Chapter 4
Advanced counting methods
4.1 The principle of Inclusion – Exclusion

Let A, B and C be sets. Then

1. $|A \cup B| = |A| + |B| - |A \cap B|$
2. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
   Where $|X|$ is cardinality X or the number of elements of set X.

Theorem:- Inclusion – Exclusion

Let $A_1,\ A_2\ , ---,\ A_n$ be finite sets. Then

$$|A_1 \cup\ A_2\ \cup A_3 \cup\ ---\ A_n| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j|$$
$$+ \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| -\ \dots +\ (-1)^{n+1}|A_1 \cap \dots \cap\ A_n|$$

Note:- 1. For 'n' number of sets, we have $2^n - 1$ number of terms.

2.The coefficient of a term is -1 if the number of sets in the intersection is even and 1 otherwise.

Exercise:- Give a formula for the number of elements in the union of four sets.

Note :- The inclusion – exclusion principle is used to solve problems that ask for the number of elements in a set that have none of 'n' properties $p_1, p_2, ---, p_n$, denoted by $N(p_1{}'\ p_2{}'\ ---\ p_n{}')$ and defined by
$N(p_1{}'\ p_2{}'\ ---\ p_n{}') = N - |A_1 \cup\ A_2\ \cup A_3 \cup\ ---\ A_n|$, where N is the no of elements which belong to the universal set (U) and $A_i \in U$ and A_i has property p_i

Example:- 12 balls are painted in the following ways: 2 are painted red, 1 is painted blue, 1 is painted white, 2 are painted red & blue, 1 is painted red & white and 3 are painted red, blue & white. How many balls are not painted?

Solution:- Let $p_1,\ p_2,\ p_3$ denotes the properties that a ball is painted red, blue, white respectively; then

$$N(p_1) = 8, \quad N(p_2) = 6, \quad N(p_3) = 5, \quad N(p_1p_2) = 5, \qquad N(p_1p_3) = 4$$
$$N(p_2p_3) = 3, \qquad N(p_1p_2p_3) =\ 3$$

Then $N(p_1{}'\ p_2{}'\ p_3{}') = N - |A_1 \cup\ A_2\ \cup A_3|$

$= N - (\ N(p_1) + N(p_2) + N(p_3) - \quad N(p_1p_2)$
$- N(p_1p_3) - \quad N(p_2p_3) +\ N(p_1p_2p_3))$ = 12– [8+6+5–4–3+3] = 2

Example:- 1. Determine the number of positive integer 'n' where $1 \le n \le 100$ and n is not divisible by 2, 3, 5.

Solution:- Here $U = \{1, 2, 3, ---, 100\}, N = 100$,
for $n \in U$, n satisfies: p_2 if n is divisible by 2.
p_3 if n is divisible by 3.
p_5 if n is divisible by 5.

Note:- $\lfloor x \rfloor$ = is the greatest integer $l\ such\ that\ l \le x$.

Now, $N(p_2) = \left\lfloor \frac{100}{2} \right\rfloor = 50$, $A_2 = \{2, 4, 6, ---\}$, $N(p_2p_3) = \left\lfloor \frac{100}{6} \right\rfloor = 16$

$N(p_3) = \left\lfloor \frac{100}{3} \right\rfloor = 33$, $A_3 = \{3, 6, 9, ---\}$, $N(p_2p_5) = \left\lfloor \frac{100}{10} \right\rfloor = 10$

$N(p_5) = \left\lfloor \frac{100}{5} \right\rfloor = 20$, $A_5 = \{5, 10, 15, ---\}$, $N(p_3p_5) = \left\lfloor \frac{100}{15} \right\rfloor = 6$

$N(p_2p_3p_5) = \left\lfloor \frac{100}{30} \right\rfloor = 3$

Then $N(p_2{}'\ p_3{}'\ p_5{}') = N - |A_2 \cup\ A_3\ \cup A_5|$

= $N - (\ N(p_2) + N(p_3) + N(p_5) - \ N(p_2p_3)$
$- N(p_2p_5) - \ N(p_3p_5) +\ N(p_2p_3p_5))$
= 100 – (50+33+20 – 10 -16 – 6 + 3)
= 26 number are not divisible by 2, 3, 5.

2. Using question (1);

i) find the number of positive integers that are not divisible by 2 nor by 5 but are divisible by 3?

ii) Find the number of positive integers that are not divisible by 5 but are divisible by 2 &3

Solution:- i) $N(p_2{}'\ p_3\ p_5{}') = N(p_3) - N(p_2p_3) - \ N(p_3p_5) +\ N(p_2p_3p_5)$ = 33-16-6+3

ii) $N(p_2\ p_3\ p_5{}') = N(p_2p_3) - N(p_2p_3p_5) = 16 - 3 = 13$

4.2. Recurrence Relation

Consider the sequence 2, 4, 8, 16, - - - can be defined recursively like

$$a_{k+1} = 2a_k, \quad a_1 = 2, k \ge 1 \qquad . \ . \ . \quad (*)$$

The equation $a_{k+1} = 2a_k$ in $(*)$, which defines one member of the sequence interms of a previous one is called a recurrence relation.

The equation $a_1 = 2$ is called an initial (boundary) condition.

Example:- 1. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}, for\ n = 2, 3, 4,\ \ . . . , a_0 = 3\ and\ a_1 = 5$. Then find a_2, a_3, a_4, a_5

2. Suppose that f is defined recursively by $f(0) = 3, f(n+1) = 2f(n) + 3$. Then find $f(1), f(2), f(3), f(4)$

3. The Fibonacci numbers $f_{0,}\ f_1,\ f_2,\ . . .$ are defined by the equations $f_0\ = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}, for\ n = 2,3,4,\ \ . . . .$ Then find $f_{2,}\ f_3,\ f_4, f_{5,}\ f_6,\ f_7$

Definition:- The general linear recurrence relation of degree k with constant coefficients has the form $c_0a_n + c_1a_{n-1} + c_2a_{n-2} + . . . + c_ka_{n-k} = f(n)$, where $c_{0,}\ c_1,\ c_2,\ . . . , c_k$ are real numbers, $c_{0,}\ c_k \neq 0$ and $k \in N$

Note:- When $f(n) = 0, \forall n \in N$, the relation is called homogeneous, otherwise non-homogeneous.

Example:- The recurrence relation

a) $p_n = 5p_{n-1}$ is linear, homogeneous with degree 1.

b) $f_n = f_{n-1} + f_{n-2}$ is linear, homogeneous with degree 2.

c) $a_n = a_{n-5}$ is linear, homogeneous with degree 5.
d) $f_n = f_{n-1} + f^3{}_{n-2}$ is not linear, it is homogeneous with constant coefficient.
e) $f_n = 2f_{n-1}f_{n-2} + 1$ is not linear, not homogeneous but it has constant coeffi.
f) $B_n = nB_{n-1}$ is linear, homogeneous but the coefficient is not constant.

4.3 Solution of recurrence relation

A. First order recurrence relation

➢ For the homogeneous case can be written as:

$a_n = ra_{n-1}, n \geq 1, a_0 = A,$ r is constant. Its general solution is $a_n = Ar^n$

1. Solve the recurrence relation
   a) $a_n = 3a_{n-1}, n \geq 1, a_0 = 5$
   b) $a_n = 7a_{n-1}, n \geq 1, a_2 = 98$
   c) $a_n = na_{n-1}, n \geq 1, a_0 = 1$

Solution:- a) Since $r = 3, A = 5$, then $a_n = 5(3^n)$ is the solution

b) Since $r = 7, A = a_0 = 2$ because $a_2 = 7a_1$ and $a_1 = 7a_0 \Rightarrow a_0 = 2$

Then $a_n = 2(7^n)$ is the solution

c) Since $= n$, which is not constant, we can not use the above formula.

$a_1 = 1, a_0 = 1, a_2 = 2a_1 = 2.1, a_3 = 3a_2 = 3.2\,, a_4 = 4a_3 = 4.3.2 = 4!$

$a_5 = 5!$. Thus $a_n = n!$

B. Second order recurrence relation

➢ For the homogeneous case can be written as:

$c_0a_n + c_1a_{n-1} + c_2a_{n-2} = 0, n \geq 2$, where $c_0, c_1, c_2 \in R$ (real n<u>o</u>), $c_0, c_2 \neq 0 \ldots (*)$

First we assume for solutions of the form $a_n = Ar^n$, where $A,$ $r \neq 0$ and substituting in to the above equation $(*)$, we have

$$c_0Ar^n + c_1Ar^{n-1} + c_2Ar^{n-2} = 0$$
$$\Rightarrow c_0r^2 + c_1r + c_2 = 0 \quad \ldots\ldots\ldots (**)$$

which is called the characteristic equation of the recurrence relation.

Then we may have two roots, r_1 and r_2 are called the characteristic roots.

For these roots we have 3 cases:

Case1:- **<u>Distinct real roots</u>**

In this case the general solution of the recurrence relation is

$a_n = b_1{r_1}^n + b_2{r_2}^n$, where $b_1,$ b_2 are arbitrary constants.

Example:- Solve the recurrence relations

a) $a_n + a_{n-1} - 6a_{n-2} = 0,$ $n \geq 2\,,$ $a_0 = -1\,,$ $a_1 = 8$
b) $a_n = a_{n-1} + 2a_{n-2},$ $n \geq 2\,,$ $a_0 = 2\,,$ $a_1 = 7$
c) $a_{n+2} = a_{n+1} + a_n,$ $n \geq 0\,,$ $a_0 = 0\,,$ $a_1 = 1$

Solution:- a) Let $a_n = cr^n$ with $c, r \neq 0$. Then substitute this on

$a_n + a_{n-1} - 6a_{n-2} = 0$, we have $cr^n + cr^{n-1} - 6cr^{n-2} = 0$

$$\Rightarrow r^2 + r - 6 = 0$$
$$\Rightarrow r = 2 \; or \; r = -3$$

Here we have two distinct real roots, so, we can $a_n = 2^n$ and $a_n = (-3)^n$ which are both solutions. They are linearly independent solutions because one is not a multiple of the other, i.e, there is no real constant k such that $(-3)^n = k(2)^n$ for all $n\epsilon\mathbb{N}$.
Therefore its general solution is the form $a_n = b_1 2^n + b_2(-3)^n$, where $b_1,\ b_2$ are arbitrary constants.
Now to determine b_1 and b_2, use $a_0 = -1, a_1 = 8$.
If $a_0 = -1 = b_1 2^0 + b_2(-3)^0 \implies -1 = b_1 + b_2$
If $a_1 = 8 = b_1 2^1 + b_2(-3)^1 \implies 8 = 2b_1 - 3b_2$
Solve this system of equations, we have $b_1 = 1,\ b_2 = -2$
$\therefore\ a_n = 2^n - 2(-3)^n$ for $n \geq 0$ is its general solution.

b) Let $a_n = cr^n$ with $c, r \neq 0$. Then substitute this on $a_n = a_{n-1} + 2a_{n-2}$,
we have $cr^n = cr^{n-1} + 2cr^{n-2}$

$$\implies r^2 - r - 2 = 0$$
$$\implies r = -1 \quad or \quad r = 2$$

Here we have two distinct real roots, so, we can $a_n = 2^n$ and $a_n = (-1)^n$ which are both solutions. They are linearly independent solutions because one is not a multiple of the other, i.e, there is no real constant k such that $(-1)^n = k(2)^n$ for all $n\epsilon\mathbb{N}$.
Therefore its general solution is the form $a_n = b_1 2^n + b_2(-1)^n$, where $b_1,\ b_2$ are arbitrary constants.
Now to determine b_1 and b_2, use $a_0 = 2, a_1 = 7$.
If $a_0 = 2 = b_1 2^0 + b_2(-1)^0 \implies 2 = b_1 + b_2$
If $a_1 = 7 = b_1 2^1 + b_2(-1)^1 \implies 7 = 2b_1 - b_2$
Solve this system of equations, we have $b_1 = 3,\ b_2 = -1$
$\therefore\ a_n = 3(2^n) - 1(-1)^n$ for $n \geq 0$ is its general solution.

c) Let $a_n = cr^n$ with $c, r \neq 0$. Then substitute this on $a_{n+2} = a_{n+1} + a_n$,
we have $cr^{n+2} = cr^{n+1} + cr^n$

$$\implies r^2 - r - 1 = 0$$
$$\implies r = \frac{1-\sqrt{5}}{2} \quad or \quad r = \frac{1+\sqrt{5}}{2}$$

Here we have two distinct real roots, so, we can $a_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$ and $a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ which are both solutions. They are linearly independent solutions because one is not a multiple of the other, i.e, there is no real constant k such that $\left(\frac{1-\sqrt{5}}{2}\right)^n = k\left(\frac{1+\sqrt{5}}{2}\right)^n$ for all $n\epsilon\mathbb{N}$.

Therefore its general solution is the form $a_n = b_1\left(\frac{1-\sqrt{5}}{2}\right)^n + b_2\left(\frac{1+\sqrt{5}}{2}\right)^n$, where $b_1,\ b_2$ are arbitrary constants.
Now to determine b_1 and b_2, use $a_0 = 0, a_1 = 1$.

If $a_0 = 0 = b_1\left(\frac{1-\sqrt{5}}{2}\right)^0 + b_2\left(\frac{1+\sqrt{5}}{2}\right)^0 \implies 0 = b_1 + b_2 \implies b_1 = -b_2$

If $a_1 = 1 = b_1\left(\frac{1-\sqrt{5}}{2}\right)^1 + b_2\left(\frac{1+\sqrt{5}}{2}\right)^1 \implies 1 = b_1\left(\frac{1-\sqrt{5}}{2}\right) + b_2\left(\frac{1+\sqrt{5}}{2}\right)$

Solve this system of equations, we have $b_1 = -\frac{1}{\sqrt{5}},\ b_2 = \frac{1}{\sqrt{5}}$

$\therefore\ a_n = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ for $n \geq 0$ is its general solution.

Case 2:- Repeated real roots

If $r_1 = r_2 = r$, then the general solution of the recurrence relation is $a_n = b_1 {r_1}^n + b_2 n {r_2}^n$ or $a_n = b_1 n {r_1}^n + b_2 {r_2}^n$, where $b_1,\ b_2$ are arbitrary constants.

Example:- Solve the recurrence relation

a) $a_{n+2} = 4\,a_{n+1} - 4a_n, n \geq 0\ \ and\ a_0 = 1, a_1 = 3$
b) $a_n = 6\,a_{n-1} - 9a_{n-2}, n \geq 2\ \ and\ a_0 = 1, a_1 = 12$
c) $a_n = 6\,a_{n-1} - 9a_{n-2}, n \geq 2\ \ and\ a_0 = 1, a_1 = 6$

Solution:- a) Let $a_n = cr^n$ with $c, r\ \neq 0$ and $n \geq 0$ Then substitute this on

$a_{n+2} = 4a_{n+1} - 4a_n$, we have $cr^{n+2} = 4cr^{n+1} - 4cr^n$

$\implies r^2 - 4r + 4 = 0 \implies r = 4$

Thus its general solution is the form $a_n = b_1 2^n + b_2 n 2^n$, where $b_1,\ b_2$ are arbitrary constants.

Now to determine b_1 and b_2, use $a_0 = 1, a_1 = 3$.

If $a_0 = 1 = b_1 2^0 + b_2(0)(2)^0 \implies 1 = b_1$

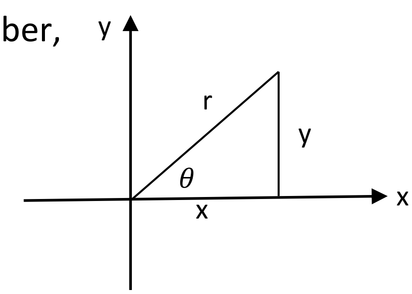
If $a_1 = 3 = b_1 2^1 + b_2(1)(2)^1 \implies 3 = 2b_1 + b_2 \implies b_2 = \frac{1}{2}$

$\therefore$ General solution $a_n = 2^n + \frac{1}{2}\, n2^n, n \geq 0$

Case3:- Complex roots

In this case the solution could be expressed in the same way as in case of distinct real roots. Before getting into the case of complex roots , we recall **Demoiver's theorem**:

- ❖ $(cos\theta + isin\theta)^n = cosn\theta + \mathrm{i}\sin n\theta\ ; n \geq 0$
- ❖ If $z = x + iy \in \mathbb{C}\ ;\ z \neq 0$, where $\mathbb{C}$ is complex number, then can write



i) For $x \neq 0$, $z = r(cos\theta + isin\theta)$, where $r = \sqrt{x^2 + y^2}$

ii) for $x = 0$ and $y > 0$,

$z = iy = r(cos\frac{\pi}{2} + isin\frac{\pi}{2}) = yisin\frac{\pi}{2}$

iii) for $x = 0$ and $y < 0$,

$z = iy = r(cos\frac{-\pi}{2} + isin\frac{-\pi}{2}) = r(cos\frac{\pi}{2} - isin\frac{\pi}{2})$

- ❖ In all case $z^n = r^n(\cos n\theta + isin\, n\theta)$, for $n \geq 0$ by **Demoiver's theorem**

Example1:- Determine $\left(1 + i\sqrt{3}\right)^{10}$

Solution:- $\left(1 + i\sqrt{3}\right)^{10} = (r(cos\theta + isin\theta))^{10}$ but $r = 2, x = 1, y = \sqrt{3}$

$$= \left(2(cos\frac{\pi}{3} + isin\frac{\pi}{3})\right)^{10} = 2^{10}\left(\cos\frac{10\pi}{3} + isin\frac{10\pi}{3}\right) = 2^{10}(\frac{-1}{2} - \frac{\sqrt{3}}{2}i)$$

Example2:- Solve the recurrence relations

a) $a_{n+2} + 4a_n = 0, n \geq 0 \text{ and } a_0 = -1, a_1 = 4$
b) $a_n = 2(a_{n-1} - a_{n-2}), n \geq 2 \text{ and } a_0 = 1, a_1 = 2$
c) $a_{n+2} + 4a_{n+1} + 16a_n = 0, n \geq 0 \text{ and } a_0 = 2, a_1 = 10$

Solution:- a) Let $a_n = br^n, where\ b, r \neq 0 \text{ and } n \geq 0.$
Then $a_{n+2} + 4a_n = 0$ becomes $br^{n+2} + 4br^n = 0$

$$\Longrightarrow r^2 + 4 = 0 \Longrightarrow a_1 = 2i, a_2 = -2i$$

It follows that the general solution of the recurrence relation is given by

$a_n = b_1(2i)^n + b_2(-2i)^n$

$\Rightarrow a_n = 2^n\left[b_1\left(cos\frac{\pi}{2} + isin\frac{\pi}{2}\right)^n + b_2\left(cos\frac{\pi}{2} - isin\frac{\pi}{2}\right)^n\right]$

$= 2^n\left[b_1\left(cos\frac{n\pi}{2} + isin\frac{n\pi}{2}\right) + b_2\left(cos\frac{n\pi}{2} - isin\frac{n\pi}{2}\right)\right]$

$= 2^n\left[(b_1 + b_2)cos\frac{n\pi}{2} + i(b_1 - b_2)sin\frac{n\pi}{2}\right]$

$= 2^n\left[c_1cos\frac{n\pi}{2} + c_2sin\frac{n\pi}{2}\right]$, where $c_1 = b_1 + b_2,\ c_2 = i(b_1 - b_2)$

But $a_0 = -1 = 2^0[c_1cos0 + c_2sin0] = c_1 \Longrightarrow c_1 = -1$

$a_1 = 4 = 2^1\left[c_1cos\frac{\pi}{2} + c_2sin\frac{\pi}{2}\right] = 2c_2 \Longrightarrow c_2 = 2$

Therefore, $a_n = 2^n\left[-cos\frac{n\pi}{2} + 2sin\frac{n\pi}{2}\right]$ is the solution.

b) Let $a_n = br^n, where\ b, r \neq 0 \text{ and } n \geq 0$. Then $a_n = 2(a_{n-1} - a_{n-2})$ becomes
$br^n = 2(br^{n-1} - br^{n-2})$

$\Longrightarrow r^n = 2(r^{n-1} - r^{n-2})$, since divide both side by b.

$\Longrightarrow r^2 - 2r + 2 = 0$

$\Longrightarrow r_1 = 1 + i,\ r_2 = 1 - i$, Now since x = 1, y = 1, then $r = \sqrt{2}$

Then the general solution is $a_n = b_1(1 + i)^n + b_2(1 - i)^n$

$\Longrightarrow a_n = \left(\sqrt{2}\right)^n\left[b_1\left(cos\frac{\pi}{4} + i\ sin\frac{\pi}{4}\right)^n + b_2\left(cos\frac{-\pi}{4} + i\ sin\frac{-\pi}{4}\right)^n\right]$

$= \left(\sqrt{2}\right)^n\left[b_1\left(cos\frac{n\pi}{4} + i\ sin\frac{n\pi}{4}\right) + b_2\left(cos\frac{n\pi}{4} - i\ sin\frac{n\pi}{4}\right)\right]$

$= \left(\sqrt{2}\right)^n\left[(b_1 + b_2)cos\frac{n\pi}{4} + (b_1 - b_2)i\ sin\frac{n\pi}{4}\right]$

$= \left(\sqrt{2}\right)^n\left[c_1cos\frac{n\pi}{4} + c_2sin\frac{n\pi}{4}\right]$, where $c_1 = b_1 + b_2$, $c_2 = (b_1 - b_2)i$

To find $c_1 \text{ and } c_2$, use initial condition, i.e.

$$a_0 = 1 = \left(\sqrt{2}\right)^0\left[c_1cos\,0 + c_2sin\,0\right] \Longrightarrow c_1 = 1$$

$$a_1 = 2 = \left(\sqrt{2}\right)^1\left[c_1cos\frac{\pi}{4} + c_2sin\frac{\pi}{4}\right] \Longrightarrow 2 = \sqrt{2}\left(c_1\frac{\sqrt{2}}{2} + c_2\frac{\sqrt{2}}{2}\right)$$

$$\Longrightarrow 2 = c_1 + c_2 \Longrightarrow c_2 = 1$$

Thus the general solution is $a_n = \left(\sqrt{2}\right)^n\left[cos\frac{n\pi}{4} + sin\frac{n\pi}{4}\right]$.

c)Let $a_n = br^n, where\ b, r \neq 0\ and\ n \geq 0$.Then $a_{n+2} + 4a_{n+1} + 16a_n = 0$ becomes

$br^{n+2} + 4br^{n+1} + 16br^n = 0$

$\Longrightarrow r^2 + 4r + 16 = 0$

$\Longrightarrow r_1 = -2 + 2\sqrt{3}\, i, r_2 = -2 - 2\sqrt{3}\, i$

Since $x = -2\ and\ y = 2\sqrt{3},\ r = 4$

Then the general solution is $a_n = b_1(-2 + 2\sqrt{3}i)^n + b_2(-2 - 2\sqrt{3}i)^n$

$$\Longrightarrow a_n = 4^n \left[b_1 \left(cos\frac{2\pi}{3} + i\ sin\frac{2\pi}{3}\right)^n + b_2 \left(cos\frac{-2\pi}{3} + i\ sin\frac{-2\pi}{3}\right)^n\right]$$

$$= 4^n \left[b_1 \left(cos\frac{2\pi}{3} + i\ sin\frac{2\pi}{3}\right)^n + b_2 \left(cos\frac{2\pi}{3} - i\ sin\frac{2\pi}{3}\right)^n\right]$$

$$= 4^n \left[b_1 \left(cos\frac{2n\pi}{3} + i\ sin\frac{2n\pi}{3}\right) + b_2 \left(cos\frac{2n\pi}{3} - i\ sin\frac{2n\pi}{3}\right)\right]$$

$$= 4^n \left[(b_1 + b_2)cos\frac{2n\pi}{3} + (b_1 - b_2)i\ sin\frac{2n\pi}{3}\right]$$

$$= 4^n \left[c_1 cos\frac{2n\pi}{3} + c_2\ sin\frac{2n\pi}{3}\right]$$, where $c_1 = b_1 + b_2\ ,\ \ c_1 = (b_1 - b_2)i$

To find the value of c_1 and c_2, use initial condition, i.e.

$a_0 = 2 = 4^0[c_1 \cos 0 + c_2 \sin 0] \ \Longrightarrow\ c_1 = 2$

$a_1 = 10 = 4^1 \left[c_1 cos\frac{2\pi}{3} + c_2\ sin\frac{2\pi}{3}\right] \Longrightarrow\ \frac{10}{4} = c_1\left(\frac{-1}{2}\right) + c_2\left(\frac{2\sqrt{3}}{4}\right)$

$$\Longrightarrow \frac{5}{2} = \frac{-1}{2}c_1 + \frac{\sqrt{3}}{2}c_2$$

$$\Longrightarrow c_2 = \frac{7\sqrt{3}}{3}$$

Hence the general solution is $a_n = 4^n \left[2cos\frac{2n\pi}{3} + \frac{7\sqrt{3}}{3}\ sin\frac{2n\pi}{3}\right]$.

C. First and second order recurrence relation for non- homogeneous:-

The general solution for non- homogeneous recurrence relation is

$a_n = {a_n}^{(h)} + {a_n}^{(p)}$, where ${a_n}^{(h)}$ is homogeneous solution,

${a_n}^{(p)}$ is particular solution.

i) Consider a non- homogeneous first order recurrence relation $a_n - ka_{n-1} = cr^n$, where c is a constant and $n \in \mathbb{Z}^+$

- If r^n is not a solution of a homogeneous relation $a_n - ka_{n-1} = 0$, then ${a_n}^{(p)} = Ar^n$, where A is constant.
- If r^n is a solution of a homogeneous relation $a_n - ka_{n-1} = 0$, then ${a_n}^{(p)} = Bnr^n$, where B is constant.

example:- Solve a) $a_n - 3a_{n-1} = 5(7^n), n \geq 1, a_0 = 2\ -------(*)$

b) $a_n - 3a_{n-1} = 5(3^n), n \geq 1, a_0 = 2\ -------(**)$

solution:- a) Let ${a_n}^{(h)} = Cr^n$, where $C, r \neq 0$. Then substitute on $a_n - 3a_{n-1} = 0$.

We have $Cr^n - 3Cr^{n-1} = 0$

$\Longrightarrow r - 3 = 0$ (by dividing both side with Cr^{n-1})

$\Longrightarrow r = 3$

Thus $a_n{}^{(h)} = C3^n$

Since 7^n is not a solution for homogeneous, $a_n{}^{(p)} = A7^n$ and substitute on $a_n - 3a_{n-1} = 5(7^n)$. We have $A7^n - 3A7^{n-1} = 5(7^n)$

$$\Longrightarrow A - \frac{3}{7}A = 5 \Longrightarrow A = \frac{35}{4}$$

Thus $a_n{}^{(h)} = \frac{35}{4}\ 7^n$

Hence the general solution of $(*)$ is $a_n = a_n{}^{(h)} + a_n{}^{(p)}$

$$\Longrightarrow a_n = C3^n + \frac{35}{4}\ 7^n$$

To find C , use initial condition $a_0 = 2 = C(3^0) + \frac{35}{4}\ (7^0) \ \ \Longrightarrow C = \frac{-27}{4}$

Therefore $a_n = \frac{-27}{4}(3^n) + \frac{35}{4}\ (7^n)$.

b) Let $a_n{}^{(h)} = Cr^n$, where $C, r \ \neq 0$. Then substitute on $a_n - 3a_{n-1} = 0$.

We have $Cr^n - 3Cr^{n-1} = 0$

$\Longrightarrow r - 3 = 0$ (by dividing both side with Cr^{n-1})

$\Longrightarrow r = 3$

Thus $a_n{}^{(h)} = C3^n$

Since 3^n is a solution for homogeneous, $a_n{}^{(p)} = Bn7^n$ and substitute on $a_n - 3a_{n-1} = 5(7^n)$. We have $A7^n - 3A7^{n-1} = 5(7^n)$

$$\Longrightarrow A - \frac{3}{7}A = 5 \Longrightarrow A = \frac{35}{4}$$

Thus $a_n{}^{(h)} = \frac{35}{4}\ 7^n$

Hence the general solution of $(*)$ is $a_n = a_n{}^{(h)} + a_n{}^{(p)}$

$$\Longrightarrow a_n = C3^n + \frac{35}{4}\ 7^n$$

To find C , use initial condition $a_0 = 2 = C(3^0) + \frac{35}{4}\ (7^0) \ \ \Longrightarrow C = \frac{-27}{4}$

Therefore $a_n = \frac{-27}{4}(3^n) + \frac{35}{4}\ (7^n)$.

ii) Consider a non- homogeneous 2nd order recurrence relation,

$c_0a_n + c_1a_{n-1} + c_2a_{n-2} = f(n), n \geq 0$, where $c_0,\ c_1,\ c_2$ are real numbers & $c_0,\ c_2 \neq 0$ ------------------------- $(*)$

Suppose $a_n = \ c_1a_{n-1} + c_2a_{n-2} + f(n)$, where $c_1,\ c_2$ are real numbers , $c_2 \neq 0$ and $f(n) = p(n)r^n$, say $p(n)$ is polynomial.

- When r is not a root of characteristic equation of homogeneous. There is $a_n{}^{(p)} = \ q(n)r^n$, where $q(n)$ is polynomial.
- When r is a root of characteristic equation of homogeneous and its multiplicity is 'm', there is $a_n{}^{(p)} = n^m\ q(n)r^n$, where $q(n)$ is polynomial.

e.g. What form ${a_n}^{(p)}$ for $a_n = 6a_{n-1} - 9a_{n-2} + f(n)$ have:
when $f(n) = 3^n, f(n) = n3^n, \ f(n) = n^2 2^n \ and \ f(n) = (n+1)^2 3^n$?
solution:- 1st find ${a_n}^{(h)}$?
Let $a_n = cr^n$, where $c, r \neq 0$ and substitute on homogeneous. We have
$cr^n = 6cr^{n-1} - 9cr^{n-2}$
$\Rightarrow \ r^n - 6r + 9 = 0$
$\Rightarrow \ (r-3)^n = 0$ has a single root with multiplicity 2. $\therefore \ {a_n}^{(h)} = b_1 3^n + b_2 n3^n$
2nd find ${a_n}^{(p)}$?
Since $r = 3$ is a root with multiplicity $m = 2$, but $r = 2$ is not a root of charactrstic equation for homogeneous.Then
${a_n}^{(p)}$ has the form $n^2 A3^n$ if $f(n) = 3^n$
${a_n}^{(p)}$ has the form $n^2(p_1 n + p_2)3^n$ if $f(n) = n3^n$
${a_n}^{(p)}$ has the form $(p_2 n^2 + p_1 n + p_0)2^n$ if $f(n) = n^2 2^n$
${a_n}^{(p)}$ has the form $n^2(p_2 n^2 + p_1 n + p_0)3^n$ if $f(n) = (n+1)^2 3^n$
$iii)$ To find ${a_n}^{(p)}$ let us use the method of undetermined coefficients, that is, assume ${a_n}^{(p)}$ for the following $f(n)$ on the table: (where c, A, A_0, A_1, A_2 are constants.)

$f(n)$	${a_n}^{(p)}$
C	A
C n	$A_0 + A_1 n$
C n^2	$A_0 + A_1 n + A_2 n^2$
$A sin\ n\theta$ or $A cos\ n\theta$	$A_1 sin\ n\theta \ + \ A_2 cos\ n\theta$
$A\, r^n sin\ n\theta$ or $A\, r^n cos\ n\theta$	$A_1 r^n sin\ n\theta \ + \ A_2 r^n cos\ n\theta$
$An^m r^n$	$r^n(A_0 + A_1 n + \ --- \ + A_m n^m)$

Example:- Find the general solution of a) $a_n = 3a_{n-1} + 2n, \ a_1 = 3$
b) $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$
solution:- a) Let ${a_n}^{(h)} = Ar^n$, where A is constant, $r \neq 0$. and substitute this on the homogeneous $a_n = 3a_{n-1}$, we have ${a_n}^{(h)} = A3^n$
since $f(n) = 2n$ is a polynomial with degree 1, ${a_n}^{(p)}$ is a linear function in n,
i.e, ${a_n}^{(p)} = c\,n + d$, where c, d are constants.Then
$a_n = 3a_{n-1} + 2n$ becomes $c\,n + d = 3(c(n-1) + d) + 2n$

$$\Rightarrow c\,n + d = 3c\,n - 3c + 3d + 2n$$
$$\Rightarrow (3c + 2 - c)n - 3c = d - 3d$$
$$\Rightarrow (2c + 2)n + (2d - 3c) = 0$$
$$\Rightarrow (2c + 2) = 1 \text{ and } (2d - 3c) = 0$$
$$\Rightarrow c = -1, \quad d = \frac{-3}{2}$$

Thus $a_n{}^{(p)} = -n - \frac{3}{2}$ is a particular solution.

All solutions are the form $a_n = a_n{}^{(h)} + a_n{}^{(p)} = b_1 3^n - n - \frac{3}{2}$

To find b_1:- use initial condition $a_1 = 3$

$a_1 = 3 = b_1 3^1 - 1 - \frac{3}{2} \quad \Longrightarrow 3b_1 = 3 + 1 + \frac{3}{2} \quad \Longrightarrow b_1 = \frac{11}{6}$

$\therefore \; a_n = \frac{11}{6} 3^n - n - \frac{3}{2}$ is a general solution.

b) To find $a_n{}^{(h)}$

Let $a_n = cr^n$, where $c, r \neq 0$ and substitute on $a_n = 5a_{n-1} - 6a_{n-2}$, we have

$cr^n = 5cr^{n-1} - 6cr^{n-2}$

$\Longrightarrow r^2 - 5r + 6 = 0$

$\Longrightarrow r = \frac{5 \pm \sqrt{25-24}}{2}$ Thus the roots are $r_1 = 3,\; r_2 = 2$

$a_n{}^{(h)} = b_1(3^n) + b_2(2^n)$

Since $f(x) = 7^n, a_n{}^{(h)} = A(7^n)$, where A is constant. Then substitute on

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$, we have $A7^n = 5A7^{n-1} - 6A7^{n-2} + 7^n$

$\Longrightarrow \; A7^2 = 5A7 - 6A + 7^2$

$\Longrightarrow \; (49 - 35 + 6)A = 49$

$\Longrightarrow A = \frac{49}{20}$

Thus $a_n{}^{(p)} = \frac{49}{20}(7^n)$

$a_n = a_n{}^{(h)} + a_n{}^{(p)} = b_1 3^n + b_2 2^n + \frac{49}{20}(7^n)$