Tutorial on Kalman Filter

September 2, 2025

1 Introduction

The Kalman Filter (KF) is a recursive algorithm that estimates the state of a linear dynamical system in the presence of noise. It is widely used in navigation, tracking, signal processing, and control systems. The KF provides an optimal estimate in the least-squares sense when the noise is Gaussian.

2 System Model

The Kalman Filter assumes a linear state-space model:

$$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{B}_{k-1}\mathbf{u}_{k-1} + \mathbf{w}_{k-1}, \tag{1}$$

$$\mathbf{z}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k, \tag{2}$$

where:

- $\mathbf{x}_k$: state vector at time k,
- $\mathbf{F}_{k-1}$: state transition matrix,
- $\mathbf{B}_{k-1}$: control-input matrix,
- $\mathbf{u}_{k-1}$: control vector,
- $\mathbf{w}_{k-1}$: process noise, $\mathbf{w}_{k-1} \sim \mathcal{N}(0, \mathbf{Q}_{k-1})$,
- $\mathbf{z}_k$: measurement vector,
- $\mathbf{H}_k$: measurement matrix,
- $\mathbf{v}_k$: measurement noise, $\mathbf{v}_k \sim \mathcal{N}(0, \mathbf{R}_k)$.

3 Kalman Filter Algorithm

The KF operates in two steps: **Prediction** and **Update**.

3.1 Prediction Step

Using the model, the state and covariance are projected forward:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_{k-1}\mathbf{u}_{k-1}, \tag{3}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_{k-1}, \tag{4}$$

where:

- $\hat{\mathbf{x}}_{k|k-1}$: predicted state estimate,
- $\mathbf{P}_{k|k-1}$: predicted covariance matrix.

3.2 Update (Correction) Step

When a new measurement $\mathbf{z}_k$ arrives, the filter updates:

$$\mathbf{y}_k = \mathbf{z}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k|k-1}, \quad \text{(innovation)} \tag{5}$$

$$\mathbf{S}_k = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k, \quad \text{(innovation covariance)} \tag{6}$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T\mathbf{S}_k^{-1}, \quad \text{(Kalman gain)} \tag{7}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k\mathbf{y}_k, \tag{8}$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k|k-1}. \tag{9}$$

4 Explanation of Terms

- **Innovation ($\mathbf{y}_k$)**: The difference between the actual measurement and the predicted measurement.
- **Innovation Covariance ($\mathbf{S}_k$)**: Measures the uncertainty of the innovation.
- **Kalman Gain ($\mathbf{K}_k$)**: Determines how much the measurements influence the updated state.
- **Updated State ($\hat{\mathbf{x}}_{k|k}$)**: Combines prediction and measurement.
- **Updated Covariance ($\mathbf{P}_{k|k}$)**: Reflects reduced uncertainty after incorporating the measurement.

5 Recursive Operation

The KF runs recursively in two steps:

1. **Predict**: Use the system model to forecast the next state.
2. **Update**: Correct the forecast using the actual measurement.

This process repeats at each time step.

6 Summary

The Kalman Filter provides:

- Optimal state estimation for linear Gaussian systems.
- A balance between prediction (model-based) and correction (measurement-based).
- Recursive formulation, efficient for real-time applications.