modified duration measures interest rate risk in terms of a change in the bond's own yield to maturity, whereas effective duration measures interest rate risk in terms of changes in the benchmark yield curve.

Bonds with an embedded option do not have a meaningful internal rate of return (YTM) because future cash flows are contingent on interest rates.

Therefore, effective duration is the appropriate interest rate risk measure, not modified duration.

LO.d: Define key rate duration and describe the use of key rate durations in measuring the sensitivity of bonds to changes in the shape of the benchmark yield curve.

Key rate duration is a measure of the price sensitivity of a bond to a change in the spot rate for a specific maturity.

Key rate durations can be used to measure a bond's sensitivity to changes in the shape of the yield curve, i.e., for non-parallel shifts in the yield curve.

LO.e: Explain how a bond's maturity, coupon, and yield level affect its interest rate risk.

The interest rate risk of a bond is measured by duration. All else equal:

- Duration increases when maturity increases.
- Duration decreases when coupon rate increases.
- Duration decreases when yield to maturity increases.

LO.f: Calculate the duration of a portfolio and explain the limitations of portfolio duration.

The weighted average of the time to receipt of aggregate cash flows

- This method is better in theory.
- Its main limitation is that it cannot be used for bonds with embedded options or for floating-rate notes.

The weighted average of the durations of individual bonds that compose the portfolio

- This method is simpler to use and quite accurate when the yield curve is relatively flat.
- Its main limitation is that it assumes a parallel shift in the yield curve.

LO.g: Calculate and interpret the money duration of a bond and price value of a basis point (PVBP).

The money duration of a bond is a measure of the price change in units of the currency in which the bond is denominated, given a change in annual yield to maturity.

$$\text{Money Duration} = \text{AnnModDur} * \text{PV}^{\text{FULL}}$$

$$\Delta\ \text{PV}^{\text{FULL}} \approx -\text{MoneyDur} * \Delta\text{yield}$$

The PVBP is an estimate of the change in the full price, given a 1bp change in the yield to maturity.

$$\text{PVBP} = \frac{\text{PV}_{-} - \text{PV}_{+}}{2}$$

LO.h: Calculate and interpret approximate convexity and compare approximate and effective convexity.

A bond's convexity can be estimated as:

$$\text{Approx. Convexity} = \frac{\text{PV}_{-} + \text{PV}_{+} - 2 * \text{PV}_0}{(\Delta\text{yield})^2 * \text{PV}_0}$$

Effective convexity of a bond is a curve convexity statistic that measures the secondary effect of a change in a benchmark yield curve. It is used for bonds with embedded options.

$$\text{Effective Convexity} = \frac{\text{PV}_{-} + \text{PV}_{+} - 2 * \text{PV}_0}{(\Delta\text{curve})^2 * \text{PV}_0}$$

LO.i: Calculate the percentage change of a bond for specified change in yield, given the bond's approximate duration and convexity.

The percentage change of a bond can be calculated using the following formula if modified duration and convexity are given:

$$\%\ \Delta\ \text{PV}^{\text{FULL}} = (-\text{AnnModDur} * \Delta\text{yield}) + [\frac{1}{2} * \text{AnnConvexity} * (\Delta\text{yield})^2]$$

LO.j: Describe how the term structure of yield volatility affects the interest rate risk of a bond.

If there is a change in a bond's YTM, there will be a corresponding change in the price of a bond. The change in the price can be explained as the product of two factors:

1. Price value of a basis point (PVBP): Impact on bond price of a one basis point change in YTM. This factor is based on the duration and convexity of the bond.
2. Number of basis points: This is the change in yield measured in basis points.

The percentage change in the price of a bond for a given change in yield can also be determined using this equation:

$$\%\ \Delta\ \text{PV}^{\text{FULL}} = (-\text{AnnModDur} * \Delta\text{yield}) + [\frac{1}{2} * \text{AnnConvexity} * (\Delta\text{yield})^2]$$

LO.k: Describe the relationships among a bond's holding period return, its duration, and the investment horizon.

The Macaulay duration can be interpreted as the investment horizon for which coupon reinvestment risk and market price risk offset each other.

Duration gap = Macaulay duration – Investment horizon

- If the investment horizon is greater than the Macaulay duration of the bond, coupon reinvestment risk dominates price risk. The investor's risk is that interest rates will

fall. The duration gap is negative.
- If the investment horizon is equal to the Macaulay duration of the bond, coupon reinvestment risk offsets price risk. The duration gap is zero.
- If the investment horizon is less than the Macaulay duration of the bond, price risk dominates coupon reinvestment risk. The investor's risk is that interest rates will rise. The duration gap is positive.

LO.l: Explain how changes in credit spread and liquidity affect yield to maturity of a bond and how duration and convexity can be used to estimate the price effect of the changes.

- The YTM on a corporate bond consists of two parts: government benchmark yield and a spread over the benchmark. Change in YTM can be due to either of these or both.
- The change in the spread can result from a change in credit risk or liquidity risk.
- Credit risk involves the probability of default and degree of recovery if default occurs.
- Liquidity risk refers to the transaction costs associated with selling a bond.
- For a traditional (option-free) fixed rate bond, the same duration and convexity statistics apply if a change occurs in the benchmark yield or a change occurs in the spread.
- A change in benchmark yield could be because of a change in the expected rate of inflation or expected real rate of interest.

LO.m: Describe the difference between empirical duration and analytical duration.

- Analytical duration approach uses mathematical formulas to estimate duration and convexity. The approach implicitly assumes that benchmark yields and credit spreads are uncorrelated with one another.
- The empirical duration approach does not rely on this assumption. It uses statistical methods and historical bond prices to derive the price-yield relationship for specific bonds or bond portfolios.
- For government bonds, both approaches will give similar results.
- For high yield bonds, the empirical duration will be lower than analytical duration.

Practice Questions

1. A buy and hold investor purchases a zero coupon bond at a discount and holds the security until it matures. Which of the following sources of return is *most likely* the largest component of the investor's total return?
   A. Capital gain.
   B. Principal payment.
   C. Reinvestment of coupon payments.

2. An investor purchases a 10-year, 6% annual coupon bond with an YTM of 8%. After the bond is purchased and before the first coupon is received, the YTM of the bond changes to 9%. Assuming coupon payments are reinvested at the YTM, the investor's return when the bond is held to maturity is:
   A. less than 8%.
   B. more than 8%.
   C. equal to 8%.

3. Holding all other characteristics the same, the bond exposed to the highest level of reinvestment risk is *most likely* the one selling at:
   A. par.
   B. premium.
   C. discount.

4. A 12% annual pay bond has 10 years to maturity. The bond is currently trading at par. Assuming a 10 basis points change in yield to maturity, the bond's approximate modified duration is *closest* to:
   A. 5.15.
   B. 5.94.
   C. 6.46.

5. The modified duration of a bond is 6.54. The approximate percentage change in price using duration only for a yield decrease of 120 basis points is *closest* to:
   A. 6.54%
   B. 7.74%
   C. 7.84%

6. Which of the following measures is *most* appropriate for measuring the interest rate risk of a bond with an embedded call option?
   A. Effective duration.
   B. Modified duration.
   C. Macaulay duration.

7. Bond A has a coupon rate of 6%. Bond B has a coupon rate of 4%. All else equal:
   A. bond A will have a higher Macaulay duration.
   B. bond B will have a higher Macaulay duration.
   C. both bonds will have the same Macaulay duration.

8. Which of the following is a limitation of calculating a bond portfolio's duration as the weighted average of the yield durations of the individual bonds that compose the portfolio?
   A. It assumes a parallel shift in the yield curve.
   B. It cannot be applied if the portfolio includes bonds with embedded options.
   C. It is less accurate when the yield curve is less steeply sloped.

9. A bond with exactly 6 years remaining until maturity offers a 4% coupon rate with annual coupons. The bond, with a yield to maturity of 5%, is priced at $94.9243 per 100 of par value. The estimated price value of a basis point for the bond is *closest* to:
   A. 0.182.
   B. 0.098.
   C. 0.049.

10. The following table shows details for a bond.

Par	PVFlat	PVFull	ModDur	EffDur
$100	$102	$105	6.30	6.27

Which of the following is *most likely* to be the approximate money duration (MoneyDur) of the bond?
A. 643.
B. 658.
C. 662.

11. A bond has a convexity of 95.6. The convexity effect, if the yield decreases by 120 basis points is *closest* to:
   A. 0.688%
   B. 0.724%
   C. 0.956%

12. A bond has an annual modified duration of 8.010 and annual convexity of 75.270. If the bonds' yield to maturity decreases by 50 basis points, the expected percentage price change is *closest* to:
   A. 4.005%
   B. 4.099%
   C. 5.105%
13. A bond is trading at 119.25. If the bond's YTM falls by 15 basis points he bond's full price

is expected to rise to 125.00. If the bond's YTM rises by 15 basis points the bond's full price is expected to fall to 117.75. The approximate convexity of the bond is *closest* to:
A. 15111.
B. 15000.
C. 15840.

14. If an investor's investment horizon is more than the Macaulay duration of the bond he owns:
A. the investor is hedged against interest rate risk.
B. reinvestment risk dominates, and the investor is at the risk of lower rates.
C. market price risk dominates, and the investor is at the risk of higher rates.

15. An investor purchases an annual coupon bond with a 6.25% coupon rate and 7 years remaining until maturity. The investor's investment horizon is two years. The Macaulay duration of the bond is 5.75 years. The duration gap at the time of purchase is *closest* to:
A. 3.75 years.
B. -3.75 years.
C. 0 years.

16. When an investor's horizon is less than Macaulay duration of the bond he owns:
A. the duration gap is negative.
B. market price risk dominates reinvestment risk.
C. market price risk offsets reinvestment risk.

17. A software company receives a ratings downgrade and the price of its fixed-rate bond decreases. The price decrease was *most likely* caused by a(n):
A. decrease in the bond's liquidity spread.
B. decrease in the bond's underlying benchmark rate.
C. increase in the bond's credit spread.

18. Due to downgrading of ratings, a fixed rate corporate bond's flat price decreases from 93.25 to 91.65 per 100 of par value. Assuming that the annual modified duration of this bond is 5.66, and that benchmark yields are unchanged, the *closest* estimated change in credit spread on the corporate bond is:
A. 30 bps.
B. 31 bps.
C. 17 bps.

19. Which of the following factors will *least likely* influence the benchmark rate on an option free, fixed rate bond?
A. Credit risk of the bond.

B. Expected inflation rate.
C. Expected real rate of interest.

Solutions

1. B is correct. A zero coupon bond makes no coupon payments hence it has no reinvestment income. Since the bond is held to maturity, there will be no capital gain/loss. The largest component of the investor's total return will be principal payment.

2. B is correct. Initially the YTM of the bond was 8%. The YTM then increased to 9%. The increase in YTM will increase the reinvestment income over the life of the bond, so the investor will earn more than 8%.

3. B is correct. A bond selling at a premium has a higher coupon rate. All else being equal, bonds with higher coupon rates have higher reinvestment risk. The reason is that the higher the coupon rate, the more dependent the bond's total dollar return will be on the reinvestment of the coupon payments in order to produce the yield to maturity at the time of purchase.

4. A is correct. The bond is priced at par which means that the initial YTM= coupon rate = 12% and V_0 = 100
   ΔYTM = 0.001
   V_- = 100.57
   N = 10, PMT = 12, FV = 100, I/Y = 11.9; CPT → PV = 100.57
   V_+ = 99.44
   I/Y = 12.1; CPT→ PV = 99.44
   $$\text{Approximate modified duration} = \frac{V_- - V_+}{2V_0\Delta\text{YTM}} = \frac{100.57 - 99.44}{2 \times 100 \times 0.001} = 5.65$$

5. C is correct. -6.54 x -0.012 = 0.07848 = 7.848%

6. A is correct. The interest rate risk of a fixed-rate bond with an embedded call option is best measured by effective duration. A callable bond's future cash flows are uncertain because they are contingent on future interest rates. The issuer's decision to call the bond depends on future interest rates. Therefore, the yield to maturity on a callable bond is not well defined. Only effective duration, which takes into consideration the value of the call option, is the appropriate interest rate risk measure. Yield durations like Macaulay and modified durations are not relevant for a callable bond because they assume no changes in cash flows when interest rates change.

7. B is correct. The coupon rate is inversely proportional to Macaulay duration.

8. A is correct. A limitation of calculating a bond portfolio's duration as the weighted average of the yield durations of the individual bonds, is that this measure implicitly

assumes a parallel shift to the yield curve (all rates change by the same amount in the same direction). In reality, interest rate changes frequently result in a steeper or flatter yield curve. This approximation of the "theoretically correct" portfolio duration is more accurate when the yield curve is flatter (less steeply sloped). An advantage of this approach is that it can be used with portfolios that include bonds with embedded options. Bonds with embedded options can be included in the weighted average using the effective durations for these securities.

9. C is correct.
PVBP = (PV- - PV+)/2 x par value x 0.01
PV- (Full price calculated by lowering the yield to maturity by one basis point)
N= 6, I/Y = 4.99, PMT=4, FV = 100. CPT PV = $94.9735
PV+(Full price calculated by raising the yield to maturity by one basis point)
N= 6, I/Y = 5.01, PMT=4, FV = 100. CPT PV = $94.8752
PVBP = ($94.9735 - $94.8752)/2 = 0.049

10. C is correct.
$\text{MoneyDur} = \text{ModDur} * \text{PV}^{\text{Full}} = 6.30 * 105 \approx 662$

11. A is correct. Convexity effect = ½ x convexity x$(\Delta \text{YTM})^2$ = ½ x 95.6 x 0.012^2 = 0.0068832 or 0.688%

12. B is correct.
Total estimated price change = duration effect + convexity effect.
= - duration x ΔYTM + ½ x convexity x $(\Delta \text{YTM})^2$
= - 8.010 x -0.005 + ½ x 75.270 x -0.005^2 = 0.04099 or 4.099%

13. C is correct.
$$\text{Approximate convexity} = \frac{\text{PV}_- + \text{PV}_+ - 2 * \text{PV}_0}{(\Delta\text{Yield})^2 * \text{PV}_0} = \frac{125 + 117.75 - (2 * 119.25)}{0.0015^2 * 119.25}$$
$\text{Approximate convexity} = 15840$

14. B is correct. The duration gap is equal to the bond's Macaulay duration minus the investment horizon. In this case, the duration gap is negative and reinvestment risk dominates price risk. The investor risk is that interest rates will fall.

15. A is correct.
Duration gap = Macaulay duration - Investment horizon = 5.75 – 2 = 3.75 years.

16. B is correct.
Duration gap = Macaulay duration – Investment horizon

- If Macaulay duration < investment horizon, the duration gap is negative: coupon reinvestment risk dominates.
- If investment horizon = Macaulay duration, the duration gap is zero: coupon reinvestment risk offsets market price risk.
- If Macaulay duration > investment horizon, the duration gap is positive: market price risk dominates.

17. C is correct. The change was most likely caused by an increase in the bond's credit spread. The increase in credit risk results in a larger credit spread.

18. A is correct.

    Percentage price change $= \dfrac{93.25 - 91.65}{93.25} = 0.01716$ or 1.716%

    Change in Yield $= \frac{-0.01716}{-5.66} = 0.003032$ or 30 bps

19. A is correct. A change in benchmark yield could be because of a change in the expected rate of inflation or expected real rate of interest. The changes in the credit risk and/or changes in the liquidity of the bond can cause the changes in the yield spread of an option-free, fixed-rate bond.

