R49 Portfolio Risk and Return: Part I

Version 1.0

1. Introduction

This reading covers:

- Investment characteristics of assets in terms of their return and risk.
- How to determine what assets are appropriate for a portfolio.
- How to construct an indifference curve and use it in the selection of an optimal portfolio using two risky assets.
- How to construct an optimal risky portfolio.

2. Investment Characteristics of Assets: Return

2.1 Return

Return can come in two forms:

- Periodic income through interest payments or dividends.
- Capital gains when the price of the asset you hold increases.

We now look at the various types of return measures and their applicability.

Holding Period Return

Holding period return is the return earned on an asset during the period it was held. It is calculated as a sum of capital gain (price appreciation) and periodic income.

$$\text{HPR} = \frac{P_T - P_0 + I_T}{P_0}$$

where:
$P_T - P_0$ = capital gain component
I_T = Income earned during period T
P_T = price at the end of the period
P_0 = price at the beginning of the period

Arithmetic or Mean Return

Arithmetic return is a simple arithmetic average of returns. Assume you have three stocks A, B, and C with returns of 10%, 20%, and 30% respectively. The collective return from the three stocks is (10 + 20 + 30)/3 = 20%.

Geometric Mean Return

Geometric mean return is the compounded rate of return earned on an investment.

$$\text{Geometric mean return} = [(1 + R_{i1}) * (1 + R_{i2}) * \ldots..* (1 + R_{iT})]^{\frac{1}{T}} - 1$$

Assume you have a stock A which returns 10%, 20%, and 30% in years 1, 2, and 3 respectively.

What is the mean return earned?

Geometric mean return = $[(1.1)\ (1.2)\ (1.3)]^{0.333} - 1$ = 19.7%

3. Money-Weighted Return or Internal Rate of Return

Money-weighted return is the internal rate of return on money invested that considers the cash inflows and cash outflows, and calculates the return on actual investment.

Money-weighted return is a useful performance measure when the investment manager is responsible for the timing of cash flows. This is often the case for private equity fund managers.

Example

Given the data below, compute the holding period return, arithmetic mean return, geometric mean return, and money-weighted return. Assume no withdrawals except at the end of year 3.

Year	Assets under management at start of year (millions of $)	Net return
1	30	10%
2	33	-5%
3	35	15%

Solution:

Holding period return:

Holding period = 3 years

Return = 1.1 x 0.95 x 1.15 = 1.20175 = 20.175%

Arithmetic mean:

Return = (10 – 5 + 15)/3 = 6.67%

Geometric mean:

Return = $(1.1 \text{ x } 0.95 \text{ x } 1.15)^{1/3} - 1 = 6.317\%$

Money-weighted return:

To calculate the money-weighted return, we must know the net cash flows (cash inflows and outflows) for every year. So, let us draw a table and fill in the values and derive some others to get the values for CF_0, CF_1, CF_2, and CF_3.

	Year 1	Year 2	Year 3
Balance from previous year	0	33.00	31.35
New investment by investor	30.00	0	3.65
Withdrawal by investor	0	0	0
Net balance at start of year	30.00	33.00	35.00
Investment return for year	10%	-5%	15%
Investment gain (loss)	3.00	(1.65)	5.25
Balance at end of year	33.00	31.35	40.25

Now that we have the cash flows for the three years, let's use the financial calculator to calculate IRR. CF_0 = -30; CF_1 = 0; CF_2 = -3.65; CF_3 = +40.25

IRR = 6.62%

4. Time-Weighted Rate of Return

The time-weighted rate of return measures the compound growth rate of $1 initially invested in the portfolio over a stated measurement period.

The time-weighted return can be calculated using the following steps:

1. Break the overall evaluation period into sub-periods based on the dates of cash inflows and outflows.
2. Calculate the holding period return on the portfolio for each sub-period.
3. Link or compound holding period returns to obtain an annual rate of return for the year (the time-weighted rate of return for the year).
4. If the investment is for more than a year, take the geometric mean of the annual returns to obtain the time-weighted rate of return over that measurement period.

Consider the following example:

Time	*Outflow*	*Inflow*
0	$20.00 to purchase the first share	
1	$22.50 to purchase the second share	$0.50 dividend received on first share
		$1.00 dividends ($0.50 x 2 shares); $47.00 from selling 2 shares @ $23.50 per share

Calculating the TWRR for this example is relatively simple because cash flows only occur at the start/end of every year. We will follow the steps mentioned earlier:

<u>Steps 1</u>: Break into evaluation period and value the portfolio at start/end of every period.

- Value of the portfolio at the start of Year 1 (t = 0) is $20.00.
- Value of portfolio at the end of Year 1 (t = 1) before the purchase of the new share is 22.50 + 0.50 = $23.00. Note that the dividend of $0.50 on the first share is received at the end of Year 1.
- Value of the portfolio at the start of Year 2 (t = 1) after the purchase of the second share is 22.50 + 22.50 = $45.00. The dividend of $0.50 from the first share is paid out and is not considered as part of the portfolio.
- Value of the portfolio at the end of Year 2 (t = 2) is 23.50 + 23.50 + 0.50 + 0.50 = $48.00. Both shares pay a dividend of $0.50 at the end of the second year.

<u>Step 2</u>: Calculate the holding period return on the portfolio for each sub-period.

- In this question the cash flows are taking place at the start/end of each period. Hence there are no sub-periods. *Scenarios involving sub-periods will be covered in the next example.*

<u>Step 3</u>: Link or compound holding period returns to obtain an annual rate of return for the year.

- The annual rate of return is based on the portfolio value at the start and end of each period.
- The portfolio value at the start of Year 1 was \$20.00 and the value at the end of Year 1 was \$23.00. Hence the holding period return was 15.00%.
- The portfolio value at the start of Year 2 was \$45.00 and the value at the end of Year 2 was \$48.00. Hence the holding period return was 6.67%.

Step 4: If the investment is for more than a year, take the geometric mean of the annual returns to obtain the time-weighted rate of return over that measurement period.

The TWRR is calculated as: $(1.15 * 1.067)^{\frac{1}{2}} - 1 = 0.1077 = 10.77\%$.

Money-weighted v/s time-weighted returns

- The money-weighted rate of return is impacted by the timing and amount of cash flows.
- The time-weighted rate of return is not impacted by the timing and amount of cash flows.
- If funds are added to an investment portfolio just before a period of relatively high returns, the money-weighted rate of return will be higher than the time-weighted rate of return, and vice versa.
- The time-weighted return is an appropriate performance measure if the portfolio manager does not control the timing and amount of investment.
- On the other hand, money-weighted return is an appropriate measure if the portfolio manager has control over the timing and amount of investment.

5. Annualized Return

Annualized return converts the returns for periods that are shorter or longer than a year, to an annualized number for easy comparison.

$$\text{Annualized return } = \left(1 + r_{\text{period}}\right)^{c} - 1$$
$$\text{Where } c = \text{number of periods in year}$$

6. Other Major Return Measures and Their Applications

Gross Return

Gross return is the return earned by an asset manager prior to deducting management fees and taxes. It measures the investment skill of a manager.

Net Return

Net return is the return earned by the investor on an investment after all managerial and administrative expenses have been accounted for. This is the measure of return that should matter to an investor.

Assume an investment manager generates \$120 for every \$100, and charges a 2% fee for management and administrative expenses. The gross return, in this case, is 20% and the net return is 18%.

Pre-tax and After-tax Nominal Return

The returns we saw till now were pre-tax nominal returns, i.e., before deducting any taxes or any adjustments for inflation. This is the default, unless otherwise stated.

After-tax nominal return is the return after accounting for taxes. The actual return an investor earns should consider the tax implications as well.

In the example that we saw above for gross and net return, 18% was the pre-tax nominal return. If the tax rate for the investor is 33.33%, then the after-tax nominal return will be 18(1 - 0.3333) = 12.0006%.

Real Return

Real return is the return after deducting taxes and inflation.

$(1 + r) = (1 + r_{real})(1 + \pi)$

where:

r_{real} = real rate

π = rate of inflation

r = nominal rate

In the previous example, the after-tax nominal return was 12%. Assume the inflation rate for the period is 10%. What is the real rate of return?

Using the above formula, (1 + 0.12) = (1 + r) (1 + 0.1). Solving for r, we get 1.818%.

Instructor's tip: If the answer choices are close to each other, use this formula to determine the correct answer. Else, you may use an approximation to solve for r quickly as nominal rate = real rate + inflation.

Leveraged Return

In cases, where an investor borrows money to invest in assets like bonds or real estate, the leveraged return is the return earned by the investor on his money after accounting for interest paid on borrowed money.

Portfolio Expected Return and Variance

For a two-asset portfolio, the expected return and variance can be computed as:

$E(R_P) = w_1R_1 + w_2R_2$

$\sigma_p^2 = w_1^2\,\sigma_1^2 + w_2^2\,\sigma_2^2 + 2\,w_1\,w_2\,\sigma_1\,\sigma_2\,\rho$

Example

My portfolio consists of two stocks X and Y. X represents 60% of the portfolio and Y the remaining 40%. X has an expected return of 12% and a standard deviation of 16%. Y has an expected return of 20% and a standard deviation of 30%. The correlation is 0.5. What is the expected return and risk of my portfolio? How does the return/risk change when the weights of X and Y change?

Solution:

$E(R_P) = w_1R_1 + w_2R_2$

$E(R_P) = 0.6 \text{ x } 12 + 0.4 \text{ x } 20 = 15.2\%$

We can calculate the portfolio variance:

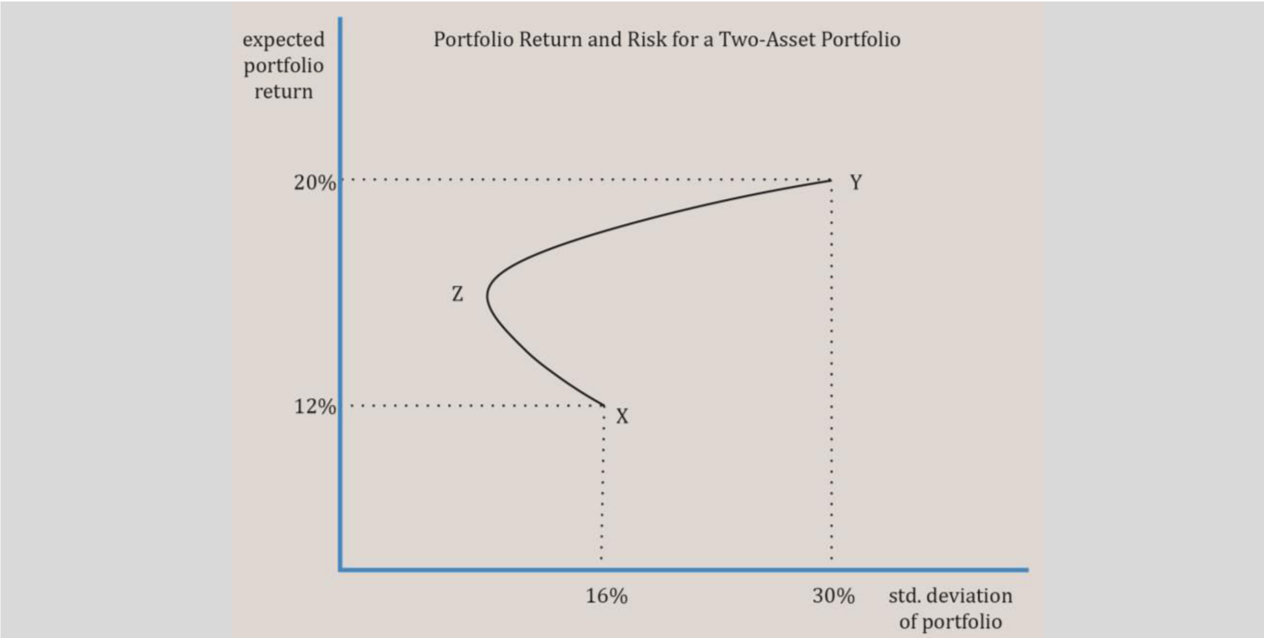
$\sigma_p^2 = (0.6)^2 (0.16)^2 + (0.4)^2 (0.3)^2 + 2 (0.6) (0.4) (0.16) (0.3) (0.5) = 0.0351$

$\sigma_p = 0.1874 = 18.74\%$

To understand how return/risk change when the weights of X and Y change, we will calculate the risk and return for different weights and plot a curve as shown in the graph below.

- Point X represents 100% of the portfolio invested in stock X with return of 12% and standard deviation of 16%.
- Point Y represents 100% of the portfolio invested in stock Y with expected return of 20% and a standard deviation of 30%.
- The curve between X and Z (to the left of X) represents the region where amount invested in stock X is decreased while the weight of Y is increased. This region is where the benefit of diversification is seen, i.e., the expected return increases while risk goes lower.
- Beyond the Point Z, when the weight of Y is increased till the point Y where 100% is invested in Y, there is no diversification benefit. For any point between Z and Y, the risk and return increase.

The graph below plots portfolio risk and return for a two-asset portfolio and shows the impact of correlation of assets on portfolio risk. As you can see, there is no risk-return trade-off when 100% is invested either in X or Y.



7. Historical Return and Risk

Historical return is the return actually earned in the past, while expected return is the return one expects to earn in the future.

Historical data shows that higher returns were earned in the past by assets with higher risk. Of the three major asset classes in the U.S., namely stocks, bonds, and T-bills, it has been observed that stocks had the highest risk and return followed by bonds, and T-bills had the lowest return and lowest risk.

Asset Class	Annual Average Return	Standard Deviation (Risk)
Small-cap stocks	High	High
Large-cap stocks	↓	↓
Long-term corporate bonds		
Long-term treasury bonds	Low	Low
Treasury bills		

8. Other Investment Characteristics

In evaluating investments using mean and variance, we make the following two assumptions:

- Assumption 1: Distributional characteristics.
  Returns are normally distributed. If this assumption does not hold, then consider skewness and kurtosis. These concepts have been covered earlier in quantitative methods.

- Assumption 2: Market characteristics.
  Markets are informationally and operationally efficient. If markets are not operationally efficient, then it leads to: 1) high transaction costs 2) wide bid-ask spreads and 3) larger price impact of trades.

9. Risk Aversion and Portfolio Selection & The Concept of Risk Aversion

9.1 The Concept of Risk Aversion

Risk aversion refers to the behavior of investors preferring less risk to more risk.

Risk tolerance is the amount of risk an investor is willing to take for an investment. High risk aversion is the same as low risk tolerance. Three types of risk profiles are outlined below:

- **Risk Seeking**: A risk seeking investor prefers high return and high risk. Given two investments, A and B, where both have the same return but investment A has higher risk, he will prefer investment A.
- **Risk Neutral**: A risk neutral investor is concerned only with returns, and is indifferent about the risk involved.
- **Risk Averse**: A risk averse investor will prefer an investment that has lower risk, all else equal. Given two investments, A and B, where both have the same return but investment A has higher risk, he will prefer investment B.

10. Utility Theory and Indifference Curves

The utility of an investment can be calculated as:

Utility **=** E(r) – 0.5 x A x σ^2

where:
A = measure of risk aversion (the marginal benefit expected by the investor in return for taking additional risk. *A* is higher for risk-averse individuals.
σ^2 = variance of the investment
E(r) = expected return

Instructor's Note:
While using this formula, use only decimal values for all parameters.

Example

An investor with A = 2 owns a risk-free asset returning 5%. What is his utility?

Solution:

Utility = 0.05 - 0.5 x 2 x 0 = 0.05

Now, he is considering an asset with σ = 10%. At what level of return will he have the same utility?

$0.05 = E(r) – 0.5 \times 2 \times 0.1^2$. Solving for E(R) we get 0.06 or 6.00%.

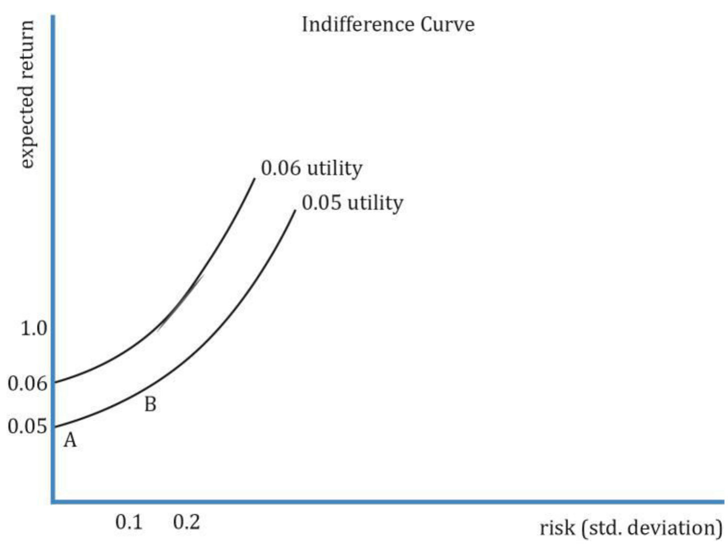
Given a choice between a risk-free asset and stock with an expected return of 10% and σ = 20%, what will he prefer?

$U = 0.1 - 0.5 * 2 * 0.2^2 = 0.06$. Since the utility number of 0.06 is higher than 0.05, the investor will prefer this investment over the previous one.

Indifference Curves

The indifference curve is a graphical representation of the utility of an investment. An indifference curve plots various combinations of risk-return pairs that an investor would accept to maintain a given level of utility. If the combinations of risk-return on a curve provide the same level of utility, then the investor would be indifferent to choosing one. Each point on an indifference curve shows that the investor is indifferent to what investment he chooses (risk/return combination) as long as the utility is the same.

To understand the indifference curve, let us plot all these numbers - expected return, standard deviation, and utility - on a graph.

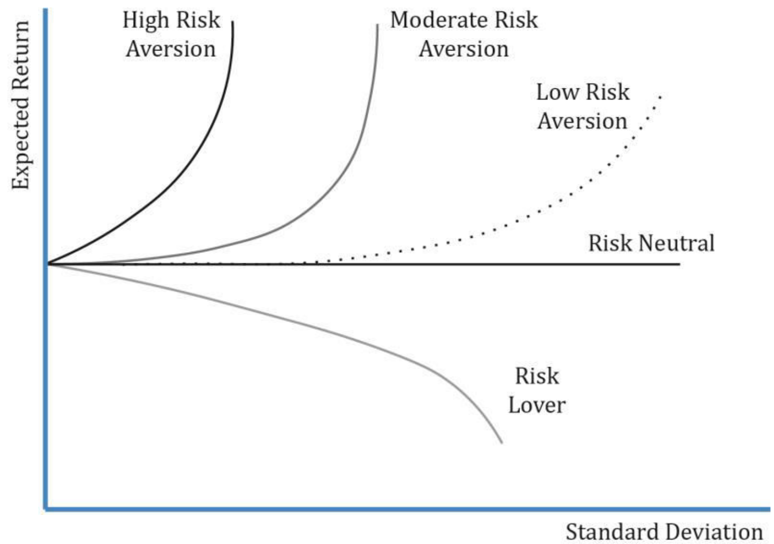


The key points to note are:

- The indifference curves run from south-west to north-east. This is intuitive because for taking on higher risk (going east), investors expect to be compensated with higher expected return (going north).
- The utility of points A and B are 0.05. This means that the investor will be indifferent about choosing an investment with E(r) of 5%, σ = 0 versus another with E(r) of 6%, σ = 10% as both have the same utility. Similarly, for assuming higher levels of risk, the investor is compensated with higher return as shown in the 0.05 utility curve. Along this curve, the investor is indifferent about choosing one point (investment) over the other.
- As the investor moves north-west, he is happier as his utility increases, in this case

from 0.05 to 0.06. Risk is compensated with higher returns and it signifies higher risk aversion.

We now consider the indifference curves for different types of investors.



The exhibit above shows the indifference curves for different types of investors with expected return on y-axis and standard deviation on x-axis. Note the following:

- Risk-neutral investor: For a risk-neutral investor, the utility is the same irrespective of risk as the investor is concerned only about the return. The expected return is constant.
- Risk-averse investor: For a risk-averse investor, the curve is upward sloping, as the investor expects additional return for taking additional risk. Finance theory assumes that most investors are risk averse, but the degree of aversion may vary. The curve is steeper for investors with high risk-aversion.
- Risk-seeking investor: The curve is downward sloping as the expected return decreases for higher levels of risk. It is not commonly seen.

11. Application of Utility Theory to Portfolio Selection

Now that we have seen utility theory and indifference curves for various investors, let us see how to apply it in portfolio selection. The simplest case is when a portfolio comprises two assets: a risk-free asset and a risky asset. For a high-risk averse investor, the choice is easy, to invest 100% in the risk-free asset but at the cost of lower returns. Similarly, for a risk-lover it would be to invest 100% in the risky asset. But is it the optimal allocation of assets, or can there be a trade-off?

Example

Consider a simple portfolio of a risk-free asset and a risky asset. Plot the expected return of the portfolio against the risk of the portfolio for different weights of the two assets.

Risk-free asset	Risky asset

R_f = 5%	R_i = 10%
σ = 0	σ_i = 20%

Solution:

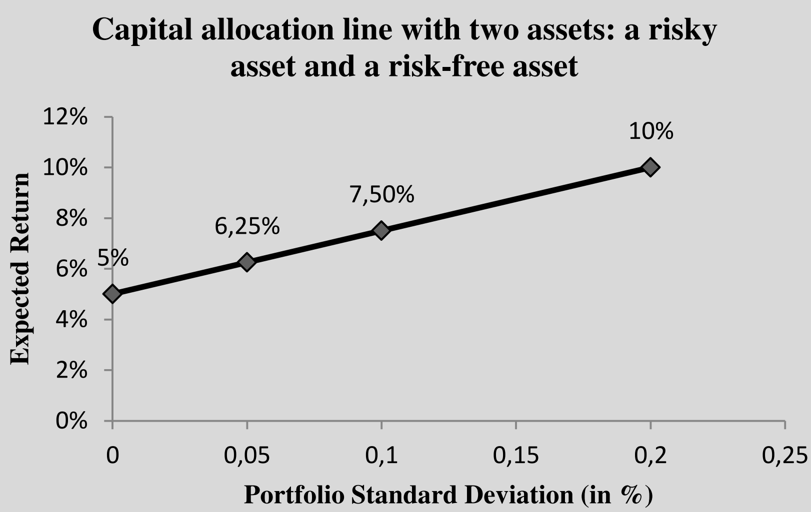
A portfolio's standard deviation is calculated as:

$$\sigma_p = \sqrt{w_1^2 * \sigma_1^2 + w_2^2 * \sigma_2^2 + 2 * w_1 * w_2 * \rho_{1,2} * \sigma_1 * \sigma_2}$$

Given that σ_1 = 0, the terms $w_1^2 * \sigma_1^2$ and $2 * w_1 * w_2 * \rho_{1,2} * \sigma_1 * \sigma_2$ become zero as the correlation between a risk-free asset with risky asset is zero.

So, the equation becomes $\sigma_p = w_2 * \sigma_2$. For different weights of the risky asset, let's calculate the values for risk and return and plot portfolio risk-return on a graph. Note that the expected return is a weighted average of the two assets.

Weight of risky asset = w_2	**σ_p (portfolio risk) = $w_2 * \sigma_2$**	**R_p (Portfolio return) = $w_1 * R_f + w_2 * R_i$**
0	0	5%
0.25	5%	6.25%
0.5	10%	7.5%
1	20%	10%



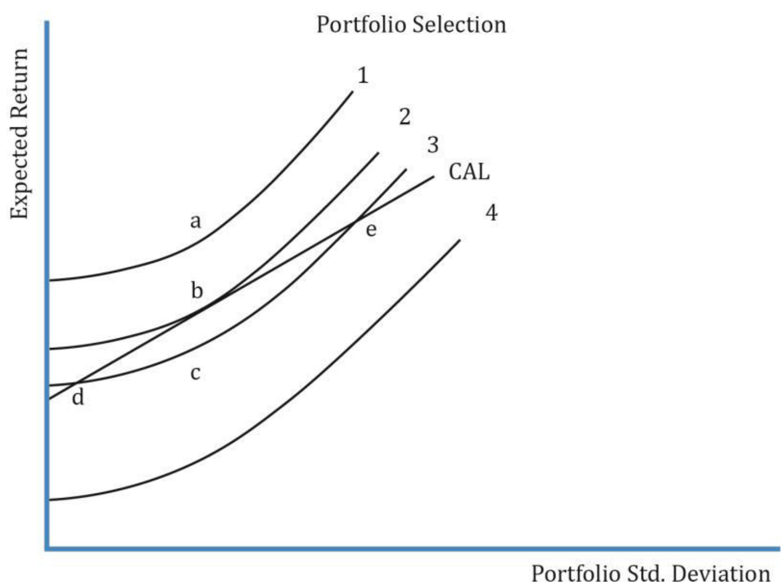
Graph interpretation:

- What you get by plotting risk and return values for different weights of the risky and risk-free asset is a straight line (linear).
- As you move north-east, the weight of the risky asset increases. The lowest point on the left corner, intersecting the y-axis with expected return of 5%, represents 100% in the risk-free asset. Similarly, the topmost point on the right corner represents 100% in the risky asset with expected return of 10%.
- The straight line you see in the chart above is called the capital allocation line which

represents the portfolios available to the investor as each point on the line is an investable portfolio.
- So, how does an investor decide which portfolio to invest in? The answer is by combining an investor's indifference curves and CAL to determine the optimal portfolio. We look at it in detail in the next section.

What is the Optimal Portfolio?

We get the optimal portfolio by combining the indifference curves and the capital allocation line. The utility theory gives us the indifference curves for an individual, while the capital allocation lines give us a set of feasible investments. The optimal portfolio for an investor will lie somewhere on the capital allocation line, i.e., some combination (weight) of the risky asset and risk-free asset. Using an investor's risk-aversion measure A, we can use the utility theory to plot the indifference curves for an individual. Assume the indifference curves for this investor look as shown in the exhibit below:



Some key points about the graph:

- Curves 1, 2, 3, and 4 are various indifference curves for an individual. The indifference curves cannot intersect each other.
- Curve 1 lies above the CAL. Curve 2 is tangential to the CAL at point *b*. Curve 3 intersects CAL at two points: *d* and *e*. Curve 4 lies below the CAL.
- Curve 1 has the highest utility while Curve 4 has the lowest utility. As we discussed before, the investor is happier when we move north-west. Of the four curves, he will be happiest with curve 1. But, as curve 1 lies outside the CAL, it is not possible to construct a portfolio at any point on this curve with the available assets.
- Points on curve 3 may be attainable with the two assets. To achieve the utility level of curve 3, the investor may choose to invest at points d and e where it intersects with CAL.

- However, curve 2 is preferred over curve 3, as it provides a higher utility. To achieve this utility level, the investor may choose to invest at point b, where the indifference curve is tangential to the capital allocation line
- This point, b, represents the optimal portfolio.

If another investor has a higher level of risk aversion, where will his optimal portfolio lie? With a higher risk aversion level, the indifference curves will be steeper. The tangential point will be closer to the risk-free asset, so it will be to the south-west of point b.

12. Portfolio Risk & Portfolio of Two Risky Assets

Portfolio risk depends on:

- Risk of individual assets.
- Weight of each asset.
- Covariance or correlation between the assets.

12.1 Portfolio of Two Risky Assets

Portfolio Return

A portfolio is usually composed of more than one asset in different proportions. Portfolio return is the weighted average of the returns of the individual investments.

Assume you have three different stocks A, B, and C in your portfolio with weights of 50%, 25%, and 25% respectively. The returns on A, B, and C are 20%, 10%, and 10% respectively. The portfolio return is the weighted average return of three stocks: 0.5 x 20 + 0.25 x 10 + 0.25 x 10 = 15%.

Risk of a two-asset portfolio is given by:

$$\sigma_p = \sqrt{w_1^2 \text{ x } \sigma_1^2 + w_2^2 \text{ x } \sigma_2^2 + 2 \text{ x } w_1 \text{ x } w_2 \text{ x } \rho_{1,2} \text{ x } \sigma_1 \text{ x } \sigma_2}$$

$$\sigma_p = \sqrt{w_1^2 \text{ x } \sigma_1^2 + w_2^2 \text{ x } \sigma_2^2 + 2 \text{ x } w_1 \text{ x } w_2 \text{ x } Cov_{1,2}}$$

$$\text{Covariance} = Cov_{1,2} = \rho_{1,2} \text{ x } \sigma_1 \text{ x } \sigma_2$$

where: $\rho_{1,2}$ = correlation coefficient that gives the correlation between returns R_1 and R_2.

Impact of correlation on portfolio risk

As correlation decreases, the diversification benefit increases, i.e., the risk of the portfolio decreases.

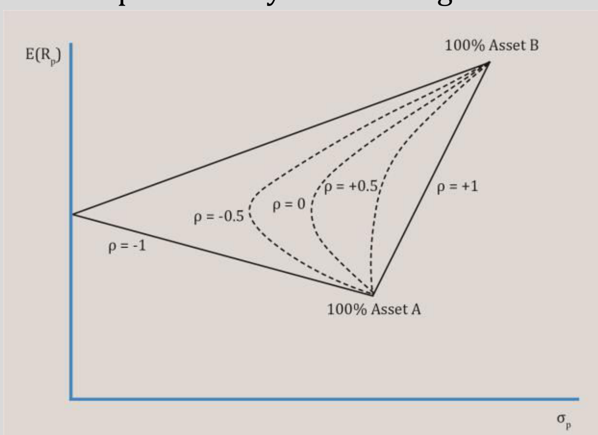
Example

Let us revisit our portfolio of X and Y. Earlier, we considered the impact of changing weights of the two assets. Now, we will also consider the impact of different correlations. The relevant data is reproduced here. X has an expected return of 12% and a standard deviation of 16%. Y has an expected return of 20% and a standard deviation of 30%. What is the expected return and risk of the portfolio for different weights assuming the following correlations: -1, -0.5, 0, 0.5, and 1?

Solution:

$R_p = w_x R_x + w_y R_y$ $\sigma_p^2 = w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \rho_{xy}\sigma_x\sigma_y$						
R_x = 12%, σ_x = 16%; R_y = 20%; σ_y = 30%						
Weight of X (%)	**Portfolio return**	Portfolio risk when correlation between X and Y is:				
		-1	**-0.5**	**0**	**0.5**	**1**
0	20	30	30	30	30	30
20	18.4	20.8	22.5	24.2	25.7	27.2
50	16	7	13	17	20.2	23
60	15.2	2.4	10.9	15.4	18.7	21.6
70	14.4	2.2	10.28	14.3	17.5	20.2
100	12	16	16	16	16	16

This chart is not exhaustive. It is plotted only for the weights used above.



- Consider the line represented with a correlation of 1. As the weight of X is reduced and weight of Y is increased, the risk and return both increase. There is no diversification benefit.
- Consider a correlation of 0. As the weight of X is reduced and weight of Y is increased, the risk initially decreases and the return increases. Hence there is clearly a diversification benefit.
- The diversification benefit increases as correlation decreases.
- With a correlation of -1, there is a certain weight of X and Y when the risk is zero.

13. Portfolio of Many Risky Assets

In the previous example, we saw how a two-asset portfolio's risk reduces as the correlation between the assets decreases. To be more generic, for a portfolio with more than two assets, the variance is given by:

$$\sigma_P^2 = \frac{\sigma^2}{N} + \frac{N-1}{N} * \text{Cov}$$

where:
σ_P^2 = portfolio variance
σ^2 = average variance
Cov = average covariance
N = number of assets in the portfolio

Instructor's tip:
The probability of being tested on this formula is low.

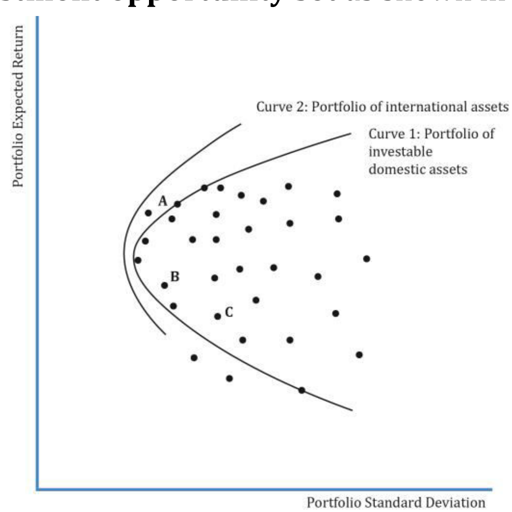
14. The Power of Diversification

When creating a diversified portfolio, the choice of assets and the correlation among assets is important. Adding perfectly positively correlated assets to a portfolio does little to reduce the portfolio's risk. Similarly, assets that have lower correlations will reduce risk. When two assets with a correlation of -1.0 are combined to form a portfolio, risk can be reduced to zero.

15. Efficient Frontier: Investment Opportunity Set & Minimum Variance Portfolios

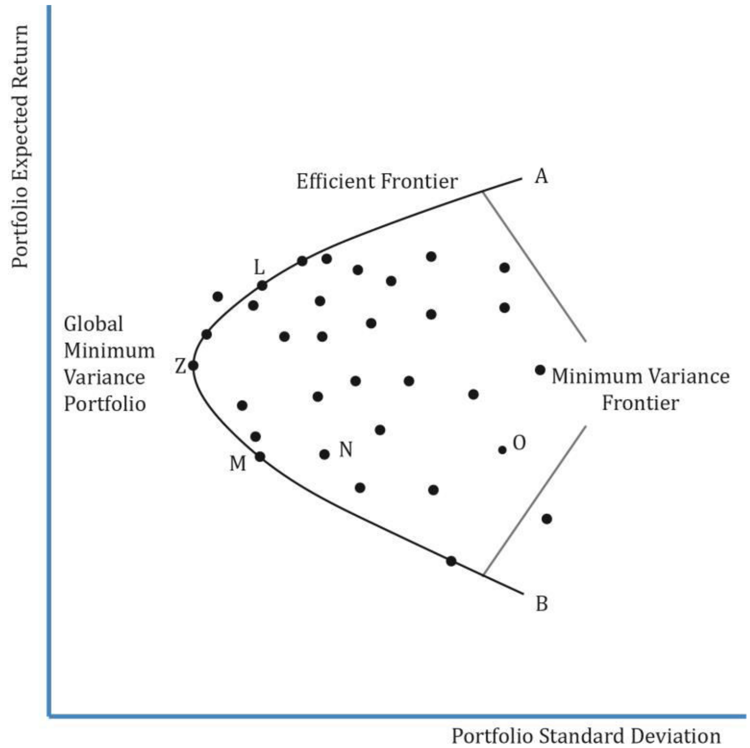
15.1 Investment Opportunity Set

Consider the universe of risky, investable assets available to an investor. These can be combined to form many portfolios. When plotted together they form a curve. This set of portfolios is called the **investment opportunity set** as shown in the exhibit below.



15.2 Minimum Variance Portfolios

At any given level of return in the investment opportunity set, there will be a portfolio with minimum variance, i.e., minimum risk for a given level of return. The line combining such portfolios with minimum variance is called the **minimum-variance frontier**.



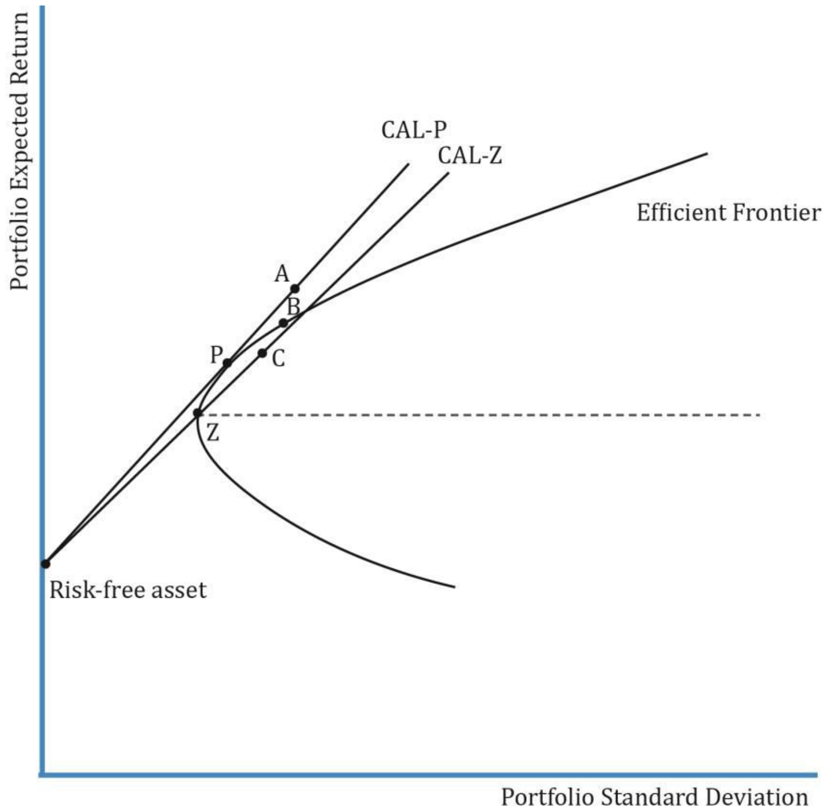
In the exhibit above, points M, N and O have the same expected return. But, investors will prefer point M over N or O because it has lower risk. The curve stretching from A to B through Z is called the **minimum-variance frontier.**

Global Minimum-Variance Portfolio: the portfolio with the lowest risk or minimum variance among portfolios of all risky assets is called the global minimum-variance portfolio. It is represented by point Z in the exhibit above. It is not possible to hold a portfolio of risky assets that has less risk than that of the global minimum-variance portfolio.

The **efficient frontier** is the part of the minimum variance frontier that is above the global minimum-variance portfolio. It represents the set of portfolios that will give the highest return at each risk level. *Remember that the efficient frontier consists of only risky assets. There is no risk-free asset.* Consider points L and M on the minimum-variance frontier. Both the portfolios have the same level of risk but portfolio L has a higher return than M. Investors will prefer the portfolio with higher return.

16. Efficient Frontier: A Risk-Free Asset and Many Risky Assets

One of the constraints of the efficient frontier is that it comprises only risky assets. We overcome this drawback with the capital allocation line: a combination of risky assets and a risk-free asset. The addition of a risk-free asset makes the investment opportunity set much richer than the investment opportunity set consisting only of risky assets. A portfolio of risky assets and a risk-free asset results in a straight line represented by the **capital allocation line**.

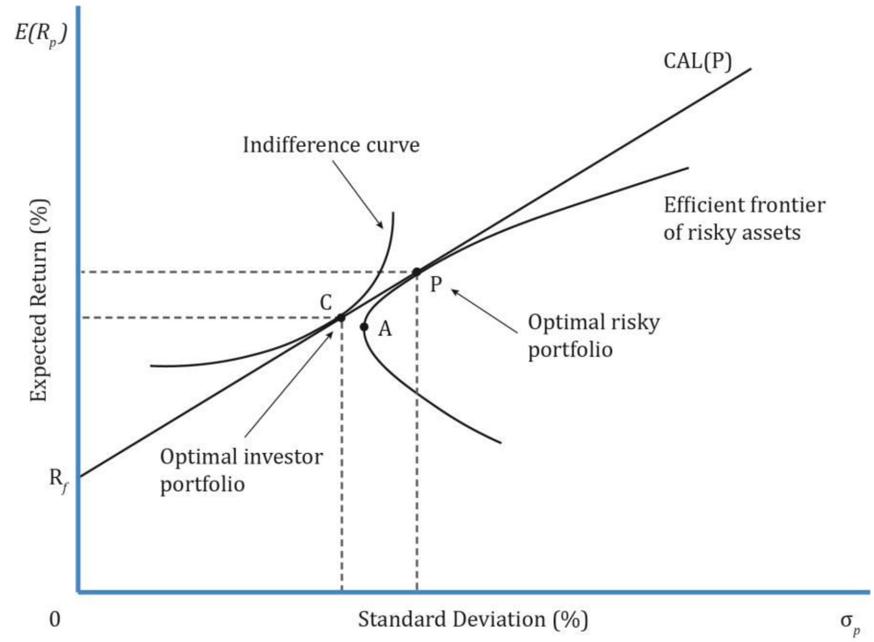


Interpretation of the diagram:

- CAL-P and CAL-Z are capital allocation lines formed by combining risky portfolios P and Z with the risk-free asset respectively.
- Portfolio P is the **optimal risky portfolio.** It is the point at which the capital allocation line is tangential to the efficient frontier of risky assets. CAL-P is the optimal capital allocation line.
- The optimal risky portfolio is based on the market and not the investor's risk preferences. Along CAL-P, the weight of risk-free assets and portfolio P varies. For instance, it is 100% in the risk-free asset at y-axis, 100% in risky portfolio P at the point P, and a mix of both in between.

17. Efficient Frontier: Optimal Investor Portfolio

To identify the optimal investor portfolio, we must consider the investor's risk preferences. The utility of each investor is best represented by his indifference curves. The optimal investor portfolio is the point at which the investor's indifference curve is tangential to the optimal allocation line.



Portfolio C in the exhibit above is the optimal investor portfolio.

Summary

LO.a: Calculate and interpret major return measures and describe their appropriate uses.

Returns can come in the form of income (interest payment and dividend) and capital gains.

Holding period return is the return earned on an asset during the period it was held.

$$\text{HPR single period} = \frac{P_T - P_0 + I}{P_0}$$

Arithmetic return is a simple arithmetic average of returns.

Geometric mean return is the compounded rate of return earned on an investment.

$$GM = [(1 + R_1) * (1 + R_2) * \ldots\ldots * (1 + R_T)]^{\frac{1}{T}} - 1$$

Money-weighted return is the internal rate of return on money invested that considers the cash inflows and cash outflows, and calculates the return on actual investment. It is synonymous with internal rate of return (IRR).

Annualized return converts the returns for periods that are shorter or longer than a year, to an annualized number for easy comparison.

Portfolio return is the weighted average of the returns of the individual investments.

Gross return is the return earned by an asset manager prior to deducting management fees and taxes. It measures investment skill.

Net return accounts for all managerial and administrative expenses is what the investor is concerned with.

Pre-tax nominal return is the return before accounting for inflation and taxes; this is the default, unless otherwise stated.

After-tax nominal return is the return after accounting for taxes.

Real return is the return after accounting for taxes and inflation.

Leveraged return is the return earned by the investor on his money after accounting for interest paid on borrowed money.

LO.b: Compare the money-weighted and time-weighted rates of return and evaluate the performance of portfolios based on these measures.

The time-weighted rate of return measures the compound growth rate of $1 initially invested in the portfolio over a stated measurement period.

Money-weighted return is the internal rate of return on money invested that considers the cash inflows and cash outflows, and calculates the return on actual investment.

Money-weighted v/s time-weighted returns

- The money-weighted rate of return is impacted by the timing and amount of cash flows.
- The time-weighted rate of return is not impacted by the timing and amount of cash flows.
- The time-weighted return is an appropriate performance measure if the portfolio manager does not control the timing and amount of investment.
- On the other hand, money-weighted return is an appropriate measure if the portfolio manager has control over the timing and amount of investment.

LO.c: Describe characteristics of the major asset classes that investors consider in forming portfolios.

Asset Class	Annual Average Return	Standard Deviation (Risk)
Small-cap stocks	High	High
Large-cap stocks	↓	↓
Long-term corporate bonds		
Long-term treasury bonds		
Treasury bills	Low	Low

LO.d: Calculate and interpret the mean, variance, and covariance (or correlation) of asset returns based on historical data.

Variance is a measure of volatility or dispersion of returns for a single variable.

Covariance is a measure of how two variables move together. It is difficult to interpret.

Correlation is a standardized measure of the linear relationship between two variables with values ranging between -1 and +1. It can be calculated using the formula:

$$\rho(\mathrm{Ri}, \mathrm{Rj}) = \mathrm{Cov}\,(\mathrm{Ri}, \mathrm{Rj})/\sigma(\mathrm{Ri})\sigma(\mathrm{Rj})$$

LO.e: Explain risk aversion and its implications for portfolio selection.

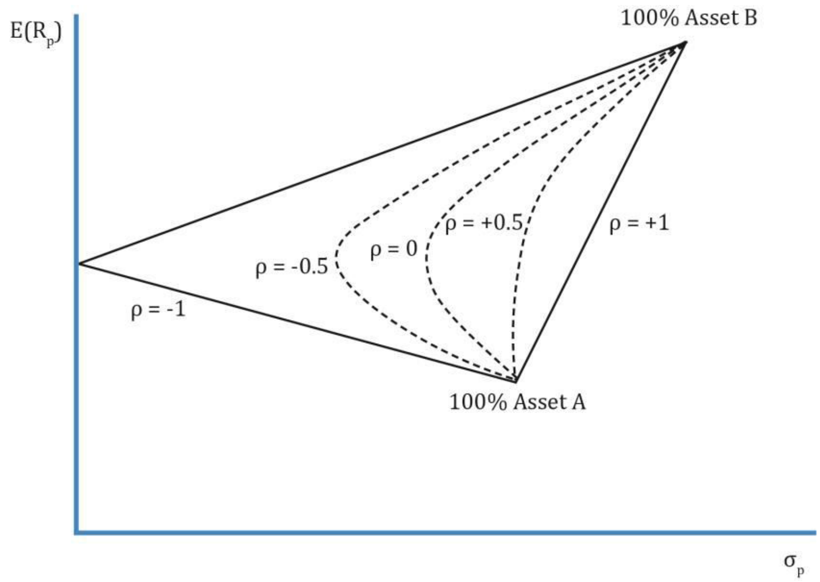
A risk-averse investor prefers less risk to more risk. In contrast, a risk seeking investor prefers more risk to less risk. If expected returns of two assets are same, a risk-averse investor will prefer asset with lower risk.

LO.f: Calculate and interpret portfolio standard deviation.

The standard deviation of a portfolio of two risky assets can be calculated using the following formula:

$$\sigma_p = \sqrt{w_1^2 * \sigma_1^2 + w_2^2 * \sigma_2^2 + 2 * w_1 * w_2 * \mathrm{Cov}_{1,2}}$$

LO.g: Describe the effect on a portfolio's risk of investing in assets that are less than perfectly correlated.



The above graph of risk and return shows the benefits of diversification. When the correlation of assets in a portfolio decreases, the risk of the portfolio decreases.

LO.h: Describe and interpret the minimum-variance and efficient frontiers of risky assets and the global minimum-variance portfolio.

Consider the universe of risky, investable assets. These can be combined to form many portfolios; when plotted together they form a curve. This set of portfolios is called the investment opportunity set.

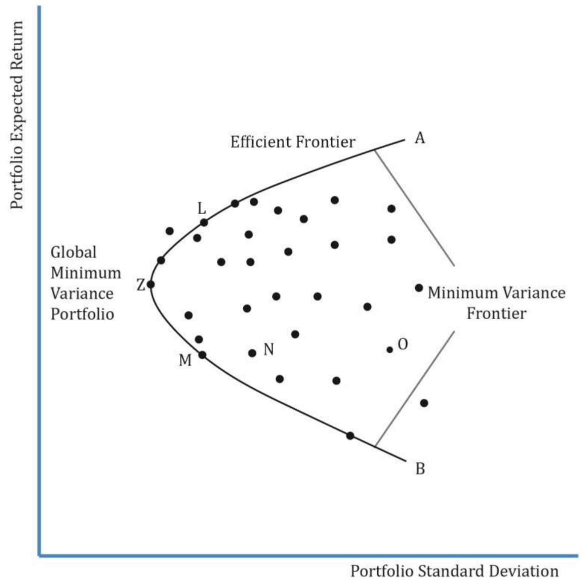
At any given level of return in the investment opportunity set, there will be a portfolio with minimum variance. The line combining such portfolios with minimum variance is called the minimum-variance frontier.

The portfolio with the lowest risk or minimum variance among portfolios of all risky assets is called the global minimum-variance portfolio.

The efficient frontier is the part of the minimum-variance frontier that is above the global minimum-variance portfolio. It represents the set of portfolios that will give the highest return at each risk level.

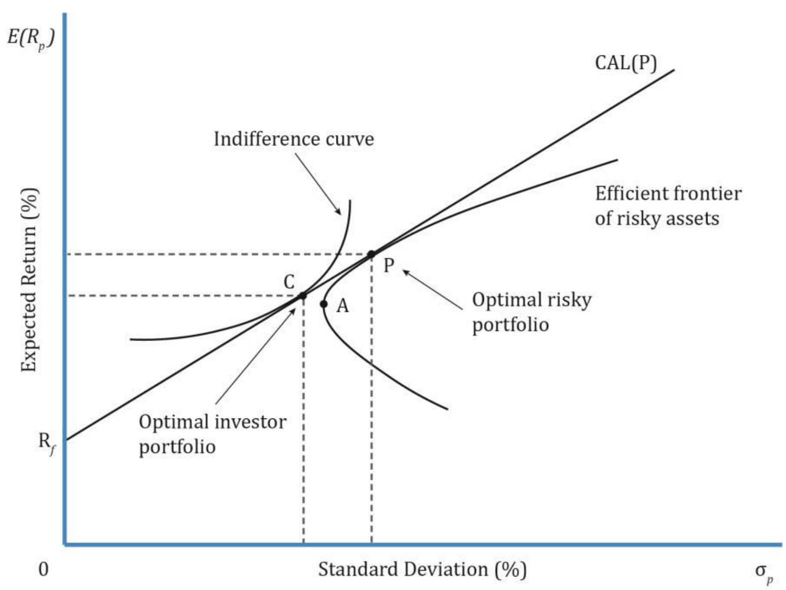
Remember that the efficient frontier consists of only risky assets. There is no risk-free asset.

The graph below shows these points:



LO.i: Discuss the selection of an optimal portfolio, given an investor's utility (or risk aversion) and the capital allocation line.

To identify the optimal investor portfolio, we must consider the investor's risk preferences. The utility of each investor is best represented by his indifference curves. The optimal investor portfolio is the point at which the investor's indifference curve is tangential to the optimal allocation line. Portfolio C in the exhibit below is the optimal investor portfolio.



CAL (P) is based on the market, not the investor's risk preferences; it represents the most efficient portfolio for each level of risk.

Practice Questions

1. An analyst has gathered the annual returns on a hedge fund:

Year	Return (%)
Year 1	26
Year 2	-12
Year 3	3

The fund's holding period return over the three-year period and its annual geometric mean return is *closest* to:

	Holding period return	Geometric mean return
A.	14.21%	4.53%
B.	14.21%	5.67%
C.	12.35%	4.53%

2. David has made investments in three securities. The returns data is as follows:

Security	Time since investment	Return since inception (%)
A	122 days	5.56
B	7 weeks	2.12
C	19 months	16.32

The security with the highest annualized rate of return is:

A. Security B.
B. Security C.
C. Security A.

3. Amy makes the following transactions in a hedge fund, which performs as given below over a three year period:

	Year 1	Year 2	Year 3
New investment at the beginning of the year	$18,000	$24,500	$32,000
Investment return for the year	-12%	15%	20%
Withdrawal by investor at the end of the year	$0	$16,000	$0

The money-weighted return that Amy will earn in this hedge fund is *closest* to:

A. 12.67%.
B. 7.67%.
C. 10.67%.

4. Gerald observes that the historic geometric returns for the US equity market are 15% while inflation is at 3% and the treasury bills are at 3.5%. The real rate of return and the risk premium for the US equity market is *closest* to:

	Real rate of return	Risk premium
A.	12.65%	11.11%
B.	11.65%	11.11%
C.	10.65%	12.22%

5. An investor purchases 100 shares of a company. The table below outlines the history of his investment:

Time	Activity	Price per Share	Dividend per Share
Start of Year 1	Buy 100 shares	$10	
End of Year 1	Buy 10 shares	$11	$1
End of Year 2		$13	$1.5
End of Year 3	Sell 110 shares	$12	

Assuming that the investor does not reinvest his dividends, which are tax-free, the time-weighted rate of return on the investment is *closest* to:
A. 12.2%.
B. 13.7%.
C. 14.8%.

6. Joe Egan has an investment account in which his portfolio manager controls the cash flows into and out of the portfolio. To evaluate the performance of the investments, Egan should *most likely* use.
A. Money-weighted return.
B. Time-weighted return.
C. Both A and B.

7. Tim Jones has compiled the historical information for two stocks, Apex and Orion:

Variance of returns for Apex	0.0653
Variance of returns for Orion	0.0927
Correlation coefficient between Apex and Orion	0.5300

The covariance calculated between Apex and Orion is *closest* to:
A. 0.0412.
B. 0.0386.
C. 0.0468.

8. An analyst observes s return series of some asset class which demonstrated return distributions concentrated to the left with frequent small losses and few large gains. This feature is *most likely* known as:
A. negative skewness.
B. positive skewness.

C. kurtosis.

9. Mike's portfolio consists of two stocks: X and Y. The standard deviation of returns is 0.15 for X and 0.095 for Y. The covariance between the returns of the two stocks is 0.0030. The correlation of returns between them is:
A. 0.2105.
B. 2.1052.
C. 4.750.

10. According to utility theory, a relatively risk-averse investor will have an indifference curve that will have:
A. higher slope coefficient.
B. lower slope coefficient.
C. higher convexity.

11. A financial planner has gathered the following information:

Investment	Expected return (%)	Expected standard deviation (%)
A	16	8
B	18	2
C	26	8
D	22	35

Susan has a risk-aversion measure of -3 while Alex has a risk aversion measure of 3. If the investor's utility function is given by $U = E(r) - 0.5 * A * \sigma^2$, the investment that Susan and Alex are most likely to select is:

	Susan	**Alex**
A.	Investment D	Investment C
B.	Investment B	Investment C
C.	Investment C	Investment D

12. The risk-return relationship for a risk seeking investor is *most likely* to be:
A. positive.
B. negative.
C. neutral.

13. The following two shares are included in a portfolio:

	Clement Corporation	**Telnet Corporation**
Expected returns	15%	11%
Std dev of returns	38%	52%
Portfolio weights	65%	35%
Correlation of returns	0.46	

The standard deviation of returns of the portfolio formed with these two stocks is *closet* to:

A. 0.2980.
B. 0.3320.
C. 0.3680.

14. The portfolio that *most likely* falls below the efficient frontier is:

Portfolio	**Expected return (%)**	**Expected std dev (%)**
A	8	9
B	12	18
C	14	15

A. Portfolio B.
B. Portfolio A.
C. Portfolio C.

15. Amy has a higher risk-aversion coefficient than Susan. Given that both Amy and Susan have selected the same risky portfolio, the optimal portfolio on the capital allocation line for Susan will have:

A. higher expected return than Amy's optimal portfolio.
B. lower expected return than Amy's optimal portfolio.
C. same expected return as that of Amy's optimal portfolio.

Solutions

1. A is correct.
   Holding period return:
   $[(1 + 0.26)(1 - 0.12)(1 + 0.03)] - 1 = 14.21\%$
   Geometric mean return:
   $[(1 + 0.26)(1 - 0.12)(1 + 0.03)]^{1/3} - 1 = 4.53\%$.

2. C is correct.
   Annualized return of Security A = $1.0556^{\left(\frac{365}{122}\right)} - 1 = 17.57\%$
   Annualized return of Security B = $1.0212^{\left(\frac{52}{7}\right)} - 1 = 16.86\%$
   Annualized return of Security A = $1.1632^{\left(\frac{12}{19}\right)} - 1 = 10.02\%$

3. A is correct.

	Year 1	Year 2	Year 3
Starting balance ($)	0	15,840	30,391
New investment ($)	18,000	24,500	32,000
Balance at the beginning of the year ($)	18,000	40,340	62,391
Investment return for the year	-12%	15%	20%
Investment gain/loss ($)	-2,160	6,051	12,478.2
Withdrawal by investor at the end of the year ($)	0	16,000	0
Balance at the end of the year ($)	15,840	30,391	74,869.2

   CF_0 = -18,000
   CF_1 = -24,500 (new investment at the beginning of year 2)
   CF_2 = -16,000 (withdrawal of 16,000 at the end of year 2, -32,000 new investment at the beginning of year 3)
   CF_3 = 74,869.2 (balance at the end of year 3)

4. B is correct.
   Real rate of return = (1 + 0.15)/ (1 + 0.03) - 1 = 11.65%
   Risk premium of return = (1 + 0.15)/ (1 + 0.035) - 1 = 11.11%

5. B is correct. TWR = $\sqrt[3]{\{[(11 + 1)/10] \times [(13 + 1.5)/11] \times [(12/13)]\}} - 1 = 0.1344$ = 13.45%.

6. A is correct. Money-weighted return is an appropriate measure if the portfolio manager has control over the timing and amount of investment.

7. A is correct.
   Covij= σi * σj * rij = Sqrt (0.0653) × Sqrt (0.0927) × 0.5300 = 0.0412

8. B is correct. A distribution is said to be right skewed or positively skewed if most is concentrated to the left, and left skewed or negatively skewed if most of the distribution is concentrated to the right. A positively skewed distribution has a long tail on the right side, which means that there will be frequent small losses and few large gains.

9. A is correct.
   $\mathrm{Cov}(R_i, R_j) = \rho(R_i, R_j) * \sigma(R_i) * \sigma(R_j)$
   $\rho(R_i, R_j) = \mathrm{Cov}(R_i, R_j) / [\sigma(R_i) * \sigma(R_j)]$
   $\rho = \frac{0.0030}{0.15 * 0.095} = 0.2105.$

10. A is correct. The more risk-averse an investor is, the higher the slope of the indifference curve. An investor with less steep indifference curves has a lower level of risk aversion, i.e., higher risk tolerance.

11. A is correct.
    Using the utility function,

Investment	Expected return (%)	Expected std dev (%)	Susan's utility function; A = - 3	Alex's utility function; A = 3
A	16	8	0.1696	0.1504
B	18	2	0.1806	0.1794
C	26	8	0.2696	0.2504
D	22	35	0.4038	0.0363

    Susan would opt for Investment D, while Alex would opt for Investment C.
    Susan is a risk-seeking investor while Alex is a risk-averse investor.

12. B is correct. A risk seeking investor the expected return decreases for higher levels of risk. Hence, the risk-return relationship is negative. A risk averse investor will prefer an investment that has lower risk, all else equal.

13. C is correct.
    $\sigma_{portfolio} = \sqrt{(0.65)^2(0.38)^2 + (0.35)^2(0.52)^2 + 2(0.65)(0.38)(0.35)(0.52)(0.46)}$
    $\sigma_{portfolio} = 0.3680$

14. A is correct. Portfolio B must be the portfolio that falls below the Markowitz efficient frontier as it has lower return and higher risk than Portfolio C.

15. A is correct. Susan has a low risk-aversion coefficient, therefore a high risk-tolerance. The indifference curve will intersect the capital allocation line at a higher level, giving an optimal portfolio that has a higher expected return than Amy's optimal portfolio.