4. Coherence

In optics, the original sense of the word coherence was attributed to the ability of radiation to produce interference phenomena. Today, the notion of coherence is defined more generally by the correlation properties between quantities of an optical field. Usual interference is the simplest phenomenon revealing correlations between light waves.

Two limiting cases of a more general description exist: temporal and spatial coherence. This is partly due to historical reasons, partly, because these limiting cases can be well realized experimentally.

Michelson introduced a technique for measuring the temporal coherence. The instrument nowadays is called the Michelson interferometer (Sect. 4.1). Spatial coherence is best illustrated by the double-slit experiment of *Young* (Sect. 4.2).

4.1 Temporal Coherence

We consider the path of rays in a Michelson interferometer (Fig. 4.1). The light to be investigated is divided into two beams by a beam splitter. One beam is reflected back onto itself by a fixed mirror, the other one is also reflected back by a mirror, but one that can be shifted along the beam. Both reflected beams are divided again into two by the beam splitter, whereby one beam from each mirror propagates to a screen. The idea of this arrangement is to superimpose a light wave with a time-shifted copy of itself.

We now set out for a mathematical description. On the screen, we have a superposition of two wave fields, E_1 and E_2. Let E_1 be the light wave that reaches the screen via the first mirror and E_2 the one that reaches the screen via the movable mirror. Then we have at a point on the screen, when the incoming wave has been split evenly at the beam splitter,

$$E_2(t) = E_1(t+\tau)\,, \quad \text{or} \quad E_1(t) = E_2(t-\tau)\,. \tag{4.1}$$

The wave E_2 thus has to start earlier to reach the screen at time t because of the additional path of length $2d$. The quantity τ depends on the mirror

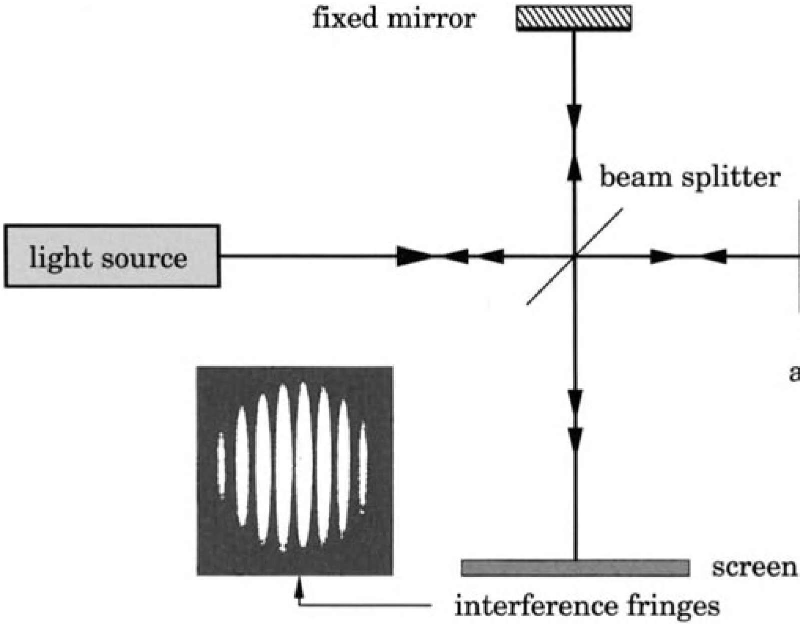

Fig. 4.1. The Michelson interferometer.

displacement d according to

$$\tau = \frac{2d}{c}. \tag{4.2}$$

On the screen, we observe the interference of both waves given by the superposition of the wave amplitudes:

$$E(t) = E_1(t) + E_2(t) = E_1(t) + E_1(t+\tau). \tag{4.3}$$

This superposition is not directly visible, but only the intensity:

$$\begin{aligned} I &= \langle EE^* \rangle = \langle (E_1 + E_2)(E_1 + E_2)^* \rangle \\ &= \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle + \langle E_2 E_1^* \rangle + \langle E_1 E_2^* \rangle \\ &= I_1 + I_2 + 2\,\mathrm{Re}\{\langle E_1^* E_2 \rangle\} \\ &= 2I_1 + 2\,\mathrm{Re}\{\langle E_1^* E_2 \rangle\}. \end{aligned} \tag{4.4}$$

It can be seen that the total intensity on the screen is given by the sum of the intensity I_1 of the first wave and I_2 of the second wave and an additional term, the interference term. The important information is contained in the expression $\langle E_1^* E_2 \rangle$. With $E_2(t) = E_1(t+\tau)$ this gives rise to the definition

$$\begin{aligned} \Gamma(\tau) &= \langle E_1^*(t) E_1(t+\tau) \rangle \\ &= \lim_{T_\mathrm{m} \to \infty} \frac{1}{T_\mathrm{m}} \int_{-T_\mathrm{m}/2}^{+T_\mathrm{m}/2} E_1^*(t) E_1(t+\tau)\,\mathrm{d}t. \end{aligned} \tag{4.5}$$

$\Gamma(\tau)$ is called the complex self coherence function. It is the autocorrelation function of the complex light wave $E_1(t)$. For the intensity $I(\tau)$ we then get

$$I(\tau) = I_1 + I_2 + 2\,\mathrm{Re}\{\Gamma(\tau)\} = 2I_1 + 2\,\mathrm{Re}\{\Gamma(\tau)\}. \tag{4.6}$$

As an example we take the harmonic wave

$$E_1(t) = E_0 \exp(-i\omega t). \tag{4.7}$$

Then

$$\begin{aligned}\Gamma(\tau) &= \lim_{T_\mathrm{m}\to\infty} \frac{1}{T_\mathrm{m}} \int_{-T_\mathrm{m}/2}^{+T_\mathrm{m}/2} E_1^*(t) E_1(t+\tau)\,\mathrm{d}t \\ &= \lim_{T_\mathrm{m}\to\infty} \frac{1}{T_\mathrm{m}} \int_{-T_\mathrm{m}/2}^{+T_\mathrm{m}/2} |E_0|^2 \exp(i\omega t) \exp\left[-i\omega(t+\tau)\right]\,\mathrm{d}t \\ &= |E_0|^2 \exp(-i\omega\tau) = I_1 \exp(-i\omega\tau),\end{aligned} \tag{4.8}$$

that is, the self coherence function harmonically depends on the time delay τ. The intensity is given, with (4.6), as

$$\begin{aligned}I(\tau) &= 2I_1 + 2\,\mathrm{Re}\{\Gamma(\tau)\} \\ &= 2I_1 + 2I_1\,\mathrm{Re}\{\exp(-i\omega\tau)\} \\ &= 2I_1 + 2I_1 \cos\omega\tau \\ &= 2I_1(1+\cos\omega\tau).\end{aligned} \tag{4.9}$$

The graph of $I(\tau)$ is plotted in Fig. 4.2. In a Michelson interferometer with a slightly tilted mirror it can be observed as a fringe pattern, see Fig. 4.1.

It is easy to envisage that it is possible to superimpose on the screen not two time-shifted light waves from the same source but two light waves from different sources whose coherence is to be tested. Interference experiments with lasers may lead to nontrivial results [4.1]. For such cases,

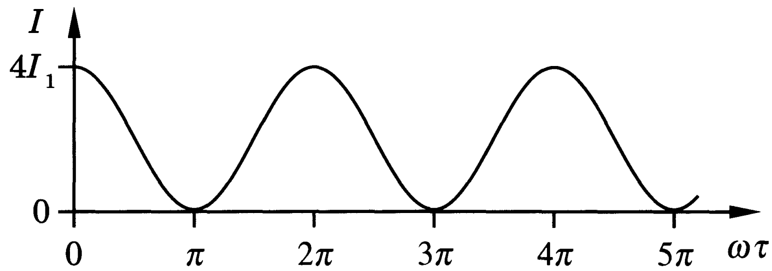

Fig. 4.2. Graph of the intensity $I(\tau)$ in a Michelson interferometer for a harmonic wave in dependence on the phase shift $\omega\tau$, where $\tau = 2d/c$, d being the mirror displacement and c being the velocity of light.

the above definition (4.5) for describing temporal coherence must be extended, leading to the definition of the cross coherence function

$$\Gamma(\tau) = \langle E_1^*(t)E_2(t+\tau)\rangle . \tag{4.10}$$

It is the crosscorrelation function of the two light waves. It is taken at one fixed location in space as is the self coherence function.

The complex self coherence function $\Gamma(\tau)$ may be normalized:

$$\gamma(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)} . \tag{4.11}$$

The magnitude $\gamma(\tau)$ is called the complex degree of self coherence. Because $\Gamma(0) = I_1$ is always real and is the largest value that occurs when we take the modulus of the autocorrelation function $\Gamma(\tau)$, we have

$$|\gamma(\tau)| \leq 1 . \tag{4.12}$$

The intensity $I(\tau)$ then reads

$$\begin{aligned} I(\tau) &= 2I_1 + 2I_1 \operatorname{Re}\{\gamma(\tau)\} \\ &= 2I_1(1 + \operatorname{Re}\{\gamma(\tau)\}) . \end{aligned} \tag{4.13}$$

The functions $\Gamma(\tau)$ and $\gamma(\tau)$ are contained in the interference term coming into existence only when we take the intensity. They are not directly obtainable. It is, however, easy to determine the contrast K between interference fringes. This quantity has already been used by *Michelson* who called it visibility and defined it via the maximum and minimum intensity $I_{\max}$ and $I_{\min}$ as

$$K = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} . \tag{4.14}$$

The contrast K obviously depends on the time shift τ between the light waves; that is, K is a function of τ. A precise definition of the contrast has to take into account the fact that the maximum and the minimum intensity of the interference fringes do not occur at the same time shift of the light waves (see Fig. 4.2). Let τ_1 and τ_2, $\tau_2 > \tau_1$, be the time shifts belonging to adjacent interference fringes of maximum and minimum intensity, $I_{\max}(\tau_1)$ and $I_{\min}(\tau_2)$. Then the contrast $K(\tau)$ is defined on the interval $[\tau_1, \tau_2)$ by

$$K(\tau) = \frac{I_{\max}(\tau_1) - I_{\min}(\tau_2)}{I_{\max}(\tau_1) + I_{\min}(\tau_2)} . \tag{4.15}$$

Usually $\tau_2 - \tau_1$, corresponding to half a mean wavelength, is small compared to the duration of the wave train to be investigated. Only in this case does the definition make sense. Then the contrast function $K(\tau)$ can be expressed in terms of the self coherence function $\Gamma(\tau)$.

We demonstrate the connection between $K(\tau)$ and $\Gamma(\tau)$ by way of example and use quasimonochromatic light, that is, light of relatively small bandwidth ($\Delta\omega/\omega \ll 1$). The typical dependence of the self coherence function $\Gamma(\tau)$ on the time shift τ for this case is given in Fig. 4.3. We observe that according to (4.6) the maximum intensity is attained at maximum $\mathrm{Re}\{\Gamma(\tau)\}$, occurring at τ_1, and the minimum intensity at minimum $\mathrm{Re}\{\Gamma(\tau)\}$, occurring at τ_2. Moreover, we see that the modulus of $\Gamma(\tau)$ practically stays constant in the interval $[\tau_1, \tau_2[$. It follows, for τ taken from this interval, that

$$\mathrm{Re}\{\Gamma(\tau_1)\} = |\Gamma(\tau)| \qquad \text{and} \qquad \mathrm{Re}\{\Gamma(\tau_2)\} = -|\Gamma(\tau)|\,. \tag{4.16}$$

This leads to the intensities

$$I_{\max}(\tau_1) = 2I_1 + 2\,\mathrm{Re}\{\Gamma(\tau_1)\} = 2I_1 + 2|\Gamma(\tau)|, \tag{4.17}$$

$$I_{\min}(\tau_2) = 2I_1 + 2\,\mathrm{Re}\{\Gamma(\tau_2)\} = 2I_1 - 2|\Gamma(\tau)|, \tag{4.18}$$

and to the contrast function

$$\begin{aligned} K(\tau) &= \frac{2I_1 + 2|\Gamma(\tau)| - 2I_1 + 2|\Gamma(\tau)|}{2I_1 + 2|\Gamma(\tau)| + 2I_1 - 2|\Gamma(\tau)|} \\ &= \frac{4|\Gamma(\tau)|}{4I_1} = \frac{|\Gamma(\tau)|}{I_1} = \frac{|\Gamma(\tau)|}{\Gamma(0)} \\ &= |\gamma(\tau)|\,. \end{aligned} \tag{4.19}$$

The contrast function thus equals the modulus of the complex degree of coherence. This is valid for two waves of equal intensity, otherwise some prefactors arise.

For quasimonochromatic light whose self coherence function slowly spirals into the origin (see Fig. 4.3) it is easily seen that a monotonously decreasing contrast function is obtained, as the modulus of $\Gamma(\tau)$ continuously decreases.

For a harmonic wave we found

$$\gamma(\tau) = \exp(-i\omega\tau)\,. \tag{4.20}$$

Therefore the contrast function is just

$$K(\tau) = |\gamma(\tau)| = 1\,. \tag{4.21}$$

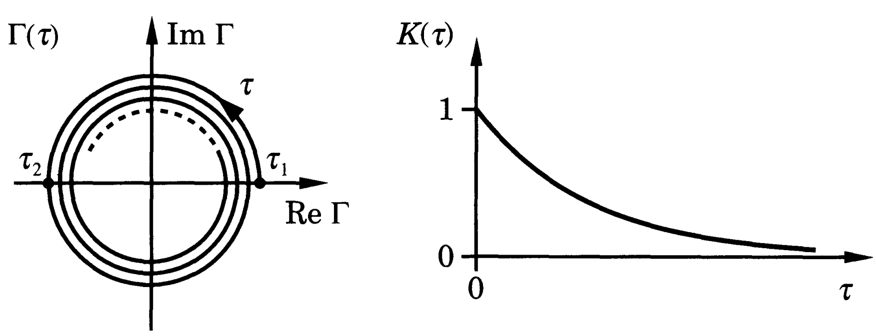

Fig. 4.3. Self coherence function $\Gamma(\tau)$ in the complex plane (*left*) and contrast function $K(\tau)$ (*right*) for quasimonochromatic light.

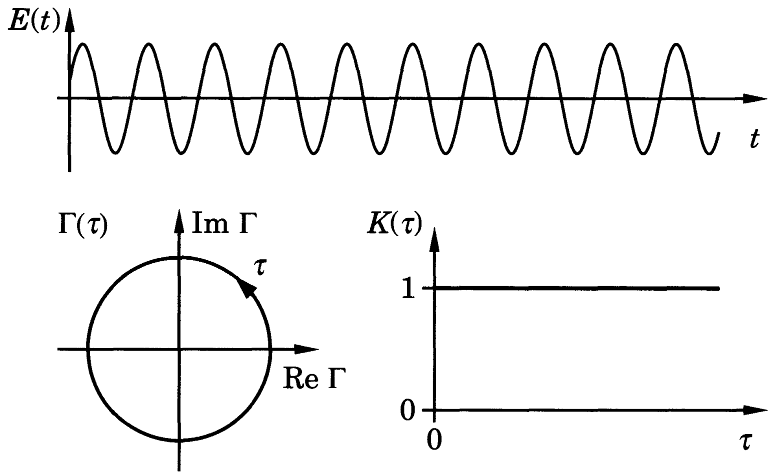

Fig. 4.4. Graph of the field amplitude E, the self coherence function Γ, and the contrast function K for completely coherent light.

A harmonic wave thus can be shifted arbitrarily in time and superimposed with itself without altering its interference properties. Light with this property is called completely coherent. This, of course, is a limiting case. It can be realized approximately, for instance, with a stabilized single-mode laser.

The graph of the contrast function can attain very different shapes. A further limiting case is completely incoherent light, characterized by $|\gamma(\tau)| = 0$ for $\tau \neq 0$ ($\gamma(0) = 1$ in all cases). The corresponding light field is made up of a mixture of light waves of all wavelengths with a statistical distribution of phases. This case, too, is realized only approximately. Good examples are daylight and incandescent light.

The two limiting cases of completely coherent and completely incoherent light are depicted in Fig. 4.4 and Fig. 4.5, respectively, with the graphs

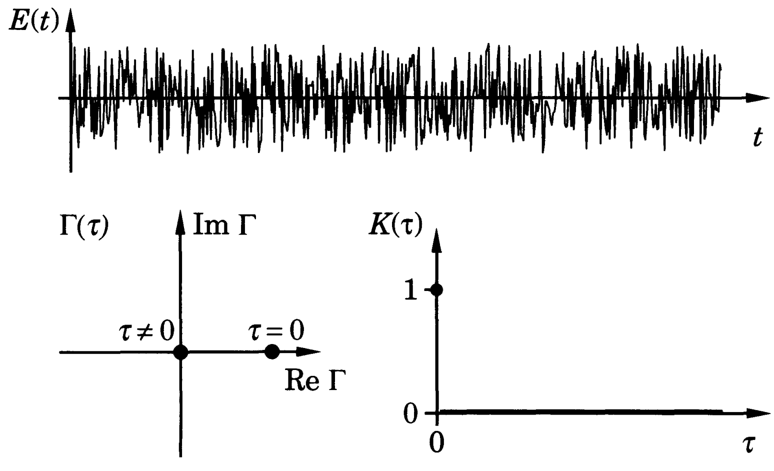

Fig. 4.5. Graph of the field amplitude E, the self coherence function Γ, and the contrast function K for completely incoherent light.

of the field amplitude versus time, the self coherence function, and the contrast function.

Light in the large range in between these two limiting cases is called partially coherent. Therefore, the following cases are distinguished ($\tau \neq 0$, $|\gamma(0)| = 1$):

$$\begin{aligned} |\gamma(\tau)| &\equiv 1 \text{ completely coherent,} \\ 0 \leq |\gamma(\tau)| &\leq 1 \text{ partially coherent,} \\ |\gamma(\tau)| &\equiv 0 \text{ completely incoherent.} \end{aligned}$$

Many natural and artificial light sources have a monotonously decreasing contrast function; for instance, the light from a spectral lamp. Figure 4.6 shows typical graphs of the field amplitude, the self coherence function, and the contrast function for light from a mercury lamp. To characterize the decay of the contrast function, the coherence time τ_c is introduced. It is defined as the time shift when the contrast function has decayed to the value 1/e. In optical arrangements, such as the Michelson interferometer, the time shift between the waves to be superimposed is effected by different optical path lengths. Thus, equivalently to the coherence time, the coherence length,

$$l_c = c\tau_c \,, \tag{4.22}$$

is used for characterizing the interference properties of light. Typical values of the coherence length are some micrometers for incandescent light and some kilometers for single-mode laser light.

The notions of coherence time and coherence length can be introduced without difficulties for all those light sources that show a monotonously decreasing contrast function (see Fig. 4.6).

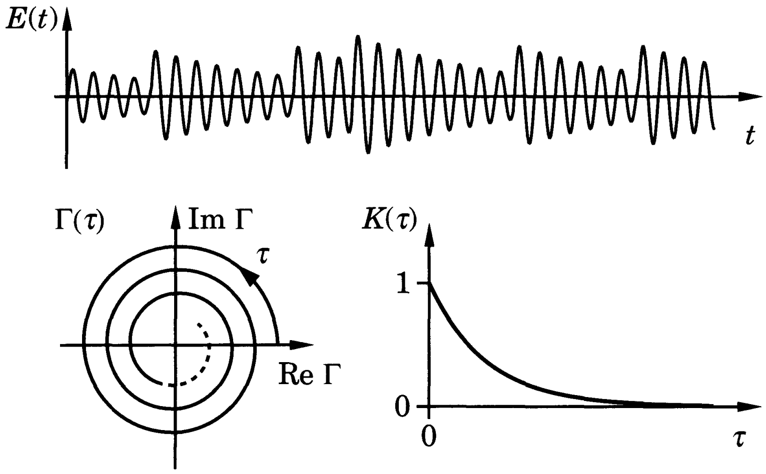

Fig. 4.6. Graph of the field amplitude E, the self coherence function Γ, and the contrast function K for light from a mercury-vapor lamp.

The decay, however, need not proceed monotonously. For instance, when we consider the superposition of two harmonic waves of different frequency, the field amplitude varies in the form of beats (Fig. 4.7). This case is approximately realized in a two-mode laser. What does the contrast function look like for this type of light? For simplicity, we consider two harmonic waves of equal amplitude:

$$E(t) = E_0 \exp(-i\omega_1 t) + E_0 \exp(-i\omega_2 t) \; . \tag{4.23}$$

Then, with (4.5), the self coherence function is given by

$$\begin{aligned}
\Gamma(\tau) &= \lim_{T_\mathrm{m}\to\infty} \frac{1}{T_\mathrm{m}} \int\limits_{-T_\mathrm{m}/2}^{+T_\mathrm{m}/2} \left[E_0^* \exp(i\omega_1 t) + E_0^* \exp(i\omega_2 t)\right] \cdot \\
&\quad \cdot \Big(E_0 \exp\left[-i\omega_1(t+\tau)\right] + E_0 \exp\left[-i\omega_2(t+\tau)\right]\Big) \mathrm{d}t \\
&= \lim_{T_\mathrm{m}\to\infty} \frac{|E_0|^2}{T_\mathrm{m}} \int\limits_{-T_\mathrm{m}/2}^{+T_\mathrm{m}/2} \Big(\exp\left[-i\omega_1\tau\right] + \exp\left[-i\omega_2\tau\right] + \\
&\quad \underbrace{\exp(-i\omega_1\tau)\exp\left[-i(\omega_1-\omega_2)t\right] + \exp(-i\omega_2\tau)\exp\left[-i(\omega_2-\omega_1)t\right]}_{\text{no contribution, since zero mean}}\Big) \mathrm{d}t \\
&= |E_0|^2 \left[\exp(-i\omega_1\tau) + \exp(-i\omega_2\tau)\right] \; .
\end{aligned} \tag{4.24}$$

With (4.11) and $\Gamma(0) = 2|E_0|^2$ we get:

$$\gamma(\tau) = \frac{1}{2}\left[\exp(-i\omega_1\tau) + \exp(-i\omega_2\tau)\right] . \tag{4.25}$$

Finally, with (4.19) we obtain:

$$\begin{aligned}
K(\tau) &= |\gamma(\tau)| = \frac{1}{2}\left|\exp(-i\omega_1\tau) + \exp(-i\omega_2\tau)\right| \\
&= \frac{1}{2}\sqrt{\left[\exp(-i\omega_1\tau) + \exp(-i\omega_2\tau)\right]\left[\exp(i\omega_1\tau) + \exp(i\omega_2\tau)\right]} \\
&= \frac{1}{2}\sqrt{2 + 2\cos(\omega_1 - \omega_2)\tau} = \frac{1}{2}\sqrt{4\cos^2\frac{(\omega_1-\omega_2)}{2}\tau} \\
&= \left|\cos\left(\frac{\omega_1-\omega_2}{2}\tau\right)\right| .
\end{aligned} \tag{4.26}$$

In this case, the contrast function $K(\tau)$ takes a periodic dependence on the time shift τ (Fig. 4.7). A coherence time or coherence length in the sense defined above does not seem meaningful as the contrast again and again attains the maximum value of unity. Here, the location of the first root or the first minimum may be taken as a measure of the coherence time or length.

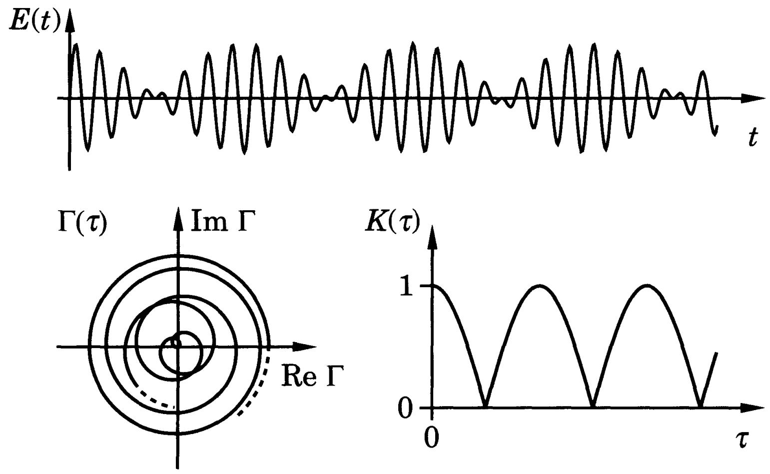

Fig. 4.7. Graph of the field amplitude E, the self coherence function Γ, and the contrast function K for light from a two-mode laser.

The result obtained for the self coherence function of two harmonic waves of different frequency can easily be extended to a sum of many harmonic waves of different frequency. Let

$$E(t) = \sum_{m=1}^{M} E_{0m} \exp(-i\omega_m t), \tag{4.27}$$

then immediately

$$\Gamma(\tau) = \sum_{m=1}^{M} |E_{0m}|^2 \exp(-i\omega_m \tau) \tag{4.28}$$

is obtained. In the limit of arbitrarily densely spaced harmonic waves we have

$$E(t) = \int_0^\infty E_0(\nu) \exp(-i2\pi\nu t) \mathrm{d}\nu. \tag{4.29}$$

Then we get for the self coherence function

$$\begin{aligned} \Gamma(\tau) &= \int_0^\infty |E_0(\nu)|^2 \exp(-i2\pi\nu\tau) \mathrm{d}\nu \\ &= \int_0^\infty W(\nu) \exp(-i2\pi\nu\tau) \mathrm{d}\nu. \end{aligned} \tag{4.30}$$

The function $W(\nu) = |E_0(\nu)|^2$ is the power spectrum of the complex light field [4.2].