Optimal Control of Nonlinear Inverted Pendulum Dynamical System with Disturbance Input using PID Controller & LQR

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Abstract— **Optimal response of the controlled dynamical systems is desired hence for that is the optimal control. Linear quadratic regulator (LQR), an optimal control method, and PID control which are generally used for control of the linear dynamical systems have been used in this paper to control the nonlinear dynamical system. The inverted pendulum, a highly nonlinear unstable system is used as a benchmark for implementing the control methods. In this paper the modeling and control design of nonlinear inverted pendulum-cart dynamic system with disturbance input using PID control & LQR have been presented. The nonlinear system states are fed to LQR which is designed using linear state-space model. Here PID & LQR control methods have been implemented to control the cart position and stabilize the inverted pendulum in vertically upright position. The MATLAB-SIMULINK models have been developed for simulation of the control schemes. The simulation results justify the comparative advantages of LQR control methods.**

Keywords—Inverted pendulum, nonlinear system, PID control, optimal control, LQR, disturbance input

I. INTRODUCTION

The Inverted Pendulum is an inherently open loop & closed loop unstable system with highly nonlinear dynamics. This is a system which belongs to the class of under-actuated mechanical systems having fewer control inputs than degrees of freedom. This renders the control task more challenging making the inverted pendulum system a classical benchmark for the design, testing, evaluating and comparing of different classical & contemporary control techniques.

The inverted pendulum is among the most difficult systems being an inherently unstable system, is a very common control problem, and so being one of the most important classical problems, the control of inverted pendulum has been a research interest in the field of control engineering. Due to its importance this is a choice of dynamic system to analyze its dynamic model and propose a control law. The aim of this case study is to stabilize the Inverted Pendulum (IP) such that the position of the cart on the track is controlled quickly and accurately so that the pendulum is always erected in its inverted position during such movements. Realistically, this simple mechanical system is representative of a class of altitude control problems whose goal is to maintain the desired vertically oriented position at all times [1-4].

In general, the control problem consists of obtaining dynamic models of systems, and using these models to determine control laws or strategies to achieve the desired system response and performance. The simplicity of control algorithm as well as to guarantee the stability and robustness in the closed-loop system is challenging task in real situations. Most of the dynamical systems such as power systems, missile systems, robotic systems, inverted pendulum, industrial processes, chaotic circuits etc. are highly nonlinear in nature. The control of such systems is a challenging task.

The Proportional-Integral-Derivative (PID) control gives the simplest and yet the most efficient solution to various real-world control problems. Both the transient and steady-state responses are taken care of with its three-term (i.e. P, I, and D) functionality. Since its invention the popularity of PID control has grown tremendously. The advances in digital technology have made the control system automatic. The automatic control system offers a wide spectrum of choices for control schemes, even though, more than 90% of industrial controllers are still implemented based around the PID algorithms, particularly at the lowest levels, as no other controllers match with the simplicity, clear functionality, applicability, and ease of use offered by the PID controller.

The performance of the dynamical systems being controlled is desired to be optimal. There are many optimization & optimal control techniques which are present in the literatures for linear & nonlinear dynamical systems [5-7]. Linear quadratic regulator (LQR), an optimal control method, and PID control which are generally used for control of the linear dynamical systems have been used in this paper to control the nonlinear inverted pendulum-cart dynamical system.

The recent development in the area of artificial intelligence (AI), such as artificial neural network (ANN), fuzzy logic theory (FL), and evolutionary computational techniques such as genetic algorithm (GA), and particle swarm optimization (PSO) etc., commonly all these are known as intelligent computational techniques which have given novel solutions to the various control system problems [8-18]. The intelligent optimal control has emerged as viable recent approach by the application of these intelligent computational techniques [10].

This paper is organized in 5 sections. Section I presents the relevance & the general introduction of the paper. Section II describes the mathematical model of the inverted pendulum-cart system with disturbance input. In section III the PID control and optimal control using LQR methods have been discussed briefly. Section IV presents MATLAB-SIMULINK modelling, and simulation results. In section V conclusion is presented. At the end a brief list of references is given.

II. Mathematical Modelling

Inverted Pendulum System Equations with Disturbance Input

The free body diagram of a motor driven cart mounted inverted pendulum system is shown in Fig. 1 [1-4, 16-20]. The system equations of this nonlinear dynamic system with disturbance input can be derived as follows [1,3,4,16,20]. It is assumed here that the pendulum rod is mass-less, and the hinge is frictionless. The cart mass and the ball point mass at the upper end of the inverted pendulum are denoted as M and m, respectively. There is an externally x-directed force on the cart, $u(t)$, and a gravity force acts on the point mass at all times. The coordinate system considered is shown in Fig. 1, where $x(t)$ represents the cart position and $\theta(t)$ is the tilt angle referenced to the vertically upward direction. Consider the inverted pendulum model with state feedback with an additional disturbance input due to wind effects. Let F_w represent the horizontal wind force on the pendulum point mass. With this additional force component, the force balance equation becomes-

$$M\frac{d^2}{dt^2}x+m\frac{d^2}{dt^2}x_G=u+F_w \tag{1}$$

which can be manipulated as to give

$$(M+m)\ddot{x}-ml\sin\theta\dot{\theta}^2+ml\cos\theta\ddot{\theta}=u+F_w \tag{2}$$

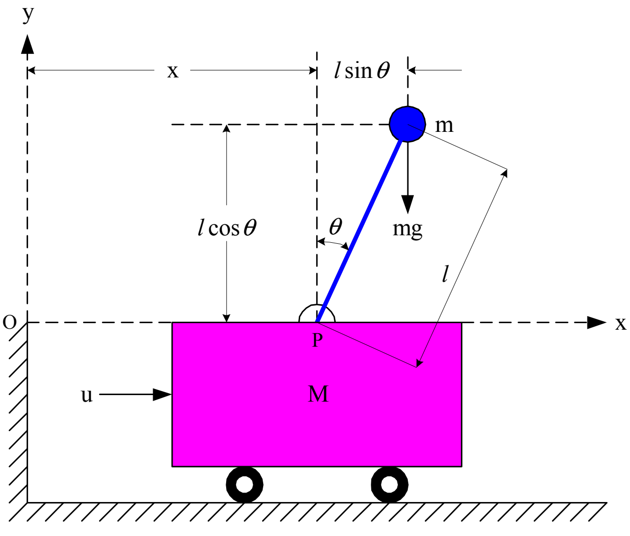

Figure 1. Motor Driven Inverted Pendulum-Cart System.

Similarly, the torque in the clockwise direction caused by the horizontal wind disturbance is $(F_w\cos\theta)l$, and this term is added to the torque balance to give

$$(F_x\cos\theta)l-(F_y\sin\theta)l=(mg\sin\theta)l+(F_w\cos\theta)l \tag{3}$$

which again can be modified to give

$$m\ddot{x}\cos\theta+ml\ddot{\theta}=mg\sin\theta+F_w\cos\theta \tag{4}$$

Equations (2) and (4) are the defining equations for this system which includes a horizontal wind disturbance.

These two nonlinear equations can be manipulated and put into standard state form. Some of the steps in this derivation process are:

1. Solve the torque balance expression for $ml\ddot{\theta}$ and put this into the force balance equation, giving

$$ml\ddot{\theta}=mg\sin\theta+F_w\cos\theta-m\ddot{x}\cos\theta \quad , \quad \text{and}$$

$$[M+m-m\cos^2\theta]\ddot{x}=u+ml\sin\theta\dot{\theta}^2-mg\sin\theta\cos\theta+F_w\sin^2\theta$$

2. Solve the torque balance equation for $\ddot{x}$ and put this into the force balance equation, giving

$$\ddot{x}=\frac{mg\sin\theta+F_w\cos\theta-ml\ddot{\theta}}{m\cos\theta} \quad , \quad \text{and}$$

$$(M+m)\frac{mg\sin\theta+F_w\cos\theta-ml\ddot{\theta}}{m\cos\theta}-ml\sin\theta\dot{\theta}^2+ml\cos\theta\ddot{\theta}$$
$$=u-F_w$$

Multiplying both sides by $\cos\theta$ gives

$$[ml\cos^2\theta-(M+m)l]\ddot{\theta}=u\cos\theta-(M+m)g\sin\theta$$
$$+ml\cos\theta\sin\theta\dot{\theta}^2-\left(\frac{M+m}{m}\right)F_w\cos\theta+F_w\cos\theta$$

Now, the state equation may be written as

$$\frac{d}{dt}\mathbf{x}=\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\frac{d}{dt}\begin{bmatrix}\theta\\\dot{\theta}\\x\\\dot{x}\end{bmatrix}=\begin{bmatrix}x_2\\f_1\\x_4\\f_2\end{bmatrix} \tag{5}$$

where,

$$f_1=\frac{u\cos x_1-(M+m)g\sin x_1+ml(\cos x_1\sin x_1)x_2^2-\frac{M}{m}F_w\cos x_1}{ml\cos^2 x_1-(M+m)l}$$

$$f_2=\frac{u+ml(\sin x_1)x_2^2-mg\cos x_1\sin x_1+F_w\sin^2 x_1}{M+m-m\cos^2 x_1}$$

If both the pendulum angle θ and the cart position x are of interest, we have

$$\mathbf{y}=\mathrm{C}\mathbf{x} \quad \text{or} \quad \mathbf{y}=\begin{bmatrix}\theta\\x\end{bmatrix}=\mathrm{C}\mathbf{x}=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}\theta\\\dot{\theta}\\x\\\dot{x}\end{bmatrix} \tag{6}$$

Equations (5) and (6) give a complete state space representation of the nonlinear inverted pendulum with disturbance input.

When the Jacobian matrices are evaluated at the reference point $\mathbf{x}_0 = 0$ and $\boldsymbol{u}_0 = 0$, a linearized model can also be developed as following:

$$\frac{d}{dt}\delta\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix}\delta\mathbf{x} + \begin{bmatrix} 0 \\ \frac{-1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix}\delta u + \begin{bmatrix} 0 \\ \frac{-1}{ml} \\ 0 \\ 0 \end{bmatrix}\delta F_w \quad (7)$$

This is the open loop linearized model for the inverted pendulum with a cart force, $\delta u(t)$, and a horizontal wind disturbance, $\delta F_w(t)$. The two inputs have been separated for convenience, thus the LTI system can be written as

$$\frac{d}{dt}\delta\mathbf{x} = \mathbf{A}\delta\mathbf{x} + \mathbf{b}_1\delta u + \mathbf{b}_2\delta F_w \quad (8)$$

III. Control Methods

Here in this paper the following control methods are presented to control the nonlinear inverted pendulum-cart dynamical system with disturbance input.

A. PID Control

To stabilize the inverted pendulum in upright position and to control the cart at desired position using PID control approach two PID controllers- angle PID controller, and cart PID controller have been designed for the two control loops of the system. The equations of PID control are given as following:

$$u_p = K_{pp}e_\theta(t) + K_{ip}\int e_\theta(t) + K_{dp}\frac{de_\theta(t)}{dt} \quad (9)$$

$$u_c = K_{pc}e_x(t) + K_{ic}\int e_x(t) + K_{dc}\frac{de_x(t)}{dt} \quad (10)$$

where, $e_\theta(t)$ and $e_x(t)$ are angle error and cart position error. Since the pendulum angle dynamics and cart position dynamics are coupled to each other so the change in any controller parameters affects both the pendulum angle and cart position which makes the tuning tedious. The tuning of controller parameters is done using trial & error method and observing the responses of SIMULINK model to be the optimal.

B. Optimal Control using LQR

Optimal control refers to a class of methods that can be used to synthesize a control policy which results in best possible behavior with respect to the prescribed criterion (i.e. control policy which leads to maximization of performance). The main objective of optimal control is to determine control signals that will cause a process (plant) to satisfy some physical constraints and at the same time extremize (maximize or minimize) a chosen performance criterion (performance index (PI) or cost function). The optimal control problem is to find a control which causes the dynamical system to reach a target or follow a state variable (or trajectory) and at the same time extremize a PI which may take several forms [1,4-7].

Linear quadratic regulator (LQR) is one of the optimal control techniques, which takes into account the states of the dynamical system and control input to make the optimal control decisions. This is simple as well as robust [1,4-7].

After linearization of nonlinear system equations about the upright (unstable) equilibrium position having initial conditions as $X_0 = [0,0,0,0]^T$, the linear state-space equation is obtained as

$$\dot{X} = AX + Bu \quad (11)$$

where, $X = [\theta, \dot{\theta}, x, \dot{x}]^T$.

The state feedback control $u = -KX$ leads to

$$\dot{X} = (A - BK)X \quad (12)$$

where, K is derived from minimization of the cost function

$$J = \int (X^T QX + u^T Ru)dt \quad (13)$$

where, Q and R are positive semi-definite and positive definite symmetric constant matrices respectively.

The LQR gain vector K is given by

$$K = R^{-1}B^T P \quad (14)$$

where, P is a positive definite symmetric constant matrix obtained from the solution of matrix algebraic reccatti equation (ARE)

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad (15)$$

IV. Simulation & Results

The MATLAB-SIMULINK models for the simulation of modelling, analysis, and control of nonlinear inverted pendulum-cart dynamical system with disturbance input have been developed. The typical parameters of inverted pendulum-cart system setup are selected as [16,20]: mass of the cart (M): 2.4 kg, mass of the pendulum (m): 0.23 kg, length of the pendulum (l): 0.36 m, length of the cart track (L): ± 0.5 m, friction coefficient of the cart & pole rotation is assumed negligible. The disturbance input parameters which has been taken in simulation are [21]: Band Limited White Noise Power = 0.001, Sample Time = 0.01, Seed = 23341.

After linearization the system matrices used to design LQR are computed as below:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 29.8615 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.9401 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1.1574 \\ 0 \\ 0.4167 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{and} \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

With the choice of

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 500 & 0 \\ 0 & 0 & 0 & 250 \end{bmatrix}, \text{and} \quad R = 1 ,$$

we obtain LQR gain vector as following:

$$K = \begin{bmatrix} -137.7896 & -25.9783 & -22.3607 & -27.5768 \end{bmatrix}$$

Here three control schemes have been implemented for optimal control of nonlinear inverted pendulum-cart dynamical system with disturbance input: 1. PID control method having two PIDs i.e. angle PID & cart PID, 2. Two PIDs (i.e. angle PID & cart PID) with LQR control method, 3. One PID (i.e. cart PID) with LQR control method.

In the optimal control of nonlinear inverted pendulum dynamical system using PID controller & LQR approach, all the instantaneous states of the nonlinear system, pendulum angle θ, angular velocity $\dot{\theta}$, cart position x, and cart velocity $\dot{x}$ have been considered available for measurement which are directly fed to the LQR. The LQR is designed using the linear state-space model of the system. The optimal control value of LQR is added negatively with PID control value to have a resultant optimal control. The tuning of the PID controllers which are used here either as PID control method or PID+LQR control methods is done by trial & error method and observing the responses achieved to be optimal. The tuned PID controller parameters of these control schemes are given as in table I.

TABLE I. PID CONTROLLER PARAMETERS

Control Schemes	Angle PID Control			Cart PID Control		
	K_{pp}	K_{ip}	K_{dp}	K_{pc}	K_{ic}	K_{dc}
PID	-40	0	-8	-1.25	0	-3.6
2 PID+LQR	1	1	1	1.5	-7.5	5
1 PID+LQR	---	---	---	1.5	-7.5	5

The SIMULINK model for control of inverted pendulum system with disturbance input using PID control method for nonlinear plant model is shown in Fig. 2. Here only pendulum angle θ and cart position x have been considered for the measurement. The band limited white noise has been added as the disturbance input to the system. The reference angle has been set to 0 (rad), and reference cart position is set to 0.1 (m). The simulation results are shown in Fig. 3. It is observed that the pendulum stabilizes in vertically upright position with minor oscillations, and also the cart position x reaches the desired position of 0.1 (m) quickly with minor oscillations. The control input u is bounded in range [-1 1]. The simulation results justify the effectiveness of the PID control.

The SIMULINK model for optimal control of nonlinear inverted pendulum-cart system with disturbance input using two PID controllers (angle PID & cart PID) with LQR control method is shown in Fig. 4. In this approach all the states of the system θ, $\dot{\theta}$, x and $\dot{x}$ are fed to LQR, which is designed using the linear state-space model of the system. Here also the angle θ & cart position x have been taken as variables of interest for control. The band limited white noise has been added as the disturbance input to the system. The reference angle is set to 0 (rad), and the reference cart position has been set to 0.1 (m). The simulation results are shown in Fig. 5. Here responses of angle θ, angular velocity $\dot{\theta}$, cart position x, cart velocity $\dot{x}$, and control u have been plotted. It is observed that the pendulum stabilizes in vertically upright position with minute oscillations, and the angular velocity oscillates by approx. +/-0.01 (rad/s) remaining at most in range approx +/-0.02 (rad/s). The cart position x reaches smoothly the desired position of 0.1 (m) quickly in approx. 6 seconds, and the cart velocity oscillates near to zero. The control input u is bounded in range [-1 1]. The simulation results justify the effectiveness of the 2 PID+LQR control.

The SIMULINK model for optimal control of nonlinear inverted pendulum-cart system with disturbance input using one PID controller (cart PID) with LQR control method is shown in Fig. 6. This control method is similar to 2 PID+LQR control method in all respect of control techniques but differs only in number of PID controllers used. Here only cart PID controller has been used, and angle PID controller has not been used. Here only cart position x has been taken as variable of interest for control. The reference cart position has been set to 0.1 (m). The desired angle to be zero is directly taken care of by state feedback control of LQR which is designed using the linear state-space model of the system with vertically upright position as reference. The band limited white noise has been added as the disturbance input to the system as same. The simulation results are shown in Fig. 7. Here also responses of angle θ, angular velocity $\dot{\theta}$, cart position x, cart velocity $\dot{x}$, and control u have been plotted. It is observed that the pendulum stabilizes in vertically upright position with minute oscillations, and the angular velocity oscillates by approx. +/-0.01 (rad/s) remaining at most in range approx +/-0.02 (rad/s). The cart position x reaches the desired position of 0.1 (m) quickly in approx. 6 seconds, and the cart velocity oscillates very near to zero. The control input u is bounded in range [-1 1]. The simulation results justify the effectiveness of the cart PID+LQR control.

Comparing the results it is observed that the responses of both alternatives of PID+LQR control method are better than PID control, which are smooth & fast also. It is also observed that the responses of 2 PID+LQR control and cart PID+LQR control are similar. Just the cart position response of 2 PID+LQR control is smoother than cart PID+LQR control and so it is slightly better, which is due to the additional degree of freedom of control added by the angle PID controller. But the cart PID+LQR control has structural simplicity in its credit. The analysis of the performances of the control schemes of PID control, 2 PID+LQR control, and cart PID+LQR control for the nonlinear inverted pendulum-cart dynamical system with disturbance input gives that these control schemes are effective & robust.

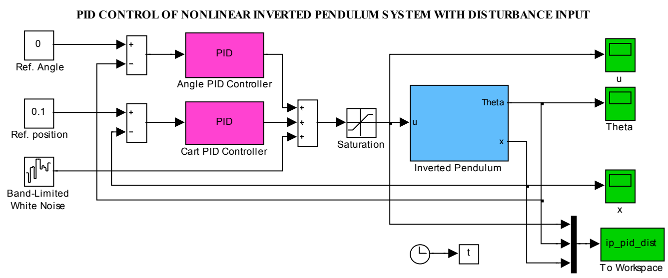

Figure 2. PID Control of Nonlinear Inverted Pendulum System with Disturbance Input.

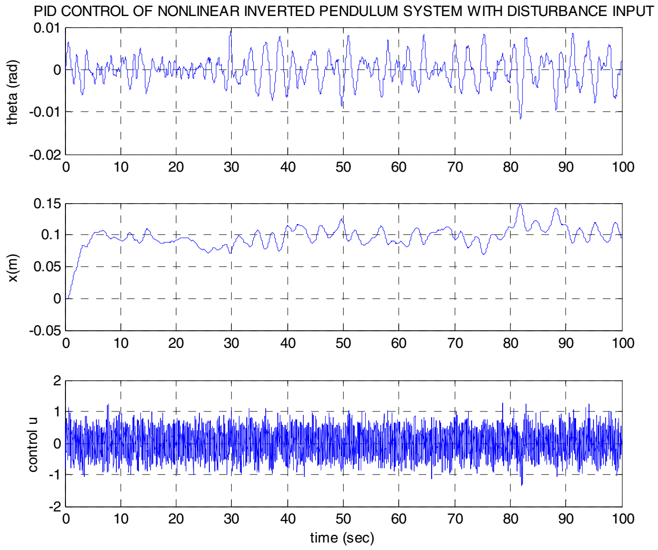

Figure 3. Responses of PID Control of Nonlinear Inverted Pendulum System with Disturbance Input.

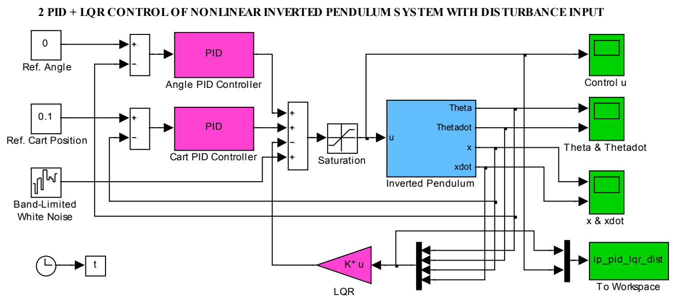

Figure 4. Angle PID, Cart PID & LQR Control of Nonlinear Inverted Pendulum System with Disturbance Input.

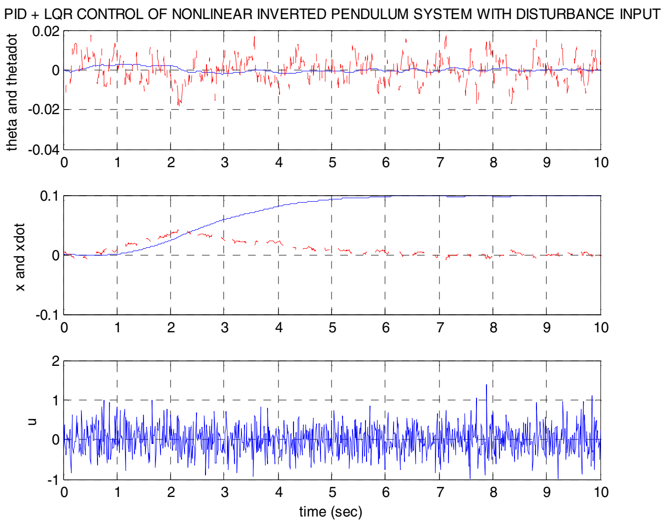

Figure 5. Responses of Angle PID , Cart PID & LQR Control of Nonlinear Inverted Pendulum System with Disturbance Input.

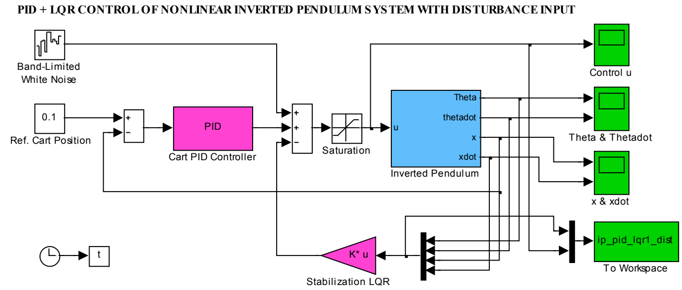

Figure 6. Cart PID & LQR Control of Nonlinear Inverted Pendulum System with Disturbance Input.

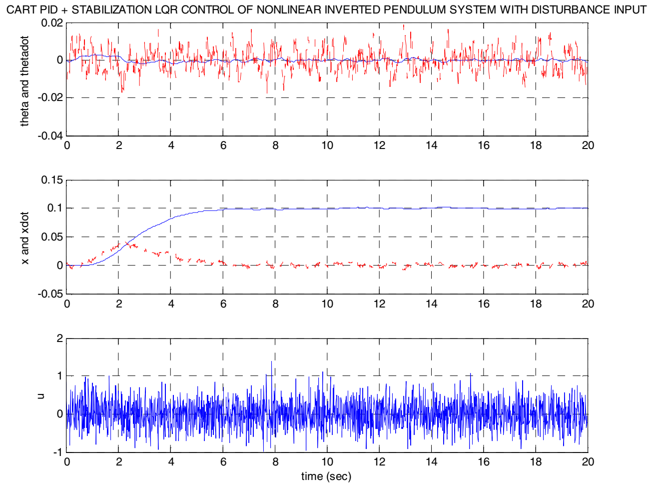

Figure 7. Responses of Cart PID & LQR Control of Nonlinear Inverted Pendulum System with Disturbance Input.

V. CONCLUSION

PID control, and LQR, an optimal control technique to make the optimal control decisions, have been implemented to control the nonlinear inverted pendulum-cart system with disturbance input. To compare the results PID control has been implemented. In the optimal control of nonlinear inverted pendulum dynamical system using PID controller & LQR approach, all the instantaneous states of the nonlinear system, are considered available for measurement, which are directly fed to the LQR. The LQR is designed using the linear state-space model of the system. The optimal control value of LQR is added negatively with PID control value to have a resultant optimal control. The MATLAB-SIMULINK models have been developed for simulation of the control schemes. The tuning of the PID controllers which are used here either as PID control method or PID+LQR control methods is done by trial & error method and observing the responses achieved to be optimal. The simulation results justify the comparative advantages of optimal control using LQR method. The pendulum stabilizes in upright position with acceptable minor oscillations and cart approaches the desired position even under the continuous disturbance input such as wind force justify that the control schemes are effective & robust. The response of PID controller using LQR is better than PID controller.

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